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ANALYSIS ON MEASURE CHAINS —

A UNIFIED APPROACH TO CONTINUOUS AND DISCRETE CALCULUS

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0. INTRODUCTION

There are a lot of well-known multiple analogies in the concepts of difference calculus with the difference operator

$$(\triangle_h f)(t) = \frac{f(t+h) - f(t)}{h}$$

on the one hand and the differential calculus with the differential operator

$$(\frac{d}{dt}f)(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

on the other hand. These analogies which are described in the relevant literature (e.g. [2],[3],[8]) lead to the idea to develop some higher ranging calculus which in special cases covers those two concepts.
In the endeavor to state a system of axioms which is sufficient for the development of an extensive generalized differential and integral calculus one arrives at the axioms 1 - 3 (cf. 1.1/1.4/2.1) with the notion of a (strong) measure chain and at the concept of differentiation which is described in section 2.4.
Besides of the main examples $\mathbb{R}$ and $h\mathbb{Z}$ any closed subset of $\mathbb{R}$ bears the structure of a measure chain in a natural manner. We notice some examples:

- Any arbitrary discrete subset of $\mathbb{R}$ is a measure chain. We will get some difference calculus with variable step width.
- We modify an example of Poulsen (cf. [4], p.7ff) from bio-mathematics (population dynamics): After birth the population of a species with non-

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overlapping generations evolves according to the logistic differential equation (Verhulst's model). After the half generation period eggs are laid, the parents' generation dies out before the offspring slips.
The mathematical conversion looks like this: The generation period is normed to 1. We choose the subset of $\mathbb{R}$:

$$P := \bigcup_{k \in Z} [k, k + ½] \tag{0.1}$$

as the appropriate measure chain. Between slipping out and laying eggs the population x evolves according to the logistic differential equation

$$x^{\cdot} = r \cdot x(K - x)/K$$

with (growth) rate r and carrying capacity K. Let the birth rate be $d > 0$:

$$x(t + 1) - x(t + ½) = d \cdot x(t + ½) \quad t \in \mathbb{Z}.$$

We shall not examine this model, but only show that the measure chain calculus allows to systematically establish the theory of dynamical equations over time scales of such a general type.

■ The foregoing example demonstrates that in general differential equations with pulses can be described within the calculus on measure chains (abbr.: m.c.calculus), if one chooses the union of disjoint closed intervals of $\mathbb{R}$ as a measure chain. One must admit strictly positive time gaps for the pulses which take place between the interval endpoints.

■ Even a time scale like the Cantor set is covered by the m.c.calculus. We mention this exotic example to illustrate the variable possibilities for choosing measure chains.

We pick up the examples $\mathbb{R}$, $h\mathbb{Z}$ and P for measure chains several times. We refer to them by the symbols ($\mathbb{R}$), ($h\mathbb{Z}$) and (P).

We shall desribe only the basis of the m.c.calculus. Nevertheless it can be extended in many directions. Especially a lot of theorems from the qualitative theory of dynamical systems can be proved in the general context of the m.c.calculus. So one can avoid the following types of proceeding which often occur in presentations of "discrete versions of topics in analysis" (cf. Mathematics Subject Classification 39A12): A parallel presentation of discrete and continuous results (cf. [1],[11]), the sometimes boring, sometimes difficult performing of a proof by forming analogies (cf. [9],[12]) or just its omission.

1. CONDITIONALLY COMPLETE CHAINS

We approach the notion of a measure chain in several steps. First of all order theoretic considerations are in the foreground.

1.1 Chains: Let $\mathbf{T}$ (for "time") be some set.

Axiom 1: There is a relation "$\leq$" on $\mathbf{T}$, which is reflexive ($t \leq t$ for all $t \in \mathbf{T}$), transitive ($r \leq s$ and $s \leq t \Longrightarrow r \leq t$ for all $r,s,t \in \mathbf{T}$), antisymmetric ($r \leq s$ and $s \leq r \Longrightarrow r = s$ for all $r,s \in \mathbf{T}$) and total ($r \leq s$ or $s \leq r$ for all $r,s \in \mathbf{T}$).

The pair $(\mathbf{T},\leq)$ is said to be a linearly ordered set or shortly a chain.
Examples are arbitrary subsets of $\mathbb{R}$, especially the sets $h\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ itself.
At once important notions from order theory are available: (Open, closed) intervals, bounds and boundedness, least upper bound (l.u.b.), greatest lower bound (g.l.b.) and isotone (order preserving) mappings. By $+\infty$ and $-\infty$, resp. we denote the universal adjoint upper and lower bound of $\mathbf{T}$, resp.

The interplay of the following "structures" decisively determines the development of the m.c.calculus.

1.2 Order topology: The Hausdorff topology on $\mathbf{T}$ which is generated by the open intervals

$$]r,s[\ := \{t \in \mathbf{T}:\ r < t < s\}\ ;\ r,s \in \mathbf{T} \cup \{\pm\infty\}$$

is called order topology.
The order topology on $\mathbb{R}$ is well-known. On $h\mathbb{Z}$ or $\mathbb{N}$ the order topology is given by the discrete topology.

1.3 Jump Operators: The mapping $\sigma:\mathbf{T} \longrightarrow \mathbf{T}$, defined by

$$\sigma(t) := \inf \{s \in \mathbf{T}:\ s > t\} \tag{1.3.1}$$

(This inf always exists!) is called jump operator. Accordingly we define the backward jump operator $\varrho:\mathbf{T} \longrightarrow \mathbf{T}$ to be the mapping

$$\varrho(t) := \sup \{s \in \mathbf{T}:\ s < t\} \tag{1.3.2}$$

A nonmaximal element $t \in \mathbf{T}$ is said to be right-scattered, if $\sigma(t) > t$, and right-dense, if $\sigma(t) = t$. We call a nonminimal element $t \in \mathbf{T}$ left-scattered, if $\varrho(t) < t$, and left-dense, if $\varrho(t) = t$.
In the case ($\mathbb{R}$) σ is the identity, in ($h\mathbb{Z}$) σ is given by $\sigma(t) = t + h$.

1.4. Conditionally complete chains

Axiom 2: The chain $(\mathbf{T},\leq)$ is conditionally complete, that is: Any nonvoid subset which is bounded above has a l.u.b.

$\mathbb{R}$ and $h\mathbb{Z}$ are conditionally complete, $\mathbb{Q}$ is not. The fundamental character of this property is expressed in the following theorem. It shows that the transfer of important features of the real line to other time scales, especially $h\mathbb{Z}$, is difficult to handle, if one tries to argue topologically, however relatively easy to handle if one involves order theoretical considerations. The handicap that arbitrary chains are not necessarily connected is just eliminated by jump operators (cf. (c)/II and (e) in the theorem).

Theorem 1.4: Let $(\mathbf{T},\leq)$ be a chain. The following properties for $\mathbf{T}$ are equivalent:

(a) $\mathbf{T}$ is conditionally complete.

(b) Any nonvoid subset $\mathbf{T}'$ which is bounded below has a g.l.b.

(c) The induction principle holds:
Assume that for a family of statements $A(t)$, $t \in [\tau,\infty[\subseteq \mathbf{T}$ the following conditions are fulfilled:
 (I) $A(\tau)$ holds true.
 (II) For each right-scattered $t \in \mathbf{T}$ one has the implication:
$$A(t) \Longrightarrow A(\sigma(t))$$
 (III) For each right-dense $t \in \mathbf{T}$ there is a neighborhood U such that:
$$A(t) \Longrightarrow A(s) \text{ for all } s \in U,\ s > t.$$
 (IV) For each left-dense $t \in \mathbf{T}$ one has :
$$A(s) \text{ for all } s \text{ with } s < t \Longrightarrow A(t).$$
Then $A(t)$ is true for all $t \in [\tau,\infty[$.

(d) A subset $\mathbf{T}' \subseteq \mathbf{T}$ is compact, if and only if it is bounded and closed.

(e) The intermediate value theorem holds:
The continuous mapping $f:[r,s] \to \mathbb{R}$, $r,s \in \mathbf{T}$, is assumed to fulfill the condition: $f(r) < 0 < f(s)$. (1.4.1)
Then there is a $\tau \in [r,s]$ with $f(\tau)\cdot f(\sigma(\tau)) \leq 0$. (1.4.2)

Remarks: Ad (c): For $\mathbf{T} = \mathbb{N}$ the conditions (III) and (IV) of the induction principle are redundant. In this case it reduces to the usual principle of mathematical induction for the natural numbers.
If $\mathbf{T}$ is a left closed interval in $\mathbb{R}$, then condition (II) becomes redundant. This induction principle then contains the (topological) principle that a nonvoid subset of an interval which is at the same time open and closed coincides with this interval. This continuous type of induction is being used by Dieudonné (cf. [5]) for proving basic results in differential and integral calculus. Therefore his presentation is a good guideline for the endeavor to develop the m.c.calculus.
Ad (e): We call a point τ with the property (1.4.2) a <u>node point</u> (cf. [9]). If τ is a node point, then f "vanishes between" τ and $\sigma(\tau)$.

Proof: (a) $\Longrightarrow$ (b): Choose $\inf \mathbf{T}' := \sup \{t \in \mathbf{T}: t < s \text{ for all } s \in \mathbf{T}'\}$.
(b) $\Longrightarrow$ (c): Assume that there is some $t \in [\tau,\infty[$ such that $A(t)$ does not hold. Then we define :
$$r := \inf \{t \in [\tau,\infty[: A(t) \text{ does not hold}\}. \tag{1.4.3}$$
(1) $A(r)$ is valid, therefore $A(t)$ holds true for all $t \in [\tau,r]$.
Case 1: r is left-scattered. $A(\varrho(r))$ holds true, with (II) it follows: $A(r) = A(\sigma(\varrho(r))$.
Case 2: r is left-dense. The assertion follows by (IV).
Case 3: r is minimal. Then we have $r = \tau$, which leads to the assertion by (I).

(2) In each of the following cases we will obtain a contradiction.
Case 1: r is right-scattered. From $A(r)$ we conclude by (II): $A(\sigma(r))$ and then $r < \sigma(r) \le r$.
Case 2: r is right-dense. Each neighborhood of $r = \sigma(r)$ contains some interval $[r,s]$ with $s > r$. From (1) we have $A(r)$ and then with (III) $A(t)$ for all $t \in [r,s]$, even for all $t \in [\tau,s]$ by (1.4.3). It follows: $r < s \le r$.
Case 3: r is maximal. With (1) we have $A(t)$ for all $t \in [\tau,r]$.
(c) $\Longrightarrow$ (d): Let $[\tau,s]$ be some bounded and closed interval and $(U_i)_{i \in I}$ an arbitrary open covering for $[\tau,s]$. We show by means of the induction principle for $t \in [\tau,s]$:

$$A(t)\text{: } [\tau,t] \text{ has a finite subcovering.}$$

The checks of (I),(II),(III) are trivial. For (IV) first of all we choose some U_j from the covering with $t \in U_j$ and then some $r \in U_j$ with $r < t$. Such an r exists, because t is left-dense. The finite subcovering of $[\tau,r]$ together with U_j forms a finite subcovering for $[\tau,t]$.
Each bounded and closed subset of $\mathbf{T}$ is a closed subset of some compact interval and therefore compact itself. If otherwise a compact $\mathbf{T}'$ is unbounded, one can immediately find an infinite open covering without a finite subcovering. Finally each compact subset is closed.
(d) $\Longrightarrow$ (e): We assume that (1.4.2) does not hold, hence:

$$f(\tau)\cdot f(\sigma(\tau)) > 0 \quad \text{for all } \tau \in \mathbf{T} \tag{1.4.4}$$

The set $A := f^{-1}(]-\infty,0[)$ and the set B of the upper bounds of A in $[r,s]$ are nonvoid. We choose $p \in A$, $q \in B$.
For a finite collection of closed intervals $[a_i,b_i] \subseteq [p,q]$ with $a_i \in A$, $b_i \in B$, $i \in J$, J finite, we have:

$$\bigcap_{i \in J} [a_i,b_i] = [\max\{a_i\}, \min\{b_i\}] \neq \emptyset, \tag{1.4.5}$$

moreover, because of the compactness of $[p,q]$:

$$\bigcap_{a \in A, b \in B} [a,b] \neq \emptyset.$$

Otherwise the dual to the Heine Borel property would guarantee that even a finite intersection of these intervals would be empty, contradicting (1.4.5). Therefore there is some $c \in [r,s]$ with

$$a \le c \le b \quad \text{for all } a \in A,\ b \in B.$$

Each neighborhood of $\sigma(c)$ contains some t with $t > c$ or $t = s$, in any case some t with $f(t) > 0$. The continuity of f has the consequence $f(\sigma(c)) \ge 0$, and then by (1.4.4): $f(c) > 0$.
c cannot be left-dense. In this case each neighborhood U of c would meet the set A and this contradicts the continuity of f in c.
If c is left-scattered, then $\rho(c)$ is in A. Then with $\tau := \rho(c)$, $\sigma(\tau) = c$ the assertion (1.4.4) cannot be claimed anymore.

(e) $\Longrightarrow$(a): Let $\mathbf{T}'$ be some nonvoid subset of $\mathbf{T}$ which is bounded above. We define B to be the nonvoid set of upper bounds of $\mathbf{T}'$ and A to be the nonvoid set of lower bounds of B. Choose $r \in A$, $s \in B$ with $r < s$ (W.l.o.g. $\mathbf{T}$ contains more than two elements). We define the mapping $f:[r,s] \to \mathbb{R}$ by

$$f(t) = \begin{cases} -1, & \text{if } t \in A, \\ +1, & \text{if } t \in B. \end{cases}$$

If A does not possess any l.u.b. - hence B any g.l.b. -, then the sets

$$f^{-1}(\{-1\}) = \bigcup_{t \in A} [r,t[\ , \qquad f^{-1}(\{+1\}) = \bigcup_{t \in B}]t,s]$$

are open in $[r,s]$ and therefore f is continuous. By (e) there is some $\tau \in [r,s]$ with $f(\tau)\cdot f(\sigma(\tau)) \le 0$. Because of the fact that f does not vanish we get $f(\tau)\cdot f(\sigma(\tau)) < 0$ and immediately: $f(\tau) < 0 < f(\sigma(\tau))$. Thus A does have a l.u.b., which is one of $\mathbf{T}'$, too. ∎

<u>**1.5 I\O-subsets of conditionally complete chains:**</u> We call a subset $\mathbf{T}'$ of a conditionally complete chain $\mathbf{T}$ an I\O-subset, if there exist an interval I and an open subset O of $\mathbf{T}$ such that $\mathbf{T}' = I\backslash O$. They are the "natural" subsets of $\mathbf{T}$ as the following theorem demonstrates:

Theorem 1.5.1: Let $\mathbf{T}$ be a conditionally complete chain and $\mathbf{T}' \subseteq \mathbf{T}$ some subset. Assume, furthermore, that $f:\mathbf{T} \to \mathbf{T}_2$ ($\mathbf{T}_2$ is an arbitrary chain) is a strongly isotone continuous mapping, for example the identity $\mathbf{T} \to \mathbf{T}$. The following statements are equivalent:

(i) $\mathbf{T}'$ is an I\O-subset.

(ii) The l.u.b. and the g.l.b. (with respect to $\mathbf{T}$) of some subset $A \subseteq \mathbf{T}'$ which is bounded in $\mathbf{T}'$ is contained in $\mathbf{T}'$.

(iii) $\mathbf{T}'$ is conditionally complete. The relative topology on $\mathbf{T}'$ which is induced by $\mathbf{T}$ and the order topology on $\mathbf{T}'$ coincide.

(iv) $\mathbf{T}'$ is conditionally complete. The restriction f' of f to $\mathbf{T}'$ is continuous (with respect to the order topology).

Proof: (i) $\Longrightarrow$(ii): The statement (ii) is valid for intervals $\mathbf{T}'$ or closed subsets $\mathbf{T}'$. Since some subset of the form I\O is closed in I, (ii) holds true even for such sets.

(ii) $\Longrightarrow$(iii): We only have to show the second statement of (iii). Since the relative topology is principally finer than the order topology, we can assume to the contrary that it is strictly finer. Then there exists some point $t \in \mathbf{T}'$ and some relative open neighborhood $U =]p,q[\cap \mathbf{T}'$, $p,q \in \mathbf{T}$, of t which is not an order open neighborhood of t. Exactly one of the two sets

$$]p,t[\cap \mathbf{T}', \quad]t,q[\cap \mathbf{T}'$$

is empty. If one assumes that both sets are not empty, then with $p' \in]p,t[\cap \mathbf{T}'$, $q' \in]t,q[\cap \mathbf{T}'$ the set $]p',q'[\cap \mathbf{T}'$ would be an order open neighborhood of t. If,

furthermore, one assumes that both sets are empty, then $\{t\}$ would be an order open neighborhood of t. Therefore both cases are absurd.
W.l.o.g. let $]p,t[\,\cap\mathbf{T}'=\emptyset$ and $q'\in\,]t,q[\,\cap\mathbf{T}'$. t cannot be a minimal element of $\mathbf{T}'$, otherwise $]t,q'[\,\cap\mathbf{T}'=\,]-\infty,q'[\,\cap\mathbf{T}'$ would be order open and a subset of U. This is impossible. Now it follows that

$$s:=\sup\,]-\infty,t[\,\cap\mathbf{T}'$$

exists and is finite. If $s\in\mathbf{T}'$, then $]s,q'[\,\cap\mathbf{T}'$ is order open and contained in U, which is a contradiction. Thus we have $s\notin\mathbf{T}'$. This contradicts our hypothesis (ii).
(iii) $\Longrightarrow$ (iv). f' is always continuous with respect to the relative topology.
(iv) $\Longrightarrow$ (i): Let I be the intersection of all intervals containing $\mathbf{T}'$ and define $O:=I\backslash\mathbf{T}'$. Then we have $\mathbf{T}'=I\backslash O$, and it remains to be proved that O is open. For this purpose let J be some maximal interval of O, i.e. for any other interval J' of $\mathbf{T}$ with $J\subseteq J'\subseteq O$ we have $J=J'$. J' is bounded and therefore it is exactly of one of the following four types ($r,s\in\mathbf{T}$):

A: $J=\,]r,s[$; B: $J=[r,s]$, r left-dense, s right-dense;
C: $J=[r,s[$, r left-dense; D: $J=\,]r,s]$, s right-dense.

The assertion is proved, if we can exclude the types B,C,D.
If J is of type B, then the set $]-\infty,r[\,\cap\mathbf{T}'$ has no l.u.b. in $\mathbf{T}'$, $\mathbf{T}'$ cannot be conditionally complete.
If J is of type C, then f' is not continuous in s, since for the inverse image of the open interval $]f(r),\infty[\,\subseteq\mathbf{T}_2$ we have:

$$(f')^{-1}(]f(r),\infty[)=f^{-1}(]f(r),\infty[)\cap\mathbf{T}'=\,]r,\infty[\,\cap\mathbf{T}'=[s,\infty[\,\cap\mathbf{T}'.$$

This interval is not open (w.r.t. the order topology), since one cannot find a $\mathbf{T}'$-order open neighborhood for s in $[s,\infty[\,\cap\mathbf{T}'$.
For type D one concludes analogously. ■

Theorem 1.5.2: We assume that $\mathbf{T}_1$ and $\mathbf{T}_2$ are conditionally complete chains and that $m:\mathbf{T}_1\longrightarrow\mathbf{T}_2$ is isotone and continuous.
(i) The inverse image of an I\O-subset is an I\O-subset.
(ii) The image of an I\O-subset is an I\O-subset.

Proof: (i): Let $I_2\backslash O_2\subseteq\mathbf{T}_2$ be an I\O-subset. Then, as a consequence of the identity

$$f^{-1}(I_2\backslash O_2)=f^{-1}(I_2)\setminus f^{-1}(O_2)$$

the inverse image of $I_2\backslash O_2$ is an I\O-subset in $\mathbf{T}_1$.
(ii): It is enough to show that $m(\mathbf{T}_1)$ is an I\O-subset of $\mathbf{T}_2$. Let A be some nonvoid subset of $m(\mathbf{T}_1)$ which is bounded in $m(\mathbf{T}_1)$. Define $s:=\sup A\in\mathbf{T}_2$. We will show that $s\in m(\mathbf{T}_1)$. One can analogously show that $\inf A\in\mathbf{T}_2$; with Theorem 1.5.1 the claimed statement follows.
In case $s\notin m(\mathbf{T}_1)$ the set

$$B := m^{-1}(A) = m^{-1}((-\infty,s])$$

would be a closed interval and bounded by an arbitrary inverse image of an upper bound of A. This yields:

$$b \le \sup B \in B \text{ for all } b \in B,$$

furthermore we have

$$m(b) \le m(\sup B) \in m(B) = A \text{ for all } b \in B.$$

For each $a \in A$ there exists a $b \in B$ with $m(b) = a$. So we have:

$$a \le m(\sup B) \le s \text{ for all } a \in A.$$

This implies $s = m(\sup B) \in m(\mathbf{T}_1)$ and therefore a contradiction. ■

2. MEASURE CHAINS - THE NOTION OF DIFFERENTIATION

2.1 Growth calibration - measure chains: It is our aim to create a notion of differentiation for functions which are defined on conditionally complete chains. Therefore we must be able to measure distances between two elements of $\mathbf{T}$ in any way. The following axiom proves to be suitable and sufficient.

Axiom 3: On the conditionally complete chain $\mathbf{T}$ there exists a mapping $\mu:\mathbf{T}\times\mathbf{T}\to\mathbf{R}$ with the following properties (for all $r,s,t \in \mathbf{T}$):

- *$\mu(r,s)+\mu(s,t)=\mu(r,t)$* *(Cocycle property)* (2.1.1)
- *If $r > s$, then: $\mu(r,s) > 0$* *(Strong isotony)* (2.1.2)
- *μ is continuous* *(Continuity).*

Because of the cocycle property we have: $\mu(r,s) = \mu(r,t) - \mu(s,t)$ for $r,s,t \in \mathbf{T}$. Therefore the continuity of μ is equivalent to the fact that $\mu(\cdot,t)$ is continuous for some arbitrary but fixed $t \in \mathbf{T}$.
One can show that each μ with the specified properties induces a measure on $\mathbf{T}$ by the assignment $\nu([r,s[) = \mu(s,r)$. We call the triple $(\mathbf{T},\le,\mu)$ or - better yet - the pair $(\mathbf{T},\mu)$ a <u>(strong) measure chain</u> with <u>time scale</u> $\mathbf{T}$ and <u>growth calibration</u> μ.

Theorem 1.5.1 (Replace f by $\mu(\cdot,\tau)$, τ fixed!) shows that for some subset $\mathbf{T}'\subseteq\mathbf{T}$ and $\mu' := \mu|\mathbf{T}'\times\mathbf{T}'$ the pair $(\mathbf{T}',\mu')$ forms a measure chain, if and only if $\mathbf{T}'$ is an I\O-subset of $\mathbf{T}$. $(\mathbf{T}',\mu')$ is called a <u>measure subchain</u> of $\mathbf{T}$.

Examples: On $\mathbb{R}$ there is a growth calibration given by the mapping $\lambda(s,r) := s - r$. It induces the Lebesgue measure on $\mathbb{R}$. For each I\O-subset $\mathbb{R}'$ of $\mathbb{R}$ $(\mathbb{R}',\lambda')$, $\lambda' := \lambda|\mathbb{R}'\times\mathbb{R}'$, forms a measure subchain. For $\mathbb{R}' = \mathbb{R}$ or $\mathbb{R}' = h\mathbf{Z}$ we get this way the measure chains, on which continuous and discrete dynamical systems are modelled, respectively. Even P and all the other examples mentioned in Chapter 0 carry structures of measure chains in a natural manner.

Suppose $(\mathbf{T}_1,\mu_1)$ and $(\mathbf{T}_2,\mu_2)$ are two measure chains. A mapping $m:\mathbf{T}_1 \longrightarrow \mathbf{T}_2$ is called a measure chain homomorphism, if:

$$\mu_2(m(s),m(r)) = \mu_1(s,r) \text{ for all } r,s \in \mathbf{T}_1.$$

Then m is strongly isotone and continuous, too (One easily sees that the inverse image of an open interval $]p,q[\subseteq \mathbf{T}_2$ is given by

$$m^{-1}(]p,q[) = (\mu_1(\cdot,\tau))^{-1}(]\mu_2(p,m(\tau)),\mu_2(q,m(\tau))[),\ \tau \in \mathbf{T}_1 \text{ fixed},$$

and hence it is open).

It is immediately clear that a surjective measure chain homomorphism has an inverse with the same properties. Therefore it is a measure chain isomorphism.

By Theorem 1.5.2 the image of a measure chain $\mathbf{T}_1$ under a measure chain homomorphism m is an $I\backslash O$-subset of the image range $\mathbf{T}_2$, hence a measure subchain of $\mathbf{T}_2$. $\mathbf{T}_1$ is isomorphic to its image.

Since for each measure chain $(\mathbf{T},\mu)$ and for fixed $\tau \in \mathbf{T}$ the mapping $\mu(\cdot,\tau)$ is a measure chain homomorphism to $(\mathbb{R},\lambda)$, we have proved:

Theorem 2.1: Each measure chain is isomorphic to some measure chain $(I\backslash O,\lambda')$, where I is an interval and O is an open subset of $\mathbb{R}$, λ' is given by $\lambda'(r,s) = r - s$.

Now it would be possible to develop further the m.c.calculus from this point of view. We, however, want to proceed with the concept formulated in the axioms 1,2,3. It is notionally clearer, because it is independent of an embedding in $\mathbb{R}$.

2.2 Topological properties of measure chains: The existence of a growth calibration has consequences concerning the topology on $\mathbf{T}$:

Theorem 2.2: If $(\mathbf{T},\mu)$ is a measure chain, then $\mathbf{T}$ – equipped with the order topology – is metrizable and a K_σ-space, i.e. it is a union of at most denumerably many compact sets. The metric is given by:

$$d(r,s) := |\mu(r,s)| \qquad (2.2.1)$$

Proof: For a fixed $\tau \in \mathbf{T}$ we abbreviate: $\mu(\cdot) := \mu(\cdot,\tau)$.

(1) An open neighborhood $]r,s[$ of τ contains an open ε-ball $U_\varepsilon(\tau)$ with $\varepsilon := \min\{|\mu(r,\tau|,|\mu(s,\tau|\}$. Thus the metric topology is finer than the order topology. On the other hand by the continuity of μ there is always an order open inverse image of the set $]-\varepsilon,+\varepsilon[$ contained in an ε-ball $U_\varepsilon(\tau)$.

(2) For $n \in \mathbb{N}$ we define the real numbers:

$$a_n := \begin{cases} -n \\ a + 1/n, \text{ if } \inf \mu(\mathbf{T}) = \\ a \end{cases} \begin{cases} -\infty \\ a \notin \mu(\mathbf{T}) \\ a \in \mu(\mathbf{T}), \end{cases}$$

$$b_n := \begin{cases} +n \\ b - 1/n, \text{ if } \sup \mu(\mathbf{T}) = \\ b \end{cases} \begin{cases} +\infty \\ b \notin \mu(\mathbf{T}) \\ b \in \mu(\mathbf{T}). \end{cases}$$

Then for each $n \in \mathbf{N}$ the set $\mu^{-1}([a_n,b_n])$ is bounded and closed, hence compact, and we have:

$$\mathbf{T} = \bigcup_{n=1}^{\infty} \mu^{-1}([a_n,b_n]). \qquad \blacksquare$$

Observing the K_σ-property we can form the expressions $\lim_{t \to \pm\infty} f(t) = a$ for a function f with values in a metric space by means of sequences.

<u>**2.3 æ-operator and grainyness:**</u> For an arbitrary conditionally complete chain $\mathbf{T}$ we define:

$$\mathbf{T}^{æ} := \{t \in \mathbf{T}\colon t \text{ nonmaximal or left-dense}\}. \qquad (2.3.1)$$

Thus this <u>æ-operator</u> cuts off an eventually existing isolated maximum of $\mathbf{T}$. The function $\mu^*:\mathbf{T}^{æ} \longrightarrow \mathbf{R}_0^+$, defined by

$$\mu^*(t) := \mu(\sigma(t),t). \qquad (2.3.2)$$

is called <u>grainyness</u>. For $(\mathbf{R})$ we have $\mu^* \equiv 0$, for $(h\mathbf{Z})$ we have $\mu^* \equiv h$. Therefore time scales on which dynamical systems are modelled have the special feature of constant grainyness. The further development of the m.c.calculus will show that one can even drop the postulate for continuity of the grainyness, which, for example, occurs in the points $k+½$, $k \in \mathbf{Z}$ of P (cf. section 4.1).

<u>**2.4 Dynamical triples and the notion of differentiation:**</u> We call a triple $(\mathbf{T},\mu,X)$ a <u>dynamical triple</u>, if $(\mathbf{T},\mu)$ is a measure chain and X is a $\mathbf{K}$-Banach space, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. It contains exactly the data which are necessary for modelling dynamical processes by means of mathematical equations (e.g. difference or differential equations).
Let f be a mapping $\mathbf{T} \longrightarrow X$. At $t \in \mathbf{T}$ f has the <u>derivative</u> $f_t^{\wedge} \in X$, if for each $\varepsilon > 0$ there exists a neighborhood U of t such that for all $s \in U$:

$$|f(\sigma(t)) - f(s) - f_t^{\wedge}\mu(\sigma(t),s)| \leq \varepsilon|\mu(\sigma(t),s)|. \qquad (2.4.1)$$

f is called <u>differentiable</u> in $t \in \mathbf{T}$, if f has exactly one derivative $f_t^{\wedge}$ in t.

One can easily check that in case $(\mathbf{R})$ these notions coincide with those of ordinary differential calculus. For $(h\mathbf{Z})$ each mapping $f:h\mathbf{Z} \longrightarrow X$ is differentiable in each $t \in h\mathbf{Z}$, we have:

$$f_t^{\wedge} = \Delta_h f(t) = [f(t+h) - f(t)]/h \qquad (2.4.2)$$

(cf. Theorem 2.5 (v)). We get the difference operator. In the literature on difference calculus and difference equations often the Bernoulli shift-operator $f(\cdot) \longmapsto f(\cdot+1) = f(\sigma(\cdot))$ is considered. This, however, is not suitable for the generalized difference- and differential calculus.
The variety of examples for this notion of differentiation is not exhausted by these two operators. Observe that it is defined for all measure chains, especially for all examples mentioned in the introduction.

2.5 Basic properties of the derivative

Theorem 2.5: Let f be a mapping $\mathbb{T} \longrightarrow X$ and $t \in \mathbb{T}$.

(i) If $t \in \mathbb{T}^{\kappa}$, f has at most one derivative in t.

(ii) If $t \notin \mathbb{T}^{\kappa}$, f has each $x \in X$ as a derivative.

(iii) If f has a derivative in t, then f is continuous in t.

(iv) The mapping $\mu(\cdot,\tau)$ is differentiable in t with:

$$\mu(\cdot,\tau)^{\Delta}_t = 1. \tag{2.5.1}$$

(v) If t is right-scattered and f is continuous in t, then f is differentiable in t. Then we have:

$$f^{\Delta}_t = \frac{f(\sigma(t)) - f(t)}{\mu^*(t)} \tag{2.5.2}$$

Proof: (i) Assume that f has two derivatives $x, y \in X$ in t. By definition of $\mathbb{T}^{\kappa}$ each neighborhood of t contains some $s \in \mathbb{T}^{\kappa}$ with $s \neq \sigma(t)$. Hence, for each $\varepsilon > 0$ there exists some $s \neq \sigma(t)$ with:

$$|f(\sigma(t)) - f(s) - x\mu(\sigma(t),s)| \leq \varepsilon/2 \cdot |\mu(\sigma(t),s)|$$

$$|f(\sigma(t)) - f(s) - y\mu(\sigma(t),s)| \leq \varepsilon/2 \cdot |\mu(\sigma(t),s)|.$$

This implies:

$$|x - y| \cdot |\mu(\sigma(t),s)| = |(x - y)\mu(\sigma(t),s)| \leq$$

$$|x\mu(\sigma(t),s) - [f(\sigma(t)) - f(s)]| + |f(\sigma(t)) - f(s) - y\mu(\sigma(t),s)| \leq \varepsilon \cdot |\mu(\sigma(t),s)|.$$

Observing that $\mu(\sigma(t),s) \neq 0$, we get: $|x - y| \leq \varepsilon$, hence $x = y$.

(ii) In this case for each $\varepsilon > 0$ and for each $x \in X$ there exists a neighborhood U (namely $U = \{t\}$) of $t = \sigma(t)$ such that for all $s \in U$ (hence $s = t$) we have:

$$|f(t) - f(s) - x\mu(t,s)| = 0 \leq \varepsilon \cdot |\mu(t,s)|.$$

(iii) Let $t \in \mathbb{T}$ and U be a compact neighborhood of t. We define:

$$M := \max\left(\{|\mu(\sigma(t),s)| : s \in U\} \cup \{|f^{\Delta}_t|\}\right) > 0.$$

For $\varepsilon > 0$ there exists a neighborhood V of t such that for all $s \in V$:

$$|f(\sigma(t)) - f(s) - f^{\Delta}_t\mu(\sigma(t),s)| \leq (\varepsilon/4M) \cdot |\mu(\sigma(t),s)|$$

$$|\mu(\sigma(t),s) - \mu(\sigma(t),t)| \leq \varepsilon/2M.$$

Thus for $s \in U \cap V$:

$|f(s) - f(t)| =$

$|f(s) - f(\sigma(t)) + f^{\Delta}_t\mu(\sigma(t),s) + f(\sigma(t)) - f(t) - f^{\Delta}_t\mu(\sigma(t),t) + f^{\Delta}_t[\mu(\sigma(t),t) - \mu(\sigma(t),s)]| \leq$

$\varepsilon/4M \cdot (|\mu(\sigma(t),s)| + |\mu(\sigma(t),t)|) + |f^{\Delta}_t||\mu(\sigma(t),t) - \mu(\sigma(t),s)| \leq (\varepsilon/4M) \cdot 2M + (\varepsilon/2M) \cdot M = \varepsilon.$

(iv) immediately follows from the cocycle property of μ.

(v) The mapping $g(\cdot) := \dfrac{f(\sigma(t)) - f(\cdot)}{\mu(\sigma(t),\cdot)}$ is defined and continuous in a neighbor hood of t. For $\varepsilon > 0$ there exists a neighborhood U of t such that for $s \in U$:

$$\left| f(\sigma(t)) - f(s) - \frac{f(\sigma(t)) - f(t)}{\mu^*(t)} \cdot \mu(\sigma(t),s) \right| =$$

$$\left| \frac{f(\sigma(t)) - f(s)}{\mu(\sigma(t),s)} - \frac{f(\sigma(t)) - f(t)}{\mu(\sigma(t),t)} \right| \cdot |\mu(\sigma(t),s)| \leq \varepsilon \cdot |\mu(\sigma(t),s)|$$

The uniqueness of the derivative follows from (i). ∎

<u>**2.6 Differentiation rules:**</u> The algebraic properties (such as linearity, Leibniz' product rule and quotient rule) for the derivative in the $\mathbb{R}$-differential calculus can be generalized:

Theorem 2.6:

(i) Let f and $g:\mathbf{T} \to X$ be differentiable in $t \in \mathbf{T}^{\kappa}$. Then for $\alpha, \beta \in \mathbf{K}$ the mapping $\alpha f + \beta g$ is differentiable in t, and we have:

$$(\alpha f + \beta g)^{\wedge}_t = \alpha f^{\wedge}_t + \beta g^{\wedge}_t. \tag{2.6.1}$$

(ii) Let X_1, X_2, X_3 be $\mathbf{K}$-Banach spaces and $\cdot : X_1 \times X_2 \to X_3$ be some bilinear and continuous mapping. If $f:\mathbf{T} \to X_1$ and $g:\mathbf{T} \to X_2$ are differentiable in $t \in \mathbf{T}^{\kappa}$, so is $f \cdot g$, and we have:

$$(f \cdot g)^{\wedge}_t = f(\sigma(t)) \cdot g^{\wedge}_t + f^{\wedge}_t \cdot g(t). \tag{2.6.2}$$

(iii) Let X be a $\mathbf{K}$-Banach algebra with 1. The mappings $f, g:\mathbf{T} \to X$ are assumed to be algebraically invertible with $f \cdot g = 1$. If f is differentiable in $t \in \mathbf{T}^{\kappa}$, then g is differentiable, too, and we get:

$$g^{\wedge}_t = -g(\sigma(t)) \cdot f^{\wedge}_t \cdot g(t). \tag{2.6.3}$$

Remarks: (1) A chain rule can be formulated and proved by establishing an even more generalizing structure which covers both measure chains and Banach spaces. We do not pursue this idea.
(2) The formulae (2.6.2/2.6.3) turn out not to be as "nice" as the corresponding ones in differential calculus. This is due to the existence of jumps in the measure chain. This trend will continue in the Chapters 6 and 7.

In Chapters 3 – 5 we shall see that on the other hand the left- or right-dense points of measure chains will have unfavourable effects in theorems about estimates and existence: Stronger conditions, more difficult proofs and though weaker statements. Altogether we can state the following rough, but striking philosophy:
The m.c.calculus is

- easier to handle in (right- or left-)scattered points (especially on $h\mathbb{Z}$) when it comes to topological and analytical considerations,
- easier to handle in right-dense points (especially on $\mathbb{R}$) in an algebraic context.

(In points which are at the same time left-dense and right-scattered the combination of jump operator and topology leads to additional difficulties.)

Proof: (i) is trivial.

(ii) Let an arbitrary $\varepsilon > 0$ be given: Choose $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that:

$$|f(\sigma(t))|\varepsilon_1 + (|g(t)| + \varepsilon_3)\varepsilon_2 + |f_t^{\wedge}|\varepsilon_3 \leq \varepsilon$$

and some neighborhood U of t such that for all $s \in U$:

$$|g(s) - g(\sigma(t)) - g_t^{\wedge}\cdot\mu(s,\sigma(t))| \leq \varepsilon_1|\mu(s,\sigma(t))|$$

$$|f(s) - f(\sigma(t)) - f_t^{\wedge}\cdot\mu(s,\sigma(t))| \leq \varepsilon_2|\mu(s,\sigma(t))|$$

$$|g(s) - g(t)| \leq \varepsilon_3.$$

Hence, for $s \in U$:

$$|f(s)\cdot g(s) - f(\sigma(t))\cdot g(\sigma(t)) - [f(\sigma(t))g_t^{\wedge} + f_t^{\wedge}g(t)]\mu(s,\sigma(t))| \leq$$

$$|f(\sigma(t))\cdot[g(s) - g(\sigma(t)) - g_t^{\wedge}\mu(s,\sigma(t))] +$$
$$[f(s) - f(\sigma(t)) - f_t^{\wedge}\mu(s,\sigma(t))][g(t) + g(s) - g(t)] + f_t^{\wedge}[g(s) - g(t)]\mu(s,\sigma(t))| \leq$$

$$[|f(\sigma(t))|\varepsilon_1 + (|g(t)| + \varepsilon_3)\varepsilon_2 + |f_t^{\wedge}|\varepsilon_3]|\mu(s,\sigma(t))| \leq \varepsilon\cdot|\mu(s,\sigma(t))|.$$

(iii) The mapping $X^* \longrightarrow X^*$, $x \longmapsto x^{-1}$ within the multiplicative group of X is continuous (cf. [6], 15.2.4 (i)). Therefore $g(\cdot)$ is continuous in t just like $f(\cdot)$. For an arbitrary $\varepsilon > 0$ there exist $\varepsilon_1, \varepsilon_2$ such that:

$$|g(\sigma(t))|\,\{\varepsilon_2(\varepsilon_1 + |g(t)|) + \varepsilon_1|f_t^{\wedge}|\} \leq \varepsilon$$

and some neighborhood U of t such that for $s \in U$:

$$|g(s) - g(t)| \leq \varepsilon_1, \qquad |f(s) - f(\sigma(t)) - f_t^{\wedge}\mu(s,\sigma(t))| \leq \varepsilon_2\cdot|\mu(s,\sigma(t))|.$$

Now we get for $s \in U$:

$$|g(s) - g(\sigma(t)) + g(\sigma(t))f_t^{\wedge}g(t)\cdot\mu(s,\sigma(t))| =$$

$$|-g(\sigma(t))\cdot\{[f(s) - f(\sigma(t)) - f_t^{\wedge}\mu(s,\sigma(t))]g(s) + f_t^{\wedge}[g(s) - g(t)]\mu(s,\sigma(t))\}| \leq$$

$$|g(\sigma(t))|\cdot\{\varepsilon_2(\varepsilon_1 + |g(t)|) + \varepsilon_1|f_t^{\wedge}|\}\cdot|\mu(s,\sigma(t))| \leq \varepsilon\cdot|\mu(s,\sigma(t))|.$$ ■

3. MEAN VALUE THEOREM AND CONCLUSIONS

3.1 Pre-differentiability: The mapping $f:\mathbf{T} \longrightarrow X$ is called pre-differentiable (on $\mathbf{T}$) with (region of differentiation) D, if for the pair (f,D) the following conditions hold:

- $D \subseteq \mathbf{T}^{\kappa}$, $\mathbf{T}^{\kappa}\backslash D$ is at most denumerable and contains no right-scattered elements of $\mathbf{T}$.
- f is continuous on $\mathbf{T}$ and differentiable in each $t \in D$.

In other words: f is continuous and everywhere differentiable with the exception of at most denumerable many right-dense points.

The mapping f is called differentiable (on $\mathbf{T}$), if it is pre-differentiable with $D = \mathbf{T}^{\kappa}$.

The mapping $f^\Delta: D \to X$, $t \mapsto f^\Delta_t$ is called pre-derivative (with range of definition) D resp. derivative of f.

Example: If $\mathbf{T}$ is an arbitrary interval of $\mathbf{R}$, then a function which is differentiable on $\mathbf{T}$, is differentiable in the interior of $\mathbf{T}$ and differentiable with respect to one side in end points belonging to $\mathbf{T}$.

The notions of pre-differentiability were created only in view of the proof of Theorem 4.4 which assures the existence of antiderivatives. We will be able to forget them, when this theorem will be proved. Nevertheless the theorems following immediately are the basis for the development of the analysis on measure chains when applying them for differentiable functions.

3.2 The mean value theorem

Theorem 3.2: Let the mappings $f:\mathbf{T} \to X$, $g:\mathbf{T} \to \mathbf{R}$ be pre-differentiable with D, and assume that :

$$|f^\Delta(t)| \le g^\Delta(t),\ t \in D. \tag{3.2.1}$$

Then for $r,s \in \mathbf{T}$, $r \le s$:

$$|f(s) - f(r)| \le g(s) - g(r). \tag{3.2.2}$$

Proof: There is a surjective mapping $\mathbf{N} \to [r,s[\setminus D$, $n \mapsto t_n$. Let $\varepsilon > 0$ be given. We apply the induction principle in $[r,s]$ to the statements

$$A(t):\ |f(t) - f(r)| \le g(t) - g(r) + \varepsilon\cdot[\mu(t,r) + \sum_{t_n<t} 2^{-n}].$$

From A(s) for arbitrary $\varepsilon > 0$ the claimed inequality (3.2.2) follows.

Ad (I): The statement A(r) is trivially satisfied.

Ad (II): Let t be right-scattered, hence $t \in D$. By (2.5.2) we have:

$$f(\sigma(t)) - f(t) = f^\Delta(t)\cdot\mu^*(t) \quad \text{and} \quad g(\sigma(t)) - g(t) = g^\Delta(t)\cdot\mu^*(t),$$

whence, observing that $|f^\Delta(t)| \le g^\Delta(t)$:

$$|f(\sigma(t)) - f(t)| \le g(\sigma(t)) - g(t).$$

The addition of this inequality to that of the induction condition A(t) yields $A(\sigma(t))$.

Ad (III): Let t be right-dense, hence $\sigma(t) = t \neq s$.

Case 1: $t \in D$: Then there exists some neighborhood U of t such that for all $p \in U$ the inequalities:

$$|f(t) - f(p) - f^\Delta(t)\mu(t,p)| \le \varepsilon/2\cdot|\mu(t,p)|$$

$$|g(t) - g(p) - g^\Delta(t)\mu(t,p)| \le \varepsilon/2\cdot|\mu(t,p)|$$

hold true, whence the estimates:

$$|f(p) - f(t)| \le (|f^\Delta(t)| + \varepsilon/2)\cdot|\mu(t,p)|$$

$$g(p) - g(t) + \varepsilon/2|\mu(t,p)| \ge g^\Delta(t)\cdot\mu(p,t)$$

follow. For all $p \in U$, $p \ge t$ we get:

$$|f(p) - f(t)| \leq (|f^{\Delta}(t)| + \varepsilon/2)\cdot\mu(p,t) \leq$$
$$(g^{\Delta}(t) + \varepsilon/2)\mu(p,t) \leq g(p) - g(t) + \varepsilon\cdot\mu(p,t).$$

<u>Case 2:</u> $t \in [r,s[\setminus D$: There exists an $m \in \mathbf{N}$ with $t = t_m$ and some neighborhood U of t such that for $p \in U$ we have:

$$|f(p) - f(t)| \leq \varepsilon/2\cdot 2^{-m} \quad \text{and} \quad |g(p) - g(t)| \leq \varepsilon/2\cdot 2^{-m}.$$

From the second inequality we derive:

$$0 \leq g(p) - g(t) + \varepsilon/2\cdot 2^{-m}.$$

An addition of this inequality to the first one yields:

$$|f(p) - f(t)| \leq g(p) - g(t) + \varepsilon\cdot 2^{-m}.$$

Therefore <u>in both cases</u> there exists some neighborhood U of t such that for $p \in U$, $p \geq t$ we have:

$$|f(p) - f(t)| \leq g(p) - g(t) + \varepsilon(\mu(p,t) + \sum_{t \leq t_n < p} 2^{-n}).$$

Taking into account the triangle inequality we get the statements $A(p)$ for all $p \in U$, $p \geq t$ by adding this inequality to the inequality of the induction condition $A(t)$.

Ad (IV): Observing that the $\leq$-inequality between continuous functions is valid on $[r,t[$, we can conclude that it is valid on $[r,t]$, too. ■

3.3 Two conclusions

Corollary 3.3:

(i) Let $f:\mathbf{T} \to X$ be pre-differentiable with D and U be some compact interval of $\mathbf{T}$ with endpoints r and s. Then we have:

$$|f(s) - f(r)| \leq \sup_{t \in U^{\varkappa} \cap D} \{|f^{\Delta}(t)|\}\cdot|\mu(s,r)|. \tag{3.3.1}$$

(ii) If $f:\mathbf{T} \to X$ is pre-differentiable with D and $f^{\Delta}(t) = 0$ for all $t \in D$, then f is constant.

Proof: For (i) we can assume w.l.o.g. that $r \leq s$. Consider the right side as a function g of s and apply the mean value theorem. (ii) immediately follows from (i). ■

3.4 Interchanging limits and differentiation

Theorem 3.4: Let the functions $f_n:\mathbf{T} \to X$ of the sequence $(f_n)_{n\in\mathbf{N}}$ be pre-differentiable with D. Assume that for each $t \in \mathbf{T}^{\varkappa}$ there exists a compact interval neighborhood U_t such that the sequence of the pre-derivatives $(f_n^{\Delta})_{n\in\mathbf{N}}$ converges uniformly on $U_t \cap D$. Then:

(i) The convergence of $(f_n)_{n\in\mathbf{N}}$ in one $t_K \in \mathbf{T}$ implies the uniform convergence of this sequence on each U_t.

(ii) $f := \lim_{n\to\infty} f_n$ is pre-differentiable with D, and we have:

$$f^{\Delta}(t) = \lim_{n\to\infty} f_n^{\Delta}(t), \quad t \in D. \tag{3.4.1}$$

Proof: We can assume that $\sigma(t) \in U_t$ for each t.
(1) Let $t \in \mathbf{T}^{\varkappa}$. We show: If $(f_n(\tau))_{n\in\mathbf{N}}$ converges for one $\tau \in U_t$, then $(f_n)_{n\in\mathbf{N}}$ converges uniformly on U_t. First of all there exists an $n_0 \in \mathbf{N}$ such that

$$\sup_{s\in U_t\cap D} \{|(f_n - f_m)^{\Delta}(s)|\}$$

exists for all $n,m \geq n_0$. For an arbitrary $r \in U_t$, $n,m \geq n_0$ by Corollary 3.3 (i) we have:

$$|f_n(r) - f_m(r)| - |f_n(\tau) - f_m(\tau)| \leq |[f_n(r) - f_m(r)] - [f_n(\tau) - f_m(\tau)]| \leq \tag{3.4.2}$$
$$\sup_{s\in U_t\cap D} \{|(f_n - f_m)^{\Delta}(s)|\}\cdot|\mu(\tau,r)|.$$

This implies:

$$|f_n(r) - f_m(r)| \leq |f_n(\tau) - f_m(\tau)| + \sup_{s\in U_t\cap D} \{|f_n^{\Delta}(s) - f_m^{\Delta}(s)|\cdot\mu(\max U_t, \min U_t)\}.$$

The right side of this inequality can be made arbitrarily small, if n,m are large enough. The numbers n,m can be chosen independently of $r \in U_t$, thus we get the claimed assertion, since X is complete.
(2) We want to show that for each $t \in \mathbf{T}$ $(f_n(t))_{n\in\mathbf{N}}$ converges.
To this purpose we apply the induction principle in $[t_k,\infty[$ to the statements

$$A(t):\ (f_n(t))_{n\in\mathbf{N}} \text{ converges.}$$

Ad (I): In fact: $(f_n(t_k))_{n\in\mathbf{N}}$ converges.
Ad (II): Let t be right-scattered, hence $t \in D$. With (2.5.2) we have for $n \in \mathbf{N}$:

$$f_n(\sigma(t)) = f_n(t) + f_n^{\Delta}(t)\cdot\mu^*(t).$$

By the assumption of the theorem $(f_n^{\Delta}(t))_{n\in\mathbf{N}}$ converges. Therefore from A(t) the statement $A(\sigma(t))$ follows.
Ad (III) and (IV): These conditions can easily be checked by the statement (1). For (IV) observe that U_t always contains some s with $s < t$, if t is left-dense. Hence, we have A(t) for $t \geq t_k$.
For $t \leq t_k$ the assertion can be shown in the same way. One applies the dual version of the induction principle for the negative direction. Thereby the roles of ϱ and σ change. Only when proving condition (II) one has to state equation (2.5.2) as follows:

$$f_n(\varrho(t)) = f_n(t) - f_n^{\Delta}(\varrho(t))\cdot\mu^*(\varrho(t)).$$

(3) Ad (ii):
(3a) f is continuous as a limit function of a locally uniformly convergent sequence of continuous functions. Let $g: D \longrightarrow X$ be defined by:

$$g(s) := \lim_{n\to\infty} f_n^\Delta(s).$$

Now, let $t \in D$ and $\varepsilon > 0$ be given. Then there is some $n_1 \geq n_0$ (cf (1)) such that for $n,m \geq n_1$ we have:

$$\sup_{s\in U_t \cap D} \{|f_n^\Delta(s) - f_m^\Delta(s)| \leq \varepsilon/3.$$

By inequality (3.4.2) – replace τ by $\sigma(t)$ – we have for all $r \in U_t$ and $n,m \geq n_1$:

$$|[f_n(r) - f_m(r)] - [f_n(\sigma(t)) - f_m(\sigma(t))]| \leq$$

$$\sup_{s\in U_t \cap D} \{|f_n^\Delta(s) - f_m^\Delta(s)|\} \cdot |\mu(r,\sigma(t))| \leq \varepsilon/3 \cdot |\mu(r,\sigma(t))|.$$

Especially as $m \to \infty$, $n \geq n_1$, it follows:

$$|[f_n(r) - f(r)] - [f_n(\sigma(t)) - f(\sigma(t))]| \leq \varepsilon/3 \cdot |\mu(r,\sigma(t))|. \tag{3.4.3}$$

(3b) There exists some $j \geq n_1$ such that:

$$|f_j^\Delta(\sigma(t)) - g(\sigma(t))| \leq \varepsilon/3. \tag{3.4.4}$$

(3c) f_j is differentiable in t, therefore we have a neighborhood W of t such that for all $r \in W$:

$$|f_j(\sigma(t)) - f_j(r) - f_j^\Delta(t) \cdot \mu(\sigma(t),r)| \leq \varepsilon/3 \cdot |\mu(r,\sigma(t))|. \tag{3.4.5}$$

(4) Now, altogether for $r \in U_t \cap W$ it follows from (3.4.3),(3.4.4),(3.4.5):

$$|f(\sigma(t)) - f(r) - g(t)\cdot\mu(\sigma(t),r)| \leq$$

$$|[f(\sigma(t)) - f(r)] - [f_j(\sigma(t)) - f_j(r)]| + |f_j(\sigma(t)) - f_j(r) - f_j^\Delta(t)\mu(\sigma(t),r)| +$$
$$|[f_j^\Delta(t) - g(t)]\cdot\mu(\sigma(t),r)| \leq$$

$$(\varepsilon/3 + \varepsilon/3 + \varepsilon/3)\cdot|\mu(\sigma(t),r)| = \varepsilon\cdot|\mu(\sigma(t),r)|.$$ ■

4. INTEGRAL AND ANTIDERIVATIVE

4.1 Regulated and rd-continuous functions: Let X be an arbitrary topological space, $\mathbf{T}$ some measure chain and $t \in \mathbf{T}$ left-dense. If for a mapping $g: \mathbf{T} \to X$ the limit

$$\lim_{s\nearrow t} g(s) = \lim_{s\to t, s<t} g(s) =: g(t-)$$

exists, then we call it the left-sided limit of g at t. Accordingly we define the notion of the right-sided limit $g(t+)$ for right-dense points t.
The mapping g is called regulated, if in each left-dense $t \in \mathbf{T}$ the left sided, and in each right-dense $t \in \mathbf{T}$ the right sided limit exists.
Finally the mapping g is called rd-continuous, if it

- is continuous in each right-dense or maximal $t \in \mathbf{T}$ and
- the left sided limit $g(t-)$ exists in each left-dense t.

By $C_{rd}(\mathbf{T},X)$ we denote the set of rd-continuous mappings $\mathbf{T} \rightarrow X$.
The following implications are immediate:

$$\text{continuous} \Longrightarrow \text{rd-continuous} \Longrightarrow \text{regulated}$$

Only, if $\mathbf{T}$ contains at the same time left-dense and right-scattered points - we shortly call them ldrs-points -, then the first implication is not invertible. On a discrete measure chains all three notions coincide.

Theorem 4.1: Let the mapping $g:\mathbf{T} \rightarrow X$ be given.

(i) If g is regulated or rd-continuous, resp., then the Bernoulli-shift $g \circ \sigma$ has this property, too.

(ii) Let Y be a topological space and $h:X \rightarrow Y$ be continuous. If g is regulated or rd-continuous, resp., then so is $h \circ g$.

(iii) If the mappings $g_n:\mathbf{T} \rightarrow X$ of the sequence $(g_n)_{n \in \mathbf{N}}$ are regulated or rd-continuous and locally uniformly convergent to $g:\mathbf{T} \rightarrow X$, then g is regulated or rd-continuous, resp., too.

(iv) For $i = 1,\dots,n$ let X_i be some topological space and let the mapping g_i be regulated or rd-continuous, resp. Then so is $g := (g_1,\dots,g_n):\mathbf{T} \rightarrow X_1 \times \dots \times X_n$.

It follows that the grainyness $\mu^*(\cdot) = \mu^*(\sigma(\cdot),\tau) - \mu^*(\cdot,\tau)$ is rd-continuous. In ldrs-points (e.g. in the points $k+\frac{1}{2}$, $k \in \mathbf{Z}$, of P), however, it is not continuous.

Proof: For (i): It is enough to consider the one sided limits. Let t be right-dense or left-dense, resp., and let $(s_n)_{n \in \mathbf{N}}$ be some sequence with $s_n > t$ or $s_n < t$, resp., converging to t. Then $(r_n)_{n \in \mathbf{N}}$, $r_n := \sigma(s_n)$, is a sequence with the same properties. It follows that

$$(g \circ \sigma)(t\pm) = \lim_{n \to \infty} g \circ \sigma(s_n) = \lim_{n \to \infty} g(r_n) = g(t\pm).$$

If t is right-dense and g rd-continuous, then we have:

$$(g \circ \sigma)(t\pm) = g(t\pm) = g(t) = g \circ \sigma(t).$$

For (ii),(iii) and (iv) we only mention the fact that limits and the respective operations under considerations commute. ∎

4.2 Pre-antiderivatives

Theorem 4.2: Let $\tau \in \mathbf{T}$, $x \in X$ and a regulated mapping $g:\mathbf{T}^\kappa \rightarrow X$ be given. Then there exists exactly one function - the so-called pre-antiderivative - which is pre-differentiable and fulfills the identities

$$f^\Delta(t) = g(t) \quad \text{for } t \in D \quad \text{and} \quad f(\tau) = x.$$

Proof: (1) The uniqueness immediately follows from Corollary 3.2 (ii).
(2) Let some fixed $n \in \mathbf{N}$ be given. For $t \in [\tau,\infty[$ we prove by induction the statements:

$A(t)$: There exists a function $f_{n,t}:[\tau,t] \longrightarrow X$ which is pre-differentiable with $D_{n,t}$ such that

$$f_{n,t}(\tau) = x \quad \text{and} \quad |f_{n,t}^{\Delta}(s) - g(s)| \leq 1/n \text{ for } s \in D_{n,t}.$$

Ad (I): We set $D_{n,\tau} := \emptyset$ and $f_{n,\tau}(\tau) := x$.

Ad (II): Let t be right-scattered, the function $f_{n,t}$ and its domain of differentiation are assumed to be constructed. We define

$$D_{n,\sigma(t)} := D_{n,t} \cup \{t\} \quad \text{and}$$

$$\begin{aligned} f_{n,\sigma(t)}:[\tau,\sigma(t)] &\longrightarrow X \\ s &\longmapsto \begin{cases} f_{n,t}(s), & \text{if } s \in [\tau,t] \\ f_{n,t}(t) + g(t)\mu^*(t), & \text{if } s = \sigma(t). \end{cases} \end{aligned}$$

Then we have (cf. 2.5.2):

$$|f_{n,\sigma(t)}^{\Delta}(s) - g(s)| \begin{cases} \leq 1/n, & \text{if } s \in D_{n,t}, \\ = 0, & \text{if } s = t, \end{cases}$$

hence, altogether: $A(\sigma(t))$.

Ad (III): Let t be right-dense. By the induction condition we assume that $D_{n,t}$ and $f_{n,t}$ are given. Then there is a neighborhood U of t such that

$$|g(s) - g(t+)| \leq 1/n, \quad s \in U, \ s > t.$$

For $r \in U$, $r > t$ we define:

$$D_{n,r} := (D_{n,t} \backslash \{t\}) \cup]t,r]^{\kappa}$$

$$\begin{aligned} f_{n,r}:[\tau,r] &\longrightarrow X \\ s &\longmapsto \begin{cases} f_{n,t}(s), & \text{if } s \in [\tau,t] \\ f_{n,t}(t) + g(t+)\mu(s,t), & \text{if } s \in]t,r]. \end{cases} \end{aligned}$$

Then $f_{n,r}$ is pre-differentiable with $D_{n,r}$; by (2.5.1) we have for $s \in]t,r]^{\kappa}$:

$$|f_{n,r}^{\Delta}(s) - g(s)| = |g(t+) - g(s)| \leq 1/n.$$

Together with the induction condition $A(t)$ this yields $A(r)$, $r \in U$, $r > t$.

Ad (IV): Let t be left-dense, for each $r < t$ $D_{n,r}$ and $f_{n,r}$ are assumed to be given. Then there exists some neighborhood U of t such that

$$|g(s) - g(t-)| \leq 1/n, \quad s \in U, \ s < t.$$

Fix some $r \in U$, $r < t$. We define:

$$D_{n,t} := \begin{cases} D_{n,r} \cup]r,t[, & \text{if } r \text{ is right-dense} \\ D_{n,r} \cup [r,t[, & \text{if } r \text{ is right-scattered} \end{cases}$$

and

$$\begin{aligned} f_{n,t}:[\tau,t] &\longrightarrow X \\ s &\longmapsto \begin{cases} f_{n,r}(s), & \text{if } s \in [\tau,r] \\ f_{n,r}(t) + g(t-)\mu(s,r), & \text{if } s \in]r,t] \end{cases} \end{aligned}$$

Then $f_{n,t}$ is pre-differentiable with $D_{n,t}$ and we have for $s \in D_{n,t}$, $s \geq r$:

$$|f_{n,t}^{\Delta}(s) - g(s)| = |g(t-) - g(s)| \leq 1/n,$$

hence $A(t)$.

(3) By means of dual considerations one can prove $A(t)$ for $t \le \tau$, too. For each fixed $n \in \mathbb{N}$ we then have found a function $f_n:\mathbf{T} \to X$ with domain of definition D_n such that

$$|f_n^{\Delta}(t) - g(t)| \le 1/n \text{ for } t \in D_n.$$

(4) We have $f_n(\tau) = x$ for all $n \in \mathbb{N}$. Now Theorem 3.4 yields a function f which is pre-differentiable with $D := \bigcap_{n \in \mathbb{N}} D_n$, such that

$$f^{\Delta}(t) = g(t), \quad t \in D. \qquad \blacksquare$$

<u>**4.3 The Cauchy-integral:**</u> We do not want to derive an integral notion by means of measure theoretical considerations. We follow the presentation in [5] and introduce integration as the inverse of differentiation: For a regulated function $g:\mathbf{T}^{\varkappa} \to X$ let $f:\mathbf{T} \to X$ be the pre-antiderivative given in Theorem 4.2. By

$$\int_r^s g(t)\Delta t := f(s) - f(r) \in X \tag{4.3.1}$$

we define the <u>Cauchy-integral</u> from r to s of the function g.

In the case ($\mathbb{R}$) each regulated real valued function is Riemann-integrable, too. The Riemann-integral coincides with the Cauchy-integral. The advantage of the Cauchy-integral is that it is defined for functions with values in an arbitrary Banach space.

For ($h\mathbb{Z}$) one easily proves the relation

$$\int_r^s g(t)\Delta t = \begin{cases} \sum\limits_{i=r/h}^{s/h-1} g(ih), & \text{if } s > r, \\ 0, & \text{if } s = r, \\ -\sum\limits_{i=s/h}^{r/h-1} g(ih), & \text{if } s < r. \end{cases}$$

For (P) and $r,s \in P$ we have:

$$\int_r^s g(t)\Delta t = \int_r^{[r]} g(t)dt + \sum_{k=[r]}^{[s]-1} \left[\int_k^{k+1/2} g(t)dt + \tfrac{1}{2} g(k+\tfrac{1}{2})\right] + \int_{[s]}^s g(t)dt$$

Here the sum has to be furnished with a minus sign, if $[s]-1 < [r]$ (Gauß-bracket).

Now, the different properties of differentiation can be translated into the integral language:

Theorem 4.3

(i) Suppose the mappings f and $g:\mathbf{T}^{\varkappa} \to X$ are regulated. Then we have for $r,s \in \mathbf{T}$, $\alpha,\beta \in \mathbb{K}$:

$$\int_r^s [\alpha f(t) + \beta g(t)]\Delta t = \alpha \cdot \int_r^s f(t)\Delta t + \beta \cdot \int_r^s g(t)\Delta t. \tag{4.3.2}$$

(ii) Partial integration: In the situation of Theorem 2.6 (ii) the mappings $f:\mathbf{T}^{\varkappa} \longrightarrow X_1$ and $g:\mathbf{T} \longrightarrow X_2$ are assumed to be differentiable with regulated derivatives f^{Δ} and g^{Δ}. Then the stated integrals exist and we have for $r,s \in \mathbf{T}$:

$$\int_r^s f(\sigma(t))g^{\Delta}(t)\Delta t + \int_r^s f^{\Delta}(t)g(t)\Delta t = f(s)g(s) - f(r)g(r). \tag{4.3.3}$$

In the case $(\mathbf{Z})$ this is called Abel summation.

(iii) Let the mappings $f:\mathbf{T}^{\varkappa} \longrightarrow X$, $g:\mathbf{T}^{\varkappa} \longrightarrow \mathbf{R}_0^+$ be regulated. If:

$$|f(t)| \leq g(t) \qquad \text{for } t \in [r,s]^{\varkappa},$$

then:

$$\left| \int_r^s f(t)\Delta t \right| \leq \int_r^s g(t)\Delta t \tag{4.3.4}$$

(iv) If the sequence $(g_n)_{n\in\mathbf{N}}$ of regulated functions $\mathbf{T}^{\varkappa} \longrightarrow X$ converges uniformly on $[r,s[$ to the regulated function g, then the sequence

$$\left(\int_r^s g_n(t)\Delta t\right)_{n\in\mathbf{N}} \text{ converges to } \int_r^s g(t)\Delta t \text{ in } X.$$

Proof: We only have to show that the integrands are regulated. This, however, follows immediately from Theorem 4.1. ■

<u>4.4 Antiderivative of a rd-continuous function:</u> It is well-known from continuous analysis that the mapping

$$t \longmapsto \int_\tau^t g(s)ds \tag{4.4.1}$$

is differentiable in $t \in \mathbf{R}$ with derivative $g(t)$, if g is continuous in t. We already know that (4.4.1) is possibly not differentiable in the right-dense points of $\mathbf{T}^{\varkappa}\setminus D$. Therefore, if one wants to guarantee the differentiability of (4.4.1), it remains to postulate the continuity of g only for these points: We arrive at the notion of rd-continuity as is described in 4.1.

Let g be a mapping $\mathbf{T}^{\varkappa} \longrightarrow X$. The mapping $f:\mathbf{T} \longrightarrow X$ is called <u>antiderivative of g on $\mathbf{T}$</u>, if it is differentiable on $\mathbf{T}$ and satisfies

$$f^{\Delta}(t) = g(t) \qquad \text{for } t \in \mathbf{T}^{\varkappa}.$$

Theorem 4.4: If $g:\mathbf{T}^{\varkappa} \longrightarrow X$ is rd-continuous, then g has the antiderivative

$$f:t \longmapsto \int_\tau^t g(s)\Delta s \ .$$

Proof: Suppose $t \in \mathbf{T}$. From the definition of the integral in 4.3 and Theorem 4.2 it follows that f is pre-differentiable with D such that $f^{\Delta}(t) = g(t)$, $t \in D$. Hence let $t \in \mathbf{T}^{\varkappa}\setminus D$, then t is right-dense: $\sigma(t) = t$. For $\varepsilon > 0$ there exists a neighborhood U of t such that:

$$|g(s) - g(t)| \leq \varepsilon \quad \text{for all } s \in U$$

The function $h(\cdot) := f(\cdot) - g(t)\mu(\cdot,\tau)$, $\tau \in \mathbf{T}$ fixed, is pre-differentiable with D, we have for $s \in U \cap D$:

$$|h^{\Delta}(s)| = |g(s) - g(t)| \le \varepsilon$$

With Corollary 3.3(i) we get for $r \in U$:

$$|f(t) - f(r) - g(t)\mu(t,r)| = |[f(t) - g(t)\mu(t,t)] - [f(r) - g(t)\mu(r,t)]| =$$

$$|h(t) - h(r)| \le \sup_{s \in U^{*} \cap D} \{|h_{\tau}^{\Delta}(s)|\} \cdot |\mu(t,r)| \le \varepsilon \cdot |\mu(t,r)|$$

Hence f has a derivative in t. It is uniquely determined by Theorem 2.5 (i). ■

5. DYNAMICAL EQUATIONS

5.1 Definitions: Let the dynamical triple $(\mathbf{T},\mu,X)$ be given as well as a mapping (the so-called right-hand side) $f:\mathbf{T}^{*} \times X \to X$. The mapping $x:\mathbf{T} \to X$ is called a solution of the dynamical equation

$$x^{\Delta} = f(t,x) \tag{5.1.1}$$

(on $\mathbf{T}$) or solution of the dynamical equation $x^{\Delta} = f(t,x)$, $t \in \mathbf{T}$, if x is an antiderivative of $f(\cdot,x(\cdot))$ on $\mathbf{T}$.
If x additionally satisfies $x(\tau) = \eta$, then x is called a solution of the initial value problem (IVP)

$$x^{\Delta} = f(t,x),\ t \in \mathbf{T},\quad x(\tau) = \eta. \tag{5.1.2}$$

5.2 rd-continuous right-hand side: First of all we discuss some important properties of the right-hand side of a dynamical equation.
The mapping $f:\mathbf{T}^{*} \times X \to X$ is called (generalizing Definition 4.1) rd-continuous, if it

- is continuous at each (t,x) with right-dense or maximal t, and
- the limits $f(t-,x) := \lim_{(s,y)\to(t,x),s<t} f(s,y)$ and $\lim_{y\to x} f(t,y)$ exist at each (t,x) with left-dense t.

Hence, in general for left-dense t the function $f(t,\cdot):X \to X$ is in no way a continuous continuation of the mapping $f:]-\infty,t[\ \times X \to X$ to the point t.

Example: Given a rd-continuous function $g:\mathbf{T} \to X_1$ (in the sense of 4.1), a continuous function $h:X_2 \to X_3$ and another continuous function $f:X_1 \times X_2 \to X_3$ the composite function $f(g(\cdot),h(\cdot))$ is rd-continuous (in the above sense).

A tool which is solely important for the proof of the Global Existence Theorem 5.7 is given by means of the mapping $f^{\tau]}:]-\infty,\tau]\ \times X \to X$ which is defined for a fixed $\tau \in \mathbf{T}^{*}$:

$$f^{\tau]}(t,x) = \begin{cases} f(t,x), & \text{if } (t,x) \in]-\infty,\tau[\ \times X \\ f(\tau-,x), & \text{if } (t,x) \in \{\tau\} \times X. \end{cases} \tag{5.2.1}$$

$f^{\tau]}$ does not necessarily coincide with f on $]-\infty,\tau]$, if τ is a ldrs-point (cf. 4.1), otherwise it does. The purpose of this definition (5.2.1) will become clear in the Lemmas 5.2 and 5.3. This notion will be discussed in (5.4) by means of the concrete example (P).

Lemma 5.2: For each continuous function $x:\mathbf{T}\to X$ the mapping $f^{\tau]}(\cdot,x(\cdot))$ is rd-continuous (with respect to the interval $]-\infty,\tau]$).

Proof: Obviously $f_1(\cdot) := f^{\tau]}(\cdot,x(\cdot))$ is continuous in right-dense points t. If t is left-dense, then we have:

$$f_1(t-) = \lim_{s\nearrow t} f_1(s) = \lim_{s\nearrow t} f^{\tau]}(s,x(s)) = f(t-,x(t)), \tag{5.2.2}$$

i.e. the left-hand limit exists. If t is maximal in $]-\infty,\tau]$ and left-dense (hence $t = \tau$), then we have:

$$f_1(\tau-) = f(\tau-,x(\tau)) = f^{\tau]}(\tau,x(\tau)) = f_1(\tau) \tag{5.2.3}$$

by (5.2.2), hence f_1 is continuous in τ. ∎

For the solution of a dynamical equation with rd-continuous right-hand side f we have $x^{\Delta}(t) = f(t,x(t))$. This implies that x is rd-continuously differentiable, i.e. differentiable with rd-continuous derivative.

5.3 Composition and restriction of solutions

Theorem 5.3: Suppose $\tau,r\in\mathbf{T}$ with $\tau\leq r$. The mapping $x:\mathbf{T}\to X$ is a (unique) solution of the IVP

$$x^{\Delta} = f(t,x),\ t\in\mathbf{T},\quad x(\tau) = \eta, \tag{5.3.1}$$

if and only if the restriction $x_1 := x|]-\infty,r]$ is a (unique) solution of the IVP

$$x^{\Delta} = f^{r]}(t,x),\ t\in\,]-\infty,r],\quad x(\tau) = \eta \tag{5.3.2}$$

and the restriction $x_2 := x|[r,\infty[$ is a (unique) solution of the IVP

$$x^{\Delta} = f(t,x),\ t\in[r,\infty[,\quad x(r) = x_1(r) \tag{5.3.3}$$

A corresponding statement holds, if $\tau\in[r,\infty[$.

Proof: "$\Longrightarrow$" Let x be a solution of (5.3.1).
By Lemma 5.2 the mapping $f_1(\cdot) := f^{r]}(\cdot,x(\cdot))$ is rd-continuous w.r.t. $]\infty,r]$. By Theorem 4.5 it has a continuous antiderivative $y:]\infty,r]\to X$ with $y(\tau) = \eta = x(\tau)$. Thus we have:

$$y^{\Delta}(t) = f_1(t) = f(t,x(t)) = x^{\Delta}(t) \quad \text{for } t\in\,]-\infty,r[, \tag{5.3.4}$$

whence immediately $x(t) = y(t)$ follows for $t\in\,]-\infty,r[$. If r is left-scattered, then from $x_{\tau}^{\Delta}(\rho(r)) = y_{\tau}^{\Delta}(\rho(r))$ the coincidence of x and y in r follows, for left-dense t it is a consequence of the continuity of x and y. Hence x is an antiderivative of f_1, and therefore a solution of the IVP (5.3.2).

The fact that x_2 is a solution of the IVP (5.3.3) one can easily see with the help of a distinction of the cases r right-dense/right-scattered for the only problematic point r.

"$\Longleftarrow$" With x_1 and x_2 x is continuous, too. Let $t \in \mathbf{T}^{\varkappa}$:

Case 1: t is right-scattered. Then x is differentiable in t (cf. Theorem 2.5(v)). The set $\{t,\sigma(t)\}$ either is entirely contained in $]-\infty,r]$ or entirely contained in $[r,\infty[$. Hence x coincides with one of the mappings x_i, $i = 1$ or 2, in t and $\sigma(t)$. It follows:

$$x^{\Delta}(t) = x_i^{\Delta}(t) = f(t,x_i(t)) = f(t,x(t))$$

Case 2: t is right-dense or maximal in $\mathbf{T}$: Here only the point $t = r$ is problematic. The differentiability of x in r directly follows from the definition of differentiation, if one observes that the union of a $]-\infty,r]$-neighborhood of r and a $[r,\infty[$-neighborhood of r is a $\mathbf{T}$-neighborhood of r.

Now, we have shown that the property of being a solution is transfered in both directions. From this one can easily derive that x is a unique solution, if and only if x_1 and x_2 are unique solutions. ■

5.4. Example: We consider the IVP:

$$x^{\Delta} = f(t,x) = \mu^*(t)\cdot x, \quad x(0) = 1 \tag{5.4.1}$$

on the dynamical triple $(P,\lambda,\mathbb{R})$, $\lambda(s.r) = s - r$. It has the solution $x(t) = (5/4)^{[t]}$ ($[\cdot]$: Gauß-bracket). For each $k \in \mathbf{Z}$ the restriction of x to $P' := \{t \in P:\ t \le k+\frac{1}{2}\}$ is not a solution of (5.4.1) on P' any more, but a solution of the dynamical equation

$$x^{\Delta} = f^{[k+\frac{1}{2}]}(t,x) \quad = \begin{cases} \mu^*(t)\cdot x, & \text{if } t < k+\frac{1}{2}, \\ 0, & \text{if } t = k+\frac{1}{2}. \end{cases} \tag{5.4.2}$$

The reason for this phenomenon is that the jump operator σ changes its value, when one cuts off the measure chain at an ldrs-point $k+\frac{1}{2}$. Therefore the derivative at such a point is no longer the difference quotient involving the right neighbor $k+1$, but the left sided derivative at the point under consideration $k+\frac{1}{2}$.

5.5 Existence of solutions under global Lipschitz-condition

Theorem 5.5: Let $\mathbf{T} = [r,s]$ be some compact measure chain and let L be a non-negative constant with $L\cdot\mu(s,r) < 1$. If the right-hand side in the IVP

$$x^{\Delta} = f(t,x), \qquad x(\tau) = \eta, \tag{5.5.1}$$

is rd-continuous and satisfies the Lipschitz-condition

$$|f(t,x_1) - f(t,x_2)| \le L\cdot|x_1 - x_2|, \ t \in \mathbf{T}^{\varkappa},\ x_1,x_2 \in X, \tag{5.5.2}$$

then the IVP has exactly one solution.

Proof: Let $C(\mathbf{T},X)$ be the Banach space of continuous mappings $\mathbf{T} \longrightarrow X$, equipped with the sup norm $\|\ \|$. The mapping

$$\begin{aligned} J: \mathcal{C}(\mathbf{T},X) &\longrightarrow \mathcal{C}(\mathbf{T},X) \\ x(\cdot) &\longmapsto \eta + \int_r^{(\cdot)} f(t,x(t))\Delta t \end{aligned}$$

is well-defined by Theorem 4.5, and, as will be shown, a contraction. By Theorem 4.3 (iii) we have for $t \in \mathbf{T}$:

$$|Jx_1(t) - Jx_2(t)| = \left| \int_r^s [f(t,x_1(t)) - f(t,x_2(t))]\Delta t \right| \leq$$

$$\int_r^s L|x_1(t) - x_2(t)|\Delta t \leq L\cdot\mu(s,r)\cdot\|x_1(\cdot) - x_2(\cdot)\|,$$

hence, altogether:

$$\|Jx_1(\cdot) - Jx_2(\cdot)\| \leq L\cdot\mu(s,r)\cdot\|x_1(\cdot) - x_2(\cdot)\|.$$

The fixed points of J are exactly the solutions of the IVP. Hence the assertion follows from the Banach fixed point theorem.

5.6 Regressivity: We consider the IVP:

$$x^{\Delta} = -x, \qquad x(\tau) = \eta. \tag{5.6.1}$$

on the dynamical triple $(\mathbb{N},\lambda,\mathbb{R})$. For $\eta = 0$ it has infinitely many solutions, for $\eta \neq 0$ it has none. Hence, when jumps exist in the measure chain, it may happen that in spite of the harmlessness of the right-hand side the continuation of solutions in backward direction is not possible in a unique manner or not possible at all. This phenomenon necessitates the following definition: The mapping $f:\mathbf{T}^{\varkappa} \times X \longrightarrow X$ or the corresponding dynamical equation or IVP is called <u>regressive in $t \in \mathbf{T}^{\varkappa}$</u>, if the mapping

$$\mathrm{id} + f(t,\cdot)\mu^*(t): X \longrightarrow X \tag{5.6.2}$$

is invertible. f is called <u>regressive (on $\mathbf{T}$)</u>, if f is regressive in each $t \in \mathbf{T}^{\varkappa}$.

One can easily verify by means of the foregoing theorem that a mapping $f:\mathbf{T}^{\varkappa} \times X \longrightarrow X$ which fulfills the condition

$$|f(t,x_1) - f(t,x_2)| \leq L\cdot|x_1 - x_2|\ , \quad x_1,x_2 \in X$$

for some $t \in \mathbf{T}^{\varkappa}$, $L\cdot\mu^*(t) < 1$, is regressive in t.

5.7 The global existence theorem

Theorem 5.7: The right-hand side $f:\mathbf{T}^{\varkappa} \times X \longrightarrow X$ in the IVP

$$x^{\Delta} = f(t,x), \qquad x(\tau) = \eta \tag{5.7.1}$$

is assumed to satisfy the following conditions:

(a) f is rd-continuous.

(b) For each $t \in \mathbf{T}^{\varkappa}$ there exists some compact neighborhood U_t such that $f^{t]}$ in $U_t^{\varkappa} \times X$ satisfies a Lipschitz-condition w.r.t. the second argument:

$$|f^{t]}(s,x_1) - f^{t]}(s,x_2)| \leq L_t\cdot|x_1 - x_2|;\ (s,x_1) \in U_t^{\varkappa} \times X. \tag{5.7.2}$$

(c) In each $t < \tau$ f is regressive.

Then the IVP (5.7.1) admits exactly one solution $x^*(\cdot) = x^*(\cdot;\tau,\eta)$.

Condition (b) is automatically fulfilled for points t which are at the same time left- and right-scattered. Then $\{t\}$ is a neighborhood of t, and we have $\{t\}^* = \emptyset$. (c) always holds for right-dense t ($\mu^*(t) = 0$).

Proof: (1) In $[\tau,\infty[$ we apply the induction principle to the following statements:

$$A(r)\text{: The IVP } x^\Delta = f^{r]}(t,x),\ t \in [\tau,r],\ x(\tau) = \eta \qquad (5.7.4/r)$$
admits exactly one solution $x_r(\cdot)$.

Ad (I): In fact there exists only one mapping $x_\tau : \{\tau\} \longrightarrow X$ with $x_\tau(\tau) = \eta$ and $x_\tau^\Delta(t) = f^{r]}(t,x_\tau(t))$ for $t \in \{\tau\}^* = \emptyset$.

Ad (II): Let r be right-scattered. The IVP (5.7.4/r) has - according to the induction condition - exactly one solution $x_r(\cdot)$. We define the mapping $x_{\sigma(r)} : [\tau,\sigma(r)] \longrightarrow X$ by:

$$x_{\sigma(r)}(t) = \begin{cases} x_r(t) , & \text{if } t \in [\tau,r] \\ x_r(r) + f(r,x_r(r))\mu^*(r), & \text{if } t = \sigma(r). \end{cases}$$

It is continuous and (by Theorem 5.3) the only solution of the IVP (5.7.4/$\sigma(r)$), since its restriction to $[\tau,r]$ is the only solution of the IVP (5.7.4/r) and its restriction to $[r,\sigma(r)]$ is the only solution of the IVP

$$x^\Delta = f(t,x), \qquad x(r) = x_r(r)$$

on $[r,\sigma(r)]$.

Ad (III): Let r be right-dense. By the induction condition there exists exactly one solution $x_r(\cdot)$ of (5.7.4/r). Let $V_r \subseteq U_r$ be a compact neighborhood of r such that $L_r \cdot |\mu(s,r)| < 1$ for all $s \in V_r$. By Theorem 5.5 and condition (b) for each $s \in V_r$, $s \geq t$ the IVP

$$x^\Delta = f^{s]}(t,x), \quad t \in [r,s], \quad x(r) = x_r(r)$$

admits exactly one solution $y_s(\cdot)$. The mapping x_s, defined by

$$x_s(t) := \begin{cases} x_r(t) , & \text{if } t \in [\tau,r] \\ y_s(t) , & \text{if } t \in [r,s], \end{cases}$$

is (by Theorem 5.3) the unique solution of the IVP (5.7.4/s). Hence we have $A(s)$ for all $s \in V_r$, $s \geq r$.

Ad (IV): Let r be left-dense, and choose V_r as above, then there is a $s \in V_r$ with $s < r$. With the help of the induction condition $A(s)$ and Theorem 5.5 existence and uniqueness of a solution $x_r(\cdot)$ of (5.7.4/r) can be shown exactly in the same way as under (III). Hence we have $A(r)$.

Since there is a solution on each interval $[\tau,r]$, $r \geq \tau$, there is one on $[\tau,\infty[$, too.

(2) Now, with the help of the dual version of the induction principle we show for $r \in]-\infty,\tau]$:

$\mathcal{B}(r)$: The IVP $x^\Delta = f(t,x),\ t \in [r,\infty[,\ x(\tau) = \eta$ (5.7.5/r)
admits exactly one solution $x_r:[r,\infty[\longrightarrow X$.

Ad (I): This has been shown in (1).
Ad (II): Let r be left-scattered. According to the induction condition the IVP (5.7.5/r) admits exactly one solution $x_r(\cdot)$.
We define the mapping $x_{\rho(r)}:[\rho(r),\infty] \longrightarrow X$ by:

$$x_{\rho(r)}(t) = \begin{cases} x_r(t), \text{ if } t \in [r,\infty] \\ [(id + f(\rho(r),\cdot)\mu^*(\rho(r))]^{-1}(x_r(r)), \text{ if } t = \rho(r). \end{cases}$$

Hence, here we need condition (b). Theorem 5.3 – applied to the intervals $[\rho(r),r]$ and $[r,\infty[$ – yields the existence and uniqueness of $x_{\rho(r)}$ as a solution of the IVP (5.7.5/$\rho(r)$).
Ad (III): Let r be left-dense. On $[r,\infty[$ there exists exactly one solution $x_r(\cdot)$. There is a compact neighborhood $V_r \subseteq U_r$ such that $L_r \cdot |\mu(s,r)| < 1$ for all $s \in V_r$. By Theorem 5.5 the IVP

$$x^\Delta = f^{r]}(t,x), \quad t \in [s,r], \qquad x(r) = x_r(r)$$

admits exactly one solution $y_s(\cdot)$ for each $s \in V_r$, $s \leq t$. With the aid of Theorem 5.3 we can compose x_r and y_s to a (unique) solution on $[s,\infty[$. Hence we have $\mathcal{B}(s)$ for all $s \in V_r$, $s < r$.
Ad (IV): Let r be right-dense. If $s > r$ is sufficiently close to r, then by means of the Theorems 5.3 and 5.5 we can uniquely continue the solution $x_s(\cdot)$ already existing until r. Thereby Theorem 5.5 has to be applied to the IVP

$$x^\Delta = f^{s]}(t,x),\ t \in [r,s], \quad x(s) = x_s(s).$$

Now, $\mathcal{B}(r)$ is valid for all $r \leq \tau$, therefore a unique solution of the IVP (5.7.1) on $\mathbf{T}$ exists. ∎

Remarks: (1) That one cannot apply the fixed point method for the check of (II) (in both time directions) on intervals about the relevant point t is due to the fact that in this case one cannot choose an interval about t of arbitrarily small measure. This, however, is necessary for pushing the contraction constant below 1.
(2) The question about locally unique solutions only arises for right- or left-dense times. A result in this direction can easily be derived from Theorem 5.5 by means of retraction mappings.
(3) Without further effort one can see that a solution can have finite escape time only "before" left-dense points t. Only their neighborhoods contain infinitely many points at the left side of t.
(4) A closer look at the proof with respect to condition (b) in the theorem exactly clears why non-continuability of solutions in backward time direction can only occur in – at least partially – discrete time scales. From the definition in Section 5.6 one can learn that "regressivity increases as grainyness (step width) decreases".

(5) Addenda to the global existence theorem about continuous dependence of solutions on initial conditions and parameters can be shown by methods known from the theory of differential equations. Thereby the criterion from Theorem 1.4 (d) about compactness plays an important role.
(6) After formulating an existence theorem the question arises on how to solve dynamical equations explicitly. Since these methods strongly make use of the algebraic features of the differential operator involved, which are - according to the underlying measure chain - quite different (cf. (2.6.2)/(2.6.3)), a transfer of these methods from the continuous theory of differential equations is only possible within a small margin. This, for example, already shows the analysis of the logistic dynamical equation

$$x^{\Delta} = rx(K - x)/K.$$

In the case ($\mathbb{R}$) it can be explicitely solved by separation of variables, in the case ($h\mathbb{Z}$), however, this is not possible. This consideration strongly emphasizes the importance of qualitative methods in investigating dynamical equations on arbitrary measure chains.
The special case of linear dynamical equations is, as the Chapters 6/7 will show, relatively easy to handle with respect to explicit solutions.

6. LINEAR DYNAMICAL EQUATIONS

6.1 The regressive group: In the dynamical triple $(\mathbb{T},\mu,\mathcal{L})$ let $\mathcal{L}$ be some $\mathbb{K}$-Banach algebra with unity, e.g. the algebra of endomorphisms of some $\mathbb{K}$-Banach space or the matrix algebra of some $\mathbb{K}^n$.

We call a mapping $A:\mathbb{T}^{\kappa} \longrightarrow \mathcal{L}$ regressive, if the right-hand side $f(t,x) = A(t)\cdot x$ defined by it is regressive. This occurs, if and only if $A(t)\mu^*(t) + 1$ is invertible in $\mathcal{L}$ for each $t \in \mathbb{T}^{\kappa}$. We define a subset of $C_{rd}(\mathbb{T}^{\kappa},\mathcal{L})$ by

$$C_{rd}R(\mathbb{T}^{\kappa},\mathcal{L}) := \{A:\mathbb{T}^{\kappa} \longrightarrow \mathcal{L}\text{: A rd-continuous and regressive}\}, \tag{6.1.1}$$

and in it we define a composition by the following equation:

$$(A \mathbin{\tilde{+}} B)(t) := A(t)\cdot B(t)\mu^*(t) + A(t) + B(t). \tag{6.1.2}$$

One easily verifies that in this way a group structure on $C_{rd}R(\mathbb{T}^{\kappa},\mathcal{L})$ arises. It is commutative, if $\mathcal{L}$ is. Its neutral element is given by the zero mapping. In order to form the inverse we can state:

$$\tilde{-}A(t) = -A(t)[A(t)\mu^*(t) + 1]^{-1} = -[A(t)\mu^*(t) + 1]^{-1}\cdot A(t) \tag{6.1.3}$$

$$(A \mathbin{\tilde{-}} B)(t) = [A(t) - B(t)][B(t)\mu^*(t) + 1]^{-1}. \tag{6.1.4}$$

The importance of this group structure is expressed in (viii) and (ix) of Theorem 6.2. Subsequent to this theorem we shall present concretizations with respect ot the examples ($\mathbb{R}$) and ($h\mathbb{Z}$).

6.2 Principal solutions and transition mappings: The right-hand side in the IVP

$$x^\Delta = A(t)x, \quad x(\tau) = 1 \tag{6.2.1}$$

is assumed to be rd-continuous and regressive. By Theorem 5.7 it admits exactly one solution $\Phi_A(\tau) := x(\cdot;\tau,1)$. We call it principal solution (of the dynamical equation) (w.r.t. the initial time τ). The transition mapping (from τ to t) is given by:

$$\Phi_A(t,\tau) := \Phi_A(\tau)(t) \tag{6.2.2}$$

The following theorem lists its properties:

Theorem 6.2: Suppose $r,s,\tau \in \mathbf{T}$ and $A,B \in \mathit{CrdR}(\mathbf{T}^\varkappa,\mathfrak{L})$. Then the following statements hold true:

(i) If $x(\cdot)$ is any solution of the IVP

$$x^\Delta = A(t)x, \quad x(\tau) = \eta,$$

then we have $x(\cdot) = \Phi_A(\tau)\cdot\eta$. If η is invertible, then in addition we have:

$$\Phi_A(\cdot,\tau) = \Phi_A(\tau) = x(\cdot)\eta^{-1}. \tag{6.2.3}$$

(ii) The cocycle property is valid:

$$\Phi_A(r,\tau) = \Phi_A(r,s)\cdot\Phi_A(s,\tau). \tag{6.2.4}$$

(iii) $\Phi_A(\tau,s)$ is invertible, and we have:

$$\Phi_A(\tau,s)\cdot\Phi_A(s,\tau) = 1, \quad \Phi_A(\tau,\cdot)^\Delta(s) = -\Phi_A(\tau,\sigma(s))\cdot A(s), \; s \in \mathbf{T}^\varkappa \tag{6.2.5}$$

(iv) Let $C:\mathbf{T} \longrightarrow \mathfrak{L}$ be differentiable. If for $t \in \mathbf{T}^\varkappa$ one has:

$$C^\Delta(t) = A(t)C(t) - C(\sigma(t))A(t),$$

then it follows that:

$$C(\cdot)\Phi_A(\tau) = \Phi_A(\tau)C(\tau). \tag{6.2.6}$$

(v) If $C(\cdot) = C$ is constant and $C\cdot A(t) = A(t)\cdot C$, $t \in \mathbf{T}^\varkappa$, then it follows:

$$C\cdot\Phi_A(\tau) = \Phi_A(\tau)\cdot C. \tag{6.2.7}$$

(vi) The constant function $C := B(\cdot) - A(\cdot)$ is assumed to commute with the function $\Phi_A(\tau)$. Then it follows:

$$C\cdot\int_r^s \Phi_A(\tau,\sigma(t))\Phi_B(t,\tau)\Delta t = \Phi_A(\tau,s)\Phi_B(s,\tau) - \Phi_A(\tau,r)\Phi_B(r,\tau). \tag{6.2.8}$$

(vii) We have: $$\Phi_A(\sigma(\tau),\tau) = A(\tau)\mu^*(\tau)+1. \tag{6.2.9}$$

(viii) (Functional equation) If the condition

$$B(t)\Phi_A(t,\tau) = \Phi_A(t,\tau)B(t), \quad t \in \mathbf{T}^\varkappa \tag{6.2.10}$$

holds - in particular, this is true, if $\mathfrak{L}$ is commutative -, then the following functional equation holds true:

$$\Phi_{A*B}(\tau) = \Phi_A(\tau)\cdot\Phi_B(\tau) \quad \text{resp.} \quad \Phi_{A*B}(s,\tau) = \Phi_A(s,\tau)\cdot\Phi_B(s,\tau). \tag{6.2.11}$$

It follows:

$$\Phi_{\ominus A}(\tau) = \Phi_A(\tau)^{-1} \quad \text{resp.} \quad \Phi_{\ominus A}(s,\tau) = \Phi_{\ominus A}(s,\tau)^{-1} = \Phi_{\ominus A}(\tau,s). \tag{6.2.12}$$

(ix) (Adjoint equation) If there is an involution $A \longmapsto A^T$ (e.g. the transposition on a matrix algebra) on $\mathfrak{L}$, then $\Phi_A(\tau)^{-T}$ is the solution of the IVP

$$x^\Delta = \ominus A(t)^T\cdot x = (\ominus A(t))^T\cdot x, \quad x(\tau) = 1. \tag{6.2.13}$$

Proof: Since the multiplication in $\mathfrak{L}$ is continuous and bilinear, we can apply the product rule (2.6.2) and the quotient rule (2.6.3).

(i) We have: $[\Phi_A(\tau)\eta]^\Delta = \Phi_A(\tau)^\Delta\eta = A(t)\Phi_A(\tau)\eta$ and $\Phi_A(\tau,\tau)\eta = \eta$. This yields our assertion.

(ii) Both of the mappings $\Phi_A(s)\Phi_A(s,\tau)$ and $\Phi_A(\tau)$ are solutions of the IVP

$$x^\Delta = A(t)x, \qquad x(s) = \Phi_A(s,\tau),$$

and therefore identical. Now insert r!

Replacing r by τ one sees that (iii) is a consequence of (ii). The formula for the inverse follows from the quotient rule.

(iv) For $t \in \mathbf{T}^\kappa$ we have:

$$[C\Phi_A(\tau)]^\Delta(t) = C(\sigma(t))\Phi_A(\tau)^\Delta(t) + C^\Delta(t)\Phi_A(t,\tau) =$$

$$[C(\sigma(t))A(t) + C^\Delta(t)]\Phi_A(t,\tau) = A(t)C(t)\Phi_A(t,\tau) = A(t)[C\Phi_A(\tau)](t),$$

and

$$[C\Phi_A(\tau)](\tau) = C(\tau).$$

Hence $C\Phi_A(\tau)$ is a solution of the IVP

$$x^\Delta = A(t)x, \quad x(\tau) = C(\tau)$$

therfore, by (i) it equals $\Phi_A(\tau)C(\tau)$.

(v) immediately follows from (iv) with $C(\cdot) = C$.

(vi) C commutes with $\Phi_A(\tau,\sigma(t))$ for each $t \in [r,s]$. By (iii) and partial integration (4.3.3) we get:

$$C\cdot\int_r^s \Phi_A(\tau,\sigma(t))\Phi_B(t,\tau)\Delta t =$$

$$\int_r^s \Phi_A(\tau,\sigma(t))B(t)\Phi_B(t,\tau)\Delta t + \int_r^s \Phi_A(\tau,\sigma(t))(-A(t))\Phi_B(t,\tau)\Delta t =$$

$$\Phi_A(\tau,s)\Phi_B(s,\tau) - \Phi_A(\tau,r)\Phi_B(r,\tau) \quad .$$

(vii) For right-dense τ ($\sigma(\tau) = \tau$, $\mu^*(\tau) = 0$) the assertion is clear. For right-scattered τ we have, taking into account (2.5.2):

$$A(\tau)\Phi_A(\tau,\tau) = \Phi_A(\cdot,\tau)^\Delta(\tau) = \frac{\Phi_A(\sigma(\tau),\tau) - \Phi_A(\tau,\tau)}{\mu^*(\tau)}$$

This establishes our assertion.

(viii) By means of the product rule and of (6.2.9) one sees:

$$[\Phi_A(\tau)\Phi_B(\tau)]^\Delta(t) = \Phi_A(\sigma(t),\tau)B(t)\Phi_B(t,\tau) + A(t)\Phi_A(t,\tau)\Phi_B(t,\tau) =$$

$$[A(t)\mu^*(t)+1]\cdot\Phi_A(t,\tau)B(t)\Phi_B(t,\tau) + A(t)\Phi_A(t,\tau)\Phi_B(t,\tau) =$$

$$[A(t)B(t)\mu^*(t) + A(t) + B(t)]\cdot\Phi_A(t,\tau)\Phi_B(t,\tau)$$

Hence, the mapping $\Phi_A(\tau)\Phi_B(\tau)$ solves the IVP

$$x^\Delta = (A \mp B)(t)x, \qquad x(\tau) = 1.$$

(ix) By (6.2.4),(6.2.5) and (6.2.9) we have:

$$(\Phi_A(\tau)^{-T})^\Delta(t) = [(\Phi_A(\tau)^{-1})^\Delta(t)]^T = [\Phi_A(\tau,\sigma(t))A(t)]^T =$$

$$[\Phi_A(\tau,t)\cdot(A(t)\mu^*(t) + 1)^{-1}A(t)]^T = \simeq A(t)^T\cdot\Phi_A(t,\tau)^{-T}$$ ■

<u>**6.3 Example:**</u> For $(\mathbb{R})$ and $\mathfrak{L}$ commutative it is well-known from the theory of ordinary differential equations that

$$\Phi_A(t,\tau) = \exp\left(\int_\tau^t A(s)ds\right), \tag{6.3.1}$$

where one can define the function $\exp: X \longrightarrow X$ as a power series within the Banach algebra. For $(h\mathbb{Z})$ it is easy to see that

$$\Phi_A(t,\tau) = \begin{cases} [A(t-h)h+1]\cdot\ldots\cdot[A(\tau+h)h+1]\cdot[A(\tau)h+1], & \text{if } t \geq \tau \\ [A(t)h+1]^{-1}\cdot[A(t+h)h+1]^{-1}\cdot\ldots\cdot[A(\tau-h)h+1]^{-1}, & \text{if } t \leq \tau. \end{cases} \tag{6.3.2}$$

If X is commutative and $A,B \in X$ are constant, then we can directly verify the functional equation (6.2.11) for this example:

$$\Phi_A(t,\tau)\Phi_B(t,\tau) = (Ah+1)^{(t-\tau)/h}\cdot(Bh+1)^{(t-\tau)/h} =$$

$$([ABh+A+B]h+1)^{(t-\tau)/h} = \Phi_{A\mp B}(t,\tau) \ .$$

<u>**6.4 Transformation and variation of constants:**</u> Let $(\mathbf{T},\mu,X)$ be a dynamical triple and $\mathfrak{L}(X)$ be the Banach algebra with unity of the linear continuous endomorphisms on X. A dynamical equation is called <u>linear homogeneous</u> or <u>linear inhomogeneous</u>, resp. if it has the form:

$$x^\Delta = A(t)x \tag{6.4.1}$$

or

$$x^\Delta = A(t)x + b(t). \tag{6.4.2}$$

The right-hand side is rd-continuous and regressive, if and only if $A \in C_{rd}R(\mathbf{T}^*,\mathfrak{L}(X))$ (and b is rd-continuous).

Theorem 6.4: We consider the inhomogeneous IVP with rd-continuous and regressive right-hand side:

$$x^\Delta = A(t)x + b(t), \qquad x(\tau) = \eta \tag{6.4.3}$$

(i) Transformation rule: Let $T:\mathbf{T} \longrightarrow \mathfrak{L}(X)$ be an algebraically invertible and rd-continuously differentiable mapping. Assume that T establishes a transformation between the two functions $x,y:\mathbf{T} \longrightarrow X$ in the following manner:

$$y(t) = T(t)^{-1}x(t); \quad x(t) = T(t)y(t), \quad t \in \mathbf{T}$$

Then x is a solution of the IVP (6.4.3), if and only if y is a solution of the inhomogeneous IVP (6.4.4):

$$\begin{aligned} y^{\Delta} &= T(\sigma(t))^{-1}[A(t)T(t) - T^{\Delta}(t)]y + T(\sigma(t))^{-1}b(t), \\ y(\tau) &= T(\tau)\eta \end{aligned} \tag{6.4.4}$$

(ii) Variation of constants: The solution of the IVP (6.4.3) is given by:

$$x(t) = \Phi_A(t,\tau)\eta + \int_\tau^t \Phi_A(t,\sigma(s))b(s)\Delta s \tag{6.4.5}$$

From this presentation the dependences of the solution of the IVP (6.4.3) on the initial point η (affine/linear) and on the inhomogeneous term $b(\cdot)$ (principle of superposition) are evident. Especially, the space of solutions of the dynamical equation (6.4.1) or (6.4.2), is linear or affine isomorphic, resp. to the state space X.

Proof: The mapping $\mathfrak{L}(X) \times X \longrightarrow X$, $(A,x) \longmapsto Ax$, is bilinear and continuous so the product rule (2.6.2) can be applied.

(i) Let x be some solution of the IVP (6.4.3). Then we have for $t \in \mathbf{T}^{\kappa}$:

$$\begin{aligned} &y^{\Delta}(t) = (T^{-1}x)^{\Delta}(t) = T(\sigma(t))^{-1}x^{\Delta}(t) + (T^{-1})^{\Delta}(t)x(t) = \\ &T(\sigma(t))^{-1}[A(t)x(t)+b(t)] - T(\sigma(t))^{-1}T^{\Delta}(t)T(t)^{-1}x(t) = \\ &T(\sigma(t))^{-1}[A(t)T(t) - T^{\Delta}(t)]y(t) + T(\sigma(t))^{-1}b(t). \end{aligned}$$

Otherwise we have:

$$\begin{aligned} &x^{\Delta}(t) = (Ty)^{\Delta}(t) = T(\sigma(t))y^{\Delta}(t) + T^{\Delta}(t)y(t) = \\ &[A(t)T(t) - T^{\Delta}(t)]y(t) + b(t) + T^{\Delta}(t)y(t) = \\ &A(t)T(t)y(t) + b(t) = A(t)x(t) + b(t). \end{aligned}$$

(ii) By setting:

$$T(t) := \Phi_A(t,\tau) \quad \text{and} \quad y(t) = \eta + \int_\tau^t \Phi_A(\tau,\sigma(s))b(s)\Delta s$$

for $t \in \mathbf{T}$, (ii) is a consequence of (i). In this case the linear part of (6.4.4) vanishes, y is an antiderivative of the inhomogeneous part.

7. THE COMPLEX EXPONENTIAL FUNCTION ON MEASURE CHAINS

7.1 Preparations: For the considerations in this chapter a number of technical notions has to be given. We present them in a concise form. Let $h \geq 0$. The set

$$R_h := \{z \in \mathbb{C}\colon zh + 1 \neq 0\} \tag{7.1.1}$$

together with the composition

$$z +_h w := z + w + zwh \tag{7.1.2}$$

forms an abelian (topological) group. For $h = 0$ we have $(R_h, +_h) = (\mathbb{C}, +)$. The set

$$Q_h := \{z \in \mathbb{C}\colon \operatorname{Im}(zh) = (2k+1)\pi \text{ for a } k \in \mathbb{Z}\} \tag{7.1.3}$$

forms a (topological) subgroup of $\mathbb{C}$. The factor group

$$S_h := \mathbb{C}/Q_h \tag{7.1.4}$$

is canonically isomorphic to the topological group, which emerges from the subset of $\mathbb{C}$

$$\{z \in \mathbb{C}\colon \operatorname{Im}(zh) \in [-i\pi, i\pi[\},$$

if one adds $\bmod 2\pi ih$ in it and (for $h > 0$) glues it together along the two horizontal border lines $\mathbb{R} - i\pi/h$ and $\mathbb{R} + i\pi/h$. An infinitely long cylinder is the result. For $h = 0$ we have $(S_h, +) = (\mathbb{C}, +)$.

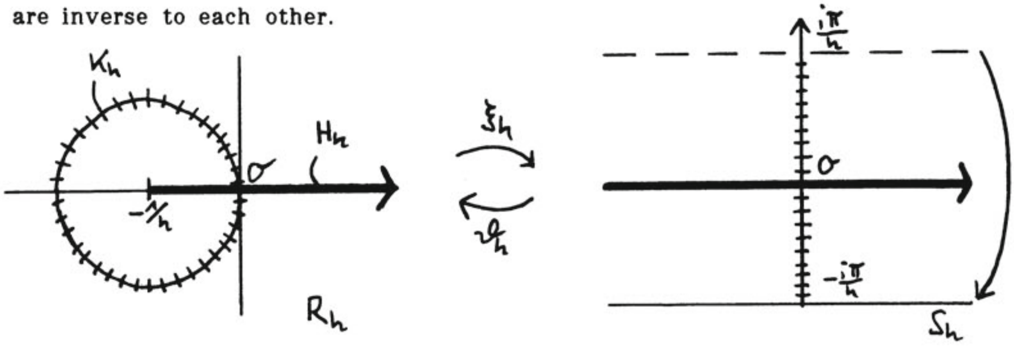
One easily verifies that by the mappings $\xi_h\colon R_h \longrightarrow S_h$

$$\xi_h(z) := \begin{cases} \dfrac{\operatorname{Ln}(zh+1)}{h}, & \text{if } h > 0 \\ z, & \text{if } h = 0 \end{cases} \tag{7.1.5}$$

(Ln is the principal branch of the complex logarithm with image $[-i\pi, i\pi[$)
and $\vartheta_h\colon S_h \longrightarrow R_h$

$$\vartheta_h(z) := \begin{cases} \dfrac{\exp(zh)-1}{h}, & \text{if } h > 0 \\ z, & \text{if } h = 0 \end{cases} \tag{7.1.6}$$

two conformal isomorphisms of the groups $(R_h, +_h)$ and $(S_h, +)$ are defined, which are inverse to each other.



By ξ_h the ray

$$H_h := \{z: zh + 1 > 0\} \subseteq R_h \tag{7.1.1}$$

is mapped to the real axis of S_h, the image of the circle about $-1/h$ through 0 (this is just the imaginary axis, if $h = 0$):

$$K_h := \{z: \lim_{k \searrow h} (|zk+1|-1)/k = 0\} \subseteq R_h \tag{7.1.8}$$

is exactly the imaginary axis of S_h.

By ξ_h and ϑ_h we transfer the projection onto the real axis Re and the projection onto the imaginary axis $i\cdot\mathrm{Im}$ which are defined in S_h to R_h. Thus we define:

$$PH_h: R_h \longrightarrow H_h \quad \text{by} \quad PH_h(z) := t_h \circ \mathrm{Re} \circ \xi_h(z) = \lim_{k \searrow h} (|zk+1|-1)/k, \tag{7.1.9}$$

$$PK_h: R_h \longrightarrow K_h \quad \text{by} \quad PK_h(z) := t_h \circ (i\cdot\mathrm{Im}) \circ \xi_h(z) = \lim_{k \searrow h} ((zk+1)^{\wedge}-1)/k. \tag{7.1.10}$$

(For $z \in \mathbf{C}\setminus\{0\}$ we use the definition $z^{\wedge} := z/|z|$.) For $h > 0$ PH_h projects a point $z \in R_h$ onto the ray $]-1/h, \infty[$ along the circle through z with center $-1/h$, PK_h projects it onto the circle through the origin with center $-1/h$ along the radius through it. For $h = 0$ we get back the projections onto the real and imaginary axis, respectively.

7.2. Transformations for the complex regressive group: We consider three function spaces, each of which is equipped with the structure of an abelian group:

- The set

  $$C_{rd}\mathcal{R}(\mathbf{T}^{\kappa}, \mathbf{C}) := \{a: \mathbf{T}^{\kappa} \longrightarrow \mathbf{C}: \text{ rd-continuous, } a(t) \in R_{\mu^*(t)} \text{ for } t \in \mathbf{T}^{\kappa}\}, \tag{7.2.1}$$

  containing the regressive rd-continuous mappings $\mathbf{T}^{\kappa} \longrightarrow \mathbf{C}$, together with the composition $\mp$ from 6.1 forms the complex regressive group. The composition $\mp$ is given by $+_{\mu^*(t)}$ for each single point t.

- The set

  $$C_{rd}\mathcal{S}(\mathbf{T}^{\kappa}, \mathbf{C}) := \{p: \mathbf{T}^{\kappa} \longrightarrow \mathbf{C}: \text{ rd-continuous, } p(t) \in S_{\mu^*(t)} \text{ for } t \in \mathbf{T}^{\kappa}\} \tag{7.2.2}$$

  forms an abelian group, if one realizes the composition $p + q$ for each point $t \in \mathbf{T}^{\kappa}$ in $S_{\mu^*(t)}$, or, alternatively, defines it by the postulate

  $$(p + q)(t) = p(t) + q(t) + 2k(t)\pi i\mu^*(t)$$

  with some function $k: \mathbf{T}^{\kappa} \longrightarrow \mathbf{Z}$.

- Observing (2.6.2/2.6.3) and Theorem 4.1 (i) for some fixed $\tau \in \mathbf{T}$ the set

  $$E_\tau(\mathbf{T}, \mathbf{C}) := \{e \in C(\mathbf{T}, \mathbf{C}\setminus\{0\}): \text{ rd-continuously differentiable, } e(\tau) = 1\}. \tag{7.2.3}$$

  is an abelian group with respect to the pointwise multiplication.

We define the four transformations in the following diagram

$$\begin{tikzcd}
& E_\tau(\mathbf{T},\mathbb{C}) & \\
Crd\mathcal{R}(\mathbf{T}^\kappa,\mathbb{C}) \arrow[ur, "\varphi"] \arrow[rr, bend left, "\xi"] & & CrdS(\mathbf{T}^\kappa,\mathbb{C}) \arrow[ul, "\psi"'] \arrow[ll, bend left, "\vartheta"]
\end{tikzcd}$$

by

$$(\varphi a)(t) := e_a(t,\tau) := \Phi_a(t,\tau),\ t \in \mathbf{T} \tag{7.2.4}$$

(solution of the IVP $x^\Delta = a(t)x$, $x(\tau) = 1$; cf. Ch.6),

$$(\psi p)(t) := \exp(\int_\tau^t p(r)\Delta r),\ t \in \mathbf{T} \tag{7.2.5}$$

(integral exponentiation),

$$(\xi a)(t) := \xi_{\mu^*(t)}(a(t)),\ t \in \mathbf{T}^\kappa, \tag{7.2.6}$$

$$(\vartheta p)(t) := \vartheta_{\mu^*(t)}(p(t)),\ t \in \mathbf{T}^\kappa \tag{7.2.7}$$

and discuss the following issues:

■ The tranformations are well-defined: φa as a solution of a dynamical equation is rd-continuously differentiable. Directly from the definitions of ξ_h and ϑ_h and from Theorem 4.1 (ii) it follows that ξa and ϑp have one-sided limites in all points $t \in \mathbf{T}^\kappa$ with $\mu^*(t) > 0$. For right-dense $t \in \mathbf{T}^\kappa$ the continuity of these two mappings is seen from the following power series representations for ξ_h resp. ϑ_h

$$\xi_h(z) = z\cdot\sum_{k=0}^{\infty} \frac{(-1)^k\cdot(hz)^k}{k+1},\ \text{for } |hz| < 1, \tag{7.2.8}$$

$$\vartheta_h(z) = z\cdot\sum_{k=0}^{\infty} \frac{(hz)^k}{(k+1)!}\quad,\ \text{for } |hz| < \infty. \tag{7.2.9}$$

That ψp is rd-continuously differentiable will follow from the next theorem.

■ The transformations preserve the group structures: The functional equation (6.2.11) shows that φ is a group homomorphism. Because of the fact that exp, ξ_h and ϑ_h preserve the group structures on $\mathbb{C}, R_h$ and S_h ψ, ξ and ϑ are group homomorphisms, too.

■ The transformations are invertible: ξ and ϑ are inverse to one another. Likewise, φ is invertible with inverse mapping $\varphi^{-1}: e \mapsto e^\Delta/e$.

7.3 Explicit representation of the complex exponential function

Theorem 7.3: We have

$$\varphi = \psi\circ\xi \quad \text{and} \quad \psi = \varphi\circ\vartheta, \tag{7.3.1}$$

i.e. the solution of the IVP $x^\Delta = a(t)x$, $x(\tau) = 1$ is given by:

$$e_a(t,\tau) := \exp(\int_\tau^t \xi a(r)\Delta r),\ t \in \mathbf{T} \tag{7.2.5}$$

For ($\mathbb{R}$) this representation of the solution is well-known from the theory of ordinary differential equations. For ($h\mathbb{Z}$) and constant $a \in R_h$ we evaluate:

$$e_a(t,\tau) = \exp\left(\int_\tau^t \frac{\mathrm{Ln}(ah+1)}{h} \Delta s\right) = \exp\left(\frac{\mathrm{Ln}(ah+1)}{h} \cdot (t-\tau)\right) = (ah+1)^{(t-\tau)/h},$$

a result which we know from 6.2. The importance of the above theorem is its validity for arbitrary measure chains $(\mathbf{T},\mu)$. Especially, for its proof no additional structure on the measure chain (e.g. a group structure) is needed.

Proof: It suffices to prove the second statement of the theorem. It is obvious that $\psi\xi a$ fulfills the initial condition.
If $\sigma(t) > t$, then we have by (2.5.2):

$$(\psi\xi a)^\Delta(t) = \mu^*(t)^{-1}\left[\exp\left(\int_\tau^{\sigma(t)} \xi a(r)\Delta r\right) - \exp\left(\int_\tau^t \xi a(r)\Delta r\right)\right] =$$

$$\mu^*(t)^{-1}\left[\exp\left(\int_t^{\sigma(t)} \xi a(r)\Delta r\right) - 1\right]\cdot \exp\left(\int_\tau^t \xi a(r)\Delta r\right) =$$

$$\mu^*(t)^{-1}[\exp(\xi a(t)\mu^*(t)) - 1]\cdot\psi\xi a(t) = \vartheta\xi a(t)\psi\xi a(t) = a(t)\psi\xi a(t).$$

In the case $\sigma(t) = t$ ξa is continuous in t. For an arbitrary ε', $\varepsilon := \varepsilon'/|\psi\xi a(t)|$, there exists a neighborhood U of t such that for all $r \in U$:

$$|\xi a(r) - a(t)| = |\xi a(r) - \xi a(t)| \le \varepsilon/2, \tag{7.3.3}$$

whence:

$$|\xi a(r)| \le |\xi a(t)| + \varepsilon/2. \tag{7.3.4}$$

Because of

$$\lim_{z\to 0} \frac{\exp(z)-(1+z)}{z} = 0,$$

there is another neighborhood $V \subseteq U$ of t such that for $s \in V$:

$$\left|\exp\left(\int_t^s \xi a(r)\Delta r\right) - \left(1 + \int_t^s \xi a(r)\Delta r\right)\right| \le$$

$$\frac{\varepsilon}{2|a(t)|+\varepsilon} \left|\int_t^s \xi a(r)\Delta r\right| \underset{\substack{\uparrow \\ \text{with } (7.3.4)}}{\le} \varepsilon/2 \;\; |\mu(t,s)|. \tag{7.3.5}$$

Now, we have for $s \in V$ by (7.3.3)/(7.3.5):

$$|\psi\xi a(t) - \psi\xi a(s) - a(t)\psi\xi a(t)\mu(t,s)| \le$$

$$|\psi\xi a(t)|\left\{\left|1 - \exp\left(\int_t^s \xi a(r)\Delta r\right) + \int_t^s \xi a(r)\Delta r\right| + \left|\int_t^s [a(r) - a(t)]\Delta r\right|\right\} \le$$

$$|\psi\xi a(t)|\cdot\{\tfrac{1}{2}\varepsilon\cdot|\mu(t,s)| + \tfrac{1}{2}\varepsilon\cdot|\mu(t,s)|\} = \varepsilon'|\mu(t,s)|. \qquad \blacksquare$$

7.4 Special exponential functions: In accordance with (7.2.1) and (7.1.7/8) we define the two sets of functions

$$C_{rd}H(\mathbf{T}^\kappa,\mathbb{C}) := \{a:\mathbf{T}^\kappa \to \mathbb{C}: \text{rd-continuous}, \; a(t) \in H_{\mu^*(t)} \text{ for } t \in \mathbf{T}^\kappa\}, \tag{7.4.1}$$

$$C_{rd}K(\mathbf{T}^\kappa,\mathbb{C}) := \{a:\mathbf{T}^\kappa \to \mathbb{C}: \text{rd-continuous}, \; a(t) \in K_{\mu^*(t)} \text{ for } t \in \mathbf{T}^\kappa\}. \tag{7.4.2}$$

Furthermore, we define the operators:

$$\mathrm{PH}:\ \mathit{CrdR}(\mathbf{T}^{\varkappa},\mathbf{C}) \longrightarrow \mathit{CrdH}(\mathbf{T}^{\varkappa},\mathbf{C}) \text{ by } \mathrm{PHa}(t) = \mathrm{PH}_{\mu^*(t)}(a(t)) \tag{7.4.3}$$

and

$$\mathrm{PK}:\ \mathit{CrdR}(\mathbf{T}^{\varkappa},\mathbf{C}) \longrightarrow \mathit{CrdK}(\mathbf{T}^{\varkappa},\mathbf{C}) \text{ by } \mathrm{PKa}(t) = \mathrm{PK}_{\mu^*(t)}(a(t)) \tag{7.4.4}$$

for rd-continuous mappings $\mathbf{T}^{\varkappa} \longrightarrow \mathbf{C}$.

With the help of the considerations at the end of section 7.1 we immediately get this corollary to Theorem 7.3:

Theorem 7.4:

(i) For $a,b \in \mathit{CrdH}(\mathbf{T}^{\varkappa},\mathbf{C})$ and $t \in \mathbf{T}$ we have $e_a(t,\tau) > 0$.

$$a(t) \le b(t),\ t \in \mathbf{T}^{\varkappa} \Longrightarrow \begin{cases} e_a(t,\tau) \le e_b(t,\tau), & t \ge \tau \\ e_a(t,\tau) \ge e_b(t,\tau), & t \le \tau. \end{cases} \tag{7.4.5}$$

(ii) For $a \in \mathit{CrdK}(\mathbf{T}^{\varkappa},\mathbf{C})$ and $t \in \mathbf{T}$ we have: $|e_a(t,\tau)| = 1$. (7.4.6)

(iii) For an arbitrary $a \in \mathit{CrdR}(\mathbf{T}^{\varkappa},\mathbf{C})$ the function can be uniquely separated into a growth- and an oscillation part. More precisely we have:

$$e_a(t,\tau) = e_{\mathrm{PHa}*\mathrm{PKa}}(t,\tau) = e_{\mathrm{PHa}}(t,\tau)\cdot e_{\mathrm{PKa}}(t,\tau) \tag{7.4.7}$$

The first factor is positive, the second has constant modulus 1.

Hence this theorem provides important and basic tools for a qualitative analysis of dynamical systems.
Part (i) can be shown directly with the aid of the intermediate value theorem 1.4 (e) (cf. [10]).

8. OUTLOOK

Up to this point the foundations of the analysis on measure chains are described; we are able to use it to enter the qualitative theory of dynamical equations. We shortly report some results which have been established as yet:

■ Taking into account the properties of the generalized exponential function one can easily formulate and prove some version of the Gronwall lemma for arbitrary measure chains:
Let a and b be continuous functions $\mathbf{T} \longrightarrow \mathbf{R}$, $c:\mathbf{T} \longrightarrow \mathbf{R}$ be an rd-continuous and regressive function with $c(t) \ge 0$ on $\mathbf{T}$. Then from the implicit estimate

$$a(t) \le b(t) + \int_\tau^t c(s)a(s)\Delta s \quad \text{for all } t \in \mathbf{T}$$

one gets the explicit estimate:

$$a(t) \le b(t) + \int_\tau^t c(s)e_c(t,\sigma(s))b(s)\Delta s \quad \text{for all } t \in \mathbf{T}.$$

The (h$\mathbf{Z}$)-version of the Gronwall lemma is used in the numerics of ordinary differential equations: With its help the linear dependence of the global discretization error on the local discretization error can be derived. Now, within the calculus on measure chains one can easily prove that this dependence persists at approximation methods on arbitrary measure chains, especially on discrete ones with variable grainyness (step width).

■ The Bernoulli inequality

$$(1 + a)^t \geq 1 + at, \quad a > -1, \ t \in \mathbf{Z}$$

has a generalization in the calculus on measure chains:

$$e_a(t,\tau) \geq 1 + a\mu(t,\tau), \quad a\mu^*(t) + 1 > 0, \ t \in \mathbf{T}.$$

Hence, it allows an estimate of the exponential function against the growth calibration.

■ From the theories of ordinary differential equations and difference equations it is known that according to the stability of linear equations with constant coefficients the imaginary axis and (when presenting the equations with the Bernoulli shift:) the unit circle, resp. separates the relevant regions of eigenvalues. This can be described within the calculus on measure chains as follows:

First of all one can prove the estimate

$$|\Phi_a(t,\tau)| \leq e_\alpha(t,\tau), \ t \geq \tau \tag{8.1}$$

for an arbitrary rd-continuous mapping $A:\mathbf{T} \longrightarrow \mathfrak{L}$ ($\mathfrak{L}$ Banach algebra of arbitrary dimension). Thereby the logarithmic norm (cf. [12], S.98) α is defined by

$$\alpha(t) := \lim_{h \searrow \mu^*(t)} [|A(t)h + 1| - 1]/h. \tag{8.2}$$

If dim $\mathfrak{L} < \infty$ and the matrix A is constant, then for each $\varepsilon > 0$, perhaps even $\varepsilon = 0$, one can find some norm $|\cdot|_\varepsilon$ on $\mathfrak{L}$, such that

$$\alpha_\varepsilon(t) \leq \max PH_{\mu^*(t)}(\Sigma(A)) + \varepsilon$$

(cf. (7.1.9)). Therefore in this case an α for the estimate (8.1) can be derived solely from the spectrum $\Sigma(A)$ of A.

■ The direct method of Lyapunov including the LaSalle invariance principle admit a generalization. This possibility depends on the existence of a chain rule (cf. the remark (1) to Theorem 2.6).

■ In [10] on the basis of the calculus on measure chains the existence of integral manifolds (stable ~, unstable ~, center stable ~, center unstable ~ and center manifolds) for linearly perturbed systems has been shown in a very general form. Closely related are results about topological equivalence of these systems to their linearizations. They are valid in the framework of the calculus on measure chains.

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