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Local behavior of Oromion curve with application to Lamé curve

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Contents

1. Introduction

In this paper, we study the local behavior between Oromion curve, Lamé curve, and the Gamma function. In the real case, (in the next research will study the complex case with the notion of foliation) will show that the area of positive measure for the closed Oromion curve of length L are given in terms of the Gamma function $\Gamma(z)$.

For a give Oromion curve, we ask when it has the structure of an algebraic curve. Such map is called *Oromion curve.*

In Section 1, we give a definition of Oromion curve and their formulas. The codimension are defined in Section 2. With these preparations, in Section 2 we prove that area of Oromion curve have positive measure.

In Section 3, we review the definitions about singularities and will find the linear differential equation whose solution is the Oromion curve for certain boundary conditions as well as its stability.

Finally, in Section 4, we give give some examples in the real life, in particular, in the art.

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Are these quotations necessary?

2. Background material

In this section, we present some familiar concepts concerning the Oromion curve, and the Lamé curve.

3. Algebraic curve

An algebraic curve is an infinite points and ....

We are particularly interested in an **integral transform**, where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s,t) f(t)\, dt$ is defined as a limit

$$\int_0^\infty K(s,t) f(t)\, dt = \lim_{b \to \infty} \int_0^b K(s,t) f(t)\, dt \tag{1}$$

If the limit in 1 exists, then we say that the integral exists or is **convergent;** if the limit does not exist, the integral does not exist and is **divergent.** The limit in 1 will, in general, exist for only certain values of the variable $s \in \mathbb{R}$.

The function $K(s,t)$ in 1 is called the **kernel** of the transform. The choice $K(s,t) = e^{-st}$ as the kernel gives us an especially important integral transform.

Definition 3.1 (Laplace Transform). Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{2}$$

is said to be the **Laplace transform** of f, provided that the integral converges.

Definition 3.2 (Gamma function). Euler's integral definition of the gamma function is

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx. \tag{3}$$

convergence of the integral requires that $x > 0$.

The recurrence relation

$$\Gamma(z+1) = z \cdot \Gamma(z), \tag{4}$$

4 can be obtained from 3 with integration by parts, see also [8].

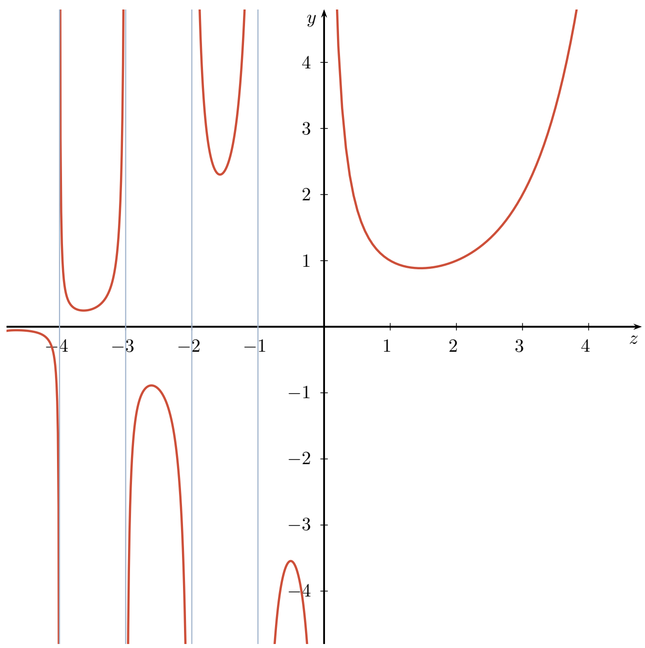

Figure 1. Graph of $\Gamma(z)\ \forall z \in \mathbb{R} - \mathbb{Z}_0^-$

Now when $z = 1, \Gamma(1) = \displaystyle\int_0^\infty e^{-x}\,dx = 1$, and thus 4 gives

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1$$
$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1$$
$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1,$$

and so on. In this manner, it is seen that when $n \in \mathbb{Z}^+$, $\Gamma(n+1) = n!$ For this reason, the gamma function is often called the **generalized factorial function**.

We used the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. This result can be derived from 3 by setting $z = \frac{1}{2}$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\,dx. \tag{5}$$

When we let $t = u^2$, 5 can be written as $\Gamma\left(\frac{1}{2}\right) = 2\displaystyle\int_0^\infty e^{-u^2}\,du$. But $\int_0^\infty e^{-u^2}\,du = \int_0^\infty e^{-v^2}\,dv$, so

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left(2\int_0^\infty e^{-u^2}\,du\right)\left(2\int_0^\infty e^{-v^2}\,dv\right) = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv.$$

Switching to polar coordinates $u = r\cos\theta, v = r\sin\theta$ enables us to evaluate the double integral:

$$4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\,du\,dv = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,dr\,d\theta = \pi.$$

Hence,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \qquad \text{or} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

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