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Author(s): Janusz Adamus

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NATURAL BOUND IN KWIECIŃSKI'S CRITERION FOR FLATNESS

JANUSZ ADAMUS

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ABSTRACT. Kwieciński has proved a geometric criterion for flatness: A morphism $f : X \rightarrow Y$ of germs of analytic spaces is not flat if and only if its i -fold fibre power $f^{\{i\}} : X^{\{i\}} \rightarrow Y$ has a vertical component, for *some* i . We show how to bound i using Hironaka's local flattener: If f is not flat, then $f^{\{d\}}$ has a vertical component, where d is the minimal number of generators of the ideal in \mathcal{O}_Y of the flattener of X .

1. INTRODUCTION

Let $f : Z \rightarrow Y$ be a morphism of germs of analytic spaces, and let W be an irreducible component of Z . Let \mathcal{P} be the associated prime of the zero ideal in the local ring \mathcal{O}_Z , corresponding to W . We say that the component W is **vertical** if there exists a nonzero $a \in \mathcal{O}_Y$ with $f^*a \in \mathcal{P}$ (see also the remark following Lemma 3.1).

In [11, Thm. 1.1], Kwieciński shows that assuming Y is irreducible, i.e. the local ring \mathcal{O}_Y is a domain, the following conditions are equivalent:

- (i) $f : X \rightarrow Y$ is flat;
- (ii) for any $i \geq 1$, the canonical map $\underbrace{X \times_Y \dots \times_Y X}_{i \text{ times}} \rightarrow Y$ has no (isolated or embedded) vertical components.

From now on we will write $X^{\{i\}}$ for the space $X \times_Y \dots \times_Y X$ (i times) and $f^{\{i\}}$ for the canonical map $X^{\{i\}} \rightarrow Y$.

Note that if f is flat, then $f^{\{i\}}$ is flat for all i , since flatness is preserved by any base change ([9, §6, Prop. 8]) and the composition of flat maps is flat. Therefore the implication (i) \Rightarrow (ii) is an immediate consequence of the definition of flatness in terms of relations (see e.g. [2, Prop. 7.3]). This in fact is the only place where the irreducibility assumption is needed (cf. the example in this section below).

Implication (ii) \Rightarrow (i) does not require irreducibility of Y , by Kwieciński's Lemma 3.1 below. Thus, for any nonflat map $f : X \rightarrow Y$ of germs of analytic spaces there is a positive integer i such that $X^{\{i\}}$ has a *vertical* component. The proof given by Kwieciński is based on Hironaka's criterion for flatness (Thm. 2.2 below, see also [2, Thm. 7.9]). Hironaka uses this criterion to prove the existence of the *local flattener* (see [2, Thm. 7.12]), which we use to give a precise power

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needed in condition (ii) of Kwieciński's theorem. The *local flattener* for a morphism $f : X \rightarrow Y$ of germs of analytic spaces is, by definition, the maximal subgerm P of Y such that $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_P$ is \mathcal{O}_P -flat, i.e. $f|_{f^{-1}(P)} : f^{-1}(P) \rightarrow P$ is a flat morphism (cf. [2, Thm. 7.12]). Our main result is the following:

Theorem 1.1. *Let $f : X \rightarrow Y$ be a nonflat morphism of germs of analytic spaces. Let Q be the ideal in \mathcal{O}_Y of the flattener of X , and let d be the minimal number of generators of Q . Then the canonical map $f^{\{d\}} : X^{\{d\}} \rightarrow Y$ has a vertical component.*

Of course, this leaves open the question of calculation. First results have been given by Vasconcelos [15], and Galligo and Kwieciński [6] under certain restrictions on X and Y : Galligo and Kwieciński assert that, assuming X, Y are reduced, X is of pure dimension, and Y is smooth of dimension n , $f : X \rightarrow Y$ is flat if and only if the canonical map $f^{\{n\}} : X^{\{n\}} \rightarrow Y$ has no *geometric vertical* components ([6, Thm. 6.1]). An analogous result in the algebraic category, but without the pure dimension assumption on X , was obtained by Vasconcelos in the case $\dim Y = 2$ ([15, Prop. 6.1]).

By a *geometric vertical* component of a morphism $f : Z \rightarrow Y$ we mean a component W of Z such that, for arbitrarily small representatives $\widetilde{W}, \widetilde{Y}, \widetilde{f}$ of W, Y, f , respectively, the image $\widetilde{f}(\widetilde{W})$ has empty interior in \widetilde{Y} with transcendental topology (cf. [6] and [11]). Note that although the notions of *vertical* and *geometric vertical* coincide in the algebraic case over irreducible germ Y (as the image of an algebraic set under a polynomial morphism is always constructible), they are not the same in the analytic setup.

Clearly, over irreducible Y , every *vertical* component is *geometric vertical*, but the converse is false in general. Consider for instance the Osgood mapping $f : \mathbb{C}_0^2 \rightarrow \mathbb{C}_0^3$ defined as $(x, y) \mapsto (x, xy, xye^y)$ (see e.g. [7]). Then for an arbitrary neighbourhood U of the origin in \mathbb{C}^2 , $f(U)$ has empty interior in \mathbb{C}^3 , but there is no proper analytic subgerm of \mathbb{C}_0^3 containing $(f(U))_0$, and hence f has no *vertical* components in our sense.

Observe that in general, i.e. without the irreducibility assumption on Y , the equivalence from Kwieciński's theorem is no longer valid. Consider for instance the identity mapping on the space $X = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$, which is obviously flat while each of the irreducible components of X is vertical.

2. DIAGRAM OF INITIAL EXPONENTS AND HIRONAKA'S CRITERION FOR FLATNESS

We briefly recall here basic facts regarding the diagram of initial exponents. For details we refer to [2].

Let A be a local analytic \mathbb{C} -algebra, say $A = \mathbb{C}\{y_1, \dots, y_m\}/J$, with the maximal ideal \mathfrak{m} . Let L be a total ordering of monomials in $t = (t_1, \dots, t_n)$ with coefficients in A which is compatible with addition of exponents. We write t^β for $t_1^{\beta_1} \dots t_n^{\beta_n}$, where $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$. Let $A\{t\} = \mathbb{C}\{y, t\}/J \cdot \mathbb{C}\{y, t\}$ be the ring of convergent power series in t with coefficients in A . For a series $F = \sum_{\beta \in \mathbb{N}^n} a_\beta t^\beta \in A\{t\}$ define its *evaluation at 0* as $F(0) = \sum_{\beta \in \mathbb{N}^n} a_\beta(0) t^\beta \in A/\mathfrak{m}\{t\} = \mathbb{C}\{t\}$, and for an ideal I in $A\{t\}$ define $I(0) = \{F(0) : F \in I\}$, the *evaluated ideal*. The *support* of F is defined

as $\text{supp } F = \{\beta \in \mathbb{N}^n : a_\beta \neq 0\}$, and $\nu_L(F) = \min_L\{\beta \in \text{supp } F\}$ denotes the *initial exponent* of F (with respect to L). Similarly, $\text{supp } F(0) = \{\beta \in \mathbb{N}^n : a_\beta(0) \neq 0\}$ and $\nu_L(F(0)) = \min_L\{\beta \in \text{supp } F(0)\}$, for the evaluated series.

Let I be an ideal in $A\{t\}$. The *diagram of initial exponents* of I (with resp. to L) is defined as $N_L(I) = \{\nu_L(F) : F \in I\} \subset \mathbb{N}^n$.

Let $f : X \rightarrow Y$ be a morphism of germs of analytic spaces. Without loss of generality can assume that X is a subgerm of \mathbb{C}_0^n , for some $n \geq 1$. We can then embed X into $Y \times \mathbb{C}_0^n$ via the graph of f . Therefore the local ring \mathcal{O}_X of X can be thought of as a quotient of the local ring of $Y \times \mathbb{C}_0^n$, i.e. $\mathcal{O}_X = \mathcal{O}_Y\{t\}/I$, for some ideal I in $\mathcal{O}_Y\{t\}$, where $t = (t_1, \dots, t_n)$. Let $\Delta = \mathbb{N}^n \setminus N_L(I(0))$ be the complement of the diagram of initial exponents of the evaluated ideal $I(0)$, and define $\mathcal{O}_Y\{t\}^\Delta = \{F \in \mathcal{O}_Y\{t\} : \text{supp } F \subset \Delta\}$. Now consider the canonical projection $\mathcal{O}_Y\{t\} \rightarrow \mathcal{O}_X$ and its restriction to $\mathcal{O}_Y\{t\}^\Delta$, called κ . The two results below, due to Hironaka, are crucial for our considerations.

Proposition 2.1 ([9, §6, Prop.9]). *The natural map $\kappa : \mathcal{O}_Y\{t\}^\Delta \rightarrow \mathcal{O}_X = \mathcal{O}_Y\{t\}/I$ is surjective.*

Theorem 2.2 ([9, §6, Prop.10]). *With the notations above, the map $f : X \rightarrow Y$ is flat if and only if κ is bijective.*

Observe that $\ker \kappa = \{F \in \mathcal{O}_Y\{t\}^\Delta : F \in I\}$, i.e. $\ker \kappa$ consists of these elements of the ideal I whose supports lie entirely in Δ .

Remark 2.3. Let Q be the ideal in \mathcal{O}_Y of the flattener of X . Then by the proof of [2, Thm. 7.12], Q is generated by all the coefficients a_β of all the series $F = \sum_{\beta \in \Delta} a_\beta t^\beta$

from $\ker \kappa$.

3. KWIECIŃSKI'S LEMMA

Our proof of Theorem 1.1 is based on the following result due to Kwieciński ([11, Lemma 3.2]).

Lemma 3.1. *Let $f : X \rightarrow Y$ be a nonflat morphism of germs of analytic spaces. Then there is a positive integer i such that there exists a nonzero $b \in \mathcal{O}_{X^{(i)}}$ and a nonzero $a \in \mathcal{O}_Y$, with $ab = 0$.*

Observe that according to our definition, the condition above is equivalent to $X^{(i)}$ having a vertical component. Indeed, for $a \in \mathcal{O}_Y$ is a zerodivisor in $\mathcal{O}_{X^{(i)}}$ iff it belongs to some of the associated primes of the zero ideal in the local ring $\mathcal{O}_{X^{(i)}}$.

Note also that without any assumptions on Y , flatness implies the following condition: For any $i \geq 1$, if $a \in \mathcal{O}_Y$ is a zerodivisor in $\mathcal{O}_{X^{(i)}}$, then it is a zerodivisor in \mathcal{O}_Y . (By the definition of flatness in terms of relations.)

We will now sketch the main steps of the proof of Lemma 3.1: Since f is not flat, Theorem 2.2 together with Proposition 2.1 imply that $\ker \kappa \neq \{0\}$. Pick any non-trivial $F = \sum_{\beta \in \Delta} a_\beta t^\beta$ from $\ker \kappa$. Let $a_{\beta_1}, \dots, a_{\beta_i}$ be distinct nonzero coefficients of F which generate the ideal in \mathcal{O}_Y of all the coefficients of the series F . One then shows that $F = a_{\beta_1} F_1 + \dots + a_{\beta_i} F_i$, where $F_j(t) = t^{\beta_j} + \sum_{\beta \in \Delta \setminus \{\beta_1, \dots, \beta_i\}} f_\beta t^\beta \in \mathcal{O}_Y\{t\}$,

$j = 1, \dots, i$. Define $a_j = a_{\beta_j}$ and $h_j = \kappa(F_j)$ for $j = 1, \dots, i$. It follows that $a_1 h_1 + \dots + a_i h_i = 0$, but $h_1(0), \dots, h_i(0)$ are linearly independent (over \mathbb{C}) in $\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X = \mathbb{C}\{t\}/I(0)$, where \mathfrak{m} is the maximal ideal in \mathcal{O}_Y .

Next consider the following commutative diagram of canonical maps of \mathcal{O}_Y -modules, where both tensor products are taken i times and we assume that Y is a germ at 0:

$$\begin{array}{ccccc} \bigwedge_{\mathcal{O}_Y}^i \mathcal{O}_X & \xrightarrow{\rho} & \mathcal{O}_X \otimes_{\mathcal{O}_Y} \dots \otimes_{\mathcal{O}_Y} \mathcal{O}_X & \xrightarrow{\lambda} & \mathcal{O}_{X^{(i)}} \\ \downarrow & & \downarrow & & \downarrow \mu \\ \bigwedge_{\mathbb{C}}^i \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X & \xrightarrow{\bar{\rho}} & \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X & \xrightarrow{\bar{\lambda}} & \mathcal{O}_{(f^{-1}(0))^i} \end{array}$$

Put $a = a_1$ and $b = \lambda \circ \rho(h_1 \wedge \dots \wedge h_i)$. Since $a_1 h_1$ is an \mathcal{O}_Y -linear combination of h_2, \dots, h_i , then $ab = \lambda \circ \rho((a_1 h_1) \wedge h_2 \cdots \wedge h_i) = 0$. Finally $b \neq 0$, because $\mu(b) = \bar{\lambda} \circ \bar{\rho}(h_1(0) \wedge \dots \wedge h_i(0))$ is nonzero, as $h_1(0), \dots, h_i(0)$ are linearly independent and $\bar{\rho}, \bar{\lambda}$ are injective.

4. PROOF OF THEOREM 1.1

Let d be the minimal number of generators of the flattener ideal Q , i.e. $d = \dim_{\mathbb{C}} Q/\mathfrak{m}Q$, where \mathfrak{m} is the maximal ideal in the local ring \mathcal{O}_Y . We begin with the following lemma.

Lemma 4.1. *Assume $N_L(I(0)) = \mathbb{N} \times D$ for some $D \subset \mathbb{N}^{n-1}$. Then there is a series $F \in \ker \kappa$ such that the coefficients of F generate the ideal Q .*

Remark 4.2. The condition $N_L(I(0)) = \mathbb{N} \times D$ means that the diagram $N_L(I(0))$ is trivial in the β_1 direction. As can readily be seen from the proof below, we do not really need that much but only triviality in *some* of the β_1, \dots, β_n directions. The point is that triviality in the β_j direction implies that for a series $F \in \mathcal{O}_Y\{t\}$ with $\text{supp } F \subset \Delta$ and for any power k , the series $t_j^k F$ has support contained in Δ again, since $\text{supp}(t_j^k F) = \text{supp } F + (0, \dots, k, \dots, 0)$, with k in the j 'th place.

Proof of Lemma 4.1. Suppose to the contrary that for any series $F = \sum a_\beta t^\beta$ from $\ker \kappa$, the coefficients a_β of F do not generate Q . Note that by Remark 2.3, all the coefficients of F belong to Q .

For a series $F \in \ker \kappa$ define $d(F)$ as the maximal number of its coefficients linearly independent (over $\mathbb{C} = \mathcal{O}_Y/\mathfrak{m}\mathcal{O}_Y$) modulo $\mathfrak{m}Q$. It follows that for any F , $d(F) < d$, because otherwise the coefficients of some series would generate Q , by Nakayama's Lemma. Let $s = \max\{d(F) \mid F \in \ker \kappa\}$. Pick any $F_1 = \sum_{\beta \in \Delta} a_\beta t^\beta$ from $\ker \kappa$ with $d(F_1) = s$, and let $a_{\beta_1}, \dots, a_{\beta_s}$ be its coefficients linearly independent modulo $\mathfrak{m}Q$. Then by the definition of $d(F_1)$, for any $\beta \in \text{supp } F_1 \setminus \{\beta_1, \dots, \beta_s\}$ there exist $r_\beta^1, \dots, r_\beta^s \in \mathbb{C}$ and $q_\beta \in \mathfrak{m}Q$ such that $a_\beta = r_\beta^1 a_{\beta_1} + \dots + r_\beta^s a_{\beta_s} + q_\beta$, i.e. all the other coefficients of F are \mathbb{C} -linear combinations of $a_{\beta_1}, \dots, a_{\beta_s}$ modulo $\mathfrak{m}Q$.

Since $s < d$, Remark 2.3, together with Nakayama's Lemma, implies that there exists a series $F_2 = \sum_{\gamma \in \Delta} b_\gamma t^\gamma \in \ker \kappa$ such that for some $\gamma_0 \in \text{supp } F_2$, the coefficient b_{γ_0} of F_2 is linearly independent from $a_{\beta_1}, \dots, a_{\beta_s}$ modulo $\mathfrak{m}Q$. Fix such a series F_2 and take a positive integer k satisfying inequality

(*) $(k, 0, \dots, 0) > \max\{\beta_1, \dots, \beta_s\}$

with respect to the total ordering (induced by) L in \mathbb{N}^n .

Define a new series $F_0 = F_1 + t_1^k F_2$ and observe that $F_0 \in \ker \kappa$. Indeed, $F_2 \in \ker \kappa$ if and only if $F_2 \in I$ and $\text{supp } F_2 \subset \Delta$. Therefore Remark 4.2 yields $\text{supp } (t_1^k F_2) \subset \Delta$ (and obviously $t_1^k F_2 \in I$), whence $t_1^k F_2 \in \ker \kappa$ and $F_0 \in \ker \kappa$.

Put $F_0 = \sum_{\beta \in \Delta} c_\beta t^\beta$. By the inequality (*), $\text{supp } (t_1^k F_2) > \max \{\beta_1, \dots, \beta_s\}$, so in particular $c_{\beta_1} = a_{\beta_1}, \dots, c_{\beta_s} = a_{\beta_s}$. Moreover, if $\beta_0 = \gamma_0 + (k, 0, \dots, 0)$, then $\beta_0 \neq \beta_j$, $j = 1, \dots, s$, and

$$c_{\beta_0} = r_{\beta_0}^1 a_{\beta_1} + \dots + r_{\beta_0}^s a_{\beta_s} + q_{\beta_0} + b_{\gamma_0},$$

where $r_{\beta_0}^j \in \mathbb{C}$, $j = 1, \dots, s$, $q_{\beta_0} \in \mathfrak{m}Q$. But b_{γ_0} is linearly independent from $a_{\beta_1}, \dots, a_{\beta_s}$, which implies that c_{β_0} is linearly independent from $c_{\beta_1}, \dots, c_{\beta_s}$. Thus $c_{\beta_0}, c_{\beta_1}, \dots, c_{\beta_s}$ are $s+1$ coefficients of F_0 linearly independent modulo $\mathfrak{m}Q$, whence $d(F_0) \geq s+1$, a contradiction. \square

Proof of Theorem 1.1. Suppose first that $N_L(I(0)) = \mathbb{N} \times D$, as in Lemma 4.1. Then we can find $F \in \ker \kappa$, $F = \sum_{\beta \in \Delta} a_\beta t^\beta$ such that among its coefficients are $a_{\beta_1}, \dots, a_{\beta_d}$ linearly independent modulo $\mathfrak{m}Q$. By Nakayama's Lemma, $a_{\beta_1}, \dots, a_{\beta_d}$ generate Q (recall that $d = \dim_{\mathbb{C}} Q/\mathfrak{m}Q$), so in particular they generate all the coefficients of F . Therefore by applying Kwieciński's Lemma 3.1 to this F one obtains a nonzero $a \in \mathcal{O}_Y$ and a nonzero $b \in \mathcal{O}_{X^{\{d\}}}$ with $ab = 0$, i.e. a vertical component in the d 'th fibre power $X^{\{d\}}$ (cf. the remark following Lemma 3.1).

Next suppose the diagram $N_L(I(0))$ is not trivial in any direction. Define $\tilde{X} = \mathbb{C}_0 \times X$, and $\tilde{f}: \tilde{X} \rightarrow Y$ as $\tilde{f} = f \circ \pi$, where $\pi: \tilde{X} \rightarrow X$ is a canonical projection. Let $\tilde{I} = I \cdot \mathcal{O}_Y\{t_0, t_1, \dots, t_n\}$ be the extended ideal, so that $\mathcal{O}_{\tilde{X}} = \mathcal{O}_Y\{t_0, t_1, \dots, t_n\}/\tilde{I}$. Since every element of \tilde{I} can be expressed in the form $\sum_{i \in \mathbb{N}} F_i(t_1, \dots, t_n) \cdot t_0^i$, with $F_i \in I$, it follows that $N_L(\tilde{I}(0)) = \mathbb{N} \times N_L(I(0))$. Moreover, the flattener ideal for $\tilde{f}: \tilde{X} \rightarrow Y$ is generated by the same elements as the flattener ideal Q , so in particular its minimal sets of generators consist of d elements.

Now Lemma 4.1 applies to the diagram $N_L(\tilde{I}(0))$, as it is trivial in the β_0 direction, and following the first part of this proof we obtain that $\tilde{X}^{\{d\}}$ has a vertical component. But clearly every irreducible component \tilde{Z} of $\tilde{X}^{\{d\}}$ is of the form $Z \times (\mathbb{C}^d)_0$ for some irreducible component Z of $X^{\{d\}}$, which implies that $X^{\{d\}}$ itself has a vertical component. This completes the proof of our theorem. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA M5S 3G3

E-mail address: `adamus@math.toronto.edu`