Result.Math. 35 (1999) 3-22
0378-6218/99/020003-20 $ 1.50+0.20/0

Results in Mathematics

Basic Calculus on Time Scales and some of its Applications

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Abstract. The study of dynamic systems on time scales not only unifies continuous and discrete processes, but also helps in revealing diversities in the corresponding results. In this paper we shall develop basic tools of calculus on time scales such as versions of Taylor's formula, l'Hôspital's rule, and Kneser's theorem. Applications of these results in the study of asymptotic and oscillatory behavior of solutions of higher order equations on time scales are addressed. As a further application of Taylor's formula, Abel-Gontscharoff interpolating polynomial on time scales is constructed and best possible error bounds are offered. We have also included notes at the end of each section which indicate further scope of the calculus developed in this paper.

Keywords. Time scale, Taylor's theorem, l'Hôspital's rule, Kneser's theorem, Abel-Gontscharoff interpolation

AMS Subject Classifications. 39A10, 93C70

1 Introduction

A powerful basic tool in mathematical analysis is *Taylor's formula*

$$f(t)=\sum_{k=0}^{n-1}\frac{(t-a)^k}{k!}f^{(k)}(a)+\frac{1}{(n-1)!}\int_a^t(t-\tau)^{n-1}f^{(n)}(\tau)\mathrm{d}\tau, \tag{1}$$

which is valid for any n-times differentiable function $f:\ \mathbb{R}\to\mathbb{R}$. In recent years the study of asymptotic and oscillatory behavior of solutions of higher order difference equations has been attracting much attention, and a discrete analog of Taylor's formula was prepared for this purpose in [3]: For any sequence $\{u_m:\ m\in\mathbb{Z}\}\subset\mathbb{R}$, we have

$$u_m=\sum_{k=0}^{n-1}\frac{(m-M)^{(k)}}{k!}\Delta^k u_M+\frac{1}{(n-1)!}\sum_{\tau=M}^{m-n}(m-\tau-1)^{(n-1)}\Delta^n u_\tau, \tag{2}$$

[1]The author is grateful to Alexander von Humboldt Foundation for awarding him a Feodor Lynen Research Fellowship to support this work.

where the usual factorial notation

$$t^{(k)} = t \cdot (t-1) \cdot \ldots \cdot (t-k+1)$$

is used. This formula together with its various consequences has proved to be a very useful tool in the study of discrete boundary value problems (see also [9]).

The main result of this paper is a unification of the continuous and discrete Taylor's formulae, which is at the same time an extension to the case of a so-called *time scale*. The theory of such time scales (or *measure chains*) was initiated by S. Hilger in [12] (see also [10]), and an up-to-date monograph on the subject has been published recently by B. Kaymakçalan, V. Lakshmikantham, and S. Sivasundaram [13]. A time scale $\mathbf{T}$ is a closed subset of the reals, and for a function $f : \mathbf{T} \to \mathbb{R}$ it is possible to introduce a derivative f^{Δ} and an integral $\int_a^b f(\tau)\Delta\tau$ in such a manner that $f^{\Delta} = f'$ and $\int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)d\tau$ in the case $\mathbf{T} = \mathbb{R}$, and $f^{\Delta} = \Delta f$ and $\int_a^b f(\tau)\Delta\tau = \sum_{\tau=a}^{b-1} f(\tau)$ in the case $\mathbf{T} = \mathbb{Z}$. Given a time scale $\mathbf{T}$, we present the construction of certain functions $h_k : \mathbf{T} \times \mathbf{T} \to \mathbb{R}$ such that the Taylor's formula on time scales

$$f(t) = \sum_{k=0}^{n-1} h_k(t,a) f^{\Delta^k}(a) + \int_a^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau \tag{3}$$

holds for any n-times differentiable function $f : \mathbf{T} \to \mathbb{R}$. In order that the reader sees how (1) and (2) follow from (3), it is at this point only necessary to know that

$$h_k(t,a) = \frac{(t-a)^k}{k!}, \quad \rho(t) = \sigma(t) = t \quad \text{if} \quad \mathbf{T} = \mathbb{R}$$

and

$$h_k(t,a) = \frac{(t-a)^{(k)}}{k!}, \quad \rho(t) = t-1,\ \sigma(t) = t+1 \quad \text{if} \quad \mathbf{T} = \mathbb{Z}.$$

Besides the advantage that (3) unifies formulae (1) and (2), it can also be applied to time scales $\mathbf{T}$ that are different from $\mathbb{R}$ and $\mathbb{Z}$; for example, $\mathbf{T} = h\mathbb{Z}$ with $h > 0$, or

$$\mathbf{T} = \{a_k :\ 0 \le k \le m\} \subset \mathbb{R} \quad \text{with} \quad m \in \mathbb{N}$$

(in this case formula (3) reduces to the well-known Newton's divided differences formula), or

$$\mathbf{T} = \left\{a^k :\ k \in \mathbb{Z}\right\} \cup \{0\} \quad \text{with} \quad a > 1$$

(see Example 1 below). Furthermore, there might be other time scales that we cannot appreciate at this moment due to our current lack of "real-world" examples.

Of course formula (3) offers the possibility of studying various analytic aspects on time scales. Another purpose of this paper is to apply the established Taylor's formula to obtain some results in interpolation theory, and to examine asymptotic properties of solutions of higher order equations on time scales. We supplement these applications by studying oscillatory properties of solutions of higher order equations; for this, *l'Hôspital's rule* and *Kneser's theorem* on time scales are developed.

The paper is organized as follows: In the next section we give an introduction to the theory of time scales and the basic properties that are needed in the subsequent parts of this paper.

Section 3 contains two versions of Taylor's formula, while Section 4 presents l'Hôspital's rule and Kneser's theorem. The last three sections are devoted to applications of our results developed in Sections 3 and 4. Specifically, in Section 5 we apply Taylor's formula to study Abel-Gontscharoff interpolation on time scales; here we offer the best possible estimate for the maximal error between a given function and its Abel-Gontscharoff interpolating polynomial. As a further application of Taylor's formula we shall study asymptotic behavior of solutions of higher order equations on time scales in Section 6. Finally, in Section 7 we use Kneser's theorem to establish a result on oscillatory behavior of solutions of higher order equations on time scales.

We wish to emphasize at this point that our results in Sections 5-7 should be viewed as a *sample* of applications that could be achieved by using Sections 3 and 4. We neither make an effort to write down a big number of similar applications nor do we state the results in their most general forms. However, at the end of each section we include certain *notes* concerning possible extensions of our results.

2 Preliminaries about Time Scales

A time scale $\mathbf{T}$ is defined to be any closed subset of $\mathbb{R}$. Hence the *jump operators* $\sigma, \rho : \mathbf{T} \to \mathbf{T}$

$$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbf{T} : s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbf{T}$ and $\sup \emptyset := \inf \mathbf{T}$) are well-defined. The point $t \in \mathbf{T}$ is called *left-dense, left-scattered, right-dense, right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. The *graininess* $\mu : \mathbf{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) = \sigma(t) - t$. By $\mathcal{T}$ we shall always denote a closed subset of $\mathbf{T}$. We define

$$\mathcal{T}^\kappa = \begin{cases} \mathcal{T} & \text{if } \mathcal{T} \text{ is unbounded above} \\ \mathcal{T} \setminus (\rho(\max \mathcal{T}), \max \mathcal{T}] & \text{otherwise.} \end{cases}$$

We say that a function $f : \mathbf{T} \to \mathbb{R}$ is differentiable at $t \in \mathbf{T}^\kappa$ provided

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where} \quad s \to t, \quad s \in \mathbf{T} \setminus \{\sigma(t)\}$$

exists. The function f is called differentiable on $\mathcal{T}$ if $f^\Delta(t)$ exists for all $t \in \mathcal{T}$. The following lemma contains results for this derivative. In what follows, we shall write f^σ for $f \circ \sigma$.

Lemma 1. Let $f, g : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}^\kappa$. Then the following hold:

(i) If $f^\Delta(t)$ exists, then f is continuous at t;

(ii) if t is right-scattered and f is continuous at t, then $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$;

(iii) if $f^\Delta(t)$ exists, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$;

(iv) if $f^\Delta(\rho(t))$ exists and if t is left-scattered, then $f(\rho(t)) = f(t) - \mu(\rho(t)) f^\Delta(\rho(t))$;

(v) if $f^\Delta(t), g^\Delta(t)$ exist and $(fg)(t)$ is defined, then $(fg)^\Delta(t) = f(\sigma(t))g^\Delta(t) + f^\Delta(t)g(t)$;

(vi) if f^Δ exists on $\mathcal{T}^\kappa$ and f is invertible on $\mathcal{T}$, then $(f^{-1})^\Delta = -(f^\sigma)^{-1}f^\Delta f^{-1}$ on $\mathcal{T}^\kappa$.

Proof. For (i) and (ii) see [10, Theorem 3] or [13, Theorem 1.2.2]. While (iii) and (iv) follow (note that $\mu(t) = 0$ if t is right-dense) from (i) and (ii), (v) and (vi) are from [10, Theorem 4] or [13, Theorem 1.2.3]. ■

Let $f : \mathbf{T} \to \mathbb{R}$ be a function. If there exists a function $F : \mathbf{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathcal{T}^\kappa$, then F is said to be an *antiderivative* of f. In this case the *Cauchy integral*

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \quad \text{for} \quad a, b \in \mathcal{T}$$

is well-defined (see [13, Section 1.4]). The function f is called *rd-continuous* on $\mathcal{T}$ provided it is continuous at all right-dense points of $\mathcal{T}$ and has a left-sided limit at all left-dense points of $\mathcal{T}$.

Lemma 2. Let $f : \mathcal{T} \to \mathbb{R}$ be rd-continuous on $\mathcal{T}$. Then f possesses an antiderivative on $\mathcal{T}$. Moreover, if f is continuous on $\mathcal{T}$, then f^σ is rd-continuous on $\mathcal{T}$, and hence possesses an antiderivative on $\mathcal{T}$.

Proof. While the second statement is trivial (note that σ is rd-continuous), we refer to [10, Theorem 6] or [13, Theorem 1.4.4] for the first statement. ■

In our study higher order derivatives of a function $f : \mathcal{T} \to \mathbb{R}$ are involved. We shall write $f^{\Delta^2} = (f^\Delta)^\Delta$ if f^Δ is differentiable on $(\mathcal{T}^\kappa)^\kappa = \mathcal{T}^{\kappa^2}$, and similarly we define f^{Δ^n} and $\mathcal{T}^{\kappa^n}$. For $t \in \mathbf{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$ and $\rho^2(t) = \rho(\rho(t))$, and $\sigma^n(t)$ and $\rho^n(t)$ are defined accordingly. For convenience we also put $\sigma^0(t) = \rho^0(t) = t$.

Notes. The calculus on time scales has been initiated by S. Hilger in his PhD thesis in the year 1988. It serves to unify both differential and difference calculus. Indeed, when $\mathrm{T} = \mathbb{R}$, then we have $\rho(t) = \sigma(t) = t$ for each $t \in \mathbb{R}$, and the derivative and integral are easily seen to be the "usual" derivative and integral, respectively. Further, when we choose $\mathrm{T} = \mathbb{Z}$, then $\rho(t) = t - 1$ and $\sigma(t) = t + 1$ for each $t \in \mathbb{Z}$, while $f^\Delta(t) = f(t+1) - f(t) = \Delta f(t)$ and $\int_a^b f(\tau)\Delta\tau = \sum_{\tau=a}^{b-1} f(\tau)$ when $a < b$. Hence all results that are proved on the general time scale include results for both differential and difference equations. The material that is known about time scales up to now is collected in the recently published monograph [13]. Further references of this subject are [5, 10, 11, 12].

3 Taylor's Formula

Lemma 3. Suppose f is n-times differentiable and g_k, $0 \le k \le n-1$, are differentiable at $t \in \mathbf{T}^{\kappa^n}$ with

$$g_{k+1}^\Delta(t) = g_k(\sigma(t)) \quad \text{for all} \quad 0 \le k \le n-2.$$

Then at t we have

$$\left[\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k} g_k\right]^\Delta = f g_0^\Delta + (-1)^{n-1} f^{\Delta^n} g_{n-1}^\sigma.$$

Proof. Using Lemma 1 (v) we find that

$$\begin{aligned}\left[\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k} g_k\right]^{\Delta} &= \sum_{k=0}^{n-1}(-1)^k\left\{f^{\Delta^{k+1}} g_k^{\sigma} + f^{\Delta^k} g_k^{\Delta}\right\} \\ &= \sum_{k=0}^{n-2}(-1)^k f^{\Delta^{k+1}} g_k^{\sigma} + (-1)^{n-1} f^{\Delta^n} g_{n-1}^{\sigma} + f g_0^{\Delta} + \sum_{k=1}^{n-1}(-1)^k f^{\Delta^k} g_k^{\Delta} \\ &= f g_0^{\Delta} + (-1)^{n-1} f^{\Delta^n} g_{n-1}^{\sigma} + \sum_{k=0}^{n-2}(-1)^k f^{\Delta^{k+1}}\left\{g_k^{\sigma} - g_{k+1}^{\Delta}\right\} \\ &= f g_0^{\Delta} + (-1)^{n-1} f^{\Delta^n} g_{n-1}^{\sigma}\end{aligned}$$

holds at t. This proves the lemma. ■

Lemma 4. Let $n \in \mathbb{N}$ and $t \in \mathbf{T}$. If f is $(n-1)$-times differentiable at $\rho^{n-1}(t)$, then

$$\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k}(\rho^{n-1}(t)) g_k(\rho^{n-1}(t)) = f(t),$$

where the g_k are defined in Lemma 3 with $g_0(\tau) \equiv 1$ on $\mathbf{T}$ and $g_k(t) = 0$ for all $k \in \mathbb{N}$.
Proof. First, it is easy to see (use Lemma 1 (iv)) that

$$g_m(\rho^k(t)) = 0 \text{ whenever } 1 \le m \le n \text{ and } 0 \le k \le m-1. \tag{4}$$

Now, for $m = 1$, the statement $\sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^{m-1}(t)) g_k(\rho^{m-1}(t)) = f(t)$ is obviously true. Suppose it holds for some $m \in \{1, 2, \ldots, n-1\}$. Then we consider two cases. First, assume $\rho^{m-1}(t)$ is left-dense. Then $\rho^m(t) = \rho(\rho^{m-1}(t)) = \rho^{m-1}(t)$ so that (apply (4))

$$\begin{aligned}\sum_{k=0}^{m}(-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^m(t)) &= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^{m-1}(t)) g_k(\rho^{m-1}(t)) \\ &\quad + (-1)^m f^{\Delta^m}(\rho^{m-1}(t)) g_m(\rho^{m-1}(t)) = f(t).\end{aligned}$$

Second, assume $\rho^{m-1}(t)$ is left-scattered. Then $\sigma(\rho^m(t)) = \sigma(\rho(\rho^{m-1}(t))) = \rho^{m-1}(t)$ so that (apply (4) and Lemma 1 (iv))

$$\begin{aligned}\sum_{k=0}^{m}(-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^m(t)) &= \sum_{k=1}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))\left[g_k(\rho^{m-1}(t)) - \mu(\rho^m(t)) g_k^{\Delta}(\rho^m(t))\right] \\ &\quad + f(\rho^m(t)) \\ &= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^{m-1}(t)) - \sum_{k=1}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))\mu(\rho^m(t)) g_{k-1}(\rho^{m-1}(t)) \\ &= \sum_{k=0}^{m-1}(-1)^k\left[f^{\Delta^k}(\rho^m(t)) + \mu(\rho^m(t)) f^{\Delta^{k+1}}(\rho^m(t))\right] g_k(\rho^{m-1}(t)) \\ &= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^{m-1}(t)) g_k(\rho^{m-1}(t)) = f(t).\end{aligned}$$

An application of the Principle of Mathematical Induction finishes the proof. ■

Theorem 1 *(Taylor's formula).* Suppose f is n-times differentiable on $\mathbf{T}^{\kappa^n}$, let $t \in \mathbf{T}$, $\alpha \in \mathbf{T}^{\kappa^{n-1}}$, and $g_0(\tau) \equiv 1$ on $\mathbf{T}$. Then, the recursively defined solutions $g_{k+1} = g_{k+1}(\cdot, t)$ of

$$g_{k+1}^{\Delta} = g_k^{\sigma}, \quad g_{k+1}(t) = 0 \quad \text{for all} \quad 0 \le k \le n-2$$

satisfy

$$f(t) = \sum_{k=0}^{n-1} (-1)^k g_k(\alpha, t) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}(\sigma(\tau), t) f^{\Delta^n}(\tau) \Delta\tau.$$

Proof. First we note that the g_k are well-defined due to Lemma 2. By Lemma 3 we have

$$\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} g_k\right]^{\Delta} (\tau) = (-1)^{n-1} f^{\Delta^n}(\tau) g_{n-1}^{\sigma}(\tau)$$

for all $\tau \in \mathbf{T}^{\kappa^n}$ so we may integrate both sides from α to $\rho^{n-1}(t)$ since $\alpha, \rho^{n-1}(t) \in \mathbf{T}^{\kappa^{n-1}}$, i.e.,

$$\begin{aligned} \int_{\alpha}^{\rho^{n-1}(t)} (-1)^{n-1} f^{\Delta^n}(\tau) g_{n-1}^{\sigma}(\tau) \Delta\tau &= \sum_{k=0}^{n-1} (-1)^k \left(f^{\Delta^k} g_k\right) (\rho^{n-1}(t)) - \sum_{k=0}^{n-1} (-1)^k \left(f^{\Delta^k} g_k\right)(\alpha) \\ &= f(t) - \sum_{k=0}^{n-1} (-1)^k f^{\Delta^k}(\alpha) g_k(\alpha), \end{aligned}$$

where we have used Lemma 4. ■

Our first application of Theorem 1 gives another representation of Taylor's formula.

Lemma 5. Let $g_0 = h_0 = 1$ on $\mathbf{T} \times \mathbf{T}$ and define for $k \in \mathbb{N}_0$ functions g_{k+1}, h_{k+1} on $\mathbf{T} \times \mathbf{T}$ recursively by

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s) \Delta\tau \quad \text{and} \quad h_{k+1}(t,s) = \int_s^t h_k(\tau, s) \Delta\tau. \tag{5}$$

Then for $n \in \mathbb{N}_0$ we have

$$h_n(t,s) = (-1)^n g_n(s,t) \quad \text{for all} \quad t \in \mathbf{T},\ s \in \mathbf{T}^{\kappa^n}.$$

Proof. First note that according to Lemma 2 all functions involved are well-defined. We let $n \in \mathbb{N}$, $s \in \mathbf{T}^{\kappa^n}$, and apply Theorem 1 with $f = h_n(\cdot, s)$. Since $f^{\Delta^k} = h_{n-k}(\cdot, s)$ for all $0 \le k \le n$, $f^{\Delta^{n+1}} = 0$, $f^{\Delta^k}(s) = h_{n-k}(s,s) = 0$ for all $0 \le k \le n-1$, and $f^{\Delta^n}(s) = 1$, it follows that

$$\begin{aligned} f(t) &= \sum_{k=0}^{n} (-1)^k g_k(s,t) f^{\Delta^k}(s) + \int_s^{\rho^n(t)} (-1)^n g_n(\sigma(\tau), t) f^{\Delta^{n+1}}(\tau) \Delta\tau \\ &= (-1)^n g_n(s,t) \end{aligned}$$

for all $t \in \mathbf{T}$. ■

Theorem 2 *(Taylor's formula).* Let f be n-times differentiable on $\mathbf{T}^{\kappa^n}$, $t \in \mathbf{T}$, and $\alpha \in \mathbf{T}^{\kappa^{n-1}}$. Then with the functions h_k defined in Lemma 5, i.e.,

$$h_0(r,s) \equiv 1 \quad \text{and} \quad h_{k+1}(r,s) = \int_s^r h_k(\tau, s) \Delta\tau \text{ for } k \in \mathbb{N}_0,$$

we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_\alpha^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Proof. Let $k \in \{0,1,\dots,n-1\}$. Then according to Lemma 5

$$h_k(t,s) = (-1)^k g_k(s,t) \quad \text{for all} \quad s \in \mathbf{T}^{\kappa^k} \supset \mathbf{T}^{\kappa^{n-1}}.$$

Hence our result follows from Theorem 1. ∎

Corollary 1. Let f be n-times differentiable on $\mathbf{T}^{\kappa^n}$ and $m \in \mathbb{N}$ with $m < n$. Then we have for all $\alpha \in \mathbf{T}^{\kappa^{n-1+m}}$ and $t \in \mathbf{T}^{\kappa^m}$

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t,\alpha) f^{\Delta^{k+m}}(\alpha) + \int_\alpha^{\rho^{n-m-1}(t)} h_{n-m-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Proof. The above representation is the same as in Theorem 2 with n and f substituted by $n-m$ and f^{Δ^m}, respectively. ∎

Corollary 2. Suppose f is n-times differentiable on $\mathbf{T}^{\kappa^n}$ and put

$$R_n(t,\alpha) = f(t) - \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha).$$

If $M = \max_{\tau \in \mathbf{T}^{\kappa^n}} \left| f^{\Delta^n}(\tau) \right|$ exists (e.g., if f^{Δ^n} is continuous), then

$$|R_n(t,\alpha)| \le M\,|h_n(t,\alpha)| \quad \text{for all} \quad t \in \mathbf{T},\ \alpha \in \mathbf{T}^{\kappa^{n-1}}.$$

Proof. This is also a consequence of Theorem 2. ∎

Remark 1. Note that the functions g_k and h_k satisfy

$$h_k(t,s) \ge 0 \quad \text{and} \quad g_k(t,s) \ge 0 \quad \text{for all} \quad t \ge s. \tag{6}$$

Moreover, it is easy to obtain the inequality

$$h_k(t,s) \le (t-s)^k \quad \text{for all} \quad t \ge s \quad \text{and} \quad k \in \mathbb{N}_0 \tag{7}$$

which we shall need later. Furthermore, for all t, s the formulae

$$g_n(t,s) = \int_s^t g_{n-1}(t,\tau)\Delta\tau \quad \text{and} \quad h_n(t,s) = \int_s^t h_{n-1}(t,\sigma(\tau))\Delta\tau \tag{8}$$

follow immediately from Lemma 5.

Example 1. Let $\mathbf{T} = \left\{a^k \mid k \in \mathbb{Z}\right\} \cup \{0\}$ for some fixed $a > 1$. In this example we have

$$\sigma(t) = at \quad \text{for all} \quad t \in \mathbf{T}.$$

We now claim that

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - a^\nu s}{\sum_{m=0}^{\nu} a^m}. \tag{9}$$

We abbreviate the right hand side of (9) by $\tilde{h}_k(t,s)$. Of course we have $\tilde{h}_0(t,s) = 1 = h_0(t,s)$ and $\tilde{h}_1(t,s) = t - s = h_1(t,s)$. If $h_p = \tilde{h}_p$ holds for all $p \le k$ with some $k \in \mathbb{N}$, then

$$\begin{aligned}
\tilde{h}_{k+1}^\Delta(t,s) = \left[\tilde{h}_k(t,s)\frac{t - a^k s}{\sum_{m=0}^{k} a^m}\right]^\Delta &= \left\{\sum_{m=0}^{k} a^m\right\}^{-1} \left[h_k(t,s)\left(t - a^k s\right)\right]^\Delta \\
&= \left\{\sum_{m=0}^{k} a^m\right\}^{-1} \left\{h_k^\Delta(t,s)\left(\sigma(t) - a^k s\right) + h_k(t,s)\right\} \\
&= \left\{\sum_{m=0}^{k} a^m\right\}^{-1} \left\{h_{k-1}(t,s)\left(at - a^k s\right) + h_k(t,s)\right\} \\
&= \left\{\sum_{m=0}^{k} a^m\right\}^{-1} \left\{\tilde{h}_{k-1}(t,s)\left(t - a^{k-1}s\right)a + \tilde{h}_k(t,s)\right\} \\
&= \left\{\sum_{m=0}^{k} a^m\right\}^{-1} \left\{\tilde{h}_k(t,s)\sum_{m=0}^{k-1} a^m a + \tilde{h}_k(t,s)\right\} \\
&= \tilde{h}_k(t,s)\left\{\sum_{m=0}^{k} a^m\right\}^{-1} \left\{\sum_{m=0}^{k-1} a^{m+1} + 1\right\} = \tilde{h}_k(t,s).
\end{aligned}$$

Since $\tilde{h}_{k+1}(s,s) = 0$, we have $\tilde{h}_{k+1} = h_{k+1}$, and hence the formula (9) holds for all $k \in \mathbb{N}_0$.

As a special case we consider $a = 2$, then for example we have

$$h_2(t,s) = \frac{(t-s)(t-2s)}{3}.$$

Later we will need a certain integral representation of the remainder in Taylor's formula, and this is our last result in this section.

Lemma 6. The remainder $R_n(t,\alpha)$ given in Corollary 2 satisfies for all $\alpha \in \mathbf{T}^{\kappa^{n-1}}$, $t \in \mathbf{T}$ and $n \in \mathbb{N} \setminus \{1\}$,

$$R_n(t,\alpha) = \int_\alpha^t \int_\alpha^{\tau_1} \cdots \int_\alpha^{\tau_{n-1}} f^{\Delta^n}(\tau_n)\Delta\tau_n\Delta\tau_{n-1}\ldots\Delta\tau_1.$$

Proof. We first let $\alpha \in \mathbf{T}^\kappa$ and $t \in \mathbf{T}$. Then

$$\begin{aligned}
\int_\alpha^t \int_\alpha^{\tau_1} f^{\Delta^2}(\tau_2)\Delta\tau_2\Delta\tau_1 &= \int_\alpha^t \left\{f^\Delta(\tau_1) - f^\Delta(\alpha)\right\}\Delta\tau_1 \\
&= f(t) - f(\alpha) - (t-\alpha)f^\Delta(\alpha) = R_2(t,\alpha)
\end{aligned}$$

so that the representation holds for $n = 2$. Now, if the representation is true for $n > 1$, then we have for $\alpha \in \mathbf{T}^{\kappa^n}$ and $t \in \mathbf{T}$

$$\int_\alpha^t \int_\alpha^{\tau_1} \int_\alpha^{\tau_2} \cdots \int_\alpha^{\tau_n} f^{\Delta^{n+1}}(\tau_{n+1})\Delta\tau_{n+1}\Delta\tau_n\ldots\Delta\tau_1$$

$$
\begin{aligned}
&= \int_\alpha^t \left\{ \int_\alpha^\tau \int_\alpha^{\tau_1} \cdots \int_\alpha^{\tau_{n-1}} (f^\Delta)^{\Delta^n}(\tau_n)\Delta\tau_n\Delta\tau_{n-1}\ldots\Delta\tau_1 \right\} \Delta\tau \\
&= \int_\alpha^t \left\{ f^\Delta(\tau) - \sum_{k=0}^{n-1} h_k(\tau,\alpha)(f^\Delta)^{\Delta^k}(\alpha) \right\} \Delta\tau \\
&= f(t) - f(\alpha) - \sum_{k=0}^{n-1} f^{\Delta^{k+1}}(\alpha) \int_\alpha^t h_k(\tau,\alpha)\Delta\tau \\
&= f(t) - f(\alpha) - \sum_{k=0}^{n-1} f^{\Delta^{k+1}}(\alpha) h_{k+1}(t,\alpha) \\
&= f(t) - f(\alpha) - \sum_{k=1}^{n} f^{\Delta^{k}}(\alpha) h_{k}(t,\alpha) \\
&= f(t) - \sum_{k=0}^{n} f^{\Delta^{k}}(\alpha) h_{k}(t,\alpha) = R_{n-1}(t,\alpha),
\end{aligned}
$$

where we have used (5). ■

Notes. Of course we have

$$g_n(t,s) = h_n(t,s) = \frac{(t-s)^n}{n!}$$

in the case $\mathbf{T} = \mathbb{R}$, while

$$g_n(t,s) = \frac{(t-s+n-1)^{(n)}}{n!} \quad \text{and} \quad h_n(t,s) = \frac{(t-s)^{(n)}}{n!}$$

can be checked immediately in the case $\mathbf{T} = \mathbb{Z}$. Hence formulae (1) and (2) follow as special cases of (3). Formula (2) and its consequences are contained in [3, Lemmas 1 and 2]; see also [1, Lemma 2.1] and [4, Formula (1.7.6)]. Several applications of Corollary 1 in the discrete case (see [9, Lemma 5.1]) are available in the recent monograph on "Advanced Topics in Difference Equations" [9]. There, Green's functions for certain boundary value problems are needed, and to construct such Green's functions on time scales still remains to be done.

4 L'Hôspital's Rule and Kneser's Theorem

Throughout this section we let

$$\bar{\mathbf{T}} = \mathbf{T} \cup \{\sup \mathbf{T}\} \cup \{\inf \mathbf{T}\}.$$

If $\infty \in \bar{\mathbf{T}}$, we call ∞ left-dense, and $-\infty$ is called right-dense provided $-\infty \in \bar{\mathbf{T}}$. For any left-dense $t_0 \in \mathbf{T}$ and any $\varepsilon > 0$, the set

$$L_\varepsilon(t_0) = \{t \in \mathbf{T} :\ 0 < t_0 - t < \varepsilon\}$$

is nonempty, and so is $L_\varepsilon(\infty) = \left\{t \in \mathbf{T} :\ t > \frac{1}{\varepsilon}\right\}$ if $\infty \in \bar{\mathbf{T}}$. The sets $R_\varepsilon(t_0)$ for right-dense $t_0 \in \bar{\mathbf{T}}$ and $\varepsilon > 0$ are defined accordingly. For a function $h : \mathbf{T} \to \mathbb{R}$ we define

$$\liminf_{t\to t_0^-} h(t) = \lim_{\varepsilon\to 0+} \inf_{t\in L_\varepsilon(t_0)} h(t) \text{ for left-dense } t_0 \in \bar{\mathbf{T}},$$

and $\liminf_{t\to t_0^+} h(t)$, $\limsup_{t\to t_0^-} h(t)$, $\limsup_{t\to t_0^+} h(t)$ are defined analogously.

Theorem 3 *(L'Hôspital's rule).* Assume f, g are differentiable on $\mathbf{T}$ with

(10) $$\lim_{t\to t_0^-} f(t) = \lim_{t\to t_0^-} g(t) = 0 \quad \text{for some left-dense} \quad t_0 \in \bar{\mathbf{T}}.$$

Suppose there exists $\varepsilon > 0$ with

(11) $$g(t) > 0,\ g^\Delta(t) < 0 \quad \text{for all} \quad t \in L_\varepsilon(t_0).$$

Then we have

$$\liminf_{t\to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} \le \liminf_{t\to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t\to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t\to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)}.$$

Proof. Let $\delta \in (0, \varepsilon]$ and put $\alpha = \inf_{\tau\in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}$, $\beta = \sup_{\tau\in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}$. Then

$$\alpha g^\Delta(\tau) \ge f^\Delta(\tau) \ge \beta g^\Delta(\tau) \quad \text{for all} \quad \tau \in L_\delta(t_0)$$

by (11) and hence

$$\int_s^t \alpha g^\Delta(\tau)\Delta\tau \ge \int_s^t f^\Delta(\tau)\Delta\tau \ge \int_s^t \beta g^\Delta(\tau)\Delta\tau \text{ for all } s, t \in L_\delta(t_0),\ s < t$$

so that

$$\alpha g(t) - \alpha g(s) \ge f(t) - f(s) \ge \beta g(t) - \beta g(s) \text{ for all } s, t \in L_\delta(t_0),\ s < t.$$

Now, letting $t \to t_0^-$, we find from (10)

$$-\alpha g(s) \ge -f(s) \ge -\beta g(s) \quad \text{for all} \quad s \in L_\delta(t_0)$$

and hence by (11)

$$\inf_{\tau\in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)} = \alpha \le \inf_{s\in L_\delta(t_0)} \frac{f(s)}{g(s)} \le \sup_{s\in L_\delta(t_0)} \frac{f(s)}{g(s)} \le \beta = \sup_{\tau\in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}.$$

Letting $\delta \to 0^+$ yields our desired result. ■

Theorem 4 *(L'Hôspital's rule).* Assume f, g are differentiable on $\mathbf{T}$ with

(12) $$\lim_{t\to t_0^-} g(t) = \infty \quad \text{for some left-dense} \quad t_0 \in \bar{\mathbf{T}}.$$

Suppose there exists $\varepsilon > 0$ with

(13) $$g(t) > 0,\ g^\Delta(t) > 0 \quad \text{for all} \quad t \in L_\varepsilon(t_0).$$

Then $\lim_{t\to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} = r \in \bar{\mathbb{R}}$ implies $\lim_{t\to t_0^-} \frac{f(t)}{g(t)} = r$.
Proof. First suppose $r \in \mathbb{R}$. Let $c > 0$. Then there exists $\delta \in (0, \varepsilon]$ such that

$$\left|\frac{f^\Delta(\tau)}{g^\Delta(\tau)} - r\right| \le c \quad \text{for all} \quad \tau \in L_\delta(t_0)$$

and hence by (13)

$$-cg^{\Delta}(\tau) \le f^{\Delta}(\tau) - rg^{\Delta}(\tau) \le cg^{\Delta}(\tau) \quad \text{for all} \quad \tau \in L_\delta(t_0).$$

We integrate as in the proof of Theorem 3 and use (13) again to obtain

$$(r-c)\left(1-\frac{g(s)}{g(t)}\right) \le \frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \le (r+c)\left(1-\frac{g(s)}{g(t)}\right) \text{ for all } s,t \in L_\delta(t_0); s<t.$$

Letting $t \to t_0^-$ and applying (12) yields

$$r-c \le \liminf_{t\to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t\to t_0^-} \frac{f(t)}{g(t)} \le r+c.$$

Now we let $c \to 0^+$ to see that $\lim_{t\to t_0^-} \frac{f(t)}{g(t)}$ exists and equals r.

Next, if $r=\infty$ (and similarly if $r=-\infty$), let $c>0$. Then there exists $\delta \in (0,\varepsilon]$ with

$$\frac{f^{\Delta}(\tau)}{g^{\Delta}(\tau)} \ge \frac{1}{c} \quad \text{for all} \quad \tau \in L_\delta(t_0)$$

and hence by (13)

$$f^{\Delta}(\tau) \ge \frac{1}{c} g^{\Delta}(\tau) \quad \text{for all} \quad \tau \in L_\delta(t_0).$$

We integrate again to get

$$\frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \ge \frac{1}{c}\left(1-\frac{g(s)}{g(t)}\right) \quad \text{for all} \quad s,t \in L_\delta(t_0); s<t.$$

Thus, letting $t \to t_0^-$ and applying (12), we find $\liminf_{t\to t_0^-} \frac{f(t)}{g(t)} \ge \frac{1}{c}$, and then, letting $c \to 0^+$, we obtain $\lim_{t\to t_0^-} \frac{f(t)}{g(t)} = \infty = r$. ■

Lemma 7. Let $m \in \mathbb{N}$ and f be m-times differentiable on $\mathbf{T}$. Then, if $\infty \in \bar{\mathbf{T}}$

- $\liminf_{t\to\infty} f^{\Delta^m}(t) > 0 \implies \lim_{t\to\infty} f^{\Delta^i}(t) = \infty$ for all $0 \le i \le m-1$;
- $\limsup_{t\to\infty} f^{\Delta^m}(t) < 0 \implies \lim_{t\to\infty} f^{\Delta^i}(t) = -\infty$ for all $0 \le i \le m-1$.

Proof. Suppose $0 < \lim_{\delta\to 0} \inf_{t\in L_\delta(\infty)} f^{\Delta^m}(t) = \alpha \in \mathbb{R}$. Then there exists $\varepsilon > 0$ with

$$f^{\Delta^m}(\tau) \ge \frac{\alpha}{2} \quad \text{for all} \quad \tau \in L_\varepsilon(\infty).$$

Thus

$$\int_s^t f^{\Delta^m}(\tau)\Delta\tau \ge \int_s^t \frac{\alpha}{2}\Delta\tau \quad \text{for all} \quad s,t \in L_\varepsilon(\infty); s<t$$

and hence

$$f^{\Delta^{m-1}}(t) - f^{\Delta^{m-1}}(s) \ge \frac{\alpha}{2}(t-s) \quad \text{for all} \quad s,t \in L_\varepsilon(\infty); s<t.$$

Therefore $\lim_{t\to\infty} f^{\Delta^{m-1}}(t) = \infty$. Next, if $\alpha = \infty$, then for all $c>0$ there exists $\varepsilon > 0$ with

$$f^{\Delta^m}(\tau) \ge \frac{1}{c} \quad \text{for all} \quad \tau \in L_\varepsilon(\infty).$$

Hence, again as above, $\lim_{t\to\infty} f^{\Delta^{m-1}}(t) = \infty$. The rest of the proof of the first statement follows by induction, and the second statement can be proved similarly. ■

Theorem 5 *(Kneser's theorem).* Let $n \in \mathbb{N}$ and f be n-times differentiable on $\mathbf{T}$. Assume $\infty \in \bar{\mathbf{T}}$. Suppose there exists $\varepsilon > 0$ such that

$$f(t) > 0, \ \operatorname{sgn}\left(f^{\Delta^n}(t)\right) \equiv s \in \{-1, +1\} \quad \text{for all} \quad t \in L_\varepsilon(\infty) \tag{14}$$

and $f^{\Delta^n}(t) \not\equiv 0$ on $L_\delta(\infty)$ for any $\delta > 0$. Then there exists $p \in [0, n] \cap \mathbb{N}_0$ such that $n + p$ is even for $s = 1$ and odd for $s = -1$ with

$$\begin{cases} (-1)^{p+j} f^{\Delta^j}(t) > 0 \text{ for all } t \in L_\varepsilon(\infty), \ j \in [p, n-1] \cap \mathbb{N}_0 \\ f^{\Delta^j}(t) > 0 \text{ for all } t \in L_{\delta_j}(\infty) \text{ (with } \delta_j \in (0, \varepsilon)), \ j \in [1, p-1] \cap \mathbb{N}_0. \end{cases}$$

Proof. First we consider the case $s = -1$. For achieving a contradiction we assume $f^{\Delta^{n-1}}(t_1) \le 0$ for some $t_1 \in L_\varepsilon(\infty)$. Then there exists $t_2 \in L_\varepsilon(\infty)$ with $t_2 > t_1$ such that

$$f^{\Delta^{n-1}}(t) \le f^{\Delta^{n-1}}(t_2) < f^{\Delta^{n-1}}(t_1) \quad \text{for all} \quad t \in L_\varepsilon(\infty), \ t > t_2.$$

Hence $\limsup_{t\to\infty} f^{\Delta^{n-1}}(t) < 0$ and by Lemma 7 $\lim_{t\to\infty} f(t) = -\infty$, which is impossible because of (14). Therefore

$$f^{\Delta^{n-1}}(t) > 0 \quad \text{for all} \quad t \in L_\varepsilon(\infty).$$

Define

$$M = \left\{m \in \mathbb{N}_0 : \ (-1)^{m+j} f^{\Delta^j}(t) > 0 \text{ for all } t \in L_\varepsilon(\infty), \ j \in [m, n-1] \cap \mathbb{N}_0; \ n + m \text{ odd}\right\}.$$

Then $n - 1 \in M$ so that $p = \min M$ exists. If $p \in \{0, 1\}$, then we are done. Assume $p \ge 2$. Seeking a contradiction, we assume that $f^{\Delta^{p-1}}(t) < 0$ for all $t \in L_\varepsilon(\infty)$. But then Lemma 7 yields as before $f^{\Delta^{p-2}}(t) > 0$ for all $t \in L_\varepsilon(\infty)$; in fact we now have

$$(-1)^{p-2+j} f^{\Delta^j}(t) > 0 \quad \text{for all} \quad t \in L_\varepsilon(\infty), \ j \in [p-2, n-1] \cap \mathbb{N}_0,$$

in contradiction to the definition of p. Hence there exists $t_3 \in L_\varepsilon(\infty)$ with $f^{\Delta^{p-1}}(t_3) \ge 0$. Since $f^{\Delta^p}(t) > 0$ for all $t \in L_\varepsilon(\infty)$, it follows that

$$f^{\Delta^{p-1}}(t) > 0 \quad \text{for all} \quad t \in L_\varepsilon(\infty) \text{ with } t > t_3.$$

Hence $\liminf_{t\to\infty} f^{\Delta^{p-1}}(t) > 0$ so that by Lemma 7

$$\lim_{t\to\infty} f^{\Delta^j}(t) = \infty \quad \text{for all} \quad 1 \le j \le p-2,$$

and this implies our claim.

Finally, if $s = 1$, we assume there exists $t_4 \in L_\varepsilon(\infty)$ such that $f^{\Delta^{n-1}}(t_4) \ge 0$. Again we may show as in the first part of the proof that $\liminf_{t\to\infty} f^{\Delta^{n-1}}(t) > 0$, and hence by Lemma 7 $\liminf_{t\to\infty} f^{\Delta^j}(t) = \infty$ for all $1 \le j \le n-2$ so that $p = n$ does the required job. If, however, $f^{\Delta^{n-1}}(t) < 0$ for all $t \in L_\varepsilon(\infty)$, then, as before from Lemma 7,

$$f^{\Delta^{n-2}}(t) > 0 \quad \text{for all} \quad t \in L_\varepsilon(\infty)$$

so that we may proceed as in the first part to finish the proof. ∎

Notes. Of course corresponding results hold for right hand limits also with the assumptions modified appropriately. Some continuous versions of the results in this section are contained in [14] and [16] whereas discrete versions are available in [3, Lemmas 3-7] and in [4, 1.7.7-1.7.11]. The paper [3] also contains several consequences (see [3, Corollaries 1-3 and Lemma 8]) of these results which are useful for applications. However, in order not to extend the size of our paper, we do not include versions of these results on time scales.

5 Abel-Gontscharoff Interpolation

Throughout this section we let $n \in \mathbb{N}$ and $a, b \in \mathbf{T}$ with $a < \rho^{n-1}(b)$, and choose a sequence $\{t_\nu : 1 \le \nu \le n\} \subset \mathbf{T}$ with

$$a \le t_1 \le t_2 \le \ldots \le t_n \le \rho^{n-1}(b).$$

Let f be a function on $\mathcal{T} = [a, b] \cap \mathbf{T}$. The unique polynomial P_{n-1} of degree n satisfying

$$P_{n-1}^{\Delta^i}(t_{i+1}) = f^{\Delta^i}(t_{i+1}) \quad \text{for all} \quad 0 \le i \le n-1$$

is called the *Abel-Gontscharoff interpolating polynomial.* The representation

$$P_{n-1}(t) = \sum_{i=0}^{n-1} T_i(t) f^{\Delta^i}(t_{i+1})$$

with $T_0(t) \equiv 1$ and

$$T_i(t) = \int_{t_1}^{t} \int_{t_2}^{\tau_1} \cdots \int_{t_i}^{\tau_{i-1}} \Delta\tau_i \Delta\tau_{i-1} \ldots \Delta\tau_1, \qquad 1 \le i \le n-1$$

can be verified easily. In this section we shall bound the error $e = f - P_{n-1}$ in terms of

$$M = \max_{t \in \mathcal{T}^{\kappa^n}} \left| f^{\Delta^n}(t) \right|.$$

Our main result concerning the best possible estimate for $\max_{t \in \mathcal{T}} |e(t)|$ reads as follows.

Theorem 6 *(Abel-Gontscharoff interpolation on time scales).* Let

$$c_m = (-1)^{m-1} \sum_{k=0}^{m-1} g_k(a, b) g_{n-k}(\rho^{n-1}(b), a) \quad \text{for} \quad 1 \le m \le n$$

and $c = \max_{1 \le m \le n} c_m$. Then we have

$$\max_{t \in \mathcal{T}} |e(t)| \le Mc,$$

where the constant c is the best possible.

Proof. The relation

$$e(t) = \int_{t_1}^{t} \int_{t_2}^{\tau_1} \cdots \int_{t_n}^{\tau_{n-1}} f^{\Delta^n}(\tau_n) \Delta\tau_n \Delta\tau_{n-1} \ldots \Delta\tau_1 \quad \text{for all} \quad t \in \mathbf{T} \tag{15}$$

is trivially true. From (15) we obtain for all $0 \le m \le n-1$

$$e^{\Delta^m}(t) = \int_{t_{m+1}}^{t} \int_{t_{m+2}}^{\tau_{m+1}} \cdots \int_{t_n}^{\tau_{n-1}} f^{\Delta^n}(\tau_n) \Delta\tau_n \Delta\tau_{n-1} \ldots \Delta\tau_1 \quad \text{for all} \quad t \in \mathbf{T}^{\kappa^m}. \tag{16}$$

Let $s \in [a, t_{m+1}] \cap \mathbf{T}$ and $0 \le m \le n-1$. By (16) and (8) we have

$$\begin{aligned} \left| e^{\Delta^m}(s) \right| &\le M \int_{s}^{t_{m+1}} \int_{\tau_{m+1}}^{t_{m+2}} \cdots \int_{\tau_{n-1}}^{t_n} \Delta\tau_n \Delta\tau_{n-1} \ldots \Delta\tau_1 \\ &\le M \int_{s}^{\rho^{n-1}(b)} \int_{\tau_{m+1}}^{\rho^{n-1}(b)} \cdots \int_{\tau_{n-1}}^{\rho^{n-1}(b)} \Delta\tau_n \Delta\tau_{n-1} \ldots \Delta\tau_1 = M g_{n-m}(\rho^{n-1}(b), s), \end{aligned}$$

i.e.,

$$\left|e^{\Delta^m}(s)\right| \le M g_{n-m}(\rho^{n-1}(b), s) \quad \text{for all} \quad s \in [a, t_{m+1}] \cap \mathbf{T},\ 0 \le m \le n-1. \tag{17}$$

Next, the formula

$$e(t) = \int_{t_1}^{t}\int_{t_2}^{\tau_1}\dots\int_{t_m}^{\tau_{m-1}} e^{\Delta^m}(\tau_m)\Delta\tau_m\Delta\tau_{m-1}\dots\Delta\tau_1 \quad \text{for all} \quad t \in \mathbf{T},\ 0 \le m \le n \tag{18}$$

can be easily checked. For $s \in [t_n, b] \cap \mathbf{T}$ we have by (18) and (5)

$$\begin{aligned} |e(s)| &= \left|\int_{t_1}^{s}\int_{t_2}^{\tau_1}\dots\int_{t_n}^{\tau_{n-1}} e^{\Delta^n}(\tau_n)\Delta\tau_n\Delta\tau_{n-1}\dots\Delta\tau_1\right| \\ &\le M\int_{a}^{s}\int_{a}^{\tau_1}\dots\int_{a}^{\tau_{n-1}} \Delta\tau_n\Delta\tau_{n-1}\dots\Delta\tau_1 = Mh_n(s,a), \end{aligned}$$

i.e.,

$$|e(s)| \le Mh_n(s,a) \quad \text{for all} \quad s \in [t_n, b] \cap \mathbf{T}. \tag{19}$$

Let $s \in [t_m, t_{m+1}] \cap \mathbf{T}$ and $0 \le m \le n-1$. By (18), (17), and (4) we have

$$\begin{aligned} |e(s)| &\le M\int_{a}^{b}\int_{a}^{\tau_1}\dots\int_{a}^{\tau_{m-1}} g_{n-m}(\rho^{n-1}(b), \tau_m)\Delta\tau_m\Delta\tau_{m-1}\dots\Delta\tau_1 = M(-1)^m R_m(b,a) \\ &= M(-1)^m\left\{g_n(\rho^{n-1}(b), b) - \sum_{k=0}^{m-1} h_k(b,a)(-1)^k g_{n-k}(\rho^{n-1}(b), a)\right\} \\ &= M(-1)^{m-1}\sum_{k=0}^{m-1} g_k(a,b)g_{n-k}(\rho^{n-1}(b), a) = Mc_m, \end{aligned}$$

where we have used (8), Lemma 6, and Theorem 2 with

$$f(t) = g_n(\rho^{n-1}(b), t). \tag{20}$$

Hence

$$|e(s)| \le Mc_m \quad \text{for all} \quad s \in [t_m, t_{m+1}] \cap \mathbf{T},\ 0 \le m \le n-1. \tag{21}$$

We note that we have for all $s \in [a, t_1] \cap \mathbf{T}$

$$|e(s)| \le Mg_n(\rho^{n-1}(b), s) \le Mg_n(\rho^{n-1}(b), a) = Mc_1$$

due to (17) with $m = 0$ (use also (8) and (6)), i.e.,

$$|e(s)| \le Mc_1 \quad \text{for all} \quad s \in [a, t_1] \cap \mathbf{T}. \tag{22}$$

By Theorem 2 with f defined by (20) we have

$$\begin{aligned} c_n &= (-1)^{n-1}\sum_{k=0}^{n-1} g_k(a,b)g_{n-k}(\rho^{n-1}(b), a) \\ &= (-1)^{n-1}\sum_{k=0}^{n} g_k(a,b)(-1)^k f^{\Delta^k}(a) - (-1)^{n-1}g_n(a,b)(-1)^n f^{\Delta^n}(a) \\ &= (-1)^{n-1}\sum_{k=0}^{n} h_k(b,a)f^{\Delta^k}(a) - (-1)^{n-1}g_n(a,b) \\ &= (-1)^{n-1}f(b) + (-1)^n g_n(a,b) = h_n(b,a) \end{aligned}$$

(apply (4)) so that for all $s \in [t_n, b] \cap \mathbf{T}$ by (19) (use also (5) and (6)),

$$|e(s)| \leq M h_n(s,a) \leq M h_n(b,a) = M c_n,$$

i.e.,

(23) $$|e(s)| \leq M c_n \quad \text{for all} \quad s \in [t_n, b] \cap \mathbf{T}.$$

Therefore it follows from (22), (21), and (23) that

$$|e(s)| \leq M \max_{1 \leq m \leq n} c_m = Mc \quad \text{for all} \quad s \in [a,b] \cap \mathbf{T} = \mathcal{T},$$

i.e.,

$$\max_{s \in \mathcal{T}} |e(s)| \leq Mc.$$

In order to prove that this constant is the best possible, we pick $\mu \in \{1, \ldots, n\}$ with $c_\mu = c$ and define

$$t_1 = t_2 = \ldots = t_\mu = a, \qquad t_{\mu+1} = \ldots = t_n = \rho^{n-1}(b)$$

and

$$f(t) = \sum_{k=\mu}^{n} h_{n-k}(a, \rho^{n-1}(b)) h_k(t,a) \quad \text{for} \quad t \in \mathbf{T}.$$

Then we have

$$f^{\Delta^i}(t) = \sum_{k=\max\{i,\mu\}}^{n} h_{n-k}(a, \rho^{n-1}(b)) h_{k-i}(t,a) \quad \text{for all} \quad 0 \leq i \leq n-1,\ t \in \mathcal{T}^{\kappa^i}$$

while $f^{\Delta^n}(t) \equiv 1$, so that $M = 1$. Now we have by Theorem 2 for all $\mu \leq i \leq n-1$

$$\begin{aligned} f^{\Delta^i}(t) &= \sum_{k=i}^{n} h_{n-k}(a, \rho^{n-1}(b)) h_{k-i}(t,a) = \sum_{k=0}^{n-i} h_{n-k-i}(a, \rho^{n-1}(b)) h_k(t,a) \\ &= \sum_{k=0}^{n-i} h_k(t,a) h_{n-i}^{\Delta^k}(a, \rho^{n-1}(b)) = h_{n-i}(t, \rho^{n-1}(b)) = (-1)^{n-i} g_{n-i}(\rho^{n-1}(b), t) \end{aligned}$$

and for all $0 \leq i \leq \mu - 1$

$$\begin{aligned} f^{\Delta^i}(t) &= \sum_{k=\mu}^{n} h_{n-k}(a, \rho^{n-1}(b)) h_{k-i}(t,a) = \sum_{k=\mu-i}^{n-i} h_{n-k-i}(a, \rho^{n-1}(b)) h_k(t,a) \\ &= \sum_{k=0}^{n-i} h_{n-k-i}(a, \rho^{n-1}(b)) h_k(t,a) - \sum_{k=0}^{\mu-i-1} h_{n-k-i}(a, \rho^{n-1}(b)) h_k(t,a) \\ &= h_{n-i}(t, \rho^{n-1}(b)) + (-1)^{n-i-1} \sum_{k=0}^{\mu-i-1} g_{n-k-i}(\rho^{n-1}(b), a) g_k(a,t). \end{aligned}$$

Thus $f^{\Delta^i}(a) = 0$ for all $0 \leq i \leq \mu - 1$ and $f^{\Delta^i}(\rho^{n-1}(b)) = 0$ for all $\mu \leq i \leq n-1$ and hence

$$f^{\Delta^i}(t_{i+1}) = 0 \quad \text{for all} \quad 0 \leq i \leq n-1$$

so that the corresponding Abel-Gontscharoff interpolating polynomial P_{n-1} is identically zero. Thus $e = f$ and (for the computation of $f(b)$ use the above calculation with $i = 0$)

$$\begin{aligned} Mc &= c \geq \max_{s\in\mathcal{T}} |e(s)| = \max_{s\in\mathcal{T}} |f(s)| \\ &\geq |f(b)| = \left| h_n(b, \rho^{n-1}(b)) + (-1)^{n-1} \sum_{k=0}^{\mu-1} g_{n-k}(\rho^{n-1}(b), a) g_k(a, b) \right| \\ &= \left| (-1)^{n-1} \sum_{k=0}^{\mu-1} g_{n-k}(\rho^{n-1}(b), a) g_k(a, b) \right| \\ &= (-1)^{\mu-1} \sum_{k=0}^{\mu-1} g_{n-k}(\rho^{n-1}(b), a) g_k(a, b) = c_\mu = c = Mc \end{aligned}$$

so that the upper bound Mc is really attained by $|e|$, and this completes the proof. ∎

Notes. In the case $\mathbf{T} = \mathbb{R}$ we have

$$\begin{aligned} c_m &= (-1)^{m-1} \sum_{k=0}^{m-1} (-1)^k \frac{(b-a)^k}{k!} \frac{(b-a)^{n-k}}{(n-k)!} \\ &= \frac{(b-a)^n}{n!} \sum_{k=0}^{m-1} (-1)^{m-k+1} \binom{n}{k} = \frac{(b-a)^n}{n!} \binom{n-1}{m-1} \\ &= h_n(b, a) \binom{n-1}{m-1} \end{aligned}$$

(see also [15] and [8, Theorem 1.1]) while in the case $\mathbf{T} = \mathbb{Z}$ (see [17, Theorem 3.1]),

$$\begin{aligned} c_m &= (-1)^{m-1} \sum_{k=0}^{m-1} (-1)^k \frac{(b-a)^{(k)}}{k!} \frac{(b-n+1-a+n-k-1)^{(n-k)}}{(n-k)!} \\ &= (-1)^{m-1} \sum_{k=0}^{m-1} \frac{(-1)^k}{k!(n-k)!} (b-a) \cdot \ldots \cdot (b-a-k+1) \cdot (b-a-k) \cdot \ldots \cdot (b-n+1-a) \\ &= \frac{(b-a)^{(n)}}{n!} \sum_{k=0}^{m-1} (-1)^{m-k+1} \binom{n}{k} = h_n(b, a) \binom{n-1}{m-1}. \end{aligned}$$

We also note that similar estimates for the derivatives of the error

$$\max_{t\in\mathcal{T}^{\kappa^i}} \left| e^{\Delta^i}(t) \right|, \qquad 0 \leq i \leq n-1$$

can be obtained rather easily. For other works related to the results of this section we refer to [6, 7, 8, 15, 17].

6 Asymptotic Behavior

In this section we shall study further applications of Taylor's formula. Throughout we let $n \in \mathbb{N}$, $a \in \mathbf{T}$, $\{\alpha_\nu : 0 \leq \nu \leq n-1\} \subset \mathbb{R}$, and assume that $\mathbf{T}$ is unbounded above. Our main result of this section deals with the initial value problem

$$(24) \qquad y^{\Delta^n} = f\left(t, y, y^\Delta, \ldots, y^{\Delta^{n-1}}\right), \qquad y^{\Delta^i}(a) = \alpha_i \text{ for } 0 \leq i \leq n-1$$

where $f:\mathbf{T}\times\mathbb{R}^n\to\mathbb{R}$ is supposed to satisfy

$$|f(t,u_0,\ldots,u_{n-1})|\le\sum_{i=0}^{n-1}p_i(t)|u_i| \text{ for all } t\in\mathbf{T},\ \{u_i:\ 0\le i\le n-1\}\subset\mathbb{R} \tag{25}$$

with certain nonnegative functions $p_i:\mathbf{T}\to\mathbb{R}$, $0\le i\le n-1$. Let

$$p(t)=\sum_{i=0}^{n-1}p_i(t)(t-a)^{n-i-1},$$

and let $\tilde{p}(t;\tau)$ satisfy

$$\tilde{p}^{\Delta}(t;\tau)=p(t)\tilde{p}(t;\tau),\qquad \tilde{p}(\tau;\tau)=1.$$

Note that $\tilde{p}$ exists uniquely according to [13, Section 2.5]. The following is our main result concerning asymptotic behavior of solutions of (24).

Theorem 7 *(Asymptotic Behavior of Solutions of (24)).* Suppose (25) holds and assume that there exists $K>0$ such that

$$\left|\int_a^{\rho^{n-m-1}(t)}p(\tau)\tilde{p}(t,\sigma(\tau))\Delta\tau\right|\le K \quad\text{for all}\quad t\in[a+1,\infty)\cap\mathbf{T},\ 0\le m\le n-1.$$

Then there exists $\gamma>0$ such that every solution y of (24) satisfies

$$\left|y^{\Delta^m}(t)\right|\le\gamma(t-a)^{n-m-1}\quad\text{for all}\quad t\in[a+1,\infty)\cap\mathbf{T},\ 0\le m\le n-1.$$

Proof. Let y be a solution of (24). Then by Corollary 1, for $0\le m\le n-1$ we have

$$\begin{aligned}
\left|y^{\Delta^m}(t)\right| &= \left|\sum_{k=0}^{n-m-1}h_k(t,a)y^{\Delta^{k+m}}(a)+\int_a^{\rho^{n-m-1}(t)}h_{n-m-1}(t,\sigma(\tau))y^{\Delta^n}(\tau)\Delta\tau\right|\\
&= \left|\sum_{k=0}^{n-m-1}h_k(t,a)\alpha_{k+m}+\int_a^{\rho^{n-m-1}(t)}h_{n-m-1}(t,\sigma(\tau))f\left(\tau,y(\tau),y^{\Delta}(\tau),\ldots,y^{\Delta^{n-1}}(\tau)\right)\Delta\tau\right|\\
&\le \sum_{k=0}^{n-m-1}h_k(t,a)|\alpha_{k+m}|+\int_a^{\rho^{n-m-1}(t)}h_{n-m-1}(t,\sigma(\tau))\sum_{i=0}^{n-1}p_i(\tau)\left|y^{\Delta^i}(\tau)\right|\Delta\tau\\
&\le \alpha\sum_{k=0}^{n-m-1}(t-a)^k+\int_a^{\rho^{n-m-1}(t)}(t-a)^{n-m-1}\sum_{i=0}^{n-1}p_i(\tau)\left|y^{\Delta^i}(\tau)\right|\Delta\tau\\
&\le (t-a)^{n-m-1}\left\{\alpha n+\int_a^{\rho^{n-m-1}(t)}\sum_{i=0}^{n-1}p_i(\tau)\left|y^{\Delta^i}(\tau)\right|\Delta\tau\right\}\\
&= (t-a)^{n-m-1}F(t)
\end{aligned}$$

for all $t\in[a+1,\infty)\cap\mathbf{T}$, using (7) for the second inequality and putting $\alpha=\max_{0\le k\le n-1}|\alpha_k|$ and

$$F(t)=\alpha n+\int_a^{\rho^{n-m-1}(t)}\sum_{i=0}^{n-1}p_i(\tau)\left|y^{\Delta^i}(\tau)\right|\Delta\tau,$$

i.e.,

$$\left|y^{\Delta^m}(t)\right|\le(t-a)^{n-m-1}F(t)\quad\text{for all}\quad t\in[a+1,\infty)\cap\mathbf{T},\ 0\le m\le n-1. \tag{26}$$

Since in view of (26) we have

$$\begin{aligned} F(t) &\leq \alpha n + \int_a^{\rho^{n-m-1}(t)} \sum_{i=0}^{n-1} p_i(\tau)(\tau-a)^{n-i-1} F(\tau)\Delta\tau \\ &= \alpha n + \int_a^{\rho^{n-m-1}(t)} p(\tau)F(\tau)\Delta\tau, \end{aligned}$$

Gronwall's inequality (see [13, Lemma 2.5.1]) yields

$$\begin{aligned} F(t) &\leq \alpha n + \int_a^{\rho^{n-m-1}(t)} p(\tau)\tilde{p}(t;\sigma(\tau))\alpha n \Delta\tau \\ &= \alpha n \left\{ 1 + \int_a^{\rho^{n-m-1}(t)} p(\tau)\tilde{p}(t;\sigma(\tau))\Delta\tau \right\} \leq \alpha n(1+K), \end{aligned}$$

and hence our claim follows from (26) with $\gamma = \alpha n(1+K)$. ■

Notes. Discrete versions of related results are contained in [1, Theorem 3.1] and [2, Theorem 3.6]. Using our version of Taylor's formula it is possible to develop further results on time scales similar to those established in [1, 2]; see also [4, Sections 6.17 and 6.18].

7 Oscillatory Behavior

Only one of the numerous applications of our results in Section 4 will be examined in this last section. We are concerned with the equation of *even* order $n \in \mathbb{N}$

$$y^{\Delta^n} + \sum_{i=1}^{m} f_i(t)F_i\left(y, y^{\Delta}, \ldots, y^{\Delta^{n-1}}\right) = 0, \tag{27}$$

where all f_i are assumed to be nonnegative functions on $\mathbf{T}$, and we suppose

$$u_1 F_i(u_1, \ldots, u_n) > 0 \quad \text{for all} \quad 1 \leq i \leq m,\ \{u_k :\ 1 \leq k \leq n\} \subset \mathbb{R},\ u_1 > 0. \tag{28}$$

As in Section 6 we assume that $\mathbf{T}$ is unbounded above. The following is our main result on oscillatory behavior of solutions of (27).

Theorem 8 *(Oscillatory Behavior of Solutions of (27)).* Suppose (28) holds and assume there exists $j \in \{1, \ldots, m\}$ such that F_j is continuous at $(u_1, 0, \ldots, 0)$,

$$F_j(\lambda u_1, \ldots, \lambda u_n) = \lambda^{2\alpha+1} F_j(u_1, \ldots, u_n) \quad \text{for all} \quad \lambda \in \mathbb{R}$$

for some $\alpha \geq 0$ and $\int_{t_1}^{\infty} f_j(\tau)\Delta\tau = \infty$ for some $t_1 \in \mathbf{T}$. Then (27) has no solution which is eventually of fixed sign.

Proof. Suppose for the sake of achieving a contradiction that there exists a solution y of (27) such that y is eventually positive (the other case can be treated similarly), i.e.,

$$y(t) > 0 \quad \text{for all} \quad t \in [a, \infty) \cap \mathbf{T},$$

where $a \in \mathbf{T}$. Then for all $t \in [a,\infty) \cap \mathbf{T}$ we have

$$y^{\Delta^n}(t) = -\sum_{i=1}^{m} f_i(t) F_i\left(y(t), y^{\Delta}(t), \ldots, y^{\Delta^{n-1}}(t)\right) \le 0.$$

Now by Kneser's theorem (Theorem 5), there exists $p \in \{0,\ldots,n\}$ with $n+p$ odd such that

$$\begin{cases} (-1)^{p+k} y^{\Delta^k}(t) > 0 \text{ for all } t \in [a,\infty)\cap\mathbf{T},\ k \in [p,n-1]\cap\mathbb{N}_0 \\ y^{\Delta^k}(t) > 0 \text{ eventually, } k \in [1,p-1]\cap\mathbb{N}_0. \end{cases}$$

Since n is even and $n+p$ odd we find $1 \le p \le n-1$ and

$$y^{\Delta^{n-1}}(t) > 0 \text{ for all } t \in [a,\infty)\cap\mathbf{T} \quad \text{and} \quad y^{\Delta}(t) > 0 \text{ for all } t \in [t_2,\infty)\cap\mathbf{T}$$

with $t_2 \in \mathbf{T}$, $t_2 \ge a$. Hence there exists $c > 0$ with

$$y(t) \ge c \quad \text{for all} \quad t \in [t_2,\infty)\cap\mathbf{T}.$$

By putting $v = \frac{y^{\Delta^{n-1}}}{y}$, we have for all $t \in [t_2,\infty)\cap\mathbf{T}$

$$\begin{aligned} v^{\Delta}(t) &= \frac{y^{\Delta^n}(t)}{y(t)} - \frac{y^{\Delta^{n-1}}(\sigma(t))y^{\Delta}(t)}{y(\sigma(t))y(t)} < \frac{y^{\Delta^n}(t)}{y(t)} \\ &= -\sum_{i=1}^{m} f_i(t) \frac{F_i\left(y(t), y^{\Delta}(t), \ldots, y^{\Delta^{n-1}}(t)\right)}{y(t)} \\ &= -\sum_{i=1}^{m} f_i(t) \frac{(y(t))^{2\alpha+1}}{y(t)} F_i\left(1, \frac{y^{\Delta}(t)}{y(t)}, \ldots, \frac{y^{\Delta^{n-1}}(t)}{y(t)}\right) \\ &\le -c^{2\alpha} f_j(t) F_j\left(1, \frac{y^{\Delta}(t)}{y(t)}, \ldots, \frac{y^{\Delta^{n-1}}(t)}{y(t)}\right). \end{aligned}$$

Now, as in [3, Lemma 8], one can find that

$$\lim_{t\to\infty} \frac{y^{\Delta^i}(t)}{y(t)} = 0 \quad \text{for all} \quad 1 \le i \le n-1.$$

Hence for $\varepsilon = 2F_j(1,0,\ldots,0) > 0$ there exists $t_3 \in (t_2,\infty)\cap\mathbf{T}$ with

$$\left| F_j\left(1, \frac{y^{\Delta}(t)}{y(t)}, \ldots, \frac{y^{\Delta^{n-1}}(t)}{y(t)}\right) - F_j(1,0,\ldots,0)\right| < \varepsilon \quad \text{for all} \quad t \in [t_3,\infty)\cap\mathbf{T}.$$

Thus for all $t \in [t_3,\infty)\cap\mathbf{T}$,

$$\begin{aligned} v(t) - v(t_3) &= \int_{t_3}^{t} v^{\Delta}(\tau)\Delta\tau \le -\int_{t_3}^{t} c^{2\alpha} f_j(\tau) F_j\left(1, \frac{y^{\Delta}(\tau)}{y(\tau)}, \ldots, \frac{y^{\Delta^{n-1}}(\tau)}{y(\tau)}\right)\Delta\tau \\ &\le c^{2\alpha}(\varepsilon - F_j(1,0,\ldots,0)) \int_{t_3}^{t} f_j(\tau)\Delta\tau \\ &= c^{2\alpha} F_j(1,0,\ldots,0) \int_{t_3}^{t} f_j(\tau)\Delta\tau \to \infty, \quad t\to\infty \end{aligned}$$

while $\lim_{t\to\infty} v(t) = 0$, a contradiction. ∎

Notes. Discrete versions of related results are contained in [3, Section 3]. Using our version of Kneser's theorem it is possible to develop further results on time scales similar to those presented in [3]. However, e.g. for including the case *odd* n to Theorem 8 it is necessary to first develop further consequences of Kneser's theorem as is done in [3] (see the notes at the end of Section 4). For further related results we refer to [4, Section 6.19].

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Eingegangen am 23. August 1997