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Author(s): Michael A. Buchner
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SIMPLICIAL STRUCTURE OF THE REAL ANALYTIC CUT LOCUS

MICHAEL A. BUCHNER

ABSTRACT. This note shows how to generalize to arbitrary dimensions the result of S.B. Myers that the cut locus in a real analytic Riemannian surface is triangulable. The basic tool is Hironaka's theory of subanalytic sets.

In 1935 and 1936 there appeared in the Duke Mathematical Journal two interesting papers by Sumner B. Myers [3], [4] on the cut locus for a Riemannian manifold. He determined completely the local structure of the cut locus on a two-dimensional real analytic manifold. It turned out that for compact real analytic surfaces the cut locus is a one-dimensional finite simplicial complex. On the other hand in [4] he remarks that "it seems difficult to prove that the locus in the analytic n-dimensional case is homeomorphic to a finite $(n-1)$-dimensional complex". The argument that follows will establish this conjecture.

On the one hand we have Hironaka's theory [1], [2] of subanalytic sets which he proved to be triangulable. On the other hand Morse theory gives us a characterization of the cut locus in terms of the behavior of an analytic function on an analytic manifold fibered over the given manifold (the "energy function"). It seemed reasonable then to try to prove the cut locus is subanalytic. This is what the following argument establishes by rather simple reasoning.

Let M be a compact analytic Riemannian manifold. Let $p \in M$. If γ is a geodesic parametrized by arclength such that $\gamma(0) = p$ then the cut point of (M,p) along γ is defined to be the first point $\gamma(t_0)$ $(t_0 > 0)$ such that for $t > t_0$, γ no longer minimizes arclength from p to $\gamma(t)$. The set of all such cut points as γ varies over all possible geodesics from p is denoted $C(p)$.

It is standard that $C(p)$ is characterized by $C(p) = \{x \in M \mid$ either x is the first conjugate point on a length minimizing geodesic starting at p and going through x or there are at least two length minimizing geodesics from p to $x\}$ [6].

Let $\Omega(M)$ be the space of piecewise analytic paths starting at p and let $E\colon \Omega(M) \to \mathbf{R}$ (the energy function) be defined by $E(\gamma) = \int_0^1 \|d\gamma(t)/dt\|^2\,dt$ (where we assume now that all paths are parametrized by the unit interval). Choose $\varepsilon > 0$ so that whenever $d(x,y) < \varepsilon$ there is a unique geodesic from x to y of length $< \varepsilon$ and so that the geodesic depends analytically on the

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endpoints x and y. ($d(x,y)$ denotes the length from x to y induced by the Riemannian metric.)

Suppose the diameter of M is $< \sqrt{s}$. Let $t_0, t_1, \ldots, t_k$, be a subdivision of $[0, 1]$ so that each $t_{i+1} - t_i < \varepsilon^2/s$. Let $\Omega(t_0, t_1, \ldots, t_k)$ be the space of broken geodesics starting at p (i.e. such that $\omega \in \Omega(t_0, t_1, \ldots, t_k)$ if and only if each $\omega|[t_i, t_{i+1}]$ is a geodesic). Let $\Omega(t_0, t_1, \ldots, t_k)^s$ be defined by $\Omega(t_0, t_1, \ldots, t_k) \cap E^{-1}([0, s))$. It is easy to verify that if $\omega \in \Omega(t_0, t_1, \ldots, t_k)^s$ then

$$d(\omega(t_i), \omega(t_{i+1})) < \varepsilon$$

so that we have a bijection

$$\Omega(t_0, t_1, \ldots, t_k)^s \to \left\{ (x_1, \ldots, x_k) \in M \times M \times \cdots \times M \,\middle|\, \sum_{i=1}^{k} \frac{[d(x_{i-1}, x_i)]^2}{t_i - t_{i-1}} < s \right\} \quad \text{where } x_0 = p.$$

REMARK. Many of the assertions concerning these path spaces can be found, with details, in Milnor [5].

The space $\Omega(t_0, \ldots, t_k)^s$ thus acquires an analytic structure and

$$E|\Omega(t_0, \ldots, t_k)^s$$

is an analytic map to $\mathbf{R}$. Let $\pi\colon \Omega(t_0, \ldots, t_k)^s \to M$ be the endpoint projection, i.e. $\pi(\omega) = \omega(1)$. Then π is an analytic submersion.

In view of the characterization of the cut locus given earlier we have $C(p) = \{x \in M \,|\, E|\pi^{-1}(x)$ has a degenerate minimum or at least two minima$\}$. (Here degenerate means the Hessian of $E|\pi^{-1}(x)$ at the minimum is not of full rank.)

Let us denote $\Omega(t_0, \ldots, t_k)^s$ by B_s. By $B_s \times_M B_s$ we mean the fibered product of B_s with itself over M, i.e. $B_s \times_M B_s$ is the disjoint union $\bigcup_{x \in M} \pi^{-1}(x) \times \pi^{-1}(x)$ with an obvious analytic manifold structure. The map $F\colon B_s \times_M B_s \to \mathbf{R}$ given by $(b_1, b_2) \mapsto E(b_1) - E(b_2)$ is an analytic map and so the set $V \subset B_s \times_M B_s$ defined by $V = \{(b_1, b_2) \,|\, F(b_1, b_2) > 0\}$ is semianalytic in $B_s \times_M B_s$.

Let t be a number satisfying (diameter of $M)^2 < t < s$. Then $B_t \times_M B_t$ is an open relatively compact subset of $B_s \times_M B_s$ and is also semianalytic. It follows that $V \cap (B_t \times_M B_t)$ is an open relatively compact semianalytic subset of $B_s \times_M B_s$.

At this point we need some facts about subanalytic sets (the basic reference being Hironaka [1]). Recall that if (X, θ_X) is a real analytic space (θ_X is the structure sheaf) then $A \subset X$ is subanalytic if we can find real analytic spaces V_{ij} ($i = 1, \ldots, k, j = 1, 2$) and proper real analytic maps $g_{ij}\colon V_{ij} \to X$ such that $A = \bigcup_{i=1}^{k} \{ g_{i1}(V_{i1}) - g_{i2}(V_{i2}) \}$. We wish to make use of the basic fact that the proper real analytic image of a subanalytic set is subanalytic. Although we will not necessarily be dealing with proper real analytic maps, we have the following substitute: Let $f\colon (X, \theta_X) \to (Y, \theta_Y)$ be a real analytic map between real analytic spaces where X is paracompact and suppose

$A \subset X$ is subanalytic and relatively compact; then $f(A)$ is subanalytic. To see this note that Hironaka proves that if $A \subset X$ is subanalytic and X is paracompact there is a single real analytic space (V, θ_V) and a proper real analytic map g such that $g(V) = \bar{A}$ (the closure of A). By compactness of $\bar{A}$ and properness of g it follows that $f \circ g$ is proper. Moreover $g^{-1}(A)$ is subanalytic so $f \circ g(g^{-1}(A)) = f(A)$ is subanalytic.

Let p_1 denote projection of $B_s \times_M B_s$ on the first factor. We then obtain, in view of the above remarks, that $p_1(V \cap (B_t \times_M B_t))$ is subanalytic in B_s. (Here we are using the fact that semianalytic sets are subanalytic.) The set $p_1(V \cap (B_t \times_M B_t))$ consists of broken geodesic paths starting at p, having energy less than t, and for which there is another such path with the same endpoints and yet smaller energy. If $b_1 \in p_1(V \cap (B_t \times_M B_t))$ then there is some point $b_2 \in B_t$ in $\pi^{-1}(\pi(b_1)) \cap B_t$ with $E(b_2) < E(b_1)$. Hence the complement of $p_1(V \cap (B_t \times_M B_t))$ in B_t consists of those points $b \in B_t$ such that $E|\pi^{-1}(\pi(b))$ is minimum at b relative to $\pi^{-1}(\pi(b)) \cap B_t$. But since (diameter $M)^2 < t$ we have that this complement consists exactly of $\mu = \{b \in B_s | E|\pi^{-1}(\pi(b))$ is minimum at $b\}$. This set μ is thus a subanalytic subset of B_s.

The subset $\mu \times_M \mu \subset B_s \times_M B_s$ is easily seen to be subanalytic. The diagonal of $B_s \times_M B_s$, which we denote by Δ, is subanalytic. It follows that $\mu \times_M \mu - \Delta$ is subanalytic. As it is relatively compact (the energy of any minimum must be $\leqslant$ (diameter $M)^2 < s$) we observe that

$$(\pi \times_M \pi)(\mu \times_M \mu - \Delta)$$

is subanalytic in M, where $\pi \times_M \pi$ is the map from $B_s \times_M B_s \to M$ determined by π. This set is precisely the set of $x \in M$ such that $E|\pi^{-1}(x)$ has at least two minima.

Since π is a submersion and μ is relatively compact it follows that μ is covered by a finite number of local products $U_1, U_2, \ldots, U_k$, i.e. each U_i is diffeomorphic to $V_i \times D_i$ where V_i is open in M and D_i is open in $\mathbf{R}^q$ (where q is the dimension of the fiber of the map π: $B_s \to M$) and we have the commutative diagram

$$\begin{tikzcd} V_i \times D_i \arrow[d, "\sigma_i"'] \arrow[r, phantom, "\approx"] & U_i \arrow[dl, "\pi"] \\ V_i & \end{tikzcd}$$

where σ_i is projection on the first factor. The above isomorphism and E define a function on $V_i \times D_i$ which we denote by $E_i = E_i(x, d)$. Let $H(E_i, x, d)$ denote the Hessian matrix of $E_{i,x}$ at d where $E_{i,x}(d) = E_i(x, d)$. Then the set $\zeta_i = \{(x, d) | H(E_i, x, d)$ has rank $< q\}$ is semianalytic. We view ζ_i as a subset of U_i and obtain a semianalytic subset $\zeta_i \cap \mu$ in U_i. These $\{\zeta_i \cap \mu\}$ piece together to give a semianalytic set in B_s (observe that the rank of the Hessian is invariantly defined at a critical point). Finally because the $\bigcup_i(\zeta_i \cap \mu)$ is relatively compact in B_s we obtain $\pi(\bigcup_i(\zeta_i \cap \mu))$ is subanalytic in M. But this set is precisely the set $\{x \in M \mid E|\pi^{-1}(x)$ has a degenerate minimum$\}$. We may now conclude that the cut locus itself is subanalytic.

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Sonderforschungsbereich, Theoretische Mathematik, Beringstrasse 1, 5300 Bonn, Federal Republic of Germany