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DISCRETE POLYNOMIAL INTERPOLATION, GREEN'S FUNCTIONS, MAXIMUM PRINCIPLES, ERROR BOUNDS AND BOUNDARY VALUE PROBLEMS

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Abstract—We construct discrete interpolating polynomials, provide explicit representations of discrete Green's functions, give several identities and inequalities for these Green's functions, use the explicit forms of the interpolating polynomials and that of Green's functions to establish several maximum principles. Further, we obtain error bounds in discrete polynomial interpolation and use them to study existence and uniqueness of the discrete boundary value problems. These bounds are also used to provide sufficient conditions for the convergence of the Picard's method, the approximate Picard's method, quasilinearization and the approximate quasilinearization. The monotone convergence of the Picard's iterative method is also analysed.

1. INTRODUCTION

The landmark paper of Hartman [1] has resulted in the tremendous interest in establishing discrete analogs of the known results for the ordinary differential equations. Although several results in the discrete case are similar to those already known in the continuous case, the adaptation from the continuous case to the discrete case is not direct but requires some special devices. For the linear difference equations disconjugacy, right disconjugacy, left disconjugacy, right disfocality, eventual disconjugacy and eventual right disfocality have been recently introduced, and for each such concept, necessary and sufficient conditions have been provided by Hartman [1], Eloe [2–4], Eloe and Henderson [5]. Further, for the linear difference equations a classification of solutions based on their behavior in a neighborhood of infinity is given by Hankerson and Peterson [6]. For the nonlinear difference equations, oscillatory theory is developed in [7]. Here, necessary discrete calculus is also discussed. Boundary value problems for higher order difference equations has been the subject matter of several recent publications, e.g., [8–18]; however, it is far from complete. The motivation of the present paper comes from these advances in the theory of difference equations.

The plan of this paper is as follows: Section 2 contains necessary notations which are used throughout the paper, certain discrete and combinatorial identities, variation of constants formulae, and the contraction mapping theorem. In Section 3, we introduce various types of boundary conditions and provide explicit representations of polynomials passing through these conditions. Such polynomials are called discrete interpolating polynomials. In Section 4, we give explicit representations of Green's functions for several higher order boundary value problems. In Section 5, we establish several identities and inequalities for these Green's functions. Related results for several other boundary value problems are available in [19–22]. The continuous analogs of these results have proved to be very useful in providing disconjugacy tests and distance between

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX consecutive zeros of the solutions of higher order differential equations, see, for example [23–25]. The explicit forms of interpolating polynomials and those of Green's functions help in establishing maximum principles for functions satisfying higher order inequalities. In Section 6, we prove some such maximum principles. The continuous analogs of these results are available in [23,26]. In Section 7, we have included several results which provide error estimates in discrete polynomial interpolation. These estimates are used in Section 8 to provide easily verifiable sets of necessary and sufficient conditions so that the corresponding nonlinear boundary value problems have at least one solution. These estimates are further used in Sections 9 and 10 to provide sufficient conditions which ensure the convergence of the constructive methods: Picard's method, the approximate Picard's method, quasilinearization, and the approximate quasilinearization, for the nonlinear boundary value problems. Finally, in Section 11 the monotonic convergence of the Picard's iterative method is analysed.

2. PRELIMINARIES

Throughout, we shall use some of the following notation: $N = \{0, 1, \dots\}$ the set of natural numbers including zero; $N(a, b-1) = \{a, a+1, \dots, b-1\}$, where $a < b-1 < \infty$ and $a, b \in N$. Let $f(k)$ be a function defined on $N(a, b-1)$, then for all $k_1, k_2 \in N(a, b-1)$ and $k_1 > k_2$, $\sum_{\ell=k_1}^{k_2} f(\ell) = 0$ and $\prod_{\ell=k_1}^{k_2} f(\ell) = 1$, i.e., empty sums and products are taken to be 0 and 1, respectively. If k and $k+1$ are in $N(a, b-1)$, then for this function $f(k)$ we define the forward operator Δ as $\Delta f(k) = f(k+1) - f(k)$. The higher order differences for a positive integer m are defined as $\Delta^m f(k) = \Delta\left[\Delta^{m-1} f(k)\right] \cdot I$ be the identity operator, i.e., $If(k) = f(k)$. As usual, $\mathbb{R}$ denotes the real line and $\mathbb{R}^+$ the set of nonnegative reals. For $t \in \mathbb{R}$ and m a nonnegative integer the factorial expression $(t)^{(m)}$ is defined as $(t)^{(m)} = \prod_{i=0}^{m-1}(t-i)$. Thus, in particular for each $k \in N$, $(k)^{(k)} = k!$.

The function $Q_{n-1}(k) = \sum_{i=0}^{n-1} a_i (k)^{(i)}$ $(n \ge 1)$, $k \in N$ is called a discrete polynomial of degree $n-1$. Using Stirling numbers, this polynomial can be written as $Q_{n-1}(k) = \sum_{i=0}^{n-1} b_i\, k^i$. It is obvious that $Q_{n-1}(k)$ can have at most $n-1$ zeros in N. However, if $Q_{n-1}(k)$ vanishes at n distinct $k_i \in N$, $1 \le i \le n$ then $Q_{n-1}(k) \equiv 0$.

LEMMA 2.1 [27]. *For the functions $u(k)$ and $v(k)$ defined on $N(a, b-1+n)$ the following relations hold*

(i) $u(k+n) = (I+\Delta)^n u(k) = \sum_{i=0}^{n} \binom{n}{i} \Delta^i u(k), \quad k \in N(a, b-1);$

(ii) $\Delta^n \left[u(k)\, v(k)\right] = \sum_{i=0}^{n} \binom{n}{i} \Delta^{n-i} u(k)\, \Delta^i v(k+n-i), \quad k \in N(a, b-1).$

LEMMA 2.2 [28]. *For positive integers m and n the following identities hold*

(i) $\sum_{\ell=0}^{n} (-1)^\ell \binom{m+n-\ell}{n-\ell} \binom{m}{\ell} = 1;$

(ii) $\sum_{\ell=0}^{n} (-1)^\ell \binom{m+n-\ell-1}{n-\ell} \binom{m}{\ell} = 0.$

LEMMA 2.3 [27]. *Let $v_1(k), \dots, v_n(k)$ be n linearly independent solutions of the homogeneous difference equation*

$$L[u] = \sum_{i=0}^{n} a_i(k)\, u(k+i) = 0, \qquad k \in N(a, b-1), \tag{2.1}$$

where $a_n(k) = 1$ and $a_0(k) \ne 0$, and let $\phi(k)$ be any solution of the nonhomogeneous difference equation

$$L[u] = b(k), \qquad k \in N(a, b-1) \tag{2.2}$$

(these solutions exist on $N(a, b-1+n)$), then the general solution $u(k)$ of (2.2) can be written as

$$u(k) = \sum_{i=1}^{n} c_i\, v_i(k) + \phi(k), \qquad k \in N(a, b-1+n), \tag{2.3}$$

where c_i, $1 \le i \le n$ are arbitrary constants.

An explicit representation of $\phi(k)$ in terms of $v_1(k), \ldots, v_n(k)$ appears as

$$\phi(k) = \sum_{\ell=a}^{k-n} G(k,\ell+1)\, b(\ell), \qquad k \in N(a, b-1+n), \tag{2.4}$$

where the function $G(k,\ell)$ is defined as

$$G(k,\ell) = \begin{vmatrix} v_1(\ell) & \cdots & v_n(\ell) \\ \vdots & & \vdots \\ v_1(\ell+n-2) & \cdots & v_n(\ell+n-2) \\ v_1(k) & \cdots & v_n(k) \end{vmatrix} \div \begin{vmatrix} v_1(\ell) & \cdots & v_n(\ell) \\ \vdots & & \vdots \\ v_1(\ell+n-1) & \cdots & v_n(\ell+n-1) \end{vmatrix},$$
$$(k,\ell) \in N(a, b-1+n) \times N(a,b). \tag{2.5}$$

The following properties of $G(k,\ell)$ are immediate.

(i) $G(k,\ell) = 0$ for all $\ell \in N(k-n+2, k)$ and $k \in N(a, b-1+n)$;
(ii) $G(k,\ell) = 0$ for all $k \in N(\ell, \ell+n-2)$ and $\ell \in N(a,b)$, and $G(\ell+n-1, \ell) = 1$;
(iii) for a fixed $\ell \in N(a,b)$, $\omega(k) = G(k,\ell)$ is a solution of (2.1);
(iv) $G(k,\ell)$ is independent of the set of linearly independent solutions $v_i(k)$, $1 \le i \le n$ of (2.1).

LEMMA 2.4 [7]. *The general solution of the difference equation*

$$\Delta^n u(k) = b(k), \qquad k \in N(a, b-1), \tag{2.6}$$

can be written as

$$u(k) = Q_{n-1}(k) + \frac{1}{(n-1)!} \sum_{\ell=a}^{k-n} (k-\ell-1)^{(n-1)}\, b(\ell), \qquad k \in N(a, b-1+n), \tag{2.7}$$

where $Q_{n-1}(k)$ is a polynomial of degree $n-1$.

THEOREM 2.5 [23]. *Let B be a Banach space and let $0 < r \in \mathbb{R}$, $\bar{S}(u_0, r) = \{u \in B : \|u - u_0\| \le r\}$. Let T map $\bar{S}(u_0, r)$ into B and*

(i) *for all $u, v \in \bar{S}(u_0, r)$, $\|Tu - Tv\| \le \alpha \|u - v\|$, where $0 < \alpha < 1$;*
(ii) *$r_0 = (1-\alpha)^{-1} \|Tu_0 - u_0\| \le r$.*

Then, the following hold

(1) *T has a fixed point u^* in $\bar{S}(u_0, r_0)$;*
(2) *u^* is the unique fixed point of T in $\bar{S}(u_0, r)$;*
(3) *the sequence $\{u_m\}$ defined by $u_{m+1} = Tu_m$, $m = 0, 1, \ldots$ converges to u^* with $\|u^* - u_m\| \le \alpha^m r_0$;*
(4) *for any $u \in \bar{S}(u_0, r_0)$, $u^* = \lim_{m \to \infty} T^m u$;*
(5) *any sequence $\{\bar{u}_m\}$ such that $\bar{u}_m \in \bar{S}(u_m, \alpha^m r_0)$, $m = 0, 1, \ldots$ converges to u^*.*

3. INTERPOLATING POLYNOMIALS

THEOREM 3.1. *The unique polynomial $P_{n-1}(k)$ of degree $n-1$ satisfying conjugate boundary conditions*

$$P_{n-1}(k_i) = u(k_i) = A_i, \qquad 1 \le i \le n, \tag{3.1}$$

where $a = k_1 < k_2 < \cdots < k_n = b-1+n$ and each $k_i \in N(a, b-1+n)$ can be written as

$$P_{n-1}(k) = \sum_{i=1}^{n} \ell_i(k)\, A_i, \tag{3.2}$$

where

$$\ell_i(k) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{k-k_j}{k_i-k_j}\right), \qquad 1 \le i \le n. \tag{3.3}$$

PROOF. It suffices to note that $\ell_i(k)$ is a polynomial of degree at most $n-1$ and $\ell_i(k_j) = \delta_{ij}$, $1 \le i,\ j \le n$. The uniqueness part is obvious. ∎

THEOREM 3.2. *The unique polynomial $P_{2m-1}(k)$ of degree $2m-1$ satisfying osculatory boundary conditions*

$$P_{2m-1}(k_i) = u(k_i) = A_i, \quad \Delta P_{2m-1}(k_i) = \Delta u(k_i) = B_i, \qquad 1 \le i \le m, \tag{3.4}$$

where $a = k_1 < k_1+2 < k_2 < k_2+2 < \cdots < k_{m-1} < k_{m-1}+2 < k_m < k_m+1 = b-1+2m$ and each $k_i \in N(a, b-1+2m)$ can be written as

$$P_{2m-1}(k) = \sum_{i=1}^{m} h_i(k)\, A_i + \sum_{i=1}^{m} \bar{h}_i(k)\, B_i, \tag{3.5}$$

where

$$h_i(k) = \left[1 - \left(1 + \prod_{j=1}^{m}\left(\frac{k_i-1-k_j}{k_i+1-k_j}\right)\right)(k-k_i)\right] \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{(k-k_j)^{(2)}}{(k_i-k_j)^{(2)}}, \tag{3.6}$$

and

$$\bar{h}_i(k) = -\left(\prod_{j=1}^{m}\left(\frac{k_i-1-k_j}{k_i+1-k_j}\right)\right)(k-k_i) \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{(k-k_j)^{(2)}}{(k_i-k_j)^{(2)}}. \tag{3.7}$$

PROOF. Since for each $1 \le i \le m$, $h_i(k)$ as well as $\bar{h}_i(k)$ is a polynomial of degree at most $2m-1$, it suffices to show that $h_i(k_j) = \delta_{ij}$, $\Delta h_i(k_j) = 0$, $\bar{h}_i(k_j) = 0$, $\Delta \bar{h}_i(k_j) = \delta_{ij}$, $1 \le i,$ $j \le m$. For this, $h_i(k_j) = \delta_{ij}$ and $\bar{h}_i(k_j) = 0$ is obvious. Also, since

$$\begin{aligned}\Delta h_i(k) = &\left[1 - \left(1 + \prod_{j=1}^{m}\left(\frac{k_i-1-k_j}{k_i+1-k_j}\right)\right)(k+1-k_i)\right] \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{(k+1-k_j)(k-k_j)}{(k_i-k_j)^{(2)}} \\ &- \left[1 - \left(1 + \prod_{j=1}^{m}\left(\frac{k_i-1-k_j}{k_i+1-k_j}\right)\right)(k-k_i)\right] \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{(k-k_j)(k-1-k_j)}{(k_i-k_j)^{(2)}},\end{aligned}$$

$\Delta h_i(k_j) = 0,\ i \neq j$ is immediate. Further, we have

$$\Delta h_i(k_i) = \left[1 - 1 - \prod_{j=1}^{m}\left(\frac{k_i-1-k_j}{k_i+1-k_j}\right)\right] \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{(k_i+1-k_j)}{(k_i-1-k_j)} - 1 = 0.$$

The proof of $\Delta \bar{h}_i(k_j) = \delta_{ij}$ is also clear. ∎

THEOREM 3.3. *The unique polynomial $P_{2m-1}(k)$ of degree $2m-1$ satisfying two point Taylor boundary conditions*

$$\Delta^i P_{2m-1}(a) = \Delta^i u(a) = A_i, \quad \Delta^i P_{2m-1}(b+m) = \Delta^i u(b+m) = B_i, \qquad 0 \le i \le m-1 \tag{3.8}$$

can be written as

(i)

$$P_{2m-1}(k) = (k-a)^{(m)} \sum_{i=0}^{m-1} \frac{(k-b-m)^{(i)}}{i!} \beta_i + (k-b-m)^{(m)} \sum_{i=0}^{m-1} \frac{(k-a)^{(i)}}{i!} \alpha_i, \tag{3.9}$$

where

$$\alpha_i = \Delta^i \left[\frac{P_{2m-1}(k)}{(k-b-m)^{(m)}} \right] \Bigg|_{k=a}, \quad \beta_i = \Delta^i \left[\frac{P_{2m-1}(k)}{(k-a)^{(m)}} \right] \Bigg|_{k=b+m}, \qquad 0 \le i \le m-1 \tag{3.10}$$

(In view of Lemma 2.1 (ii), each $\alpha_i(\beta_i)$ is explicitly known in terms of $A_j(B_j)$, $0 \le j \le i$);

(ii)

$$P_{2m-1}(k) = (k-a)^{(m)} \sum_{i=0}^{m-1} \left(\sum_{j=i}^{m-1} \binom{j}{i} \frac{(k-b-m)^{(j)} (k-b-m-j-1)^{(m-j-1)}}{j!\,(-1)^{m-j-1} (m-j-1)!\,(b+m+j-a)^{(m)}} \right) B_i$$

$$+ (k-b-m)^{(m)} \sum_{i=0}^{m-1} \left(\sum_{j=i}^{m-1} \binom{j}{i} \frac{(k-a)^{(j)} (k-a-j-1)^{(m-j-1)}}{j!\,(-1)^{m-j-1} (m-j-1)!\,(a+j-b-m)^{(m)}} \right) A_i; \tag{3.11}$$

(iii)

$$P_{2m-1}(k) = \sum_{i=0}^{m-1} q_i(k)\, A_i + \sum_{i=0}^{m-1} \bar{q}_i(k)\, B_i, \tag{3.12}$$

where $q_i(k)$ and $\bar{q}_i(k)$, $0 \le i \le m-1$ are the polynomials of degree $2m-1$ satisfying $\Delta^r q_i(a) = \delta_{ir}$, $\Delta^r q_i(b+m) = 0$, $\Delta^r \bar{q}_i(a) = 0$, $\Delta^r \bar{q}_i(b+m) = \delta_{ir}$, $0 \le i,\ r \le m-1$ and appear as

$$q_i(k) = (b+2m-k-1)^{(m)} \sum_{j=0}^{m-i-1} \binom{m+j-1}{j} \frac{(k-a)^{(i+j)}}{i!\,(b+2m-i-1-a)^{(m+j)}}, \tag{3.13}$$

$$\bar{q}_i(k) = (-1)^i (k-a)^{(m)} \sum_{j=0}^{m-i-1} \binom{m+j-1}{j} \frac{(b+m+i+j-k-1)^{(i+j)}}{i!\,(b+m+i+j-a)^{(m+j)}}, \qquad 0 \le i \le m-1. \tag{3.14}$$

PROOF.

(i) The polynomial $P_{2m-1}(k)$ in (3.9) is obviously of degree at most $2m-1$. Thus, it suffices to show that this $P_{2m-1}(k)$ indeed satisfies the conditions (3.8). For this, we rewrite $P_{2m-1}(k)$ as

$$\frac{P_{2m-1}(k)}{(k-b-m)^{(m)}} = \frac{(k-a)^{(m)}}{(k-b-m)^{(m)}} \sum_{i=0}^{m-1} \frac{(k-b-m)^{(i)}}{i!} \beta_i + \sum_{i=0}^{m-1} \frac{(k-a)^{(i)}}{i!} \alpha_i.$$

From Lemma 2.1 (ii), it is clear that the first term of the right side and all of its differences up to the order $m-1$ vanish when $k=a$. Further, the i^{th} difference of the second term when $k=a$ is α_i. Thus, α_i must be the same as given in (3.10). The same observation holds for each β_i.

(ii) From Theorem 3.1, the unique polynomial of degree $2m-1$ satisfying the boundary conditions $P_{2m-1}(a+i) = u(a+i)$, $P_{2m-1}(b+m+i) = u(b+m+i)$, $0 \le i \le m-1$ can be written as

$$P_{2m-1}(k) = \sum_{i=0}^{m-1} \frac{(k-a)^{(i)} (k-a-i-1)^{(m-i-1)} (k-b-m)^{(m)}}{i!\,(-1)^{m-i-1} (m-i-1)!\,(a+i-b-m)^{(m)}} u(a+i)$$

$$+ \sum_{i=0}^{m-1} \frac{(k-a)^{(m)} (k-b-m)^{(i)} (k-b-m-i-1)^{(m-i-1)}}{i!\,(-1)^{m-i-1} (m-i-1)!\,(b+m+i-a)^{(m)}} u(b+m+i). \tag{3.15}$$

However, in view of Lemma 2.1 (i), $u(a+i) = \sum_{j=0}^{i} \binom{i}{j} A_j$ and $u(b+m+i) = \sum_{j=0}^{i} \binom{i}{j} B_j$. Using these relations in (3.15) and rearranging the terms, the required polynomial (3.11) follows.

(iii) It is clear that $q_i(k)$ and $\bar{q}_i(k)$ are the polynomials of degree at most $2m-1$. Thus, it suffices to show that $\Delta^r q_i(a) = \delta_{ir}$, $\Delta^r q_i(b+m) = 0$, $\Delta^r \bar{q}_i(a) = 0$, $\Delta^r \bar{q}_i(b+m) = \delta_{ir}$, $0 \le i,\ r \le m-1$. For this, we note that $\Delta^r q_i(b+m) = \Delta^r \bar{q}_i(a) = 0,\ 0 \le r \le m-1$ is obvious. Further, in view of Lemma 2.1 (ii), we have

$$\Delta^r q_i(k) = \sum_{j=0}^{m-i-1} \binom{m+j-1}{j} \frac{1}{i!\,(b+2m-i-1-a)^{(m+j)}}$$
$$\times \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{(i+j)!}{(i+j-r+\ell)!} (k-a)^{(i+j-r+\ell)} (-1)^m \frac{m!}{(m-\ell)!} (k+r-\ell-b-m)^{(m-\ell)},$$

and hence, $\Delta^r q_i(a) = 0$ if $0 \le r \le i-1$, and

$$\Delta^i q_i(a) = \frac{1}{i!\,(b+2m-i-1-a)^{(m)}}\, i!\,(-1)^m\,(a+i-b-m)^{(m)} = 1.$$

Also, for $i+1 \le r \le m-1$, we have

$$\Delta^r q_i(a) = \sum_{j=0}^{r-i} \binom{m+j-1}{j} \frac{1}{i!\,(b+2m-i-1-a)^{(m+j)}} \binom{r}{r-i-j} (i+j)!$$
$$\times (-1)^m \frac{m!}{(m-r+i+j)!} (a+i+j-b-m)^{(m-r+i+j)}$$
$$= \frac{r!}{i!\,(b+2m-i-1-a)^{(r-i)}} \sum_{j=0}^{r-i} (-1)^{r-i-j} \binom{m}{r-i-j} \binom{m+j-1}{j}$$
$$= \frac{r!}{i!\,(b+2m-i-1-a)^{(r-i)}} \sum_{j=0}^{r-i} (-1)^{j} \binom{m+r-i-j-1}{r-i-j} \binom{m}{j},$$

which, in view of Lemma 2.2 (ii), is zero. Thus, $\Delta^r q_i(a) = \delta_{ir}$.

Now we shall show that $\Delta^r \bar{q}_i(b+m) = \delta_{ir}$. For this, once again from Lemma 2.1 (ii), we have

$$\Delta^r \bar{q}_i(k) = (-1)^i \sum_{j=0}^{m-i-1} \binom{m+j-1}{j} \frac{1}{i!\,(b+m+i+j-a)^{(m+j)}}$$
$$\times \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{(i+j)!\,(-1)^{r-\ell}}{(i+j-r+\ell)!} (b+m+i+j-k-1-r+\ell)^{(i+j-r+\ell)} \frac{m!}{(m-\ell)!} (k+r-\ell-a)^{(m-\ell)},$$

and hence, $\Delta^r \bar{q}_i(b+m) = 0$ if $0 \le r \le i-1$, and

$$\Delta^i \bar{q}_i(b+m) = (-1)^i \frac{1}{i!\,(b+m+i-a)^{(m)}}\, i!\,(-1)^i\,(-1)^{(0)}\,(b+m+i-a)^{(m)}$$
$$= 1.$$

Also, for $i+1 \le r \le m-1$, we have

$$\Delta^r \bar{q}_i(b+m) = (-1)^i \sum_{j=0}^{r-i} \binom{m+j-1}{j} \frac{1}{i!\,(b+m+i+j-a)^{(m+j)}}$$
$$\times \binom{r}{r-i-j} (i+j)!\,(-1)^{i+j} \frac{m!}{(m-r+i+j)} (b+m+i+j-a)^{(m-r+i+j)}$$
$$= \frac{r!}{i!\,(b-a+r)^{(r-i)}} \sum_{j=0}^{r-i} (-1)^j \binom{m}{r-i-j} \binom{m+j-1}{j}$$
$$= 0. \qquad \blacksquare$$

REMARK 3.1. From (3.13) and (3.14), it is clear that $q_i(k) \ge 0$, $(-1)^i \bar{q}_i(k) \ge 0$, $k \in N(a, b+m)$, $0 \le i \le m-1$. Also, since $q(k) = q_0(k) + \bar{q}_0(k)$ is a polynomial of degree at most $2m-1$, satisfying $q(a) = 1$, $\Delta^r q(a) = 0$, $1 \le r \le m-1$, $q(b+m) = 1$, $\Delta^r q(b+m) = 0$, $1 \le r \le m-1$ it is necessary that

$$q(k) = q_0(k) + \bar{q}_0(k) = 1. \tag{3.16}$$

THEOREM 3.4. *The unique polynomial $P_{n-1}(k)$ of degree $n-1$ satisfying Hermite (r point) boundary conditions*

$$\Delta^j P_{n-1}(k_i) = \Delta^j u(k_i) = A_{i,j}, \qquad 1 \le i \le r, \quad 0 \le j \le p_i, \tag{3.17}$$

where $a = k_1 < k_1 + p_1 + 1 < k_2 < k_2 + p_2 + 1 < \cdots < k_{r-1} < k_{r-1} + p_{r-1} + 1 < k_r \le k_r + p_r = b - 1 + n$ and each $k_i \in N(a, b-1+n)$, $p_i \ge 0$, $\sum_{i=1}^{r} p_i + r = n$ can be written as

$$P_{n-1}(k) = \sum_{j=1}^{r} \sum_{\ell=0}^{p_j} \sum_{s=\ell}^{p_j} \binom{s}{\ell} \prod_{\substack{i=1\\ i\ne j}}^{r} \frac{(k-k_i)^{(p_i+1)}}{(k_j + s - k_i)^{(p_i+1)}} \, \frac{(k-k_j)^{(s)}(k-k_j-s-1)^{(p_j-s)}}{s!\,(-1)^{p_j-s}\,(p_j-s)!} \, A_{j,\ell}. \tag{3.18}$$

PROOF. From Theorem 3.1, the unique polynomial $P_{n-1}(k)$ of degree $n-1$ satisfying $P_{n-1}(k_j + s) = u(k_j + s)$, $1 \le j \le r$, $0 \le s \le p_j$ can be written as

$$P_{n-1}(k) = \sum_{j=1}^{r} \sum_{s=0}^{p_j} \prod_{\substack{i=1\\ i\ne j}}^{r} \frac{(k-k_i)^{(p_i+1)}}{(k_j + s - k_i)^{(p_i+1)}} \, \frac{(k-k_j)^{(s)}(k-k_j-s-1)^{(p_j-s)}}{s!\,(-1)^{p_j-s}\,(p_j-s)!} \, u(k_j+s). \tag{3.19}$$

However, in view of Lemma 2.1 (i), $u(k_j + s) = \sum_{\ell=0}^{s} \binom{s}{\ell} A_{j,s}$. Using this relation in (3.19) and rearranging the terms, equation (3.18) follows. ∎

THEOREM 3.5. *The unique polynomial $P_{n-1}(k)$ of degree $n-1$ satisfying Abel-Gontscharoff (right focal point) boundary conditions*

$$\Delta^i P_{n-1}(k_{i+1}) = \Delta^i u(k_{i+1}) = A_i, \qquad 0 \le i \le n-1, \tag{3.20}$$

where $k_1 \le k_2 \le \cdots \le k_n$ ($k_n > k_1$) and each $k_i \in N(a,b)$ can be written as

$$P_{n-1}(k) = \sum_{i=0}^{n-1} T_i(k) A_i, \tag{3.21}$$

where

$$T_i(k) = \frac{1}{1!\,2!\cdots i!} \begin{vmatrix} 1 & (k_1)^{(1)} & (k_1)^{(2)} & \cdots & (k_1)^{(i-1)} & (k_1)^{(i)} \\ 0 & 1 & 2(k_2)^{(1)} & \cdots & (i-1)(k_2)^{(i-2)} & i(k_2)^{(i-1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (i-1)! & i!\,(k_i)^{(1)} \\ 1 & (k)^{(1)} & (k)^{(2)} & \cdots & (k)^{(i-1)} & (k)^{(i)} \end{vmatrix}. \tag{3.22}$$

PROOF. It suffices to note that $T_j(k)$ is a polynomial of degree i and that $\Delta^j T_i(k_{j+1}) = 0$, $0 \le j \le i-1$, $\Delta^i T_i(k_{i+1}) = 1$. ∎

REMARK 3.2. An alternative representation of $T_i(k)$ is in terms of iterated summations

$$T_i(k) = \oint_{\ell_1=k_1}^{k-1} \oint_{\ell_2=k_2}^{\ell_1-1} \cdots \oint_{\ell_i=k_i}^{\ell_{i-1}-1} 1, \tag{3.23}$$

where, for the integers p and q and any function $u(k)$,

$$\oint_{\ell=p}^{q-1} u(\ell) = \begin{cases} \sum_{\ell=p}^{q-1} u(\ell), & \text{if } q \geq p, \\ -\sum_{\ell=q}^{p-1} u(\ell), & \text{if } p \geq q. \end{cases}$$

Further, in particular,

$$\begin{aligned} T_0(k) &= 1, \\ T_1(k) &= \left[(k)^{(1)} - (k_1)^{(1)}\right], \\ T_2(k) &= \frac{1}{2!}\left[\left((k)^{(2)} - (k_1)^{(2)}\right) - 2(k_2)^{(1)}\left((k)^{(1)} - (k_1)^{(1)}\right)\right], \\ T_3(k) &= \frac{1}{3!}\left[\left((k)^{(3)} - (k_1)^{(3)}\right) - 3(k_3)^{(1)}\left((k)^{(2)} - (k_1)^{(2)}\right)\right. \\ &\quad \left. + \left(6(k_2)^{(1)}(k_3)^{(1)} - 3(k_2)^{(2)}\right)\left((k)^{(1)} - (k_1)^{(1)}\right)\right]. \end{aligned}$$

THEOREM 3.6. *The unique polynomial $P_{n-1}(k)$ of degree $n-1$ satisfying two point right focal boundary conditions*

$$\begin{aligned} \Delta^i P_{n-1}(a) &= \Delta^i u(a) = A_i, \quad & 0 \leq i \leq p-1, \quad (1 \leq p \leq n-1, \text{ but fixed}), \\ \Delta^i P_{n-1}(b) &= \Delta^i u(b) = A_i, \quad & p \leq i \leq n-1, \end{aligned} \tag{3.24}$$

can be written as

$$\begin{aligned} P_{n-1}(k) &= \sum_{i=0}^{p-1} \frac{(k-a)^{(i)}}{i!} A_i \\ &\quad + \sum_{i=0}^{n-p-1} \left(\sum_{j=0}^{i} \frac{(k-a)^{(p+j)}}{(p+j)!} \frac{(-1)^{i-j}}{(i-j)!} (b-a+i-j-1)^{(i-j)} \right) A_{p+i}. \end{aligned} \tag{3.25}$$

PROOF. That $P_{n-1}(k)$ defined in (3.25) is a polynomial of degree $n-1$ is obvious. Further, since $\Delta^r P_{n-1}(a) = A_r$, $0 \leq r \leq p-1$ is straightforward, it suffices to show that

$$L = \Delta^{p+\ell} \left(\sum_{j=0}^{i} \frac{(k-a)^{(p+j)}}{(p+j)!} \frac{(-1)^{i-j}}{(i-j)!} (b-a+i-j-1)^{(i-j)} \right) \Bigg|_{k=b} = \delta_{i\ell},$$
$$0 \leq i,\ \ell \leq n-p-1.$$

For this, once again if $i < \ell$ then $L = 0$, and if $i = \ell$ then $L = 1$ is immediate, and for $i > \ell$ we have

$$\begin{aligned} L &= \sum_{j=\ell}^{i} \frac{(b-a)^{(j-\ell)}}{(j-\ell)!} \frac{(-1)^{i-j}}{(i-j)!} (b-a+i-j-1)^{(i-j)} \\ &= \sum_{j=0}^{i-\ell} \frac{(b-a)^{(j)}}{j!} (-1)^{i-\ell-j} \frac{(b-a+i-\ell-j-1)^{(i-\ell-j)}}{(i-\ell-j)!} \\ &= (-1)^{i-\ell} \sum_{j=0}^{i-\ell} (-1)^j \binom{b-a+i-\ell-j-1}{i-\ell-j} \binom{b-a}{j}, \end{aligned}$$

which, in view of Lemma 2.2 (ii), is zero. ∎

The proof of the following two results is similar.

THEOREM 3.7. *The unique polynomial $P_{n-1}(k)$ of degree $n-1$ satisfying (n,p) boundary conditions*

$$\begin{aligned}\Delta^i P_{n-1}(a) &= \Delta^i u(a) = A_i, && 0 \le i \le n-2,\\ \Delta^p P_{n-1}(b-1+n-p) &= \Delta^p u(b-1+n-p) = B, && (0 \le p \le n-1, \text{ but fixed}),\end{aligned} \tag{3.26}$$

can be written as

$$P_{n-1}(k) = \sum_{i=0}^{n-2} \frac{(k-a)^{(i)}}{i!} A_i + \left[B - \sum_{i=0}^{n-p-2} \frac{(b+n-p-a-1)^{(i)}}{i!} A_{p+i}\right] \times \frac{(n-p-1)!}{(n-1)!} \frac{(k-a)^{(n-1)}}{(b+n-p-a-1)^{(n-p-1)}}. \tag{3.27}$$

THEOREM 3.8. *The unique polynomial of degree $n-1$ satisfying (p,n) boundary conditions*

$$\begin{aligned}\Delta^p P_{n-1}(a) &= \Delta^p u(a) = B, && (0 \le p \le n-1, \text{ but fixed}),\\ \Delta^i P_{n-1}(b+1) &= \Delta^i u(b+1) = A_i, && 0 \le i \le n-2,\end{aligned} \tag{3.28}$$

can be written as

$$P_{n-1}(k) = \sum_{i=0}^{n-2} \frac{(b+i-k)^{(i)}}{i!} (-1)^i A_i + \left[B - \sum_{i=0}^{n-p-2} \frac{(b+i-a)^{(i)}}{i!} (-1)^i A_{p+i}\right] \times \frac{(n-p-1)!}{(n-1)!} (-1)^p \frac{(b+n-1-k)^{(n-1)}}{(b+n-p-a-1)^{(n-p-1)}}. \tag{3.29}$$

4. GREEN'S FUNCTIONS

Consider the difference equation (2.2) together with the linearly independent boundary conditions

$$\ell_i[u] = \sum_{\tau=0}^{n-1} \alpha_{i\tau}\, u(k_i+\tau) = A_i, \qquad 1 \le i \le n, \tag{4.1}$$

where $a \le k_1 \le \cdots \le k_n \le b$ and $\alpha_{i\tau}$, A_i, $1 \le i \le n$, $0 \le \tau \le n-1$ are the known constants. Obviously, in view of Lemma 2.1 (i), all the boundary conditions considered in the previous section are particular cases of (4.1). The solution $u(k)$ of (2.2) defined in (2.3) satisfies these boundary conditions (4.1) if and only if the system

$$A_i = \ell_i\left[\sum_{j=1}^{n} c_j\, v_j + \phi\right] = \sum_{j=1}^{n} c_j\, \ell_i[v_j] + \ell_i[\phi], \qquad 1 \le i \le n,$$

has a unique solution. Thus, the boundary value problem (2.2), (4.1) has a unique solution if and only if $\det(\ell_i[v_j]) \ne 0$. Further, in such a case, the existence of the fundamental system of solutions $\bar v_j(k)$, $1 \le j \le n$ of (2.1) satisfying $\ell_i[\bar v_j] = \delta_{ij}$ is assured (as $\det(\ell_i[\bar v_j]) = 1$).

For convenience, we shall write $D_i(\ell) =$ cofactor of $\bar v_i(\ell+n-1)$ in the det $V(\ell) = \det(\bar v_i(\ell+j))$; $1 \le i \le n$, $0 \le j \le n-1$. Further, let $k_0 = a$, $k_{n+1} = b$, and $D_0(k) = D_{n+1}(k) = \bar v_0(k) = \bar v_{n+1}(k) = 0$ on $N(a, b-1+n)$. Then, in view of (2.3) and (2.4), the general solution $u(k)$ of (2.2) can be written as

$$u(k) = \sum_{j=1}^{n} c_j\, \bar v_j(k) + \sum_{\ell=a}^{k-n} \frac{1}{\det V(\ell+1)} \sum_{j=0}^{n+1} D_j(\ell+1)\, \bar v_j(k)\, b(\ell), \qquad k \in N(a, b-1+n). \tag{4.2}$$

Since, from the properties of $G(k,\ell)$ defined in (2.5),

$$u(k_i+\tau)=\sum_{j=1}^{n}c_j\,\bar v_j(k_i+\tau)+\sum_{\ell=a}^{k_i-1}\frac{1}{\det V(\ell+1)}\sum_{j=0}^{n+1}D_j(\ell+1)\,\bar v_j(k_i+\tau)\,b(\ell),\qquad 0\le\tau\le n-1,$$

boundary conditions (4.1) can be used to determine the constants c_j, $1\le j\le n$ which appear as

$$c_j=A_j-\sum_{\ell=a}^{k_j-1}\frac{1}{\det V(\ell+1)}D_j(\ell+1)\,b(\ell),\qquad 1\le j\le n.$$

Thus, the solution of (2.2), (4.1) can be written as

$$\begin{aligned}u(k)&=\sum_{j=1}^{n}A_j\,\bar v_j(k)-\sum_{j=0}^{n+1}\sum_{\ell=k_0}^{k_j-1}\frac{1}{\det V(\ell+1)}D_j(\ell+1)\,\bar v_j(k)\,b(\ell)\\&\quad+\sum_{\ell=k_0}^{k-n}\frac{1}{\det V(\ell+1)}\sum_{j=0}^{n+1}D_j(\ell+1)\,\bar v_j(k)\,b(\ell)\\&=\sum_{j=1}^{n}A_j\,\bar v_j(k)-\sum_{i=0}^{n}\sum_{\ell=k_i}^{k_{i+1}-1}\frac{1}{\det V(\ell+1)}\sum_{j=i+1}^{n+1}D_j(\ell+1)\,\bar v_j(k)\,b(\ell)\\&\quad+\sum_{\ell=k_0}^{k-n}\frac{1}{\det V(\ell+1)}\sum_{j=0}^{n+1}D_j(\ell+1)\,\bar v_j(k)\,b(\ell)\\&=\sum_{j=1}^{n}A_j\,\bar v_j(k)+\sum_{\ell=a}^{b-1}g(k,\ell)\,b(\ell),\end{aligned}$$

where, for $k_{i+1}-k_i\ge 1$,

$$g(k,\ell)=\begin{cases}\dfrac{1}{\det V(\ell+1)}\displaystyle\sum_{j=0}^{i}D_j(\ell+1)\,\bar v_j(k), & k_i\le\ell\le k-n,\\[2ex] -\dfrac{1}{\det V(\ell+1)}\displaystyle\sum_{j=i+1}^{n+1}D_j(\ell+1)\,\bar v_j(k), & k-n+1\le\ell\le k_{i+1}-1,\ 0\le i\le n,\end{cases}\tag{4.3}$$

and for $k_{i+1}-k_i<1$, $g(k,\ell)=0$.

This function $g(k,\ell)$ is called the Green's function of the boundary value problem (2.1),

$$\ell_i[u]=0,\qquad 1\le i\le n,\tag{4.4}$$

and is uniquely determined on $N(a,b-1+n)\times N(a,b-1)$. The following properties of $g(k,\ell)$ are fundamental.

(i) $\Delta^\tau g(k,\ell)$, $0\le\tau\le n-1$ exists on $N(a,b-1+n-\tau)\times N(a,b-1)$;

(ii) $g(k,\ell)$ as a function of k satisfies

$$L[g(k,\ell)]=\sum_{i=0}^{n}a_i(k)\,g(k+i,\ell)=\delta_{k\ell},\qquad k\in N(a,b-1);$$

(iii) $g(k,\ell)$ as a function of k satisfies the homogeneous boundary conditions (4.4);

(iv) for any $b(k)$ defined on $N(a,b-1)$, the unique solution of the boundary value problem (2.2), (4.4) is given by

$$u(k)=\sum_{\ell=a}^{b-1}g(k,\ell)\,b(\ell).$$

THEOREM 4.1. *Let $u(k)$ be a function defined on $N(a, b-1+n)$, and satisfy the conjugate boundary conditions (3.1). Then, the following holds*

$$u(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\, \Delta^n u(\ell), \qquad k \in N(a, b-1+n), \tag{4.5}$$

where $P_{n-1}(k)$ is the conjugate interpolating polynomial defined in (3.2); and $g(k,\ell)$ is the Green's function of the boundary value problem

$$\Delta^n u(k) = 0, \tag{4.6}$$

$$u(k_i) = 0, \qquad 1 \le i \le n, \tag{4.7}$$

which can be written as

$$g(k,\ell) = -\frac{1}{(n-1)!} \begin{cases} g_1(k,\ell) - (k-\ell-1)^{(n-1)}, & a \le k_r - n + 1 \le \ell \le k-n, \\ g_1(k,\ell), & k-n+1 \le \ell \le k_{r+1} - n, \\ & \qquad 1 \le r \le n-1, \end{cases} \tag{4.8}$$

where

$$g_1(k,\ell) = \sum_{i=r+1}^{n} \prod_{\substack{j=1 \\ j\ne i}}^{n} \left(\frac{k-k_j}{k_i - k_j} \right) (k_i - \ell - 1)^{(n-1)}.$$

PROOF. Let $b(k) = \Delta^n u(k)$ in Lemma 2.4 so that the function $u(k)$ can be written as

$$u(k) = \sum_{i=1}^{n} \ell_i(k)\, c_i + \frac{1}{(n-1)!} \sum_{\ell=a}^{k-n} (k-\ell-1)^{(n-1)}\, \Delta^n u(\ell), \qquad k \in N(a, b-1+n),$$

where $\ell_i(k)$ is defined in (3.3). This function satisfies (3.1) if and only if

$$A_i = c_i + \frac{1}{(n-1)!} \sum_{\ell=a}^{k_i-n} (k_i - \ell - 1)^{(n-1)}\, \Delta^n u(\ell), \qquad 1 \le i \le n.$$

Therefore, it follows that

$$\begin{aligned} u(k) &= \sum_{i=1}^{n} \ell_i(k)\, A_i - \frac{1}{(n-1)!} \sum_{i=1}^{n} \ell_i(k) \sum_{\ell=a}^{k_i-n} (k_i - \ell - 1)^{(n-1)}\, \Delta^n u(\ell) \\ &\quad + \frac{1}{(n-1)!} \sum_{\ell=a}^{k-n} (k-\ell-1)^{(n-1)}\, \Delta^n u(\ell) \\ &= \sum_{i=1}^{n} \ell_i(k)\, A_i - \frac{1}{(n-1)!} \left[\sum_{r=1}^{n-1} \sum_{\ell=k_r-n+1}^{k_{r+1}-n} \sum_{j=r+1}^{n} \ell_j(k)\, (k_j - \ell - 1)^{(n-1)}\, \Delta^n u(\ell) \right. \\ &\quad \left. - \sum_{\ell=a}^{k-n} (k-\ell-1)^{(n-1)}\, \Delta^n u(\ell) \right], \end{aligned}$$

which is the same as (4.5). ∎

COROLLARY 4.2. *The Green's function $g(k,\ell)$ of the two point boundary value problem*

$$\Delta^2 u(k) = 0, \tag{4.9}$$

$$u(a) = 0, \quad u(b+1) = 0, \tag{4.10}$$

can be written as

$$g(k,\ell) = -\frac{1}{b+1-a}\begin{cases} (b+1-k)(\ell+1-a), & a \le \ell \le k-2, \\ (k-a)(b-\ell), & k-1 \le \ell \le b-1. \end{cases} \tag{4.11}$$

The proof of the following results is analogous to that of Theorem 4.1.

THEOREM 4.3. *Let $u(k)$ be a function defined on $N(a, b-1+2m)$, and satisfy the osculatory boundary conditions (3.4). Then, the following holds*

$$u(k) = P_{2m-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\,\Delta^{2m} u(\ell), \qquad k \in N(a, b-1+2m), \tag{4.12}$$

where $P_{2m-1}(k)$ is the osculatory interpolating polynomial defined in (3.5), and $g(k,\ell)$ is the Green's function of the boundary value problem

$$\Delta^{2m} u(k) = 0, \tag{4.13}$$

$$u(k_i) = \Delta u(k_i) = 0, \qquad 1 \le i \le m, \tag{4.14}$$

which can be written as

$$g(k,\ell) = -\frac{1}{(2m-1)!}\begin{cases} g_1(k,\ell) - (k-\ell-1)^{(2m-1)}, & a \le k_r - 2m + 2 \le \ell \le k - 2m, \\ g_1(k,\ell), & k - 2m + 1 \le \ell \le k_{r+1} - 2m + 1, \\ & 1 \le r \le m-1, \end{cases} \tag{4.15}$$

where

$$g_1(k,\ell) = \sum_{j=r+1}^{m} \left[h_j(k)\,(k_j - \ell - 2m + 1) + \bar{h}_j(k)\,(2m-1)\right]\,(k_j - \ell - 1)^{(2m-2)},$$

and $h_i(k)$, $\bar{h}_j(k)$ are defined in (3.6) and (3.7).

THEOREM 4.4. *Let $u(k)$ be a function defined on $N(a, b-1+2m)$, and satisfy the two point Taylor boundary conditions (3.8). Then, equation (4.12) holds, where $P_{2m-1}(k)$ is the two point Taylor interpolating polynomial defined in (3.12), and $g(k,\ell)$ is the Green's function of the boundary value problem (4.13),*

$$\Delta^i u(a) = \Delta^i u(b+m) = 0, \qquad 0 \le i \le m-1, \tag{4.16}$$

which can be written as

$$g(k,\ell) = -\frac{1}{(2m-1)!}\begin{cases} g_1(k,\ell) - (k-\ell-1)^{(2m-1)}, & a \le \ell \le k-2m, \\ g_1(k,\ell), & k-2m+1 \le \ell \le b-1, \end{cases} \tag{4.17}$$

where

$$g_1(k,\ell) = \sum_{i=0}^{m-1} (2m-1)^{(i)}\,(b+m-\ell-1)^{(2m-i-1)}\,\bar{q}_i(k),$$

and $\bar{q}_i(k)$ is defined in (3.14).

THEOREM 4.5. *Let $u(k)$ be a function defined on $N(a, b-1+n)$, and satisfy the two point right focal point boundary conditions (3.24). Then, equation (4.5) holds, where $P_{n-1}(k)$ is the two point right focal interpolating polynomial defined in (3.25); and $g(k,\ell)$ is the Green's function of the boundary value problem (4.6),*

$$\begin{aligned} \Delta^i u(a) &= 0, \quad 0 \le i \le p-1, \quad (1 \le p \le n-1, \text{ but fixed}), \\ \Delta^i u(b) &= 0, \quad p \le i \le n-1, \end{aligned} \tag{4.18}$$

which can be written as

$$g(k,\ell) = (-1)^{n-p} \begin{cases} \displaystyle\sum_{\tau=a}^{\ell} g_0(k,\ell,\tau), & a \le \ell \le k-1, \\ \displaystyle\sum_{\tau=a}^{k-1} g_0(k,\ell,\tau), & k \le \ell \le b-1, \end{cases} \tag{4.19}$$

where

$$g_0(k,\ell,\tau) = \frac{(k-\tau-1)^{(p-1)}\,(\ell+n-p-1-\tau)^{(n-p-1)}}{(p-1)!\,(n-p-1)!}. \tag{4.20}$$

THEOREM 4.6. *Let $u(k)$ be a function defined on $N(a, b-1+n)$, and satisfy the (n,p) boundary conditions (3.26). Then, equation (4.5) holds, where $P_{n-1}(k)$ is the (n,p) interpolating polynomial defined in (3.27); and $g(k,\ell)$ is the Green's function of the boundary value problem (4.6),*

$$\begin{aligned} \Delta^i u(a) &= 0, \quad 0 \le i \le n-2, \\ \Delta^p u(b-1+n-p) &= 0, \quad (0 \le p \le n-1, \text{ but fixed}), \end{aligned} \tag{4.21}$$

which can be written as

$$g(k,\ell) = -\frac{1}{(n-1)!} \begin{cases} g_1(k,\ell) - (k-\ell-1)^{(n-1)}, & a \le \ell \le k-n, \\ g_1(k,\ell), & k-n+1 \le \ell \le b-1, \end{cases} \tag{4.22}$$

where

$$g_1(k,\ell) = \frac{(k-a)^{(n-1)}\,(b+n-p-\ell-2)^{(n-p-1)}}{(b+n-p-a-1)^{(n-p-1)}}.$$

THEOREM 4.7. *Let $u(k)$ be a function defined on $N(a, b-1+n)$, and satisfy the (p,n) boundary conditions (3.28). Then, equation (4.5) holds, where $P_{n-1}(k)$ is the (p,n) interpolating polynomial defined in (3.29); and $g(k,\ell)$ is the Green's function of the boundary value problem (4.6),*

$$\begin{aligned} \Delta^p u(a) &= 0, \quad (0 \le p \le n-1, \text{ but fixed}), \\ \Delta^i u(b+1) &= 0, \quad 0 \le i \le n-2, \end{aligned} \tag{4.23}$$

which can be written as

$$g(k,\ell) = -\frac{(-1)^{n+1}}{(n-1)!} \begin{cases} g_1(k,\ell), & a \le \ell \le k-1, \\ g_1(k,\ell) - (\ell+n-1-k)^{(n-1)}, & k \le \ell \le b-1, \end{cases} \tag{4.24}$$

where

$$g_1(k,\ell) = \frac{(b-1+n-k)^{(n-1)}\,(\ell+n-p-1-a)^{(n-p-1)}}{(b-1+n-p-a)^{(n-p-1)}}.$$

For a fixed $1 \le j \le m$, we recursively define

$$\begin{aligned} g_j^j(k,\ell) &= g_j(k,\ell), && N(a, b+2j-1) \times N(a, b+2j-3), \\ g_{i+1}^j(k,\ell) &= \sum_{k_1=a}^{b+2i-1} g_{i+1}(k,k_1)\, g_i^j(k_1,\ell), && N(a, b+2i+1) \times N(a, b+2j-3), \\ & && i = j, j+1, \dots, m-1, \end{aligned} \tag{4.25}$$

where, for each $1 \le i \le m$,

$$g_i(k,\ell) = -\frac{1}{b-1+2i-a}\begin{cases}(b-1+2i-k)(\ell+1-a), & a \le \ell \le k-2,\\ (k-a)(b+2i-2-\ell), & k-1 \le \ell \le b+2i-3,\end{cases} \tag{4.26}$$

which, in view of Corollary 4.2, is the Green's function of the boundary value problem (4.9),

$$u(a) = u(b-1+2i) = 0. \tag{4.27}$$

THEOREM 4.8. *Let* $u(k)$ *be a function defined on* $N(a, b-1+2m)$, *and satisfy the Lidstone boundary conditions*

$$\begin{aligned}\Delta^{2i}u(a) &= A_{2i},\\ \Delta^{2i}u(b-1+2m-2i) &= B_{2i}, \qquad 0 \le i \le m-1.\end{aligned} \tag{4.28}$$

Then, for all $k \in N(a, b-1+2m)$, *the following holds*

$$u(k) = P_{2m-1}(k) + \sum_{\ell=a}^{b-1} g_m^1(k,\ell)\,\Delta^{2m}u(\ell), \tag{4.29}$$

where $g_m^1(k,\ell)$ *is the Green's function of the Lidstone boundary value problem (4.13),*

$$\Delta^{2i}u(a) = \Delta^{2i}u(b-1+2m-2i) = 0, \qquad 0 \le i \le m-1, \tag{4.30}$$

and $P_{2m-1}(k)$ *is the Lidstone interpolating polynomial defined as*

$$\begin{aligned}P_{2m-1}(k) = {} & \left(\frac{k-a}{b-1+2m-a}\right)B_0 + \left(1 - \frac{k-a}{b-1+2m-a}\right)A_0\\ & + \sum_{i=0}^{m-2}\ \sum_{\ell=a}^{b+2m-2i-3} g_m^{m-i}(k,\ell)\left[\left(\frac{\ell-a}{b+2m-2i-3-a}\right)B_{2i+2}\right.\\ & \left. + \left(1 - \frac{\ell-a}{b+2m-2i-3-a}\right)A_{2i+2}\right].\end{aligned} \tag{4.31}$$

5. INEQUALITIES AND EQUALITIES FOR GREEN'S FUNCTIONS

THEOREM 5.1. *For the Green's function* $g(k,\ell)$ *of the conjugate boundary value problem (4.6), (4.7) defined in (4.8), the following hold*

(i) $(-1)^{n+\sigma(k)}g(k,\ell) \ge 0$, $(k,\ell) \in N(a, b-1+n) \times N(a, b-1)$, *where* $\sigma(k) =$ card$\{i : k_i < k,\ 1 \le i \le n\}$;

(ii) $|g(k,\ell)| \le \left(\frac{n-1}{n}\right)^{n-1}\frac{(b-1+n-a)^{n-1}}{n!}$;

(iii) $\left|\Delta^i g(k,\ell)\right| \le \frac{i}{n-1}\left(\frac{n-i-1}{n-1}\right)^{(n-i-1)/i}\frac{(b-1+n-a)^{n-i-1}}{(n-i-1)!}$, $1 \le i \le n-1$;

(iv) $\sum_{\ell=a}^{b-1}|g(k,\ell)| = \frac{1}{n!}\prod_{i=1}^{n}|k-k_i| \le \frac{(n-1)^{n-1}}{n^n}\frac{(b-1+n-a)^n}{n!}$.

PROOF. Part (i) is a particular case of the more general result proved by Hartman [1]. Parts (ii) and (iii) are established by Teptin [29]. To prove Part (iv), we note that $u(k) = 1/n!\prod_{i=1}^{n}(k-k_i)$ and $u(k) = \sum_{\ell=a}^{b-1} g(k,\ell)$ are two different representations of the unique solution of the boundary value problem $\Delta^n u(k) = 1$, (4.7). Therefore, it follows that

$$\sum_{\ell=a}^{b-1} g(k,\ell) = \frac{1}{n!}\prod_{i=1}^{n}(k-k_i). \tag{5.1}$$

However, since $(-1)^{n+\sigma(k)}\prod_{i=1}^{n}(k-k_i)\geq 0$, and $g(k,\ell)$ as a function of k has the same zeros as $\prod_{i=1}^{n}(k-k_i)$, it follows that $g(k,\ell)/\prod_{i=1}^{n}(k-k_i)\geq 0$, $(k,\ell)\in N(a,b-1+n)\times N(a,b-1)$. Thus, from (5.1) we conclude that

$$\sum_{\ell=a}^{b-1}|g(k,\ell)|=\frac{1}{n!}\prod_{i=1}^{n}|(k-k_i)|. \tag{5.2}$$

For a continuous variable $t\in[a,b-1+n]$ in [24], it is shown that $\prod_{i=1}^{n}|t-k_i|\leq((n-1)^{n-1}/n^n)\times(b-1+n-a)^n$ from which the required inequality in Part (iv) follows. ∎

COROLLARY 5.2. *For the Green's function $g(k,\ell)$ of the osculatory boundary value problem (4.13), (4.14) defined in (4.15), the following hold*

(i) $g(k,\ell)\geq 0,\ (k,\ell)\in N(a,b-1+2m)\times N(a,b-1)$;
(ii) $\sum_{\ell=a}^{b-1}|g(k,\ell)|=\frac{1}{(2m)!}\prod_{i=1}^{m}(k-k_i)^{(2)}\leq\frac{(2m-1)^{2m-1}}{(2m)^{2m}}\frac{(b-1+2m-a)^{2m}}{(2m)!}$.

COROLLARY 5.3. *For the Green's function $g(k,\ell)$ of the two point Taylor boundary value problem (4.13), (4.16) defined in (4.17), the following hold*

(i) $(-1)^m g(k,\ell)\geq 0,\ (k,\ell)\in N(a,b-1+2m)\times N(a,b-1)$;
(ii) $\sum_{\ell=a}^{b-1}|g(k,\ell)|=\frac{1}{(2m)!}(k-a)^{(m)}(b-1+2m-k)^{(m)}\leq\left(\frac{1}{4}\right)^m\frac{(b+m-a)^{2m}}{(2m)!}$.

THEOREM 5.4. *For the Green's function $g(k,\ell)$ of the two point right focal boundary value problem (4.6), (4.18) defined in (4.19), the following hold*

(i) $(-1)^{n-p}\Delta^i g(k,\ell)\geq 0,\ (k,\ell)\in N(a,b-1+n-i)\times N(a,b-1),\ 0\leq i\leq p-1$;
(ii) $(-1)^{n-p+i}\Delta^{i+p}g(k,\ell)\geq 0,\ (k,\ell)\in N(a,b-1+n-i-p)\times N(a,b-1),\ 0\leq i\leq n-p-1$;
(iii)

$$\begin{aligned}\sum_{\ell=a}^{b-1}|\Delta^i g(k,\ell)|&=\left|\sum_{j=0}^{n-p}(-1)^j\binom{k-a}{n-i-j}\binom{b-a+j-1}{j}\right|\\&\leq\left|\sum_{j=0}^{n-p}(-1)^j\binom{b-a}{n-i-j}\binom{b-a+j-1}{j}\right|=C_{n,i},\qquad 0\leq i\leq p-1;\end{aligned}$$

(iv)

$$\begin{aligned}\sum_{\ell=a}^{b-1}|\Delta^{i+p}g(k,\ell)|&=\frac{(b+n-p-1-i-k)^{(n-p-i)}}{(n-p-i)!}\\&\leq\frac{(b+n-p-1-i-a)^{(n-p-i)}}{(n-p-i)!}=C_{n,i+p},\qquad 0\leq i\leq n-p-1.\end{aligned}$$

PROOF. From (4.19), it is clear that for $0\leq i\leq p-1$,

$$\Delta^i g(k,\ell)=(-1)^{n-p}\begin{cases}\displaystyle\sum_{\tau=a}^{\ell}g_i(k,\ell,\tau), & a\leq\ell\leq k-1,\\ \displaystyle\sum_{\tau=a}^{k-1}g_i(k,\ell,\tau), & k\leq\ell\leq b-1,\end{cases} \tag{5.3}$$

where

$$g_i(k,\ell,\tau)=\frac{(k-\tau-1)^{(p-i-1)}(\ell+n-p-1-\tau)^{(n-p-1)}}{(p-i-1)!\,(n-p-1)!}, \tag{5.4}$$

and for $0 \le i \le n-p-1$,

$$\Delta^{i+p} g(k,\ell) = (-1)^{n-p+i} \begin{cases} 0, & a \le \ell \le k-1, \\ \dfrac{(\ell+n-p-1-i-k)^{(n-p-i-1)}}{(n-p-i-1)!}, & k \le \ell \le b-1. \end{cases} \tag{5.5}$$

Parts (i) and (ii) now immediately follow from (4.19), (4.20), and (5.3)–(5.5). To prove Part (iii), we note that

$$u(k) = \sum_{j=0}^{n-p} (-1)^j \binom{k-a}{n-j} \binom{b-a+j-1}{j}$$

is the unique solution of the boundary value problem $\Delta^n u(k) = 1$, $\Delta^i u(a) = 0$, $0 \le i \le p-1$, $\Delta^{p+i} u(b) = 0$, $0 \le i \le n-p-1$. For this, it suffices to note that for $0 \le i \le n-p-1$,

$$\begin{aligned} \Delta^{p+i} u(b) &= \sum_{j=0}^{n-p-i} (-1)^j \binom{b-a}{n-p-i-j} \binom{b-a+j-1}{j} \\ &= (-1)^{n-p-i} \sum_{j=0}^{n-p-i} (-1)^j \binom{b-a}{j} \binom{b-a+n-p-i-j-1}{n-p-i-j}, \end{aligned}$$

which, in view of Lemma 2.2 (ii), is zero. Further, since $u(k) = \sum_{\ell=a}^{b-1} g(k,\ell)$ is another representation of the same solution, it follows that

$$\sum_{\ell=a}^{b-1} g(k,\ell) = \sum_{j=0}^{n-p} (-1)^j \binom{k-a}{n-j} \binom{b-a+j-1}{j}.$$

The required equalities in Part (iii) now directly follow from Part (i), whereas the inequalities are obvious.

For Part (iv), from (5.5) we have

$$\begin{aligned} \sum_{\ell=a}^{b-1} \left| \Delta^{i+p} g(k,\ell) \right| &= \sum_{\ell=k}^{b-1} \frac{(\ell+n-p-1-i-k)^{(n-p-i-1)}}{(n-p-i-l)!} \\ &= \left. \frac{(\ell+n-p-i-k)^{(n-p-i)}}{(n-p-i)!} \right|_{\ell=k}^{b} \\ &= \frac{(b+n-p-i-k)^{(n-p-i)}}{(n-p-i)!}. \end{aligned}$$

∎

THEOREM 5.5. *For the Green's function $g(k,\ell)$ of the (n,p) boundary value problem (4.6), (4.21) defined in (4.22), the following hold*

(i) $-\Delta^i g(k,\ell) \ge 0$, $(k,\ell) \in N(a, b-1+n-i) \times N(a, b-1)$, $0 \le i \le p$;

(ii)

$$\begin{aligned} \sum_{\ell=a}^{b-1} \left| \Delta^i g(k,\ell) \right| &= \frac{1}{(n-i-1)!} (k-a)^{(n-i-1)} \left[\frac{b-a}{n-p} - \frac{k-a-n+i+1}{n-i} \right] \\ &\le \left. \begin{cases} \dfrac{(p-i)}{(n-p)} \dfrac{(b-p+n-a)^{(n-i)}}{(n-i)!}, & 0 \le i \le p-1 \\ \dfrac{(n-p-1)^{n-p-1}}{(n-p)^{n-p}} \dfrac{(b-1+n-p-a)^{n-p}}{(n-p)!}, & i = p \end{cases} \right\} = D_{n,i}, \\ & \qquad 0 \le i \le p. \end{aligned}$$

PROOF. The proof is similar to that of Theorem 5.4. ∎

THEOREM 5.6. *For the Green's function $g(k,\ell)$ of the (p,n) boundary value problem (4.6), (4.23) defined in (4.24), the following hold*

(i) $(-1)^{n+i+1}\,\Delta^i g(k,\ell) \ge 0,\ \ (k,\ell) \in N(a, b-1+n-i) \times N(a, b-1),\ \ 0 \le i \le p;$

(ii)

$$\sum_{\ell=a}^{b-1} \left|\Delta^i g(k,\ell)\right| = \frac{(b-1+n-i-k)^{(n-i-1)}}{(n-i-1)!}\left[\frac{b-a}{n-p} - \frac{b-k}{n-i}\right]$$

$$\le \left\{\begin{array}{ll} \dfrac{(p-i)}{(n-p)}\,\dfrac{(b-1+n-i-a)^{(n-i)}}{(n-i)!}, & 0 \le i \le p-1 \\[2ex] \dfrac{(n-p-1)^{n-p-1}}{(n-p)^{n-p}}\,\dfrac{(b-1+n-p-a)^{n-p}}{(n-p)!}, & i = p \end{array}\right\} = E_{n,i},$$

$$0 \le i \le p.$$

PROOF. The proof is similar to that of Theorem 5.4. ∎

THEOREM 5.7. *For the Green's function $g_m^1(k,\ell)$ of the Lidstone boundary value problem (4.13), (4.30) the following hold*

(i) $(-1)^m\, g_m^1(k,\ell) \ge 0,\ \ (k,\ell) \in N(a, b-1+2m) \times N(a, b-1);$

(ii) $\sum_{\ell=a}^{b-1} \left|g_m^1(k,\ell)\right| \le \left(\frac{1}{8}\right)^m \prod_{i=1}^m (b+2i-1-a)^2.$

PROOF. In view of (4.26) and (4.25), Part (i) is immediate. Further, since

$$\sum_{\ell=a}^{b+2i-3} |g_i(k,\ell)| = \frac{(k-a)\,(b+2i-1-k)}{2} \le \frac{1}{8}\,(b+2i-1-a)^2,$$

Part (ii) also follows from (4.25). ∎

6. MAXIMUM PRINCIPLES

Results in this section are motivated by the following theorem.

THEOREM 6.1. *If $u(k)$ is defined in $N(a, b+1)$, and $\Delta^2 u(k) \ge 0$, $k \in N(a, b-1)$, and attains its maximum at some $k^* \in N(a+1, b)$, then $u(k)$ is identically constant on $N(a, b+1)$.*

PROOF. Suppose $k^* \in N(a+1, b)$ is such that $u(k^*) \ge u(k)$ for all $k \in N(a, b+1)$. If $u(k)$ is not a constant, then either there exists an integer $i \ge 0$ such that $u(k^*) = u(k^*+1) = \cdots = u(k^*+i)$, $k+i \in N(a+1, b)$ and $u(k^*+i+1) < u(k^*+i)$; or there exists an integer $j \le 0$ such that $u(k^*+j) = u(k^*+j+1) = \cdots = u(k^*)$, $k+j \in N(a+1, b)$, and $u(k^*+j-1) < u(k^*+j)$. But then in the first case, $2\,u(k^*+i) > u(k^*+i+1) + u(k^*+i-1)$, i.e., $\Delta^2 u(k^*+i-1) < 0$, which contradicts $\Delta^2 u(k^*+i-1) \ge 0$. Similarly, in the latter case $\Delta^2 u(k^*+j-1) < 0$, which contradicts $\Delta^2 u(k^*+j-1) \ge 0$. ∎

REMARK 6.1. As a consequence of Theorem 6.1, $u(k) \le \max\{u(a), u(b+1)\}$, $k \in N(a, b+1)$.

REMARK 6.2. Theorem 6.1 holds if we reverse the inequality and replace "maximum" by "minimum."

The maximum principle stated in Theorem 6.1 does not necessarily hold for functions satisfying higher order inequalities. For example, let $u(k) = -(k/10-1)^2$, $k \in N(0, 20)$. For this function $\Delta^4 u(k) \ge 0$, $k \in N(0, 16)$, but $u(k)$ attains its maximum at $k = 10$, which is a point in $N(1, 19)$. An extension of Theorem 6.1 is embodied in the following theorem.

THEOREM 6.2. *Let $u(k)$ be defined on $N(a, b-1+2m)$, and*

$$\Delta^{2m} u(k) \ge 0, \qquad k \in N(a, b-1), \tag{6.1}$$

$$(-1)^m\,\Delta^i u(a) \ge 0, \quad (-1)^{m+i}\,\Delta^i u(b+m) \ge 0, \qquad 1 \le i \le m-1, \tag{6.2}$$

then in the case m even (m odd) $u(k)$, $k \in N(a, b+m)$ attains its minimum (maximum) at either a or $b+m$.

PROOF. We shall consider the case when m is even. For this, in view of (3.12) and Theorem 4.4, the function $u(k)$ can be written as

$$u(k) = \sum_{i=0}^{m-1} q_i(k)\,\Delta^i u(a) + \sum_{i=0}^{m-1} \bar{q}_i(k)\,\Delta^i u(b+m) + \sum_{\ell=a}^{b-1} g(k,\ell)\,\Delta^{2m} u(\ell), \tag{6.3}$$

where $q_i(k)$, $\bar{q}_i(k)$, $0 \le i \le m-1$ are defined in (3.13), (3.14); and $g(k,\ell)$ is the Green's function of the boundary value problem (4.13), (4.16) and is defined in (4.17). Using Remark 3.1, Corollary 5.3 (i), and equations (6.1) and (6.2) in (6.3), we obtain for all $k \in N(a, b+m)$ that

$$\begin{aligned} u(k) &\ge q_0(k)\,u(a) + \bar{q}_0(k)\,u(b+m) \\ &\ge (q_0(k) + \bar{q}_0(k))\,\min\{u(a), u(b+m)\} \\ &= \min\{u(a), u(b+m)\}. \end{aligned}$$

∎

REMARK 6.3. When the inequalities in (6.1), (6.2) are reversed, the result remains true provided the word maximum (minimum) is replaced by minimum (maximum).

THEOREM 6.3. *Let $u(k)$ be defined on $N(a, b-1+n)$, and*

$$\Delta^n u(k) \ge 0, \qquad k \in N(a, b-1), \tag{6.4}$$

$$\Delta^i u(a) \le 0, \qquad 1 \le i \le n-2, \tag{6.5}$$

then $u(k)$ attains its maximum either at a or $b-1+n$.

PROOF. In view of (3.27) and Theorem 4.6 with $p=0$, the function $u(k)$ can be written as

$$\begin{aligned} u(k) = \sum_{i=0}^{n-2} \frac{(k-a)^{(i)}}{i!} &\left[1 - \frac{(k-a-i)^{(n-i-1)}}{(b+n-a-i-1)^{(n-i-1)}}\right] \Delta^i u(a) \\ &+ \frac{(k-a)^{(n-1)}}{(b+n-a-1)^{(n-1)}}\, u(b-1+n) + \sum_{\ell=a}^{b-1} g(k,\ell)\,\Delta^n u(\ell), \end{aligned} \tag{6.6}$$

where $g(k,\ell)$ is the Green's function of the boundary value problem (4.6), (4.21) with $p=0$ and is defined in (4.22). Since for each $1 \le i \le n-2$, the coefficient of $\Delta^i u(a)$ in (6.6) is nonnegative on $N(a, b-1+n)$, using Theorem 5.5 (i) with $p=0$ and Equations (6.4) and (6.5) in (6.6), we obtain for all $k \in N(a, b-1+n)$ that

$$u(k) \le \left[1 - \frac{(k-a)^{(n-1)}}{(b+n-a-1)^{(n-1)}}\right] u(a) + \frac{(k-a)^{(n-1)}}{(b+n-a-1)^{(n-1)}}\, u(b-1+n).$$

In the above inequality, the coefficients of $u(a)$ and $u(b-1+n)$ are nonnegative on $N(a, b-1+n)$ and their sum is 1. Therefore, it follows that $u(k) \le \max\{u(a), u(b-1+n)\}$. ∎

THEOREM 6.4. *Let $u(k)$ be defined in $N(a, b-1+n)$, and satisfy the inequality (6.4). Further, let*

$$(-1)^i\,\Delta^i u(b+1) \ge 0, \qquad 1 \le i \le n-2, \tag{6.7}$$

then in the case n odd (n even), $u(k)$ attains its minimum (maximum) at a or $b+1$.

PROOF. The proof is similar to that of Theorem 6.3. Further, if the inequalities (6.4), (6.5) and (6.7) are reversed, then a remark similar to Remark 6.3 holds. ∎

7. ERROR ESTIMATES IN POLYNOMIAL INTERPOLATION

Combining Theorems 4.1 and 5.1, we obtain the following theorem.

THEOREM 7.1. *Let $u(k)$ and $P_{n-1}(k)$ be as in Theorem 4.1. Then, for all $k \in N(a, b-1+n)$, the following inequality holds*

$$|u(k) - P_{n-1}(k)| \le \frac{(n-1)^{n-1}}{n^n} \frac{(b-1+n-a)^n}{n!} \max_{k \in N(a,b-1)} |\Delta^n u(k)|. \tag{7.1}$$

Combining Theorem 4.3 and Corollary 5.2, we get:

THEOREM 7.2. *Let $u(k)$ and $P_{2m-1}(k)$ be as in Theorem 4.3. Then, for all $k \in N(a, b-1+2m)$, the following inequality holds*

$$|u(k) - P_{2m-1}(k)| \le \frac{(2m-1)^{2m-1}}{(2m)^{2m}} \frac{(b-1+2m-a)^{2m}}{(2m)!} \max_{k \in N(a,b-1)} \left|\Delta^{2m} u(k)\right|. \tag{7.2}$$

A combination of Theorem 4.4 and Corollary 5.3 leads to:

THEOREM 7.3. *Let $u(k)$ and $P_{2m-1}(k)$ be as in Theorem 4.4. Then, for all $k \in N(a, b-1+2m)$, the following inequality holds*

$$|u(k) - P_{2m-1}(k)| \le \left(\frac{1}{4}\right)^m \frac{(b+m-a)^{2m}}{(2m)!} \max_{k \in N(a,b-1)} \left|\Delta^{2m} u(k)\right|. \tag{7.3}$$

Similarly, combining Theorems 4.5 and 5.4; Theorems 4.6 and 5.5; Theorems 4.7 and 5.6; and Theorems 4.8 and 5.7; we respectively find:

THEOREM 7.4. *Let $u(k)$ and $P_{n-1}(k)$ be as in Theorem 4.5. Then, the following holds*

$$\left|\Delta^i \left(u(k) - P_{n-1}(k)\right)\right| \le C_{n,i} \max_{k \in N(a,b-1)} |\Delta^n u(k)|; \qquad k \in N(a, b-1+n-i), \quad 0 \le i \le n-1, \tag{7.4}$$

where $C_{n,i}$ are defined in Theorem 5.4.

THEOREM 7.5. *Let $u(k)$ and $P_{n-1}(k)$ be as in Theorem 4.6. Then, the following holds*

$$\left|\Delta^i \left(u(k) - P_{n-1}(k)\right)\right| \le D_{n,i} \max_{k \in N(a,b-1)} |\Delta^n u(k)|; \qquad k \in N(a, b-1+n-i), \quad 0 \le i \le p, \tag{7.5}$$

where $D_{n,i}$ are defined in Theorem 5.5.

THEOREM 7.6. *Let $u(k)$ and $P_{n-1}(k)$ be as in Theorem 4.7. Then, the following holds*

$$\left|\Delta^i \left(u(k) - P_{n-1}(k)\right)\right| \le E_{n,i} \max_{k \in N(a,b-1)} |\Delta^n u(k)|; \qquad k \in N(a, b-1+n-i), \quad 0 \le i \le p, \tag{7.6}$$

where $E_{n,i}$ are defined in Theorem 5.6.

THEOREM 7.7. *Let $u(k)$ and $P_{2m-1}(k)$ be as in Theorem 4.8. Then, for all $k \in N(a, b-1+2m)$, the following inequality holds*

$$|u(k) - P_{2m-1}(k)| \le \left(\frac{1}{8}\right)^m \prod_{i=1}^{m} (b+2i-1-a)^2 \max_{k \in N(a,b-1)} \left|\Delta^{2m} u(k)\right|. \tag{7.7}$$

8. EXISTENCE AND UNIQUENESS OF BOUNDARY VALUE PROBLEMS

Inequalities obtained in Section 7 will be used here to provide easier tests for the local existence and uniqueness of the solutions of the n^{th} order nonlinear difference equation

$$\Delta^n u(k) = f\left(k, u(k), u(k+1), \dots, u(k+n-1)\right), \qquad k \in N(a, b-1), \tag{8.1}$$

and its variant

$$\Delta^n u(k) = f\left(k, u(k), \Delta u(k), \dots, \Delta^{n-1} u(k)\right), \qquad k \in N(a, b-1). \tag{8.2}$$

THEOREM 8.1. *With respect to the conjugate boundary value problem (8.1), (3.1), we assume that*

(i) $M > 0$ *is a given real number and the function* $f(k, u_0, u_1, \dots, u_{n-1})$ *is continuous on the compact set* $N(a, b-1) \times D_0$, *where*

$$D_0 = \left\{(u_0, u_1, \dots, u_{n-1}) : |u_i| \le 2M, \ \ 0 \le i \le n-1\right\},$$

and

$$\max_{N(a,b-1)\times D_0} |f(k, u_0, u_1, \dots, u_{n-1})| \le Q\,;$$

(ii) $\max_{N(a,b-1+n)} |P_{n-1}(k)| \le M$, *where* $P_{n-1}(k)$ *is the conjugate interpolating polynomial defined in (3.2);*

(iii) $\frac{(n-1)^{n-1}}{n^n} \frac{(b-1+n-a)^n}{n!}\, Q \le M$.

Then, the problem (8.1), (3.1) has a solution in D_0.

PROOF. In view of (4.5), the problem (8.1), (3.1) is equivalent to the equation

$$u(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k, \ell)\, f\left(\ell, u(\ell), u(\ell+1), \dots, u(\ell+n-1)\right), \tag{8.3}$$

where $g(k, \ell)$ is the Green's function of the conjugate boundary value problem (4.6), (4.7) defined in (4.8). Let $S(a, b-1+n)$ be the space of all real functions defined on $N(a, b-1+n)$. We shall equip the space $S(a, b-1+n)$ with the norm $\|u\| = \max_{N(a,b-1+n)} |u(k)|$, so that it becomes a Banach space. Now define an operator $T : S(a, b-1+n) \longrightarrow S(a, b-1+n)$ as follows

$$Tu(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k, \ell)\, f\left(\ell, u(\ell), u(\ell+1), \dots, u(\ell+n-1)\right). \tag{8.4}$$

Obviously, $u(k)$ is a solution of (8.1), (3.1) if and only if $u(k)$ is a fixed point of T. The set $S_1 = \{u(k) \in S(a, b-1+n) : \|u\| \le 2M\}$ is a closed convex subset of the Banach space $S(a, b-1+n)$. Since

$$\Delta^n \left[\sum_{\ell=a}^{b-1} g(k, \ell)\, f\left(\ell, u(\ell), u(\ell+1), \dots, u(\ell+n-1)\right)\right] = f\left(k, u(k), u(k+1), \dots, u(k+n-1)\right),$$

for any $u(k) \in S_1$, in view of (8.4) and (7.1), it follows that

$$|Tu(k) - P_{n-1}(k)| \le \frac{(n-1)^{n-1}}{n^n} \frac{(b-1+n-a)^n}{n!}\, Q,$$

and therefore,

$$\begin{aligned} \|Tu\| &\le \max_{N(a,b-1+n)} |P_{n-1}(k)| + \frac{(n-1)^{n-1}}{n^n} \frac{(b-1+n-a)^n}{n!}\, Q, \\ &\le M + M = 2M. \end{aligned}$$

Thus, T maps S_1 into itself and $\overline{T(S_1)}$ is compact. By the Schauder fixed point theorem, the operator T has a fixed point in S_1. Thus, the boundary value problem (8.1), (3.1) has a solution in D_0. ∎

THEOREM 8.2. *With respect to the osculatory boundary value problem (8.1), with $n = 2m$, (3.8), we assume that*

(i) *$M > 0$ is a given real number and the function $f(k, u_0, u_1, \dots, u_{2m-1})$ is continuous on the compact set $N(a, b-1) \times D_0$, where*

$$D_0 = \{(u_0, u_1, \dots, u_{2m-1}) : |u_i| \le 2M, \ \ 0 \le i \le 2m-1\},$$

and

$$\max_{N(a,b-1)\times D_0} |f(k, u_0, u_1, \dots, u_{2m-1})| \le Q;$$

(ii) $\max_{N(a,b-1+2m)} |P_{2m-1}(k)| \le M$, *where $P_{2m-1}(k)$ is the osculatory interpolating polynomial defined in (3.5);*

(iii) $\frac{(2m-1)^{2m-1}}{(2m)^{2m}} \frac{(b-1+2m-a)^{2m}}{(2m)!} Q \le M$.

Then, the problem (8.1), with $n = 2m$, (3.4) has a solution in D_0.

THEOREM 8.3. *With respect to the two point Taylor boundary value problem (8.1), with $n = 2m$, (3.8), we assume*

(i) *condition (i) of Theorem 8.2;*

(ii) *condition (ii) of Theorem 8.2 with $P_{2m-1}(k)$ as the two point Taylor interpolating polynomial defined in (3.12);*

(iii) $\left(\frac{1}{4}\right)^m \frac{(b+m-a)^{2m}}{(2m)!} Q \le M$.

Then, the problem (8.1), with $n = 2m$, (3.8) has a solution in D_0.

THEOREM 8.4. *With respect to the two point right focal boundary value problem (8.2), (3.24), we assume that*

(i) *$M_i > 0$, $0 \le i \le n-1$ are given real numbers and the function $f(k, u_0, u_1, \dots, u_{n-1})$ is continuous on the compact set $N(a, b-1) \times D_0$, where*

$$D_0 = \{(u_0, u_1, \dots, u_{n-1}) : |u_i| \le 2M_i, \ \ 0 \le i \le n-1\},$$

and

$$\max_{N(a,b-1)\times D_0} |f(k, u_0, u_1, \dots, u_{n-1})| \le Q;$$

(ii) $\max_{N(a,b-1+n-i)} \left|\Delta^i P_{n-1}(k)\right| \le M_i$, $0 \le i \le n-1$, *where $P_{n-1}(k)$ is the two point right focal interpolating polynomial defined in (3.25);*

(iii) $C_{n,i}\, Q \le M_i$, $0 \le i \le n-1$, *where $C_{n,i}$ are defined in Theorem 5.4.*

Then, the problem (8.2), (3.24) has a solution in D_0.

PROOF. For the problem (8.2), (3.24), equations corresponding to (8.3) and (8.4) are

$$u(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k, \ell)\, f\left(\ell, u(\ell), \Delta u(\ell), \dots, \Delta^{n-1} u(\ell)\right), \tag{8.5}$$

and

$$Tu(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k, \ell)\, f\left(\ell, u(\ell), \Delta(\ell), \dots, \Delta^{n-1} u(\ell)\right), \tag{8.6}$$

where $g(k, \ell)$ is the Green's function of the two point right focal boundary value problem (4.6), (4.18) defined in (4.19). The space $S(a, b-1+n)$ we shall equip with the norm $\|u\| = \max\left\{\|\Delta^i u(k)\|, \ 0 \le i \le n-1\right\}$, where $\|\Delta^i u(k)\| = \max_{N(a,b-1+n-i)} \left|\Delta^i u(k)\right|$. The set $S_1 =$

$\{u(k) \in S(a, b-a+n) : \|\Delta^i u(k)\| \le 2M_i,\ 0 \le i \le n-1\}$ is a closed convex subset of the Banach space $S(a, b-1+n)$, and as in Theorem 8.1, in view of (7.4), for any $u(k) \in S_1$ it follows that

$$\begin{aligned}\|\Delta^i Tu(k)\| &\le \max_{N(a,b-1+n-i)} |\Delta^i P_{n-1}(k)| + C_{n,i}\, Q \\ &\le 2M_i, \qquad 0 \le i \le n-1,\end{aligned}$$

from which the conclusion is immediate. ∎

THEOREM 8.5. *With respect to the (n,p) boundary value problem (8.1), (3.26), we assume*

(i) *condition (i) of Theorem 8.1;*
(ii) *condition (ii) of Theorem 8.1 with $P_{n-1}(k)$ as the (n,p) interpolating polynomial defined in (3.27);*
(iii) *$D_{n,0}\, Q \le M$, where $D_{n,0}$ is defined in Theorem 5.5.*

Then, the problem (8.1), (3.26) has a solution in D_0.

THEOREM 8.6. *With respect to the (p,n) boundary value problem (8.1), (3.28), we assume*

(i) *condition (i) of Theorem 8.1;*
(ii) *condition (ii) of Theorem 8.1 with $P_{n-1}(k)$ as the (p,n) interpolating polynomial defined in (3.29);*
(iii) *$E_{n,0}\, Q \le M$, where $E_{n,0}$ is defined in Theorem 5.6.*

Then, the problem (8.1), (3.28) has a solution in D_0.

THEOREM 8.7. *With respect to the Lidstone boundary value problem (8.1), $n = 2m$, (4.28), we assume*

(i) *condition (i) of Theorem 8.2;*
(ii) *condition (ii) of Theorem 8.2 with $P_{2m-1}(k)$ as the Lidstone interpolating polynomial defined in (4.31);*
(iii) $\left(\frac{1}{8}\right)^m \prod_{i=1}^m (b+2i-1-a)^2\, Q \le M$.

Then, the problem (8.1), with $n = 2m$, (4.28) has a solution in D_0.

Hereafter, we shall prove results only for the two point right focal boundary value problem (8.2), (3.24) whereas, for the other problems, analogous results can easily be stated.

THEOREM 8.8. *Suppose that the function $f(k, u_0, u_1, \dots, u_{n-1})$ is continuous and that on $N(a, b-1) \times \mathbb{R}^n$,*

$$|f(k, u_0, u_1, \dots, u_{n-1})| \le L + \sum_{i=0}^{n-1} L_i\, |u_i|^{\alpha(i)}, \tag{8.7}$$

where $0 \le \alpha(i) < 1$, L and L_i, $0 \le i \le n-1$ are nonnegative constants. Then, the problem (8.2), (3.24) has a solution.

PROOF. We shall show that the conditions of Theorem 8.4 are satisfied. For this, the inequality (8.7) implies that on $N(a, b-1) \times D_0$,

$$|f(k, u_0, u_1, \dots, u_{n-1})| \le L + \sum_{i=0}^{n-1} L_i\, (2M_i)^{\alpha(i)} \equiv Q_1.$$

Thus, it suffices to choose M_i, $0 \le i \le n-1$ so large that the condition (ii) of Theorem 8.4 holds and $C_{n,i}\, Q_1 \le M_i$, $0 \le i \le n-1$. ∎

Theorem 8.4 is a local existence result, whereas Theorem 8.8 does not require any condition on the constants $C_{n,i}$ or the boundary conditions. The question, what happens if $\alpha(i) = 1$, $0 \le i \le n-1$ in (8.7), is considered in the next result.

THEOREM 8.9. *Suppose that the function $f(k, u_0, u_1, \dots, u_{n-1})$ is continuous and that on $N(a, b-1) \times D_1$,*

$$|f(k, u_0, u_1, \dots, u_{n-1})| \le L + \sum_{i=0}^{n-1} L_i\, |u_i|, \tag{8.8}$$

where

$$D_1 = \left\{(u_0, u_1, \ldots, u_{n-1}) : |u_i| \le \max_{N(a,b-1+n-i)} |\Delta^i P_{n-1}(k)| + C_{n,i}\, \frac{L+c}{1-\theta},\ 0 \le i \le n-1\right\},$$

and

$$c = \sum_{i=0}^{n-1} L_i \max_{N(a,b-1+n-i)} |\Delta^i P_{n-1}(k)|,$$

$$\theta = \sum_{i=0}^{n-1} C_{n,i}\, L_i < 1.$$

Then, the problem (8.2), (3.24) has a solution in D_1.

PROOF. The boundary value problem (8.2), (3.24) can be written as

$$\Delta^n u(k) = f\left(k, v(k) + P_{n-1}(k),\ \Delta v(k) + \Delta P_{n-1}(k), \ldots,\ \Delta^{n-1} v(k) + \Delta^{n-1} P_{n-1}(k)\right), \tag{8.9}$$

$$\begin{aligned} \Delta^i v(a) &= 0, \qquad 0 \le i \le p-1, \\ \Delta^i v(b) &= 0, \qquad p \le i \le n-1. \end{aligned} \tag{8.10}$$

We define $S_2(a, b-1+n)$ as the space of all real functions defined on $N(a, b-1+n)$ satisfying the boundary conditions (8.10). If we introduce in $S_2(a, b-1+n)$ the norm $\|v\| = \max_{N(a,b-1)} |\Delta^n v(k)|$, then it becomes a Banach space. We shall show that the mapping $T : S_2(a, b-1+n) \longrightarrow S_2(a, b-1+n)$ defined by

$$Tv(k) = \sum_{\ell=a}^{b-1} g(k,\ell)\, f\left(\ell, v(\ell) + P_{n-1}(\ell), \ldots\right), \tag{8.11}$$

maps the ball $S_3 = \{v(k) \in S_2(a, b-1+n) : \|v\| \le (L+c)/(1-\theta)\}$ into itself. For this, let $v(k) \in S_3$. Then, from Theorem 7.4 on $N(a, b-1+n-i)$, we have

$$|\Delta^i v(k)| \le C_{n,i}\, \frac{L+c}{1-\theta}, \qquad 0 \le i \le n-1,$$

and hence, on $N(a, b-1+n-i)$,

$$|\Delta^i v(k) + \Delta^i P_{n-1}(k)| \le \max_{N(a,b-1+n-i)} |\Delta^i P_{n-1}(k)| + C_{n,i}\, \frac{L+c}{1-\theta}, \qquad 0 \le i \le n-1,$$

which implies that $\left(k, v(k) + P_{n-1}(k),\ \Delta v(k) + \Delta P_{n-1}(k), \ldots,\ \Delta^{n-1} v(k) + \Delta^{n-1} P_{n-1}(k)\right) \in N(a, b-1) \times D_1$.

Further, from (8.11) we have

$$\|Tv\| = \max_{N(a,b-1)} |f\left(k, v(k) + P_{n-1}(k), \ldots\right)|,$$

and hence, in view of (8.8), it follows that

$$\begin{aligned} \|Tv\| &\le L + \sum_{i=0}^{n-1} L_i \max_{N(a,b-1)} |\Delta^i v(k) + \Delta^i P_{n-1}(k)| \\ &\le L + c + \sum_{i=0}^{n-1} L_i\, C_{n,i}\, \frac{L+c}{1-\theta} \\ &= L + c + \theta\, \frac{L+c}{1-\theta} \\ &= \frac{L+c}{1-\theta}. \end{aligned}$$

Thus, the operator T has a fixed point in S_3. This fixed point $v(k)$ is a solution of (8.9), (8.10) and hence, the problem (8.2), (3.24) has a solution $u(k) = v(k) + P_{n-1}(k)$. ∎

THEOREM 8.10. *Suppose that the boundary value problem (8.2), (4.18) has a nontrivial solution $u(k)$ and the condition (8.8) with $L = 0$ is satisfied on $N(a, b-1) \times D_2$, where*

$$D_2 = \{(u_0, u_1, \ldots, u_{n-1}) : |u_i| \le C_{n,i}\, M, \quad 0 \le i \le n-1\},$$

and $M = \max_{N(a,b-1)} |\Delta^n u(k)|$. Then, it is necessary that $\theta \ge 1$.

PROOF. Since $u(k)$ is a nontrivial solution of (8.2), (4.18), it is necessary that $M \ne 0$, and Theorem 7.4 implies that $\left(k, u(k), \Delta u(k), \ldots, \Delta^{n-1} u(k)\right) \in N(a, b-1) \times D_2$. Thus, we have

$$\begin{aligned} M = \max_{N(a,b-1)} |\Delta^n u(k)| &= \max_{N(a,b-1)} \left| f\left(k, u(k), \Delta u(k), \ldots, \Delta^{n-1} u(k)\right)\right| \\ &\le \sum_{i=0}^{n-1} L_i \max_{N(a,b-1+n-i)} \left|\Delta^i u(k)\right| \\ &\le \sum_{i=0}^{n-1} L_i\, C_{n,i}\, M \\ &= \theta\, M, \end{aligned}$$

and hence, $\theta \ge 1$. ∎

Conditions of Theorem 8.10 ensure that in (8.8) at least one of the L_i, $0 \le i \le n-1$ will not be zero, otherwise on $N(a, b-1+n)$ the solution $u(k)$ will coincide with a polynomial of degree at most $n-1$ and will not be a nontrivial solution of (8.2), (4.18). Further, $u(k) \equiv 0$ is obviously a solution of (8.2), (4.18). If $\theta < 1$, then it is also unique.

THEOREM 8.11. *Suppose that for all $(k, u_0, u_1, \ldots, u_{n-1})$, $(k, v_0, v_1, \ldots, v_{n-1}) \in N(a, b-1) \times D_1$ the function f satisfies the Lipschitz condition*

$$|f(k, u_0, u_1, \ldots, u_{n-1}) - f(k, v_0, v_1, \ldots, v_{n-1})| \le \sum_{i=0}^{n-1} L_i\, |u_i - v_i|, \tag{8.12}$$

where $L = \max_{N(a,b-1)} |f(k, 0, 0, \ldots, 0)|$. Then, the boundary value problem (8.2), (3.24) has a unique solution in D_1.

PROOF. The Lipschitz condition (8.12) in particular implies (8.8) and the continuity of f on $N(a, b-1) \times D_1$. Therefore, the existence of a solution of (8.2), (3.24) follows from Theorem 8.9. To show the uniqueness, let $u(k)$ and $v(k)$ be two solutions of (8.2), (3.24) in D_1. Then, in view of (8.5) and (7.4), it follows that

$$\begin{aligned} |\Delta^n (u(k) - v(k))| &\le \max_{N(a,b-1)} \sum_{i=0}^{n-1} L_i \left|\Delta^i (u(k) - v(k))\right| \\ &\le \sum_{i=0}^{n-1} L_i\, C_{n,i}\, |\Delta^n (u(k) - v(k))| \\ &= \theta\, |\Delta^n (u(k) - v(k))|. \end{aligned}$$

Since $\theta < 1$, we find that $\Delta^n (u(k) - v(k)) = 0$, $k \in N(a, b-1)$. But, then $u(k) \equiv v(k)$, $k \in N(a, b-1+n)$ follows from the boundary conditions (3.24). ∎

9. PICARD'S AND APPROXIMATE PICARD'S METHODS

In the last few years, in [23,30], Picard's and Approximate Picard's methods have been successfully used to construct the solutions of the continuous boundary value problems. These methods have an important characteristic, that bounds of the difference between iterates and the solution are easily available. In this section, we shall discuss these methods only for the boundary value problem (8.2), (3.24). For other problems, analogous results can be stated without much difficulty. For this, we need the following definition.

DEFINITION 9.1. *A function $\bar{u}(k)$ defined on $N(a, b-1+n)$ is called an approximate solution of (8.2), (3.24) if there exist δ and ϵ nonnegative constants such that*

$$\max_{N(a,b-1)} \left|\Delta^n \bar{u}(k) - f\left(k, \bar{u}(k), \Delta\bar{u}(k), \dots, \Delta^{n-1}\bar{u}(k)\right)\right| \le \delta, \tag{9.1}$$

and

$$\max_{N(a,b-1+n-i)} \left|\Delta^i P_{n-1}(k) - \Delta^i \bar{P}_{n-1}(k)\right| \le \epsilon C_{n,i}, \qquad 0 \le i \le n-1, \tag{9.2}$$

where $P_{n-1}(k)$ and $\bar{P}_{n-1}(k)$ are the two point right focal interpolating polynomials satisfying (3.24) and

$$\begin{aligned} \Delta^i \bar{P}_{n-1}(a) &= \Delta^i \bar{u}(a), \qquad 0 \le i \le p-1, \\ \Delta^i \bar{P}_{n-1}(b) &= \Delta^i \bar{u}(b), \qquad p \le i \le n-1, \end{aligned} \tag{9.3}$$

respectively, and the constants $C_{n,i}$ are defined in Theorem 5.4.

Inequality (9.1) means that there exists a function $\eta(k)$, $k \in N(a, b-1)$ such that

$$\Delta^n \bar{u}(k) = f\left(k, \bar{u}(k), \Delta\bar{u}(k), \dots, \Delta^{n-1}\bar{u}(k)\right) + \eta(k), \qquad k \in N(a, b-1),$$

where $\max_{N(a,b-1)} |\eta(k)| \le \delta$. Thus, the approximate solution $\bar{u}(k)$ can be expressed as

$$\bar{u}(k) = \bar{P}_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) \left[f\left(\ell, \bar{u}(\ell), \Delta\bar{u}(\ell), \dots, \Delta^{n-1}\bar{u}(\ell)\right) + \eta(\ell)\right]. \tag{9.4}$$

In what follows, we shall consider the Banach space $S(a, b-1+n)$ and for $u(k) \in S(a, b-1+n)$ the norm is $\|u\| = \max\left\{\|\Delta^i u(k)\| / C_{n,i},\ 0 \le i \le n-1\right\}$.

THEOREM 9.1. *With respect to the boundary value problem (8.2), (3.24), we assume that there exists an approximate solution $\bar{u}(k)$ and that*

(i) *the function f satisfies the Lipschitz condition (8.12) on $N(a, b-1) \times D_3$ where*

$$D_3 = \left\{(u_0, u_1, \dots, u_{n-1}) : \left|u_i - \Delta^i \bar{u}(k)\right| \le \mu C_{n,i},\ \ k \in N(a, b-1+n-i),\ \ 0 \le i \le n-1\right\};$$

(ii) *$\theta < 1$;*
(iii) *$(1-\theta)^{-1}(\epsilon + \delta) \le \mu$.*

Then, the following hold

(1) *there exists a solution $u^*(k)$ of (8.2), (3.24) in $\bar{S}(\bar{u}, \mu_0)$;*
(2) *$u^*(k)$ is the unique solution of (8.2), (3.24) in $\bar{S}(\bar{u}, \mu)$;*
(3) *the Picard iterative sequence $\{u_m(k)\}$ defined by*

$$\begin{aligned} u_{m+1}(k) &= P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\, f\left(\ell, u_m(\ell), \Delta u_m(\ell), \dots, \Delta^{n-1} u_m(\ell)\right), \\ u_0(k) &= \bar{u}(k), \qquad m = 0, 1, \dots, \end{aligned} \tag{9.5}$$

converges to $u^*(k)$ with $\|u^* - u_m\| \le \theta^m \mu_0$;

(4) *for $u_0(k) = u(k) \in \bar{S}(\bar{u}, \mu_0)$, the iterative process (9.5) converges to $u^*(k)$;*
(5) *any sequence $\{\bar{u}_m(k)\}$ such that $\bar{u}_m(k) \in \bar{S}(u_m, \theta^m \mu_0)$, $m = 0, 1, \dots$ converges to $u^*(k)$.*

PROOF. We shall show that the operator $T : \bar{S}(\bar{u},\mu) \longrightarrow S(a, b-1+n)$ defined in (8.6) satisfies the conditions of Theorem 2.5. Let $u(k) \in \bar{S}(\bar{u},\mu)$, then from the definition of norm, we have $\|u-\bar{u}\| = \max\left\{\max_{N(a,b-1+n-i)} \left|\Delta^i u(k) - \Delta^i \bar{u}(k)\right|/C_{n,i},\ 0 \le i \le n-1\right\} \le \mu$, which implies that $\left|\Delta^i u(k) - \Delta^i \bar{u}(k)\right| \le \mu C_{n,i}$, $k \in N(a, b-1+n-i)$, $0 \le i \le n-1$. Thus, $(u(k), \Delta u(k), \ldots, \Delta^{n-1}u(k)) \in D_3$. Further, if $u(k)$, $v(k) \in \bar{S}(\bar{u},\mu)$, then $Tu(k) - Tv(k)$ satisfies the conditions of Theorem 7.4 with $P_{n-1}(k) \equiv 0$, and we get

$$\begin{aligned}\left|\Delta^j Tu(k) - \Delta^j Tv(k)\right| &\le C_{n,j} \max_{N(a,b-1)} \left|f(k, u(k), \ldots) - f(k, v(k), \ldots)\right| \\ &\le C_{n,j} \sum_{i=0}^{n-1} L_i \max_{N(a,b-1+n-i)} \left|\Delta^i u(k) - \Delta^i v(k)\right| \\ &\le C_{n,j} \sum_{i=0}^{n-1} L_i\, C_{n,i}\, \|u-v\|, \qquad 0 \le j \le n-1,\end{aligned}$$

and hence,

$$\frac{1}{C_{n,j}} \left|\Delta^j Tu(k) - \Delta^j Tv(k)\right| \le \theta\, \|u-v\|, \qquad 0 \le j \le n-1,$$

from which it follows that $\|Tu - Tv\| \le \theta\, \|u-v\|$.

Next, from (8.6) and (9.4), we have

$$\begin{aligned}T\bar{u}(k) - \bar{u}(k) &= Tu_0(k) - u_0(k) \\ &= P_{n-1}(k) - \bar{P}_{n-1}(k) - \sum_{\ell=a}^{b-1} g(k,\ell)\, \eta(\ell).\end{aligned} \tag{9.6}$$

The function $w(k) = -\sum_{\ell=a}^{b-1} g(k,\ell)\, \eta(\ell)$ satisfies the conditions of Theorem 7.4 with $P_{n-1}(k) \equiv 0$, $\Delta^n w(k) = -\eta(k)$. Thus,

$$\max_{N(a,b-1)} \left|\Delta^n w(k)\right| = \max_{N(a,b-1)} \left|\eta(k)\right| \le \delta,$$

and hence,

$$\left|\Delta^j w(k)\right| \le C_{n,j}\, \delta, \qquad 0 \le j \le n-1.$$

Using these inequalities and (9.2) in (9.6), we obtain

$$\left|\Delta^j Tu_0(k) - \Delta^j u_0(k)\right| \le (\epsilon+\delta)\, C_{n,j}, \qquad 0 \le j \le n-1,$$

which is the same as

$$\frac{1}{C_{n,j}} \left|\Delta^j Tu_0(k) - \Delta^j u_0(k)\right| \le (\epsilon+\delta), \qquad 0 \le j \le n-1,$$

and hence $\|Tu_0 - u_0\| \le (\epsilon+\delta)$. Thus, from the hypothesis (ii) it follows that $(1-\theta)^{-1}\, \|Tu_0 - u_0\| \le (1-\theta)^{-1}\, (\epsilon+\delta) \le \mu$.

Hence, the conditions of Theorem 2.5 are satisfied and conclusions (1)–(5) follow. ∎

In Theorem 9.1, the conclusion (3) ensures that the sequence $\{u_m(k)\}$ obtained from (9.5) converges to the solution $u^*(k)$ of (8.2), (3.24). However, in practical evaluation, this sequence is approximated by the computed sequence $\{v_m(k)\}$. To find $v_{m+1}(k)$, the function f is approximated by f_m. Therefore, the computed sequence $\{v_m(k)\}$ satisfies the recurrence relation

$$\begin{aligned}v_{m+1}(k) &= P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\, f_m\left(\ell, v_m(\ell), \Delta v_m(\ell), \ldots, \Delta^{n-1} v_m(\ell)\right), \\ v_0(k) &= u_0(k) = \bar{u}(k), \qquad m = 0, 1, \ldots.\end{aligned} \tag{9.7}$$

With respect to f_m, we shall assume the following condition.

CONDITION (c_1). For all $k \in N(a,b-1)$ and $\Delta^i v_m(k)$, $0 \le i \le n-1$ obtained from (9.7), the following inequality is satisfied

$$|f(k, v_m(k), \dots) - f_m(k, v_m(k), \dots)| \le \nu\, |f(k, v_m(k), \dots)|, \tag{9.8}$$

where ν is a nonnegative constant.

Inequality (9.8) corresponds to the relative error in approximating f by f_m for the $(m+1)^{\text{th}}$ iteration.

THEOREM 9.2. *With respect to the boundary value problem (8.2), (3.24), we assume that there exists an approximate solution $\bar{u}(k)$ and the condition (c_1) is satisfied. Further, we assume*

(i) *condition (i) of Theorem 9.1;*
(ii) $\theta_1 = (1+\nu)\,\theta < 1$;
(iii) $\mu_1 = (1-\theta_1)^{-1}(\epsilon + \delta + \nu F) \le \mu$, *where* $F = \max_{N(a,b-1)} \left|f\left(k, \bar{u}(k), \Delta\bar{u}(k), \dots, \Delta^{n-1}\bar{u}(k)\right)\right|$.

Then, the following hold

(1) *all the conclusions (1)–(5) of Theorem 9.1 are valid;*
(2) *the sequence $\{v_m(k)\}$ obtained from (9.7) remains in $\bar{S}(\bar{u}, \mu_1)$;*
(3) *the sequence $\{v_m(k)\}$ converges to $u^*(k)$, the solution of (8.2), (3.24) if and only if $\lim_{m\to\infty} w_m = 0$ where*

$$w_m = \left\|v_{m+1}(k) - P_{n-1}(k) - \sum_{\ell=a}^{b-1} g(k,\ell)\, f\left(\ell, v_m(\ell), \dots, \Delta^{n-1} v_m(\ell)\right)\right\|, \tag{9.9}$$

and

$$\|u^* - v_{m+1}\| \le (1-\theta)^{-1}\left[\theta\, \|v_{m+1} - v_m\| + \nu \max_{N(a,b-1)} |f(k, v_m(k), \dots)|\right]. \tag{9.10}$$

PROOF. Since $\theta_1 < 1$ implies $\theta < 1$ and obviously $\mu_0 \le \mu_1$, the conditions of Theorem 9.1 are satisfied and conclusion (1) follows.

To prove (2), we note that $\bar{u}(k) \in \bar{S}(\bar{u}, \mu_1)$ and from (9.4), (9.7), we find

$$v_1(k) - \bar{u}(k) = P_{n-1}(k) - \bar{P}_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\, \left[f_0(\ell, \bar{u}(\ell), \dots) - f(\ell, \bar{u}(\ell), \dots) - \eta(\ell)\right].$$

Thus, from Theorem 7.4, we get

$$\left|\Delta^j v_1(k) - \Delta^j \bar{u}(k)\right| \le (\epsilon + \delta)\, C_{n,j} + C_{n,j}\, \nu F, \qquad 0 \le j \le n-1,$$

and hence,

$$\|v_1 - \bar{u}\| \le (\epsilon + \delta + \nu F) \le \mu_1.$$

Now we assume that $v_m(k) \in \bar{S}(\bar{u}, \mu_1)$ and will show that $v_{m+1}(k) \in \bar{S}(\bar{u}, \mu_1)$. From (9.4) and (9.7), we have

$$v_{m+1}(k) - \bar{u}(k) = P_{n-1}(k) - \bar{P}_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\, \left[f_m(\ell, v_m(\ell), \dots) - f(\ell, \bar{u}(\ell), \dots) - \eta(\ell)\right],$$

and Theorem 7.4 provides

$$\begin{aligned}
\left|\Delta^j v_{m+1}(k) - \Delta^j \bar{u}(k)\right| &\le (\epsilon+\delta)\, C_{n,j} + C_{n,j} \max_{N(a,b-1)} \left[\,|f_m(k, v_m(k), \dots) - f(k, v_m(k), \dots)|\right. \\
&\qquad \left. + |f(k, v_m(k), \dots) - f(k, \bar{u}(k), \dots)|\,\right] \\
&\le C_{n,j}\left[\epsilon + \delta + \nu F + (1+\nu) \max_{N(a,b-1)} |f(k, v_m(k), \dots) - f(k, \bar{u}(k), \dots)|\right] \\
&\le C_{n,j}\left[\epsilon + \delta + \nu F + (1+\nu) \sum_{i=0}^{n-1} L_i \max_{N(a,b-1+n-i)} \left|\Delta^i v_m(k) - \Delta^i \bar{u}(k)\right|\right] \\
&\le C_{n,j}\left[\epsilon + \delta + \nu F + (1+\nu)\,\theta\, \|v_m - \bar{u}\|\right], \qquad 0 \le j \le n-1.
\end{aligned}$$

Hence, we get

$$\frac{1}{C_{n,j}} \left|\Delta^j v_{m+1}(k) - \Delta^j \bar{u}(k)\right| \le \epsilon + \delta + \nu F + \theta_1 \|v_m - \bar{u}\|, \qquad 0 \le j \le n-1,$$

which gives

$$\|v_{m+1} - \bar{u}\| \le (1-\theta_1)\mu_1 + \theta_1 \mu_1 = \mu_1.$$

This completes the proof of (2).

From the definitions of $u_{m+1}(k)$ and $v_{m+1}(k)$, we have

$$u_{m+1}(k) - v_{m+1}(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) f(\ell, v_m(\ell), \dots) - v_{m+1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) \left[f(\ell, u_m(\ell), \dots) - f(\ell, v_m(\ell), \dots)\right],$$

and hence, as earlier, we find

$$\|u_{m+1} - v_{m+1}\| \le w_m + \theta \|u_m - v_m\|.$$

Since $u_0(k) = v_0(k)$, the above inequality provides

$$\|u_{m+1} - v_{m+1}\| \le \sum_{i=0}^{m} \theta^{m-i} w_i.$$

Thus, from the triangle inequality, we get

$$\|u^* - v_{m+1}\| \le \sum_{i=0}^{m} \theta^{m-i} w_i + \|u^* - u_{m+1}\|. \tag{9.11}$$

In equation (9.11), Theorem 9.1 ensures that $\lim_{m\to\infty} \|u^* - u_{m+1}\| = 0$. Thus, the condition $\lim_{m\to\infty} w_m = 0$ is necessary and sufficient for the convergence of the sequence $\{v_m(k)\}$ to $u^*(k)$, which follows from the Toeplitz lemma "for any $0 \le \alpha < 1$, let $s_m = \sum_{i=0}^{m} \alpha^{m-i} d_i$, $m = 0, 1, \dots$, then $\lim_{m\to\infty} s_m = 0$ if and only if $\lim_{m\to\infty} d_m = 0$."

Finally, to prove (9.10), we note that

$$u^*(k) - v_{m+1}(k) = \sum_{\ell=a}^{b-1} g(k,\ell) \left[f(\ell, u^*(\ell), \dots) - f(\ell, v_m(\ell), \dots) + f(\ell, v_m(\ell), \dots) - f_m(\ell, v_m(\ell), \dots)\right],$$

and as earlier, we find

$$\begin{aligned} \|u^* - v_{m+1}\| &\le \theta \|u^* - v_m\| + \nu \max_{N(a,b-1)} |f(k, v_m(k), \dots)| \\ &\le \theta \|u^* - v_{m+1}\| + \theta \|v_{m+1} - v_m\| + \nu \max_{N(a,b-1)} |f(k, v_m(k), \dots)|, \end{aligned}$$

which is the same as (9.10). ∎

In our next result, we shall assume the following condition.

CONDITION (c_2). For all $k \in N(a, b-1)$ and $\Delta^i v_m(k)$, $0 \le i \le n-1$ obtained from (9.7), the following inequality is satisfied

$$|f(k, v_m(k), \dots) - f_m(k, v_m(k), \dots)| \le \nu_1, \tag{9.12}$$

where ν_1 is a nonnegative constant.

Inequality (9.12) corresponds to the absolute error in approximating f by f_m for the $(m+1)^{\text{th}}$ iteration.

THEOREM 9.3. *With respect to the boundary value problem (8.2), (3.24), we assume that there exists an approximate solution $\bar{u}(k)$ and the condition (c_2) is satisfied. Further, we assume*

(i) *condition (i) of Theorem 9.1;*
(ii) *condition (ii) of Theorem 9.1;*
(iii) $\mu_2 = (1-\theta)^{-1}(\epsilon + \delta + \nu) \le \mu$.

Then, the following hold

(1) *all the conclusions (1)–(5) of Theorem 9.1 are valid;*
(2) *the sequence $\{v_m(k)\}$ obtained from (9.7) remains in $\bar{S}(\bar{u}, \mu_2)$;*
(3) *the condition $\lim_{m\to\infty} w_m = 0$ is necessary and sufficient for the convergence of $\{v_m(k)\}$ to the solution $u^*(k)$ of (8.2), (3.24), where w_m is defined in (9.9), and $\|u^* - v_{m+1}\| \le (1-\theta)^{-1}\left[\theta\,\|v_{m+1} - v_m\| + \nu_1\right]$.*

PROOF. The proof is contained in Theorem 9.2. ∎

10. QUASILINEARIZATION AND APPROXIMATE QUASILINEARIZATION

Newton's method when applied to boundary value problems for higher order differential equations has been labelled as quasilinearization by Agarwal [23,31] and Agarwal and Wong [32]. Here, once again we shall discuss this method only for the discrete boundary value problem (8.2), (3.24), whereas analogous results for the other problems can be stated easily. For this, following the notations and definitions of the previous section, we shall provide sufficient conditions so that the sequence $\{u_m(k)\}$ generated by the quasilinear iterative scheme

$$\begin{aligned}\Delta^n u_{m+1}(k) &= f\left(k, u_m(k), \Delta u_m(k), \dots, \Delta^{n-1}u_m(k)\right)\\ &\quad + \sum_{i=0}^{n-1}\left(\Delta^i u_{m+1}(k) - \Delta^i u_m(k)\right)\frac{\partial}{\partial \Delta^i u_m(k)} f\left(k, u_m(k), \dots\right),\end{aligned} \tag{10.1}$$

$$\begin{aligned}\Delta^i u_{m+1}(a) &= A_i, \qquad 0 \le i \le p-1,\\ \Delta^i u_{m+1}(b) &= A_i, \qquad p \le i \le n-1, \quad m = 0, 1, \dots,\end{aligned} \tag{10.2}$$

with $u_0(k) = \bar{u}(k)$, converges to the unique solution $u^*(k)$ of the boundary value problem (8.20), (3.24).

THEOREM 10.1. *With respect to the boundary value problem (8.2), (3.24), we assume that there exists an approximate solution $\bar{u}(k)$ and that*

(i) *the function $f(k, u_0, u_1, \dots, u_{n-1})$ is continuously differentiable with respect to all u_i, $0 \le i \le n-1$ on $N(a, b-1) \times D_3$;*
(ii) *there exist L_i, $0 \le i \le n-1$ nonnegative constants such that for all $(k, u_0, u_1, \dots, u_{n-1}) \in N(a, b-1) \times D_3$, we have*

$$\left|\frac{\partial}{\partial u_i} f(k, u_0, u_1, \dots, u_{n-1})\right| \le L_i\,;$$

(iii) $3\theta < 1$;
(iv) $\mu_3 = (1-3\theta)^{-1}(\epsilon + \delta) \le \mu$.

Then, the following hold

(1) *the sequence $\{u_m(k)\}$ generated by the process (10.1), (10.2) remains in $\bar{S}(\bar{u}, \mu_3)$;*
(2) *the sequence $\{u_m(k)\}$ converges to the unique solution $u^*(k)$ of (8.2), (3.24);*
(3) *a bound on the error is given by*

$$\|u_m - u^*\| \le \left(\frac{2\theta}{1-\theta}\right)^m \left(1 - \frac{2\theta}{1-\theta}\right)^{-1} \|u_1 - \bar{u}\| \tag{10.3}$$

$$\le \left(\frac{2\theta}{1-\theta}\right)^m \left(1 - \frac{2\theta}{1-\theta}\right)^{-1} (1-\theta)^{-1}(\epsilon + \delta). \tag{10.4}$$

PROOF. First, we shall show that the sequence $\{u_m(k)\}$ remains in $\bar{S}(\bar{u},\mu_3)$. We define an implicit operator T as follows

$$Tu(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\left[f(\ell,u(\ell),\dots) + \sum_{i=0}^{n-1}\left(\Delta^i Tu(\ell) - \Delta^i u(\ell)\right)\frac{\partial}{\partial \Delta^i u(\ell)} f(\ell,u(\ell),\dots)\right], \tag{10.5}$$

whose form is patterned on the summation equation representation of (10.1), (10.2).

Since $\bar{u}(k) \in \bar{S}(\bar{u},\mu_3)$, it is sufficient to show that if $u(k) \in \bar{S}(\bar{u},\mu_3)$, then $Tu(k) \in \bar{S}(\bar{u},\mu_3)$. For this, if $u(k) \in \bar{S}(\bar{u},\mu_3)$, then $\left(u(k),\Delta u(k),\dots,\Delta^{n-1}u(k)\right) \in D_3$ and from (9.4) and (10.5), we have

$$Tu(k) - \bar{u}(k) = P_{n-1}(k) - \bar{P}_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell)\left[f(\ell,u(\ell),\dots) + \sum_{i=0}^{n-1}\left(\Delta^i Tu(\ell) - \Delta^i u(\ell)\right)\frac{\partial}{\partial \Delta^i u(\ell)} f(\ell,u(\ell),\dots) - f(\ell,\bar{u}(\ell),\dots) - \eta(\ell)\right].$$

Thus, an application of Theorem 7.4 provides

$$\left|\Delta^j Tu(k) - \Delta^j \bar{u}(k)\right| \le \epsilon\, C_{n,j} + C_{n,j} \max_{N(a,b-1)}\left[\left|f(k,u(k),\dots) - f(k,\bar{u}(k),\dots)\right| + \sum_{i=0}^{n-1} L_i \left\{\left|\Delta^i Tu(k) - \Delta^i\bar{u}(k)\right| + \left|\Delta^i u(k) - \Delta^i \bar{u}(k)\right|\right\} + \delta\right],$$

and hence, we get

$$\frac{1}{C_{n,j}}\left|\Delta^j Tu(k) - \Delta^j\bar{u}(k)\right| \le (\epsilon+\delta) + \sum_{i=0}^{n-1} C_{n,i} L_i \left[\|Tu - \bar{u}\| + 2\|u - \bar{u}\|\right], \qquad 0 \le j \le n-1.$$

From the above inequality, we find

$$\|Tu - \bar{u}\| \le (\epsilon+\delta) + \theta\,\|Tu - \bar{u}\| + 2\theta\,\|u - \bar{u}\|,$$

which gives

$$\|Tu - \bar{u}\| \le (1-\theta)^{-1}\left[(\epsilon+\delta) + 2\theta\mu_3\right].$$

Thus, $\|Tu - \bar{u}\| \le \mu_3$ follows from the definition of μ_3.

Next, we shall show the convergence of the sequence $\{u_m(k)\}$. From (10.1), (10.2) we have

$$u_{m+1}(k) - u_m(k) = \sum_{\ell=a}^{b-1} g(k,\ell)\left[f(\ell,u_m(\ell),\dots) - f(\ell,u_{m-1}(\ell),\dots) + \sum_{i=0}^{n-1}\left\{\left(\Delta^i u_{m+1}(\ell) - \Delta^i u_m(\ell)\right)\frac{\partial}{\partial\Delta^i u_m(\ell)} f(\ell,u_m(\ell),\dots) - \left(\Delta^i u_m(\ell) - \Delta^i u_{m-1}(\ell)\right)\frac{\partial}{\partial \Delta^i u_{m-1}(\ell)} f(\ell,u_{m-1}(\ell),\dots)\right\}\right]. \tag{10.6}$$

Thus, from Theorem 7.4 and the fact that $\{u_m(k)\} \subseteq \bar{S}(\bar{u},\mu_3)$, we get

$$\left|\Delta^j u_{m+1}(k) - \Delta^j u_m(k)\right| \le C_{n,j} \max_{N(a,b-1)}\left[2\sum_{i=0}^{n-1} L_i \left|\Delta^i u_m(k) - \Delta^i u_{m-1}(k)\right| + \sum_{i=0}^{n-1} L_i \left|\Delta^i u_{m+1}(k) - \Delta^i u_m(k)\right|\right],$$

and hence,

$$\frac{1}{C_{n,j}}\left|\Delta^j u_{m+1}(k)-\Delta^j u_m(k)\right| \le 2\theta\,\|u_m-u_{m-1}\|+\theta\,\|u_{m+1}-u_m\|, \qquad 0\le j\le n-1,$$

which provides

$$\|u_{m+1}-u_m\| \le 2\theta\,\|u_m-u_{m-1}\|+\theta\,\|u_{m+1}-u_m\|,$$

or

$$\|u_{m+1}-u_m\| \le \frac{2\theta}{1-\theta}\,\|u_m-u_{m-1}\|,$$

and by an easy induction, we get

$$\|u_{m+1}-u_m\| \le \left(\frac{2\theta}{1-\theta}\right)^m \|u_1-\bar{u}\|. \tag{10.7}$$

Since $3\theta<1$, inequality (10.7) implies that $\{u_m(k)\}$ is a Cauchy sequence and hence converges to some $u^*(k)\in\bar{S}(\bar{u},\mu_3)$. This $u^*(k)$ is the unique solution of (10.1), (10.2) and can easily be verified.

The error bound (10.3) follows from (10.7) and the triangle inequality

$$\begin{aligned}\|u_{m+p}-u_m\| &\le \|u_{m+p}-u_{m+p-1}\|+\|u_{m+p-1}-u_{m+p-2}\|+\cdots+\|u_{m+1}-u_m\|\\ &\le \left[\left(\frac{2\theta}{1-\theta}\right)^{m+p-1}+\left(\frac{2\theta}{1-\theta}\right)^{m+p-2}+\cdots+\left(\frac{2\theta}{1-\theta}\right)^{m}\right]\|u_1-\bar{u}\|\\ &\le \left(\frac{2\theta}{1-\theta}\right)^m\left(1-\frac{2\theta}{1-\theta}\right)^{-1}\|u_1-\bar{u}\|,\end{aligned}$$

and now let $p\longrightarrow\infty$.

Next, from (9.4), (10.1), (10.2) we have

$$\begin{aligned}u_1(k)-u_0(k) &= P_{n-1}(k)-\bar{P}_{n-1}(k)\\ &\quad-\sum_{\ell=a}^{b-1} g(k,\ell)\left[\sum_{i=0}^{n-1}\left(\Delta^i u_1(\ell)-\Delta^i u_0(\ell)\right)\frac{\partial}{\partial\Delta^i u_0(\ell)} f(\ell,u_0(\ell),\dots)-\eta(\ell)\right],\end{aligned}$$

and as earlier, we find

$$\|u_1-u_0\| \le (1-\theta)^{-1}(\epsilon+\delta). \tag{10.8}$$

Using equation (10.8) in (10.3), the inequality (10.4) follows. ∎

THEOREM 10.2. *Let the conditions of Theorem 10.1 be satisfied. Also, let $f(k,u_0,u_1,\dots,u_{n-1})$ be continuously twice differentiable with respect to all u_i, $0\le i\le n-1$ on $N(a,b-1)\times D_3$ and*

$$\left|\frac{\partial^2}{\partial u_i\,\partial u_j} f(k,u_0,u_1,\dots,u_{n-1})\right| \le L_i\,L_j\,\tau, \qquad 0\le i,\ j\le n-1.$$

Then, the following hold

$$\begin{aligned}\|u_{m+1}-u_m\| \le \alpha\,\|u_m-u_{m-1}\|^2 &\le \frac{1}{\alpha}\left(\alpha\,\|u_1-u_0\|\right)^{2^m}\\ &\le \frac{1}{\alpha}\left[\frac{1}{2}\tau(\epsilon+\delta)\left(\frac{\theta}{1-\theta}\right)^2\right]^{2^m},\end{aligned} \tag{10.9}$$

where $\alpha=\left(\tau\,\theta^2/2(1-\theta)\right)$. Thus, the convergence is quadratic if $\frac{1}{2}\tau(\epsilon+\delta)\,(\theta/1-\theta)^2<1$.

PROOF. From $\{u_m(k)\} \subseteq \bar{S}(\bar{u}, \mu_3)$ it follows that for all m, $\left(u_m(k), \Delta u_m(k), \dots, \Delta^{n-1} u_m(k)\right) \in D_3$. Further, since f is twice continuously differentiable, we have

$$\begin{aligned} f(k, u_m(k), \dots) &= f(k, u_{m-1}(k), \dots) \\ &\quad + \sum_{i=0}^{n-1} \left(\Delta^i u_m(k) - \Delta^i u_{m-1}(k)\right) \frac{\partial}{\partial \Delta^i u_{m-1}(k)} f(k, u_{m-1}(k), \dots) \\ &\quad + \frac{1}{2} \left[\sum_{i=0}^{n-1} \left(\Delta^i u_m(k) - \Delta^i u_{m-1}(k)\right) \frac{\partial}{\partial p_i(k)}\right]^2 f(k, p_0(k), p_1(k), \dots, p_{n-1}(k)), \end{aligned} \tag{10.10}$$

where $p_i(k)$ lies between $\Delta^i u_{m-1}(k)$ and $\Delta^i u_m(k)$, $0 \le i \le n-1$.

Using equation (10.10) in (10.6), we get

$$\begin{aligned} u_{m+1}(k) - u_m(k) = \sum_{\ell=a}^{b-1} g(k, \ell) &\left\{ \sum_{i=0}^{n-1} \left(\Delta^i u_{m+1}(\ell) - \Delta^i u_m(\ell)\right) \frac{\partial}{\partial \Delta^i u_m(\ell)} f(\ell, u_m(\ell), \dots) \right. \\ &\left. + \frac{1}{2} \left[\sum_{i=0}^{n-1} \left(\Delta^i u_m(\ell) - \Delta^i u_{m-1}(\ell)\right) \frac{\partial}{\partial p_i(\ell)}\right]^2 f(\ell, p_0(\ell), p_1(\ell), \dots, p_{n-1}(\ell)) \right\}. \end{aligned}$$

Thus, Theorem 7.4 provides

$$\begin{aligned} \left|\Delta^j u_{m+1}(k) - \Delta^j u_m(k)\right| \le C_{n,j} &\left[\sum_{i=0}^{n-1} L_i C_{n,i} \|u_{m+1} - u_m\| \right. \\ &\left. + \frac{1}{2} \left(\sum_{i=0}^{n-1} L_i C_{n,i}\right)^2 \tau \|u_m - u_{m-1}\|^2 \right], \end{aligned}$$

and hence,

$$\|u_{m+1} - u_m\| \le \theta \|u_{m+1} - u_m\| + \frac{1}{2} \tau \theta^2 \|u_m - u_{m-1}\|^2,$$

which is the same as the first part of the inequality (10.9). The second part of (10.9) follows by an easy induction. Finally, the last part is an application of (10.8). ∎

In Theorem 10.1 the conclusion (3) ensures that the sequence $\{u_m(k)\}$ generated from (10.1), (10.2) converges linearly to the unique solution $u^*(k)$ of the boundary value problem (8.2), (3.24). Theorem 10.2 provides sufficient conditions for its quadratic convergence. However, in practical evaluation this sequence is approximated by the computed sequence, say, $\{v_m(k)\}$ which satisfies the recurrence relation

$$\begin{aligned} \Delta^n v_{m+1}(k) &= f_m\left(k, v_m(k), \Delta v_m(k), \dots, \Delta^{n-1} v_m(k)\right) \\ &\quad + \sum_{i=0}^{n-1} \left(\Delta^i v_{m+1}(k) - \Delta^i v_m(k)\right) \frac{\partial}{\partial \Delta^i v_m(k)} f_m(k, v_m(k), \dots), \end{aligned} \tag{10.11}$$

$$\begin{aligned} \Delta^i v_{m+1}(a) &= A_i, \qquad 0 \le i \le p-1, \\ \Delta^i v_{m+1}(b) &= A_i, \qquad p \le i \le n-1, \quad m = 0, 1, \dots, \end{aligned} \tag{10.12}$$

where $v_0(k) = u_0(k) = \bar{u}(k)$.

With respect to f_m, we shall assume the following condition.

CONDITION (d_1).

(i) The function $f_m(k, u_0, u_1, \dots, u_{n-1})$ is continuously differentiable with respect to all u_i, $0 \le i \le n-1$ on $N(a, b-1) \times D_3$ and

$$\left|\frac{\partial}{\partial u_i} f_m(k, u_0, u_1, \dots, u_{n-1})\right| \le L_i, \qquad 0 \le i \le n-1;$$

(ii) condition c_1 is satisfied.

Theorem 10.3. *With respect to the boundary value problem (8.2), (3.24), we assume that there exists an approximate solution $\bar{u}(k)$ and the condition d_1 is satisfied. Further, we assume*

(i) *conditions (i) and (ii) of Theorem 10.1;*
(ii) $\theta_2 = (3+\nu)\,\theta < 1$;
(iii) $\mu_4 = (1-\theta_2)^{-1}(\epsilon + \delta + \nu F) \le \mu$, *where* $F = \max_{N(a,b-1)} \left|f\left(k, \bar{u}(k), \Delta\bar{u}(k), \dots, \Delta^{n-1}\bar{u}(k)\right)\right|$.

Then, the following hold

(1) *all the conclusions (1)–(3) of Theorem 10.1 are valid;*
(2) *the sequence $\{v_m(k)\}$ obtained from (10.11), (10.12) remains in $\bar{S}(\bar{u}, \mu_4)$;*
(3) *the sequence $\{v_m(k)\}$ converges to $u^*(k)$, the solution of (8.2), (3.24), if and only if $\lim_{m\to\infty} w_m = 0$, where w_m are defined in (9.9), and*

$$\|u^* - v_{m+1}\| \le (1-\theta)^{-1}\left[2\theta\,\|v_{m+1} - v_m\| + \nu \max_{N(a,b-1)} |f(k, v_m(k), \dots)|\right]. \tag{10.13}$$

Proof. Since $\theta_2 < 1$ implies $3\theta < 1$ and obviously $\mu_3 \le \mu_4$, the conditions of Theorem 10.1 are satisfied and part (1) follows.

To prove (2), we note that $\bar{u}(k) \in \bar{S}(\bar{u}, \mu_4)$ and from (9.4), (10.11), (10.12) we have

$$\begin{aligned} v_1(k) - \bar{u}(k) = P_{n-1}(k) - \bar{P}_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) \Bigg[& f_0(\ell, v_0(\ell), \dots) \\ & + \sum_{i=0}^{n-1} \left(\Delta^i v_1(\ell) - \Delta^i v_0(\ell)\right) \frac{\partial}{\partial \Delta^i v_0(\ell)} f_0(\ell, v_0(\ell), \dots) - f(\ell, v_0(\ell), \dots) - \eta(\ell)\Bigg] \end{aligned}$$

and Theorem 7.4 provides

$$\|v_1 - \bar{u}\| \le (\epsilon + \delta + \nu F) + \theta\,\|v_1 - v_0\|,$$

and hence,

$$\|v_1 - \bar{u}\| \le (1-\theta)^{-1}(\epsilon + \delta + \nu F) \le \mu_4. \tag{10.14}$$

Thus, $v_1(k) \in \bar{S}(\bar{u}, \mu_4)$. Next, we assume that $v_m(k) \in \bar{S}(\bar{u}, \mu_4)$ and will show that $v_{m+1}(k) \in \bar{S}(\bar{u}, \mu_4)$. From (9.4), (10.11), (10.12) we have

$$\begin{aligned} v_{m+1}(k) - \bar{u}(k) = P_{n-1}(k) - \bar{P}_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) \Bigg[& f_m(\ell, v_m(\ell), \dots) \\ & + \sum_{i=0}^{n-1} \left(\Delta^i v_{m+1}(\ell) - \Delta^i v_m(\ell)\right) \frac{\partial}{\partial \Delta^i v_m(\ell)} f_m(\ell, v_m(\ell), \dots) - f(\ell, v_0(\ell), \dots) - \eta(\ell)\Bigg], \end{aligned}$$

and from Theorem 7.4, we get

$$\begin{aligned} \left|\Delta^j v_{m+1}(k) - \Delta^j \bar{u}(k)\right| \le (\epsilon + \delta)\, C_{n,j} + C_{n,j} \max_{N(a,b-1)} \Bigg[& \sum_{i=0}^{n-1} L_i \left|\Delta^i v_{m+1}(k) - \Delta^i v_m(k)\right| \\ & + (1+\nu)\, |f(k, v_m(k), \dots) - f(k, v_0(k), \dots)| + \nu\, |f(k, v_0(k), \dots)| \Bigg] \end{aligned}$$

and hence, we find

$$\begin{aligned} \|v_{m+1} - \bar{u}\| &\le (\epsilon + \delta + \nu F) + \theta\,\|v_{m+1} - v_m\| + (1+\nu)\,\theta\,\|v_m - v_0\| \\ &\le (\epsilon + \delta + \nu F) + (2+\nu)\,\theta\,\|v_m - v_0\| + \theta\,\|v_{m+1} - v_0\|. \end{aligned}$$

From the last inequality, we obtain

$$\begin{aligned}\|v_{m+1} - \bar{u}\| &\le (1-\theta)^{-1}\ [(\epsilon + \delta + \nu F) + (2+\nu)\,\theta\,\mu_4]\\ &= \mu_4.\end{aligned}$$

This completes the proof of part (2).

Next, from the definitions of $u_{m+1}(k)$ and $v_{m+1}(k)$, we have

$$\begin{aligned}u_{m+1}(k) - v_{m+1}(k) = P_{n-1}(k) &+ \sum_{\ell=a}^{b-1} g(k,\ell)\, f(\ell, v_m(\ell),\dots) - v_{m+1}(k)\\ &+ \sum_{\ell=a}^{b-1} g(k,\ell)\left[f(\ell, u_m(\ell),\dots) - f(\ell, v_m(\ell),\dots)\right.\\ &\left.+ \sum_{i=0}^{n-1} (\Delta^i u_{m+1}(\ell) - \Delta^i u_m(\ell))\,\frac{\partial}{\partial \Delta^i u_m(\ell)} f(\ell, u_m(\ell),\dots)\right],\end{aligned}$$

and hence, as earlier, we find

$$\|u_{m+1} - v_{m+1}\| \le w_m + \theta\,\|u_m - v_m\| + \theta\,\|u_{m+1} - u_m\|. \tag{10.15}$$

Using (10.7) in (10.15), we get

$$\|u_{m+1} - v_{m+1}\| \le w_m + \theta\,\|u_m - v_m\| + \theta\left(\frac{2\theta}{1-\theta}\right)^m \|u_1 - \bar{u}\|.$$

Since $u_0(k) = v_0(k) = \bar{u}(k)$, the above inequality provides

$$\|u_{m+1} - v_{m+1}\| \le \sum_{i=0}^{m} \theta^{m-i}\left[w_i + \theta\left(\frac{2\theta}{1-\theta}\right)^i \|u_1 - \bar{u}\|\right]. \tag{10.16}$$

Using (10.16) in the triangle inequality, we obtain

$$\|v_{m+1} - u^*\| \le \|u_{m+1} - u^*\| + \sum_{i=0}^{m} \theta^{m-i}\left[w_i + \theta\left(\frac{2\theta}{1-\theta}\right)^i \|u_1 - \bar{u}\|\right]. \tag{10.17}$$

In (10.17), Theorem 10.1 ensures that $\lim_{m\to\infty} \|u_{m+1} - u^*\| = 0$. Thus, from the Toeplitz lemma, $\lim_{m\to\infty} \|v_{m+1} - u^*\| = 0$ if and only if $\lim_{m\to\infty} [w_m + \theta\,(2\theta/1-\theta)^m\,\|u_1 - \bar{u}\|] = 0$. However, $\lim_{m\to\infty} (2\theta/1-\theta)^m = 0$, and hence, if and only if $\lim_{m\to\infty} w_m = 0$.

Finally, to prove (10.13) we note that

$$\begin{aligned}u^*(k) - v_{m+1}(k) = \sum_{\ell=a}^{b-1} g(k,\ell) &\left[f(\ell, u^*(\ell),\dots) - f(\ell, v_m(\ell),\dots)\right.\\ &+ f(\ell, v_m(\ell),\dots) - f_m(\ell, v_m(\ell),\dots)\\ &\left.- \sum_{i=0}^{n-1} (\Delta^i v_{m+1}(\ell) - \Delta^i v_m(\ell))\,\frac{\partial}{\partial \Delta^i v_m(\ell)} f_m(\ell, v_m(\ell),\dots)\right],\end{aligned}$$

and hence,

$$\begin{aligned}\|u^* - v_{m+1}\| &\le \theta\,\|u^* - v_m\| + \theta\,\|v_{m+1} - v_m\| + \nu \max_{N(a,b-1)} |f(k, v_m(k),\dots)|\\ &\le 2\theta\,\|v_{m+1} - v_m\| + \nu \max_{N(a,b-1)} |f(k, v_m(k),\dots)| + \theta\,\|u^* - v_{m+1}\|,\end{aligned}$$

which is the same as (10.13). ∎

THEOREM 10.4. *Let the conditions of Theorem 10.3 be satisfied. Further, let $f_m = f_0$ for all $m = 1, 2, \dots$ and $f_0(k, u_0, u_1, \dots, u_{n-1})$ be continuously twice differentiable with respect to all u_i, $0 \le i \le n-1$ on $N(a, b-1) \times D_3$ and*

$$\left|\frac{\partial^2}{\partial u_i\,\partial u_j} f_0(k, u_0, u_1, \dots, u_{n-1})\right| \le L_i L_j \tau, \qquad 0 \le i,\ j \le n-1.$$

Then, the following hold

$$\begin{aligned}\|v_{m+1} - v_m\| \le \alpha \|v_m - v_{m-1}\|^2 &\le \frac{1}{\alpha}\left(\alpha\|v_1 - v_0\|\right)^{2^m} \\ &\le \frac{1}{\alpha}\left[\frac{1}{2}\tau(\epsilon + \delta + \nu F)\left(\frac{\theta}{1-\theta}\right)^2\right]^{2^m},\end{aligned} \tag{10.18}$$

where α is the same as in Theorem 10.2.

PROOF. As in the proof of Theorem 10.2, we have

$$\begin{aligned} v_{m+1}(k) - v_m(k) = \sum_{\ell=a}^{b-1} g(k,\ell) &\left\{ \sum_{i=0}^{n-1} \left(\Delta^i v_{m+1}(\ell) - \Delta^i v_m(\ell)\right) \frac{\partial}{\partial \Delta^i v_m(\ell)} f_0(\ell, v_m(\ell), \dots) \right. \\ &\left. + \frac{1}{2}\left[\sum_{i=0}^{n-1} \left(\Delta^i v_m(\ell) - \Delta^i v_{m-1}(\ell)\right) \frac{\partial}{\partial p_i(\ell)}\right]^2 f_0(\ell, p_0(\ell), p_1(\ell), \dots, p_{n-1}(\ell)) \right\}, \end{aligned}$$

where $p_i(k)$ lies between $\Delta^i v_{m-1}(k)$ and $\Delta^i v_m(k)$, $0 \le i \le n-1$.

Thus, as earlier, we get

$$\|v_{m+1} - v_m\| \le \theta\|v_{m+1} - v_m\| + \frac{1}{2}\tau\theta^2\|v_m - v_{m-1}\|^2,$$

which is the same as the first part of (10.18). The last part of (10.18) follows from (10.14). ∎

11. MONOTONE CONVERGENCE

In Sections 9 and 10, we have respectively discussed the linear and quadratic convergence of Picard's and Newton's iterative methods. However, from the computational point of view, monotone convergence has superiority over ordinary convergence [10,33,34]. Therefore, here we shall provide sufficient conditions for the monotone convergence of Picard's iterative method. For the boundary value problem (8.2), (3.24), we need to consider the following four cases: (i) n is even, p is odd; (ii) n is even, p is even; (iii) n is odd, p is odd; (iv) n is odd, p is even. We shall consider only the case (i), whereas results for the other three cases can be stated analogously.

In the space $S(a, b-1+n)$, we introduce the partial ordering $\le_P$ as follows: for $u, v \in S(a, b-1+n)$ we say that $u \le_P v$ if and only if $\Delta^i u(k) \le \Delta^i v(k)$, $k \in N(a, b-1+n-i)$, $i \in J_1 = \{j : 0 \le j \le p\} \cup \{j : p < j\,(\text{odd}) \le n-1\}$, and $\Delta^i u(k) \ge \Delta^i v(k)$, $k \in N(a, b-1+n-i)$, $i \in J_2 = \{j : p < j\,(\text{even}) \le n-1\}$. Thus, from Theorem 5.4, $\Delta^i g(k,\ell) \le 0$, $(k,\ell) \in N(a, b-1+n-i) \times N(a, b-1)$, $i \in J_1$, and $\Delta^i g(k,\ell) \ge 0$, $(k,\ell) \in N(a, b-1+n-i) \times N(a, b-1)$, $i \in J_2$.

THEOREM 11.1. *With respect to the boundary value problem (8.2), (3.24), we assume that n is even, p is odd, and*

(i) *$f(k, u_0, u_1, \dots, u_{n-1})$ is continuous on $N(a, b-1) \times \mathbb{R}^n$, and nonincreasing in u_i for all $i \in J_1$ and nondecreasing in u_i for all $i \in J_2$;*

(ii) *there exist functions $v_0(k)$ and $w_0(k)$ in the Banach space $S(a, b-1+n)$ (with the norm $\|u\| = \max\{\|\Delta^j u(k)\| = \max_{N(a,b-1+n-i)} |\Delta^i u(k)|,\ 0 \le i \le n-1\}$) such that*

$$v_0 \le_P w_0, \tag{11.1}$$

$$\Delta^n w_0(k) - f\left(k, w_0(k), \Delta w_0(k), \dots, \Delta^{n-1} w_0(k)\right) \le 0 \le$$
$$\Delta^n v_0(k) - f\left(k, v_0(k), \Delta v_0(k), \dots, \Delta^{n-1} v_0(k)\right), \qquad k \in N(a, b-1), \tag{11.2}$$
$$P_{n-1,v_0} \le P_{n-1} \le_P P_{n-1,w_0}, \tag{11.3}$$

where $P_{n-1}(k)$ is defined in (3.25), and $P_{n-1,v_0}(k)$ and $P_{n-1,w_0}(k)$ are the polynomials of degree $n-1$ satisfying

$$\begin{aligned} \Delta^i P_{n-1,v_0}(a) &= \Delta^i v_0(a), \qquad 0 \le i \le p-1, \\ \Delta^i P_{n-1,v_0}(b) &= \Delta^i v_0(b), \qquad p \le i \le n-1, \end{aligned} \tag{11.4}$$

and

$$\begin{aligned} \Delta^i P_{n-1,w_0}(a) &= \Delta^i w_0(a), \qquad 0 \le i \le p-1, \\ \Delta^i P_{n-1,w_0}(b) &= \Delta^i w_0(b), \qquad p \le i \le n-1, \end{aligned} \tag{11.5}$$

respectively.

Then, the sequences $\{v_m\}$, $\{w_m\}$ where $v_m(k)$ and $w_m(k)$ are defined by the iterative schemes

$$v_{m+1}(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) f\left(\ell, v_m(\ell), \Delta v_m(\ell), \dots, \Delta^{n-1} v_m(\ell)\right),$$
$$w_{m+1}(k) = P_{n-1}(k) + \sum_{\ell=a}^{b-1} g(k,\ell) f\left(\ell, w_m(\ell), \Delta w_m(\ell), \dots, \Delta^{n-1} w_m(\ell)\right), \qquad m = 0, 1, \dots,$$

converge in $S(a, b-1+n)$ to the solutions $v(k)$ and $w(k)$ of (8.2), (3.24). Further,

$$v_0 \le_P v_1 \le_P \cdots \le_P v_m \le_P \cdots \le_P v \le_P w \le_P \cdots \le_P w_m \le_P \cdots \le_P w_1 \le_P w_0.$$

Also, each solution $z(k)$ of this problem which is such that $v_0 \le_P z \le_P w_0$ satisfies $v \le_P z \le_P w$.

Proof. Let $S(v_0, w_0) = \{u \in S(a, b-1+n) : v_0 \le_P u \le_P w_0\}$. Obviously, $S(v_0, w_0)$ is a closed convex subset of the Banach space $S(a, b-1+n)$. We shall show that the continuous operator $T : S(a, b-1+n) \longrightarrow S(a, b-1+n)$ defined in (8.6) maps $S(v_0, w_0)$ into itself.

Suppose $u, v \in S(v_0, w_0)$ and $u \le_P v$. Then, in view of the partial ordering $\le_P$, the sign properties of the Green's function $g(k,\ell)$, and the monotonic nature of the function f, we have

$$\Delta^i g(k,\ell) f\left(\ell, u(\ell), \Delta u(\ell), \dots, \Delta^{n-1} u(\ell)\right) \le \Delta^i g(k,\ell) f\left(\ell, v(\ell), \Delta v(\ell), \dots, \Delta^{n-1} v(\ell)\right),$$
$$(k,\ell) \in N(a, b-1+n-i) \times N(a, b-1), \quad i \in J_1,$$

and

$$\Delta^i g(k,\ell) f\left(\ell, u(\ell), \Delta u(\ell), \dots, \Delta^{n-1} u(\ell)\right) \ge \Delta^i g(k,\ell) f\left(\ell, v(\ell), \Delta v(\ell), \dots, \Delta^{n-1} v(\ell)\right),$$
$$(k,\ell) \in N(a, b-1+n-i) \times N(a, b-1), \quad i \in J_2.$$

From these inequalities, $Tu \le_P Tv$ is obvious. Thus, the operator T is monotone in $S(v_0, w_0)$ with respect to $\le_P$.

We shall now show that $v_0 \le_P Tv_0$ and $Tw_0 \le_P w_0$, and then it will follow that T maps $S(v_0, w_0)$ into itself. For this, we note that

$$\Delta^i v_0(k) = \Delta^i P_{n-1,v_0}(k) + \sum_{\ell=a}^{b-1} \Delta^i g(k,\ell)\, \Delta^n v_0(\ell),$$

and hence, if $i \in J_1$, then

$$\begin{aligned} \Delta^i v_0(k) &\le \Delta^i P_{n-1}(k) + \sum_{\ell=a}^{b-1} \Delta^i g(k,\ell) f\left(\ell, v_0(\ell), \Delta v_0(\ell), \dots, \Delta^{n-1} v_0(\ell)\right) \\ &= \Delta^i T v_0(k), \end{aligned}$$

and similarly, if $i \in J_2$, then

$$\Delta^i v_0(k) \ge \Delta^i T v_0(k).$$

This completes the proof of $v_0 \le_P Tv_0$. The inequality $Tw_0 \le w_0$ can be proved analogously.

The existence of a fixed point u of T in $S(v_0, w_0)$ now follows as an application of the Schauder fixed point theorem. The conclusions of the theorem are now immediate from the established monotone property of the operator T in $S(v_0, w_0)$ with respect to the partial ordering $\le_P$. ∎

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