Back to Menu

CHAPTER

10

MODULE 1

Difference equations

- What is a difference equation?
- How do we use a difference equation to generate terms in a sequence?
- How do we graph the terms of a sequence generated by a difference equation?
- What is a first-order difference equation?
- How do we recognise when a first-order difference equation defines:
  - an arithmetic sequence?
  - a geometric sequence?
- How do we solve first-order difference equations?
- What is a Fibonacci sequence and how is it generated?
- How do we use difference equations to model a range of real-world situations?

10.1 Introduction

In this chapter you will learn about a new mathematical tool, the difference equation. This will enable you to analyse and model more complex real-world situations than you can with just arithmetic and geometric sequences.

The difference equation

One way of specifying the terms in a sequence is to write down a general rule for the nth term. For example, for an arithmetic sequence with first term $t_1 = 10$ and common difference $d = 5$, the nth term, t_n, is given by:

$$t_n = 10 + (n - 1)5$$

and the sequence is:

$$10,\ 15,\ 20,\ 25,\ 30, \ldots$$

Another way of specifying a sequence is to construct a rule that enables the value of each term in the sequence to be determined from the value of the previous term. For example, for the sequence above, a rule that would generate the sequence is:

'to find successive terms in the sequence, take the current term, starting with 10, and add 5'.

Applying this rule we have:

$$\begin{array}{cccccc} 10, & 15, & 20, & 25, & 30, & \ldots \\ & (10+5) & (15+5) & (20+5) & (25+5) & \end{array}$$

A compact way of writing this rule is as follows:

$$t_{n+1} = t_n + 5 \text{ where } t_1 = 10$$

Here t_{n+1}, represents the $(n + 1)$th term in the sequence, while t_n represents the nth term in the sequence. Such a rule is known as a **difference equation**.

Using this rule, we can generate the sequence as follows

$$\begin{aligned} t_1 &= 10 \\ t_2 &= t_1 + 5 = 15 \\ t_3 &= t_2 + 5 = 20 \\ t_4 &= t_3 + 5 = 25 \\ t_5 &= t_4 + 5 = 30 \quad \text{and so on.} \end{aligned}$$

Definition of a difference equation

A difference equation has two parts:

- a rule that links successive terms in a sequence
- the value of one or more terms in the sequence

For example: $\underbrace{t_{n+1} = 3t_n + 5}_{\text{rule linking terms}}$ where $\underbrace{t_1 = 6}_{\text{value of a term}}$ defines a difference equation.

A comment on notation

An alternative way of writing this difference equation is:

$$t_n = t_{n-1} + 5 \quad \text{where} \quad t_1 = 10$$

where t_n and t_{n-1} are used to represent successive terms. You will also sometimes see different symbols used to represent successive terms in a sequence, and brackets rather than subscripts to represent term numbers. For example, the graphics calculator used in this book uses $u(n - 1)$ and $u(n)$. While it is necessary for you to be able to work with alternative notations, for consistency purposes, we will always use t_{n+1} and t_n in all theory development in this chapter.

Back to Menu

Example 1 **Generating a sequence from a difference equation**

Generate and graph the first five terms of the sequence defined by the difference equation
$t_{n+1} = 3t_n - 1$ where $t_1 = 2$.

Solution

1 Write down the rule for the difference equation and the value of the first term.

$t_{n+1} = 3t_n - 1$

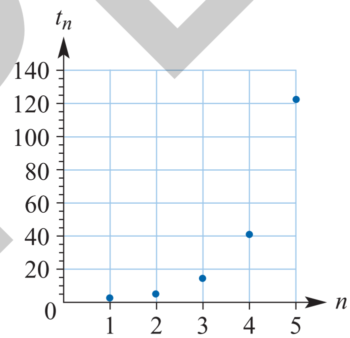
2 In words, the rule says, to find the next term in the sequence
'multiply (the current term) by 3 and subtract 1'.
Applying the rule:

$$t_2 = 3 \times t_1 - 1 = 3 \times 2 - 1 = 5$$
$$t_3 = 3 \times t_2 - 1 = 3 \times 5 - 1 = 14$$

and so on.

$t_1 = 2$
$t_2 = 3 \times 2 - 1 = 5$
$t_3 = 3 \times 5 - 1 = 14$
$t_4 = 3 \times 14 - 1 = 41$
$t_5 = 3 \times 41 - 1 = 122$
and so on.

3 Graph the sequence.



Using a graphics calculator to generate a sequence from a difference equation

While you need to be able to generate and graph the terms of a sequence from a difference equation by hand, it can be a tedious process. When more than just a few terms of the sequence are needed, you can use a graphics calculator. One way of doing this is to use recursion (repeated recalculation) on the Home screen. This method is most useful when you just want to quickly generate a few terms of the sequence. The other way, which you met in the previous chapter, uses Sequence Mode. This enables you to automatically generate the terms in a sequence starting with a difference equation and display them in either a table or a graph. This is very useful when solving application problems.

Back to Menu

How to generate a sequence defined by a difference equation using the TI-Nspire CAS

Generate the first five terms of the sequence defined by the difference equation $t_{n+1} = 3t_n - 1$ where $t_1 = 2$.

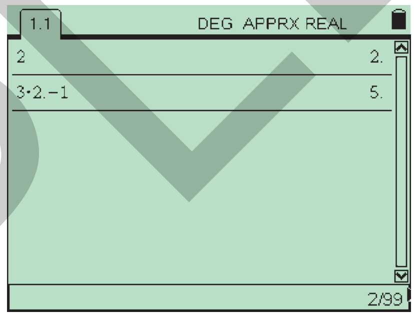
Method 1 (using recursion)

1 Write down the rule for the difference equation and the value of the first term.

$t_{n+1} = 3t_n - 1$ where $t_1 = 2$

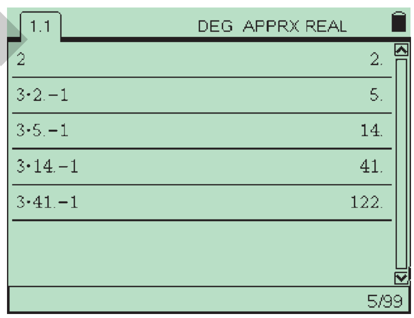
2 Start a new document by pressing ctrl + N. Select **1:Add Calculator**.

3 Type **2**, the value of the first term. Press enter. The calculator stores the value 2 as **Ans**wer. (You can't see this yet.)



4 Now type **3**× **Ans** −**1**, using the keystrokes 3 × ctrl (−) − 1, then press enter. The second term in the sequence is 5. This value is now stored as **Ans**.

5 Press enter to generate the next term. Continue pressing enter until the required number of terms is generated.



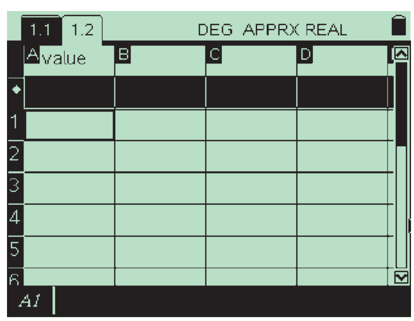
Method 2 (using Sequence mode)

1 Write down the rule for the difference equation and the value of the first term.

$t_{n+1} = 3t_n - 1$ where $t_1 = 2$.

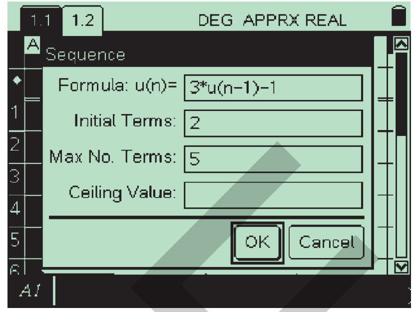
2 Open a **Lists and Spreadsheet** application page and name column A *value*.

Note: We are naming the column now because we want to graph the sequence later.



Back to Menu

3 With the cursor in column A, press (menu)/**3:Data/1:Generate Sequence** to display the pop-up screen shown opposite.
Type in the entries as shown. Use (tab) to move between entry boxes. The **Ceiling Value** (i.e. highest value) can be left blank.

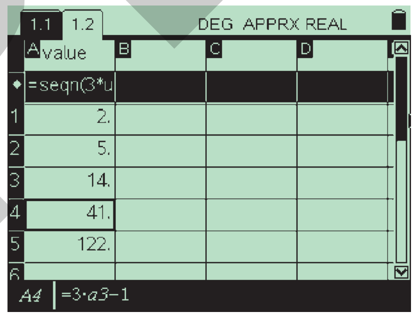


Notes:

1 The calculator uses $u\,(n-1)$ and $\mathrm{u}(n)$ to represent successive terms in the sequence.

2 $t_{n+1} = 3t_{n-1}$ can also be written as $t_n = 3t_{n-1} - 1$; hence, we use $\mathrm{u}\,(n) = 3 \times \mathrm{u}\,(n-1) - 1$ on the calculator.

4 Press (enter) to close the pop-up screen and list the first five terms of the sequence.



Notes:

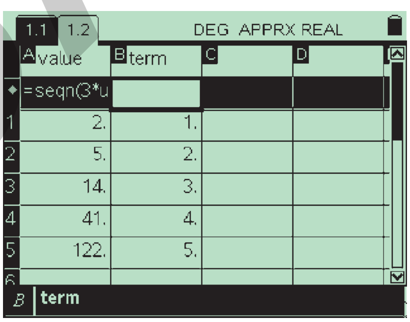
1 The ▼ arrow can be used to move down through the sequence.

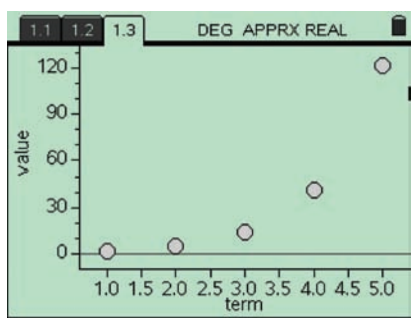
2 The term number can be read directly from the row number (left-hand side) of the spreadsheet. For example, the 4th term in this sequence is 41.

5 The sequence is displayed best using a scatterplot.

a As we are plotting the first five terms only, enter the numbers **1** to **5** in the list called *term* (below left).

b Construct a scatterplot with *term* on the horizontal axis and *value* on the vertical axis to obtain the plot shown below right.





How to generate a sequence defined by a difference equation using the ClassPad

Generate the first five terms of the sequence defined by the difference equation

$t_{n+1} = 3t_n - 1$ where $t_1 = 2$.

Method 1 (using recursion)

1 Write down the rule for the difference equation and the value of the first term.

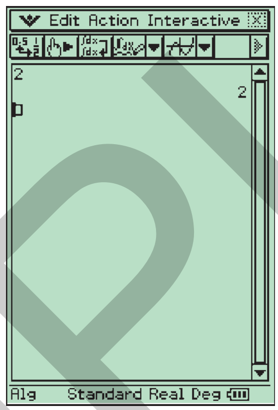
$t_{n+1} = 3t_n - 1$ where $t_1 = 2$

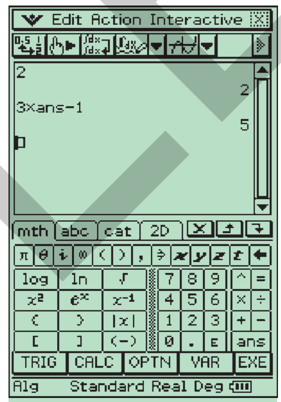
2 From the application menu screen, open the **Main** application, Main.

a Starting with a clean screen, enter the value of the first term, **2**.

Then press EXE.

The calculator stores the value 2 as **Ans**wer. (You can't see this yet.)

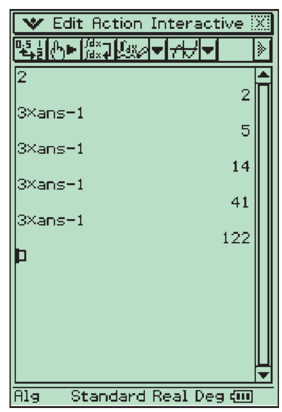




b Press Keyboard and locate the button ans. Enter **3**× **Ans** **−1** (see opposite). Now press EXE.

The second term in the sequence, **5**, is generated and stored as **Ans**wer.

c Pressing EXE again will generate the next term, **14**. Keep pressing EXE until the required number if terms is generated.



Method 2 (using Sequence mode)

1 Write down the rule for the difference equation and the value of the first term.

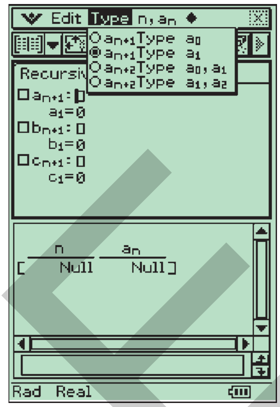
$t_{n+1} = 3t_n - 1$ where $t_1 = 2$.

Back to Menu

2 From the application menu screen, open the **Sequence** application.

The **Sequence** application opens with two half-screens. The top half is used to input the rule defining the sequence. The bottom half displays the sequence in a table format.

To set-up the input screen, first tap the **Recursive** tab, then select **Type** from the menu bar. For **Type**, choose a_{n+1} **Type** a_1.



3 Enter the difference (or recursive) equation as follows:

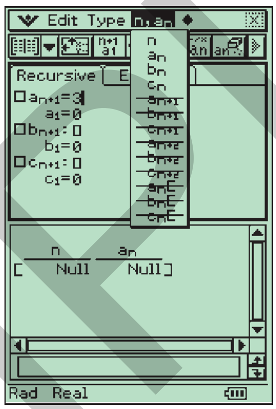
For a_{n+1}: enter $3a_n - 1$ (i.e. the expression for t_{n+1}).

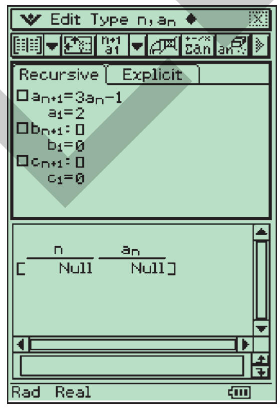
Note: Tapping the $[n, a_n]$ item from the top menu bar will display a sub-menu that contains a_n.

Press EXE to confirm your equation.

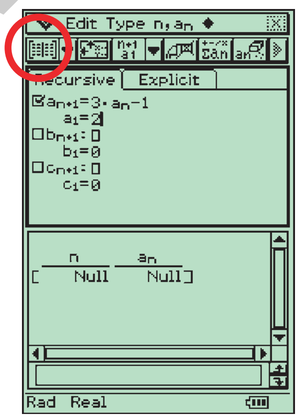
For a_1: enter **2**.

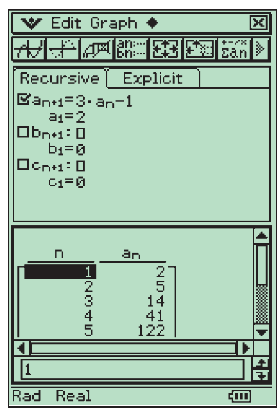
Press EXE to confirm the first value.





4 Tap the table icon from the left-hand side of the toolbar to display the terms of the sequence in table format. The first column n displays the term numbers 1,2,3, . . . and the second column a_n displays the values of the terms in the sequence 2, 5, 14, 41, 122, 365, . . .



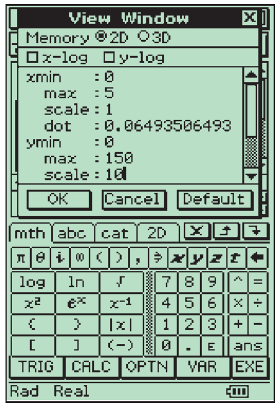


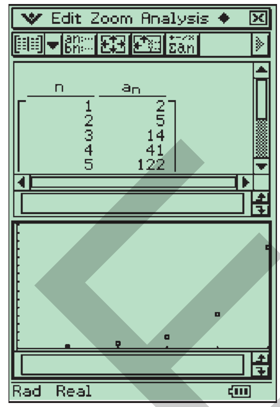
Back to Menu

5 To graph the sequence values, Tap in the toolbar to open the **View Window** dialog box. For

- ***x*min:** type **0**
- **max:** type **5**
- **scale:** type **1**
- ***y*min:** type **0**
- **max:** type **150**
- **scale:** type **10**

Tap OK to confirm your settings.





6 From the toolbar, tap to plot the sequence.

Note: Because the terms of the sequence increase so rapidly, only the first five terms are plotted.

Example 2 Determining a difference equation from a sequence

A difference equation of the form

$t_{n+1} = rt_n + d$ where $t_1 = 10$

generates the sequence 10, 24, 52, . . . Find the value of r and d.

Solution

1 Write down the difference equation rule.
$$t_{n+1} = rt_n + d$$

2 From the sequence, $t_1 = 10$, $t_2 = 24$ and $t_3 = 52$.
$$t_1 = 10, \ t_2 = 24, \ t_3 = 52$$

3 Substitute the values for t_1, t_2 and t_3 into the rule $t_{n+1} = rt_n + d$ to obtain two simultaneous equations with r and d as the unknowns
$$t_2 = rt_1 + d$$
$$\therefore 24 = 10r + d \quad (1)$$
$$t_3 = rt_2 + d$$
$$\therefore 52 = 24r + d \quad (2)$$

4 Solve for r and d.

Subtract (1) from (2)

$\therefore 28 = 14r$ or $r = 2$

Substitute $r = 2$ in (1)

$\therefore 24 = 10 \times 2 + d$ or $d = 4$

5 Write down your answer.

The difference equation is:

$t_{n+1} = 2t_n + 4$ where $t_1 = 10$.

Exercise 10A

1 Generate the first five terms of the sequence defined by the following difference equations.

a $t_{n+1} = t_n + 3$ where $t_1 = 2$

b $t_n = t_{n-1} - 5$ where $t_1 = 50$

c $u_{n+1} = u_n - 3$ where $u_1 = 27$

d $t_{n+1} = 3t_n$ where $t_1 = 1$

e $t_{n+1} = -2t_n$ where $t_1 = 1$

f $t_n = 0.5t_{n-1}$ where $t_1 = 32$

g $R_n = 4R_{n-1} + 3$ where $R_0 = 2$

h $t_{n+1} = 0.25t_n - 1$ where $t_1 = 64$

i $v_n = 4v_{n-1} - 5$ where $v_0 = 5$

j $t_{n+1} = 5t_n - 8$ where $t_1 = 2$

2 Find the values of the unknowns in the difference equation (r, d, a), for each of the following difference equations and sequences.

a $t_{n+1} = 5t_n + d$, where $t_1 = 2$, that generates the sequence 2, 6, 26, . . .

b $P_n = rP_{n-1} + 2$, where $P_1 = 5$, that generates the sequence 5, 7, 9, . . .

c $t_{n+1} = 0.25t_n + 8$, where $t_1 = a$, that generates the sequence a, 64, 24, . . .

d $v_n = rv_{n-1} + d$, where $v_0 = 500$, that generates the sequence 500, 650, 875, . . .

e $T_{n+1} = rT_n + d$, where $T_1 = 1000$, that generates the sequence 1000, 100, −350, . . .

f $t_{n+1} = 4t_n + 2$, where $t_1 = a$, that generates the sequence a, 22, 90, . . .

10.2 The relationship between arithmetic and geometric sequences and difference equations

Arithmetic sequences and difference equations

The defining characteristic of an arithmetic sequence is that successive terms in the sequence differ by a constant amount, called the common difference, d.

Thus, for an **arithmetic sequence** we can write:

$$t_{n+1} - t_n = d \qquad \text{or} \qquad t_{n+1} = t_n + d$$

Difference equation for an arithmetic sequence

The first term of an arithmetic sequence is a and its common difference is d. The difference equation that generates the sequence is:

$$t_{n+1} = t_n + d \quad \text{where} \quad t_1 = a$$

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Example 3 **Finding the difference equation that generates an arithmetic sequence**

Write down a difference equation that generates the arithmetic sequence 20, 15, 10, 5, . . .
Check your result.

Solution

1 Write down the first few terms of the arithmetic sequence, and the values of a and d.

$20, 15, 10, 5, \ldots$ arithmetic sequence
$a = 20 = t_1$, $d = -5$

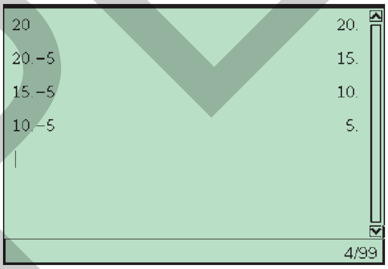
2 For an arithmetic sequence $t_{n+1} = t_n + d$.

$\therefore\ t_{n+1} = t_n + (-5)$
$\quad = t_n - 5$

3 Write down your answer.

The difference equation that generates the sequence is:
$t_{n+1} = t_n - 5$ where $t_1 = 20$

4 Check by calculator.



Geometric sequences and difference equations

The defining characteristic of a geometric sequence is that the ratio of successive terms in the sequence is a constant, which we call the common ratio r.

Thus, for a **geometric sequence** we can write:

$$\frac{t_{n+1}}{t_n} = r \qquad \text{or} \qquad t_{n+1} = r t_n$$

Difference equation for a geometric sequence

The first term of a geometric sequence is a and its common ratio is r. The difference equation that generates the sequence is:

$$t_{n+1} = r t_n \quad \text{where} \quad t_1 = a$$

Example 4 **Finding the difference equation that generates a geometric sequence**

Write down the difference equation that defines the geometric sequence 20, 10, 5, 2.5, . . .
Check your result.

Solution

1 Write down the first few terms of the sequence, and the values of a and r.

$20, 10, 5, 2.5, \ldots$ (Geometric sequence)

$a = 20 = t_1$

$r = \dfrac{t_{n+1}}{t_n} = \dfrac{10}{20} = 0.5$

2 For a geometric sequence $t_{n+1} = rt_n$.

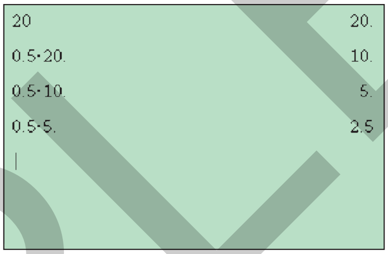
$\therefore\ t_{n+1} = 0.5\,t_n$

3 Write down your answer.

The difference equation that generates the sequence is:

$t_{n+1} = 0.5t_n$ where $t_1 = 20$

4 Check by calculator.



Exercise 10B

1 The following are arithmetic sequences. Write down difference equations that can be used to generate the sequences, and check your results with a graphics calculator.

a 5, 10, 15, 20, 25, . . . **b** 12, 8, 4, 0, . . . **c** 0.1, 0.11, 0.12, 0.13, . . .
d 15, 30, 45, 60, . . . **e** −9, 0, 9, . . . **f** 1.1, 1.2, 1.3, . . .

2 The following are geometric sequences. Write down difference equations that can be used to generate the sequences, and check your results with a graphics calculator.

a 2, 4, 8, 16, . . . **b** 1, 1.1, 1.21, . . . **c** 100, 80, 64, . . .
d 0.1, 1, 10, 100, . . . **e** 12, 144, 1728, . . . **f** −4, −6, −9, . . .

3 The following are either arithmetic or geometric sequences. Write down difference equations that can be used to generate the sequences, and check your results with a graphics calculator.

a 6, 12, 24, . . . **b** 10, 12, 14, . . . **c** 100, 90, 80, 70, . . .
d 1, 1.2, 1.44, . . . **e** 1.8, 1.62, 1.458, . . . **f** 8, −16, 32, . . .

10.3 First-order difference equations

The difference equations generating arithmetic and geometric sequences are special cases of what are called **first-order** difference equations. They are called first-order difference equations because the rule links terms that are only **one step** apart in the sequence.

First-order difference equations

First-order difference equations are defined by rules of the form:

$$t_{n+1} = rt_n + d \quad \text{where} \quad t_1 = a$$

Here r, d and a, are constants.

The following are examples of first-order difference equations.

- $t_{n+1} = 2t_n + 3 \quad$ where $\quad t_1 = 2$ is a first-order difference equation with $r = 2, d = 3$
- $t_{n+1} = t_n + 10 \quad$ where $\quad t_1 = 20$ is a first-order difference equation with $r = 1, d = 10$

  You should also recognise that this difference equation generates an **arithmetic** sequence with a common difference $d = 10$ and first term $a = 20$. A first-order difference equation with $r = 1$ will generate an arithmetic sequence.
- $t_{n+1} = 0.5t_n \quad$ where $\quad t_1 = 64$ is a first-order difference equation with $r = 0.5, d = 0$

  You should also recognise that this difference equation represents a **geometric** sequence with common ratio $r = 0.5$ and first term $a = 64$. A first-order difference equation with $d = 0$ will generate a geometric sequence.

The following is **not** an example of first-order difference equations:

- $t_n = t_{n-2} + t_{n-1} \quad$ where $\quad t_1 = t_2 = 1$

  This is not a a first-order difference equation because it links terms that are up to **two steps** apart. Because of this, it is known as a **second-order** difference equation. This difference equation generates what is known as a Fibonacci sequence. You will learn more about Fibonacci sequences later in the chapter.

What is special about first-order difference equations?

First-order difference equations are special in that it is always possible to solve a first-order difference equation. That is, for a first-order difference equation it is always possible to write down an expression for the nth term of the sequence generated by the difference equation.

Sometimes, solving the general form of the first-order difference equation is quite complex. However, if the sequence represented by the difference equation is arithmetic or geometric, we can solve the difference equation using our knowledge of arithmetic or geometric sequences. Thus we can save ourselves much work if we can recognise, right from the start, whether the sequence represented by the difference equation is arithmetic or geometric.

Exercise 10C

1 Which of the following could be rules for first-order difference equations?

a $t_{n+1} = 3t_n - 4$ **b** $t_{n+1} = t_n + 2$ **c** $t_{n+1} = 5n^2$ **d** $t_{n+1} = t_{n-1} + t_{n-2}$

e $u_n = 5u_{n-1} - 3u_{n-2}$ **f** $t_{n+1} = 1$ **g** $t_{n+1} = 5t_n$ **h** $t_n - 4t_{n-1} = 2$

2 The general equation for a first-order difference equation is

$$t_{n+1} = rt_n + d \quad \text{where } t_1 = a.$$

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For each of the following first-order difference equations, write down the values of r, d and a.

a $t_{n+1} = 3t_n - 4, t_1 = 3$ **b** $t_{n+1} = t_n + 2, t_1 = 4$ **c** $t_{n+1} = 5t_n, t_1 = 1$
d $t_{n+1} = t_n + 2, t_1 = 0$ **e** $u_{n+1} = 5u_n - 3, u_1 = -1$ **f** $G_{n+1} = -0.5G_n, G_1 = 4$
g $t_{n+1} = 2 - t_n, t_1 = 3$ **h** $2t_{n+1} + t_n = 4, t_1 = 3$ **i** $t_{n+1} + 2t_n = -2, t_1 = 0$

3 The general equation for a first-order difference equation is

$$t_{n+1} = rt_n + d \quad \text{where} \quad t_1 = a.$$

There are two special cases:

- when $r = 1$, the difference equation generates an arithmetic sequence
- when $d = 0$, the difference equation generates a geometric sequence

For each of the following first-order difference equations, identify it as generating:

i an arithmetic sequence **ii** a geometric sequence **iii** neither

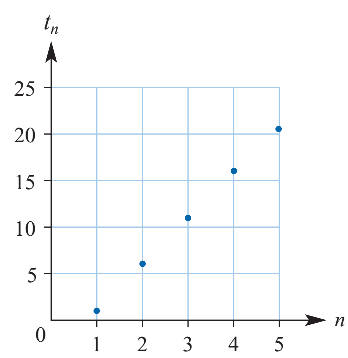
a $t_{n+1} = 3t_n - 4, t_1 = 3$ **b** $t_{n+1} = t_n + 2, t_1 = 4$ **c** $t_{n+1} = 5t_n, t_1 = 1$
d $t_{n+1} = t_n - 3, t_1 = 0$ **e** $u_{n+1} = 5u_n - 3, u_1 = -1$ **f** $G_{n+1} = -0.5G_n, G_1 = 4$
g $t_{n+1} = 2 - t_n, t_1 = 3$ **h** $t_{n+1} - t_{n-1} = 5, t_1 = 3$ **i** $t_{n+1} + 2t_n = -2, t_1 = 0$

10.4 Solving first-order difference equations that generate arithmetic sequences

The first-order difference equation:

$$t_{n+1} = t_n + 5 \quad \text{where} \quad t_1 = 1$$

generates the **arithmetic** sequence 1, 6, 11, 16, 21 . . .



When terms of the sequence, t_n, are plotted against term number $\boldsymbol{n}$, the points lie on a **straight line**. This shows that the terms in the sequence increase in value in a linear fashion. We can confirm this by determining an expression for the nth term.

Remember that for an arithmetic sequence with first term a and common difference d, the nth term is given by:

$$t_n = a + (n - 1)d$$

For this sequence $a = 1$ and $d = 5$, so that the nth term is given by:

$$t_n = 1 + (n - 1)5 \quad \text{or} \quad t_n = 5n - 4$$

which is a linear equation in n. Thus the terms in the sequence follow a **linear** pattern.

Solution of first-order difference equations of the form $t_{n+1} = t_n + d$ $(r = 1)$

Difference equations of the form $t_{n+1} = t_n + d$, where $t_1 = a$, generate arithmetic sequences.

The solution is given by the nth term of this sequence: $t_n = a + (n - 1)d$.

Example 5 **Solving a difference equation that generates an arithmetic sequence**

Solve the difference equation $t_{n+1} = t_n - 3$ where $t_1 = 2$.

Solution

1 The sequence generated by the difference equation is arithmetic, with $a = 2$ and $d = -3$.

Difference equation generates an arithmetic sequence with $a = 2$ and $d = -3$

2 The solution of the difference equation is given by the nth term of an arithmetic sequence: $t_n = a + (n - 1)d$. Substitute $a = 2$ and $d = -3$ into this expression. Simplify.

$$\therefore t_n = 2 + (n - 1) \times (-3)$$
$$= 2 - 3n + 3$$
$$= 5 - 3n$$

3 Write down your answer.

The solution of the difference equation is $t_n = 5 - 3n$

Skillsheet TEACHER

Exercise 10D

1 Which of the following first-order difference equation rules generate arithmetic sequences?

a $t_{n+1} = 5t_n - 4$ b $t_{n+1} = t_n - 4$ c $t_{n+1} = 5t_n$ d $t_n = t_{n-1} + 2$
e $t_n = 6 - t_{n-1}$ f $G_{n+1} = G_n - 3$ g $u_n = u_{n-1} + 3$ h $t_n = t_{n-1} + 3$

2 The following first-order difference equations generate arithmetic sequences. Write down their solutions.

a $t_{n+1} = t_n + 1, t_1 = 5$ b $t_{n+1} = t_n - 4, t_1 = 25$ c $t_n = t_{n-1} + 3, t_1 = 5$
d $G_{n+1} = G_n - 2, G_1 = 10$ e $u_n = u_{n-1} - 1, u_1 = 4$ f $t_n = t_{n-1} - 5, t_1 = 10$

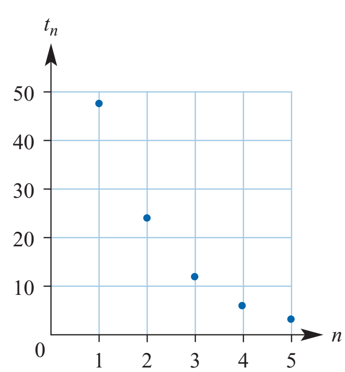
10.5 Solving difference equations that generate geometric sequences

The difference equation:

$$t_{n+1} = 0.5t_n \quad \text{where} \quad t_1 = 48$$

represents the **geometric** sequence 48, 24, 12, 6, 3, . . .

When terms of the sequence, t_n, are plotted against term number, n, the points lie on a curve. The **graph** shows that the terms in the sequence decrease in value in a **non-linear** fashion. We can be more precise about the curve by finding an expression for the nth term.



Remember that the nth term of a geometric sequence with first term a and common ratio r is given by:

$$t_n = ar^{n-1}$$

For this sequence $a = 48$ and $r = 0.5$, so that the nth term is given by:

$$t_n = 48(0.5)^{n-1}$$

which is an **exponential** equation in n. Thus the terms in the sequence follow a **non-linear** pattern.

Solution of first order difference equations of the form $t_{n+1} = rt_n$ ($d = 0$)

Difference equations of the form $t_{n+1} = rt_n$, where $t_1 = a$, generate geometric sequences. The solution is given by the nth term of this sequence: $t_n = ar^{n-1}$.

Example 6 Solving a difference equation that generates a geometric sequence

Solve the difference equation $t_{n+1} = 2t_n$ where $t_1 = 3$.

Solution

1 The sequence generated by the difference equation is geometric, with $a = 3$ and $r = 2$.

2 The solution of the difference equation is given by the nth term of a geometric sequence: $t_n = ar^{n-1}$. Substitute $a = 3$ and $r = 2$ into this expression. Write down your answer.

Difference equation generates a geometric sequence with $a = 3$ and $r = 2$

$\therefore t_n = 3 \times 2^{n-1}$

A solution of the difference equation is $t_n = 3 \times 2^{n-1}$.

Exercise 10E

1 Which of the following first order difference equation rules generate geometric sequences?

a $t_{n+1} = 5t_n - 4$ b $t_{n+1} = t_n - 4$ c $t_{n+1} = 5t_n$ d $t_n = -4t_{n-1}$

e $t_n = 6 - t_{n-1}$ f $G_{n+1} = 0.5G_n$ g $u_n = 1.01u_{n-1}$ h $t_n = t_{n-1} + 2$

2 The following first-order difference equations generate geometric sequences. Write down their solutions.

a $t_{n+1} = 2t_n$, $t_1 = 10$ b $t_{n+1} = 1.5t_n$, $t_1 = 8$ c $t_n = -2t_{n-1}$, $t_1 = 12$

d $G_{n+1} = 0.90G_n$, $G_1 = 10$ e $u_n = -u_{n-1}$, $u_1 = 4$ f $t_n = 5t_{n-1}$, $t_1 = 10$

10.6 Solution of general first-order difference equations (optional)

Finding an expression for the nth term of a general first-order difference equation

Consider the difference equation:

$$t_{n+1} = rt_n + d \quad \text{where} \quad t_1 = a$$

For $n = 1$

$$t_2 = rt_1 + d = ar + d$$

For $n = 2$

$$t_3 = rt_2 + d = r(ar + d) + d = ar^2 + d(1 + r)$$

For $n = 3$

$$t_4 = rt_3 + d = r(ar^2 + d(r + 1)) + d = ar^3 + d(1 + r + r^2)$$

so that, generalising:

$$t_n = ar^{n-1} + d(1 + r + r^2 + \cdots + r^{n-2})$$

Noting that $1 + r + r^2 + \cdots + r^{n-2}$ is the sum to $n - 1$ terms of a geometric sequence with $a = 1$ and common ratio r, we can rewrite t_n as:

$$t_n = ar^{n-1} + d\frac{(r^{n-1} - 1)}{r - 1} \quad r \neq 1$$

Solution of general first-order difference equations of the form $t_{n+1} = rt_n + d \; (r \neq 1)$

Difference equations of the form $t_{n+1} = rt_n + d$ where $t_1 = a$, generate a sequence that is neither arithmetic nor geometric.

The solution is given by the nth term of this sequence: $t_n = ar^{n-1} + d\dfrac{(r^{n-1} - 1)}{r - 1} \quad r \neq 1$

Example 7 **Using the formula to solve a general first-order difference equation**

Solve the difference equation $t_{n+1} = 2t_n - 6$ where $t_1 = 7$.

Solution

1 This is a general first-order linear difference equation with solution:

$$t_n = ar^{n-1} + d\frac{(r^{n-1} - 1)}{r - 1} \quad r \neq 1,$$

with $a = 7, r = 2$ and $d = -6$

$$t_n = ar^{n-1} + d\frac{(r^{n-1} - 1)}{r - 1} \quad r \neq 1$$

$a = 7$, $r = 2$ and $d = -6$

2 Substitute these values in the equation for t_n and evaluate.

$$\begin{aligned} \therefore t_n &= 7 \times 2^{n-1} + (-6)\left(\frac{2^{n-1} - 1}{2 - 1}\right) \\ &= 7 \times 2^{n-1} - 6 \times (2^{n-1} - 1) \\ &= 7 \times 2^{n-1} - 6 \times 2^{n-1} + 6 \\ &= 2^{n-1} + 6 \end{aligned}$$

3 Write down your answer.

The solution of the difference equation is $t_n = 2^{n-1} + 6$.

Back to Menu

Exercise 10F

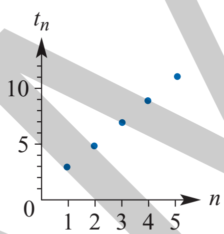
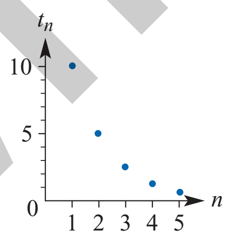
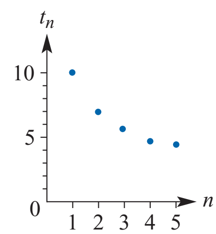
1 The following are general first-order difference equations. Write down their solutions.

a $t_{n+1} = 1.5t_n - 8,\ t_1 = 32$ **b** $t_{n+1} = 0.5t_n + 14,\ t_1 = 20$

c $t_{n+1} = 0.5t_n - 10,\ t_1 = 20$

10.7 Summary of first-order difference equations

An important skill is the ability to differentiate between the three types of first-order difference equations by the sequences they generate. The table below should help you in this task.

Difference equation	$t_{n+1} = t_n + d,\ t_1 = a$	$t_{n+1} = rt_n,\ t_1 = a$	$t_{n+1} = rt_n + d,\ t_1 = a$
Sequence type	arithmetic	geometric	neither arithmetic nor geometric
***n*th term**	$t_n = a + (n-1)d$	$t_n = ar^{n-1}$	$t_n = ar^{n-1} + d\dfrac{(r^{n-1} - 1)}{r - 1}$
Example			
equation	$t_{n+1} = t_n + 2,\ t_1 = 3$	$t_{n+1} = 0.5t_n,\ t_1 = 10$	$t_{n+1} = 0.5t_n + 2,\ t_1 = 10$
sequence	3, 5, 7, . . .	10, 5, 2.5, . . .	10, 7, 5.5, . . .
graph			
***n*th term**	$t_n = 3 + (n-1)2 = 2n + 1$	$t_n = 10(0.5)^{n-1}$	$t_n = 6 \times 0.5^{n-1} + 4$

Exercise 10G

1 For each of the following first-order difference equations:

a identify as generating

i an arithmetic sequence **ii** a geometric sequence

iii neither an arithmetic nor a geometric sequence

b write out the first five terms of the sequences they define

c plot a graph of t_n against n for $1 \le n \le 5$

d if *arithmetic or geometric*, write down an expression for t_n in terms of n

A	$t_{n+1} = t_n - 5$	where	$t_1 = 35$	**B**	$t_{n+1} = 0.25t_n$	where	$t_1 = 64$
C	$t_{n+1} = t_n + 5$	where	$t_1 = 0$	**D**	$t_{n+1} = 1.1t_n$	where	$t_1 = 100$
E	$t_{n+1} = t_n + 0.5$	where	$t_1 = 0$	**F**	$t_{n+1} = 1.5t_n - 8$	where	$t_1 = 32$
G	$t_{n+1} = 0.5t_n + 10$	where	$t_1 = 20$	**H**	$t_{n+1} = 0.5t_n + 14$	where	$t_1 = 20$
I	$t_{n+1} = 0.5t_n - 10$	where	$t_1 = 20$	**J**	$t_{n+1} = 0.5t_n + 0.5$	where	$t_1 = 1$
K	$t_{n+1} = 10t_n$	where	$t_1 = 0.1$	**L**	$t_{n+1} = 2t_n - 2$	where	$t_1 = 1$

10.8 Applications of first-order difference equations

First-order difference equations are very useful mathematical tools for describing growth and decay in such things as animal populations, investments and loans, and the values of goods and services. You will meet some of these applications in this section.

First-order difference equations can be solved by finding the value of the nth term. However, in application problems it is far more convenient and acceptable to use your calculator to generate and manipulate the required terms in the sequences.

Example 8 **Applications: managing a budget**

Jarrad has moved to an interstate university to study law. Over the summer break he has accumulated \$3635. He wants to use the money to pay his living expenses while studying. He plans to allow himself \$165 per week to spend on general living expenses. Assume that Jarrad sticks to his plan.

a Write down a difference equation to describe the reduction in Jarrad's savings week by week.

b Determine the number of weeks his money will last.

Solution

a Write down a difference equation to describe the reduction in Jarrad's savings week by week.

1. Identify, name and define the key variable.

   Let S_n be the value of Jarrad's savings at the start of week n.

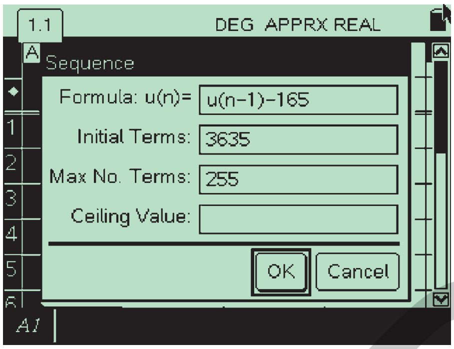
2. Jarrad plans to spend \$165 per week. At the start of week 1 he has \$3635. The sequence is arithmetic. Use this information to write down a difference equation.

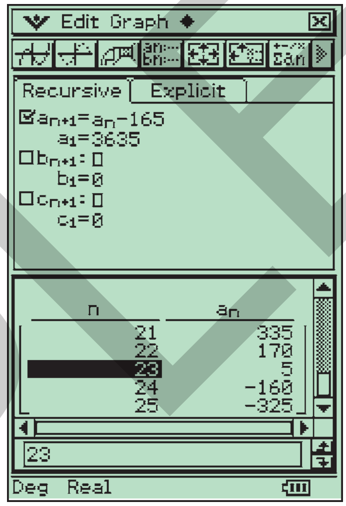
   arithmetic sequence:
   $d = -165,\ a = 3635$
   $\therefore S_{n+1} = S_n - 165$ where $S_1 = 3635$

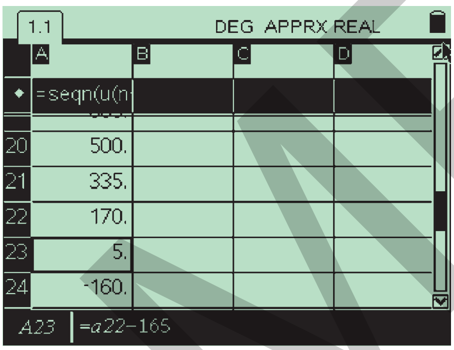
Back to Menu

b Determine the number of weeks his money will last.

1 Use your calculator to list the sequence of terms generated by this difference equation. Find the first term in the sequence that has a value less than 165. This is the 23rd term, which has a value of 5, meaning that Jarrad has only $5 left to spend at the start of the 23rd week. Thus, Jarrad can only spend $165 per week for 22 weeks.







2 Use the down arrow key (▾) to find when $\mathbf{u_n}$ first has a value below 165. This is week 23, when there is only $5 left to spend. So Jarrad can only spend $165 per week for 22 weeks.

3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	. . .	21	22	23 . . .
S_n	3635	. . .	335	170	5 . . .

If Jarrad needs $ 165 a week, his money will last 22 weeks.

Example 9 Applications: car depreciation

Your neighbour has just bought a new car for $29 790. When you look up a motoring magazine on secondhand car prices you find out that this particular model of car loses, on average, 17% of its value each year.

a Write down a difference equation to describe the decreasing value of the car each year.

b What will be the secondhand value of the car after your neighbour has owned it for six years, that is, at the start of the seventh year? Give your answer correct to the nearest dollar.

Back to Menu

Solution

a Write down a difference equation to describe the decreasing value of the car each year.

1 Identify, name and define the key variable.

Let V_n be the value of the car at the start of the nth year

2 Use the fact that the car loses 17% of its value each year to write down an expression for V_{n+1} in terms of V_n.

The value of the car at the start of year 1 is \$29 790, so $V_1 = 29\,790$.

Combine this information to write down the difference equation.

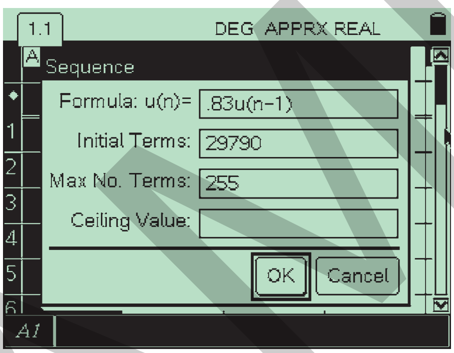
$$V_{n+1} = V_n - 17\% \text{ of } V_n$$
$$= V_n - \frac{17}{100} \times V_n$$
$$= V_n(1 - 0.17) = 0.83V_n$$

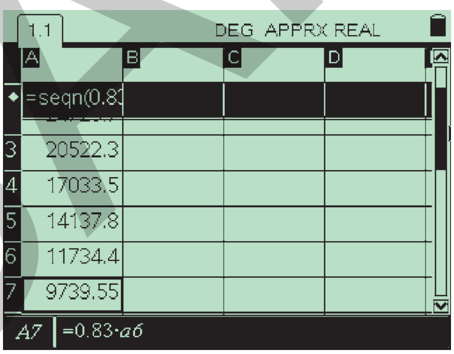
Difference equation:

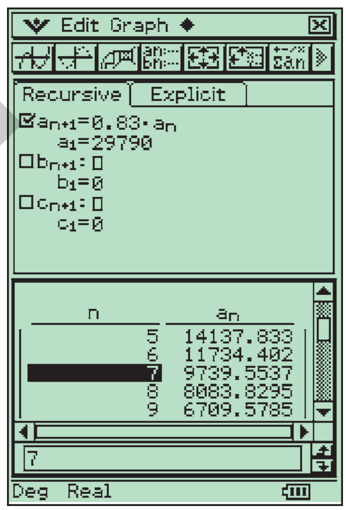
$V_{n+1} = 0.83V_n$ where $V_1 = 29\,790$

b What will be the secondhand value of the car after your neighbour has owned it for six years, that is, at the start of the seventh year? Give your answer correct to the nearest dollar.

1 Use your calculator to list the sequence of terms generated by this difference equation. Find the 7th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer

n	1	. . .	6	7
V_n	29 790	. . .	11 734	9739.6

The value of the car after six years is \$9740.

Example 10 Applications: Population dynamics

When first investigated, a lake in a national park contained 10 000 trout. If left to natural forces, the trout numbers in the lake would increase, on average, by 20% per year. On this basis, the park authorities give permission for 1800 trout to be fished from the lake each year.

a Write down a difference equation that determines the number of trout in the lake each year.
b Under these conditions, how long will it take for trout numbers in the lake to double?
c Graph trout numbers against year for 15 years. Comment on the pattern of growth.
d Suppose that the park authorities had allowed 2200 trout to be fished from the lake each year. Write down a difference equation that determines the number of trout in the lake each year.
e Graph trout numbers against year for 15 years. Comment on the pattern of growth.
f If 2200 trout are fished from the lake each year, the trout will disappear. In which year?
g Suppose the park authorities allow 2000 trout to be fished from the lake each year. Investigate.

Solution

a Write down a difference equation that determines the number of trout in the lake each year.

1. Identify, name and define the key variable.

   Let T_n be the number of trout in the lake at the start of the nth year.

2. Two factors influence trout numbers in the lake:
   - a natural increase of 20% per year
   - a decrease of 1800 trout per year due to fishing

   Use this information to write down an expression for T_{n+1} in terms of T_n.

   $$T_{n+1} = T_n + 20\% \text{ of } T_n - 1800$$
   $$= T_n + 0.2T_n - 1800$$
   $$= 1.2T_n - 1800$$

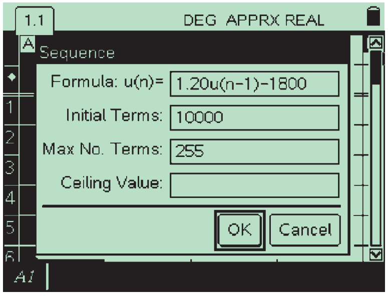
3. The number of trout in the lake at the start of Year 1 is 10 000, so $T_1 = 10\,000$. Combine this information with the expression T_{n+1} to write down the required difference equation.

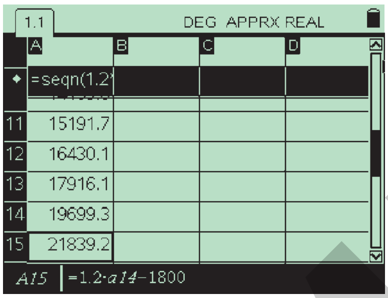
   Difference equation
   $T_{n+1} = 1.2T_n - 1800, \ T_1 = 10\ 000$

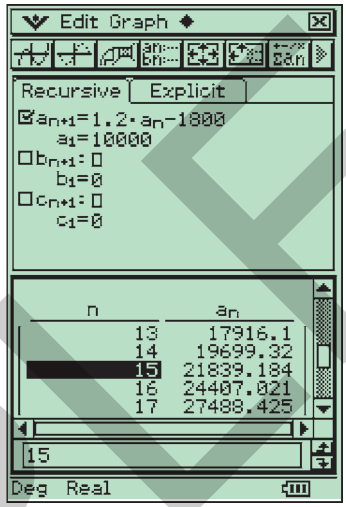
Back to Menu

b Under these conditions, how long will it take for trout numbers in the lake to double?

1 Use your calculator to list the sequence of terms generated by this difference equation. Find the first term in the sequence that exceeds 20 000.







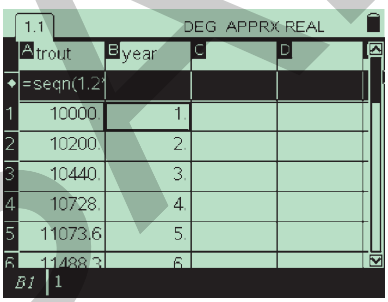
2 Write down key values in the sequence (to show how you solved the problem) and your answer.

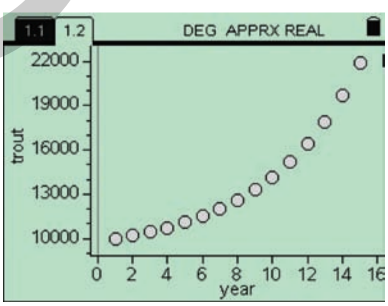
$n =$	1	...	14	15	...
$T_n =$	10 000	...	19 699	21 839	...

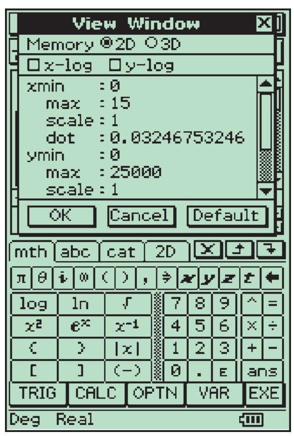
Trout numbers double during the 14 th year.

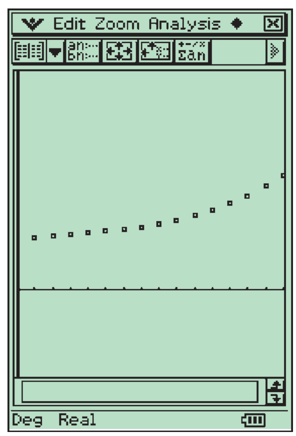
c Graph trout numbers against year for 15 years. Comment on the pattern of growth.

1 Plot the sequence.









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2 Use the plot to comment on the growth in trout numbers.

Over the 15-year period, trout numbers increase in a non-linear manner.

d Suppose that the park authorities had allowed 2200 trout to be fished from the lake each year. Write down a difference equation that determines the number of trout in the lake each year.

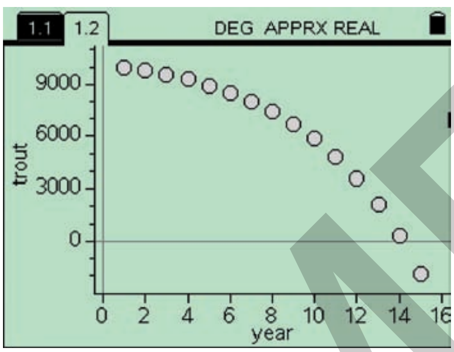
1 To investigate the effect of allowing 2200 trout to be fished from the lake, change the 1800 in the original difference equation to 2200.

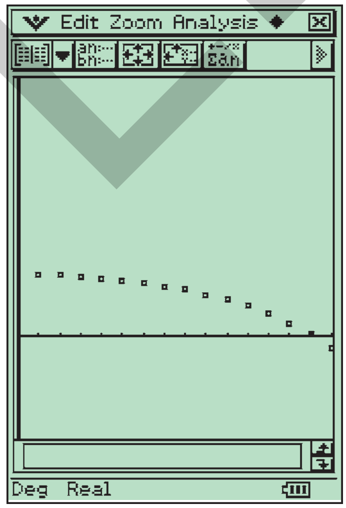
Difference equation:

$T_{n+1} = 1.2T_n - 2200, T_1 = 10\ 000$

e Graph trout numbers against year for 15 years. Comment on the pattern of growth.

1 Replace the value of 1800 in the original difference equation stored in your calculator with 2200 and replot the sequence.



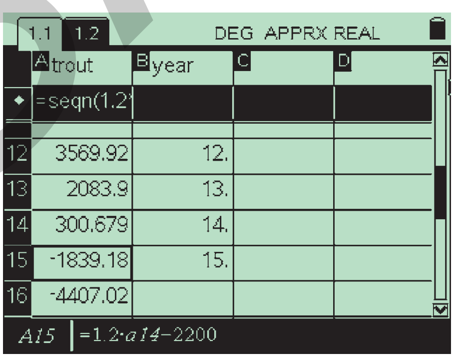


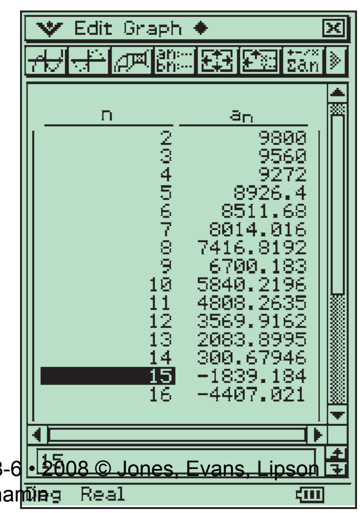
2 Use the plot to comment on the growth in trout numbers.

If the number of trout that can be fished from the lake is increased to 2200 per year, trout numbers will decrease.

f If 2200 trout are fished from the lake each year, the trout will disappear. In which year?

1 List the terms of the sequence on your calculator and determine the value of n when a term first becomes zero or negative.





2 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	...	14	15	...
Tn	10 000	...	300.68	−1839	...

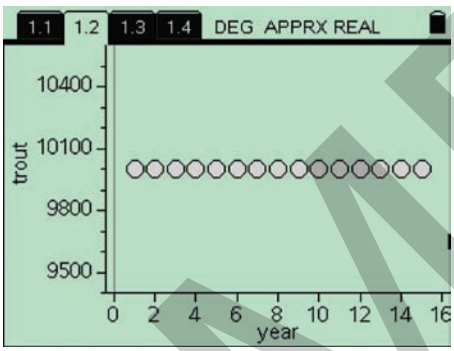
The trout will disappear from the lake during the 14th year.

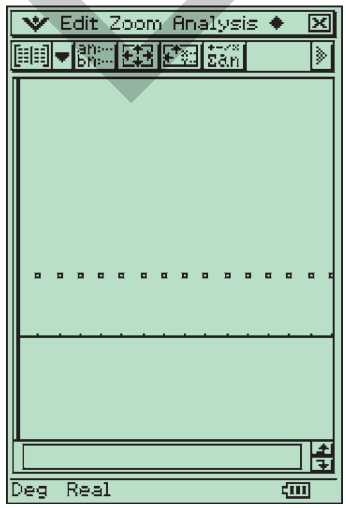
g Suppose the park authorities allow 2000 trout to be fished from the lake each year. Investigate.

1 The effect of allowing 2000 fish to be taken from the lake by anglers can be investigated by changing the 2200 in the original difference equation to 2000.

Difference equation:
$T_{n+1} = 1.2T_n - 2200$, $T_1 = 10\,000$

Replace the value of 2200 in the original difference equation stored in your calculator with 2000 and replot the sequence.





2 Use the plot to comment on the growth in trout numbers.

Trout numbers will remain constant.

Exercise 10H

1 Rob is offered a job at \$275 per week with yearly increments of \$45 per week.

a Write down a difference equation of the form $W_{n+1} = W_n + d$ where $W_1 = a$ that can be used to describe the growth in Rob's weekly wage year by year. (W_n represents Rob's wage at the start of the nth year.)

b Solve the difference equation by finding an expression for W_n, Rob's wage in year n.

c Use this expression to determine Rob's wage in year 7.

2 Sarah is saving to go on a long overseas holiday. With savings, combined with money earned working as a waitress, she has accumulated \$4670. She wants to use the money to pay her living expenses while away. She plans to allow herself \$70 per week to spend on general living expenses. If Sarah sticks to her plan:

a write down a difference equation that can be used to describe the reduction in Sarah's savings by the week

b determine:

i how much Sarah will have left at the start of the 37th week

ii the last week in which she will still have \$70 to spend

3 A superball is dropped from a height of 3 metres and rebounds to 75% of its original height. If the ball is allowed to continue bouncing:

a write down a difference equation of the form $H_n = rH_{n-1}$ where $H_0 = a$ that can be used to describe this situation (H_n represents the height of the ball on its nth bounce.)

b write down an expression for the height of the bounce on the nth bounce

c calculate the height of the ball on the 6th bounce

4 You intend to buy a new car, which costs \$15 990. When you look up a motoring magazine on second-hand car prices you find that this particular model of car loses, on average, 14.5% of its value each year.

a Write down a difference equation of the form $V_{n+1} = rV_n$ where $V_1 = a$ that can be used to describe the decreasing value of the car each year. (V_n represents the value of the car at the start of the nth year.)

b determine:

i the value of the car after you have owned it for nine years (that is, at the start of the 10th year)

ii during which year the car's value will first fall below \$1000

5 Two-and-a-half million bacteria were added to a culture in a test tube. In this culture, the bacteria are able to double in number every 12 hours.

a Write down a difference equation that can be used to describe the growth in the number of bacteria in the test tube every 12 hours.

b What will be the number of bacteria in the test tube at the start of the third day?

6 Provided the conditions are right, a fish population increases its size by 60% every year. If there are 1000 fish to start with:

a write down a difference equation of the form $N_{n+1} = rN_n$ where $N_1 = a$ that can be used to describe this situation (N_n represents the number of fish at the start of the nth year.)

b write down an expression for the number of fish at the start of the nth year

c calculate the number of fish at the start of the 11th year

d determine in which year the fish population will first exceed 1 000 000

7 A large water storage tank contains 5000 litres of water when full. If water is pumped out of the tank at a constant rate of 120 litres per minute:

- **a** write down a difference equation of the form $V_{n+1} = V_n - d$ where $V_1 = a$ that can be used to describe this situation. (V_n represents the volume of water in the tank at the start of the nth minute.)
- **b** write down an expression for the volume of water in the tank at the start of the nth minute
- **c** calculate the volume of water in the tank after 15 minutes
- **d** to the nearest minute, calculate the time taken to empty the tank

8 Wild deer are causing a problem in a nature reserve. Under normal conditions, the deer population grows at a rate of 22% per year. When counted at the start of the year there were 1356 deer in the nature reserve.

- **a** Write down a difference equation of the form $N_{n+1} = rN_n$ where $N_1 = a$ that can be used to describe the growth of the deer population in the nature reserve under normal conditions. (N_n represents the number of deer in the nature reserve at the start of the nth year.)
- **b** Use the difference equation and a graphics calculator to complete the table.

Start of year (n)	1	2	3	4	5
Number of deer (N_n)					

- **c** Plot a graph of deer numbers against year.

To try to reduce deer numbers in the nature reserve, the rangers recommend that hunters be allowed to take 250 deer from the reserve each year. The deer continue to breed at the same rate.

- **d** Write down a second difference equation of the form $N_{n+1} = rN_n - d$ where $N_1 = a$ that can be used to describe the growth of the deer population in the nature reserve when hunters are allowed to take 250 per year. (N_n represents the number of deer in the nature reserve at the start of the nth year.)
- **e** Use the difference equation and a graphics calculator to complete the table.

Start of year (n)	1	2	3	4	5
Number of deer (N_n)					

- **f** On the same graph as before, plot a graph of the number of deer against year when hunters are allowed to take 250 per year.
- **g** What impact does allowing hunters to take 250 deer per year have on the growth of the deer population?

9 A distant relative has left you \$100 000. You plan to invest the money at 6.9% per annum. Interest is paid at the end of the year. At the end of the year, and after the interest is paid, you plan to withdraw \$6500 and reinvest what remains for another year.

- **a** Write down a difference equation of the form $A_{n+1} = rA_n - d$ where $A_1 = a$ that can be used to describe the amount of your investment, A_n, at the start of the nth year.
- **b** Use the difference equation and a graphics calculator to complete the table.

Year (n)	1	2	3	4	5
Amount (A_n)					

- **c** Is your investment increasing in value or decreasing in value with time?

Back to Menu

d If you continue to invest under the same conditions, what will be the value of your investment:

i at the start of the 12th year? **ii** after 19 years?

e What is the maximum amount of money you could withdraw each year and still have an investment of $100 000?

10.9 The Fibonacci sequence

The Fibonacci sequence

The sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

is known as the **Fibonacci sequence**. It is named after the Italian merchant and mathematician, Fibonacci (Leonardo di Pisa, 1170–1250). This sequence has the property that, *after* the first two terms, each successive term is the sum of the preceding two terms:

$$\begin{aligned}
t_1 &= 1\\
t_2 &= 1\\
t_3 &= 2 = t_1 + t_2\\
t_4 &= 3 = t_2 + t_3\\
t_5 &= 5 = t_3 + t_4\\
t_6 &= 8 = t_4 + t_5 \text{ and so on.}
\end{aligned}$$

Fibonacci proposed this sequence as a way of modelling the growth in the number of rabbits produced from a single pair of breeding rabbits.

The breeding rabbit problem

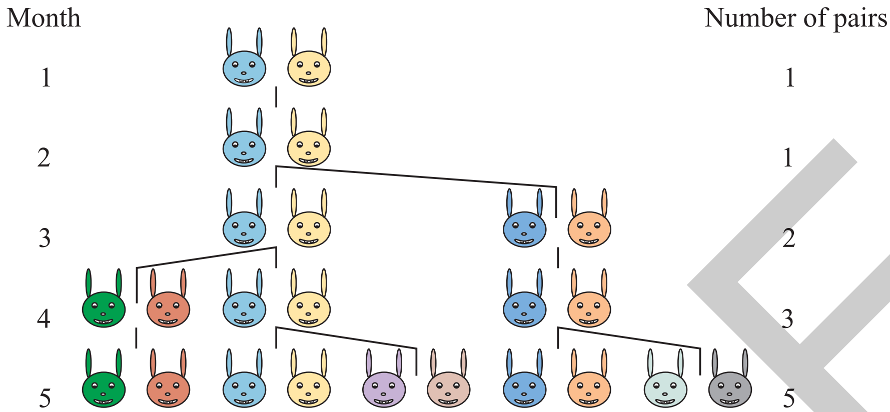
The situation he set out to model was as follows. We start out with a newly born pair of rabbits (one male, one female).

- At the start of the 2nd month, they mate.
- At the start of the 3rd month, the female produces a new pair of rabbits (one male, one female).
- At the start of the 4th month, the older female produces a new pair of rabbits (one male, one female) and the new pair mate, and so on.

Assuming the same breeding pattern continues and no rabbit dies, the problem is to predict the number of pairs of rabbits in successive generations. The situation can be represented pictorially as shown below.

As we can see, the number of pairs of rabbits follows the Fibonacci sequence: 1, 1, 2, 3, 5, . . .

While this is a highly artificial situation, people have found over the years that many situations in nature, such as spiral patterns in nautilus shells, flower petal arrangements and leaf patterns, can be modelled with the Fibonacci sequence. These are beautifully illustrated and explained in the website Fibonacci Numbers and Nature <www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html>.



The Fibonacci sequence has also turned out to have a number of interesting mathematical properties, some of which we will investigate now.

Using a difference equation to generate a Fibonacci sequence

Like most regular sequences, the Fibonacci sequence can be generated using a difference equation. Listing the terms of the sequence we have:

$$
\begin{aligned}
t_1 &= 1 \\
t_2 &= 1 \\
t_3 &= t_1 + t_2 \qquad (= 1 + 1 = 2) \\
t_4 &= t_2 + t_3 \qquad (= 1 + 2 = 3) \\
t_5 &= t_3 + t_4 \qquad (= 2 + 3 = 5) \\
&\vdots \\
t_n &= t_{n-2} + t_{n-1} \\
&\vdots
\end{aligned}
$$

Having an expression for the nth term in the sequence, we can now write down the difference equation.

Difference equation for the Fibonacci sequence

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$

Note: This difference equation differs from those you have met earlier. All the difference equations you have dealt with so far are **first-order** difference equations. That is, they link terms that are only **one step** apart in the sequence. The difference equation that generates the Fibonacci sequence is an example of a **second-order** difference equation. It links terms that are up to **two steps** apart in the sequence.

While the difference equation defining the Fibonacci sequence gives a rule for determining values of terms, simply remembering that ‘after the first two terms, each successive term is the sum of the preceding two terms’ will help you answer many questions.

Example 11 **Generating terms in a Fibonacci sequence from adjacent terms**

For the Fibonacci sequence, $t_{11} = 89$, $t_{13} = 233$ and $t_{14} = 377$:

a Determine the values of: **i** t_{15} **ii** $t_9 + t_{10}$ **iii** t_{12}

b Name the terms represented by: **i** $t_{18} + t_{19}$ **ii** $t_{31} - t_{29}$ **iii** $t_{20} + t_{21} + t_{23} + t_{24} - t_{22}$

Solution

Strategy: The key fact in answering all of these questions is that $t_n = t_{n-2} + t_{n-1}$ for $n > 2$. Or, in words, 'after the first two terms, each successive term is the sum of the preceding two terms'.

$t_{11} = 89,\ t_{13} = 233,\ t_{14} = 377$

a **i** $t_{15} = t_{13} + t_{14} = 233 + 377 = 610$

ii $t_9 + t_{10} = t_{11} = 89$

iii $t_{11} + t_{12} = t_{13}$

$\therefore t_{12} = t_{13} - t_{11} = 233 - 89 = 144$

b **i** $t_{18} + t_{19} = t_{20}$

ii $t_{31} - t_{29} = (t_{30} + t_{29}) - t_{29} = t_{30}$

iii
$$\begin{aligned} t_{20} + t_{21} + t_{23} + t_{24} - t_{22} &= (t_{20} + t_{21}) + (t_{23} + t_{24}) - t_{22} \\ &= t_{22} + t_{25} - t_{22} \\ &= t_{25} \end{aligned}$$

When wanting to generate and/or graph more than just a few terms of the Fibonacci sequence from its difference equation the usual procedure is to use a calculator.

How to generate and graph the terms of the Fibonacci sequence using the TI-Nspire CAS

Generate the terms of the Fibonacci sequence given the difference equation:

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$.

Graph the first 10 terms.

Steps

1 Write down the rule and the values of the first two terms.

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1, t_2 = 1$.

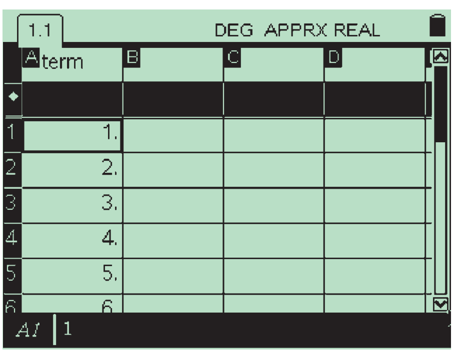
2 Start a new document by pressing ctrl + N.

a Select **3:Add Lists & Spreadsheet**.

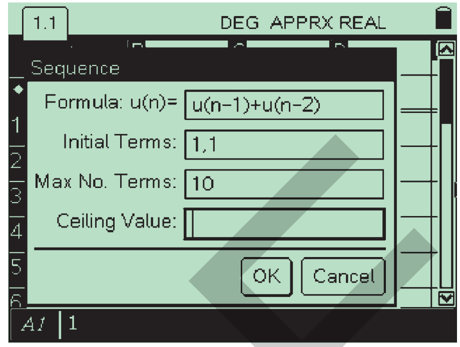
b Enter the data **1–10** into a list named *term*, as shown. This is needed later when we come to plot the sequence.

Note: You can also use the sequence command to do this.

c Name the list *value* in column B. We will use this column to list the terms of the sequence.



d Place the cursor in any cell in column B and press (menu) /**3:Data/1:Generate Sequence**. This will generate the pop-up screen opposite.

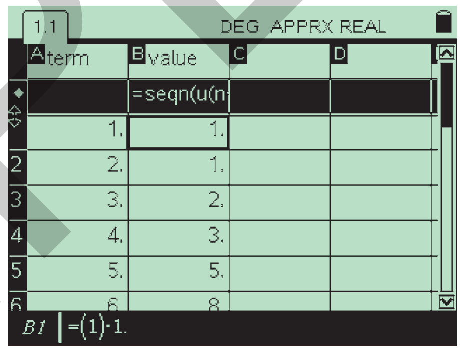


e On this screen, type in the entries as shown. Use (tab) to move between entry boxes.

Notes:

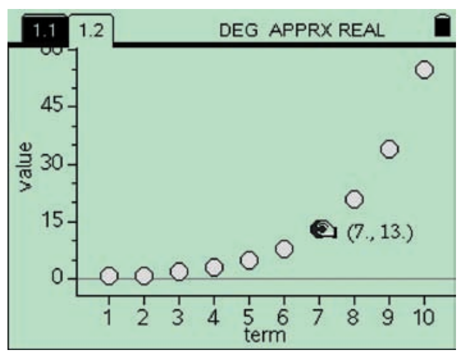
1. Second-order difference equations need two initial terms to be specified.
2. Leave the **Ceiling Value** box blank.

f Press (enter) to list the sequence of terms, as shown.



3. Graph the sequence by constructing a scatterplot using *term* as the independent variable and *value*as the dependent variable.

Note: You can read the values from the graph by placing the cursor on the data point and holding the (hand) key.



Back to Menu

How to generate and graph the terms of the Fibonacci sequence using the ClassPad

Generate the terms of the Fibonacci sequence given the difference equation:

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$.

Graph the first 10 terms.

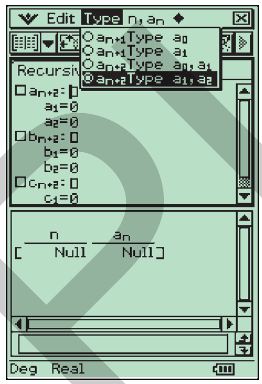
Steps

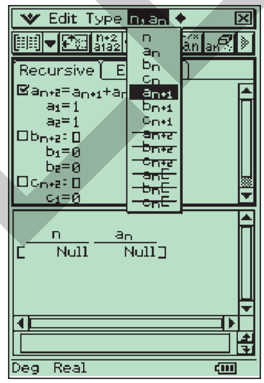
1 Write down the rule and the values of the first two terms.

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1, t_2 = 1.$

2 From the application menu screen, open the built-in **Sequence** application, Sequence.

3 Select a_{n+2} **Type a$_1$, a$_2$** from the **Type** menu item.



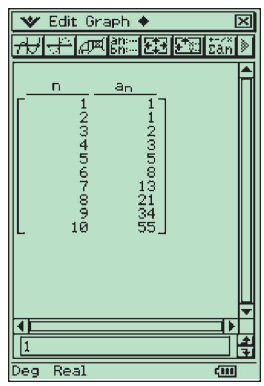


4 Use the n, a_n menu item to select the correctly formatted terms to help enter the information below. For

- a_{n+2}: type $a_{n+1} + a_n$
- a_1: type **1**
- a_2: type **1**

5 Press from the toolbar to display the sequence in tabular form.

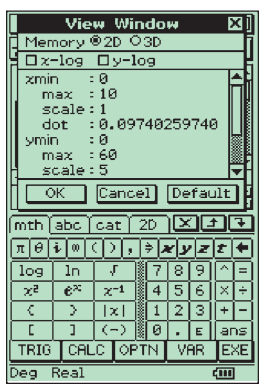
Tapping Resize from the icon panel will allow the table to fill the entire screen.

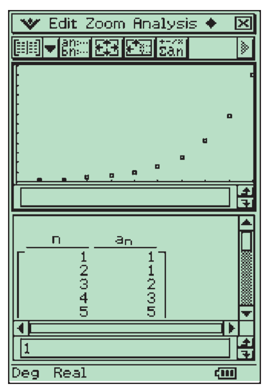


6 To graph the sequence values, tap in the toolbar to open the **View Window** dialog box. For

- **xmin:** type **0**
- **max:**type **10**
- **scale:** type **1**
- **ymin:** type **0**
- **max:**type **60**
- **scale:** type **5**

Tap OK to confirm your settings.





7 From the toolbar, tap to plot the sequence.

Exercise 10I

1 Is the difference equation defining the Fibonacci sequence a first- or second-order difference equation?

2 The Fibonacci sequence can be defined by the difference equation $t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$.

- a Use the difference equation to generate the first 10 terms by hand. Check your result by using a graphics calculator to generate these same terms.
- b Graph the first ten terms.
- c Which term in the Fibonacci sequence is first to exceed 50 000?
- d Use a graphics calculator to determine the value of the 20th, 21st and 22nd terms of the Fibonacci sequence. Show that $t_{20} + t_{21} = t_{22}$.
- e Use a graphics calculator to determine the value of the 10th, 11th and 12th terms of the Fibonacci sequence. Show that $t_{12} - t_{10} = t_{11}$.

3 In the Fibonacci sequence, $t_7 = 13$, $t_9 = 34$ and $t_{10} = 55$.

- a Use these values to determine: i t_{11} ii t_8 iii $t_5 + t_6$
- b Name the terms represented by:
  i $t_{27} + t_{28}$ ii $t_{102} - t_{100}$ iii $t_{31} + t_{32} + (t_{35} - t_{33})$ iv $t_{11} + 2t_{12} + t_{13}$

4 In the Fibonacci sequence, $t_{15} = 610$, $t_{16} = 987$ and $t_{18} = 2584$.

- a Use these values to determine: i t_{17} ii t_{14}
- b Name the terms represented by: i $t_2 + t_3$ ii $t_{200} - t_{198}$ iii $t_3 + 2t_4 + t_5$

5 The Lucas sequence can be defined by the difference equation that is almost identical to the difference equation defining the Fibonacci sequence. The sole difference is the value of t_2.

$$\text{Fibonacci}: t_n = t_{n-2} + t_{n-1} \quad \text{where} \quad t_1 = 1 \text{ and } t_2 = 1$$
$$\text{Lucas}: t_n = t_{n-2} + t_{n-1} \quad \text{where} \quad t_1 = 1 \text{ and } t_2 = 3$$

- a Use the Lucas difference equation to generate the first 10 terms of the Lucas sequence by hand. Check your result by using a graphics calculator to generate these same terms.
- b Which term in the Lucas sequence is first to exceed 10 000?
- c Use a graphics calculator to determine the value of the 20th and 21st terms of the Lucas sequence. Show that $t_{20} + t_{21} = t_{22}$.
- d Use a graphics calculator to determine the value of the 10th and 12th terms of the Lucas sequence. Show that $t_{12} - t_{10} = t_{11}$.

6 The nth term of the Fibonacci sequence is given by:

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Use this expression to show that the value of the 8th term of the Fibonacci sequence is 21. (This will test your knowledge of brackets and the graphics calculator.)

7 The nth term of the Lucas sequence is given by:

$$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Use this expression to show that the value of the 4th term of the Lucas sequence is 7. (This will also test your knowledge of brackets and the graphics calculator.)

MC TEST

D&D TEST

Key ideas and chapter summary

Difference equation

A **difference equation** has two parts:

- a rule that links successive terms in a sequence, e.g.:
  $t_{n+1} = 2t_n - 5$
- the value of one or more terms in the sequence, e.g.:
  $t_1 = 4$

First-order difference equations

A difference equation of the form $t_{n+1} = rt_n + d$ where $t_1 = a$ is known as a **first-order** difference equation, because the rule links terms that are only one step apart in the sequence.

Solving first-order difference equations

The general form of a first-order linear difference equation is
$t_{n+1} = rt_n + d$ where $t_1 = a$

First-order difference equations can be solved by finding an expression for the nth term

Case 1 : $r = 1$

When $r = 1$, the difference equation becomes:

$t_{n+1} = t_n + d$ where $t_1 = a$

This equation generates an **arithmetic sequence**.
The terms in the sequence follow a **linear pattern**.
The nth term is given by: $t_n = a + (n - 1)d$

Example: $t_{n+1} = t_n + 2$ where $t_1 = 3$
arithmetic sequence: 3, 5, 7, . . . $a = 3, d = 2$
nth term: $t_n = 3 + (n - 1)2 = 2n + 1$

Case 2 : $d = 0$

When $d = 0$, the difference equation becomes:
$t_{n+1} = rt_n$ where $t_1 = a$
This equation generates a **geometric sequence**.
The terms in the sequence follow an **exponential pattern**.
The nth term is given by: $t_n = ar^{n-1}$

Example: $t_{n+1} = 0.5t_n$ where $t_1 = 5$
geometric sequence: 5, 2.5, 1.25, . . . $a = 5, r = 0.5$
nth term: $t_n = 5(0.5)^{n-1}$

Case 3 : $d \neq 0$ and $r \neq 1$

When $d \neq 0$ and $r \neq 1$, the difference equation is:
$t_{n+1} = rt_n + d$ where $t_1 = a$

Example: $t_{n+1} = 0.5t_n + 2$ where $t_1 = 10$

sequence: 10, 7, 5.5, . . . (neither arithmetic nor geometric)

Review

The Fibonacci sequence The Fibonacci sequence is: 1, 1, 2, 3, 5, 8, . . .
It is generated by the difference equation:

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$

The Lucas sequence The Lucas sequence is: 1, 3, 4, 7, 11, 18, . . .
It is generated by the difference equation:

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 3$

Second-order linear difference equations The difference equation that generates the Fibonacci and Lucas sequences is a **second-order difference equation** because the rule links terms that are up to **two** steps apart in the sequence.

Skills check

Having completed this chapter you should be able to:

- generate a sequence from a difference equation
- generate the terms of a sequence represented by a difference equation
- graph the terms of the sequence generated by a difference equation
- recognise when a difference equation generates an arithmetic or geometric sequence
- solve first-order difference equations by finding an expression for the *n*th term
- model real-world situations using first-order linear difference equations
- recognise and work with difference equations that generate Fibonacci and Lucas sequences

Multiple-choice questions

1 Which of the following **cannot** be a rule for a first-order difference equation?

A $t_{n+1} = t_n + 4$ **B** $t_{n+1} = 2 + t_n$ **C** $t_{n+1} = n$ **D** $t_n = 2t_{n-1} + 1$
E $u_n = 5u_{n-1} - 3$

2 The general equation for a first-order difference equation is:

$t_{n+1} = rt_n + d$ where: $t_1 = a$.

In the difference equation, $t_{n+1} = 3 - t_n$ where $t_1 = 5$:

A $r = -1, d = 3, a = 5$ **B** $r = 3, d = -1, a = 5$ **C** $r = 3, d = 1, a = 5$
D $r = 1, d = 3, a = 5$ **E** $r = 5, d = 3, a = -1$

3 The difference equation $t_{n+1} = 2t_n - 5$, $t_1 = 4$, generates the sequence:

A 4, 8, 16, . . . **B** 4, 3, 1, . . . **C** 3, −1, 3, . . . **D** 2, 4, 5, . . .
E 4, −20, 100, . . .

4 The difference equation $t_{n+1} = -4t_n + 2$, $t_1 = 1$, generates the sequence:

A 1, −2, 10, −38, . . . **B** 1, −2, −10, 38, . . . **C** 1, −2, 10, 42, . . .
D 1, 2, 4, 6, . . . **E** 1, 2, 10, 42, . . .

Review

5 A difference equation is defined by $t_{n+1} = -2t_n + 10$ where $t_1 = 5$. For this difference equation, t_7 is equal to:

A -210 **B** -50 **C** 10 **D** 110 **E** 430

6 The first four terms of a sequence are:

$4, 2, 4, 2, \ldots$

A difference equation that generates this sequence is:

A $t_{n+1} = -2t_n - 6$ where $t_1 = 4$
B $t_{n+1} = -t_n + 6$ where $t_1 = 4$
C $t_{n+1} = 2t_n - 4$ where $t_1 = 4$
D $t_{n+1} = -6t_n + 4$ where $t_1 = 4$
E $t_{n+1} = 0.5t_n$ where $t_1 = 4$

7 The difference equation: $t_{n+1} = at_n + 4$, $t_1 = 8$ generates the sequence:

$8, 20, 44, \ldots$

The value of a is:

A -1 **B** -0.5 **C** 1 **D** 2 **E** 4

8 The difference equation $t_{n+1} = 2t_n + d$, $t_1 = 8$ generates the sequence:

$8, 14, 26, \ldots$

The value of d is:

A -2 **B** -1 **C** 0 **D** 6 **E** 12

9 The difference equation $t_{n+1} = 0.75t_n$, $t_1 = 100$, generates:

A a decreasing arithmetic sequence
B a decreasing geometric sequence
C neither an arithmetic nor a geometric sequence
D an increasing arithmetic sequence
E an increasing geometric sequence

10 A difference equation is defined by $t_{n+1} = t_n - 5$ where $t_1 = 55$.
The nth term generated by this difference equation, t_n, is given by:

A -5^{n-1} **B** $(-5)^{n-1}$ **C** $n - 5$ **D** $50 - 5n$ **E** $60 - 5n$

11 A difference equation is defined by $t_{n+1} = 1.2t_n$ where $t_1 = 10$.
The nth term generated by this difference equation, t_n, is given by:

A $1.2 \times 10^{n-1}$ **B** $10 \times 1.2^{n-1}$ **C** 10^{n-1} **D** 1.2^{n-1} **E** $1.2n$

12 The value of a house increases 10% per year. Its current value is \$320 000.
If H_n is the value of house in the nth year, a difference equation that can be used to determine the price of the house in future years is:

A $H_{n+1} = 0.10H_n$ where $H_1 = \$320\,000$
B $H_{n+1} = 0.90H_n$ where $H_1 = \$320\,000$
C $H_{n+1} = 1.10H_n$ where $H_1 = \$320\,000$
D $H_{n+1} = H_n + 10$ where $H_1 = \$320\,000$
E $H_{n+1} = 10H_n$ where $H_1 = \$320\,000$

Review

13 When first investigated, a dam contained 900 yabbies. If left to natural forces, the yabbie numbers in the dam would increase, on average, by 35% per year. Two hundred yabbies are taken from the dam each year.
If Y_n is the number of yabbies in the dam at the start of year n, then a difference equation that can be used to describe this situation is:

A $Y_{n+1} = 0.35Y_n - 200$ where $Y_1 = 900$
B $Y_{n+1} = 1.35Y_n - 200$ where $Y_1 = 900$
C $Y_{n+1} = 200Y_n - 35$ where $Y_1 = 900$
D $Y_{n+1} = 200Y_n + 35$ where $Y_1 = 900$
E $Y_{n+1} = 900Y_n - 200$ where $Y_1 = 900$

14 The Fibonacci sequence is generated by the difference equation $t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$.
The value of the 19th term of the Fibonacci sequence is:

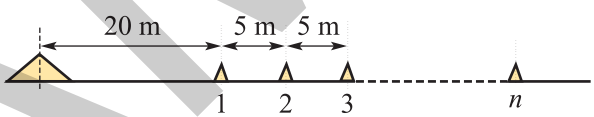
A 19 **B** 1597 **C** 2584 **D** 4181 **E** 6765

15 In the Fibonacci sequence $t_{20} =$

A $t_{22} + t_{21}$ **B** $2t_{19}$ **C** $t_{18} - t_{19}$ **D** $t_{18} + t_{19}$ **E** $t_{22} - 2t_{19}$

Extended-response questions

1 A gardener moves a load of sand by emptying barrow loads of sand at points along a straight line. The first drop-off point is 20 metres from where the sand was originally dumped. The remaining loads of sand are then dropped off at five-metre intervals along the line.



a Write down a difference equation of the form:
$L_{n+1} = L_n + d$ where $L_1 = a$
that can be used to describe this situation. (L_n represents the distance of the nth drop-off point from the load of sand.)

b Write down an expression for the distance of the nth drop-off point from the load of sand.

c Calculate the distance of the 10th drop-off point from the load of sand.

d After 15 barrow loads, all the sand has been moved. How far in total would the gardener have to walk to move all the sand and then return to the point where the sand was originally dumped?

2 When you buy a premium bottle of wine for $90 per bottle from a boutique vineyard, you are told that provided you store it properly, it will increase in value by 20% every year.

a Let V_n be the value of the bottle of wine at the start of the nth year. Write down a difference equation that describes this situation.

b Use the difference equation to complete the table.

Year	1	2	3	4	5
Value (V_n)	90				

c How much will the bottle of wine be worth after 10 years (that is, at the start of the 11th year)? Give your answer to the nearest dollar.

d You intend to keep the wine for 15 years before you sell it. How much per glass would it be worth? (Assume that you get five glasses of wine from the bottle and give your answer to the nearest dollar.)

3 When counted at the start of a year, there were 148 possums in a large suburban park. Possum numbers in the park are known to increase by 25% each year due to natural causes. Also, experience shows that 10 possums leave the park each year.

a Let P_n be the number of possums in the park at the start of the nth year. Write down a difference equation that describes this situation.

b Use the difference equation to complete the table. Give your answer to the nearest whole number.

Year	1	2	3	4	5
Number (P_n)					

c If nothing changes, how many possums would you expect to find in the park at the start of the 7th year? Give your answer to the nearest whole number.

d If nothing changes, graph the growth in possum numbers over a ten-year period.

e If nothing changes, during which year will possum numbers first exceed 1000?

To reduce possum numbers, park officers decide to relocate an extra 30 possums to another park at the start of each year.

f Write down a difference equation that describes this new situation.

g Graph the growth in possum numbers over a ten-year period on the same graph as before. Is relocating 30 possums enough to stop the population of possums growing?

h How many possums would the park officers need to relocate each year to ensure that possum numbers are stable? That is, they neither increase nor decrease.