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Function series theory of time scales[☆]

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ABSTRACT

In this paper, we extend the concept of function series to time scales, and then we present two necessary and sufficient conditions and several criteria for uniform convergence. Moreover, we demonstrate several analytical properties of function series on general time scales. Furthermore, we illustrate our conclusions through examples, respectively.

1. Introduction

The theory of time scales was initiated by Stefan Hilger [1,2] in order to unify continuous and discrete analysis, which has recently received a lot of attention. It has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in population dynamics, economics, physics and so on [2–6]. It is known that the basic theorems of derivative and integral for functions $f : \mathbb{T} \to \mathbb{R}$ are introduced in [7–9]. And many useful results such as mathematical induction and L'Hôspital's rules on time scales have been obtained [7–14]. Already, many theories of classical mathematical analysis have been extended to time scales. To the best of the authors' knowledge, function series theory on time scales has not been investigated. The aim of our paper is to develop function series theory on time scales.

In the classical mathematical analysis, researchers have studied the problem of the convergence of function series and criteria concerning convergence. Also, the fundamental problem of function series has been studied, namely, whether the sum functions maintain continuity, differentiability, integrability and so on if every term of the function series has the corresponding property [15]. Hence it is especially important to find conditions that are sufficiently convenient in practice and which guarantee that when every term of a function series converges, their derivatives or integrals also converge to the derivative or integral of the sun function. However, the corresponding results on time scales have not been developed. Motivated by the discussion above, we provide analytical properties of function series on time scales in this paper.

The rest of the paper is arranged as follows. We provide concepts concerning function series on time scales, and then further demonstrate two necessary and sufficient conditions and several criteria for uniform convergence in Section 2. Section 3 is devoted to the discussion of analytical properties of function series on time scales. Finally, we close the paper with conclusions in Section 4.

2. Function series

In this section, we start with some concepts of function series on time scales. Since the analytical properties of function series studied in this paper are built on uniform convergence, the definition and two necessary and sufficient

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conditions for uniform convergence are stated in the following. At the end of this section, criteria for uniform convergence are provided.

Let $u_n(t) : \mathbb{T} \to \mathbb{R},\ n = 1, 2, \ldots$. Then we consider the following infinite sum:

$$\sum_{n=1}^{\infty} u_n(t). \tag{1}$$

If the numerical series $\sum_{n=1}^{\infty} u_n(t_0),\ t_0 \in \mathbb{T}$ is convergent, then t_0 is called a convergent point of the function series (1). The set of all convergent points is called its convergence domain D. And if a convergence domain of a function series is not empty, then the function series is called pointwise convergence on its convergence domain. We define $U(t) = \sum_{n=1}^{\infty} u_n(t),\ t \in D$ as the sum function.

Definition 1. The function

$$U_n(t) = u_1(t) + u_2(t) + \cdots + u_n(t)$$

is called the partial sum or, more precisely, the nth partial sum of the function series (1).

Now the important definition investigated in this paper is provided in the following.

Definition 2. Suppose that the partial sum sequence $\{U_n(t)\}$ of (1) is uniformly convergent on its convergence domain, i.e., for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|U(t)_n - U(t)| < \varepsilon \quad \text{for all } t \in D.$$

Then $\{U_n(t)\}$ is called uniformly convergent to $U(t)$, i.e., $U_n(t) \rightrightarrows U(t)$ on D as $n \to \infty$.

If the partial sum sequence $U_n(t)$ of (1) is uniformly convergent to $U(t)$ on D, then we term $\sum_{n=1}^{\infty} u_n(t)$ as uniformly convergent to $U(t)$.

Making use of Definition 2, we present the following example.

Example 1. Assume that $U_n(t) = \sqrt{t^2 + \frac{1}{n^2}}$. Show that $\{U_n(t)\}$ is uniformly convergent to $U(t) = t$ on $D = \{q^k : k \in \mathbb{Z}\} \bigcup \{0\} (q > 1)$.

Since

$$|U_n(t) - U(t)| = \left| \sqrt{t^2 + \frac{1}{n^2}} - t \right| = \frac{1}{n^2 \left(\sqrt{t^2 + \frac{1}{n^2}} + t \right)} \le \frac{1}{n},$$

then for any given $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon}$ such that $n > N$ implies

$$|U_n(t) - U(t)| \le \frac{1}{n} < \varepsilon,$$

for all $t \in D$. Hence $\{U_n(t)\}$ is uniformly convergent to $U(t) = t$ on D.

Now let us take a further step by providing two necessary and sufficient conditions for uniform convergence. They will play an important role in judging the uniform convergence of the function series.

Theorem 1. *Suppose that the function sequence $\{U_n(t)\}$ is pointwise convergent to $U(t)$ on D and the distance of $U_n(t)$ and $U(t)$ is defined as*

$$d(U_n, U) = \sup_{t \in D} |U_n(t) - U(t)|.$$

Then $\{U_n(t)\}$ is uniformly convergent to $U(t)$ on D if and only if

$$\lim_{n \to \infty} d(U_n, U) = 0.$$

Proof. Assume that $\{U_n(t)\}$ converges uniformly to $U(t)$ on D. Then for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|U_n(t) - U(t)| < \frac{\varepsilon}{2} \quad \text{for all } t \in D.$$

Hence

$$d(U_n, U) \le \frac{\varepsilon}{2} < \varepsilon \quad \text{for } n > N.$$

Thus

$$\lim_{n\to\infty} d(U_n, U) = 0.$$

Conversely, let $\lim_{n\to\infty} d(U_n, U) = 0$. For any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$d(U_n, U) < \varepsilon.$$

This yields

$$|U_n(t) - U(t)| < \varepsilon,$$

for all $t \in D$. Therefore $\{U_n(t)\}$ is uniformly convergent to $U(t)$ on D and the proof is finished. □

The following example is an application of Theorem 1.

Example 2. Consider $\{U_n(t)\}$ in Example 1.

We note that

$$|U_n(t) - U(t)| = \frac{1}{n^2\left(\sqrt{t^2 + \frac{1}{n^2}} + t\right)} \le \frac{1}{n}.$$

Thus

$$d(U_n, U) = \frac{1}{n} \to 0 \quad (n \to \infty).$$

By employing Theorem 1, we get that $\{U_n(t)\}$ is uniformly convergent to $U(t) = t$ on D.

Theorem 2. *Assume that the function sequence $\{U_n(t)\}$ is pointwise convergent to $U(t)$ on D. Then $\{U_n(t)\}$ is uniformly convergent to $U(t)$ on D if and only if for an arbitrary sequence $\{t_n\},\ t_n \in D$,*

$$\lim_{n\to\infty} (U_n(t_n) - U(t_n)) = 0.$$

Proof. Necessity. Suppose that $\{U_n(t)\}$ is uniformly convergent to $U(t)$ on D. Then

$$d(U_n, U) = \sup_{t\in D} |U_n(t) - U(t)| \to 0 \quad (n \to \infty).$$

Hence for any sequence $\{t_n\},\ t_n \in D$,

$$|U_n(t_n) - U(t_n)| \le d(U_n, U) \to 0 \quad (n \to \infty),$$

i.e., $\lim_{n\to\infty}(U_n(t_n) - U(t_n)) = 0$.

The proof of sufficiency is by contradiction. Suppose that $\{U_n(t)\}$ is non-uniformly convergent to $U(t)$ on D. Hence there exists $\varepsilon_0 > 0$ such that for any given $N > 0$ there are $n > N,\ t \in D$ implying

$$|U_n(t) - U(t)| \ge \varepsilon_0.$$

Therefore we have the following:

For $N_1 = 1$, there exist $n_1 > 1,\ t_{n_1} \in D$ such that

$$|U_{n_1}(t_{n_1}) - U(t_{n_1})| \ge \varepsilon_0.$$

For $N_2 = n_1$, there exist $n_2 > n_1,\ t_{n_2} \in D$ such that

$$|U_{n_2}(t_{n_2}) - U(t_{n_2})| \ge \varepsilon_0.$$

$\cdots$.

For $N_k = n_{k-1}$, there exist $n_k > n_{k-1},\ t_{n_k} \in D$ such that

$$|U_{n_k}(t_{n_k}) - U(t_{n_k})| \ge \varepsilon_0.$$

$\cdots$.

For $m \in \mathbb{N}^+ \setminus \{n_1, n_2, \ldots, n_k, \ldots\}$, let $t_m \in D$; then we obtain the sequence $\{t_n\},\ t_n \in D$. Since the subsequence $\{t_{n_k}\}$ of $\{t_n\}$ satisfies

$$|U_{n_k}(t_{n_k}) - U(t_{n_k})| \ge \varepsilon_0,$$

this leads to a contradiction due to

$$\lim_{n\to\infty} (U_n(t_n) - U(t_n)) = 0.$$

This yields our desired result. □

Now we show the following example as an application of Theorem 2.

Example 3. Consider $U_n(t) = nt(1-t)^n$ defined on $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \bigcup \{0\}$.
$U_n(t) = nt(1-t)^n$, $U_n(t)$ is convergent to $U(t) = 0$ on D.
Choosing $t_n = \frac{1}{n} \in D$, we have

$$U_n(t_n) - U(t_n) = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n \nrightarrow 0 \quad (n \to \infty).$$

This shows that $\{U_n(t)\}$ is non-uniformly convergent to $U(t) = 0$ on D.

We can see that judging the uniform convergence of function series by using Definition 2, and Theorems 1 and 2, one needs to know its sum function: the nature of the convergence of the function series is identified with the nature of the convergence of its sequence of partial sums. But in many cases, getting the sun function is difficult or even impossible. Hence it is necessary to find principles that do not need the sum function. Now we will present and prove the following well-known criterion.

Theorem 3 (*Cauchy Criterion for Uniform Convergence of a Function Series*)**.** *The function series $\sum_{n=1}^{\infty} u_n(t)$ converges uniformly on D if and only if for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|u_{n+1}(t) + \cdots + u_m(t)| < \varepsilon$$

for all natural numbers m, n satisfying $m > n > N$ and every point $t \in D$.

Proof. Necessity. Suppose that the function series $\sum_{n=1}^{\infty} u_n(t)$ converges uniformly on D and the sum function is $U(t)$. Then for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\left|U(t) - \sum_{k=1}^{n} u_k(t)\right| < \frac{\varepsilon}{2} \quad \text{for all } t \in D.$$

Hence, for all $m > n > N$ and all $t \in D$, we have

$$\begin{aligned} |u_{n+1}(t) + \cdots + u_m(t)| &= \left|\sum_{k=1}^{m} u_k(t) - \sum_{k=1}^{n} u_k(t)\right| \\ &\le \left|\sum_{k=1}^{m} u_k(t) - U(t)\right| + \left|U(t) - \sum_{k=1}^{n} u_k(t)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Sufficiency. Suppose that for any given $\varepsilon > 0$ there exists a natural number N such that

$$|u_{n+1}(t) + \cdots + u_m(t)| = \left|\sum_{k=1}^{m} u_k(t) - \sum_{k=1}^{n} u_k(t)\right| < \frac{\varepsilon}{2}$$

for all $m > n > N$ and all $t \in D$. Fix $t \in D$. Then we obtain the general numerical series $\sum_{n=1}^{\infty} u_n(t)$. Hence $\sum_{n=1}^{\infty} u_n(t)$ satisfies the Cauchy criterion for convergence of a numerical series. Thus the series $\sum_{n=1}^{\infty} u_n(t)$ is convergent. Let

$$U(t) = \sum_{n=1}^{\infty} u_n(t) \quad \text{for all } t \in D.$$

Choose n for $\left|\sum_{k=1}^{m} u_k(t) - \sum_{k=1}^{n} u_k(t)\right| < \frac{\varepsilon}{2}$. Let $m \to \infty$; we get

$$\left|U(t) - \sum_{k=1}^{n} u_k(t)\right| \le \frac{\varepsilon}{2} < \varepsilon \quad \text{for all } t \in D.$$

Therefore $\sum_{n=1}^{\infty} u_n(t)$ converges uniformly to $U(t)$ on D. □

Remark 1. If under the hypothesis of Theorem 3 all the functions $u_n(t)$ are constant, we obtain the familiar Cauchy criterion for convergence of a numerical series $\sum_{n=1}^{\infty} u_n$.

Corollary 1 (*Necessary Condition for Uniform Convergence of a Function Series*)**.** *A necessary condition for the series $\sum_{n=1}^{\infty} u_n(t)$ to converge uniformly on D is that $u_n \rightrightarrows 0$ on D as $n \to \infty$.*

Our first application of the Cauchy criterion for uniform convergence of a function series yields the following example.

Example 4. Prove that the function series $\sum_{n=0}^{\infty} \frac{t^2}{(1+t^2)^n}$ is non-uniformly convergent on $D = \left\{ \frac{1}{\sqrt{n}} : n \in \mathbb{N} \right\} \bigcup \{0\}$.

Define $u_n(t) = \frac{t^2}{(1+t^2)^n}$. Note that

$$\sum_{k=n+1}^{2n} u_k(t) = \frac{t^2}{(1+t^2)^{n+1}} + \frac{t^2}{(1+t^2)^{n+2}} + \cdots + \frac{t^2}{(1+t^2)^{2n}} > \frac{nt^2}{(1+t^2)^{2n}}.$$

Let $\varepsilon_0 = \frac{1}{e^2} > 0$. For any $N \in \mathbb{N}$, choose $m = 2n$ $(n > N)$ and $t_n = \frac{1}{\sqrt{n}} \in D$ implying

$$\sum_{k=n+1}^{2n} u_k(t_n) > \frac{nt_n^2}{(1+t_n^2)^{2n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{2n}} > \frac{1}{e^2} = \varepsilon_0.$$

By using Theorem 3, we obtain that $\sum_{n=0}^{\infty} \frac{t^2}{1+t^{2\,n}}$ is non-uniformly convergent on D.

Theorem 4 (*The Weierstrass M-test for Uniform Convergence of a Function Series*)**.** *Suppose that every term $u_n(t)$ of the function series $\sum_{n=1}^{\infty} u_n(t)$ satisfies*

$$|u_n(t)| \le a_n \quad \textit{for } n = 1, 2, \ldots, \textit{ and all } t \in D$$

and the numerical series $\sum_{n=1}^{\infty} a_n$ is convergent. Then the function series $\sum_{n=1}^{\infty} u_n(t)$ converges uniformly on D.

Proof. Note that the numerical series $\sum_{n=1}^{\infty} a_n$ is convergent. By using the Cauchy criterion, we get that for any given $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$a_{n+1} + a_{n+2} + \cdots + a_m < \varepsilon \quad \text{for all } m > n > M.$$

For all $t \in D$ and all natural numbers $m > n$, we have

$$\begin{aligned} |u_{n+1}(t) + \cdots + u_m(t)| &\le |u_{n+1}(t)| + \cdots + |u_m(t)| \\ &\le a_{n+1} + \cdots + a_m \\ &< \varepsilon. \end{aligned}$$

Then an application of Theorem 3 implies that $\sum_{n=1}^{\infty} u_n(t)$ converges uniformly on D. □

The Weierstrass M-test is the simplest and at the same time the most frequently used sufficient condition for uniform convergence of a function series.

Example 5. Show that the function series $\sum_{n=0}^{\infty} \frac{t}{1+n^3t^2}$, $t \in D = \mathbb{P}_{1,2} = \bigcup_{k=0}^{\infty}[3k, 3k+1]$ is uniformly convergent.

Let $u_n(t) = \frac{t}{1+n^3t^2}$. When $n \ge 1$ we have

$$|u_n(t)| \le \frac{t}{2n^{\frac{3}{2}}t} = \frac{1}{2n^{\frac{3}{2}}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{3}{2}}}$ is convergent, by Theorem 4 we get that $\sum_{n=0}^{\infty} \frac{t}{1+n^3t^2}$ is uniformly convergent on D.

In order to present the next criterion, a well-known lemma is required.

Lemma 1 (*Abel Lemma*)**.** *Suppose that:*

(1) $\{u_k\}$ *is a monotonic sequence;*

(2) $\{V_k\}$ $\left(V_k = \sum_{i=1}^{k} v_i,\ k = 1, 2, \ldots\right)$ *is a bounded sequence, namely, there exists a $M > 0$ such that $|V_k| \le M$ for all k.*

Then

$$\left|\sum_{k=1}^{p} u_k v_k\right| \le M(|u_1| + 2|u_p|).$$

The following pair of related sufficient conditions for uniform convergence of a series are somewhat more specialized. But these conditions are more delicate than the Weierstrass M-test, since they make it possible to investigate series $\sum_{n=1}^{\infty} u_n(t)v_n(t),\ t \in D$, that converge.

Theorem 5. *Assume that function series $\sum_{n=1}^{\infty} u_n(t)v_n(t),\ t \in D$, satisfies one of the following conditions. Then $\sum_{n=1}^{\infty} u_n(t)v_n(t)$ is uniformly convergent on D.*

(1) *(Abel Test) The function sequence $\{u_n(t)\}$ is monotonic for each fixed $t \in D$ with respect to n and $\{u_n(t)\}$ is uniformly bounded on D:*

$$|u_n(t)| \le M \quad \text{for } t \in D \text{ and } n \in \mathbb{N}^+;$$

in addition, the function series $\sum_{n=1}^{\infty} v_n(t)$ is uniformly convergent on D.

(2) *(Dirichlet Test) The function sequence $\{u_n(t)\}$ is monotonic for each fixed $t \in D$ with respect to n and $\{u_n(t)\}$ is uniformly convergent to 0; in addition, the partial sum sequence of $\sum_{n=1}^{\infty} v_n(t)$ is uniformly bounded on D:*

$$\left|\sum_{k=1}^{n} v_k(t)\right| \le M \quad \text{for } t \in D \text{ and } n \in \mathbb{N}^+.$$

Proof. (1) Since $\sum_{n=1}^{\infty} v_n(t)$ is uniformly convergent on D, then for any given $\varepsilon > 0$ there exists a positive integer N such that

$$\left|\sum_{k=n+1}^{m} v_k(t)\right| < \varepsilon,$$

for all $m > n > N$ and all $t \in D$. By applying the Abel Lemma, we obtain

$$\left|\sum_{k=n+1}^{m} u_k(t)v_k(t)\right| \le \varepsilon(|u_{n+1}(t)| + 2|u_m(t)|) \le 3M\varepsilon,$$

for all $m > n > N$ and all $t \in D$. Employing Theorem 3, we get that $\sum_{n=1}^{\infty} u_n(t)v_n(t)$ is uniformly convergent on D. This proves the Abel Test.

(2) Note that $\{u_n(t)\}$ is uniformly convergent to 0. For any given $\varepsilon > 0$ there exists a positive integer N such that $n > N$ implies

$$|u_n(t)| < \varepsilon \quad \text{for all } t \in D.$$

The partial sum sequence of $\sum_{n=1}^{\infty} v_n(t)$ is uniformly bounded; then for all $m > n > N$,

$$\left|\sum_{k=n+1}^{m} v_k(t)\right| = \left|\sum_{k=1}^{m} v_k(t) - \sum_{k=1}^{n} v_k(t)\right| \le 2M.$$

Using the Abel Lemma, we get

$$\left|\sum_{k=n+1}^{m} u_k(t)v_k(t)\right| \le 2M(|u_{n+1}(t)| + 2|u_m(t)|) < 6M\varepsilon,$$

for all $t \in D$. By Theorem 3, $\sum_{n=1}^{\infty} u_n(t)v_n(t)$ is uniformly convergent on D. This completes the proof of the Dirichlet Test. □

Now we present two examples by using the Abel Test and the Dirichlet Test.

Example 6. Suppose that $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent. Then $\sum_{n=1}^{\infty} \frac{e^{-nt}}{n!}$ is uniformly convergent on $D = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right]$.

It is clear that $\{e^{-nt}\}$ is monotonic with respect to n and

$$|e^{-nt}| \le 1 \quad \text{for all } t \in D \text{ and all } n.$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n!}$ is a numerical series. Then the convergence of $\sum_{n=1}^{\infty} \frac{1}{n!}$ means that it is uniformly convergent with respect to t.

By the Abel Test, $\sum_{n=1}^{\infty} \frac{e^{-nt}}{n!}$ is uniformly convergent on D.

Example 7. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+t^2}$ is uniformly convergent on $D = \mathbb{N}_0 \bigcup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$.

Assume that $u_n(t) = \frac{1}{n+t^2}$ and $v_n(t) = (-1)^n$. Then $\{u_n(t)\}$ is monotonic for each fixed $t \in D$ with respect to n and it is uniformly convergent to 0 on D.

We can easily obtain that

$$\left|\sum_{k=1}^{n} v_k(t)\right| \le 1,$$

i.e., $\{v_n(t)\}$ is uniformly bounded. Using the Dirichlet Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+t^2}$ is uniformly convergent on D.

3. Statement of the fundamental problems

In Section 2 we presented some related concepts of function series on time scales and proved several principles for uniform convergence of a function series. In this section, we discuss the fundamental problems of function series, namely, what kinds of functional properties the sum function has. The most important properties for analysis are continuity, differentiability, and integrability. As before, we will present and prove several lemmas and theorems.

Lemma 2. *Let $D \in \mathbb{T}$. Assume that every term $U_n(t) : D \to \mathbb{R},\ n \in \mathbb{N}$, of the function sequence $\{U_n(t)\}$ is rd-continuous and $\{U_n(t)\}$ is uniformly convergent to function $U(t) : D \to \mathbb{R}$ on D. Then $U(t)$ is rd-continuous.*

Proof. The discontinuous points of an rd-continuous function are the points that are left-dense and right-scattered. Thus we only need to discuss this kind of points.

Let $t_0 \in D \subset \mathbb{T}$ be left-dense and right-scattered. For any given $\varepsilon > 0$ there exists a natural number M such that

$$|U_M(t) - U(t)| < \frac{\varepsilon}{3} \quad \text{for all } t \in D. \tag{2}$$

Since function $U_n(t)$ is rd-continuous, we have for the above $\varepsilon > 0$ that there exists $\delta > 0$ such that

$$|U_M(t') - U_M(t'')| < \frac{\varepsilon}{3} \quad \text{for all } t', t'' \in (t_0 - \delta, t_0)_{\mathbb{T}}. \tag{3}$$

Assume that $t_n \to t_0^-$, $n \to \infty$, $t_n \in \mathbb{T},\ n = 1, 2, \ldots$. Then for $\delta > 0$ above, there is a natural number N such that $n, m > N$ implies $t_n, t_m \in (t_0 - \delta, t_0)_{\mathbb{T}}$. Hence, by using (3), we get

$$|U_M(t_n) - U_M(t_m)| < \frac{\varepsilon}{3} \quad \text{for all } t_n, t_m \in (t_0 - \delta, t_0)_{\mathbb{T}}. \tag{4}$$

Therefore for $n, m > N$,

$$\begin{aligned} |U(t_n) - U(t_m)| &\le |U(t_n) - U_M(t_n)| + |U_M(t_n) - U_M(t_m)| + |U_M(t_m) - U(t_m)| \\ &< \varepsilon. \end{aligned}$$

Above, we have used (2) and (4). This proves that $\lim_{t \to t_0^-} U(t)$ exists and is finite. This yields our desired result. □

Theorem 6. *Assume that every term $u_n(t)$ of the function series $\sum_{n=1}^{\infty} u_n(t)$ is rd-continuous and $\sum_{n=1}^{\infty} u_n(t)$ is uniformly convergent to function $U(t)$ on D. Then $U(t)$ is rd-continuous on D.*

Proof. Consider $\{U_n(t)\}$ in Lemma 2 as the partial sum sequence of the function series $\sum_{n=1}^{\infty} u_n(t)$; this is a direct consequence of Lemma 2. □

Example 8. Consider the function series $\sum_{n=0}^{\infty} \frac{t}{1+n^3t^2}$ in Example 5.

We can easily note that, for each n, $u_n(t) = \frac{t}{1+n^3t^2}$ is rd-continuous on $D = \mathbb{P}_{1,2} = \bigcup_{k=0}^{\infty}[3k, 3k+1]$ and $\sum_{n=0}^{\infty} \frac{t}{1+n^3t^2}$ is uniformly convergent on D.

By employing the continuity theorem (Theorem 6), we get that $\sum_{n=0}^{\infty} \frac{t}{1+n^3t^2}$ is rd-continuous on $D = \mathbb{P}_{1,2}$.

We shall show that if functions that are rd-continuous on $[a, b]_{\mathbb{T}}$ converge uniformly on that interval, then the limit function is integrable and its integral over that interval equals the limit of the integrals of the original functions.

Lemma 3. *Suppose that every term $U_n(t) : [a, b]_{\mathbb{T}} \to \mathbb{R}, n \in \mathbb{N}$ of the function sequence $\{U_n(t)\}$ is rd-continuous and $\{U_n(t)\}$ is uniformly convergent to function $U(t) : [a, b]_{\mathbb{T}} \to \mathbb{R}$. Then $U(t)$ is integrable on $[a, b]_{\mathbb{T}}$ and*

$$\int_a^b U(t)\Delta t = \lim_{n\to\infty} \int_a^b U_n(t)\Delta t.$$

Proof. By Lemma 2 we know that $U(t)$ is rd-continuous, so it is integrable. Then for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|U_n(t) - U(t)| \le \frac{\varepsilon}{b - a}.$$

Hence

$$\left| \int_a^b U_n(t)\Delta t - \int_a^b U(t)\Delta t \right| \le \int_a^b |U_n(t) - U(t)|\Delta t \le \frac{\varepsilon}{b-a}(b - a) = \varepsilon.$$

Thus

$$\int_a^b U(t)\Delta t = \lim_{n\to\infty}\int_a^b U_n(t)\Delta t.$$

This proves our claim. □

Theorem 7. *Assume that the function series $\sum_{n=1}^{\infty} u_n(t) = U(t)$ is uniformly convergent and every term $u_n(t)$ of $\sum_{n=1}^{\infty} u_n(t)$ is rd-continuous on $[a,b]_{\mathbb{T}}$. Then the sum function $U(t)$ is integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b U(t)\Delta t = \int_a^b \sum_{n=1}^{\infty} u_n(t)\Delta t = \sum_{n=1}^{\infty}\int_a^b u_n(t)\Delta t.$$

Proof. Like in Theorem 6, we can get the conclusion by letting $\{U_n(t)\}$ in Lemma 3 be the partial sum sequence of the function series $\sum_{n=1}^{\infty} u_n(t)$. This gives us the desired result. □

Example 9. By Example 6, $\sum_{n=1}^{\infty}\frac{e^{-nt}}{n!}$ is uniformly convergent on $D = \left[0,\frac{1}{3}\right]\bigcup\left[\frac{2}{3},1\right]$ and, for each n, $u_n(t) = \frac{e^{-nt}}{n!}$ is rd-continuous on D.

Employing Theorem 7, we have that $\sum_{n=1}^{\infty}\frac{e^{-nt}}{n!}$ is integrable on D and

$$\begin{aligned}
\int_0^1 \sum_{n=1}^{\infty} u_n(t)\Delta t &= \sum_{n=1}^{\infty}\int_0^1 u_n(t)\Delta t \\
&= \sum_{n=1}^{\infty}\int_0^{\frac{1}{3}}\frac{e^{-nt}}{n!}dt + \sum_{n=1}^{\infty}\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{e^{-nt}}{n!}\Delta t + \sum_{n=1}^{\infty}\int_{\frac{2}{3}}^{1}\frac{e^{-nt}}{n!}dt \\
&= \sum_{n=1}^{\infty}\frac{1-e^{-\frac{n}{3}}}{n\cdot n!} + \sum_{n=1}^{\infty}\frac{e^{-\frac{n}{3}}}{3\cdot n!} + \sum_{n=1}^{\infty}\frac{e^{-\frac{2n}{3}} - e^{-n}}{n\cdot n!} \\
&= \sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{1}{n} - \frac{1}{n}e^{-\frac{n}{3}} + \frac{1}{3}e^{-\frac{n}{3}} + \frac{1}{n}e^{-\frac{2n}{3}} - \frac{1}{n}e^{-n}\right] \\
&= \sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\left[1 + \frac{n-3}{3}e^{-\frac{n}{3}} - e^{-n}\right].
\end{aligned}$$

Let us now take a further step by considering the differentiability of the limit function, and then study the corresponding properties of the sum function of the function series.

Lemma 4. *Suppose that the function sequence $\{U_n(t)\}$, $U_n(t) : [a,b]_{\mathbb{T}} \to \mathbb{R},\ n\in\mathbb{N}$, satisfies the following conditions:*

(1) *$U_n(t),\ t\in[a,b]_{\mathbb{T}},\ n\in\mathbb{N}$, is differentiable and its derived function $U_n^{\Delta}(t)$ is rd-continuous on $[a,b]_{\mathbb{T}}$;*
(2) *$U_n(t)$ converges pointwise to $U(t)$ on $[a,b]_{\mathbb{T}}$;*
(3) *$U_n^{\Delta}(t)$ is uniformly convergent to $W(t)$ on $[a,b]_{\mathbb{T}}$.*

Then $U(t)$ is differentiable on $[a,b]_{\mathbb{T}}$ and

$$U^{\Delta}(t) = W(t) \quad \text{for all } t\in[a,b]_{\mathbb{T}}.$$

Proof. By Lemma 2 we know that $W(t)$ is rd-continuous, so it is integrable. Then by Lemma 3 we get

$$\begin{aligned}
\int_a^t W(r)\Delta r &= \int_a^t \lim_{n\to\infty} U_n^{\Delta}(r)\Delta r \\
&= \lim_{n\to\infty}\int_a^t U_n^{\Delta}(r)\Delta r \\
&= \lim_{n\to\infty}[U_n(t) - U_n(a)] \\
&= U(t) - U(a).
\end{aligned}$$

The left side of the above formula is differentiable, so the right side is also differentiable and this leads to

$$U^{\Delta}(t) = W(t) \quad \text{for all } t\in[a,b]_{\mathbb{T}}.$$

This completes the proof. □

Theorem 8. *Suppose that the function series* $\sum_{n=1}^{\infty} u_n(t)$ *satisfies:*

(1) $u_n(t)$ *is differentiable and its derived function* $u_n^{\Delta}(t)$ *is rd-continuous on* $[a, b]_{\mathbb{T}}$*;*
(2) $\sum_{n=1}^{\infty} u_n(t)$ *converges pointwise to* $U(t)$ *on* $[a, b]_{\mathbb{T}}$*;*
(3) $\sum_{n=1}^{\infty} u_n^{\Delta}(t)$ *is uniformly convergent to* $V(t)$ *on* $[a, b]_{\mathbb{T}}$.

Then $U(t) = \sum_{n=1}^{\infty} u_n(t)$ *is differentiable on* $[a, b]_{\mathbb{T}}$ *and*

$$U^{\Delta}(t) = \left\{\sum_{n=1}^{\infty} u_n(t)\right\}^{\Delta} = \sum_{n=1}^{\infty} u_n^{\Delta}(t) = V(t) \quad \text{for all } t \in [a, b]_{\mathbb{T}}.$$

Proof. Like in Theorem 7, we consider $\{U_n(t)\}$ in Lemma 4 as the partial sum sequence of the function series $\sum_{n=1}^{\infty} u_n(t)$. Then an application of Lemma 4 yields our desired result. □

Example 10. Show that $\sum_{n=0}^{\infty} \frac{1}{2^n+t}$ is differentiable on $D = \mathbb{N}$.

First, we calculate the derivative of $u_n(t)$, i.e.,

$$u_n^{\Delta}(t) = \frac{u_n(\sigma(t)) - u_n(t)}{\mu(t)} = \frac{\frac{1}{2^n+(t+1)} - \frac{1}{2^n+t}}{1} = -\frac{1}{(2^n+t+1)(2^n+t)}.$$

Then we can easily get that the derived function of $u_n(t)$ is rd-continuous on D. As $\frac{1}{2^n+t} \le \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent, we obtain that $\sum_{n=0}^{\infty} \frac{1}{2^n+t}$ is pointwise convergent on D.

For all $t \in D$, we have

$$|u_n^{\Delta}(t)| = \left|\frac{1}{(2^n+t+1)(2^n+t)}\right| \le \frac{1}{2^{2n}}.$$

Since $\sum_{n=0}^{\infty} \frac{1}{2^{2n}}$ is convergent, $\sum_{n=0}^{\infty} u_n^{\Delta}(t)$ is uniformly convergent on D.

Using Theorem 8, $\sum_{n=0}^{\infty} \frac{1}{2^n+t}$ is differentiable on $D = \mathbb{N}$. Furthermore,

$$\left(\sum_{n=0}^{\infty} \frac{1}{2^n+t}\right)^{\Delta} = \sum_{n=0}^{\infty} \left(\frac{1}{2^n+t}\right)^{\Delta}.$$

We have shown that, by Theorem 6, the rd-continuous function sequence $\{U_n(t)\}$ converges to an rd-continuous function $U(t)$. However, this does not mean that $\{U_n(t)\}$ converges uniformly to $U(t)$. At the same time there is a specific situation in which the convergence of an rd-continuous function sequence to an rd-continuous function guarantees that the convergence is uniform.

Lemma 5 (*Dini Theorem*)**.** *Assume that the function sequence* $\{U_n(t)\}, U_n(t) : [a, b]_{\mathbb{T}} \to \mathbb{R},\ n \in \mathbb{N}$*, converges pointwise to function* $U(t)$ *on the closed interval* $[a, b]_{\mathbb{T}}$*. If the following conditions hold:*

(1) $U_n(t),\ n = 1, 2, \ldots,$ *is rd-continuous on* $[a, b]_{\mathbb{T}}$*;*
(2) $U(t)$ *is rd-continuous on* $[a, b]_{\mathbb{T}}$*;*
(3) *for any given* $t \in [a, b]_{\mathbb{T}}$, $\{U_n(t)\}$ *is monotonic with respect to* $n \in \mathbb{N}$,

then $\{U_n(t)\}$ *is uniformly convergent to* $U(t)$ *on* $[a, b]_{\mathbb{T}}$.

Proof. The proof is by contradiction. Suppose that $\{U_n(t)\}$ is non-uniformly convergent to $U(t)$ on $[a, b]_{\mathbb{T}}$. Then there exists $\varepsilon_0 > 0$ such that for any given $N \in \mathbb{N}$ there exist $n > N,\ t \in [a, b]_{\mathbb{T}}$ implying

$$|U_n(t) - U(t)| \ge \varepsilon_0.$$

Then for $N = 1$ there exist $n_1 > 1$, $t_1 \in [a, b]_{\mathbb{T}}$ such that

$$|U_{n_1}(t_1) - U(t_1)| \ge \varepsilon_0.$$

And for $N = n_1$ there exist $n_2 > n_1$, $t_2 \in [a, b]_{\mathbb{T}}$ such that

$$|U_{n_2}(t_2) - U(t_2)| \ge \varepsilon_0.$$

$\ldots$.

Thus for $N = n_{k-1}$ there exist $n_k > n_{k-1}$, $t_k \in [a, b]_{\mathbb{T}}$ such that

$$|U_{n_k}(t_k) - U(t_k)| \ge \varepsilon_0.$$

$\ldots$.

Hence we obtain a point sequence $\{t_k\}$, $t_k \in [a, b]_{\mathbb{T}}$. Then the sequence has a convergent subsequence. Let $t_k \to \xi, k \to \infty$. Note that $\xi \in [a, b]_{\mathbb{T}}$ since $\{t_k\} \subset [a, b]_{\mathbb{T}}$ and $[a, b]_{\mathbb{T}}$ is closed. Because $U_n(\xi) \to U(\xi), n \to \infty$, then for the above $\varepsilon_0 > 0$ there exists $N \in \mathbb{N}$ such that

$$|U_N(\xi) - U(\xi)| < \frac{\varepsilon_0}{2}.$$

Suppose that ξ is left-dense and right-scattered. Then for the above point sequence $\{t_k\} \subset [a, b]_{\mathbb{T}}$ we get $t_k \le \xi$ and $t_k \to \xi, k \to \infty$. Thus for $\varepsilon_0 > 0$ as above there exists $K \in \mathbb{N}$ such that $k > K$ implies

$$|\xi - t_k| < \varepsilon_0.$$

Since $U_N(t)$ and $U(t)$ are rd-continuous, we have

$$|(U_N(t_k) - U(t_k)) - (U_N(\xi) - U(\xi))| < \frac{\varepsilon_0}{2}.$$

Hence

$$|U_N(t_k) - U(t_k)| < \frac{\varepsilon_0}{2} + |U_N(\xi) - U(\xi)| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0.$$

Suppose that ξ is not left-dense and right-scattered. Then ξ is a continuous point of $U_N(t)$ and $U(t)$. Because $t_k \to \xi, k \to \infty$, there is a natural number K_1 such that $k > K_1$ implies

$$|U_N(t_k) - U(t_k)| < \varepsilon_0.$$

By using the monotonicity condition, we get

$$|U_n(t_k) - U(t_k)| \le |U_N(t_k) - U(t_k)| < \varepsilon_0$$

with $n > N, k > \max\{K, K_1\}$. $n_k \to \infty$, so when n is sufficiently large, $n_k > N$ and $k > \max\{K, K_1\}$ are satisfied. Thus

$$|U_{n_k}(t_k) - U(t_k)| < \varepsilon_0.$$

But this leads to a contradiction since

$$|U_{n_k}(t_k) - U(t_k)| \ge \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

This proves the lemma. □

Corresponding to the function series, we get the following theorem.

Theorem 9 (*Dini Theorem*)**.** *Assume that the function series $\sum_{n=1}^{\infty} u_n(t)$ is pointwise convergent to its sum function $U(t)$ on $[a, b]_{\mathbb{T}}$. If the following conditions hold:*

(1) *$u_n(t),\ n = 1, 2, \ldots$, is rd-continuous on $[a, b]_{\mathbb{T}}$;*
(2) *$U(t)$ is rd-continuous on $[a, b]_{\mathbb{T}}$;*
(3) *for any given $t \in [a, b]_{\mathbb{T}}$, the function series $\sum_{n=1}^{\infty} u_n(t)$ is either a positive term series or a negative term series,*

then $\sum_{n=1}^{\infty} u_n(t)$ is uniformly convergent to $U(t)$ on $[a, b]_{\mathbb{T}}$.

Proof. Consider $\{U_n(t)\}$ in Lemma 5 as the partial sum sequence of the function series $\sum_{n=1}^{\infty} u_n(t)$; an application of Lemma 5 yields our desired result. □

Example 11. Assume that $u_n(t), v_n(t)$ are rd-continuous on D and $|u_n(t)| \le v_n(t)$ for all $n \in \mathbb{N}^+$. Show that if $\sum_{n=1}^{\infty} v_n(t)$ is pointwise convergent to an rd-continuous function on D, then $\sum_{n=1}^{\infty} u_n(t)$ is also convergent to an rd-continuous function.

We note that $v_n(t) \ge 0$ and the sum function $\sum_{n=1}^{\infty} v_n(t)$ is continuous on D.
By using Theorem 9, we get that $\sum_{n=1}^{\infty} v_n(t)$ is uniformly convergent on D.
Employing Theorem 3 we obtain that for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} |u_{n+1}(t) + u_{n+2}(t) + \cdots + u_m(t)| &\le |u_{n+1}(t)| + |u_{n+2}(t)| + \cdots + |u_m(t)| \\ &\le v_{n+1}(t) + v_{n+2}(t) + \cdots + v_m(t) \\ &< \varepsilon, \end{aligned}$$

for all $m > n > N$ and all $t \in D$. This implies that $\sum_{n=1}^{\infty} u_n(t)$ is uniformly convergent on D. By Theorem 6, $\sum_{n=1}^{\infty} u_n(t)$ is rd-continuous.

4. Conclusions

Throughout this paper, we extend the concept of the function series to time scales. Two necessary and sufficient conditions and criteria for uniform convergence were stated. We further discuss the fundamental problem of function series on general time scales. Results regarding the most important properties for analysis, namely, continuity, differentiability, and integrability, are obtained. Moreover, we illustrate our conclusions through examples. Finally, we get that function series theory can be extended and applied to time scales.

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