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On codimension one foliations with prescribed cuspidal separatrix

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Abstract

In this paper, we will construct a pre-normal form for germs of codimension one holomorphic foliation having a particular type of separatrix, of cuspidal type. As an application, we will explain how this form could be used in order to study the analytic classification of the singularities via the projective holonomy, in the generalized surface case. We will also give an application to the analytic classification of singularities, and a sufficient condition, in the quasi-homogeneous, three-dimensional case, for these foliations to be of generalized surface type.

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1. Introduction and statement of the results

The objective of this paper is to find a (pre)-normal form for holomorphic differential 1-forms defining local holomorphic foliations in $(\mathbb{C}^{n+1}, \mathbf{0})$, with a fixed separatrix of a certain particular type, that we will describe later, and with a condition on the multiplicity at the origin. As an application, we will use this pre-normal form to describe a Hopf fibration, constructed after reduction of the singularities, transverse to the foliation away from the separatrices, which turns out to be a useful tool to study the analytic classification of the singularities of the foliations via the projective holonomy. As a side result, a condition in dimension three for some of these forms, that will be called cuspidal, in order that they are of generalized surface type, is obtained.

The search of normal forms for germs of holomorphic vector fields, or of holomorphic 1-forms, is carried out in several papers, either treating directly this problem, or as a need for treating another problems, such as the formal, analytic or topological classification of these objects. It turns out to be a useful tool in the study of the singularities of these objects. Let us briefly define the main concepts that we will use throughout the paper, and give an account of some of the main achievements in dimensions two and three, focusing in the case of codimension one foliations.

A germ of codimension one holomorphic foliation $\mathcal{F}$ in $(\mathbb{C}^n, \mathbf{0})$ is defined by a holomorphic 1-form ω, satisfying the Frobenius integrability condition $\omega \wedge d\omega = 0$, and such that its coefficients have no common factors. If $\omega = \sum_{i=1}^{n} a_i(\mathbf{x})\,dx_i$, the *singular set* of ω, $\mathrm{Sing}(\omega)$, is the germ of analytic set defined by the zeros of the ideal $(a_i(\mathbf{x})_{1 \leqslant i \leqslant n})$. Previous condition implies that $\mathrm{Sing}(\omega)$ has codimension at least two.

We will also consider throughout the paper meromorphic integrable 1-forms: these are integrable 1-forms $\omega = \sum_{i=1}^{n} a_i(\mathbf{x})\,dx_i$, where $a_i(\mathbf{x})$ are germs of meromorphic functions. Consider an $(n-1)$-dimensional germ of analytic set, S, defined by an equation $(f = 0)$, with $f \in \mathcal{O}$ reduced (here and throughout the paper, $\mathcal{O} = \mathcal{O}_n$ will denote the ring $\mathbb{C}\{\mathbf{x}\}$ of convergent power series in n variables, where n is usually omitted as there is no risk of confusion).

A meromorphic 1-form is called *logarithmic* along S if $f\omega$ and $f\,d\omega$ are holomorphic forms, or equivalently if $f\omega$ and $df \wedge \omega$ are holomorphic. The following result follows:

Proposition 1.1. *(See [15,31].) Let ω be a holomorphic 1-form, and f as before. The following conditions are equivalent:*

(1) *There exists a holomorphic 2-form η such that $\omega \wedge df = f\eta$.*
(2) *ω/f is logarithmic along S.*
(3) *There exist $g, h \in \mathcal{O}$, and a holomorphic 1-form α, such that $g\omega + h\,df = f\alpha$, and moreover, g, f have no common factors.*

The equivalence between (1) and (3) can be read in [15], assuming irreducibility of f. The result is stated in [31] in a more general context (q-forms), without the irreducibility assumption in the statement, but the proof provided there is only valid in the irreducible case. Nevertheless, a modification of that proof is enough to establish the result in the non-irreducible case.

If $\mathcal{F}$ is a holomorphic foliation defined by a 1-form ω satisfying the conditions of the previous theorem, we will say that S (or $f = 0$) is a separatrix for $\mathcal{F}$. Let us observe that, if $\Omega^1(\log S)$ represents the set of meromorphic 1-forms having f as separatrix (i.e. logarithmic along S), $\Omega^1(\log S)$ has a structure of $\mathcal{O}$-module. Under some condition, this $\mathcal{O}$-module is free. In fact, K. Saito proves in [31] the following result:

Proposition 1.2. *$\Omega^1(\log S)$ is a free $\mathcal{O}$-module if and only if there exist n 1-forms $\omega_1, \omega_2, \dots, \omega_n \in \Omega^1(\log S)$ such that $\omega_1 \wedge \cdots \wedge \omega_n = U(\mathbf{x})\frac{dx_1 \wedge \cdots \wedge dx_n}{f}$, where $U(\mathbf{x})$ is a unit in $\mathcal{O}$.*

When this condition is fulfilled, and $\omega_1, \omega_2, \dots, \omega_n$ are found, it is easy to represent the elements of $\Omega^1(\log S)$.

Example 1.3. Let $f = y^2 + x^n$, $\omega_1 = \frac{df}{f}$, $\omega_2 = \frac{1}{f}(2x\,dy - ny\,dx)$, $\omega_1 \wedge \omega_2 = \frac{2n}{f}\,dx \wedge dy$. So, $\{\omega_1, \omega_2\}$ is a basis of the free $\mathcal{O}$-module $\Omega^1(\log S)$.

Proposition 1.2 is somewhat related with our problem (it allows to write every 1-form having S as a separatrix in terms of a basis), but it is not of immediate application because, for instance, it does not take into account the integrability condition if $n \geqslant 3$. So, we will need to use different techniques. Firstly, we will recall first some notions and known results in dimensions two and three.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two germs of holomorphic foliations in $(\mathbb{C}^n, \mathbf{0})$, represented by integrable 1-forms ω_1, ω_2 respectively. Following [20,21] (where the dimension of the ambient space is two), we will say that $\mathcal{F}_1$, $\mathcal{F}_2$ are analytically equivalent if there exists a biholomorphism $\Phi : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $\Phi^*\omega_1 \wedge \omega_2 = 0$, or equivalently, if there exists a unit U such that $\Phi^*\omega_1 = U \cdot \omega_2$. If we only impose Φ to be formal (i.e., $\Phi : \mathbb{C}[\![\mathbf{x}]\!]^n \to \mathbb{C}[\![\mathbf{x}]\!]^n$ invertible), we will say that $\mathcal{F}_1$, $\mathcal{F}_2$ are formally equivalent.

Simple (or *reduced*) singularities are defined, for instance, in [22] (dimension two) and in [5,7] in dimension three and higher. The analytic classification of simple singularities (saddle-node and resonant cases) is the main objective of the papers of J. Martinet and J.-P. Ramis [20,21]. For non-simple singularities, let us focus, in dimension two, in the nilpotent case, i.e., defined by a 1-form such that the dual vector field has a nilpotent, non-zero, linear part, i.e., $\omega = y\,dy + \cdots$. F. Takens [33] shows that such a foliation is formally equivalent to $\omega_{n,p} = d(y^2 + x^n) + x^p U(x)\,dy$, for some integers $n \geqslant 3$, $p \geqslant 2$, and $U(x) \in \mathbb{C}[\![x]\!]$, $U(0) \neq 0$. These integers are not uniquely determined: nevertheless the conditions $2p > n$, $2p = n$, $2p < n$ are preserved under conjugation. If $2p > n$, the foliation has an invariant curve analytically equivalent to the cusp $y^2 + x^n = 0$ (for this reason these foliations are usually called *cuspidal*). In the study of these foliations, it is interesting to write down explicitly the foliation having $y^2 + x^n = 0$ as a separatrix. Following Proposition 1.2, such a foliation can be defined by a form $a(x, y)d(y^2 + x^n) + b(x, y)(ny\,dx - 2x\,dy)$. The condition about the order at the origin implies that $a(x, y)$ is a unit, so $\omega = d(y^2 + x^n) + A(x, y)(ny\,dx - 2x\,dy)$ defines such a foliation, as is stated in [11]. This form is very useful in order to study the analytic classification, via the projective holonomy, of cuspidal foliations (see [11,18,24,1,32,25]).

In the three-dimensional case, the study of the analytic classification of simple singularities of foliations is done in [9] and [12]. For general singularities, different results are available in the literature. Let us sketch some of them:

(1) If the codimension of the singular locus is at least three, B. Malgrange shows [19] that a holomorphic first integral always exists. The study reduces to the study of germs of holomorphic functions.
(2) If a transversal section (to be defined in Section 2) $i : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ exists, such that $i^*\omega$ is simple, then, either the foliation has a first integral, or it is a cylinder over a two-dimensional simple foliation [10].

(3) If the linear part is not zero, and non-tangent to the radial vector field, the foliation turns out to be a pull-back of a two-dimensional foliation, via a particular holomorphic map: this is the Preparation Theorem of F. Loray [17].

We will focus in a case that fits in (3) above. We are interested in a convenient generalization of cuspidal foliations, namely, foliations having $z^2 + \varphi(x, y) = 0$ as its separatrix. These surfaces are a particular case of the sometimes called *Zariski surfaces*, of equations $z^k + \varphi(x, y) = 0$, which, from different points of view, have been largely studied by several authors (see, for instance, [27] or [23]). The analytic classification in the *quasi-ordinary* case, i.e., when $\varphi(x, y) = x^p y^q$ (the discriminant of the projection over the plane (x, y) has normal crossings), is studied in detail in [13], under some additional assumptions. In that paper, a precise expression of foliations having $z^2 + x^p y^q = 0$ as a separatrix is needed (Proposition 1 from [13]), and it is computed directly.

In this paper, we will describe a pre-normal form for singularities of codimension one holomorphic foliations in higher dimension, of cuspidal type. By *cuspidal* we will understand non-dicritical, generalized hypersurfaces (see Section 2 for precise definitions), having $z^2 + \varphi(\mathbf{x}) = 0$ as a separatrix. The main result of the paper is the following:

Theorem 1.4. *Consider a cuspidal foliation $\mathcal{F}$ on $(\mathbb{C}^{n+1}, \mathbf{0})$, with separatrix defined by the equation $f = z^2 + \varphi(\mathbf{x}) = 0$. Then, there exist coordinates such that a generator of $\mathcal{F}$ is*

$$\omega = d\big(z^2 + \varphi'(\mathbf{x})\big) + G\big(\Psi(\mathbf{x}), z\big) \cdot \big(z \cdot \Psi(\mathbf{x})\big)\left(2\frac{dz}{z} - \frac{d\varphi'}{\varphi'}\right),$$

where $\Psi(\mathbf{x})$ is a germ of analytic function that is not a power, $\varphi'(\mathbf{x}) = \Psi(\mathbf{x})^r$ for some $r \in \mathbb{N}$, $\varphi(\mathbf{x})$ differs from $\varphi'(\mathbf{x})$ in some unit $u(\mathbf{x})$ (i.e., $\varphi'(\mathbf{x}) = u(\mathbf{x}) \cdot \varphi(\mathbf{x})$, $u(\mathbf{0}) \neq 0$), and G is a germ of holomorphic function in two variables.

In the quasi-homogeneous case, φ' can be replaced by φ in the statement of the theorem. This result can be seen as a variant of Saito's result (Proposition 1.2), but now integrability condition is taken into account. In fact, it would be interesting (and we think that it is an open problem) to characterize all hypersurfaces $f = 0$ such that the set of integrable 1-forms having f as a separatrix is a $\mathcal{O}$-free module.

The proof is based in direct computation, and integrability condition will be introduced geometrically via the mentioned Preparation Theorem of F. Loray. In some sense, we make explicit this two-dimensional form (in the particular case we are studying), putting into evidence the separatrix.

The structure of the paper will be as follows. In Section 2, for the sake of completeness and in order to help the reader, some preliminaries about singular holomorphic foliations are presented. The proof of Theorem 1.4 is done in Section 3, and the passage to the quasi-homogeneous case is the main objective of Section 4. Finally, in Section 5, an application in order to construct a Hopf fibration that allows to extend the analytic conjugation of the projective holonomy to an analytic conjugation of the foliations, is presented. It is worth to mention here that this paper is related with one of the problems mentioned in the list of *Open* *questions and related problems* proposed by F. Cano and D. Cerveau in [7], which is stated as follows:

(5) Classify the non-dicritical singular foliations in $(\mathbb{C}^3, \mathbf{0})$ generated by one 1-form with initial part of the type $x\,dx$.

2. Generalized hypersurfaces

The foliations we study in this paper will be of the generalized surface, non-dicritical type, as defined in [14]. Let us briefly precise some of these notions, generalizing them, when possible, to the n-dimensional case.

In dimension two, a germ of foliation is called *dicritical* if the following two equivalent conditions are satisfied:

(1) The foliation has an infinite number of separatrices.
(2) In the reduction of the singularities, a component of the divisor, transversal to the foliation, appears.

These two conditions are no longer equivalent in higher dimensions, as it can be seen in [7]. In particular, if the second condition is satisfied, it does not imply the existence of a separatrix. A deeper study of the dicriticalness has been done in several papers as in [4] or [3]. Let us adopt, following the survey [6], the following definition:

Definition 2.1. A germ of codimension one foliation $\mathcal{F}$, defined by a 1-form ω is called *dicritical* if there exists $\varphi : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $\varphi^*\mathcal{F} = \{dx = 0\}$, and that $\varphi(\{y = 0\})$ is invariant by $\mathcal{F}$.

Let us observe that, in dimension two, this condition is equivalent to the two equivalent conditions above, thanks to the reduction of the singularities. In dimension three, this definition is equivalent to the existence of a component of the exceptional divisor, generically transverse to the foliation, in every reduction of the singularities [7,6]. In particular, this condition does not depend on the particular reduction of the singularities chosen.

Recall now that, in dimension two, [2] define a generalized curve as a germ of holomorphic foliation such that, in its reduction of the singularities, no saddle-node appears. We will use the term generalized curve for non-dicritical, generalized curves in the above sense.

Let us generalize this notion to higher dimensions. In dimension three, where reduction of the singularities of codimension one, holomorphic foliations, is available, a non-dicritical germ of holomorphic foliation is called a *generalized surface* [14] if no saddle-nodes appear in the reduction of its singularities. This condition is independent of the chosen reduction, and it is equivalent to the following two conditions to be simultaneously verified:

(1) $\mathcal{F}$ is non-dicritical, according to the previous definition.
(2) For every $\varphi : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that the foliation $\varphi^*\mathcal{F}$ is defined (i.e., if ω is a 1-form defining $\mathcal{F}$, then $\varphi^*\omega \not\equiv 0$), then $\varphi^*\mathcal{F}$ is a generalized curve.

Remark 2.2. Foliations satisfying condition (2) above are called *complex hyperbolic* (CH), for instance in [28].

This motivates the extension of this definition to higher dimensions, even in the absence of a reduction of the singularities theorem. For completeness, let us explicit the definition of this notion in dimension n.

Definition 2.3. Let $\mathcal{F}$ be a germ of non-dicritical foliation in $(\mathbb{C}^n, \mathbf{0})$, generated by an integrable 1-form ω. $\mathcal{F}$ is a *generalized hypersurface* if every generically transverse plane section $\varphi : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ (i.e., such that $\varphi^*\omega \not\equiv 0$) defines a generalized curve.

In [22], Mattei and Moussu define a transverse section $i : (\mathbb{C}^l, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ (with a slight modification in order to cover the regular case) as an inmersion satisfying:

(1) $\operatorname{Sing}(i^*\omega) = i^{-1}(\operatorname{Sing}\omega)$.
(2) $\operatorname{codim}(\operatorname{Sing} i^*\omega) = \min(\operatorname{codim}(\operatorname{Sing}\omega), l - \min(0, \dim(\operatorname{Sing}(\omega))))$.

If $l = 2$, it implies that $\operatorname{Sing}(\varphi^*\omega) \subseteq \{\mathbf{0}\}$. Transverse sections in Mattei and Moussu sense (MM-transverse) always exist, and in the non-dicritical case, they are generic among linear inmersions. MM-transverse sections verify that $\nu_0(\mathcal{F}) = \nu_0(i^*\mathcal{F})$, ν_0 denoting the multiplicity at the origin.

In particular, this means that if $f = 0$ is an equation of the set of separatrices of $\mathcal{F}$, then $\nu_0(df) = \nu_0(\omega)$. In fact, results from [7,8] show that every separatrix of $i^*\mathcal{F}$ extends to a separatrix of $\mathcal{F}$, and moreover we have that $\nu_0(\omega) = \nu_0(i^*\mathcal{F}) = \nu_0(d(f \circ i)) = \nu_0(df)$.

As for generalized surfaces in the three-dimensional case, the following result can be proved for generalized hypersurfaces.

Theorem 2.4. *Let $\mathcal{F}$ be a generalized hypersurface and $f = 0$ an equation of its set of separatrices. If π is an inmerse reduction of the singularities of f, then $\pi^*\mathcal{F}$ has only simple singularities.*

Remark 2.5. Recall that in any dimension, the notion of simple singularity has been defined, for instance in [5].

As in smaller dimensions, the key point is the following lemma:

Lemma 2.6. *If a generalized hypersurface has exactly n smooth and transversal separatrices through a singularity P, then it is simple.*

Proof. In fact, take a transversal section i through P, such that $i^*\mathcal{F}$ has n smooth transversal curves as separatrices, so $\nu_0(i^*\omega) = n - 1 = \nu_0(\omega)$. If the foliation is generated by a 1-form $\omega = x_1 x_2 \cdots x_n \sum_{i=1}^{n} a_i(\mathbf{x}) \dfrac{dx_i}{x_i}$, this implies that some $a_i(\mathbf{0}) \neq 0$, so the singularity is pre-simple. A pre-simple generalized hypersurface, that is a corner, i.e. that has exactly n smooth, normal crossing invariant hypersurfaces, is simple, according to [5]. In fact, the non-dicriticalness condition implies the absence of resonances, and so, the singularities become simple. □

3. Proof of Theorem 1.4

We will begin establishing the following lemma, in a more general setting that the main result of the paper:

Lemma 3.1. *Let ω be a germ of 1-form in $(\mathbb{C}^{n+1}, \mathbf{0})$, logarithmic along S, hypersurface defined by $f = z^k + \varphi(\mathbf{x})$, with $\nu_0(\varphi) \geqslant k$, and such that $\nu_0(\omega) = \nu_0(df) = k - 1$. Then, there exist a unit $U(\mathbf{x}, z) \in \mathbb{C}\{\mathbf{x}, z\}$ and a germ $H(\mathbf{x}, z) \in \mathbb{C}\{\mathbf{x}, z\}$, such that $U(\mathbf{x}, z) \cdot \omega = \omega_1 + H(\mathbf{x}, z) \cdot \omega_2 + (z^k + \varphi) \cdot \omega_3$, where,*

$$\omega_1 = d\big(z^k + \varphi\big),$$
$$\omega_2 = z\, d\varphi - k\varphi\, dz,$$
$$\omega_3 = \sum_{i=2}^{n} g_i\, dx_i.$$

Proof. Denote $\omega = \sum_{i=1}^{n} A_i\, dx_i + A\, dz$. As $\omega, \omega_1, \omega_2$ are logarithmic along S, we have that the coefficients of $\omega \wedge \omega_1$, $\omega \wedge \omega_2$ are divisible by $z^k + \varphi$. In particular, considering the coefficients of $dx_i \wedge dz$, we have

$$kz^{k-1} A_i - \varphi_{x_i} A = \big(z^k + \varphi\big) \cdot H_i,$$
$$-k\varphi A_i - z\varphi_{x_i} \cdot A = \big(z^k + \varphi\big) \cdot G_i,$$

for some H_i, G_i, so

$$-kA_i = -zH_i + G_i,$$
$$-\varphi_{x_i} A = \varphi H_i + z^{k-1} G_i.$$

We have that

$$\begin{aligned}
\omega &= \sum_{i=1}^{n} \frac{1}{k} \cdot (zH_i - G_i)\, dx_i - \frac{1}{\varphi_{x_1}} \cdot \big(\varphi H_1 + z^{k-1} G_1\big)\, dz \\
&= \sum_{i=1}^{n} \left[z \cdot \frac{\varphi_{x_i}(\varphi H_1 + z^{k-1} G_1) - \varphi_{x_1} z^{k-1} G_i}{k \cdot \varphi \cdot \varphi_{x_1}} - \frac{G_i}{k} \right] dx_i - \frac{\varphi H_1 + z^{k-1} G_1}{\varphi_{x_1}}\, dz \\
&= H_1 \cdot \frac{1}{k\varphi_{x_1}} \cdot \omega_2 - G_1 \cdot \frac{1}{k} \cdot \frac{1}{\varphi_{x_1}} \cdot \omega_1 + \big(z^k + \varphi\big) \cdot \frac{1}{k\varphi} \sum_{i=2}^{n} \frac{G_1 \varphi_{x_i} - G_i \varphi_{x_1}}{\varphi_{x_1}}\, dx_i.
\end{aligned}$$

As $\nu_0(\omega) = k - 1$, necessarily $\frac{G_1}{\varphi_{x_1}}$ must be a unit. So,

$$k\frac{\varphi_{x_1}}{G_1} \omega = \omega_1 - \frac{H_1}{G_1} \cdot \omega_2 - \big(z^k + \varphi\big) \frac{1}{\varphi G_1} \sum_{i=2}^{n} \frac{G_1 \varphi_{x_i} - G_i \varphi_{x_1}}{\varphi_{x_1}}\, dx_i,$$

which turns out to be holomorphic, as ω is. So, we obtain the stated result. □

Let us start the proof of Theorem 1.4, that concerns the case $k = 2$. Consider a germ of holomorphic foliation in $(\mathbb{C}^{n+1}, \mathbf{0})$ generated by an integrable 1-form $\omega = \omega_1 + H \cdot \omega_2 + (z^2 + \varphi) \cdot \omega_3$, as in Lemma 3.1. The linear part of ω is $2z\,dz + \sum_{i=1}^{n} \frac{\partial \varphi_2}{\partial x_i}\,dx_i$, where φ_2 is the homogeneous part of degree two of φ (which is zero if the order of φ is strictly greater than two), that is not tangent to the radial vector field. Applying Theorem 1 and Corollary 3 of [17], there exists a fibered change of variables $\Phi_1(\mathbf{x}, z) = (x_1, x_2, \dots, x_n, \varphi_1(\mathbf{x}, z))$, such that $\Phi_1^* \mathcal{F}$ is generated by $\omega' = z\,dz + df_0(\mathbf{x}) + z\,df_1(\mathbf{x})$, for certain germs of functions $f_0, f_1 \in \mathbb{C}\{\mathbf{x}\}$. A second change of variables transforms ω' in $\omega'' = d(\frac{z^2}{2} + f_0(\mathbf{x})) - f_1(\mathbf{x})\,dz$. Start, now, up to a constant, with a form $\omega'' = d(z^2 + f_0(\mathbf{x})) - f_1(\mathbf{x})\,dz$. The integrability condition reads as $df_0 \wedge df_1 = 0$. As stated in [17] this means, by [22], that there exist $f(\mathbf{x})$, $h_0(t)$, $h_1(t)$, such that $f_0(\mathbf{x}) = h_0(f(\mathbf{x}))$, $f_1(\mathbf{x}) = h_1(f(\mathbf{x}))$. So, if $\rho(\mathbf{x}, z) = (f(\mathbf{x}), z)$, then $\omega'' = \rho^* \omega_0$, with $\omega_0 := d(z^2 + h_0(t)) - h_1(t)\,dz$. Let $r := \nu_0(h_0)$. The two-dimensional foliation defined by ω_0 has $z^2 + t^r + \text{h.o.t.}$ as a separatrix, that is, up to a unit, equal to $z^2 + a(t)z + b(t) = 0$. A change of variables $z \mapsto z - \frac{a(t)}{2}$ transforms it in $z^2 + c(t) = 0$, $r = \nu_0(c(t))$. Writing $c(t) = t^r u_0(t)^2$, $u_0(0) \neq 0$, the separatrix can be written as $(\frac{z}{u_0(t)})^2 + t^r = 0$: a new change of variables $z \mapsto z u_0(t)$ allows to write this separatrix as $z^2 + t^r = 0$. In this case, it is well-known (see [11]) that the foliation is generated by a 1-form $\omega_0' = d(z^2 + t^r) + ztA(z, t) \cdot (r\frac{dt}{t} - 2\frac{dz}{z})$.

Collecting all previous considerations, we see that there exists a change of variables $\Phi_0(t, z) = (t, z s_1(t) + s_0(t))$, with $s_1(0) \neq 0$, such that $\omega_0 \wedge \Phi_0^* \omega_0' = 0$. Consider the diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} & \xrightarrow{\rho} & \mathbb{C}^2 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Phi_0} \\ \mathbb{C}^{n+1} & \xrightarrow{\rho} & \mathbb{C}^2 \end{array}$$

It is commutative choosing $F(\mathbf{x}, z) = (\mathbf{x}, z s_1(f(\mathbf{x})) + s_0(f(\mathbf{x})))$, which is a diffeomorphism. The form $(F^{-1})^* \omega''$ defines the same foliation as

$$\Omega = d\big(z^2 + f(\mathbf{x})^r\big) + z f(\mathbf{x}) A\big(z, f(\mathbf{x})\big) \cdot \left(r\frac{df}{f} - 2\frac{dz}{z}\right),$$

analytically equivalent to $\mathcal{F}$, and having $z^2 + f(\mathbf{x})^r = 0$ as separatrix. Let us observe that the map that transforms ω in Ω is of the form $(\mathbf{x}, z) \mapsto (\mathbf{x}, Z(\mathbf{x}, z))$, i.e., it respects the projection over the first n coordinates. Write $Z(\mathbf{x}, z) = \sum_{k=0}^{\infty} Z_k(\mathbf{x}) z^k$, with $Z_1(\mathbf{0}) \neq 0$. There is a unit $U(\mathbf{x}, z)$ such that

$$Z(\mathbf{x}, z)^2 + \varphi(\mathbf{x}) = U(\mathbf{x}, z)\big(z^2 + f(\mathbf{x})^r\big).$$

If $U(\mathbf{x}, z) = \sum_{k=0}^{\infty} U_k(\mathbf{x}) z^k$, $U_0(\mathbf{0}) \neq 0$, the first coefficients in z of the last equality give the conditions

$$Z_0(\mathbf{x})^2 + \varphi(\mathbf{x}) = U_0(\mathbf{x}) f(\mathbf{x})^r,$$

$$2Z_0(\mathbf{x}) Z_1(\mathbf{x}) = U_1(\mathbf{x}) f(\mathbf{x})^r.$$

So,

$$\varphi(\mathbf{x}) = U_0(\mathbf{x}) f(\mathbf{x})^r - Z_0(\mathbf{x})^2 = U_0(\mathbf{x}) f(\mathbf{x})^r - \frac{U_1(\mathbf{x})^2 f(\mathbf{x})^{2r}}{4Z_1(\mathbf{x})^2}$$
$$= f(\mathbf{x})^r \left[U_0(\mathbf{x}) - \frac{U_1(\mathbf{x})^2 f(\mathbf{x})^r}{Z_1(\mathbf{x})^2} \right].$$

The expression between brackets turns out to be a unit, as required.

Remark 3.2. Let us remark that, in Theorem 1.4, the dicritical case is included. For instance, the foliation defined by

$$\omega = d\big(z^2 + \Psi(x, y)^5\big) + \Psi(x, y)^2 z\left(5\frac{d\Psi}{\Psi} - 2\frac{dz}{z}\right),$$

where $\Psi(x, y) = (y^2 + x^5)(y^2 + 2x^5)^2$, turns out to be dicritical. From Loray's Preparation Theorem, the existence of a separatrix for these foliations is granted. It is worth to mention here that if dicritical components are non-compact, or if the foliation induced on the compact dicritical components has a meromorphic first integral [29], then the foliation has a separatrix. In the above example, we are in the first situation. It deserves a closer study to verify if foliations in the conditions of the statement fall into some of these two situations.

4. The quasi-homogeneous case

A polynomial $P(x_1, \dots, x_n) = \sum_I a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[x_1, \dots, x_n]$ is said to be $(\alpha_1, \dots, \alpha_n)$-quasi-homogeneous of degree d if $a_{i_1,\dots,i_n} \neq 0$ implies $\langle \alpha, I \rangle = \sum_{k=1}^n \alpha_k i_k = d$. Equivalently, if the vector field $X = \frac{1}{d} \sum_{k=1}^n \alpha_k x_k \frac{\partial}{\partial x_k}$ verifies $X(P) = P$.

More generally, a germ of analytic function $f(x_1, \dots, x_n) \in \mathbb{C}\{x_1, \dots, x_n\}$ is called quasi-homogeneous if there exists a biholomorphism $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f \circ \Phi$ is a quasi-homogeneous polynomial. It is well-known (cf. [30]) that the following conditions are equivalent:

(1) f is quasi-homogeneous.
(2) There exists a vector field X such that $X(f) = f$.
(3) $f \in Jac(f)$, ideal generated by the partial derivatives of f.

Let us particularize Theorem 1.4 to the case when $z^2 + \varphi(\mathbf{x})$ is quasi-homogeneous. This means that there exists a vector field $\mathcal{X} = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} + Z \frac{\partial}{\partial z}$ such that $\mathcal{X}(z^2 + \varphi(\mathbf{x})) = z^2 + \varphi(\mathbf{x})$. If, in this expression, we put $z = 0$, we see that $\sum_{i=1}^n X_i(\mathbf{x}, 0)\varphi_{x_i} = \varphi$, so φ is quasi-homogeneous. In [10], the following result is shown:

Lemma 4.1. *If $\varphi(\mathbf{x})$ is quasi-homogeneous and $u(\mathbf{x})$ is a unit with $u(\mathbf{0}) = 1$, then there exists a biholomorphism Φ such that $\varphi \circ \Phi = u\varphi$.*

Let us observe that the condition $u(\mathbf{0}) = 1$ is not necessary: if φ is quasi-homogeneous and $c \in \mathbb{C} \setminus \{0\}$, there exists a biholomorphism Φ such that $\varphi \circ \Phi = c \cdot \varphi$.

So, Theorem 1.4, in the quasi-homogeneous case, implies:

Corollary 4.2. *Let $\mathcal{F}$ be a generalized hypersurface, with quasi-homogeneous separatrix $z^2 + \varphi(\mathbf{x}) = 0$. Then, there exist coordinates such that a generator of $\mathcal{F}$ is*

$$d\left(z^2 + \varphi\right) + G(\Psi, z) \cdot (z \cdot \Psi) \cdot \left(2\frac{dz}{z} - \frac{d\varphi}{\varphi}\right),$$

where $\varphi = \Psi^r$, Ψ not a power, and G a germ of holomorphic function in two variables

Proof. By Theorem 1.4, the foliation is defined by

$$d\left(z^2 + \varphi'\right) + G(\Psi, z) \cdot (z \cdot \Psi) \cdot \left(2\frac{dz}{z} - \frac{d\varphi'}{\varphi'}\right),$$

where φ and φ' differ in a unit factor. A biholomorphism Φ exists such that $\varphi' \circ \Phi = \varphi$. Applying this change of variables we obtain

$$d\left(z^2 + \varphi\right) + G(\Psi \circ \Phi, z) + (z \cdot \Psi \circ \Phi) \cdot \left(2\frac{dz}{z} - \frac{d\varphi}{\varphi}\right),$$

and the result follows. □

5. An application: Analytic classification via projective holonomy for some generalized surfaces

In previous sections we established a pre-normal form for a generator of a codimension one holomorphic foliation having $z^2 + \varphi(\mathbf{x}) = 0$ as a separatrix, but this foliation may eventually be dicritical, or have more separatrices. Let us briefly return to dimension two. According to [16], a 1-form having $y^p - x^q = 0$ as a separatrix has the form

$$\omega = d\left(y^p - x^q\right) + \Delta(x, y)(px\,dy - qy\,dx),$$

where $\Delta(x, y) \in \mathbb{C}\{x, y\}$. If $y^p - x^q = 0$ is an irreducible curve, let c be the conductor of its semigroup. Then, if $\nu_{(p,q)}(\Delta(x, y)) \geqslant c$, the reduction of the singularities of ω agrees with the reduction of $y^p - x^q = 0$. Here, $\nu_{(p,q)}(\sum_{i,j} a_{ij} x^i j^j) := \min\{pi + qj;\ a_{ij} \neq 0\}$. In general, if $\delta = \gcd(p, q) > 1$, define $\nu_{(p,q)}(\sum_{i,j} a_{ij} x^i y^j) := \min\{\frac{1}{\delta} \cdot (pi + qj);\ a_{ij} \neq 0\}$. Then, the resolutions of ω and $y^p - x^q = 0$ agree if $\nu_{(p,q)}(\Delta(x, y)) \geqslant \frac{(p-1)(q-1)}{\delta}$.

Remark 5.1. In [16], Loray states that this inequality is *necessary* and sufficient. In fact, it is not necessary, as examples like $d(y^6 - x^3) + axy(6x\,dy - 3y\,dx)$ show ($a \in \mathbb{C}^*$), where a certain arithmetic condition in a is needed, in the style of [24].

Let us give a version of this result in dimension three, where a reduction of the singularities is available. In the quasi-homogeneous case, consider a hypersurface given by the equation $f(x, y, z) = z^2 + \prod_{i=1}^{l} (y^p - a_i x^q)^{d_i}$, where $a_i \neq 0$, and $a_i \neq a_j$ if $i \neq j$. Let us observe that this is not the more general case of quasi-homogeneous surfaces of our type: such a quasi-homogeneous surface could have an equation $z^2 + x^r y^s \prod_{i=1}^{l}(y^p - a_i x^q)^{d_i} = 0$. We exclude from our considerations the case $r > 0$ or $s > 0$. Let us briefly describe a reduction of the singularities of $f(x, y, z) = 0$. We will follow the so-called Weierstrass–Jung method: first of all, the discriminant curve of the projection $(x, y, z) \mapsto (x, y)$ will be reduced to normal crossings, using punctual blow-ups, and then, we will apply usual procedures in order to desingularize quasi-ordinary surfaces.

Remark that the singular locus of $f(x, y, z) = 0$ is

$$S = \{\mathbf{0}\} \cup \{y^p - a_i x^q = 0,\ z = 0;\ d_i > 1\}.$$

Denote $r = \gcd(d_1, d_2, \ldots, d_l)$, $d_i' = d_i/r$, $d = \sum_{i=1}^{l} d_i$, $d' = \sum_{i=1}^{l} d_i'$, $\delta = \gcd(p, q)$, $\varphi(x, y) = \prod_{i=1}^{l}(y^p - a_i x^q)^{d_i} = \Psi(x, y)^r$. The reduction of the singularities will follow several steps:

Step I: Blow-up the origin a certain number of times, in order to reduce to normal crossings the curve $\varphi(x, y) = 0$. Let us see the result in the most interesting chart. For, take $m, n \geqslant 0$ such that $mp - nq = \delta$, m, n minimal (i.e. $0 \leqslant m < q/\delta$, $0 \leqslant n < p/\delta$). After the transformation

$$\begin{aligned} x &\to x^{p/\delta} y^n, \\ y &\to x^{q/\delta} y^m, \\ z &\to x^{\frac{p+q}{\delta}-1} y^{m+n-1} z, \end{aligned}$$

the surface is reduced to

$$x^{2(\frac{p+q}{\delta}-1)} y^{2(m+n-1)} \cdot \left[z^2 + x^P y^Q \prod_{i=1}^{l} (y^\delta - a_i)^{d_i} \right],$$

where

$$P = \frac{pq}{\delta} d - 2\left(\frac{p+q}{\delta} - 1\right), \qquad Q = nqd - 2(m + n - 1).$$

The singular locus of the strict transform is given now by the projective lines given locally by $z = x = 0$, $z = y = 0$, and the sets of lines $z = 0$, $y = a_i^{1/\delta}$.

Step II: Blow-up the lines $z = x = 0$, $z = y = 0$, a certain number of times. Let us describe a typical case, namely when P, Q are even numbers. In this case, blow-up $(z = x = 0)$ $Q/2$ times, and $(z = y = 0)$ $P/2$ times. The equations of this transform, in a certain appropriate chart, are

$$\begin{aligned} x &\to x, \\ y &\to y, \\ z &\to x^{P/2} y^{Q/2} z. \end{aligned}$$

The strict transform of the surface is now

$$z^2 + \prod_{i=1}^{l} \left(y^\delta - a_i\right)^{d_i}.$$

Step III: Finally, the lines $z = 0$, $y = a_i^{1/\delta}$ with $d_i > 1$, must be blown-up, according to the well-known scheme of the reduction of the singularities of cuspidal plane curves $z^2 + Y^{d_i} = 0$.

Let us follow this reduction scheme in the 1-form

$$\omega = d\left(z^2 + \varphi(x, y)\right) + G(\Psi, z) \cdot z\Psi \cdot \left(\frac{d\varphi}{\varphi} - 2\frac{dz}{z}\right),$$

with previous notations.

After Step I, the inverse image of ω is

$$\begin{aligned} &x^{2(\frac{p+q}{\delta}-1)-1} y^{2(m+n-1)-1} \cdot \left[\left(z^2 + x^P y^Q h^r\right)\omega_1 + xyd\left(z^2 + x^P y^Q h^r\right)\right. \\ &\quad \left. + x^a y^b z h G_1 \cdot \left(P\frac{dx}{x} + Q\frac{dy}{y} - 2\frac{dz}{z} + r\frac{dh}{h}\right)\right], \end{aligned}$$

where:

$$\begin{aligned} \omega_1 &= 2\left(\frac{p+q}{\delta} - 1\right) y\,dx + 2(m+n-1)x\,dy, \\ h(y) &= \prod_{i=1}^{l} \left(y^\delta - a_i\right)^{d_i'}, \\ P &= \frac{pq}{\delta}d - 2\left(\frac{p+q}{\delta} - 1\right), \\ Q &= nqd - 2(m+n-1), \\ a &= \frac{pq}{\delta}d' - \left(\frac{p+q}{\delta} - 1\right) + 1, \\ b &= nqd' - (m+n-1) + 1. \end{aligned}$$

Denote Ω the strict transform of ω, i.e., the result of dividing by a monomial $x^{2(\frac{p+q}{\delta}-1)-1} y^{2(m+n-1)-1}$. In Step II, it is necessary to blow up several projective lines. Assume, as before, that P, Q are even numbers. In the chart we are working, blow up $P/2$ times the line $(z = y = 0)$, and $Q/2$ times the line $(z = x = 0)$. The inverse image of Ω is, then,

$$x^P y^Q \cdot \left[\left(z^2 + h^r\right) \cdot \omega_2 + xyd\left(z^2 + h^r\right) + x^{a-\frac{P}{2}} y^{b-\frac{Q}{2}} z h G_2\left(-2\frac{dz}{z} + r\frac{dh}{h}\right)\right],$$

where

$$\omega_2 = \frac{pq}{\delta} d \cdot y\,dx + nqd \cdot x\,dy.$$

The singularities not still reduced are, in these coordinates, the lines $z = 0$, $y^\delta = a_i$, for every i such that $d_i > 1$ (in particular, for all i if $r > 0$). Take ξ with $\xi^\delta = a_i$, and make a translation $y \to y + \xi$. Denote $h(y + \xi) = y^{d'_i} H_i(y)$, $H_i(0) \neq 0$. In these new coordinates, the 1-form turns out to have the expression

$$\begin{aligned}&\bigl(z^2 + y^{d_i} H_i^r\bigr)\omega_3 + x(y+\xi) d\bigl(z^2 + y^{d_i} H_i^r\bigr) + x^{a-\frac{P}{2}} (y+\xi)^{b-\frac{Q}{2}} z y^{d'_i} H_i G_3 \\ &\quad \cdot \left(-2\frac{dz}{z} + d_i \frac{dy}{y} + r\frac{dH_i}{H_i}\right).\end{aligned}$$

Assume that d_i is even (the other cases are treated similarly), and blow up $(z = x = 0)\backslash d_i/2$ times. The equations are $z \to y^{d_i/2} z$, and the final result,

$$\begin{aligned}&y^{d_i-1} \cdot \Bigl[\bigl(z^2 + H_i^r\bigr)\omega_4 + xy(y+\xi) d\bigl(z^2 + H_i^r\bigr) + x^{a-\frac{P}{2}} (y+\xi)^{b-\frac{Q}{2}} y^{d'_i - \frac{d_i}{2} + 1} \\ &\quad \cdot z H_i G_4 \left(-2\frac{dz}{z} + r\frac{dH_i}{H_i}\right)\Bigr],\end{aligned}$$

where

$$\omega_4 = \frac{pq}{\delta} d(y+\xi) y\,dx + \bigl(nqd \cdot y + d_i(y+\xi)\bigr) x\,dy.$$

In all these expressions, we denoted G_1, G_2, G_3, G_4 the total transforms of $G(\Psi, z)$ by the succesive transformations. Let us consider the component of the exceptional divisor given in these coordinates by the equation $x = 0$. This component carries all the relevant information about the projective holonomy of the whole exceptional divisor, in the following sense: Consider two foliations of this type, and of the generalized surface type, having the same surface as separatrix. Assume that the projective holonomies of this component (more precisely, of the related components) of the divisor are analytically conjugated. Then, following arguments from [11] and [22], this conjugation can be lifted to a conjugation of the foliations in a neighbourhood of this component.

In order to perform this lifting, and to extend it to a neighbourhood of the whole divisor, the existence of a Hopf fibration is needed, as it is done in [11] (dimension two) and in [13] (dimension three). Such a fibration must be transverse to the foliation away from the separatrices, and these separatrices must be union of fibers.

Using the pre-normal form we have constructed in this paper, such a fibration can be defined in the coordinates after the reduction of the singularities by integrating the vector field $\frac{\partial}{\partial y}$ or a multiple thereof. Going back through the sequence of blow-ups of Steps I, II and III, we can consider the vector field

$$\mathcal{X} = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + \frac{pqd}{2} \cdot z\frac{\partial}{\partial z}.$$

A straightforward computation shows that

$$\iota_{\mathcal{X}}(\omega) = pqd \cdot \big(z^2 + \varphi(x, y)\big).$$

So, this vector field defines the desired fibration and allows to construct the analytic conjugation between the foliations. In the PhD thesis of the third author [26], precise details of this and other cases can be found.

Finally, let us observe that if $G(\Psi, z) = \sum G_{\alpha\beta}\Psi^\alpha z^\beta$, the term $\Psi^\alpha z^\beta$ is transformed, after previous steps, in

$$\big(x^{\frac{pq}{\delta}d'}(y+\xi)^{nqd'}y^{d_i'}H_i\big)^\alpha \cdot \big(x^{\frac{1}{2}\frac{pq}{\delta}d}(y+\xi)^{\frac{1}{2}nqd}y^{\frac{d_i}{2}}z\big)^\beta.$$

In order that the powers that appear are positive, it is necessary that

$$\frac{pq}{\delta}d'\left(1-\frac{r}{2}\right)+1+\frac{pq}{\delta}d'\left(\alpha+\frac{r}{2}\beta\right) \geqslant 0,$$
$$nqd'\left(1-\frac{r}{2}\right)+1+nqd'\left(\alpha+\frac{r}{2}\beta\right) \geqslant 0,$$
$$d_i'\left(1-\frac{r}{2}\right)+1+d_i'\left(\alpha+\frac{r}{2}\beta\right) \geqslant 0.$$

These conditions are satisfied if $2\alpha + \beta \geqslant r - 2$. This condition could be slightly relaxed, including additional arithmetical conditions as in the two-dimensional case, but we will not enter into details. As in the two-dimensional case, the effect in the other singular points of the total transform of the foliation is smaller, so, no new restrictions appear.

The cases P even and Q odd, or P odd and Q even are treated similarly. It is slightly different the case P and Q odd numbers, where it is needed to perform punctual blow-ups at a time of the reduction process (see [13]), but the final result turns out to be the same. Precise details of the computations are omitted. Collecting previous computations, we have shown the following result.

Theorem 5.2. *Consider a codimension one germ of holomorphic foliation in $\mathbb{C}^3$, generated by an integrable 1-form*

$$\omega = d\big(z^2+\varphi(x,y)\big) + G(\Psi, z)\cdot(z\Psi)\left(\frac{d\varphi}{\varphi} - 2\frac{dz}{z}\right),$$

where $\Psi(x,y) = \prod_{i=1}^{l}(y^p - a_i x^q)^{d_i'}$, $p, q \geqslant 2$, $a_i \neq 0$, $a_i \neq a_j$ if $i \neq j$, d_i' relatively prime, and $\varphi(x,y) = \Psi(x,y)^r$. Write $G(\Psi, z) = \sum_{\alpha,\beta} G_{\alpha,\beta}\Psi^\alpha z^\beta$, holomorphic function in two variables.

Let us denote

$$\nu_{(2,r)}\big(\Psi^\alpha z^\beta\big) := (2\alpha + r\beta)/\gcd(2,r),$$

and

$$\nu_{(2,r)}(G) = \min\big\{\nu_{(2,r)}\big(\Psi^\alpha z^\beta\big);\ G_{\alpha\beta} \neq 0\big\}.$$

Then, if $\nu_{(2,r)}(G) \geqslant (r-2)/\gcd(2,r)$, the foliation is a generalized surface.

This allows to recognize, from the pre-normal form, when the foliation is a generalized surface, situation in which the techniques concerning the analytic classification of the foliation from the holonomy of a special component of the divisor could be applied, in the form stated in this section.

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