A REMARK ON A PAPER OF M. HIRONAKA.*

By Maurice Auslander.

In [4] H. Hironaka quoted and used, but did not prove, a theorem I had communicated to him which he called Theorem 0 in his paper. In this note I present a proof of this theorem.

We begin with a brief summary of some pertinent homological facts. The reader is referred to [1] for a more complete exposition.

Let R be a local ring and M a finitely generated non-zero R-module. A sequence of non-units $x_1, \cdots, x_t$ in R is said to be an M-sequence if x_i is not a zero divisor in $M/(x_1, \cdots, x_{i-1})M$ for all $i = 1, \cdots, t-1$. It is well known that if $x_1, \cdots, x_t$ is an M-sequence, then $t \leqq \dim(M) = \dim(R/\mathfrak{a}(M))$ where $\mathfrak{a}(M)$ is the annihilator of M. Thus it makes sense to talk about maximal M-sequences. It can be shown that all maximal M-sequences have the same length and this length is called the codimension of M (notation: $\operatorname{codim}_R M$). For an R-module M we define $d(M) = \dim(M) - \operatorname{codim} M$. It follows from our previous remark that $d(M) \geqq 0$.

Now if R is a regular local ring, then it is well known that the $\operatorname{hd} M + \operatorname{codim} M = \dim R$ for any finitely generated non-zero R-module M. Therefore $d(M) = \dim(M) - (\dim R - \operatorname{hd} M) = \operatorname{hd} M - \operatorname{rank} M$ where $\operatorname{rank} M = \operatorname{rank}(\mathfrak{a}(M))$ by definition.

Proposition 1. *Let R be a regular local ring, M a non-zero finitely generated R-module and t an integer greater than or equal to 0. Then $d(M) \geqq t$ if and only if there exist integers $i \geqq 0$ and $h \geqq t$ such that $\operatorname{Ext}^i_R(M, R) \neq 0$ and $\operatorname{Ext}^{i+h}_R(M, R) \neq 0$.*

Proof. We first recall a result of D. Rees (see [3]).

If S is a noetherian ring, M a finitely generated S-module and i the smallest integer such that $\operatorname{Ext}^i_S(M, S) \neq 0$, then i is the length of the largest S-sequence contained in the $\mathfrak{a}(M)$. If S is a regular local ring, then we have that $i = \operatorname{rank}(\mathfrak{a}(M))$ (see [1, Proposition 2.7]).

Now we return to the proof of the proposition. Suppose that $d(M) \geqq t$. Then we have that the $\operatorname{hd} M \geqq \operatorname{rank}(M) + t$. By the previous remark we

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know that if we set $i = \operatorname{rank}(M)$, then $\operatorname{Ext}^i(M, R) \neq 0$. But it is well known that the hd M is the largest integer k such that $\operatorname{Ext}^k(M, R) \neq 0$ (see [2; VI, Exercise 9]). Therefore setting $h = k - i \geqq t$ we have the desired integers i and h.

On the other hand suppose that $\operatorname{Ext}^i{}_R(M, R) \neq 0$ and $\operatorname{Ext}^{i+h}{}_R(M, R) \neq 0$ with $h \geqq t$. Then it follows from what has been observed before that the rank $(M) \leqq i$ and that the hd $M \geqq i + h$. Therefore

$$d(M) = \operatorname{hd} M - \operatorname{rank} M \geqq h \geqq t.$$

We say that a noetherian ring R is regular if $R_{\mathfrak{p}}$ is a regular local ring for each prime ideal $\mathfrak{p}$ in R and $\dim R$ is finite.

THEOREM 2. *Let R be a regular ring, M a finitely generated, non-zero R-module. For each integer $t \geqq 0$, there exists an ideal $\mathfrak{c}_t \supset \mathfrak{a}(M)$ such that a prime ideal $\mathfrak{p} \supset \mathfrak{c}_t$ if and only if $d(M_{\mathfrak{p}}) \geqq t$.*

Proof. Let $\mathfrak{a}_i = \mathfrak{a}(\operatorname{Ext}^i{}_R(M, R))$ for all $i \geqq 0$. It is clear that each $\mathfrak{a}_i \supset \mathfrak{a}(M)$ and only a finite number of $\mathfrak{a}_i \neq R$ since only a finite number of $\operatorname{Ext}^i{}_R(M, R) \neq 0$. For each pair of integers $i \geqq 0$ and $h \geqq 0$ define $\mathfrak{b}(i, h) = (\mathfrak{a}_i, \mathfrak{a}_{i+h})$. It should be observed that only a finite number of the $\mathfrak{b}(i, h) \neq R$ and that each $\mathfrak{b}(i, h) \supset \mathfrak{a}(M)$.

For each integer $t \geqq 0$ define $\mathfrak{c}_t$ to be $\bigcap_{h \geqq t} \mathfrak{b}(i, h)$. The fact that $\mathfrak{c}_t \supset \mathfrak{a}(M)$ is trivial. Since $\bigcap_{h \geqq t} \mathfrak{b}(i, h)$ is really a finite intersection, we know that a prime ideal $\mathfrak{p}$ in R contains $\mathfrak{c}_t$ if and only if $\mathfrak{p} \supset \mathfrak{b}(i, b)$ for some i and some $h \geqq t$. But $\mathfrak{p} \supset \mathfrak{b}(i, h)$ if and only if $R_{\mathfrak{p}} \otimes \operatorname{Est}^i{}_R(M, R) \neq 0$ and $R_{\mathfrak{p}} \otimes \operatorname{Ext}^{i+h}{}_R(M, R) \neq 0$. Since $R_{\mathfrak{p}}$ is R-flat, it follows from [2; VI, Exercise 11] that $R_{\mathfrak{p}} \otimes \operatorname{Ext}^j{}_R(M, R) = \operatorname{Ext}^j{}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}})$ for all j. Therefore applying Proposition 1, we have that $\mathfrak{p} \supset \mathfrak{c}_t$ if and only if $d(M_{\mathfrak{p}}) \geqq t$.

The following module theoretic version of Theorem 0 of [4] is an easy consequence of Theorem 2.

COROLLARY 3. *Let S be a noetherian ring which is a factor ring of a regular ring and let M be a finitely generated non-zero S-module. Then for each integer $t \geqq 0$, the set of all prime ideals $\mathfrak{p} \in \operatorname{Spec}(S)$ such that $d(M_{\mathfrak{p}}) \geqq t$ forms a closed set.*

Proof. Let $f\colon R \to S$ be the canonical ring epimorphism. Let $\mathfrak{p}$ be a prime ideal in S and $\mathfrak{P}$ the prime ideal $f^{-1}(\mathfrak{p})$ in R. Then it is well known that $\dim(M_{\mathfrak{P}}) = \dim(M_{\mathfrak{p}})$ and $\operatorname{codim}(M_{\mathfrak{P}}) = \operatorname{codim}(M_{\mathfrak{p}})$. Therefore we have that $d(M_{\mathfrak{P}}) = d(M_{\mathfrak{p}})$.

Now given an integer $t \geqq 0$ we have by Theorem 2 that there exists an ideal $\mathfrak{c}_t$ in R which contains $\mathfrak{a}(M)$, and therefore the Ker f, such that a prime ideal $\mathfrak{P}$ in R contains $\mathfrak{c}_t$ if and only if $d(M_{\mathfrak{P}}) \geqq t$. It therefore follows by standard arguments that a prime ideal $\mathfrak{p}$ in S contains $f(\mathfrak{c}_t)$ if and only if $d(M_{\mathfrak{p}}) \geqq t$.

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REFERENCES.

[1] M. Auslander and D. Buchsbaum, "Homological dimension in local rings," *Transactions of the American Mathematical Society*, vol. 85 (1957), pp. 390-405.

[2] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton, 1956.

[3] D. Rees, "The grade of an ideal or module," *Proceedings of the Cambridge Philosophical Society*, vol. 52 (1957), pp. 28-42.

[4] H. Hironaka, "A generalized theorem of Krull Seidenberg on parameterized algebras of finite type," *American Journal of Mathematics*, vol. 82 (1960), pp. 831-850.