ANNALS OF MATHEMATICS
Vol. 79, No. 1, January, 1964
Printed in Japan

RESOLUTION OF SINGULARITIES OF AN ALGEBRAIC VARIETY OVER A FIELD OF CHARACTERISTIC ZERO: I

BY HEISUKE HIRONAKA*

(*Dedicated to Professor Oscar Zariski*)

(Received June 3, 1962)
(Revised February 14, 1963)

Due to the length of this paper, it is being published in two parts. Part II will appear at the beginning of the next issue of this journal.

BIBLIOGRAPHY

1. S. ABHYANKER, *Local uniformization on algebraic surfaces over ground fields of characteristic* $p \neq 0$, Ann. of Math., 63 (1956), 491–526.
2. M. F. ATIYAH, and W. V. D. HODGE, *Integrals of the second kind on an algebraic variety*, Ann. of Math., 62 (1955), 56–91.
3. F. BRUHAT, and H. WHITNEY, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comment. Math. Helv., 33 (1959), 132–160.
4. H. CARTAN, *Variétés analytiques complexes et cohomologie*, Colloque de Bruxelles, (1953), pp. 41–55.
5. H. GRAUERT, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math., 68 (1958), 460–472.
6. A. GROTHENDIECK, Élèments de géométrie algébrique, I et II, Publications Mathematiques, No. 4 (1960) et No. 8 (1961).
7. ———, Séminaire de géométrie algébrique, de l'Institut des Hautes Études Scientifiques, No. IV-Platitude, (1960).
8. H. HIRONAKA, *A generalized theorem of Krull-Seidenberg on parametrized algebras of finite type*, Amer. J. Math., 82 (1960), 831–850.
9. ———, *Noetherian schemes in analytic geometry*, forthcoming.
10. ———, *Fundamental problems in bimeromorphic geometry*, forthcoming.
11. B. LEVI, *Sulla riduzione delle singolarità puntuali delle superficie algebriche dello spazio ordinario per transformationi quadratiche*, Ann. Mat. pura appl. II. s. Vol. 26 (1897).
12. ———, *Risoluzione delle singolarità puntuali delle superficie algebriche*, Atti Accad. Sci. Torino, Vol. 33 (1897).
13. M. NAGATA, *Some remarks on local rings*, II, Memoirs, Kyoto Univ., 28 (1954), 109–120.
14. ———, *A Jacobian criterion of simple points*, Illinois J. Math., 1 (1957), 427–432.
15. ———, Local Rings, Interscience Tracts in Math., No. 13, New York.
16. R. NARASIMHAN, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math., 82 (1960), 917–934.
17. P. SAMUEL, *La notion de multiplicité en algèbre et en géométrie algébrique*, J. Math. pures et appl., 30 (1951), 159–274; Thèse, Paris, 1951.
18. ———, and O. ZARISKI, Commutative algebra, I and II, Van Nostrand, Princeton, 1960.

* The author was supported by N.S.F. Grant-in-aid at the Institute for Advanced Study, Princeton, during the final course of preparation of this paper.

19. R. L. E. SCHWARZENBERGER, *Vector bundles on algebraic surfaces*, Proc. London Math. Soc., 11 (1961), 601–622.
20. J-P. SERRE, *Quelques problèmes globaux relatifs aux variétés de Stein*, Colloque de Bruxelles, (1953), pp. 57–68.
21. ———, *Géométrie algébrique et géométrie analytique*, Ann. de l'Institut Fourier, 6 (1955/56), 1–42.
22. J. WALKER, *Reduction of singularities of an algebraic surface*, Ann. of Math., 36 (1935), 336–365.
23. O. ZARISKI, *The reduction of singularities of an algebraic surface*, Ann. of Math., 40 (1939), 369–689.
24. ———, *Local uniformization theorem on algebraic varieties*, Ann. of Math., 41 (1940), 852–896.
25. ———, *A simplified proof for resolution of singularities of an algebraic surface*, Ann. of Math., 43 (1942), 583–593.
26. ———, *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc., 53 (1943), 490–542.
27. ———, *Reduction of singularities of algebraic three dimensional varieties*, Ann. of Math., 45 (1944), 472–542.
28. ———, *La risoluzione delle singolarità delle superficie algebriche immerse*, Accad. Naz. dei Lincei, Roma, 31 (1962), 97–102 (Nota I) pp. 177–180 (Nota II).
29. ———, *The concept of a simple point of an abstract algebraic variety*, Trans, Math. Soc., 62 (1947), 1–52.

TABLE OF CONTENTS

Introduction

Let X be complex-(resp. real-)analytic space, i.e., an analytic $\mathbf{C}$-(resp. $\mathbf{R}$-)space in the sense defined in §1 of Chapter 0. *We ask if there exists a morphism of complex-(resp. real-)analytic spaces, say $f: \tilde{X} \to X$, such that:*

(1) *$\tilde{X}$ is a complex-(resp. real-)analytic manifold,* i.e., *a non-singular complex-(resp. real-)analytic space,*

(2) *if V is the open subspace of X which consists of the simple points of X, then $f^{-1}(V)$ is an open dense subspace of $\tilde{X}$ and f induces an isomorphism of complex-(resp. real-)analytic manifolds: $f^{-1}(V) \xrightarrow{\approx} V$, and*

(3) *f is proper,* i.e., *the preimage by f of any compact subset of X is compact in $\tilde{X}$.*

This is the problem which we call *the resolution of singularities in the category of complex-(resp. real-)analytic spaces*, or more specifically, the resolution of singularities of the given complex-(resp. real-)analytic space X. If X is a reduced *complex*-analytic space, then the open subspace V is dense in X and therefore the condition (2) implies that f is a *modification*. (The term '*reduced*' means that the structural sheaf of local rings has no nilpotent elements.) It should be noted, however, that V is not always dense if X is a reduced *real*-analytic space. So far as the resolution of singularities is concerned, we are particularly interested in the case of reduced complex-(resp. real-)analytic spaces. As for the general case in which X may not be reduced, we have a better formulation of the problem in terms of normal flatness. (See Definition 1, § 4, Ch. 0.)

The most significant result of this work is the solution of the above problem for the case in which X has *an algebraic structure*; that is to say, X is covered by a finite number of coordinate neighborhoods, each of which is isomorphic to a closed analytic subspace of a complex (resp. real) number space defined by polynomial equation and which piece together by means of rational coordinate transformation. As a matter of fact, we prove the resolution of singularities of an arbitrary algebraic scheme over a field of characteristic zero. In general, we formulate *the resolution of singularities in the category of algebraic schemes* as follows. Let X be an algebraic $\mathbf{B}$-scheme in the sense defined in § 1 of Ch. 0, where $\mathbf{B}$ may be any commutative ring with unity. *Then we ask if there exists a morphism of algebraic* $\mathbf{B}$*-schemes* $f: \tilde{X} \to X$ *such that:*

(1) $\tilde{X}$ *is non-singular in the sense that all the local rings of* $\tilde{X}$ *are regular,*

(2) *if* V *is the open subscheme of* X *which consists of the simple points of* X*, then* $f^{-1}(V)$ *is an open dense subscheme of* $\tilde{X}$*, and* f *induces an isomorphism of schemes:* $f^{-1}(V) \xrightarrow{\approx} V$*, and*

(3) f *is proper in the sense of Grothendieck.* (See § 5, Ch. II, [6].)

Main theorem I (§ 3, Ch. 0) asserts *the resolution of singularities of an arbitary reduced algebraic* $\mathbf{B}$*-scheme* X *with a field* $\mathbf{B}$ *of characteristic zero. Main theorem* I* (§ 4, Ch. 0) states a better formulation of the result for the case in which X may not be reduced. We shall see that *main theorem* I* includes *main theorem* I in the strong form under the conditions (a), (b), and (c) imposed on the resolution process, which is explained in § 3 of Ch. 0. The finest form of the resolution of singularities that we can prove in this work can be found in *main theorem* I(n) stated in § 3 of Ch. I. We note here that no substantial progress is made in this work for the resolution of singularities of an algebraic $\mathbf{B}$-scheme where $\mathbf{B}$ is a field of a positive characteristic, or $\mathbf{B}$ admits a homomorphism onto a field of a positive characteristic.[1]

In § 7 of Ch. 0, we can find some generalizations of the above result to the analytic case. A local-analytic analogue of *main theorem* I is stated in *main theorem* I′ (§ 7, Ch. 0) and as an application of this theorem we can prove *the resolution of singularities of an arbitrary reduced complex-analytic space of dimension* 3. (See Corollary 3 of *main theorem* I′.) *Main theorem* I′(n) (§ 7, Ch. 0) includes *the resolution of singularities of an arbitrary complex Stein space* and *of an arbitrary analytic subspace of a product* $\mathbf{C}^n \times P^m$ of a complex number space $\mathbf{C}^n$ and a complex projective space P^m. *Main theorem* I″(n) (§ 7, Ch. 0) asserts *the resolution of singularities of an arbitrary real-analytic space.* We note that the

[1] The resolution of singularities in the positive characteristic case is proved only for surfaces. (See Abhyankar [1]). In the unequal characteristic case (i.e., the case in which the base ring $\mathbf{B}$ has characteristic zero and admits a homomorphism into a field of positive characteristic), the problem is not solved even for surfaces.

results of this paper in the complex-analytic case is far from being satisfactory.

In §1 of Ch. 0, we recall and make precise some of the basic terminologies concerning schemes, complex-analytic (or real-analytic) spaces and others related to them. In §2 of Ch. 0, we introduce and discuss the general notion of *blowing-up*, including the notion of monoidal transformation in the algebraic geometry and in the analytic geometry. These two sections are presented for the purpose of helping the reader to avoid any confusion in the geometric language which characterizes the later discussions.

In §3 of Ch. 0, we discuss the notions of equi-multiplicity and normal flatness of a scheme along a subscheme, as well as that of an analytic space along an analytic subspace. We shall recall the *theorem of B. Levi-Zariski* which asserts an almost canonical process of resolving the singularities of an arbitrary analytic (or algebraic) surface embedded in a three-dimensional analytic (or algebraic) manifold. In the theorem of B. Levi-Zariski, the equi-multiplicity plays an important role as a condition to be imposed on the centers of monoidal transformations. In the resolution of singularities in arbitrary dimensions, we take the normal flatness as the condition to be imposed on the centers of monoidal transformations, which turns out to be equivalent to the equi-multiplicity in the hypersurface case as that of B. Levi-Zariski. There will be found a principle of the resolution of singularities in general, modelled on the theorem of B. Levi-Zariski.

In §5, Ch. 0, we discuss the problems and the results concerning *the elimination of points of indeterminacy* of a rational (or, meromorphic) application of schemes (or, analytic space) by means of finite succession of monoidal transformations with non-singular centers. The elimination of points of indeterminacy is translated as *the trivialization of coherent sheaves of ideals*. (See Corollary 1 of *main theorem* II (§5, Ch. 0).) *Main theorem* II (§5, Ch. 0) which we call *the simplification of a coherent sheaf of ideals*, includes also what we call *the simplification of an algebraic boundary*. This asserts that the complement of a Zariski-open subset of a non-singular algebraic **B**-scheme (**B** being a field of characteristic zero) can be transformed into a subscheme which has *only normal crossings* by means of a finite succession of monoidal transformations with non-singular centers. (See Corollary 3 of *main theorem* II (§5, Ch. 0) and also Definition 2, (§5, Ch. 0) for the notion of *only normal crossings*). We shall also find a *lemma of Schwarzenberger* as a corollary of *main theorem* II, which asserts a transformation of a vector bundle into an extension of line bundles.

In §6, Ch. 0, the *local uniformization theorem* of Zariski is recalled with explanation of its use in the resolution of singularities in lower dimensional cases. There will be found discussions on the problems, such as *the factorization of a birational application,* which arose in the effort to deduce the resolution of singularities from the *local uniformization theorem*.

Further discussions of the above problems in the analytic case can be found in §7 of Ch. 0.

Thus Chapter 0 is devoted to general discussions of the problems and the results in the birational (or, bimeromorphic) geometry, and most of the results are stated only with sketchy proofs. Some of the results stated there with neither proofs nor sufficient references will be published whenever needed in the future.[2] The author hopes that the discussion in Chapter 0 will serve to help the reader to a better understanding of the theorems and their proofs in the rest of this work, especially for his better understanding of the *fundamental theorems* (§1, Ch. I) which are formulated strictly in the form which suits our inductive proofs. In order to follow the logical development in the rest of this work, the reader who knows the basic language in the algebraic geometry and in the theory of local rings will need nothing from Chapter 0 but Definition 1 of §4 (the notion of *normal flatness*) and Definition 2 of §5 (the notion of *only normal crossings*).

Ch. II is a self-contained study of normal flatness (the definition of *algebraic schemes* in the introductory paragraph of Ch. I and Definition 1 (§4, Ch. 0) should be referred to) and then Ch. III follows as a local study of singularities. The reader who studied these two chapters may proceed to Ch. IV with references to §1 and §2 of Ch. I. The arguments in §3 and §4 of Ch. I depend upon some of the results in the later chapters, although the theorems stated there are by nature very close to those stated in the earlier sections of Ch. I. For this reason the reader may find it suitable to set aside the last two sections of Ch. I until he goes through the rest of Chs. I-IV.

The resolution theorems are formulated in terms of resolution data defined in §1 of Ch. I and stated as *the fundamental theorems* $\mathrm{I}_1^{N,n}$, $\mathrm{I}_2^{N,n}$, II_1^N and II_2^N in §2 of Ch. I. We shall indicate in §2 of Ch. I, the diagram of implications among the *fundamental theorems* with various integers N and n attached to them, according to which the inductive proof proceeds. The complete proof of the indicated implications will be estab-

[2] cf. the forthcoming paper, [9] and [10].

lished in the last chapter of this paper.

The normal flatness plays an important role in the statements of the *fundamental theorems* as well as in their proofs. It is one of the conditions imposed on the centers of monoidal transformations which are used for the purpose of resolving singularities, or resolving resolution data in the sense of the *fundamental theorems*. A general discussion of the normal flatness can be found in § 4, Ch. 0. Ch. II is devoted to prove certain fundamental theorems on the normal flatness of an algebraic scheme along subschemes; Theorem 1, § 1, Ch. II, shows that the normal flatness along a subscheme is an open condition on the points of the subscheme, Theorem 2, § 2, Ch. II, gives various conditions equivalent to the normal flatness, and finally Theorem 3, § 3, Ch. II, asserts a certain chain property of the normal flatness along subschemes.

Ch. III is devoted to *a local study of singularities* of an algebraic scheme, of which the central theme is the effects on singularities by a succession of *permissible* monoidal transformations, i.e., monoidal transformations whose centers are non-singular and subject to the normal flatness conditions. For some technical reason, we fix an ambient scheme, that is a non-singular algebraic scheme which contains the given scheme as a subscheme, and we try to describe the singularity of the given scheme at a point by means of the ideal on the ambient scheme at the point which defines the subscheme. We define a numerical character of an ideal in a regular local ring, denoted by ν^* (in § 1, Ch. III), and another, denoted by τ^* (in § 4, Ch. III). As a principal measurement of singularities of an algebraic scheme with a specific ambient scheme, we take the numerical characters ν^* and τ^* of the local ideals on the ambient scheme which define the subscheme. If V is an algebraic scheme with an ambient scheme X, and if V' is an algebraic scheme obtained by a monoidal transformation of V with a non-singular center, then the scheme obtained by the monoidal tranformation of X with the same center can be viewed as an ambient scheme of V' in a canonical manner. Denote by X' this ambient scheme of V'. Theorem 3 (§ 5, Ch. III) asserts that, if the monoidal transformation is permissible for V, then for every closed point x' of V' which is mapped to a closed point x of V, the character ν^* of V' at x' is at most that of V at x (with reference to X and X') and, if they are the same, the character τ^* of V' at x' is at least that of V at x (with reference to X and X'). Let us simply say that *the singularity does not become worse by any permissible monoidal transformation*. Theorem 4 (§ 6, Ch. III) asserts that, given infinite succession of permissible monoidal transformations for an algebraic scheme, those numerical characters at corresponding points are the same except for a finite portion of the succession. We may roughly express the result by saying that *the singularities can not be made infinitely better by any succession of permissible monoidal transformations.* By virtue of these theorems, the resolution of singularities is reduced to find a suitable succession of permissible monoidal transformations which makes the given singularities *better* in terms of those numerical characters. (This principle is described further in § 4, Ch. 0.) In reality, the situation is more complicated for the reason that we are forced to prove as much as the *fundamental theorems* in order to carry out the inductive proof for all dimensions. The goal of the study of singularities, locally at a point, is to prove the existence of a special coordination (called *a regular τ-frame*, Definition 10, § 7) of the ambient scheme at the point, and the existence of a special base (called *a standard base*) of the ideal defining the subscheme at the point, both of which have a certain *stability* (*or universality*) *property* with respect to successions of permissible monoidal transformations. (See § 9 for the notion of stability, and Theorem 9, § 10, for the existence.) The *existence theorem* plays the most important role in the inductive proof of the *fundamental theorems.*

Ch. IV, the last one, is devoted to the proofs of the inductive implications (indicated in § 2, Ch. I) which establish the *fundamental theorems* for all possible integers N and n. In this chapter, the arguments are comparatively of geometric nature, and all the algebraic results in the two preceding chapters are applied to the geometric situations which arise in the proofs.

The author learned the methodological foundation of the present work from the three distinguished teachers, Professor A. Grothendieck, Professor M. Nagata and Professor O. Zariski. Namely, this work is based on the theory of local rings[3] (in a rather special case), the language of schemes in the sense of Grothendieck[4], and the birational geometry of Zariski.[5] In completing the present manuscript, the author received considerable help from Professor O. Zariski, and also from Professor

[3] The two chapters, VII and VIII, of Samuel-Zariski, [18], should cover all that we need from the theory of local rings. A complete reference to the theory may be obtained from M. Nagata [15].

[4] Grothendieck [6] should be referred to.

[5] We refer to Zariski [26] and also to his works on the resolution of singularities [23] [24] [25] [27] [28] etc. The theory in this paper (except for the introductory chapter in which we give an expository background of the problem) does not depend upon any particular resolution theorem of Zariski, but the author was mostly inspired by those works of Zariski on the problem.

M. Kuranishi, through their careful reading of the first draft of this work, which was mimeographed by a research project under Professor D. C. Spencer at Princeton University. The author sincerely appreciates the financial support of this work during the past five years, especially that received at Harvard University, at Brandeis University, and at l'Institut des Hautes Études Scientifiques.

CHAPTER 0. PRELIMINARIES, MAIN PROBLEMS AND MAIN THEOREMS

1. Generalities on schemes[6] and complex-(or real-)analytic spaces

A *local-ringed space* X is by definition a pair of a topological space $\mathbf{X}$ and a sheaf of local rings (i.e., sheaf of rings whose stalks are local rings) O_X on $\mathbf{X}$ which is often referred to as the structural sheaf of X. A *morphism of local-ringed spaces* $f\colon X \to Y$, where $X = (\mathbf{X}, O_X)$ and $Y = (\mathbf{Y}, O_Y)$, is by definition a pair of a continuous mapping $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ and an $\mathbf{f}$-homomorphism $\theta\colon O_Y \to O_X$ which induces for each point x of X a *local* homomorphism $\theta_x\colon O_{Y,\mathbf{f}(x)} \to O_{X\,x}$, 'local' in the sense that it maps the maximal ideal of the first local ring into that of the second. To be precise, an $\mathbf{f}$-homomorphism $\theta\colon O_Y \to O_X$ is a homomorphism of sheaves of rings $\mathbf{f}^*(O_Y) \to O_X$, where $\mathbf{f}^*(O_Y)$ denotes the sheaf of rings on X obtained as the inverse image of O_Y by the continuous mapping $\mathbf{f}$. It is convenient that a point of $\mathbf{X}$ is referred to as a point of X, and that we write $f(x)$ for $\mathbf{f}(x)$. Let us denote by $\mathscr{R}$ *the category of local-ringed spaces.*

A *prescheme* is by definition a local-ringed space $X = (\mathbf{X}, O_X)$ such that, for each point x of X, there exists an open neighborhood $\mathbf{X}_i$ of x in $\mathbf{X}$ such that the restriction $X_i = (\mathbf{X}_i, O_X \mid \mathbf{X}_i)$ of X to $\mathbf{X}_i$ is isomorphic to an affine scheme, i.e., Spec($\mathbf{A}$) with a commutative ring $\mathbf{A}$ with unity. We denote by Spec($\mathbf{A}$) the local-ringed space $S = (\mathbf{S}, O_S)$ where $\mathbf{S}$ is the set of prime ideals in the ring $\mathbf{A}$ with Zariski topology (i.e., a subset of $\mathbf{S}$ is *open* if and only if it is of the form $\{\mathbf{P} \in \mathbf{S} \mid \mathbf{P} \not\supseteq \mathbf{I}\}$ for an ideal $\mathbf{I}$ in $\mathbf{A}$) and O_S is the sheaf of localizations of $\mathbf{A}$ on $\mathbf{S}$ (i.e., for every open subset $\mathbf{U}$ of $\mathbf{S}$, $\Gamma(\mathbf{U}, O_S) = \mathbf{A}_{T(\mathbf{U})}$ where $T(\mathbf{U})$ denotes the multiplicative set of those elements in $\mathbf{A}$ which are not contained in any prime ideal in $\mathbf{U}$). We call Spec($\mathbf{A}$) the *prime spectrum* (or, simply, the *spectrum*) *of* $\mathbf{A}$. We shall denote by *pre-$\mathcal{S}$ the category of preschemes,* which is a *full* subcategory of $\mathscr{R}$, full in the sense that, if X and Y are preschemes, then the set of morphisms, Hom (X, Y), in *pre-$\mathcal{S}$* is equal to that in $\mathscr{R}$. As is done for an arbitrary category, if S is an object in *pre-$\mathcal{S}$* (i.e., a prescheme), then we can derive a new category from *pre-$\mathcal{S}$* which is called

[6] cf. Ch. I of Grothendieck [6].

the category of preschemes over the prescheme S or the category of S-*preschemes*, and denoted by *pre*-$\mathcal{S}/S$. An S-prescheme is a pair (X, p) of a prescheme X and a morphism p of X to S, and a morphism $g: (X_1, p_1) \to (X_2, p_2)$ of S-preschemes (X_i, p_i), $i = 1, 2$, is by definition a morphism $g: X_1 \to X_2$ of preschemes such that $p_1 = p_2 \circ g$. In the category *pre*-$\mathcal{S}/S$, the *product* of any two objects exists. Namely, if (X_i, p_i), $i = 1, 2$, are S-preschemes, then there exists an S-prescheme, denoted by $(X_1 \times {}_S X_2, p_1 \times {}_S p_2)$ and called the *product*, and a pair of morphisms (π_1, π_2) from the product to (X_1, p_1) and to (X_2, p_2) respectively, called the *projections to the first factor* and *to the second* respectively, such that, if P, F_1, and F_2 denote the product, the first factor and the second respectively, we have a canonical isomorphism of sets:

$$\mathrm{Hom}\,(Q, P) \xrightarrow{\approx} \mathrm{Hom}\,(Q, F_1) \times \mathrm{Hom}\,(Q, F_2)\,, \qquad \text{for every } Q\,,$$

where this mapping is obtained by composing any given morphism in $\mathrm{Hom}\,(Q, P)$ with π_1 and π_2. In particular, if the given two S-preschemes are the same, say (X, p) then we have a canonical morphism $\Delta: (X, p) \to (X \times {}_S X, p \times {}_S p)$ which is called *the diagonal morphism* of (X, p) into $(X \times {}_S X, p \times {}_S p)$ or, simply, *the diagonal morphism* X *into* $X \times {}_S X$. This diagonal morphism is an immersion in the sense that it induces an isomorphism of X to a subscheme of $X \times {}_S X$ which is called *the diagonal of* $X \times {}_S X$. If the diagonal of $X \times {}_S X$ is a closed subscheme, then the S-prescheme (X, p) is called an S-*scheme*. In general, any prescheme X can be viewed in a unique way as a prescheme over Spec($\mathbf{Z}$), where $\mathbf{Z}$ denotes the ring of rational integers. Let us denote the canonical morphism by $p_{\mathbf{Z}}: X \to \mathrm{Spec}(\mathbf{Z})$. We say that X is a *scheme*, if $(X, p_{\mathbf{Z}})$ is an Spec($\mathbf{Z}$)-scheme. It can be proved that every affine scheme is a scheme, and that if S is a scheme and (X, p) is an S-scheme, then X is a scheme. Let us denote by $\mathcal{S}$ *the category of schemes*, which is a full subcategory of *pre*-$\mathcal{S}$ (hence, of $\mathcal{R}$).

We shall often say that X *is an* S-*prescheme*, meaning that (X, p) is an S-prescheme with a specific morphism p. We shall also say that, when two S-preschemes (X_1, p_1) and (X_2, p_2) are given, a morphism $f: X_1 \to X_2$ is *an* S-*morphism of* X_1 *to* X_2, if f is a morphism of S-preschemes $(X, p_1) \to (X_2, p_2)$. In this manner, we shall drop the symbols to indicate the *structural morphism* p (or, p_i) from X (or, X_i) to the *base* prescheme S. If $S = \mathrm{Spec}(\mathbf{B})$ with a commutative ring $\mathbf{B}$ with unity, then an S-prescheme (resp. an S-scheme) will be called a $\mathbf{B}$-*prescheme* (resp. a $\mathbf{B}$-*scheme*). We shall be particularly interested in $\mathbf{B}$-schemes where $\mathbf{B}$ is a noetherian ring. Assuming this, a $\mathbf{B}$-prescheme (or, $\mathbf{B}$-scheme) is said to be *algebraic* if it is of finite type over Spec(**B**) (i.e., the structural morphism of the **B**-prescheme to Spec(**B**) is of finite type). Thus, by definition, an algebraic **B**-prescheme (resp. **B**-scheme) is obtained by patching together a finite number of open affine subschemes of the type Spec(**A**) with **B**-algebras **A** of finite type. We denote by *pre*-$\mathcal{A}l/\mathbf{B}$ (resp. $\mathcal{A}l/\mathbf{B}$) the category of algebraic **B**-prechemes (resp. algebraic **B**-schemes), which is a full subcategory of *pre*-$\mathcal{S}$/Spec(**B**) (resp. of $\mathcal{S}$/Spec(**B**) as well as of *pre*-$\mathcal{A}l/\mathbf{B}$). In the beginning of the next chapter we shall define the terms "*algebraic scheme*" and "*morphism of algebraic schemes*" which are to be used throughout from that chapter on. An algebraic scheme will be a scheme which can be given a structure of algebraic **B**-scheme with **B** of rather special nature. This special nature of **B** will there be specified.

Another important category of local-ringed spaces is that of complex- (or real-)analytic spaces. Before we give definitions to these terms, we set up the following general terminology. If S is a local-ringed space, we denote by $\mathcal{R}/S$ the category of local-ringed spaces over S. We write $\mathcal{R}/\mathbf{B}$ for $\mathcal{R}$/Spec(**B**), where **B** is any commutative ring with unity. We say that X is an *S-local-ringed* (resp. **B**-*local-ringed*) *space*, meaning that a specific morphism $p: X \to S$ (resp. $X \to \mathrm{Spec}(\mathbf{B})$) is given (no matter whether or not mentioned). This specific morphism is called *the structural morphism* of the S-(or **B**-)local-ringed space. If $X_i (i = 1, 2)$ are given as an S-(resp. **B**-)local-ringed space, an *S-morphism* (resp. **B**-*morphism*) *of X_1 to X_2* is meant to be a morphism $f: X_1 \to X_2$ such that $p_1 = f \circ p_2$ with the structural morphism $p_i: X_i \to S$ (resp. $X_i \to \mathrm{Spec}(\mathbf{B})$), that is, a morphism in the category $\mathcal{R}/S$ (resp. in $\mathcal{R}/\mathbf{B}$).

Let us denote by **K** either the field of complex numbers **C** or that of real numbers **R**, in which the standard topology is given. Let $\mathbf{K}^n$ denote the cartesian product of n-copies of the topological space **K**, and $A_{\mathbf{K}^n}$ the sheaf of germs of regular analytic **K**-functions on $\mathbf{K}^n$, i.e., the sheaf of convergent power series rings $A_{\mathbf{K}^n,x} = \mathbf{K}\{z_1 - x_1, \cdots, z_n - x_n\}$ on $\mathbf{K}^n$, where $x = (x_1, \cdots, x_n)$ is a point of $\mathbf{K}^n$ and $(z_1, \cdots, z_n)$ the system of coordinate functions on $\mathbf{K}^n$. Let **G** be a connected open subset of $\mathbf{K}^n$ and A_G the restriction of $A_{\mathbf{K}^n}$ to **G**. Notice that $G = (\mathbf{G}, A_G)$ (in particular, $(\mathbf{K}^n, A_{\mathbf{K}^n})$) has a natural structure of **K**-local-ringed space. We consider **K**-local-ringed subspaces of G of the following type: $(S(I), (A_G/I) \mid \mathrm{S}(I))$ where I is a sheaf of ideals in A_G generated by a finite number of regular analytic **K**-functions on G, i.e., elements of $\Gamma(\mathbf{G}, A_G)$, and $\mathrm{S}(I)$ denotes the support of A_G/I. Let us call a **K**-local-ringed space obtained in this manner a *local analytic* **K**-*space*. We define a general analytic **K**-space as follows: *Analytic* **K**-*space is a* **K**-*local-ringed space* $X =$

$(\mathbf{X}, O_X)$ *such that*

(i) *for every point* x *of* $\mathbf{X}$, *there exists an open neighborbood* $\mathbf{U}$ *of* x *in* X *such that the restriction* $(\mathbf{U}, O_X \mid \mathbf{U})$ *of* X *to* $\mathbf{U}$ *is* $\mathbf{K}$-*isomorphic to a local analytic* $\mathbf{K}$-*space defined above,*

(ii) *the underlying topological space* $\mathbf{X}$ *of* X *is countable at infinity,* i.e., *a union of countably many compact subsets, and*

(iii) $\mathbf{X}$ *is a Hausdorff space, or equivalently, the diagonal in the cartesian product* $\mathbf{X} \times \mathbf{X}$ *is closed.*

A $\mathbf{K}$-local-ringed space is called *a preanalytic* $\mathbf{K}$-*space,* if it satisfies the conditions (i) and (ii) (but not necessarily (iii)). Let us call $(\mathbf{U}, h)$ *a local* $\mathbf{K}^n$-*coordination of* X at a point x of X, if $\mathbf{U}$ is an open neighborhood of x in $\mathbf{X}$ and h is a $\mathbf{K}$-morphism of $X \mid \mathbf{U}$ (the restriction of X to $\mathbf{U}$) to $(\mathbf{K}^n, A_{\mathbf{K}^n})$ which induces a $\mathbf{K}$-isomorphism of $X \mid \mathbf{U}$ to a local analytic $\mathbf{K}$-space (in $(\mathbf{K}^n, A_{\mathbf{K}^n})$), as in (i). We identify once for always $\mathbf{R}$ as a topological subfield of $\mathbf{C}$, and accordingly $\mathbf{R}^n$ as a topological subspace of $\mathbf{C}^n$. We have a natural $\mathbf{C}$-isomorphism $\alpha_n: (\mathbf{R}^n, A_{\mathbf{R}^n} \otimes_{\mathbf{R}} \mathbf{C}) \to (\mathbf{C}^n, A_{\mathbf{C}^n}) \mid \mathbf{R}^n$. Let $X = (\mathbf{X}, O_X)$ be a preanalytic $\mathbf{R}$-space. Then $(\mathbf{X}, O_X \otimes_{\mathbf{R}} \mathbf{C})$ is a $\mathbf{C}$-local-ringed space, which will be denoted by $X(\mathbf{C})$. *A preanalytic complexification* of X is by definition a pair (Y, f) of a preanalytic $\mathbf{C}$-space Y and a $\mathbf{C}$-morphism of $X(\mathbf{C})$ to Y, such that

(1) the image $f(\mathbf{X})$ of $\mathbf{X}$ is a closed subset of the underlying topological space of Y, and

(2) f induces a $\mathbf{C}$-isomorphism of $X(\mathbf{C})$ to $Y \mid f(\mathbf{X})$.

If (Y, f) is a pre-analytic complexification of an analytic $\mathbf{R}$-space X, then for every point x of X there exists an open neighborhood $\mathbf{U}$ of x in the underlying topological space of X such that any $\mathbf{R}$-morphism h of $X \mid \mathbf{U}$ to $(\mathbf{R}^n, A_{\mathbf{R}^n})$ induces a $\mathbf{C}$-morphism h' of $Y \mid \mathbf{U}'$ to $(\mathbf{C}^n, A_{\mathbf{C}^n})$, where $\mathbf{U}'$ is an open neighborhood of $f(\mathbf{U})$ in the underlying topological space of Y, such that $h' \circ (f \mid \mathbf{U}) = \alpha_n \circ h_{\mathbf{C}}$, where $h_{\mathbf{C}}$ denotes the $\mathbf{C}$-morphism, induced by h, from $X(\mathbf{C}) \mid \mathbf{U}$ to $(\mathbf{R}^n, A_{\mathbf{R}^n} \otimes_{\mathbf{R}} \mathbf{C})$. If (U, h) is a local $\mathbf{R}^n$-coordination of X at x, then for a sufficiently small neighborhood $\mathbf{U}''$ of $f(\mathbf{U})$ in $\mathbf{U}'$, the pair $(\mathbf{U}'', h' \mid \mathbf{U}'')$ is a local $\mathbf{C}^n$-coordination of Y at $f(x)$. *An analytic complexification of* X is then defined as a preanalytic complexification (Y, f) of X such that Y is an analytic $\mathbf{C}$-space. Due to H. Whitney and F. Bruhat is the following complexification theorem: *Every analytic* $\mathbf{R}$-*space admits an analytic complexification.*[7]

We shall call an analytic $\mathbf{C}$-space X *a complex Stein space,* if Theorems A and B (of H. Cartan and J-P. Serre) hold for every coherent sheaf of

[7] cf. Proposition 1 of Bruhat-Whitney [3]. Their proof is applicable to any analytic $\mathbf{R}$-space which may have singularities. (cf. [9].)

modules on X.[8] It is due to H. Grauert that, if (Y, f) is an analytic complexification of an analytic $\mathbf{R}$-space X, there exists a fundamental system of neighborhoods of $f(\mathrm{X})$ in the underlying topological space of Y such that the restrictions of Y to those neighborhoods are Stein spaces.[9] It follows that *the Theorems* A *and* B *hold for every coherent sheaf of modules on an analytic* $\mathbf{R}$-*space*.[10] A point x of a preanalytic $\mathbf{K}$-space $X = (\mathrm{X}, O_X)$ will be said to be *simple* (resp. *multiple*) if the local ring $O_{X,x}$ of x is regular (resp. not regular). Accordingly, X is said to be *non-singular* if it has no multiple points. This definition of the terms '*simple*', '*multiple*' '*non-singular*' etc., makes sense for arbitrary local-ringed spaces and they will be so meant throughout this paper. By means of jacobian criterion and the implicit function theorem, one can prove that x is a simple point of X if and only if there exists an open neighborhood U of x in X such that the restriction $X \mid \mathrm{U}$ is $\mathbf{K}$-isomorphic to a local analytic $\mathbf{K}$-space of the form $(\mathrm{G}, A_{\mathbf{K}^n} \mid \mathrm{G})$, where G is is an open subset of $\mathbf{K}^n$. It was also proved by H. Grauert that every non-singular irreducible analytic $\mathbf{R}$-space is $\mathbf{R}$-isomorphic to a closed analytic subspace of $(\mathbf{R}^n, A_{\mathbf{R}^n})$, i.e., an analytic $\mathbf{R}$-space of the form $(\mathrm{S}, (A_{\mathbf{R}^n}/I) \mid \mathrm{S})$ where I is a coherent sheaf of ideals in $A_{\mathbf{R}^n}$ and S denotes the support of the quotient sheaf $A_{\mathbf{R}^n}/I$ in $\mathbf{R}^n$.[11] This is not true in general if it has singularities. Let us denote by *pre-$\mathcal{A}n/\mathbf{K}$* (resp. *$\mathcal{A}n/\mathbf{K}$*) *the category of preanalytic* $\mathbf{K}$-*spaces* (resp. *of analytic* $\mathbf{K}$-*spaces*), which is a full subcategory of $\mathcal{R}/\mathbf{K}$ (resp. of *pre-$\mathcal{A}n/\mathbf{K}$*).

Let $\mathbf{k}$ be a subfield of $\mathbf{K}$, where $\mathbf{K}$ denotes either $\mathbf{C}$ or $\mathbf{R}$ as above. Then, every preanalytic $\mathbf{K}$-space (which is a $\mathbf{K}$-local-ringed space) can be viewed in a canonical way as a $\mathbf{k}$-local-ringed space whose structural morphism is obtained by composing that of the preanalytic $\mathbf{K}$-space to $\mathrm{Spec}(\mathbf{K})$ and the morphism $\mathrm{Spec}(\mathbf{K}) \to \mathrm{Spec}(\mathbf{k})$ associated with the inclusion homomorphism $\mathbf{k} \to \mathbf{K}$. Given a $\mathbf{k}$-local-ringed space $X = (\mathrm{X}, O_X)$, we first ask if there exists a pair $(\hat{X}, \lambda)$ of a preanalytic $\mathbf{K}$-space $\hat{X}$ and a $\mathbf{k}$-morphism $\lambda: \hat{X} \to X$, which satisfies the following universal mapping property:

[8] cf. Cartan [4] and Serre [20].

[9] cf. Grauert [5]. His proof can be easily modified and applied to a general analytic $\mathbf{R}$-space.

[10] This fact may be proved as follows: We first prove it for a real number space, then for every restriction of an analytic $\mathbf{R}$-space to a relatively compact open subset (using the fact that such a restriction admits a proper imbedding into a real number space by means of Narasimhan [16]), and finally for a general analytic $\mathbf{R}$-space (which, by definition, can be expressed as a countable increasing union of relatively compact open subspaces).

[11] cf. Grauert [15].

(*) *If* $(\hat{X}', \lambda')$ *is any pair of a preanalytic* **K**-*space* $\hat{X}'$ *and a* **k**-*morphism* $\lambda'\colon \hat{X}' \to X$, *then there exists a unique* **K**-*morphism* $\mu\colon \hat{X}' \to \hat{X}$ *such that* $\lambda' = \lambda \circ \mu$. If such a pair $(\hat{X}, \lambda)$ for X does exist, we ask if it satisfies the following conditions:

(1) *The morphism* $\lambda\colon \hat{X} \to X$ *is flat*, i.e., $O_{\hat{X},\hat{x}}$ *is* $O_{X\ \lambda(\hat{x})}$-*flat for every point* $\hat{x}$ *of* $\hat{X}$, *and*

(2) *at every point* $\hat{x}$ *of* $\hat{X}$, *there exists a local* $\mathbf{K}^n$-*coordination of* $\hat{X}$ *such that the germs of the* n *coordinate functions at* $\hat{x}$ *belong to the image of the homomorphism* $O_{X\ \lambda(\hat{x})} \to O_{\hat{X},\hat{x}}$ *induced by the morphism* λ. If a pair $(\hat{X}, \lambda)$ exists for a given X, which satisfies (*), then it is unique up to a **K**-isomorphism of $\hat{X}$, and it will be called *an analytic* **K**-*cover* of X provided it satisfies (1) and (2) in addition to (*). Let us denote by $\mathcal{A}(\mathbf{K}/\mathbf{k})$ the full subcategory of $\mathcal{R}/\mathbf{k}$ consisting of those **k**-local-ringed spaces which admit analytic **K**-covers. We can prove that *every algebraic* **k**-*prescheme admits an analytic* **K**-*cover*, i.e., it belongs to $\mathcal{A}(\mathbf{K}/\mathbf{k})$. Let X be an algebraic **k**-prescheme and $(\hat{X}, \lambda)$ the analytic **K**-cover of X. Then $\hat{X}$ is an *analytic* **K**-space if X is an algebraic **k**-scheme. The converse is true if $\mathbf{K} = \mathbf{C}$, but not in general if $\mathbf{K} = \mathbf{R}$. Let us consider the case in which $\mathbf{k} = \mathbf{K}$. We write $\mathcal{A}(\mathbf{K})$ for $\mathcal{A}(\mathbf{K}/\mathbf{K})$. In this case, a trivial observation tells us that every preanalytic **K**-space belongs to $\mathcal{A}(\mathbf{K})$. Thus *pre*-$\mathcal{A}n/\mathbf{K}$ and *pre*-$\mathcal{A}l/\mathbf{K}$ are full subcategories of $\mathcal{A}(\mathbf{K})$.

Let $\hat{X} = (\hat{\mathbf{X}}, O_{\hat{X}})$ be a complex Stein space. Let $X = \mathrm{Spec}(\Gamma(\hat{\mathbf{X}}, O_{\hat{X}}))$ and let $\lambda\colon \hat{X} \to X$ be the canonical **C**-morphism. *Then* $(\hat{X}, \lambda)$ *is an analytic* **C**-*cover of* X. Let $\hat{Y} = (\hat{\mathbf{Y}}, O_{\hat{Y}})$ be any analytic **R**-space. Let $Y = \mathrm{Spec}(\Gamma(\hat{\mathbf{Y}}, O_{\hat{Y}}))$ and let $\mu\colon \hat{Y} \to Y$ be the canonical morphism. *Then* $(\hat{Y}, \mu)$ *is an analytic* **R**-*cover of* Y. These two statements can be proved by means of Theorems A and B for coherent sheaf of modules, which hold for complex Stein spaces and analytic **R**-spaces.[12]

Let **k** and **K** be as above. Let X be an algebraic **k**-prescheme and let $X_{\mathbf{K}}$ denote the product $X \times_{\mathrm{Spec}(\mathbf{k})} \mathrm{Spec}(\mathbf{K})$ in the category *pre*-$\mathcal{S}/\mathbf{k}$, which can be viewed as an algebraic **K**-prescheme. Let $(\hat{X}, \lambda)$ be a pair of an analytic **K**-space $\hat{X}$ and a **k**-morphism $\lambda\colon \hat{X} \to X$. We then get a unique **K**-morphism $\lambda'\colon \hat{X} \to X_{\mathbf{K}}$ which gives λ and the structural morphism of $\hat{X}$ when followed by the projections of $X_{\mathbf{K}}$ to the factors. We see that $(\hat{X}, \lambda)$ *is an analytic* **K**-*cover of* X *if and only if* $(\hat{X}, \lambda')$ *is such of* $X_{\mathbf{K}}$.

We shall make reference to the category $\mathcal{A}(\mathbf{K})$ in the next section when we speak of blowing up of coherent sheaves of modules on an analytic **K**-space.

---

[12] Some basic facts about analytic **C**-covers will be presented and proved in a forthcoming paper [9].

2. Generalities on blowing-up

Let S be a local-ringed space or a commutative ring with unity, and $\mathcal{R}/S$ the category of S-local-ringed spaces. Then by *an* $(\mathcal{R}/S)$*-module*, we shall mean a pair (X, J) of an S-local-ringed space X and a sheaf of modules J on X. *A morphism of* $(\mathcal{R}/S)$*-modules*, say $(X_1, J_1) \to (X_2, J_2)$, is by definition a pair (f, φ) of an S-morphism $f: X_1 \to X_2$ and an f-epimorphism of sheaves of modules $\varphi: J_2 \to J_1$, i.e., a surjective homomorphism of sheaves of modules on X_1 from $f^*(J_2)$ to J_1. We identify in a natural manner the category $\mathcal{R}$ with $\mathcal{R}/\mathbf{Z}$, where $\mathbf{Z}$ denotes the ring of integers. For instance, an $(\mathcal{R}/\mathbf{Z})$-module will be called *an* $\mathcal{R}$*-module*. An $(\mathcal{R}/S)$-module (X, J) with $X = (\mathbf{X}, O_X)$ will be said to be *invertible* if every point of $\mathbf{X}$ has a neighborhood $\mathbf{U}$ in $\mathbf{X}$ such that the restrictions $O_X \mid \mathbf{U}$ and $J \mid \mathbf{U}$ are isomorphic to each other as sheaves of $(O_X \mid \mathbf{U})$-modules. Let us denote by $\mathcal{M}(\mathcal{R}/S)$ *the category of* $(\mathcal{R}/S)$*-modules*. We write $\mathcal{M}(\mathcal{R})$ for $\mathcal{M}(\mathcal{R}/\mathbf{Z})$. If $\mathcal{R}'$ is a subcategory of $\mathcal{R}/S$, we denote by $\mathcal{M}(\mathcal{R}')$ the subcatgory of $\mathcal{M}(\mathcal{R}/S)$ of those $(\mathcal{R}/S)$-modules (X, J) with X in $\mathcal{R}'$ and of those morphisms $(f, \varphi): (X_1, J_1) \to (X_2, J_2)$ with f in $\mathcal{R}'$. We have then a canonical covariant functor from $\mathcal{M}(\mathcal{R}')$ to $\mathcal{M}(\mathcal{R})$, induced by that from $\mathcal{M}(\mathcal{R}/S)$ to $\mathcal{M}(\mathcal{R})$, which is obtained by neglecting the structural morphisms of S-local-ringed spaces. In general, let T be a covariant functor from a category $\mathcal{M}'$ to $\mathcal{M}(\mathcal{R})$. Then, given an object M in $\mathcal{M}'$, we ask if *there exists a pair* (P, Φ) *of an object* P *in* $\mathcal{M}'$ *such that* $T(P)$ *is invertible and of a morphism* $\Phi: P \to M$ *in* $\mathcal{M}'$ *which has the following universal mapping property*:

(**) *If* (P', Φ') *is any pair of the same type, then there exists a unique morphism* $\Psi: P' \to P$ *in* $\mathcal{M}'$ *such that* $\Phi' = \Phi \circ \Psi$.

Such a pair (P, Φ) is unique up to an isomorphism of P in $\mathcal{M}'$, if it exists, and will be called *a blowing-up of* M *in the category* $\mathcal{M}'$ *with respect to* T. When the functor T is an obvious one, such as the canonical one from $\mathcal{M}(\mathcal{R}')$ (or from a subcategory of $\mathcal{M}(\mathcal{R}')$) to $\mathcal{M}(\mathcal{R})$ for some subcategory $\mathcal{R}'$ of $\mathcal{R}/S$, we shall speak of blowing-up in $\mathcal{M}'$ without mentioning the reference to T.

The basic existence theorem of blowing-up is that for the category $\mathcal{M}(\mathcal{R})$ with the identity functor to itself. Namely, we can prove that *every* $\mathcal{R}$*-module admits a blowing-up in the category of* $\mathcal{R}$*-modules* $\mathcal{M}(\mathcal{R})$. An immediate consequence is the existence theorem of blowing-up for the category $\mathcal{M}(\mathcal{R}/S)$. In general, let $\mathcal{M}'$ be a category with an obvious functor $T: \mathcal{M}' \to \mathcal{M}(\mathcal{R})$. We shall say that $\mathcal{M}'$ is *closed under blowing-up in the category* $\mathcal{M}(\mathcal{R})$, if every object M in $\mathcal{M}'$ admits a blowing-up (P, Φ) in $\mathcal{M}'$ (with respect to T) such that $(T(P), T(\Phi))$ is a blowing-up of $T(M)$ in $\mathcal{M}(\mathcal{R})$. In this sense, $\mathcal{M}(\mathcal{R}/S)$ is closed under blowing-up in $\mathcal{M}(\mathcal{R})$. We shall say an $(\mathcal{R}/S)$-module (or $\mathcal{R}$-module) (X, J) is *coherent* (resp. *quasi-coherent*) if J is coherent (resp. quasi-coherent). It can be proved that *the following categories* (*with obvious functors to* $\mathcal{M}(\mathcal{R})$) *are closed under blowing-up in* $\mathcal{M}(\mathcal{R})$:

(a) *The full subcategory of* $\mathcal{M}(S)$ (*or, of* $\mathcal{M}(pre\text{-}S)$) *of those which are quasi-coherent.*

(b) *The full subcategory of* $\mathcal{M}(\mathcal{A}l/\mathbf{B})$ (*or, of* $\mathcal{M}(pre\text{-}\mathcal{A}l/\mathbf{B})$) *of those which are coherent, where* $\mathbf{B}$ *is a noetherian ring.*

Let us denote by $\mathcal{M}^*(\mathcal{A}l/\mathbf{B})$ the full subcategory of $\mathcal{M}(\mathcal{A}l/\mathbf{B})$ of those which are coherent. Then, by the assertion (b), we have *the existence theorem of blowing-up in the category* $\mathcal{M}^*(\mathcal{A}l/\mathbf{B})$.

The proofs of the basic existence theorem of blowing-up and of the assertions of (a) and (b) are based on the following observation of A. Grothendieck:[13] Let $\mathbf{A}$ be a commutative ring with unity, and $\mathbf{J}$ an $\mathbf{A}$-module. Let $S_{\mathbf{A}}(\mathbf{J})$ denote the symmetric tensor algebra over $\mathbf{A}$, i.e., the graded $\mathbf{A}$-algebra whose homogeneous part of degree n, $S_{\mathbf{A}}(\mathbf{J})_n$, is the n^{th} symmetric tensor product of $\mathbf{J}$ with itself over $\mathbf{A}$. In particular, $S_{\mathbf{A}}(\mathbf{J})_0 \approx \mathbf{A}$ and $S_{\mathbf{A}}(\mathbf{J})_1 \approx \mathbf{J}$. Then we construct the $\mathbf{A}$-scheme $\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J})) = (\mathbf{X}, O_X)$ where $\mathbf{X}$ is the set of those homogeneous prime ideals in $S_{\mathbf{A}}(\mathbf{J})$ which do not contain $S_{\mathbf{A}}(\mathbf{J})_1$ and, if $\mathrm{Spec}(S_{\mathbf{A}}(\mathbf{J})) = (\mathbf{X}', O_{X'})$, O_X is the subsheaf of rings of the restriction $O_{X'} \mid \mathbf{X}$ which consists of homogenous elements of degree zero.[14] Let J_1 denote the sheaf of modules on $\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J}))$ obtained as the subsheaf of modules of $O_{X'} \mid \mathbf{X}$ which consists of homogeneous elements of degree one. We have then a canonical homomorphism of the module $S_{\mathbf{A}}(\mathbf{J})_1$ ($\approx \mathbf{J}$) into $\Gamma(\mathbf{X}, J_1)$ such that the image of $S_{\mathbf{A}}(\mathbf{J})_1$ generates the sheaf J_1 of O_X-modules. Thus, if we denote by J the sheaf of modules on $\mathrm{Spec}(\mathbf{A})$ generated by $\mathbf{J}$, we obtain a canonical morphism Φ of $\mathcal{R}$-modules from $(\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J})), J_1)$ to $(\mathrm{Spec}(\mathbf{A}), J)$. Due to A. Grothendieck is the fact that *the pair of the* $\mathcal{R}$*-module* $(\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J})), J_1)$ *and* Φ: $(\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J})), \mathbf{J}_1) \to (\mathrm{Spec}(\mathbf{A}), \mathbf{J})$ *is a blowing-up of* $(\mathrm{Spec}(\mathbf{A}), \mathbf{J})$ *in the category* $\mathcal{M}(\mathcal{R})$. Let us write $S(\mathbf{A}, \mathbf{J})$ for $(\mathrm{Spec}(\mathbf{A}), \mathbf{J})$ and $P(\mathbf{A}, \mathbf{J})$ for $(\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J})), \mathbf{J}_1)$.

Now, in general, let $M = (X, J)$ be an $\mathcal{R}$-module with $X = (\mathbf{X}, O_X)$. We shall construct a blowing-up (N, Φ) of M in the category $\mathcal{M}(\mathcal{R})$, where $N = (P, I)$ with $P = (\mathbf{P}, O_P)$. For each $x \in \mathbf{X}$, we write (P^x, I^x) for the restriction $P(O_x, J_x) \mid \mathbf{P}^x$ where $\mathbf{P}^x$ denotes the set of those points of $\mathrm{Proj}(S_{O_x}(J_x))$ which correspond to the unique closed point of $\mathrm{Spec}(O_x)$.

[13] cf. § 2, Ch. II, of Grothendieck [6].

[14] cf. *ibid.*, paragraph 2.3.

Let $P^x = (\mathbf{P}^x, O_{P^x})$. We define the set $\mathbf{P}$ as the disjoint union of $\mathbf{P}^x$ for all $x \in \mathbf{X}$. We shall define a topology on $\mathbf{P}$, a sheaf of local rings O_P on $\mathbf{P}$, and a sheaf of O_P-modules I on $\mathbf{P}$. Let $\mathbf{U}$ be any open subset of $\mathbf{X}$. Let $\mathbf{A}(\mathbf{U}) = \Gamma(\mathbf{U}, O_X)$, $\mathbf{J}(\mathbf{U}) = \Gamma(\mathbf{U}, J)$, and $\mathbf{S}(\mathbf{U}) = S_{\mathbf{A}(\mathbf{U})}(\mathbf{J}(\mathbf{U}))$, the symmetric tensor algebra over $\mathbf{A}(\mathbf{U})$. Let ξ be any homogeneous element of $\mathbf{S}(\mathbf{U})$. For every $x \in \mathbf{U}$, we have a canonical homomorphism of $\mathbf{S}(\mathbf{U})$ to $S_{O_x}(J_x)$. We denote by ξ_x the image of ξ by this homomorphism. We can view $\mathbf{P}^x$ as a set of homogeneous prime ideals in $S_{O_x}(J_x)$, and we take the subset $\mathbf{P}^x(\xi)$ of $\mathbf{P}^x$ which consists of those homogeneous prime ideals which do not contain ξ_x. Let us denote by $\langle \mathbf{U}, \xi \rangle$ the disjoint union of $\mathbf{P}^x(\xi)$ for all $x \in \mathbf{U}$, which is viewed as a subset of $\mathbf{P}$. We define *the topology on* $\mathbf{P}$ by taking the family of subsets of the form $\langle \mathbf{U}, \xi \rangle$ as a base of open subsets. For every pair $(\mathbf{U}, \xi)$ as above, we denote by $\mathbf{B}(\mathbf{U}, \xi)$ (resp. $\mathbf{I}(\mathbf{U}, \xi)$) the homogeneous part of degree 0 (resp. of degree 1) of the ring of fractions of $\mathbf{S}(\mathbf{U})$ with respect to the powers of ξ. In view of the definition of (P^x, I^x), we see that if $x \in \mathbf{U}$, then there exists a natural homomorphism of $\mathbf{B}(\mathbf{U}, \xi)$ (resp. of $\mathbf{I}(\mathbf{U}, \xi)$) into $\Gamma(\mathbf{P}^x(\xi), O_{P^x})$ (resp. into $\Gamma(\mathbf{P}^x(\xi), I^x)$), where $\mathbf{P}^x(\xi) = \langle \mathbf{U}, \xi \rangle \cap \mathbf{P}^x$. Let $\mathbf{W}$ be any open subset of $\mathbf{P}$. Then for every $\langle \mathbf{U}, \xi \rangle \subset \mathbf{W}$, we have natural homomorphisms α (resp. β) of $\prod_{x \in \mathbf{X}} \Gamma(\mathbf{W} \cap \mathbf{P}^x, O_{P^x})$ (resp. of $\mathbf{B}(\mathbf{U}, \xi)$) to $\prod_{x \in \mathbf{U}} \Gamma(\mathbf{P}^x(\xi), O_{P^x})$, and γ (resp. δ) of $\prod_{x \in \mathbf{X}} \Gamma(\mathbf{W} \cap \mathbf{P}^x, I^x)$ (resp. of $I(\mathbf{U}, \xi)$) to $\prod_{x \in \mathbf{U}} \Gamma(P^x(\xi), I^x)$. We define *the stuctural sheaf O_P of P* as follows: For every open subset $\mathbf{W}$ of $\mathbf{P}$, $\Gamma(\mathbf{W}, O_P)$ is the subring of $\prod_{x \in \mathbf{X}} \Gamma(\mathbf{W} \cap \mathbf{P}^x, O_{P^x})$ consisting of those elements r such that for every $x \in \mathbf{W}$, there exists $s \in \mathbf{B}(\mathbf{U}, \xi)$ with $x \in \langle \mathbf{U}, \xi \rangle \subseteqq \mathbf{W}$ and $\alpha(r) = \beta(s)$. We also define *the sheaf of modules I on P* as follows: For every open subset $\mathbf{W}$ of $\mathbf{P}$, $\Gamma(\mathbf{W}, I)$ is the submodule of $\prod_{x \in \mathbf{X}} \Gamma(\mathbf{W} \cap \mathbf{P}^x, I^x)$ consisting of those elements t such that for every $x \in \mathbf{W}$, there exists $u \in \mathbf{I}(\mathbf{U}, \xi)$ with $x \in \langle \mathbf{U}, \xi \rangle \subseteqq \mathbf{W}$ and $\gamma(t) = \delta(u)$. We see that $N = (P, I)$ with $P = (\mathbf{P}, O_P)$ is an $\mathscr{R}$-module and that there exists a natural morphism $\Phi: N \to M$ which is compatible with the canonical morphism of $P(O_x, J_x)$ to $S(O_x, J_x)$ for every $x \in X$. It is not difficult to prove that (P, Φ) *is a blowing-up of M in the category $\mathscr{M}(\mathscr{R})$*. The basic existence theorem of blowing-up can be thus proved. The assertions of (a) and (b) follow from this fact and the result stated in the preceding paragraph.

Let $\mathbf{K}$ be either $\mathbf{C}$ or $\mathbf{R}$, $\mathbf{k}$ a subfield of $\mathbf{K}$, and $\mathscr{A}(\mathbf{K}/\mathbf{k})$ the full subcategory of $\mathscr{R}/\mathbf{k}$ of those which admit analytic $\mathbf{K}$-covers. (See § 1.) Let X be a $\mathbf{k}$-local-ringed space in $\mathscr{A}(\mathbf{K}/\mathbf{k})$ and $(\hat{X}, \lambda)$ an analytic $\mathbf{K}$-cover of X. A sheaf of modules J on X will be said to be $\mathbf{X}$-*analytically coherent* if $\lambda^*(J)$ is a coherent sheaf of modules on $\hat{X}$. Accordingly, an $(\mathscr{R}/\mathbf{k})$-

module (X, J) in $\mathcal{M}(\mathcal{A}(\mathbf{K}/\mathbf{k}))$ will be said to be $\mathbf{K}$-*analytically coherent* if J is so. It can be proved that *the following category* (*with an obvious functor to* $\mathcal{M}(\mathcal{R})$) *is closed under blowing-up in* $\mathcal{M}(\mathcal{R})$:

(c) *The full subcategory of* $\mathcal{M}(\mathcal{A}(\mathbf{K}/\mathbf{k}))$ *of those which are* $\mathbf{K}$-*analytically coherent.*

Let us consider the case of $\mathbf{k} = \mathbf{K}$. Let $\mathcal{M}^*(\mathcal{A}n/\mathbf{K})$ denoted the full subcategory of $\mathcal{M}(\mathcal{A}n/\mathbf{K})$ of those which are coherent. Let $M = (X, J)$ be an $(\mathcal{R}/\mathbf{K})$-module in $\mathcal{M}^*(\mathcal{A}n/\mathbf{K})$. Let (N, Φ) be a blowing-up of M in $\mathcal{M}(\mathcal{R})$, say $N = (P, I)$ and $\Phi = (f, \varphi)$. We can view P in a natural way as a $\mathbf{K}$-local-ringed space. What we can then prove is that P admits an analytic $\mathbf{K}$-cover, say $(\hat{P}, \lambda)$, and that the sheaf of modules $\lambda^*(I)$ on $\hat{P}$ is coherent and invertible. (This is an immediate consequence of the assertion of (c), and conversely the assertion of (c) can be deduced from this.) It is easy to see that $\hat{P}$ is in fact an *analytic* $\mathbf{K}$-space, and that if ψ denotes the canonical λ-homomorphism of I to $\lambda^*(I)$, then $(\hat{N}, \Psi)$ with $\hat{N} = (\hat{P}, \lambda^*(I))$ and with $\Psi = (f\circ\lambda, \psi\circ\varphi)$ is a blowing-up of M in the category $\mathcal{M}^*(\mathcal{A}n/\mathbf{K})$. We can thus establish *the existence theorem of blowing-up in the category* $\mathcal{M}^*(\mathcal{A}n/\mathbf{K})$.

Let f: $X' \to X$ be a morphism of algebraic $\mathbf{B}$-schemes (resp. analytic $\mathbf{C}$-spaces). Suppose we have a coherent sheaf of $O_{X'}$-modules J' on X' such that the canonical homomorphism of $f_*(J')$ to J' is an f-epimorphism. Suppose that $f_*(J')$ is coherent. (The coherency of $f_*(J')$ follows that of J' if f is proper.)[14'] Let (N', Φ') and (N, Φ) be the blowings-up of (X', J') and $(X, f_*(J'))$, respectively, in the category $\mathcal{M}^*(\mathcal{A}l/\mathbf{B})$ (resp. in $\mathcal{M}^*(\mathcal{A}n/\mathbf{C})$). We then have a canonical morphism $(\tilde{f}, \tilde{\varphi})$: $N' \to N$ such that $\Phi\circ(\tilde{f}, \tilde{\varphi}) = (f, \varphi)\circ\Phi'$, where φ denotes the canonical f-epimorphism of $f_*(J')$ to J'. Let us write $N' = (\tilde{X}', \tilde{J}')$, $N = (\tilde{X}, \tilde{J})$, $\Phi' = (h', \alpha')$ and $\Phi = (h, \alpha)$. We say that f is *projective* if for every quasi-compact (resp. compact) subset $\mathbf{T}$ of the underlying topological space of X, we can find a coherent sheaf J' on X', as above, such that $J' \mid f^{-1}(\mathbf{T})$ is invertible on $X' \mid f^{-1}(\mathbf{T})$ and that $\tilde{f}$ induces an isomorphism of $\tilde{X}' \mid (f\circ h')^{-1}(\mathbf{T})$ to $X \mid h^{-1}(\mathbf{T})$. As is easily seen, if f is a morphism of algebraic $\mathbf{B}$-schemes, then we have only to consider the case in which $\mathbf{T}$ is the underlying topological space of X so that J' is required to be invertible and that $\tilde{f}$ (and h') is required to be an isomorphism. We say that an algebraic $\mathbf{B}$-scheme (resp. an analytic $\mathbf{C}$-space) X is *projective* if the structural morphism of X to $\mathrm{Spec}(\mathbf{B})$ (resp. to $\mathrm{Spec}(\mathbf{C})$) is projective. A projective

[14'] cf. Ein Theorem der analytischen Garbentheorie und die Modulräume Komplexer Structuren, by H. Grauert, Publ. Math. de l'Institut des Hautes Études Scientifiques, No. 5.

morphism of algebraic **B**-schemes (resp. of analytic **C**-spaces) $f: X' \to X$ is *proper* in the sense of Grothendieck (resp. in the sense that the pre-image by f of every compact subset of the underlying topological space of X is compact in the underlying topological space of X'). In particular, a projective analytic **C**-space X has a compact underlying topological space. It is known as Chow's theorem that, *given any projective analytic* **C**-*space* $\hat{X}$, *one can find a unique algebraic* **C**-*scheme* X *with a* **C**-*morphism* $\lambda: \hat{X} \to X$ *such that* $(\hat{X}, \lambda)$ *is an analytic* **C**-*cover of* X.[14'']

In the resolution of singularities and some other related problems, we are primarily interested in *the blowing-up in the birational* (*or bimeromorphic*) *geometry*. Namely, we consider the dlowing-up of a sheaf of ideals on a local-ringed space. To be precise, S being a local-ringed space or a commutative ring with unity, we define *an* $(\mathscr{R}/S)$-*ideal* as a pair (X, J) of an S-local-ringed space $X = (\mathbf{X}, O_X)$ and a sheaf of ideals J on X, i.e., a sheaf of O_X-modules with a specific monomorphism $J \to O_X$. *A morphism of* $(\mathscr{R}/S)$-*ideals*, $(X_1, J_1) \to (X_2, J_2)$, is a morphism of $(\mathscr{R}/S)$-modules (the $(\mathscr{R}/S)$-ideals are viewed as $(\mathscr{R}/S)$-modules in a natural way), say (f, φ) with $f = (\mathbf{f}, \theta): X_1 \to X_2$ and with a f-epimorphism $\varphi: J_2 \to J_1$, which is subject to the additional condition that φ and θ are compatible with the monomorphisms $J_2 \to O_{X_2}$ and $J_1 \to O_{X_1}$. In other words, a morphism of $(\mathscr{R}/S)$-ideals as above is uniquely determined by the morphism $f = (\mathbf{f}, \theta): X_1 \to X_2$ with $\theta: O_{X_2} \to O_{X_1}$, in such a way that $J_1 = \theta(J_2)O_{X_1}$ and φ is the natural homomorphism of $f^*(J_2)$ onto $\theta(J_2)O_{X_1}$. We shall denote by $\mathscr{I}(\mathscr{R}/S)$ *the category of* $(\mathscr{R}/S)$-*ideals* defined as above. The category $\mathscr{I}(\mathscr{R}/S)$ is not closed under blowing-up in $\mathscr{M}(\mathscr{R})$, whereas $\mathscr{M}(\mathscr{R}/S)$ is. However, the existence of blowing-up in the category $\mathscr{M}(\mathscr{R}/S)$ implies the same in the category $\mathscr{I}(\mathscr{R}/S)$. In fact, let $M = (X, J)$ be an $(\mathscr{R}/S)$-ideal with $X = (\mathbf{X}, O_X)$. Let (N, Φ) be a blowing-up of M in the category $\mathscr{M}(\mathscr{R}/S)$ with $N = (P, I)$, $P = (\mathbf{P}, O_P)$, and with $\Phi = (f, \varphi)$, $f = (\mathbf{f}, \theta)$. Let H be the union of the annihilators of $\theta(J)^n$ in O_P for all positive integers n. Let $\mathbf{P}'$ be the support of O_P/H, $O_{P'}$ the restriction of O_P/H to $\mathbf{P}'$ and $P' = (\mathbf{P}', O_{P'})$. Let I' be the sheaf of ideals in $O_{P'}$ which is generated by the natural image of $\theta(J)$. Then Φ induces a natural morphism $\Phi': (P', I') \to (X, J)$, which is a morphism of $(\mathscr{R}/S)$-ideals. It can be easily seen that (N', Φ') *with* $N' = (P', I')$ *is a blowing-up of* M *in the category* $\mathscr{I}(\mathscr{R}/S)$. Thus we obtain *the exitence theorem of blowing-up in the category* $\mathscr{I}(\mathscr{R}/S)$, where S is any local-ringed space or a commutative ring with unity.

[14''] This formulation of Chow's theorem is due to Serre [21]. (cf. *On compact complex analytic varieties*, by W. L. Chow, Amer. J. Math., 71 (1949), 893–914.)

In general, if $\mathscr{R}'$ is a subcategory of $\mathscr{R}/S$, we denote by $\mathscr{I}(\mathscr{R}')$ the subcategory of $\mathscr{I}(\mathscr{R}/S)$ of those objects (X, J) with X in $\mathscr{R}'$ and of those morphisms $(f, \varphi)\colon (X_1, J_1) \to (X_2, J_2)$ with f in $\mathscr{R}'$. We also apply the usage of the adjective "*coherent*" to $(\mathscr{R}/S)$-ideals in the same way as to $(\mathscr{R}/S)$-modules. Let us denote by $\mathscr{I}^*(\mathscr{A}l/\mathbf{B})$ *the full subcategry of* $\mathscr{I}(\mathscr{A}l/\mathbf{B})$ *of those which are coherent*, and similarly by $\mathscr{I}^*(\mathscr{A}n/\mathbf{K})$ *the full subcategory of* $\mathscr{I}(\mathscr{A}n/\mathbf{K})$ *of those which are coherent*. We can easily deduce *the existence theorem of blowing-up in the category* $\mathscr{I}^*(\mathscr{A}l/\mathbf{B})$ (resp. $\mathscr{I}^*(\mathscr{A}n/\mathbf{K})$) from that in $\mathscr{M}^*(\mathscr{A}l/\mathbf{B})$ (resp. $\mathscr{M}^*(\mathscr{A}n/\mathbf{K})$). (See the preceding paragraph where we deduce the existence theorem for $\mathscr{I}(\mathscr{R}/S)$ from that for $\mathscr{M}(\mathscr{R}/S)$.) We can also see that $\mathscr{I}^*(\mathscr{A}l/\mathbf{B})$ *is closed under blowing-up in* $\mathscr{I}(\mathscr{R}/\mathbf{B})$ in the sense that a blowing-up of an object of the first category in the second category is necessarily a blowing-up of the same object in the first category.

Let $\mathbf{A}$ be a commutative ring with unity, and $\mathbf{J}$ an ideal in $\mathbf{A}$. Then $\mathrm{S}(\mathbf{A}, \mathbf{J})$ ($= (X, J)$ with $X = \mathrm{Spec}(\mathbf{A})$ and with the sheaf of ideals J on X generated by $\mathbf{J}$) can be viewed both as an $\mathscr{R}$-module and as an $\mathscr{R}$-ideal, i.e., an $(\mathscr{R}/\mathbf{Z})$-ideal. We have remarked that a blowing-up of $S(\mathbf{A}, \mathbf{J})$ in $\mathscr{M}(\mathscr{R})$ can be obtained as $(\mathrm{Proj}(S_{\mathbf{A}}(\mathbf{J})), I_1)$ with the canonical morphism to $\mathrm{S}(\mathbf{A}, \mathbf{J})$, where I_1 is the sheaf of modules generated by $S_{\mathbf{A}}(\mathbf{J})_1$. In comparison with this, a blowing-up of the same $\mathrm{S}(\mathbf{A}, \mathbf{J})$ in $\mathscr{I}(\mathscr{R})$ ($= \mathscr{I}(\mathscr{R}/\mathbf{Z})$) can be obtained as follows: Let $S^*_{\mathbf{A}}(\mathbf{J})$ denote *the graded* $\mathbf{A}$-*algebra obtained as the direct sum of* $\mathbf{J}^n$ (the n^{th} power of $\mathbf{J}$ in $\mathbf{A}$) for all non-negative integers n. Then we have a canonical morphism of $\mathrm{Proj}(S^*_{\mathbf{A}}(\mathbf{J}))$ to $\mathrm{Spec}(\mathbf{A})$. Let I^*_1 denote the sheaf of modules on $\mathrm{Proj}(S^*_{\mathbf{A}}(\mathbf{J}))$ generated by the homogeneous part of degree one of $S^*_{\mathbf{A}}(\mathbf{J})$. Then we have an obvious homomorphism of I^*_1 to the sheaf of ideal on $\mathrm{Proj}(S^*_{\mathbf{A}}(\mathbf{J}))$ generated by $\mathbf{J}$. We can prove that *this homomorphism is bijective*. By means of this isomorphism, we view I^*_1 as a sheaf of ideals on $\mathrm{Proj}(S^*_{\mathbf{A}}(\mathbf{J}))$. We then have a natural morphism of $\mathscr{R}$-ideals from $(\mathrm{Proj}(S^*_{\mathbf{A}}(\mathbf{J})), I^*_1)$ to $S(\mathbf{A}, \mathbf{J})$. We can prove that $(\mathrm{Proj}(S^*_{\mathbf{A}}(\mathbf{J}), I^*_1))$ *with this morphism is a blowing-up of* $\mathrm{S}(\mathbf{A}, \mathbf{J})$ *in the category* $\mathscr{I}(\mathscr{R})$.

Let $f = (\mathbf{f}, \theta)\colon X_1 \to X_2$ be a morphism of local-ringed spaces, where $X_i = (\mathbf{X}_i, O_{X_i})$ for $i = 1, 2$. Let J_2 be a sheaf of ideals on X_2. Then, as was remarked above, there exists a unique sheaf of ideals J_1 on X_1 and a unique morphism of $\mathscr{R}$-ideals from (X_1, J_1) to (X_2, J_2), such that f undergoes this morphism. In fact, J_1 is the sheaf of ideals on X_1 generated by J_2 with reference to f, i.e., $\theta(J_2)O_{X_1}$, and the morphism of $\mathscr{R}$-ideals is (f, φ) with the canonical f-epimorphism φ of J_2 to $\theta(J_2)O_{X_1}$. *We shall write* $f^{-1}(J_2)$ *for* $\theta(J_2)O_{X_1}$. Let $\mathscr{I}'$ be a subcategory of $\mathscr{I}(\mathscr{R})$ (or of

$\mathcal{I}(\mathcal{R}/S)$). Suppose we have a subcategory $\mathcal{R}'$ of $\mathcal{R}$ (or of $\mathcal{R}/S$) such that every $(X, J) \in \mathcal{I}'$ has $X \in \mathcal{R}'$ and that, if $f: X' \to X$ belongs to $\mathcal{R}'$, then the canonical morphism (f, φ): $(X', f^{-1}(J)) \to (X, J)$ belongs to $\mathcal{I}'$. For instance, the categories $\mathcal{I}^*(\mathcal{A}l/\mathbf{B})$ and $\mathcal{I}^*(\mathcal{A}n/\mathbf{K})$ have this property. Assuming that we have $\mathcal{R}'$ with the property for $\mathcal{I}'$, we can define a blowing-up (N, Φ) of $(X, J) \in \mathcal{I}'$ in the category $\mathcal{I}'$, where $N = (P, I)$ and $\Phi = (f, \varphi)$, by the following universal mapping property:

(1) *P and f: $P \to X$ belong to $\mathcal{R}'$, and $f^{-1}(J)$ is invertible, and*

(2) *for every pair (P', f') of P' and f': $P' \to X$ both belonging to $\mathcal{R}'$, if $f'^{-1}(J)$ is invertible, there exists a unique g: $P' \to P$ in $\mathcal{R}'$ such that $f' = f \circ g$.*

Here (N, Φ) is determined by (P, f). In particular, when either $\mathcal{I}' = \mathcal{I}^*(\mathcal{A}l/\mathbf{B})$ with $\mathcal{R}' = \mathcal{A}l/\mathbf{B}$, or $\mathcal{I}' = \mathcal{I}^*(\mathcal{A}n/\mathbf{K})$ with $\mathcal{R}' = \mathcal{A}n/\mathbf{K}$, J is a coherent sheaf of ideals on X and determines a local-ringed subspace $D = (\mathbf{D}, O_D)$ of X, where $\mathbf{D}$ is the support of O_X/J and O_D is the restriction $(O_X/J) \mid \mathbf{D}$, such that D belongs to the same $\mathcal{R}'$. In this manner, we get a one to one correspondence between the set of coherent sheaves of ideals J on X, and the set of local-ringed subspaces D of X with $D \in \mathcal{R}'$. A local-ringed subspace $D \in \mathcal{R}'$ of $X \in \mathcal{R}'$ will be called *an algebraic subscheme of X* if $\mathcal{R}' = \mathcal{A}l/\mathbf{B}$, and *an analytic subspace of X* is $\mathcal{R}' = \mathcal{A}n/\mathbf{K}$. When J and D are in correspondence as above, we say that *D is defined by J on X* and also that *J is the sheaf of ideals of D on X*. Given X and D, the pair (D, f) of D and a morphism f: $P \to X$ satisfying the conditions (1) and (2) for J will be called *the monoidal transformation of X with center D*, where J is the sheaf of ideals of D on X.[15] We shall abuse the language by saying that f: $P \to X$ is the monoidal transformation of X with center D, if no ambiguity occurs. It should be noted, however, that *two different centers on X always give rise to two different monoidal transformations* whereas the morphism obtained may be the same. Such abuse of language will be used especially when we speak of *the transforms of data on X*, such as subschemes, sheaves, resolution data, etc., *by a monoidal transformation of X*.

Let Y be an algebraic $\mathbf{B}$-scheme (resp. an analytic $\mathbf{K}$-space), X and D algebraic subschemes (resp. analytic subspaces) of Y. *If g: $Y' \to Y$ and f: $X' \to X$ are the monoidal transformations of Y and X with center D and $D \cap X$ respectively, then there exists a unique isomorphism of X' to an algebraic subscheme (resp. an analytic subspace) X'' of Y' such*

[15] cf. Zariski [26], and also *Projective Modificationen komplexer Räume*, by N. Kuhlmann. The definition by the universal mapping property was introduced and applied to a study on the factorizations of a birational morphism, by Hironaka Ph. D. Thesis, Harvard Univ. (1960).

that g induces f. (Note that the intersection of two subschemes is the subscheme defined by the join of their sheaves of ideals.) If J denotes the sheaf of ideals of D on Y and I that of X on Y, then the sheaf of ideals of the above X'' on Y' is obtained as the union of the kernels of the canonical homomorphisms $O_{Y'} \to \mathrm{Hom}\,(g^{-1}(J)^n, O_{Y'}/g^{-1}(I))$ (defined by multiplication modulo $g^{-1}(I)$) for all positive integers n, where Hom denotes the sheaf of germs of $O_{Y'}$-homomorphisms. This fact is an immediate consequence of the universal mapping properties (1) and (2). The situation being such, X'' (resp. the sheaf of ideals of X'' on Y') is called *the strict transform of* X (resp. I) by the monoidal transformation g: $Y' \to Y$. (Compare this with the weak transform of a coherent sheaf of ideals, which will be defined in § 5.)

3. The resolution of singularities: Main theorem I

The theorem, to the proof of which this work is primarily devoted, is *the resolution of singularities of an arbitrary algebraic variety of characteristic zero.* Let $X = (\mathbf{X}, O_X)$ be a scheme. Then a point $\bar{x}$ of X is called *a generic point of* X, if the closure of the one point set $\{\bar{x}\}$ in X contains an open subset of X. We say that X is *irreducible* if it has one and only one generic point. When X is reduced and irreducible, the local ring $O_{X,\bar{x}}$ of X at the unique generic point $\bar{x}$ of X is called *the function field of* X, which is in fact a field. If X is given as a $\mathbf{B}$-scheme, where $\mathbf{B}$ is a commutative ring with unity, then the function field of X has a canonical structure of $\mathbf{B}$-algebra. Let us first consider the case in which X is an algebraic $\mathbf{B}$-scheme where $\mathbf{B}$ is a field. Let us also assume that X is reduced and irreducible. Let Ω be an algebraically closed field, and suppose $\mathbf{B}$ is given as a subfield of Ω, so that we can view Spec(Ω) as a $\mathbf{B}$-scheme in a fixed way. Let X_Ω denote the product of X and Spec(Ω) in the category of $\mathbf{B}$-schemes. Then X_Ω can be viewed as an algebraic Ω-scheme in a natural manner. Let $X(\Omega)$ denote the set of Ω-morphisms from Spec(Ω) to X_Ω, which are called *geometric points of X in Ω.* (In fact, this set is in a one to one correspondence with the set of $\mathbf{B}$-morphisms of Spec(Ω) to X, where an Ω-morphism of Spec(Ω) to X_Ω followed by the projection of X_Ω gives rise to a $\mathbf{B}$-morphism of Spec(Ω) to X.) Let f be the natural mapping of $X(\Omega)$ into the underlying topological space of X_Ω. Then f is injective and induces a structure of Ω-local-ringed space on $X(\Omega)$ which is isomorphic to the restriction of X_Ω to the image of f. An Ω-local-ringed space obtained in this manner is called *an algebraic variety* (or, more precisely, *an algebraic Ω-variety*) if X_Ω is also reduced and irreducible. This condition on X (which is assumed to be reduced and irreducible) is equivalent to saying that the field extension $\mathbf{B}(X)/\mathbf{B}$ is *regular* in the sense of A. Weil, where $\mathbf{B}(X)$ denotes the function field of X. This regularity of the field extension is independent of Ω, and if $\mathbf{B}$ has characteristic zero, it amounts to saying that $\mathbf{B}$ is algebraically closed in $\mathbf{B}(X)$. The above definition of algebraic variety conicides with that of J-P. Serre[16] (and essentially with that of A.Weil[17]). When an algebraic Ω-variety $\bar{X}$ is obtained from an algebraic $\mathbf{B}$-scheme X (such that $\mathbf{B}(X)/\mathbf{B}$ is a regular extension) in such a way as above, X is uniquely determined by $\bar{X}$ and $\mathbf{B}$ (up to a $\mathbf{B}$-isomorphism) and we say that $\bar{X}$ is *defined over* $\mathbf{B}$, or $\mathbf{B}$ is *a field of definition of* $\bar{X}$. With X and $\bar{X}$ related in this manner, *if* $\bar{X}$ *is non-singular, then* X *is necessarily non-singular. The converse is true if* Ω (*or, equivalently* $\mathbf{B}$) *has characteristic zero,* but not in general if otherwise. In fact, *if* $\mathbf{B}$ (*hence,* Ω) *has characteristic zero, the singular locus of* $\bar{X}$ (i.e., the set of multiple points of $\bar{X}$) *is equal to the preimage of the singular locus of* X *under the natural* $\mathbf{B}$*-morphism of* $\bar{X}$ *to* X. If $f\colon X_1 \to X_2$ is a $\mathbf{B}$-morphism of algebraic $\mathbf{B}$-schemes, and if an algebraic Ω-variety $\bar{X}_i$ is obtained from X_i for each $i = 1, 2$, then f gives rise to a canonical Ω-morphism of $\bar{X}_1$ to $\bar{X}_2$, i.e., the one induced by $f \times$ (identity) from $X_{1\Omega}$ to $X_{2\Omega}$. *A morphism of algebraic* Ω*-varieties,* $\bar{X}_1 \to \bar{X}_2$, is by definition an Ω-morphism, and it is obtained as above from a $\mathbf{B}$-morphism of algebraic $\mathbf{B}$-schemes, where $\mathbf{B}$ is a suitable subfield of Ω over which X_1 and X_2 are defined. If $\mathbf{B}$ is so chosen, then we say that *the morphism of algebraic* Ω*-varieties is defined over* $\mathbf{B}$. A morphism $\bar{f}\colon \bar{X}_1 \to \bar{X}_2$ is said to be *proper,* if it is obtained from a *proper* $\mathbf{B}$-morphism (in the sense of A. Grothendieck) of algebraic $\mathbf{B}$-schemes as above. This condition of being *proper* on the associated $\mathbf{B}$-morphisms for a given $\bar{f}$, is independent of the choice of $\mathbf{B}$. We note that if Ω is the field of complex numbers (hence, $\mathbf{B}$ is a subfield of $\mathbf{C}$) then an algebraic $\mathbf{B}$-scheme X_i admits an analytic $\mathbf{C}$-cover $(\hat{X}_i, \lambda_i)$, that a $\mathbf{B}$-morphism $f\colon X_1 \to X_2$ gives rise to a unique morphism $\hat{f}\colon \hat{X}_1 \to \hat{X}_2$ with $\lambda_2 \circ \hat{f} = f \circ \lambda_1$, and that f is proper if and only if $\hat{f}$ is *proper in the sense that the preimage of every compact set of points of* $\hat{X}_2$ *by* $\hat{f}$ *is compact.* (This definition of *being proper* is applicable only to morphisms of analytic $\mathbf{K}$-spaces, and not to morphisms of schemes. As for the latter, one should refer to A.Grothendieck's definition.)

Thus our result on the resolution of singularities of algebraic varieties can be stated as follows:

---

[16] cf. *Faisceaux algébrigues cohérents,* by J-P. Serre, Ann. of Math., 61 (1955), 197–278.

[17] cf. Foundations of Algebraic Geometry, by A.Weil, Amer. Math. Soc. Coll. Publ., No. 29 (1946).

MAIN THEOREM I. *Let* **B** *be a field of characteristic zero. If* X *is an algebraic* **B**-*scheme, say reduced and irreducible, then there exists an algebraic subscheme* D *of* X *such that*

(i) *the set of points of* D *is exactly the singular locus of* X, *and*

(ii) *if* $f\colon \tilde{X} \to X$ *is the monoidal transformation of* X *with center* D, *then* $\tilde{X}$ *is non-singular.*

We see that the complementary open subset of X to D, say $\mathbf{U}$, is *dense* in the underlying topological space of X and that f induces an *isomorphism* of $X \mid f^{-1}(\mathbf{U})$ to $X \mid \mathbf{U}$. By replacing the condition (i) by these conditions on D, we get *main theorem* I *in the weak form.*

Proofs of *main theorem* I in the weak form, for the case in which $\dim(X) = 2$ and $\mathbf{B} = \mathbf{C}$ (the first non-trivial case), had been proposed by various authors such as Del-Pezzo, B. Levi, Severi, Albanese etc., before the first rigorous proof was established by J. Walker [22]. O. Zariski then gave various proofs of *main theorem* I for $\dim(X) = 2$, where **B** is an arbitrary field of characteristic zero. Each of his proofs in this case has its own interest and will be referred to and discussed later. (See [23], [25], and [28].) For $\dim(X) = 3$, a proof of *main theorem* I in the weak form was established by O. Zariski [27], in which he makes an essential use of his *local uniformization theorem* [24].

We note that the following existence theorem is an immediate conseqence of our *main theorem* I (or, even, of the same theorem in the weak form).

COROLLARY. *If* **L** *is a finitely generated field over a field of characteristic zero, say* **B**, *then there exists an irreducible non-singular projective algebraic* **B**-*scheme whose function field is* **B**-*isomorphic to* **L**.

An algebraic **B**-scheme is said to be *projective* if it is of the form $\operatorname{Proj}(H)$ with a quotient ring H, graded in a natural way, of a polynomial ring with coefficients in **B**. If $\mathbf{B} = \mathbf{C}$, the assertion is equivalent to saying that there exists a complex projective manifold (i.e., a non-singular analytic subspace of a complex projective space) of which the field of meromorphic functions is **C**-isomorphic to the given **L**.

We can prove the *main therem* I *in the strong form* as follows: *The morphism* f *is obtained by a finite succession of monoidal transformations* $f_i\colon X_{i+1} \to X_i$, *where* $0 \leqq i < r$, $X = X_0$ *and* $\tilde{X} = X_r$, *which satisfy the following conditions*:

(a) *The center of* f_i, *say* D_i, *is non-singular, and*

(b) D_i *does not contain any simple point of* X_i *for* $0 \leqq i < r$.

One can see that, in general, a morphism obtained by a finite succession of monoidal transformations can be also obtained by a single monoidal transformation with suitably chosen center. In fact, let $f_1: X_2 \to X_1$ and $f_0: X_1 \to X_0$ be respectively the monoidal transformation of X_1 with center D_1, and of X_0 with center D_0. Let J_1 and J_0 denote respectively the sheaf of ideals of D_1 on X_1 and of D_0 on X_0. Then we can see that for a sufficiently large integer m there exists a coherent sheaf of ideals $J(m)$ on X_0 such that $f_0^{-1}(J(m)) = f_0^{-1}(J_0)^m J_1$. Let D' be the algebraic sucheme of X_0 defined by $J(m)J_0$. We can then easily see that if $f': X' \to X_0$ is the monoidal transformation of X_0 with center D', there exists a unique isomorphism $h: X' \to X_2$ such that $f' = f_0 \circ f_1 \circ h$.

Finally, according to our proof of *main theorem* I *in the strong form*, we may impose (and in fact we do) the following condition on the monoidal tranformations f_i in addition to the above (a) and (b):

(c) *X_i is normally flat along D_i for all i, $0 \leqq i < r$.* The notion of normal flatness along a subscheme will be defined in the following section; and there, also, we shall find some geometric significance of the notion.

4. Equi-multiplicity and normal flatness: Main theorem I*

If $\mathbf{A}$ is a commutative ring with unity, and $\mathbf{m}$ is an ideal in $\mathbf{A}$, we associate with the pair $(\mathbf{A}, \mathbf{m})$ the graded $(\mathbf{A}/\mathbf{m})$-algebra $\mathrm{gr}_{\mathbf{m}}(\mathbf{A})$ which is the direct sum of the $(\mathbf{A}/\mathbf{m})$-modules $\mathbf{m}^p/\mathbf{m}^{p+1}$ for all non-negative integers p. For each p, the direct summand $\mathbf{m}^p/\mathbf{m}^{p+1}$ is denoted by $\mathrm{gr}^p_{\mathbf{m}}(\mathbf{A})$ which is referred to as the homogeneous part of degree p of $\mathrm{gr}_{\mathbf{m}}(\mathbf{A})$. If $\mathbf{A}$ is a noetherian local ring and $\mathbf{m}$ is the maximal ideal of $\mathbf{A}$, then we have a polynomial $H(p)$ with rational coefficients such that for all sufficiently large integers p the value $H(p)$ of the polynomial is equal to the sum of ranks of the $(\mathbf{A}/\mathbf{m})$-modules $\mathrm{gr}^i_{\mathbf{m}}(\mathbf{A})$ with $0 \leqq i \leqq p$. The degree of the polynomial is equal to the Krull dimension of $\mathbf{A}$, say d, and the coefficient of the term of degree d of $H(p)$ is of the form $e(\mathbf{A})/d!$ where $e(\mathbf{A})$ is a positive integer. This integer $e(\mathbf{A})$ is called *the multiplicity of the local ring* $\mathbf{A}$. If $\mathbf{R}$ is a local ring and $\mathbf{M}$ is an ideal $\neq (0)$ in $\mathbf{R}$, we define $\nu_{\mathbf{M}}(\mathbf{J})$ for each ideal $\mathbf{J}$ in $\mathbf{R}$ to be the maximal non-negative integer ν such that $\mathbf{M}^\nu \supset \mathbf{J}$, if it exists; and ∞. When $\mathbf{M}$ is the maximal ideal of $\mathbf{R}$, we call $\nu_{\mathbf{M}}(\mathbf{J})$ *the order of the ideal* $\mathbf{J}$ and often denote it by $\nu(\mathbf{J})$. If $\mathbf{R}$ is a regular local ring and $\mathbf{J}$ is a non-zero principal ideal in $\mathbf{R}$, then we can prove that $e(\mathbf{R}/\mathbf{J}) = \nu(\mathbf{J})$.

Let $X = (X, O_X)$ be a local-ringed space. If $O_{X,x}$ is noetherian, then $e(O_{X,x})$ will be called *the multiplicity of X at the point x*. Let Y be a local-ringed subspace of X such that $O_{X,x}$ is noetherian for all points x of Y. We say that X is *equi-multiple along Y* if X has the same multiplicity at the points of Y. It is easy to see that if X belongs to $\mathscr{A}(\mathbf{K}/\mathbf{k})$, and if $(\hat{X}, \lambda)$ is an analytic **K**-cover of X, then for every point $\hat{x}$ of $\hat{X}$ the multiplicity of $\hat{X}$ at $\hat{x}$ is equal to that of X at $\lambda(\hat{x})$.

One can prove:

(A) *If X is either an analytic* **K**-*space or an algebraic* **B**-*scheme, irreducible and with a field* **B**, *then for each integer d there exists an analytic subspace or an algebraic subscheme S_d of X, respectively, such that x is a point of S_d if and only if the multiplicity of X at x is greater than or equal to d.*

(B) *Let X be as above, and assume that X is reduced and irreducible. Let D be a non-singular analytic subspace or algebraic subscheme of X, respectively, along which X is equi-multiple. Let $f: X' \to X$ be the monoidal transformation of X with center D. Then for every point x' of X' the multiplicity of X' at x' is at most equal to that of X at $f(x')$.*

(C) *Let X be the same as in* (B). *Then a point x of X is simple if and only if the multiplicity of X at x is equal to one.*

In view of these facts, we are led to the following question.

Question (D). *Let X be the same as in* (B), **W** *a quasi-compact (or compact) set of points of X, and d the maximum of the multiplicities of X at the points in* **W**, *say $d > 1$. We ask if there exists a finite succession of monoidal transformations $f_i: X_{i+1} \to X_i$ with centers D_i in X_i, say $0 \leqq i < r$ and $X_0 = X$, such that*

(i) *D_i is non-singular, and X_i is equi-multiple along D_i with multiplicity $\geqq d$ within a neighborhood of the preimage of* **W** *in X_i, for all i, and*

(ii) *if f denotes the composition of the f_i, X_r has multiplicities $< d$ at all points of $f^{-1}(\mathbf{W})$?*

In the case of an algebraic **B**-scheme X, the role played by **W** is irrelevant to the question, and we take **W** to be the set of all points of X. An affirmative answer to the *question* (D) can be obtained from the *main theorem* I in the strong form, with (a), (b), and (c) in the case of algebraic **B**-schemes with fields **B** of characteristic zero, and it seems quite plausible in general. Obviously, if the answer to (D) is affirmative, then the resolution of singularities in the form of the *main theorem* I follows.

Although the above is not the idea which underlies our proof of the *main theorem* I, we shall point out that this idea leads to a *canonical* (but not unique) resolution process in a certain special case.

Let Y be a non-singular algebraic **B**-scheme of dim 3, with a field **B** of characteristic zero, and X an irreducible and reduced algebraic subscheme of dim 2 of Y. Let d be the maximum of the multiplicities of X at its points. We assume $d > 1$. Then S_d is a finite union of irreducible and reduced algebraic subschemes of dimension one and points. Let us call them *d-fold curves* and *isolated d-fold points of* X respectively. B. Levi first proposed a proof of resolution of singularities of X in such a way as to answer (D) affirmatively, [11] and [12]. Following Levi's idea, and making precise certain points which were left ambiguous, O. Zariski answered (D) explicitly in the following form:

(1) *First, applying a finite number of monoidal transformations whose centers are points on d-fold curves, we can establish the situation in which:*

(a) *each of the d-fold curve is non-singular;*

(b) *at most two of the d-fold curves pass through each point; and*

(c) *any two of the d-fold curves passing through a point have transversal crossing at the point.*

(There exists a unique shortest process of doing this.)

(2) *After* (1) *is achieved, we take either an isolated d-fold point or an irreducible d-fold curve, which is non-singular, and we apply the monoidal transformation with it as center.* (Here we have an arbitrary choice out of a finite number of possibilities.).

Zariski proved that *the process of* (2) *can be repeated, without affecting the situation established in* (1), *until the transform of* X *has neither d-fold curve nor isolated d-fold point*, and that, as is of the most importance, *the process of* (2) *cannot be repeated infinitely many times.* (See [27, § 17, Part III].) We shall refer to this result as *theorem of B. Levi-Zariski.*

As for the resolution of singularities for the special X as above, the *theorem* of B. Levi-Zariski seems to be the best result known by the author. We do not know, however, how to generalize the above process to arbitrary algebraic **B**-schemes, say reduced, irreducible and of characteristic zero. The *main theorem* I in the strong form with (a), (b), and (c), which was stated in the previous section and will be proved later, is related to the affirmative answer to (D) by the fact that the normal flatness in (c) is stronger than the equi-multiplicity in (i) of (D). We define the normal flatness as follows:

DEFINITION 1. *Let* X *be a local-ringed space (in particular, an algebraic* **B**-*scheme, an analytic* **K**-*scheme,* etc.). *Let* D *be a local-ringed subspace (in particular, an algebraic subscheme, an analytic subspace,* etc.) *of* X *which is defined by a sheaf of ideals on* X, *say* J. *We denote by* $\mathrm{gr}^p_D(X)$ *the sheaf of modules on* D *obtained as the restriction of the quotient sheaf* J^p/J^{p+1} *to* D, *where* p *is an arbitrary non-negative integer. For a point* x *of* D, *we say that* X *is normally flat along* D *at the point* x, *if the* *stalk of* $\mathrm{gr}^p_D(X)$ *at* x *is a free* $O_{D,x}$*-module for all non-negative integers* p. *We say that* X *is normally flat along* D *if it is so at every point of* D.

The direct sum of $\mathrm{gr}^p_D(X)$ for all non-negative p, denoted by $\mathrm{gr}_D(X)$, is a sheaf of graded O_D-algebras in a natural way. The normal flatness of X along D is equivalent to saying that $\mathrm{gr}_D(X)$ is O_D-flat, if the local rings $O_{X,x}$ (hence, $O_{D,x}$) are all noetherian.

Let X be a $\mathbf{k}$-local-ringed space belonging to the category $\mathcal{A}(\mathbf{K}/\mathbf{k})$, and let $(\hat{X}, \lambda)$ be an analytic $\mathbf{K}$-cover of X. Then one can prove that, *for a local-ringed subspace* D *of* X *such that* $\lambda^{-1}(D)$ *is an analytic subspace of* $\hat{X}$, $\hat{X}$ *is normally flat along* $\lambda^{-1}(D)$ *at a point* $\hat{x}$ *if and only if* X *is normally flat along* D *at* $\lambda(\hat{x})$.

Let X be either an algebraic $\mathbf{B}$-scheme or an analytic $\mathbf{K}$-space. We can prove:

(I) *Let* D *be a non-singular algebraic subscheme, or analytic subspace, of* X *respectively. If* X *is normally flat along* D, *then* X *is equi-multiple along* D. (*Moreover, if* D *contains a simple point of* X *then every point of* D *is a simple point of* X.) *The converse is true if* X *is imbedded in a non-singular algebraic* $\mathbf{B}$*-scheme, or analytic* $\mathbf{K}$*-space,* Y *such that the sheaf of ideals of* X *on* Y *is everywhere locally principal; but not in general.*[18]

(II) *Let* D *be an algebraic subscheme, or analytic subspace, of* X. *Then there exists an algebraic subscheme, or analytic subspace,* S *of* D *such that for a point* x *of* D, X *is normally flat along* D *at the point* x *if and only if* x *is not a point of* S.[19]

In our proof of *main theorem* I, the notion of normal flatness plays an important role. As a matter of fact, we can prove the theorem only in the strong form with (a), (b), and (c). (See § 3.) The basic reason for this is simply that we do not know enough about the behavior of multiplicities under monoidal transformations, and that we have to use a

---

[18] cf. Proposition 1, § 1, Ch. II; Theorem 2, § 2, Ch. II, and also their corollaries. The assertions in the analytic case are immediate consequences of the corresponding ones in the algebraic case by means of the following fact: *Let* $\mathbf{H}$ *be a compact and holomorphically convex subset of a complex Stein space* X. *Suppose* $\mathbf{H}$ *is defined by a finite number of real-analytic inequalities locally everywhere on* X. *Then the ring* $\mathbf{A}$ *of all holomorphic functions on* $\mathbf{H}$, i.e., $\mathbf{A} = \Gamma(\mathbf{H}, O_X | \mathbf{H})$, *is noetherian, and the canonical* $\mathbf{C}$*-morphism of local-ringed spaces,* $X | \mathbf{H} \to \mathrm{Spec}\,(\mathbf{A})$, *is flat. Moreover, we have a canonical isomorphism of categories from the category of coherent sheaves of modules on* $X | \mathbf{H}$ *to the category of* $\mathbf{A}$*-modules of finite type.* Proofs of these facts will be found in a forthcoming paper [9].

[19] cf. Theorem 1, § 1, Ch. II, and its corollary. The assertion in the analytic case follows immediately for the same reason as in the previous footnote.

different and more complicated character of singularities which is closely related to normal flatness rather than to equi-multiplicity. The character of singularities will be defined in §§ 1 and 4 of Ch. III as a pair or sequences of integers, denoted by ν^* and τ^*. We do not make this precise at the present moment.

In an abstract manner of speaking, consider the class of all the triples (x, X, Y) of non-singular algebraic **B**-schemes Y, subschemes X of Y, and points x of X. Let us denote the class by C. *A character of singularities* that we shall seek to define and to use is a function, say $s_x(X/Y)$ in symbol, defined for all $(x, X, Y) \in C$, and having the values $s_x(X/Y)$ in an ordered (or semi-ordered) set. For a triple (x, X, Y), we shall call Y the ambient non-singular space of X. The character $s_x(X/Y)$ is a measurement of the singularities of X at the point x of X and is essentially independent of the ambient non-singular space Y in the sense that $s_x(X/Y')$ for a different Y' is obtained by knowing only the value $s_x(X/Y)$ and the integer $\dim(O_{Y',x}) - \dim(O_{Y,x})$, where *dim* denotes the Krull dimension. Moreover, $s_x(X/Y)$ is of local nature in the sense that it is uniquely determined by the local rings $O_{Y,x}$ and $O_{X,x}$. We require that $s_x(X/Y)$ has the following properties:

(1) *If D is a non-singular algebraic subscheme of an algebraic scheme X such that X is normally flat along D, and if $f: X' \to X$ is the monoidal transformation of X with center D, then we have $s_{x'}(X'/Y') \leqq s_{f(x')}(X/Y)$ for all points x' of X', where (Y', Y) is a arbitrary pair of non-singular algebraic* **B**-*schemes such that $Y' \supset X'$, $Y \supset X$ and f can be extended to the monoidal transformation $Y' \to Y$ with center D.*

(2) *Suppose there is given an infinite sequence of monoidal transformations $\{f_i: X_{i+1} \to X_i\}$ $(i = 0, 1 2, \cdots)$ with centers D_i in X_i such that D_i is non-singular and that X_i is normally flat along D_i for all i, and also an infinite sequence of points $\{x_i\}$ with $x_i \in X_i$ such that $f_i(x_{i+1}) = x_i$ for all i. Then there exists an integer r such that for all $i \geqq r$, $s_{x_i}(X_i/Y_i) = s_{x_r}(X_r/Y_r)$, where $\{Y_i\}$ is any sequence of non-singular algebraic* **B**-*schemes such that $Y_i \supset X_i$ and f_i can be extended to the monoidal transformation $Y_{i+1} \to Y_i$ with center D_i for all i.*

Once we have found $s_x(X/Y)$ having the above properties (1) and (2), we come to the following question analogous to (D):

Question (D*). *Let X be an algebraic* **B**-*scheme. We denote by* red(X) *the associated reduced scheme of X,* i.e., *the reduced subscheme of X which contains all the points of X. Let x be a point of X such that either it is a multiple point of* red(X) *or X is not normally flat along* red(X) *at x. Let X_0' be the restriction of X to an open neighborhood of*

x, and Y_0' a non-singular algebraic **B**-*scheme containing X_0' as a subscheme. We then ask if there exists a finite succession of monoidal transformations $\{f_i: X_{i+1} \to X_i\}$ $(0 \leqq i < r)$ with centers D_i in X_i such that*

(a) *D_i is non-singular and X_i is normally flat along D_i for all i; and*

(b) *if X_i' is the restriction of X_i to a suitable open subset such that f_i induces the monoidal transformation $f_i': X_{i+1}' \to X_i'$ with center $D_i \cap X_i'$ for all i, and if Y_i' is a non-singular algebraic* **B**-*scheme containing X_i' such that f_i' can be extended to the monoidal transformation $Y_{i+1}' \to Y_i'$ with center $D_i \cap X_i'$ for all i, then we have $s_{x_r}(X_r'/Y_r') < s_x(X_0'/Y_0')$ for all points x_r of X_r which are mapped to the point x.*

There are many other ways of posing questions similar to the above. In general, the type of question to be asked depends upon the nature of the character of singularities to be used. At any rate, the above abstract arguments indicate the most basic principle by which we can prove the following theorem which is obviously stronger than *main theorem* I in the strong form with (a), (b), and (c) stated in the previous section:

MAIN THEOREM I*. *Let X be an algebraic* **B**-*scheme with a field* **B** *of characteristic zero. Then there exists a finite succession of monoidal transformations $\{f_i: X_{i+1} \to X_i\}$ with centers D_i in X_i $(0 \leqq i < r)$, where $X = X_0$, such that*

(a) *D_i is non-singular;*

(b) *every point of D_i is either a multiple point of* red (X_i), *or such that X_r is not normally flat along* red (X_i) *at the point;*

(c) *X_i is normally flat along D_i, for all i; and*

(d) red (X_r) *is non-singular, and X_r is normally flat along* red (X_r).

5. Simplification of coherent sheaves of ideals: Main theorem II

Let X and Y be S-local-ringed spaces, where S is a local-ringed space or a commutative ring with unity which is fixed. Let **U** and **V** be open dense subsets of the underlying topological space of X, and let f: $X|\mathbf{U} \to Y$ and g: $X|\mathbf{V} \to Y$ be S-morphisms of the respective restrictions of X to Y. We say that f and g are *equivalent* to each other if they induce the same morphism of $X|\mathbf{U} \cap \mathbf{V}$ to Y. An equivalence class of S-morphisms of such restrictions of X to Y is called *a rational S-application of X to Y*, or simply *rational application of X to Y* whenever no confusion can arise.[20]

We are interested in the case in which X and Y are either algebraic **B**-schemes or analytic **K**-spaces. A rational application of analytic **K**-spaces is called *a meromorphic application*.[21] We say that a rational

[20] cf. Grothendieck [6]; in particular, § 7, Ch. I.

[21] cf. *Holomorphe und meromorphe Abbildungen komplexer Räume*, by R. Remmert, Math. Ann., 133 (1957), 328–370. For further references, one can find: *Über meromorphe Abbildungen komplexer Räume*, by W. Stoll, Math. Ann., 136 (1958).

application f of X to Y is *proper*, if there exists an algebraic subscheme (or analytic subspace) T of the product $X \times Y$ (either in $\mathscr{A}l/\mathbf{B}$ or in $\mathscr{A}n/\mathbf{K}$) and also a morphism $\mathrm{f}: X \mid \mathbf{U} \to Y$ belonging to the class f such that the projections p_1 and p_2 of $X \times Y$ to X and Y induce proper morphisms of T, and that p_1 induces an isomorphism of $T \mid p_1^{-1}(\mathbf{U})$ to $X \mid \mathbf{U}$ while p_2 induces the composition of this isomorphism and the morphism $\mathrm{f}: X \mid \mathbf{U} \to Y$.

In general, if f is a rational application of an S-local-ringed space X into another Y, there exists the maximal open subset $\mathbf{W}$ of the underlying topological space of X such that for every $x \in \mathbf{W}$, we can find a morphism $\mathrm{f}: X \mid \mathbf{U} \to Y$ with $x \in \mathbf{U} \subset \mathbf{W}$ which belongs to the class f. We call such $\mathbf{W}$ *the domain of definition of* f. We say that f is *defined at a point* x of X, if x belongs to the domain of definition of f. If X is a reduced pre-scheme (or reduced preanalytic $\mathbf{K}$-space) and Y is a scheme (resp. an analytic $\mathbf{K}$-space), then for a rational application f of X to Y these exists a unique morphism $\mathrm{f}: X \mid \mathbf{W} \to Y$, with the domain of definition $\mathbf{W}$ of f, which belongs to the class f. In this case, we identify f with f and, by abuse of language, call f a rational application of X to Y. For instance, we say that *the rational application f is a morphism*, meaning that the domain of definition $\mathbf{W}$ of f contains all the points of X and there exists a unique morphism $\mathrm{f}: X \to Y$ belonging to the class. In general, a point of X is called *a point of indeterminacy of a rational application f* of X to Y, if it is not contained in the domain of definition of f. If f (resp. g) is a rational application of X to Y (resp. Y to Z), and if f contains a morphism $\mathrm{f}: X \mid \mathbf{U} \to Y$ such that $f^{-1}(\mathbf{V})$ is open and dense for every open dense subset $\mathbf{V}$ of the underlying topological space of Y, then we get a unique rational application of X to Z as the class of composed morphisms of those in f and those in g. This rational application of X to Z will be called *the composition of f* and g, or *the composed application* of them.

Let f be a rational application of algebraic $\mathbf{B}$-schemes (resp. analytic $\mathbf{K}$-spaces) from X to Y. Let D be an algebraic subscheme (resp. an analytic subspace) of X, and $\mathrm{h}: X' \to X$ the monoidal transformation of X with center D. If the set of points of D is nowhere dense in the underlying topological space of X, then we get a unique rational application f' of X' to Y which is the composition $f \circ h$ (i.e., the composition of f and the rational application of X' to X to which h belongs.). We shall only consider the case in which X is reduced, so that X' is necessarily reduced for every monoidal transformation h as above. We ask the following question, which is known as the problem of *the elimination of points of indeterminacy*:

Question (E). *Let* $f\colon X \to Y$ *be any rational application as above. We assume that* X *is non-singular and that* f *is proper. Let* S_0 *be any quasi-compact (resp. compact) subset of the underlying topological space of* X. *We then ask if there exists a finite succession of monoïdal transformations* $\mathrm{h}_i\colon X_{i+1} \to X_i$ *with centers* D_i *in* X_i, *where* $0 \leqq i < r$ *and* $X_0 = X$, *such that if* $S_{i+1} = \mathrm{h}_i^{-1}(\mathrm{S}_i)$ *for all* i, *then*

(i) D_i *is irreducible and has no multiple points within a neighborhood of* S_i *in* X_i;

(ii) *if* f_i *denotes the composition of* f *and those* h_j *with* $j < i$, *every point of* D_i *is a point of indeterminacy of* f_i, *and*

(iii) *the domain of the definition of* f_r *contains* S_r.

For the case of algebraic **B**-schemes, it suffices (and it is most interesting) to ask the question in which S_0 contains all the points of X so that (iii) asserts that f_r is a morphism.

The assumption that f *is proper may be replaced by the weaker one that the projection of* $X \times Y$ *to* X *induces a proper morphism of the graph of* f *to* X. For instance, the latter is satisfied (but not in general the former) by an arbitrary rational application of X into a proper **B**-scheme (resp. a compact analytic **K**-space) Y. (In general, a **B**-scheme is said to be *proper* if the structural morphism of itself to Spec(**B**) is proper.) However, we gain nothing at all by weakening the assumption in this manner. In fact, in general, the question about f is equivalent to the question about the corresponding rational application of X to the graph of f.

In this paper, we shall find the affirmative answer to the *question* (E), at least for the case in which X and Y are algebraic **B**-schemes, where **B** is a field of characteristic zero.[22] For the proof of this fact, it suffices to consider the case in which the inverse of the given rational application $f\colon X \to Y$ is a proper morphism of Y to X. In fact, if otherwise, replace Y by the graph of f. If f^{-1} is a morphism obtained by a monoidal transformation of X with center D, then *the elimination of points of indeterminacy of* f can be translated into what we call *the trivialization of a coherent sheaf of ideals* J on X, where J is the sheaf of ideals defining the subscheme D of X. The sense of the trivialization will be made precise, and we shall see that it is a corollary to the *main theorem* II. In general, the given morphism f^{-1} may not be obtained by a monoidal transformation of X. (In fact, this possibility is seen by simple examples.) However, one can always find a finite system of coherent sheaves of ideals on X in terms of which a similar translation of the

[22] Zariski gave an affirmative answer to the question in 3-dimensional case. (cf. [27].)

problem is obtained. In this respact, we note the following fact:

($*$) *Let f be a rational application of a non-singular algebraic* **B**-*scheme X to another algebraic* **B**-*scheme Y. We assume that f is proper. Then there exists a finite system of coherent sheaves of ideals $\neq (0)$, say $J_1, J_2, \cdots, J_d$, on X, such that: If $h: X' \to X$ is any morphism of a non-singular algebraic* **B**-*scheme X' to X, and if we can write $J'_i P_i = h^{-1}(J_i)$ for each i where P_i is an invertible sheaf of ideals on X' and J'_i is a sheaf of ideals $\neq (0)$ on X' which is not divisible by any invertible sheaf of ideals (except $O_{X'}$), then the domain of definition of the composition $f \circ h$ is exactly equal to the set of those points x of X' at which $J'_{ix} = O_{X',x}$ for at least one i.*

The number d can be one if the morphism of the graph of f to X is obtained by a monoidal transformation of X. We remark that the same translation of the problem can be done even in the analytic case if the morphism of the graph of f to X is obtained by a monoidal transformation of X.

In order to state the *main theorem* II, we need to introduce *the notion of normal crossings*. This notion will play one of the most important roles throughout this paper.

DEFINITION 2. *Let X be a non-singular algebraic* **B**-*scheme (resp. a non-singular analytic* **K**-*space). Let E be a reduced subscheme (resp. a reduced analytic subspace) of X. Assume that the sheaf of ideals of E on X is invertible (or, equivalently, E is everywhere of codimension one on X in the sense specified in a later chapter). Let D be a subscheme (resp. an analytic subspace) of X. We then say that E has only normal crossings with D (or, D has only normal crossings with E) at a point x of D, if there exists a regular system of parameters of the local ring $O_{X,x}$ of X at x, say $(z_1, z_2, \cdots, z_n)$, such that the ideal in $O_{X,x}$ of each irreducible component (i.e., a maximal reduced irreducible subscheme, resp. such an analytic subspace) of E containing x is generated by one of the z_i, and that the ideal of D in $O_{X,x}$ is generated by some of the z_i. If this is the case for $D = X$, we simply say that E has only normal crossings at x. We also say that E has only normal crossings (resp. the same with D), with no reference to a point, if it is the case at every point of X (resp. the same with D at every point of D).*

Here we remark that this definition applies, in an obvious manner, to an arbitrary non-singular scheme X and its subschemes.

Let $f: X' \to X$ be a morphism of algebraic **B**-schemes (resp. analytic **K**-spaces). If D is an algebraic subscheme (resp. an analytic subspace) of X, then we denote by $f^{-1}(D)$ the algebraic subscheme (resp. analytic subspace) of X' defined by the coherent sheaf of ideals $f^{-1}(J)$, i.e., the one generated by J with reference to f, where J is the sheaf of ideals of D on X. The subscheme (resp. analytic subspace) $f^{-1}(D)$ of X' is called *the total transform of D by the morphism f*. If f is the monoidal transformation of X with center D, and if X and D are non-singular, then we know that X' and $f^{-1}(D)$ are also non-singular. If E and F are algebraic subschemes (resp. analytic subspaces) of X, then we shall denote by $E \cup F$ the subscheme (or, the analytic subspace) of X defined by the intersection of the sheaves of ideals defining E and F on X. Incidentally, we denote by $E \cap F$ the one defined by the sheaf of ideals generated by those of E and F on X.

Let X be a non-singular algebraic $\mathbf{B}$-scheme (resp. a non-singular analytic $\mathbf{K}$-space), and J a coherent sheaf of non-zero ideals on X. For each point x of X, we have defined an integer $\nu(J_x)$ as the maximal integer m such that the m^{th} power of the maximal ideals of $O_{X,x}$ contains J_x. Let D be a non-singular irreducible subscheme (resp. such an analytic subspace) of X, and $f\colon X' \to X$ the monoidal transformation of X with center D. If $m = \nu(J_x)$ for the generic point x of D, then we have $m = \nu(f^{-1}(J)_y)$ for the generic point y of $f^{-1}(D)$, or equivalently, $f^{-1}(J)$ is divisible by the m^{th} power of the sheaf of ideals of $f^{-1}(D)$ on X'. The sheaf of ideals on X' obtained by this division will be called *the weak transform of J by the monoidal transformation f*. (Compare with the strict transform of a coherent sheaf of ideals, which was defined near the end of § 2.) Note that this transform is determined by the monoidal transformation (in other words, by the center) but not by the morphism.

We shall prove in this work the following

MAIN THEOREM II. *Let X be a non-singular algebraic $\mathbf{B}$-scheme with a field $\mathbf{B}$ of characteristc zero. Let J be a coherent sheaf of non-zero ideals on X. Let d be the maximum of $\nu(J_x)$ for all the points x of X. Let E_0 be a reduced subscheme of everywhere codimension one of X, which has only normal crossings. Then there exists a finite succession of monoidal transformations $\{f_i\colon X_{i+1} \to X_i\}$ with centers D_i, where $0 \leqq i < r$ and $X = X_0$, which has the following properties:*

(i) *D_i is non-singular and irreducible;*

(ii) *if J_i denotes the sheaf of ideals on X_i such that J_{i+1} is the weak transform of J_i by the monoidal transformation f_i, for $0 \leqq i < r$ and $J = J_0$, then $\nu(J_{ix(i)}) \geqq d$ for all the points $x(i)$ of D_i for $0 \leqq i < r$;*

(iii) *if E_i denotes the subscheme of X_i such that*

$$E_{i+1} = \mathrm{red}\big(f_i^{-1}(E_i) \cup f_i^{-1}(D_i)\big), \qquad \textit{for } 0 \leqq i < r\ ,$$

where red ( ) *denotes the associated reduced scheme, then* E_i *has only normal crossings with* D_i; *and*

(iv) E_r *has only normal crossings, and we have* $\nu(J_{ry}) < d$ *for every point* y *of* X_r.

We remark that, in the above theorem, the first property of (iv) is an obvious consequence of (iii); in fact, all the E_i have only normal crossings by (iii). Moreover, it follows from (ii) that, for every i with $0 \leqq i < r$, d is the maximum of $\nu(J_{iz})$ for all the points z of X_i, so that in particular we have $\nu(J_{ix(i)}) = d$. More precisely, we have the following theorem, analogous to (B) of § 4:

(B*) *Let* X *be a non-singular algebraic* **B**-*scheme* (*resp. analytic* **K**-*space*), J *a coherent sheaf of non-zero ideals on* X, *and* D *a non-singular subscheme* (*resp. a non-singular analytic subspace*) *of* X, *say irreducible. Let* $f\colon X' \to X$ *be the monoidal transformation with center* D, *and* J *the weak transform of* J *by this* f. *If* $\nu(J_x)$ *is constant for* $x \in D$, *then we have* $\nu(J'_{x'}) \leqq \nu(J_{f(x')})$ *for all* $x' \in X'$.

In order to see that the *main theorem* II implies the affirmative answer to the *question* (E) for the case in which X and Y are algebric **B**-schemes with a field **B** of characteristic zero, the following corollary of *main theorem* II is sufficient. This corollary will be referred to as *the trivialization of a system of coherent sheaves of ideals* (or, *the trivialization of a coherent sheaf of ideals*, in case $e = 1$).

COROLLARY 1. *Let* X *be a non-singular algebraic* **B**-*scheme with a field* **B** *of characteristic zero. Let* $(J_1, J_2, \cdots, J_e)$ *with* $e \geqq 1$ *be a finite system of coherent sheaves of non-zero ideals on* X. *Then there exists a finite succession of monoidal transformations* $f_i\colon X(i+1) \to X(i)$ *with non-singular irreducible centers* $D(i)$, *where* $0 \leqq i < r$ *and* $X(0) = X$, *which has the following property: If* $J_j(i+1)$ *denotes the weak transform of* $J_j(i)$ *by* f_i, *where* $1 \leqq j \leqq e$, $0 \leqq i < r$, *and* $J_j(0) = J_j$, *then*

(1) $\nu\big(J_j(i)_y\big)$ *is a positive constant for the points* y *of* $D(i)$, *for every pair* (i, j); *and*

(2) *for every point* x *of* $X(r)$, *we have* $J_j(r)_x = O_{X(r),x}$ *for at least one* j.

To prove this corollary, let d be the maximal integer which is attained by $\nu\big((\prod_{j=1}^{e} J_j)_x\big)$ for those points x of X such that $(J_j)_x \neq O_{X,x}$ for all j. Here we assume that such points x of X exist. (If otherwise, the assertion is trivial.) We shall prove that the integer d can be reduced by means of a finite succession of monoidal transformations with non-singular irreducible centers satisfying the condition (1), until we achieve (2). Let

$\mathbf{T} = \{x \in X \mid \nu((\prod_{j=1}^{e} J_j)_x) \geqq d\}$ and $\mathbf{S} = \{x \in \mathbf{T} \mid (J_j)_x = O_{X,x}$ for at least one $j\}$. Then we can prove that $\mathbf{T}$, $\mathbf{S}$ and $\mathbf{T} - \mathbf{S}$ are all closed in the underlying topological space of X. Let $\bar{X}$ and $\bar{J}$ be the restrictions of X and $\prod_{j=1}^{e} J_j$, respectively, to the complement of $\mathbf{S}$ in the underlying topological space of X. By applying *main theorem* II to the coherent sheaf of ideals $\bar{J}$ on $\bar{X}$, we get a finite succession of monoidal transformations $\bar{f}_i$: $\bar{X}(i+1) \to \bar{X}(i)$ with non-singular irreducible centers $\bar{D}(i)$ in $\bar{X}(i)$, which has the properties stated in *main theorem* II, where E_0 is the empty subscheme of $\bar{X} = \bar{X}(0)$; say $0 \leqq i < \bar{r}$. By (ii) of *main theorem* II, we see that all the $\bar{D}(i)$ are mapped into the closed subset $\mathbf{T} - \mathbf{S}$ of the underlying topological space of X. Thus we get a finite succession of monoidal transformations f_i: $X(i+1) \to X(i)$ with non-singular irreducible centers $D(i)$ in $X(i)$, where $X(0) = X$ and $0 \leqq i < \bar{r}$, such that there exists an open immersion u_i: $\bar{X}(i) \to X(i)$ for each i having the properties that $u_i(\bar{D}(i)) = D(i)$ and $u_i \circ \bar{f}_i = f_i \circ u_{i+1}$ for all i, where u_0 is the canonical immersion. As is easily seen, by this succession (satisfying the condition (1)), either the integer d gets reduced, or (2) is achieved. Corollary 1 is thus proved.

Let $\mathbf{B}$ be a field of characteristic zero as above, and $\mathbf{L}$ a field containing $\mathbf{B}$ and finitely generated over $\mathbf{B}$. Let us consider the pairs (X, θ) of reduced and irreducible algebraic $\mathbf{B}$-schemes X and $\mathbf{B}$-isomorphisms θ of $\mathbf{L}$ to the functions fields of X. If (X_1, θ_1) and (X_2, θ_2) are two of such pairs, then there exists a unique rational application f of X_1 to X_2 which induces the isomorphism $\theta_1 \circ \theta_2^{-1}$ of the function field of X_2 to that of X_1. We call f *the rational application of* (X_1, θ_1) *to* (X_2, θ_2). If f is a morphism, then we say that (X_1, θ_1) *dominates* (X_2, θ_2). If (X, θ) is as above, and if f: $X' \to X$ is a monoidal transformation with center nowhere dense in X, we have a unique $\mathbf{B}$-isomorphism θ' of $\mathbf{L}$ to the function field of X' such that $\theta \circ \theta'^{-1}$ is induced by f. By abuse of language, we say that f: $(X', \theta') \to (X, \theta)$ is the monoidal transformation of (X, θ). Let $\mathfrak{M}(\mathbf{L}/\mathbf{B})$ denote the class of those pairs (X, θ) with *proper* X (i.e., the structural morphisms of X to $\mathrm{Spec}(\mathbf{B})$ are proper). As consequences of Corollary 1, we can prove:

(1) *Let* (X_1, θ_1) *and* (X_2, θ_2) *belong to* $\mathfrak{M}(\mathbf{L}/\mathbf{B})$. *Then we can obtain* $(X_1', \theta_1') \in \mathfrak{M}(\mathbf{L}/\mathbf{B})$ *from* (X_1, θ_1) *by means of a finite succession of monoidal transformations with non-singular centers, such that* (X_1', θ_1') *domiates* (X_2, θ_2). *If* X_1 *is non-singular, so is* X_1'. (See also *main theorem* I in the strong form.)

(2) *Let us suppose that* (X_1, θ_1) *dominates* (X_2, θ_2) *and that both* X_1 *and* X_2 *are non-singular. Let* f *be the rational application of* (X_1, θ_1) *to*

(X_2, θ_2), *which is a morphism of* X_1 *to* X_2. *Then the direct images* $R^i f(O_{X_1})$ *of* O_{X_1} *by* f *are zero sheaves on* X_2 *except* $R^0 f(O_{X_1})$ *which is* O_{X_2}. (In fact, it follows from the same result for a monoidal transformation of a non-singular algebraic **B**-scheme with non-singular center, and from (1) applied twice.)

(3) *Let* $(X_1, \theta_1) \in \mathfrak{M}(\mathbf{L}/\mathbf{B})$. *Then we can obtain* (X', θ') *from it by means of a finite succession of monoidal transformations with non-singular centers, such that* X' is *non-singular and projective*. (We may assume that X_1 is non-singular by *main theorem* I in the strong form. Then we apply (1) to (X_2, θ_2) which *dominates* (X_1, θ_1) and of which X_2 is *projective*. The existence of such (X_2, θ_2) is obvious. Note that the algebraic **B**-scheme X_1' obtained in (1) is then necessarily projective.)

It has been suggested by Schwarzenberger [19], that every vector bundle on a non-singular projective variety X may be obtained from an extension of line bundles on a non-singular projective variety X' which is obtained from X by a finite succession of monoidal transformations with non-singular centers. He gave a proof of this statement when X is a surface. We can prove the statement for the case in which X is an arbitrary non-singular projective variety defined over a field of characteristic zero. We state the result as follows:

COROLLARY 2. *Let* X *be the same as in the above Corollary* 1. *We assume that* X *is irreducible. Let* F *be a locally free sheaf of* O_X*-modules which is coherent. Then there exists a morphism* $f: X' \to X$ *obtained by a finite succession of monoidal transformation with non-singular and nowhere dense centers, such that the inverse image* $f^*(F)$ *contains an invertible subsheaf of* $O_{X'}$*-modules* L *with a locally free quotient sheaf* $f^*(F)/L$.

In fact, this follows from the above Corollary 1 of *main theorem* II. To see this, we first remark that any locally free coherent sheaf F contains an invertible sheaf of O_X-modules M. Of course, in general, the quotient sheaf F/M may not be locally free. Let J be the coherent sheaf of ideals on X which is defined as follows: For each point x of X, we choose a free base $(c_1, c_2, \cdots, c_r)$ (resp. d) of the $O_{X,x}$-module F_x (resp. M_x), and we write $d = \sum_{j=1}^{r} h_j c_j$ with elements h_j in $O_{X,x}$. Then we see that the ideal $(h_1, h_2, \cdots, h_r)O_{X,x}$ is independent of the choice of those bases. Now J is defined in such a way that $J_x = (h_1, h_2, \cdots, h_r)O_{X,x}$. To this sheaf of ideals J on X, we apply Corollary 1 of *main theorem* II in which $e = 1$. We let $f: X' \to X$ be the composition of a succession of monoidal transformations which we find according to Corollary 1. We then obtain an invertible subsheaf L of $f^*(F)$ such that $f^*(M) = f^{-1}(J)L$, where $f^*(M)$

is viewed as a subsheaf of $f^*(F)$ in a natural manner. We can easily see that $f^*(F)/L$ is locally free on X'.

Finally we state the following immediate but important consequence of *main theorem* II, which will be referred to as *the simplification of an algebraic boundary*:

COROLLARY 3.[22'] *Let X be the same as in main theorem* II. *Let W be an algebraic subscheme of X which is nowhere dense in X. Then there exists a finite succession of monoidal transformations $f_i: X_{i+1} \to X_i$ with non-singular centers D_i, where $0 \leqq i < r$ and $X_0 = X$, such that:*

(i) X_r is non-singular;

(ii) if $\bar{f}_i$ denotes the composition of the f_j for $0 \leqq j < i$, D_i is contained in $\bar{f}_i^{-1}(W)$ for all i; and

(iii) the associated reduced scheme $\mathrm{red}(\bar{f}_r^{-1}(W))$ *of $\bar{f}_r^{-1}(W)$ is defined by an invertible sheaf of ideals on X_r and has only normal crossings.*

Since X is assumed to be non-singular, (i) is a necessary consequence of the non-singularity of the D_i. In view of *main theorem* I in the strong form, *this assumption of Corollary* 2 *may be replaced, without affecting the conclusion, by the weaker one that singular locus of X is contained in W.* (Corollary 3 was conjectured and used by Atiyah and Hodge [2].)

6. Local uniformization theorem and schematization

Let $f: X' \to X$ be a rational application of reduced algebraic **B**-schemes (resp. reduced analytic **K**-spaces). We say that f is *a modification* if f induces an isomorphism of the restrictions of X' and X to open dense subsets of their underlying topological spaces. A modification is said to be *proper* if it is so as a rational application. For a fixed reduced algebraic **B**-scheme (resp. a reduced analytic **K**-space) X, we consider the class $\mathfrak{M}(X)$ of all pairs (X', f) of reduced algebraic **B**-schemes (resp. reduced analytic **K**-spaces) X and proper modifications f of X' to X. For $(X_1, f_1) \in \mathfrak{M}(X)$ and $(X_2, f_2) \in \mathfrak{M}(X)$, there exists $(X_3, f_3) \in \mathfrak{M}(X)$ such that the rational applications $f_i^{-1} \circ f_3$ $(i = 1, 2)$ are both *morphisms*. We can prove that there exists a local-ringed space $\mathfrak{Z}(X)$ with morphisms $h(X', f): \mathfrak{Z}(X) \to X'$ for $(X', f) \in \mathfrak{M}(X)$, which has the following properties:

(i) If $(X_i, f_i) \in \mathfrak{M}(X)$, for $i = 1, 2$, such that $f_1^{-1} \circ f_2$ is a morphism, $h(X_1, f_1) = (f_1^{-1} \circ f_2) \circ h(X_2, f_2)$, and

(ii) if another $\mathfrak{Z}(X)'$ with $h(X', f)': \mathfrak{Z}(X) \to X'$ for $(X', f) \in \mathfrak{M}(X)$ has the property (i), then there exists a unique morphism $g: \mathfrak{Z}(X)' \to \mathfrak{Z}(X)$

[22'] The original formulation of this fact as a problem is due to Zariski and can be found in his paper, *Le problème de la réduction des singularités d'une variété algébrique*, Bull. Sci. Math. France (2) 78 (1954), 31-40.

such that $h(X', f)' = h(X', f) \circ g$ for all $(X', f) \in \mathfrak{M}(X)$.

We see that $\mathfrak{Z}(X)$ with $h(X', f)$ is unique up to an isomorphism of $\mathfrak{Z}(X)$. We call $\mathfrak{Z}(X)$ *the Zariski space of* X, and $h(X', f)$ *the canonical morphism of* $\mathfrak{Z}(X)$ *to* (X', f). We write $h(X)$ for $h(X, \mathrm{id}_X)$. This morphism $h(X)$ of $\mathfrak{Z}(X)$ to X determines all the other morphisms $h(X', f)$ by the equality $h(X) = f \circ h(X', f)$, which makes sense in an open dense subset of $\mathfrak{Z}(X)$. If $g\colon Y \to X$ is a morphism of reduced algebraic **B**-schemes (resp. reduced analytic **K**-spaces), then we have a unique morphism $\mathfrak{Z}(g)\colon \mathfrak{Z}(Y) \to \mathfrak{Z}(X)$ such that $h(X) \circ \mathfrak{Z}(g) = g \circ h(Y)$. If g is a proper modification, then $\mathfrak{Z}(g)$ is an isomorphism of local-ringed spaces. In fact, a canonical isomorphism $\mathfrak{Z}(g)$ is obtained for an arbitrary proper modification g, not necessarily a morphism. Some further comments will be made on Zariski spaces in the following section, especially for the analytic case.

Let $f\colon X' \to X$ be a morphism of algebraic **B**-*pre*schemes (resp. *pre*analytic **K**-spaces). In the case of preanalytic **K**-spaces, f is said to be *proper* if every point of X admits a neighborhood (closed) whose preimage in X' by f is compact and Hausdorff. The same term in the case of algebraic **B**-preschemes will be used in the sense of Grothendieck, [6, II, Ch. II, § 5]. Let us assume that X' and X are both reduced. We then say that f is *complete* (or, X' is *complete over* X by the morphism f), if the following conditions are satisfied:

(i) *The morphism f is surjective* (i.e., so is the underlying continuous mapping),

(ii) *for every point x' of X', we can find a* 4^{ple} $(\mathbf{U}, \bar{X}', \bar{f}, j)$ *consisting of an open dense subset* **U** *of the underlying topological space of X' which contains x', a reduced algebraic* **B**-*prescheme* (*resp. reduced preanalytic* **K**-*space*) $\bar{X}'$, *a proper morphism* $\bar{f}\colon \bar{X}' \to X$, *and a morphism* $j\colon X' \mid \mathbf{U} \to \bar{X}'$ *which induces an isomorphism of the same to the restriction of $\bar{X}'$ to an open dense subset of its underlying topological space and such that* $\bar{f} \circ j = f \mid \mathbf{U}$; *and*

(iii) *every point x of X admits an open neighborhood* **V** *in the underlying topological space of X, such that $X \mid \mathbf{V}$ is an algebraic* **B**-*scheme* (*resp. an analytic* **K**-*space*) *and that, if we identify in a canonical way the Zariski spaces* $\mathfrak{Z}(\bar{X}' \mid \bar{f}^{-1}(\mathbf{V}))$ *for all* 4^{ples} $(\mathbf{U}, \bar{X}', \bar{f}, j)$ *of* (ii) *and call it* $\mathfrak{Z}(X' \mid \mathbf{V})$, *then the underlying topological space of* $\mathfrak{Z}(X' \mid \mathbf{V})$ *is equal to the union of* $h(\bar{X}' \mid \bar{f}^{-1}(\mathbf{V}))^{-1}(j(\mathbf{U}) \cap \bar{f}^{-1}(\mathbf{V}))$ *for all* $(\mathbf{U}, \bar{X}', \bar{f}, j)$.

The *local uniformization theorem* of Zariski [24], can be stated as follows:

THEOREM. *Let X be a reduced algebraic* **B**-*scheme* (*resp. a reduced analytic* **C**-*space*), *where* **B** *is a field of characteristic zero. Let x be a* *point of X. Then there exist an open neighborhood* $\mathbf{V}$ *of x in the underlying topological space of X, a non-singular algebraic* $\mathbf{B}$*-prescheme (resp. a non-singular preanalytic* $\mathbf{C}$*-space) X', and a morphism $f: X' \to X \mid \mathbf{V}$ which is complete and induces an isomorphism of their restrictions to some open dense subsets.*

(Zariski's proof of the theorem applies to the case of analytic $\mathbf{R}$-spaces, as well, and establishes an analogous theorem in this case. But we omit this to avoid any further complication.)

Zariski's proofs of resolution of singularities in two and three dimensions (except the theorem of B. Levi-Zariski, § 3, and his most recent one [28]) are essentially based on his *local uniformization theorem*. In the *local uniformization theorem*, we require that $\mathbf{V}$ contain all the points of X for the algebraic case. Clearly this requirement leaves the statement equivalent in the algebraic case (but not in the analytic case by any trivial means). With this modification, the theorem will be denoted by (LU). By requiring that X' be an algebraic $\mathbf{B}$-*scheme* (resp. an *analytic* $\mathbf{C}$-space) in (LU), we obtain a statement which we shall denote by (RS), and which is the one called the reduction of singularities by Zariski in the algebraic case, and proved by him for $\dim(X) \leqq 3$

In the two-dimensional case, the deduction of (RS) from (LU) is *strikingly simple*, according to Zariski [25]. The same, in the three-dimensional case, was accomplished also by Zariski [27], but the proof is highly complicated, and is not of the type to be generalized by any simple means to any higher dimensional case. In this work the *local uniformization theorem* plays no role at all, but we like to examine a certain problem, interesting in its own right, which was raised in the effort to deduce (RS) from (LU).

The deduction of (RS) from (LU) is immediate if the following question is answered affirmatively:

Question (F). *Let $f: X_1 \to X_2$ be a proper modification (not necessarily a morphism) of reduced algebraic* $\mathbf{B}$*-schemes (resp. reduced analytic* $\mathbf{C}$*-spaces). Let $\mathfrak{Z} = \mathfrak{Z}(X_1) = \mathfrak{Z}(X_2)$, the Zariski spaces canonically identified. Let $h_i = h(X_i): \mathfrak{Z} \to X_i$ be the canonical morphisms for $i = 1, 2$. Let* $\mathbf{G}$ *be a quasi-compact (resp. compact) subset of the underlying topological space of $\mathfrak{Z}$. Let $\mathbf{U}_i$ be an open subset of the underlying topological space of X_i such that $X_i \mid \mathbf{U}_i$ is non-singular, for $i = 1, 2$. We assume that $\mathbf{G} \subset h_1^{-1}(\mathbf{U}_1) \cup h_2^{-1}(\mathbf{U}_2)$. Then we ask if there exist morphisms $X_i' \to X_i$ $(i = 1, 2)$ obtained by finite successions of monoidal transformations $X_{ij+1} \to X_{ij}$ with centers D_{ij}, nowhere dense in X_{ij}, which have the following properties:*

(1) *D_{ij} has no multiple points contained in $h_{ij}(\mathbf{G}) \cap f_{ij}^{-1}(\mathbf{U}_i)$, where $h_{ij}: \mathfrak{Z} \to X_{ij}$ and $f_{ij}: X_{ij} \to X_i$ are the canonical morphisms, for all (i, j)*; and

(2) *for every point z of $\mathbf{G}$ such that $h_i(z) \in \mathbf{U}_i$ for $i = 1, 2$, the canonical rational application $f': X_1' \to X_2'$ is defined at $h_1'(z)$ as well as f'^{-1} at $h_2'(z)$, where $h_i': \mathfrak{Z} \to X_i'$ denotes the canonical morphism, for $i = 1, 2$.*

In the algebraic case, we are interested in the case in which $\mathbf{G}$ is the underlying topological space of $\mathfrak{Z}$. In this case, (1) says that the morphisms $g_i: X_i' \to X_i$ induce $X_i' \mid g_i^{-1}(\mathbf{U}_i) \to X_i \mid \mathbf{U}_i$ which are obtained by finite successions of monoidal transformations with non-singular and nowhere dense centers, and (2) says that $X_1' \mid g_1^{-1}(\mathbf{U}_1)$ and $X_2' \mid g_2^{-1}(\mathbf{U}_2)$ patch together and make a non-singular algebraic $\mathbf{B}$-scheme X' with canonical morphisms to X_1 and X_2 which are both proper modifications.

As a special case of *question* (F), we ask the following

Question (F′). *Let $f: X_1 \to X_2$ be a proper modification as in question* (F). *We assume that both X_1 and X_2 are non-singular and that, especially in the analytic case, X_1 and hence X_2 are compact. Can one find a proper modification $h_i: X_i' \to X_i$ for each i, which is obtained by a finite succession of monoidal transformations with non-singular centers, such that the rational application $h_2^{-1} \circ f \circ h_1$ is an isomorphism of X_1' to X_2'?*

It is quite desirable to have an answer to the *question* (F′). We have no answer to it for $\dim(X_1) = \dim(X_2) \geqq 3$. For $\dim(X_1) = \dim(X_2) = 2$, we have an affirmative answer to *question* (F), hence, to (F′). In fact, it can be easily derived from the following fact:

THEOREM.[23] *Let X be a non-singular algebraic $\mathbf{B}$-scheme (or, a non-singular analytic $\mathbf{C}$-space) of dimension 2. Then every proper morphism $f: X' \to X$, such that X' is non-singular and f is a modification, is obtained by a succession of monoidal transformations whose centers are points.*

The same statement for $\dim(X) \geqq 3$ is false. In order to deduce (RS) from (LU) in the three-dimensional case, Zariski proves the following fact by means of the above theorem and the theorem of B. Levi-Zariski:

THEOREM. *Let $f: X_1 \to X_2$, $\mathfrak{Z}$ and h_i, $i = 1, 2$, be the same as in the question* (F). *We assume that X_1 and X_2 are reduced algebraic $\mathbf{B}$-schemes of dimension 3, where $\mathbf{B}$ is a field of characteristic zero. Let $\mathbf{U}_i$ be the set of simple points of X_i for $i = 1, 2$. Then for each i, there exists a modification $g_i: X_i' \to X_i$, which is a proper morphism, such that if*

[23] This theorem is due to Zariski. (See the Lemma [27, p. 538], and notice that his proof applies to the general case.) One may also refer to: *Schlichte Abbildungen und lokale Modificationen 4-dimensionaler komplexer Mannigfaltigkeiten*, by H. Hopf, Comment. Math. Helv., 29 (1955), 132–155.

$\mathbf{U}_i' = g_i^{-1}(\mathbf{U}_i)$ *and* $h_i'\colon \mathfrak{Z} \to X_i'$ *is the canonical morphism, then*

(1) $X_i' \mid \mathbf{U}_i' \to X_i \mid \mathbf{U}_i$, *induced by* g_i, *is obtained by a finite succession of monoidal transformations with non-singular centers, for* $i = 1, 2$.

(2) $\mathbf{V}_i' = h_i'(h_1^{-1}(\mathbf{U}_1) \cap h_2^{-1}(\mathbf{U}_2))$ *is an open subset of* $\mathbf{U}_i'$ *for* $i = 1, 2$, *and*

(3) *the canonical modification* $p\colon X_1' \mid \mathbf{V}_1' \to X_2' \mid \mathbf{V}_2'$ *is a proper morphism such that* p^{-1} *has only a finite number of points of indeterminacy.*

Let $\mathbf{U}_2''$ be the complement of the points of indeterminacy of p^{-1} in $\mathbf{U}_2'$. Then we see that $X_1' \mid \mathbf{U}_1'$ and $X_2' \mid \mathbf{U}_2''$ patch together by means of p^{-1} and give rise to a *non-singular* algebraic $\mathbf{B}$-*scheme* whose Zariski space is canonically identified with the restriction of $\mathfrak{Z}(X_1) = \mathfrak{Z}(X_2)$ to the open subset consisting of those points which are mapped either to a simple point of X_1 or (and) to a simple point of X_2. Using this theorem a finite number of times repeatedly, we can deduce (RS) from (LU) in the three-dimensional algebraic case. A better proof of the last theorem can be obtained by taking Corollary 1 of *main theorem* II with $\dim(X) = 3$, instead of the theorem of B. Levi-Zariski.

7. Generalizations and problems in the analytic case

It was proved by Zariski [23], that the resolution of singularities of an algebraic surface (over a field of characteristic zero) can be obtained by applying alternately and repeatedly the normalizations and the monoidal transformations with singular points as their centers. The same proof can be carried out for complex-analytic surfaces. More precisely, we can state his theorem as follows:

THEOREM (a). *Let* X *be a reduced algebraic* $\mathbf{B}$-*scheme* (*or a reduced analytic* $\mathbf{C}$-*space*) *of dimension* 2, *where* $\mathbf{B}$ *is a field of characteristic zero. We then take the sequence of modifications* $f_i\colon X_{i+1} \to X_i$ ($X = X_0$, *and* $i = 0, 1, 2, \cdots$) *such that for even* i, f_i *is the normalization of* X_i; *and for odd* i, f_i *is the monoidal transformation of* X_i *whose center is the singular locus of* X_i (i.e., *the reduced subscheme* (*or reduced analytic subspace*) *of* X_i *whose points are exactly the multiple points of* X_i). *Then the sequence is necessarily finite over any quasi-compact* (*or compact*) *subset* $\mathbf{W}$ *of the underlying topological space of* X, i.e., *if* $\mathbf{W}_0 = \mathbf{W}$ *and* $\mathbf{W}_{i+1} = f_i^{-1}(\mathbf{W}_i)$ *for all* i, *then* X_i *has no multiple points in* $\mathbf{W}_i$ *for some integer* i. *Thus the sequence is finite in the algebraic case, and in both cases we have the projective limit* $h\colon X' \to X$ *of the system of modifications* $h_i\colon X_i \to X$ *such that* X' *is a non-singular algebraic* $\mathbf{B}$-*scheme* (*or, a non-singular analytic* $\mathbf{C}$-*space*) *and* h *is a proper modification.*

This is the most canonical process of resolving singularities, and moreover, according to Zariski's proof, at least when X admits a morphism into a non-singular X_0 which is a modification, the morphism $h: X' \to X$ obtained in Theorem (a) is *the smallest modification* with non-singular X' in the sense that, for any morphism $h^*: X^* \to X$ which is a modification with a non-singular X^*, there exists a unique morphism $g: X^* \to X'$ such that $h^* = h \circ g$. In the case of $\dim(X) = 2$, *we have always the smallest modification* $h: X' \to X$ *which is a proper morphism of a non-singular* X' *to* X, while this is no longer true for $\dim(X) \geqq 3$. Notice that the singular locus of a normal algebraic **B**-scheme (or, a normal analytic **C**-space) of dimension two is a discrete set of points; this makes the situation quite simple in comparison with any higher dimensional case.

Zariski's proof of singularities in 3-dimensional case works only in the algebraic case [27]. Recently, Kuhlmann in Würzburg succeeded in proving the resolution of singularities in the 3-dimensional analytic case *with the restriction* that the given analytic **C**-space X, reduced and of dimension 3, admits a finite morphism $X \to X_0$ where X_0 has no singularities.

We can generalize the resolution of singularities in the analytic case to the extent that includes an arbitrary reduced analytic **C**-space of dimension 3, an arbitrary reduced analytic subspace of a product $S \times P$, where S is a complex Stein space and P is a complex projective space, and an arbitrary reduced analytic **R**-space.

We first remark that *main theorems* I, I*, II, *stated in the preceding sections, remain verified if the field* **B** *is replaced by any local ring belonging to a certain class of local rings*, which we shall specify in the beginning paragraph of Chapter I, and denote by $\mathscr{B}$. Here we simply remark that the class $\mathscr{B}$ contains local rings of an arbitrary **K**-analytic space (**K** is either **C** or **R**), localizations of the ring of analytic functions $\Gamma(X, O_X)$ with respect to the maximal ideals corresponding to the points of X where $X = (X, O_X)$ is an arbitrary complex Stein space or an arbitrary analytic **R**-space, and so on. We also remark that if V is an analytic subspace of a product $S \times P$, where S is a complex Stein space, then for every point x of V the class $\mathscr{B}$ contains the subring of $O_{V,x}$ which consists of the holomorphic germs of global meromorphic functions on V, for $\mathscr{B}$ has the property that for any $\mathbf{B} \in \mathscr{B}$ and any **B**-algebra of finite type **A**, the local rings of Spec(**A**) belong to $\mathscr{B}$.

The *main theorem* I in the strong form with (a), (b), and (c) for a local ring **B** of an analytic **C**-space implies immediately a local-analytic analogue of the theorem which can be stated as follows:

MAIN THEOREM I′. *Let X be a reduced analytic subspace of $S \times P$, where S is a complex Stein space and P is a complex projective space. Let x be a point of S. Then there exists an open neighborhood* **V** *of $p^{-1}(x)$ in the* *underlying topological space of X, where p denotes the morphism $X \to S$ induced by the projection of $S \times P$, such that the restriction $V = X \mid \mathbf{V}$ has the following property: There exists a finite succession of monoidal transformations f_i: $V_{i+1} \to V_i$ with centers D_i, where $0 \leqq i < r$ and $V = V_0$, such that:*

(a) *the center D_i is non-singular;*

(b) *D_i does not contain any simple point of V_i;*

(c) *V_i is normally flat along D_i, for all i; and finally*

(d) *V_r is non-singular.*

The *main theorem* II for a local ring $\mathbf{B}$ of an analytic C-space implies immediately a local-analytic analogue of the theorem which can be stated as follows:

MAIN THEOREM II′. *Let X, $S \times P$, p be as in main theorem* I′, *J a coherent sheaf of non-zero ideals on X, and F a reduced analytic subspace of X. We assume that X is non-singular, and that F is everywhere of codimension one and has only normal crossings. Let x be a point of S. Then there exists an open neighborhood $\mathbf{Y}$ of $p^{-1}(x)$ in the underlying topological space of X, such that $Y = X \mid \mathbf{Y}$ has the following property: There exists a finite succession of monoidal transformations f_i: $Y_{i+1} \to Y_i$ with centers D_i, where $0 \leqq i < r$ and $Y = Y_0$, such that:*

(i) *D_i is non-singular and irreducible;*

(ii) *if I_{i+1} is the weak transform of I_i by f_i for all i, where $I_0 = J \mid Y$, then $\nu(I_{iy})$ is constant for $y \in D_i$, positive and equal to its maximum for all $y \in Y_i$;*

(iii) *if E_{i+1} denotes the reduced analytic subspace* $\mathrm{red}(f_i^{-1}(E_i) \cup f_i^{-1}(D_i))$ *of Y_{i+1} for all i, where E_0 is the restriction $F \mid Y_0$, then E_i has only normal crossings with D_i; and finally*

(iv) *E_r has only normal crossings, and $I_r = O_{Y_r}$.*

These results are sufficient in most cases for the local study of analytic singularities, in particular for the study of an isolated multiple point of an analytic C-space. In the local study of modification of an analytic C-space, the following lemma is useful:

LEMMA[24] (Local Chow's Lemma) *Let f: $X' \to X$ be a modification of reduced analytic* C*-spaces, which is a proper morphism. Let x be a point of X. Then there exists an open neighborhood $\mathbf{Y}$ of x in the underlying topological space of X, such that the restriction $Y = X \mid \mathbf{Y}$ has the following property: There exists an analytic subspace D of Y such that*

[24] The algebro-geometrical analogue of this lemma is trivial to prove and often referred to as Chow's lemma. The lemma thus formulated in the analytic case seems no longer trivial, and we shall give a proof of this lemma in a forthcoming paper [10].

(i) *the points of* D *are exactly those points of* Y *at which the meromorphic application* f^{-1} *is not defined; and*

(ii) *if* $h: Y' \to Y$ *is the monoidal transformation of* Y *with center* D, *then the meromorphic application* $(f^{-1} \mid Y) \circ h$ *is a morphism of* Y' *to* X'.

Using this lemma, we can deduce the following corollary of the *main theorems* I′ and II′:

COROLLARY 1. *Let* $f: X' \to X$ *be a modification which is a proper morphism. Assume that there exists a unique point* x *of* X *at which* f^{-1} *is not defined, and that* $f^{-1}(x)$ *contains all the multiple points of* X'. *Then there exists a finite succession of monoidal transformations* $h_i: X_{i+1} \to X_i$ *with centers* D_i, *where* $0 \leqq i < r$ *and* $X = X_0$, *such that*

(i) D_i *is non-singular, and every point of* D_i *is mapped to* x *by the composed morphism* $X_i \to X$;

(ii) *if* h *is the composed morphism* $X_r \to X$, *then the meromorphic application* $f^{-1} \circ h: X_r \to X'$ *is a morphism;*

(iii) X_r *is non-singular; and*

(iv) $\mathrm{red}(h^{-1}(x))$ *is everywhere of codimension one and has only normal crossings.*

This corollary seems to be useful in the study of isolated singularities on analytic C-spaces, imbeddings of compact analytic C-spaces with negative (or weakly negative) normal bundles, and so on.[25]

Using the same lemma as above, we also get another corollary of the *main theorem* II′ as follows:

COROLLARY 2.[26] *Let* $f: X' \to X$ *be a modification of analytic* C-*spaces which is a proper morphism. If both* X *and* X' *are non-singular, then we have the vanishing of higher direct images* $R^q f(O_{X'})$ *for all* $q > 0$. (*Clearly,* $R^0 f(O_{X'}) = O_X$.)

In fact, in view of *local Chow's lemma* and *main theorem* II′, the proof of Corollary 2 can be reduced to the case in which f is a monoidal transformation with non-singular center. In this case, the assertion is known and can be given a direct proof. We remark that in the above assertion the non-singularity assumption on both X and X' is essential.

Moreover, combining Zariski's methed of proving Theorem (a) with the result in *main theorem* I′, we get

COROLLARY 3. *Let* X *be a reduced analytic* C-*space of dimension* 3. *Let* U *be the set of simple points of* X, *which is an open dense subset of the*

---

[25] cf. *Über Modificationen und exzeptionelle analytische Mengen,* by H. Grauert, Math. Ann., 146 (1962).

[26] This fact was conjectured by Grothendieck in his talk at the International Congress for Mathematicians, Edinburgh, 1958. For a detailed proof, see a forthcoming paper [10].

underlying topological space of X. Then there exists a non-singular analytic $\mathbf{C}$*-space X' with a modification f: $X' \to X$ which is a proper morphism and induces an isomorphism of $X' \mid f^{-1}(\mathrm{U})$ to $X \mid \mathrm{U}$.*

In fact, we first take the succession of modifications f_i: $X_{i+1} \to X_i$, where $i = 0, 1, 2, \cdots$, and $X = X_0$, as follows: For even i, f_i is the normalization of X_i, and for odd i, f_i is the monoidal transformation whose center is the union of those reduced irreducible components of the singular locus of X_i which are not mapped to points by the composed morphism $X_i \to X$. This sequence may be infinite, but it is always *locally finite* in the sense that every point of X has a neighborhood $\mathbf{N}$ such that the preimage of $\mathbf{N}$ in X_i is contained in the domain of definition of f_i^{-1} for all but a finite number of i. Thus, in any case, we get a proper morphism g: $X'' \to X$ which is the inverse limit of the composed morphisms $X_i \to X$ for all i. What is more, every irreducible reduced component of the singular locus of X'' is mapped to a point of X by g. The proofs of these results are essentially the same as that of Theorem (a).[26'] Now, take any point x of X which is in the image of the singular locus of X''. We have only a discrete set of such points on X. There exists an open neighborhood of x, say S, such that $S = X \mid \mathrm{S}$ is a complex Stein space, and $X'' \mid g^{-1}(\mathrm{S})$ can be imbedded in a product $S \times P$ with a complex projective space P in such a way that the projection $S \times P \to S$ induces $g \mid g^{-1}(\mathrm{S})$. We may assume that all the multiple points of X'' in $g^{-1}(\mathrm{S})$ are mapped into x by g. We then apply *main theorem* I′ to $(g \mid g^{-1}(\mathrm{S}))$: $X'' \mid g^{-1}(\mathrm{S}) \to X \mid \mathrm{S}$. We can do this for all x as above simultaneously without affecting each other, and finally get a non-singular analytic $\mathbf{C}$-space X' with a proper morphism h: $X' \to X''$ which is a modification. The pair (X', f) with $f = g \circ h$ has the property asserted in Corollary 3.

Main theorems I′ and II″ can be strengthened and proved if we take X itself instead of its restrictions V and Y, although the conclusions must be accordingly modified to the extent that an infinite (countable and locally finite) succession of monoidal transformations (whose centers satisfy the conditions stated in the theorems) should be admitted. For this purpose, however, the *main theorems* I in the strong form with (a)–(c) and II (for $\mathbf{B} \in \mathscr{B}$) are not sufficient, whereas they were for *main theorems* I′ and II′. As a matter of fact, we need certain stronger and more detailed theorems which we shall actually be forced to prove in order to carry out our inductive method of resolving singularities in all dimensions. We state below

[26'] In his paper [23], Zariski uses some results of his paper, *Polynomial ideals defined by infinitely near base points*, Amer. J. of Math., 60 (1938), 151–204. The results of this paper have been generalized to arbitrary regular local rings of dimension two. (See Samuel-Zariski [18, Vol. II, Appendix 5].) For our purpose, we need this generalization.

in precise forms an analytic analogue I$'(n)$ of the *main theorem* I(n), which will be stated and proved (§ 3, Ch. I), and also an analytic analogue II$'(N)$ of the *main theorem* II(N), which will also be stated and proved (§ 3, Ch. I).

Let Λ be a well-ordered set. We shall denote by 0 the smallest element of Λ, and by $\lambda + 1$ the successor of each $\lambda \in \Lambda$, whenever no ambiguity arises. By *a succession of monoidal transformations* $\{f_\lambda: X_{\lambda+1} \to X_\lambda\}$ for $\lambda \in \Lambda$, we shall mean that f_λ is a monoidal transformation of X_λ (with a certain center) for each $\lambda \in \Lambda$ with $\lambda + 1 \in \Lambda$ and that, for each $\lambda \in \Lambda$ whose predecessor does not exist, X_λ is the projective limit of the system $\{f_\mu: X_{\mu+1} \to X_\mu,\ \mu < \lambda\}$. According to this definition, we have a canonical morphism of X_λ to X_0 for each $\lambda \in \Lambda$. We say that a succession of monoidal transformations as above is *locally finite*, if every point of X_0 has a neighborhood $\mathbf{N}$ in the underlying topological space of X_0 such that the center of f_λ meets $\bar{f}_\lambda^{-1}(\mathbf{N})$ only a finite number of $\lambda \in \Lambda$, where $\bar{f}_\lambda$ denotes the canonical morphism of X_λ to X_0.

MAIN THEOREM I$'(n)$. *Let X be an analytic* $\mathbf{C}$-*space, W a reduced analytic subspace of X, and $V_i (0 \leqq i \leqq a)$ a finite number of analytic subspaces of X each of which contains W. Let n denote the dimension of W. We assume that there exists a projective morphism of X to a complex Stein space.* (*See* § 1 *for the definition of a complex Stein space, and* § 2 *for the definition of a projective morphism.*) *Then there exists a well-ordered set Λ, countable and with the maximal element γ, and a locally finite succession of monoidal transformations $\{f_\lambda: X(\lambda+1) \to X(\lambda)\}$ with centers $D(\lambda)$ for $\lambda \in \Lambda$ and for $X(0) = X$, which has the following properties:*

(i) *we have a reduced analytic subspace $W(\lambda)$ and analytic subspaces $V_i(\lambda)$ $(0 \leqq i \leqq a)$ of $X(\lambda)$, for $\lambda \in \Lambda$, such that $D(\lambda)$ is contained in $W(\lambda)$, that $W(\lambda)$ is contained in all $V_i(\lambda)$, and that the succession f_λ induces successions of monoidal transformations $\{W(\lambda + 1) \to W(\lambda)\}$ and $\{V_i(\lambda + 1) \to V_i(\lambda)\}$ with centers $D(\lambda)$ for $\lambda \in \Lambda$;*

(ii) *$D(\lambda)$ is non-singular and does not contain any simple point of $W(\lambda)$ at which $X(\lambda)$ and $V_i(\lambda)$ are all normally flat along $W(\lambda)$, for all $\lambda \in \Lambda$;*

(iii) *$X(\lambda)$ and $V_i(\lambda)$ are all normally flat along $D(\lambda)$, for all $\lambda \in \Lambda$; and finally*

(iv) *$W(\gamma)$ is non-singular, and all the $V_i(\gamma)$ and $X(\gamma)$ are normally flat along $W(\gamma)$.*

Notice that we have a canonical modification $f: X(\gamma) \to X$ which is a proper morphism, and that it induces such modifications $W(\gamma) \to W$ and $V_i(\gamma) \to V_i$, for all i. In fact, if $\mathbf{U}$ is the maximal open subset of the underlying topological space of W such that every $x \in \mathbf{U}$ is a simple point of W, and that X and V_i are all normally flat along W at every point of $\mathbf{U}$, then $\mathbf{U}$ is dense and equal to the domain of definition of $f^{-1}|W$.

Let Λ be a well-ordered set, and $\{f_\lambda: X_{\lambda+1} \to X_\lambda\}$ a succession of monoidal transformations of non-singular analytic $\mathbf{C}$-spaces X_λ with non-singular irreducible centers D_λ for $\lambda \in \Lambda$. Suppose the sequence is locally finite. Then, given a sheaf of ideals J_0 on X_0 (or, an analytic subspace E_0 of X_0), a sequence of sheaves of ideals J_λ on X_λ (or, analytic subspaces E_λ of X_λ) is uniquely determined by specifying the way of deriving $J_{\lambda+1}$ from J_λ (or, $E_{\lambda+1}$ from E_λ) for all $\lambda \in \Lambda$, in addition to a general agreement that, if λ has no predecessor, J_λ (or, E_λ) is the projective limit of $J_{\lambda'}$ (or, $E_{\lambda'}$) for all $\lambda' < \lambda$.

MAIN THEOREM II$'(N)$. *Let X be a non-singular analytic $\mathbf{C}$-space of dimension N, E_0 a reduced analytic subspace everywhere of codimension one of X, and J_0 a coherent sheaf of non-zero ideals on X. We assume that X admits a projective morphism of itself to a complex Stein space, and that E_0 has only normal crossings. Then there exists a well-ordered set Λ, countable and with the maximal element γ, and a locally finite succession of monoidal transformations $f_\lambda: X_{\lambda+1} \to X_\lambda$ with centers D_λ for $\lambda \in \Lambda$, where $X_0 = X$, such that*

(i) *D_λ is non-singular and irreducible;*

(ii) *if $J_{\lambda+1}$ is the weak transform of J_λ by f_λ for all $\lambda \in \Lambda$, then $\nu(J_{\lambda y})$ is a positive constant for $y \in D_\lambda$;*

(iii) *if $E_{\lambda+1}$ is the reduced analytic subspace* $\mathrm{red}(f_\lambda^{-1}(E_\lambda) \cup f_\lambda^{-1}(D_\lambda))$ *of $X_{\lambda+1}$ for all $\lambda \in \Lambda$, then E_λ has only normal crossings with D_λ; and*

(iv) *E_γ has only normal crossings, and $J_\gamma = O_{X_\gamma}$.*

Let $p: X \to S$ be a projective morphism of an analytic $\mathbf{C}$-space X to a complex Stein space $S = (\mathbf{S}, O_S)$. Let s be a point of $\mathbf{S}$, and $\mathbf{B}_s$ the local ring of $\mathrm{Spec}(\Gamma(\mathbf{S}, O_S))$ at the point corresponding to s. (We have a canonical morphism of S to the prime spectrum such that the pair of S and this morphism is an analytic $\mathbf{C}$-cover of the prime spectrum. See § 1.) Since p is projective, we have a projective $\mathbf{B}_s$-scheme $X_{(s)}$ and a canonical morphism $c_s: X_s \to X_{(s)}$, where X_s is the restriction of X to $p^{-1}(s)$, such that

(1) c_s induces a bijection of the set of points of X_s to the the set of those points of $X_{(s)}$ which are closed and mapped to the closed point of $\mathrm{Spec}(\mathbf{B}_s)$; and

(2) if x is any point of X_s, and if $y = c_s(x)$, then $O_{X_{(s)},x} (= O_{X,x})$ is faithfully flat over $O_{X_{(s)},y}$ and the maximal ideal of the latter generates that of the former.

The pair $(X_{(s)}, c_s)$ is uniquely determined by these properties, up to an isomorphism of $X_{(s)}$. We see that every analytic subspace V of X induces an algebraic subscheme $V_{(s)}$ of $X_{(s)}$ such that the relation between V and $V_{(s)}$ is the same as that between X and $X_{(s)}$, or that the sheaf of ideals of $V_{(s)}$ on $X_{(s)}$ generates the sheaf of ideals of V on X within the set of $p^{-1}(s)$. Moreover, every algebraic subscheme of $X_{(s)}$ is induced in this manner by an analytic subspace of X.

Now, we assert that the *main theorem* I$'(n)$ can be deduced from the *theorems* I(n') with $n' \leqq n$ (§ 3, Ch. I) by induction on n. Namely, take a projective morphism $p: X \rightarrow S$ with a complex Stein space S, which exists by assumption. Take any point s of S, and let $X_{(s)}$ (resp. $W_{(s)}$, $V_{i(s)}$) be the algebraic $\mathbf{B}_s$-scheme (resp. algebraic subschemes of $X_{(s)}$) which is obtained as above from X (resp. W, V_i). Then $W_{(s)}$ is a reduced subscheme of $X_{(s)}$ whose dimension, say n', is at most n. We apply the *theorem* I(n') to $(W_{(s)}, V_{i(s)}, X_{(s)})$. If the succession of monoidal transformations in the *theorem* I(n') can be induced by a succession of monoidal transformations applied to X, W, V_i which satisfy the conditions (i), (ii), and (iii) of the *main theorem* I$'(n)$, then we see that this succession of monoidal transformations transforms Y, W, V_i to X', W', V'_i with a canonical morphism $f: X' \rightarrow X$, such that if $p' = p \circ f$, then W' and V'_i, X' have the property (iv) of the *theorem* I$'(n)$ within a neighborhood of $p'^{-1}(s)$. We replace X, W, V_i by X', W', V'_i respectively, and repeat the same process if necessary. As is easily seen, a suitably chosen repetition (at most countably many times) of this process establishes the *theorem* I$'(n)$. Now, to show that the succession of monoidal transformations of the *theorem* I(n') for $X_{(s)}$, $W_{(s)}$, $V_{i(s)}$ can be induced by a succession of monoidal transformations satisfying (i), (ii), and (iii) of the *theorem* I$'(n)$ for X, W, V_i, we use the induction assumption in the following manner. Let $f_0: X_{(s),1} \rightarrow X_{(s)}$ be a monoidal transformation of $X_{(s)}$ with center $D_{(s)}$, which satisfies the conditions of the *theorem* I(n') for $X_{(s)}$, $V_{i(s)}$, $W_{(s)}$. We then have a reduced analytic subspace D of X which induces $D_{(s)}$. If we choose D to be the smallest one with the property, then we see that D does not contain any simple point of W at which all the X and V_i are normally flat along W. Moreover, if x is a point of D which is contained in $p^{-1}(s)$, then x is a simple point of D at which all W, V_i, X are normally flat along D. Let $n'' = \dim(D)$, which is smaller than n, and apply the *theorem* I$'(n'')$ (an induction assumption) to X, D, W, V_i. Note that the succession of monoidal transformations in the *theorem* I$'(n'')$ for X, D, W, V_i must induce a trivial succession (i.e., a succession of monoidal transformations with empty centers) in $X_{(s)}$, and that if X', D', W', V'_i are the final transforms of

X, D, W, V_i by this succession, then the monoidal transformation of X' with center D' satisfies the conditions (ii) and (iii) of I (n), and the succession followed by this monoidal transformation induces the given monoidal transformation f_0: $X_{(s),1} \to X_{(s)}$ with center $D_{(s)}$. The same argument can be easily extended to any succession of monoidal transformations applied to $X_{(s)}$, $W_{(s)}$, $V_{i(s)}$ which satisfy the conditions in the *theorem* I(n'). *Main theorem* I'(n) is thus established.

The deduction of *main theorem* II'(N) is done by the same principle, but is slightly more complicated. The simplest way to do this will be as follows: We first formulate and prove an analytic analogue of the *fundamental theorem* $\mathrm{I}_2^{N,n}$, (§ 2, Ch. I). This can be done by induction on (N, n) in the same way as in the above deduction of the *theorem* I'(n). Using this result, we can formulate and prove an analytic analogue of the *theorem* III(N, n), (§ 4, Ch. I). This is also done in the same way as above. We then see that *main theorem* II (N) follows from the analytic analogue of III(N, n) for the case in which $W = X$ and $b = 1$. (Compare the proof of *main theorem* II(N), (§ 3, Ch. I).)

The key fact in the above deduction of the *main theorems* I'(n) and II'(N) is that the analytic **C**-space X in these theorems admits a morphism c: $X \to X_*$ for a *scheme* X_* (to be precise, a **C**-scheme X_*) such that (X, c) is an *analytic* **C**-*cover of* X_*, where the scheme X_* should have a morphism p_*: $X_* \to S_*$ such that S_* is a scheme whose local rings at the points in the image of $p_* \circ c$ belong to the class $\mathscr{B}$ in the beginning of Ch. I; and that, for any of these points s of S_*, the restriction of p_* to $p_*^{-1}(s)$ is of finite type.

Let X be any analytic **R**-space. If **A** denotes the **R**-algebra $\Gamma(X, O_X)$, where $X = (X, O_X)$, then (X, c) is an analytic **R**-cover of Spec(**A**), where c denotes the canonical morphism, and all the local rings of Spec(**A**) at the points in the image of c belong to the class $\mathscr{B}$. The analogues of I(n) and II(N) can be formulated and proved for arbitrary analytic **R**-spaces and analytic subspaces of them. More precisely,

MAIN THEOREM I''(n). *Let X be an arbitrary analytic* **R**-*space, W a reduced analytic subspace of dimension n of X, and V_i $(0 \leqq i \leqq a)$ a finite number of analytic subspaces of X which contain W. Then there exist Λ and a succession of monoidal transformations f_λ: $X(\lambda + 1) \to X(\lambda)$ for $\lambda \in \Lambda$, which have the properties stated in the main theorem* I'(n).

MAIN THEOREM II''(N). *Let X be a non-singular analytic* **R**-*space, E_0 a reduced analytic subspace everywhere of codimension one of X, and J_0 a coherent sheaf of non-zero ideals on X. We assume that E_0 has only normal crossings. Then there exist Λ and a succession of monoidal trans-*

formations $f_\lambda\colon X_{\lambda+1} \to X_\lambda$ *for* $\lambda \in \Lambda$, *which have the properties stated in the main theorem* II$'(N)$.

Our results in the real-analytic case, as in the above theorems, are satisfactory, while those in the complex-analytic case are far from being so. In fact, in the complex-analytic case, we need the assumption that the given analytic C-space X admits a projective morphism of itself to a complex Stein space; and, for instance, *we do not know any proof of resolution of singularities of compact analytic* C*-spaces, say reduced and irreducible, of dimensions greater than* 3. The author hopes that the complete generalization of his results in the complex-analytic case will be established in near future. The difficulties in the complex-analytic case are laid up in the passage from the local analysis of singularities to the global resolution of singularities. We simply remark that the *main theorems* I$'(n)$ and II$'(N)$ can be proved without the assumption on the existence of a projective morphism of X to a complex Stein space if $n \leqq 3$ and $N \leqq 3$. Their proofs are non-trivial, and will be presented on another occasion. In this way, a better resolution theorem than Corollary 3 of *main theorem* I$'$ is obtained for reduced analytic C-spaces of dimensions $\leqq 3$.

Finally, we pose the following question, which has not been answered in general, and seems to be of importance in the theory of meromorphic modifications.

Question (M)[27]. *Let X be a reduced analytic* C*-space, and $f\colon X \to Y$ a proper meromorphic application of X to an analytic* C*-space Y. Can one find a projective morphism $h\colon X' \to X$, which is a modification and such that the meromorphic modification $f \circ h$ is a morphism?*

In § 6, we defined the Zariski space $\mathfrak{Z}(X)$ of X with the canonical morphism $h(X)\colon \mathfrak{Z}(X) \to X$, as the projective limit of the reduced analytic C-spaces X' with a proper modification $f\colon X' \to X$. By taking only those pairs (X', f) subject to the condition that f are obtained by monoidal transformations of X with centers nowhere dense in X, we obtain another projective limit $\mathfrak{Z}_0(X)$ with $h_0(X)\colon \mathfrak{Z}_0(X) \to X$. It is clear that we obtain the same limit $(\mathfrak{Z}_0(X), h_0(X))$ by taking those (X', f) subject to a weaker condition that f are projective morphisms. Obviously, we have a canonical morphism $t(X)\colon \mathfrak{Z}(X) \to \mathfrak{Z}_0(X)$ such that $h(X) = h_0(X) \circ t(X)$. The *question* (M) *is equivalent to asking if $t(X)$ is an isomorphism of local-ringed spaces*. In this manner, the *question* (M) *is answered affirmatively for an arbitrary X with* $\dim(X) \leqq 3$, *for X which admits a projective*

---

27 This problem will be discussed in more detail and to larger extent in a forthcoming paper [10].

morphism of itself to a complex Stein space, and for X which is compact irreducible and has independent n meromorphic functions where $n = \dim(X)$.[28] For instance, using the affirmative answer to the *question* (M) for $\dim(X) \leqq 3$ and *main theorem* II'(3) without the assumption on the existence of projective morphism $X \to S$ with a complex Stein space S, we can easily deduce the following fact: *Let X be a compact complex manifold* (i.e., *a compact non-singular analytic* C-*space*) *of dimension* 3. *If X admits a proper modification of itself to a complex Kähler manifold* (*resp. X has independent n meromorphic functions where $n = \dim(X)$*), *then there exists a finite succession of monoidal transformations f_i: $X_{i+1} \to X_i$ with non-singular centers D_i, which are nowhere dense on X_i, where $0 \leqq i < r$ and $X_0 = X$, such that X_r is a complex Kähler manifold* (*resp. a complex projective manifold*). A complete proof of this statement will be given on another occasion. We note here that, given any compact complex-analytic manifold X of dimension $\geqq 3$, there exists a morphism f: $X' \to X$ such that X' is non-singular and non-kählerian, that f is a proper modification and that for some point x' of X' with $x = f(x')$, there exists no local coordinate system of functions on X' at x' which are local meromorphic functions on X at x. (This last property of f: $X' \to X$ implies that f is not projective.) Such an example of X' with f is obtained as follows: Let g: $X_i \to X$ be the monoidal transformation with a point x as its center. Then $g^{-1}(x)$ with the induced structure of a complex-analytic space is isomorphic to a complex projective space P of dimension $= \dim(X) - 1$. Let us take an irreducible curve D with one and only one double point y in P, hence in X_1, such that D has two analytic branches D_1 and D_2 with distinct tangents in the neighborhood U of y in X_1. We view U as a complex-analytic manifold with the induced structure. Let h_1: $U' \to U$ be the morphism obtained by a succession of two monoidal transformations, the first one having the center D_1 in U, and the second one having the center which is the strict transform of D_2. Let V be the complement of y in X_1, and h_2: $V' \to V$ the monoidal transformation of V with center $V \cap D$. It then can be easily seen that there exists a complex-analytic manifold X' with a morphism h: $X' \to X_1$ such that there exist canonical X_1-isomorphisms of $h^{-1}(V) \to V'$ and $h^{-1}(U) \to U'$. Let $f = g \circ h$. Then we see that f: $X' \to X$ has the properties stated above.

CHAPTER I. AN INDUCTIVE FORMULATION OF RESOLUTION PROBLEMS

In Ch. 0, we saw two *main theorems* in the birational geometry, which include the resolution of singularities, the elimination of the points of

[28] For the proofs of these statements, see [10].

indeterminacy, the simplification of an algebraic boundary, etc., that are closely related to each other and that we propose to prove simultaneously in this work. The proof of the *main theorems* will be established by reducing it to the proof of certain inductive implications among suitably reformulated resolution theorems. The purpose of this chapter is to present in precise terminology the reformulated resolution theorems, and to indicate the inductive implications among them according to which the proofs will be carried out in a later chapter.

As the very basic assumption that underlies the geometric arguments throughout this work, we fix once for always *the particular type of schemes* with which we shall be concerned. Let $\mathscr{B}$ denote the class of those local rings S which have the following properties:

(i) S *is a noetherian local ring with the residue field of characteristic zero; and*

(ii) *if* $\hat{\mathrm{S}}$ *denotes the completion of the local ring* S, *then for every* S-*algebra of finite type* A, *the singular locus of* $\mathrm{Spec}(\mathrm{A} \otimes_{\mathrm{S}} \hat{\mathrm{S}})$ *is the preimage of that of* Spec(A) *under the canonical morphism of the first spectrum to the second.*[29]

Under the assumptions (i) and (ii), one can prove the following:

(iii) *For every* S-*algebra of finite type* A, *the singular locus of* Spec(A) *is closed* (*with reference to Zariski topology*).[30]

Moreover, one can prove that the class $\mathscr{B}$ has the following properties:

(a) *Every field of characteristic zero belongs to* $\mathscr{B}$;

(b) *the completion of any* $\mathrm{S} \in \mathscr{B}$ *belongs to* $\mathscr{B}$; *and*

(c) *if* A *is an algebra of finite type over* $\mathrm{S} \in \mathscr{B}$, *then every local ring of* Spec(A) *belongs to* $\mathscr{B}$.[31]

The schemes that we consider in this paper are only those which admit

---

[29] Throughout this paper, the term 'simple' (accordingly, 'multiple' or 'singular') is used only in the absolute sense of Grothendieck. Hence, the singular locus of a scheme should be understood as the set of those points at which the local rings of the scheme are not regular. (cf. paragraphs below.)

[30] A proof of this fact can be obtained by reducing it (by virtue of (ii)) to the case in which S is complete, hence, it is a homomorphic image of a formal power series ring over a field of characteristic zero. Namely, we have only to prove (iii) for the case in which A is a homomorphic image of a polynomial ring over such a formal power series ring. For this case, will suffice the jacobian criterion for simplicity. (cf. Nagata, [14], or Nagata, § 46, Ch. VII of [15].) We can also make use of Theorem 1 (with its proof in § 1) of: *On the closedness of singular loci*, by M. Nagata, Publ. Math. de l'Inst. des Hautes Etudes Scientifiques, No. 2, 1959. The author likes to point out that the present formulation of the class $\mathscr{B}$ is due to Grothendieck. (cf. Ch. IV in the series of [6].)

[31] Reduce the proof to the case in which S is complete, and then apply the jacobian criterion.

at least one morphism of finite type to Spec(S) with $S \in \mathscr{B}$. For the sake of convenience in regard to the purpose of this paper, a scheme of this type will be called an *algebraic scheme*. Thus, if V is an algebraic scheme, there exists a morphism of finite type $p: V \to \mathrm{Spec}(S)$ for some $S \in \mathscr{B}$. If p and S are such, we shall call S a *base ring* of the algebraic scheme V, and p a *structural morphism of* V *over* S. However, in most cases we shall not be concerned with particular p and S for a given algebraic scheme V, and all we need will be the special nature of V which comes from their existence. A *morphism of algebraic schemes* can be any one in the category of schemes, which may not be of finite type in general. We simply remark that all the morphisms of algebraic schemes actually considered are those which are compatible with certain morphisms of spectra of certain base rings.

The class of local rings $\mathscr{B}$, defined above, includes not only the local rings in the algebraic geometry (and the formal algebraic geometry) over fields of characteristic zero, but also those in the complex-(or real-)analytic geometry; for instance, local rings of complex-(or real-)analytic spaces, the localizations of the ring of holomorphic functions on a complex Stein space with respect to the maximal ideals which correspond to points of the space, etc.. Thus our results on *algebraic schemes* have applications not only to algebraic geometry but also to complex-(or real-)analytic geometry, at least to the local study of analytic modifications, analytic singularities, etc. (§ 7, Ch. 0.)

All we need in the later discussions are the properties (i)—(iii) and (a)—(c) of the class $\mathscr{B}$, so that we may replace $\mathscr{B}$ by any other (i.e., smaller) class of local rings having these properties, without affecting any reasoning in the rest of the paper. For example, *we may choose* $\mathscr{B}$ *to be the smallest class of local rings which has the properties* (a), (b), *and* (c), i.e., the class of those local rings which are obtained, by localization with respect to prime ideals, from algebras of finite type over formal power series rings with coefficient fields of characteristic zero. It is immediately seen that the local rings of this class have the properties (i), (ii), and (iii).[32] In this manner, one can avoid the proof of (a), (b), and (c) for the class $\mathscr{B}$ as long as he is strictly interested in the algebro-geometrical consequences of our results. It should be noted in any case, however, that our inductive proof can not be carried out if we restrict our considerations only to algebraic schemes over *fields* of characteristic zero. In other words, according to our proof, the resolution of *algebraic* singularities requires at least the same of *algebroid* singularities to be done simultaneously.

[32] cf. Footnote[30] above.

To gain familiarity in the language of schemes, one should refer to Grothendieck [6]. In this paper, however, we do not need much more than the definition of schemes. We recall here a few definitions in order to avoid any possible confusion. We shall use the notion of simplicity only in the absolute sense. Namely, we say that a point of an algebraic scheme is *simple* (resp. *multiple*) if the local ring of the scheme at the point is regular (resp. not regular). We sometimes say that an algebraic scheme is *singular* at a point, if the point is a multiple point of the scheme. The *singular locus* of an algebraic scheme V is by definition the set of multiple points of V, which is a closed subset of V and denoted by $s(V)$. We shall often denote by O_V the structural sheaf of local rings of V, and by $O_{V,x}$ the stalk of O_V at a point x of V; i.e., the local ring of V at x. The Krull-dimension of $O_{V,x}$ will be called *the dimension of V at the point x*, and denoted by $\dim_x(V)$. The maximum of $\dim_x(V)$ for all the points x of an algebraic scheme V is a non-negative integer, provided V is not empty. This integer will be called *the dimension of V*, denoted by $\dim(V)$. We take the integer -1 as *the dimension of the empty scheme* (which is an algebraic scheme). Let W be a subscheme of V. (Throughout this paper, by a *subscheme* we mean a closed subscheme.) We remark that W is necessarily algebraic because V is so. If W is not empty, then we define *the codimension of W in V* to be the minimum of $\dim_x(V)$ for all the points x of W. If x is a point of W, then the *codimension* of *W in V at x* is defined to be the codimension of $\operatorname{Spec}(O_{W,x})$ in $\operatorname{Spec}(O_{V,x})$ where the first spectrum is viewed in a canonical way as a subscheme of the second. We can see that the codimension of W in V is equal to the minimum of the codimensions of W in V at the points of W, provided W is not empty. We shall say that W is *everywhere of codimension s in V*, if W has codimension s in V at every point of W. We shall find it convenient in our inductive proof of resolution theorems, to agree that *any integer can be the codimension of the empty subscheme of an algebraic scheme*. According to this agreement, *we shall often view the empty subscheme as a non-singular irreducible subscheme which is everywhere of codimension one.*

1. Resolution data and permissible transformations

In this section, X denotes once for always a non-singular irreducible algebraic scheme. Before we can present precise statements of the resolution theorems so formulated as to be proved inductively, we need several preparatory definitions concerning *the objects to be resolved* and *the processes to be employed for resolution*. Namely, we shall first define, and denote by $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$, $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$, $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$, and $\mathfrak{R}_{\mathrm{II}}^{N}$, the four different types of *resolution data*, i.e., objects on X which we want to resolve. We shall then make clear what we mean by saying that a given resolution datum on X is *resolved*. In order to *resolve* a given datum on X, i.e., to transform it into a resolved datum, we employ a finite succession of certain elementary transformations of X and the datum on X. An elementary transformation of X to be employed there will be defined and called a *permissible monoidal transformation* for the given datum. In other words, it is a monoidal transformation of X which is *permissible* for the given datum, and the permissibility will be defined depending on the type of datum. We shall also define the *transform* of a given datum on X by a permissible monoidal transformation of X for the datum, where the transform is a resolution datum on the algebraic scheme (necessarily non-singular and irreducible) obtained from X by the transformation. We shall thus be able to speak of successively permissible monoidal transformations, i.e., a *permissible succession of monoidal transformations*, of X for a given datum on X. The resolution theorems will thus make sense, and will be stated in the following section.

Definitions 1 and 2 of Chapter 0 should be recalled; the notions of *normal flatness* and *normal crossings*, defined there, play important roles in the following consideration.

DEFINITION 3. *Let X be a non-singular irreducible algebraic scheme as above. We always assume that* $\dim(X) \geqq 0$.

(I) *A resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ on X is an object of the following form:*

$$\mathfrak{R}_{\mathrm{I}}^{N,n} = (E;\ V_1,\ V_2,\ \cdots,\ V_\beta;\ W)$$

where

(i) *E is a reduced subscheme of X which is everywhere of codimension one and has only normal crossings;*

(ii) *β is a non-negative integer, and the V_j $(1 \leqq j \leqq \beta)$ are subschemes of X;*

(iii) *W is a reduced subscheme of X which is contained in $\bigcap_{j=1}^{\beta} V_j$; and*

(iv) $N = \dim(X)$ *and* $n = \dim(W)$.

(II) *A resolution datum of type $\mathfrak{R}_{\mathrm{II}}^{N}$ on X is an object of the following form:*

$$\mathfrak{R}_{\mathrm{II}}^{N} = \left(\begin{matrix} E_1, & \cdots, & E_\alpha \\ a_1, & \cdots, & a_\alpha \end{matrix}\ \middle|\ J, b\right)$$

where

(i) *α and a_i $(1 \leqq i \leqq \alpha)$ are non-negative integers, and b is a positive integer;*

(ii) *E_i $(1 \leqq i \leqq \alpha)$ are non-singular irreducible subschemes of X of codimension one, such that $\bigcup_{i=1}^{\alpha} E_i$ has only normal crossings on X;*

(iii) *J is a coherent sheaf of non-zero ideals on X; and*

(iv) *$N = \dim(X)$.*

We remark that the *intersection* (resp. the *union*) of subschemes is by definition the subscheme defined by the union (resp. the intersection) of their sheaves of ideals. We say also that one subscheme is *contained in* another, if the sheaf of ideals of the first subscheme contains that of the second. Note that, if $\beta = 0$ in the above definition, then the condition (iii) of (I) is trivially satisfied by any reduced subscheme W of X. We also remark that, according to our agreement on codimensions of the empty subscheme, the empty subscheme of X may appear and may even be repeated among E and E_i $(1 \leqq i \leqq \alpha)$ of the above definition.

Let $\mathbf{S}$ be a commutative ring with unity, $\mathbf{M}$ and $\mathbf{J}$ ideals in $\mathbf{S}$. Then we denote by $\nu_{\mathbf{M}}(\mathbf{J})$ the largest integer ν, if it exists, such that $\mathbf{M}^{\nu} \supseteqq \mathbf{J}$. (If it does not exist, we let $\nu_{\mathbf{M}}(\mathbf{J})$ be infinity, ∞ in symbol.) In particular, if $\mathbf{S}$ is local and $\mathbf{M}$ is the maximal ideal of $\mathbf{S}$, then we write $\nu(\mathbf{J})$ for $\nu_{\mathbf{M}}(\mathbf{J})$.

DEFINITION 4. *Let the notation and the assumptions be as in Definition 3.*

(I) *We say that the resolution datum $\mathfrak{R}_{\mathrm{I}}^{N,n}$ is resolved at a point x of W, if the following conditions are satisfied:*

(i) *x is a simple point of W;*

(ii) *all the V_j $(1 \leqq j \leqq \beta)$ are normally flat along W at the point x; and*

(iii) *E has only normal crossings with W at x on X.*

(II) *We say that the resolution datum $\mathfrak{R}_{\mathrm{II}}^{N}$ is resolved at a point x of X, if the following inequality holds:*

$$\textstyle\sum_{x \in E_i} a_i + \nu(J_x) < b \ ,$$

where J_x denotes the stalk of J at x which is an ideal in the local ring $O_{X,x}$.

We introduce the following symbols:

$$S(\mathfrak{R}_{\mathrm{II}}^{N}) = \{x \in X \mid \textstyle\sum_{x \in E_i} a_i + \nu(J_x) \geqq b\}$$

$$\nu(\mathfrak{R}_{\mathrm{II}}^{N}) \begin{cases} = \max_{x \in S(\mathfrak{R}_{\mathrm{II}}^{N})}\{\nu(J_x)\} & \text{if } S(\mathfrak{R}_{\mathrm{II}}^{N}) \text{ is not empty ,} \\ = 0 & \text{if } S(\mathfrak{R}_{\mathrm{II}}^{N}) \text{ is empty .} \end{cases}$$

$$S_{\nu}(\mathfrak{R}_{\mathrm{II}}^{N}) = \{x \in S(\mathfrak{R}_{\mathrm{II}}^{N}) \mid \nu(J_x) \geqq \nu\}$$

$$S_{*}(\mathfrak{R}_{\mathrm{II}}^{N}) = S_{\nu}(\mathfrak{R}_{\mathrm{II}}^{N}) \qquad \text{with } \nu = \nu(\mathfrak{R}_{\mathrm{II}}^{N}) \ .$$

DEFINITION 5. (I. 1) *A resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ with closed restric-*

tion F on X is a pair

$$(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$$

of a resolution datum $\mathfrak{R}_{\mathrm{I}}^{N,n}$ *on* X, *as was defined in Definition* 3, *and a non-singular irreducible subscheme* F *of* X *of codimension one, such that*

(i) $E \cup F$ *has only normal crossings on* X, *and*

(ii) $W - W \cap F$ *is dense in* W.

(I.2) *A resolution datum of type* $\mathfrak{R}_{\mathrm{I}}^{N,n}$ *with open restriction* U *on* X *is a pair*

$$(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$$

of a resolution datum $\mathfrak{R}_{\mathrm{I}}^{N,n}$ *on* X, *as was defined in Definition* 3, *and a dense open subset* U *of* W, *such that the datum* $\mathfrak{R}_{\mathrm{I}}^{N,n}$ *is resolved at every point of* U.

(II.1) *A resolution datum of type* $\mathfrak{R}_{\mathrm{II}}^{N}$ *with closed restriction* F *on* X *is a pair*

$$(\mathfrak{R}_{\mathrm{II}}^{N}, F)$$

of a resolution datum $\mathfrak{R}_{\mathrm{II}}^{N}$ *on* X, *as was defined in Definition* 3, *and a non-singular irreducible subscheme* F *of* X *of codimension one, such that* $E_1 \cup E_2 \cup \cdots \cup E_\alpha \cup F$ *has only normal crossings on* X.

The above three types of resolution data with restrictions and the resolution data of type $\mathfrak{R}_{\mathrm{II}}^{N}$ *without restriction* will be the four basic objects in terms of which the resolution theorems will be formulated and proved inductively.

We recall the definition of monoidal transformation. Let V be an algebraic scheme, and B a subscheme of V. Then *the monoidal transformation* $f: V' \to V$ (as a morphism of algebraic schemes) *of* V *with center* B can be characterized (up to an isomorphism of V') by the following universal mapping property:

(i) *If* I *denotes the sheaf of ideals of* B *on* V, *then* $f^{-1}(I)$ (*which denotes the sheaf of ideals on* V' *generated by* I) *is invertible*; *and*

(ii) *if* $f': V'' \to V$ *is any morphism of algebraic schemes, which satisfies* (i), *then there exists a unique morphism* $h: V'' \to V'$ *such that* $f' = f \circ h$.

The existence can be proved by taking as V' the projective scheme $\mathrm{Proj}(S(I))$ over V where $S(I)$ denotes the quasi-coherent sheaf of graded O_V-algebras $\sum_{m=0}^{\infty} I^m$ (the direct sum of the m^{th} powers of I in O_V for all non-negative integers m).[33] We note that the above universal mapping property remains equivalent if we take arbitrary schemes (or preschemes) in the places of V' and V''.

[33] cf. § 3, Ch. II, of Grothendieck, [6]. Also, refer to § 2, Ch. 0, of this paper.

Let us consider the case in which V is given as a subscheme of an algebraic scheme X. Let $g\colon X' \to X$ be the monoidal transformation of X with center B. Then, immediately from the above universal mapping property, it follows that

(a) *V' can be in a unique way identified with a subscheme of X' such that the following diagram commutes:*

$$\begin{array}{ccc} g\colon X' & \to & X \\ \cup & & \cup \\ f\colon V' & \to & V \end{array}$$

(b) *V' (as a subscheme of X') is the strict transform of V by the monoidal transformation g with center B.*

In general, if $g\colon X' \to X$ is a monoidal transformation with center B as above, and if V is any subscheme of X, we define the *strict transform of V by g* to be the smallest subscheme V' of X', such that the restriction of V' to $X' - g^{-1}(B)$ is equal to the preimage of $V - B$ by the morphism g (which induces an isomorphism of $X' - f^{-1}(B)$ to $X - B$).[34] We define in an obvious manner *the strict transform of a subscheme by a finite succession of monoidal transformations.*

We remark that, $g\colon X' \to X$ and B being as above, if both X and B are non-singular, then X' is necessarily non-singular. Throughout this paper, as is so in the above definition of strict transform, *it is important to note that whenever we speak of a monoidal transformation, say $g\colon X' \to X$, we mean the pair of the morphism g and a specific subscheme of X which is taken as the center of the monoidal transformation,* even if we do not name the center explicitly. Thus the set of monoidal transformations of an algebraic scheme X is in a one-to-one correspondence with the set of subschemes of X, although two distinct subschemes of X may give rise to the same transform of X, and the same morphism from it to X.

DEFINITION 6. *Let X be as in Definition* 3. *Let B be a non-singular irreducible subscheme of X, and $f\colon X' \to X$ the monoidal transformation of X with center B. Let $\mathfrak{R}_{\mathrm{I}}^{N,n}$ and $\mathfrak{R}_{\mathrm{II}}^{N}$ be resolution data as in Definition* 3. *Let* $m = \dim(B)$. *We then say that f (the monoidal transformation) is permissible for $\mathfrak{R}_{\mathrm{I}}^{N,n}$ if the object of the following form:*

$$(E;\ V_1,\ V_2,\ \cdots,\ V_\beta,\ W;\ B)$$

is a resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,m}$ on X and resolved everywhere, i.e., *resolved at every point of B. We say that f is permissible for $\mathfrak{R}_{\mathrm{II}}^{N}$ if B is contained in $S_*(\mathfrak{R}_{\mathrm{II}}^{N})$ and, at the same time, $E_1 \cup \cdots \cup E_\alpha$ has only*

[34] cf. the last paragraph of § 2, Ch. 0, and also § 2, Ch. III.

normal crossings with B on X. Let $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$, $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$, and $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$ be as in Definition 5. We say that f is permissible for $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$ $\big($resp. $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)\big)$ if it is so for $\mathfrak{R}_{\mathrm{I}}^{N,n}$ and, at the same time, B is contained in $W \cap F$ (resp. $W - U$). We say that f is permissible for $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$ if it is so for $\mathfrak{R}_{\mathrm{II}}^{N}$ and, at the same time, B is contained in F.

In the above definition, B may be the empty subscheme of X, in which case the monoidal transformation is permissible for any resolution datum on X.

DEFINITION 7. *Let $\mathfrak{R}$ be any one of those resolution data, $\mathfrak{R}_{\mathrm{I}}^{N,n}$, $\mathfrak{R}_{\mathrm{II}}^{N}$, $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$, $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$, and $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$, which we defined in Definitions 3 and 5. Let B be a non-singular irreducible subscheme of X, such that the monoidal transformation $f\colon X' \to X$ of X with center B is permissible for the given datum $\mathfrak{R}$. Then we define the transform, $f^*(\mathfrak{R})$ in symbol, of $\mathfrak{R}$ by the monoidal transformation f as follows: The notation being as in Definitions 3 and 5, let*

E' = *the strict transform of E by f,*
E_i' = *the strict transform of E_i by f $(1 \leqq i \leqq \alpha)$,*
V_j' = *the strict transform of V_j by f $(1 \leqq j \leqq \beta)$,*
W' = *the strict transform of W by f,*
F' = *the strict transform of F by f,*
$E_{\alpha+1}' = f^{-1}(B)$,[35] *and* $U' = f^{-1}(U)$.
Then

$$f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}) = (E' \cup E_{\alpha+1}';\ V_1', V_2', \cdots, V_\beta';\ W')$$
$$f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}, F) = \big(f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}), F'\big)$$
$$f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}, U) = \big(f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}), U'\big)$$
$$f^*(\mathfrak{R}_{\mathrm{II}}^{N}) = \left(\begin{matrix} E_1', \cdots, E_\alpha', E_{\alpha+1}' \\ a_1, \cdots, a_\alpha, a_{\alpha+1} \end{matrix}\middle| f^{-1}(J)P^{-\nu}, b\right)$$

and

$$f^*(\mathfrak{R}_{\mathrm{II}}^{N}, F) = \big(f^*(\mathfrak{R}_{\mathrm{II}}^{N}), F'\big)\,,$$

where $\nu = \nu(\mathfrak{R}_{\mathrm{II}}^{N})$, $a_{\alpha+1} = \big(\sum_{B \subseteqq E_i} a_i\big) + \nu - b$, P = *the sheaf of ideals of* $E_{\alpha+1}'$ *on* X', *and* $f^{-1}(J)$ = *the sheaf of ideals generated by J on X'.*

REMARK 1. Let $\mathfrak{R}$ be a resolution datum on X as in Definition 7, and $f\colon X' \to X$ a monoidal transformation of X with center B which is permis-

[35] Let $f\colon X' \to X$ be a morphism of schemes (or a monoidal transformation, or a finite succession of monoidal transformations). Let B be any subscheme of X. Then we define the *total transform* of B in X' (or, by f) and denote it by $f^{-1}(B)$, to be the subscheme of X' defined by the sheaf of ideals on X' generated by that of B on X.

sible for $\mathfrak{R}$. Then we have $\dim(X') = \dim(X) = N$, unless $X = B$. If $X = B$, then X' is the empty algebraic scheme. This can happen only if $\mathfrak{R} = \mathfrak{R}_{\mathrm{I}}^{N,n}$ (as in Definition 3) and $X = V_i = W$ for all i, in which case the datum $\mathfrak{R}_{\mathrm{I}}^{N,n}$ is clearly resolved everywhere. *We exclude once for always this uninteresting case in which $X = W$.* Let W' be as in Definition 7. Then we have $\dim(W') = \dim(W) = n$, unless B contains an irreducible component of W. Note that B can not contain any irreducible component of W if $\mathfrak{R}$ has restriction either closed or open. We then assert that *the transform $f^*(\mathfrak{R})$ of the resolution datum $\mathfrak{R}$ by the monoidal transformation f is again a resolution datum on X' of the same type with the same kind of restriction*, except for the case in which $\mathfrak{R} = \mathfrak{R}_{\mathrm{I}}^{N,n}$, and B contains some of the irreducible components of W. In this last case, $f^*(\mathfrak{R})$ is in general a resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n'}$ with an integer $n' \leqq n$.

To prove the assertion on $f^*(\mathfrak{R})$, all that may be non-trivial is the normal crossing property of $E' \cup E'_{\alpha+1}$ (or $E' \cup F' \cup E'_{\alpha+1}$ or $E'_1 \cup \cdots \cup E'_\alpha \cup E'_{\alpha+1}$ or $E'_1 \cup \cdots \cup E'_{\alpha+1} \cup F'$). This can be seen as follows: Let x' be any point of X', and $x = f(x')$. Then we can find a regular system of parameters $(z_1, \cdots, z_r, z_{r+1}, \cdots, z_s)$ of the local ring O_x of X at x, such that every irreducible component F_i of E containing x is defined by $z_j = 0$ at x for some j, and that B is defined by $z_{r+1} = \cdots = z_s = 0$ at x. Then the local ring $O_{x'}$ of X' at x' is a ring of fractions of $\mathbf{A} = O_x[z_{r+1}z_t^{-1}, \cdots, z_s z_t^{-1}]$ for some t $(r + 1 \leqq t \leqq s)$ with respect to a prime ideal $\mathbf{M}$ in $\mathbf{A}$. Let $z'_v = z_v$ for $1 \leqq v \leqq r$, $z'_t = z_t$ and $z'_v = z_v z_t^{-1}$ for $r + 1 \leqq v\ (\neq t) \leqq s$. It is then easily seen that, (F_i, z_j) being as above, the strict transform F'_i of F_i contains the point x' if and only if $j \neq t$ and $z'_j \in \mathbf{M}$, in which case F'_i is defined by $z'_j = 0$ at x', that $E'_{\alpha+1}$ is defined by $z'_t = 0$ at x', provided $x' \in E'_{\alpha+1}$, and that a system of those z'_v which are in $\mathbf{M}$ can be extended to a regular system of parameters of $O_{x'} = \mathbf{A}_{\mathbf{M}}$. From these follows the normal crossing property of $E' \cup E'_{\alpha+1}$ (and similarly for the others).

DEFINITION 8. *Let $\mathfrak{R}$ be a resolution datum on X as in Definition 7. Then a succession of monoidal transformations*

$$f = \{f_i \colon X_{i+1} \longrightarrow X_i\} \qquad (0 \leqq i < m, \text{ and } X_0 = X)$$

with centers B_i in X_i will be said to be permissible for $\mathfrak{R}$, if there exists a resolution datum $\mathfrak{R}_i$ on X_i for $0 \leqq i \leqq m$, such that

(i) $\mathfrak{R} = \mathfrak{R}_0$;

(ii) *f_i is a permissible monoidal transformation for $\mathfrak{R}_i$; and*

(iii) $\mathfrak{R}_{i+1} = f_i^*(\mathfrak{R}_i)$ *for* $0 \leqq i < m$.

Moreover, if f and $\mathfrak{R}_m$ are such, we shall call $\mathfrak{R}_m$ the transform of $\mathfrak{R}$ by the permissible succession f and denote it by $f^*(\mathfrak{R})$.

We shall often use the same symbol both for a succession f as above and for the composed morphism $X_m \to X$, if the sense is made clear by the context. For instance, whenever we speak of a succession (permissible or not) of monoidal transformations, say $f\colon \tilde{X} \to X$, we mean a *specific factorization* of the morphism f into monoidal transformations with *specific centers*. The above transform $f^*(\mathfrak{R})$ depends not only upon the morphism f but also upon the specific factorization of f with specific centers.

2. The fundamental theorems $\mathrm{I}_1^{N,n}$, $\mathrm{I}_2^{N,n}$, II_1^N and II_2^N

In this section, we denote by X any non-singular irreducible algebraic scheme of dimension $N \geqq 0$. In this section, and often in the remainder of this paper, we use the symbol $\mathfrak{R}_{\mathrm{I}}^{N,n}$ (resp. $\mathfrak{R}_{\mathrm{II}}^{N}$) to denote a resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ (resp. of type $\mathfrak{R}_{\mathrm{II}}^{N}$). By a resolution datum $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$, resp. $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$, resp. $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$ etc., on X, we shall mean a resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ with closed restriction F, resp, a resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ with open restriction U, resp. a resolution datum of type $\mathfrak{R}_{\mathrm{II}}^{N}$ with closed restriction F, etc. The following four theorems will be referred to as the *fundamental theorems* of this paper, each of which depends upon either a pair of integers (N, n) or a single integer N such that $N > n \geqq -1$. The remainder of this paper is devoted primarily to the proofs of these four *fundamental theorems* for all (N, n) and N. If $(E; V_1, V_2, \cdots, V_\beta; W)$ is any resolution datum of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ on a non-singular irreducible algebraic scheme, then we shall call W *the principal subscheme of the resolution datum.*

THEOREM $\mathrm{I}_1^{N,n}$. *Given any resolution datum $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$ on X, there exists a finite succession of monoidal transformations, say $f\colon X' \to X$, which is permissible for $(\mathfrak{R}_{\mathrm{I}}^{N,n}, F)$, such that if $f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}, F) = \big(f^*(\mathfrak{R}_{\mathrm{I}}^{N,n}), F'\big)$ then F' has no common points with the principal subscheme of $f^*(\mathfrak{R}_{\mathrm{I}}^{N,n})$.*

THEOREM $\mathrm{I}_2^{N,n}$. *Given any resolution datum $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$ on X, there exists a finite succession of monoidal transformations, say $f\colon X' \to X$, which is permissible for $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$, such that $f^*(\mathfrak{R}_{\mathrm{I}}^{N,n})$ is resolved everywhere* (i.e., at every point of the principal subscheme of $f^*(\mathfrak{R}_{\mathrm{I}}^{N,n})$).

THEOREM II_1^N. *Given any resolution datum $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$ on X such that $\nu(\mathfrak{R}_{\mathrm{II}}^{N}) > 0$, there exists a finite succession of monoidal transformations, say $f\colon X' \to X$, which is permissible for $(\mathfrak{R}_{\mathrm{II}}^{N}, F)$, such that if $f^*(\mathfrak{R}_{\mathrm{II}}^{N}, F) = \big(f^*(\mathfrak{R}_{\mathrm{II}}^{N}), F'\big)$, then F' has no points contained in $S_\nu\big(f^*(\mathfrak{R}_{\mathrm{II}}^{N})\big)$ where $\nu = \nu(\mathfrak{R}_{\mathrm{II}}^{N})$.*

THEOREM II_2^N. *Given any resolution datum $\mathfrak{R}_{\mathrm{II}}^{N}$ without restriction on*

X, there exists a finite succession of monoidal transformations, say $f\colon X' \to X$, which is permissible for $\mathfrak{R}^N_{\mathrm{II}}$, such that $f^*(\mathfrak{R}^N_{\mathrm{II}})$ is resolved everywhere.

In each of the above theorems, we do not exclude the empty succession of monoidal transformations. If the succession is the empty one, then the morphism f and the transformation f^* of the data are the identities. In other words, we can take the empty succession of monoidal transformations in each of the theorems if the given resolution datum has the properties required on its transform by the succession.

Theorems $\mathrm{I}_1^{N,n}$ and $\mathrm{I}_2^{N,n}$ are trivially true if $n \leqq 0$; and so are *theorems* II_1^N and II_2^N if $N = 0$. As for *theorem* II_1^N with $N = 1$, the only non-trivial case is the one in which F is not empty, hence it is a point, and it is contained in $S_*(\mathfrak{R}^N_{\mathrm{II}}) = S_\nu(\mathfrak{R}^N_{\mathrm{II}})$. In this case, the monoidal transformation $f\colon X' \to X$ of X with center F will do for the theorem. As for *theorem* II_2^N with $N = 1$, if $\nu = \nu(\mathfrak{R}^N_{\mathrm{II}}) > 0$, then we take any point x of $S_\nu(\mathfrak{R}^N_{\mathrm{II}})$ (which is a finite number of points) and apply the monoidal transformation with center x to X and to $\mathfrak{R}^N_{\mathrm{II}}$. By repeating this process (in fact, exactly s times, where s is the number of points in $S_1(\mathfrak{R}^N_{\mathrm{II}})$), we necessarily come to the situation that $\nu(\bar{\mathfrak{R}}^N_{\mathrm{II}}) = 0$ where $\bar{\mathfrak{R}}^N_{\mathrm{II}}$ denotes the transform of $\mathfrak{R}^N_{\mathrm{II}}$. Thus we may assume that $\nu(\mathfrak{R}^N_{\mathrm{II}}) = 0$ for the case of $N = 1$. After assuming this, we take any point x of $S(\mathfrak{R}^N_{\mathrm{II}})$ and apply the monoidal transformation of X with center x to the datum. By repeating this (necessarily a finite number of times), we come to the situation where *theorem* II_2^N with $N = 1$ is established. Thus the four theorems for $N \leqq 1$ are more or less trivial and verified directly. However, according to the implication diagram of induction stated below and to be proved in the last chapter of this paper, *all that must be maintained at this moment are theorems* $\mathrm{I}_1^{N,-1}$, $\mathrm{I}_2^{N,-1}$, II_1^0, *and* II_2^0 *for all non-negative integers N, which are all absolutely trivial.*

It can easily be seen that the *theorems* $\mathrm{I}_1^{N,n}$, $\mathrm{I}_2^{N,n}$, II_1^N, and II_2^N will be established for all possible integers N and n ($N > n \geqq -1$ and $N \geqq 0$) if we can prove the following four inductive implications:

The implication (A). *Given* (N, n) *with* $0 \leqq n < N$,

$$\left.\begin{array}{lll} \text{Th. } \mathrm{I}_1^{N',n'} & \textit{with} & n' < N' < N \\ \text{Th. } \mathrm{I}_2^{N,n''} & \textit{with} & n'' < n \\ & & \\ \text{Th. } \mathrm{II}_2^{N'} & \textit{with} & N' < N \end{array}\right\} \Longrightarrow \text{Th. } \mathrm{I}_1^{N,n}\ .$$

and

The implication (B). *Given* (N, n) *with* $0 \leqq n < N$,

$$\left.\begin{array}{lll} \text{Th. I}_2^{N',n'} & \textit{with} & n' < N' < N \\ \text{Th. I}_2^{N,n''} & \textit{with} & n'' < n \\ \text{Th. I}_1^{N^*,n^*} & \textit{with} & \begin{cases} n^* < N^* \leqq N \\ n^* \leqq n \end{cases} \\ \textit{and} & & \\ \text{Th. II}_2^{N'} & \textit{with} & N' < N \end{array}\right\} \Longrightarrow \text{Th. I}_2^{N,n}\,.$$

The implication (C). *Given* $N \geqq 1$,

$$\left.\begin{array}{lll} \text{Th. II}_1^{N'} & \textit{with} & N' < N \\ \text{Th. I}_2^{N^*,n^*} & \textit{with} & n^* < N^* \leqq N \\ \textit{and} & & \\ \text{Th. II}_2^{N'} & \textit{with} & N' < N \end{array}\right\} \Longrightarrow \text{Th. II}_1^{N}\,.$$

The implication (D). *Given* $N \geqq 1$,

$$\left.\begin{array}{lll} \text{Th .II}_2^{N'} & \textit{with} & N' < N \\ \text{Th. I}_2^{N^*,n^*} & \textit{with} & n^* < N^* \leqq N \\ \textit{and} & & \\ \text{Th. II}_1^{N} & & \end{array}\right\} \Longrightarrow \text{Th. II}_2^{N}\,.$$

It should be checked, and this is easily done, that these implications (A), (B), (C), and (D) are sufficient for the *fundamental theorems*. These four implications will be proved in Chapter IV.

3. Main theorems I(n) and II(N)[36]

Let X be a non-singular irreducible algebraic scheme of dimension N. Let V_i $(1 \leqq i \leqq \beta)$ be a finite number of subschemes of X, $\beta \geqq 0$, and W a reduced subscheme of X of dimension n such that $W \subsetneqq \bigcap_{i=1}^{\beta} V_i$. Then, by taking the empty subscheme of X as E, which is everywhere of codimension one on X and has only normal crossings, we obtain a resolution datum $(E; V_1, \cdots, V_\beta; W)$ of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ on X. Let us denote this datum by $\mathfrak{R}_{\mathrm{I}}^{N,n}$. Let U be the subset of W consisting of those points at which $\mathfrak{R}_{\mathrm{I}}^{N,n}$ is resolved. Then we can prove that *U in an open dense subset of W.* This is done by showing first that the set of simple points of a reduced algebraic scheme is open and dense (cf. the property (iii) of the class $\mathscr{B}$ in the leading paragraph of this chapter); and secondly that, if V is an algebraic scheme and W a reduced subscheme of V, the set of those points

[36] In this section and the next, we shall make use of some results proven in later chapters. No reference to any of these sections will be made in the later arguments, and the reader may skip them and come back afterwards.

of W at which V is normally flat along W is an open dense subset of W (cf. Th. 1, § 1, Ch. II). Thus we get a resolution datum $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$ of type $\mathfrak{R}_{\mathrm{I}}^{N,n}$ with open restriction U on X. By the *fundamental theorem* $\mathrm{I}_2^{N,n}$, we have a finite succession of monoidal transformations f: $X' \to X$, which is permissible for $(\mathfrak{R}_{\mathrm{I}}^{N,n}, U)$, such that $f^*(\mathfrak{R}_{\mathrm{I}}^{N,n})$ is resolved everywhere. Let us write the succession f as $\{f_i: X(i+1) \to X(i)\}$, where $0 \leqq i < r$, $X(0) = X$, and $X(r) = X'$. Let B_i be the center of f_i for each i. Let $\mathfrak{R}_{\mathrm{I}}^{N,n}(i)$ be the resolution datum on $X(i)$ such that $\mathfrak{R}_{\mathrm{I}}^{N,n}(\mathrm{i}+1) = f_i^*(\mathfrak{R}_{\mathrm{I}}^{N,n}(i))$ for all i and for $\mathfrak{R}_{\mathrm{I}}^{N,n}(0) = \mathfrak{R}_{\mathrm{I}}^{N,n}$. Let $\mathfrak{R}_{\mathrm{I}}^{N,n}(i) = (E(i); V_1(i), \cdots, V_\beta(i); W(i))$ for all i. Then we know that f_i induces the monoidal transformations $V_j(i+1) \to V_j(i)$ and $W(i+1) \to W(i)$ with the same center B_i as f_i for $0 \leqq i < r$. (See the paragraphs preceding Definition 6, § 1.) According to the *fundamental theorem* $\mathrm{I}_2^{N,n}$, we have the following facts:

(1) *B_i is a non-singular irreducible subscheme of $W(i)$ which is nowhere dense;*

(2) *all the $V_j(i)$ $(1 \leqq j \leqq \beta)$ and $W(i)$ are normally flat along the subscheme B_i; and*

(3) *$W(r)$ is a non-singular subscheme of $V_j(r)$ for $1 \leqq j \leqq \beta$, and all the $V_j(r)$ are normally flat along $W(r)$.*

The *fundamental theorem* $\mathrm{I}_2^{N,n}$ asserts the normal crossing property of the E_r in addition to the above facts, but we draw back our attention from this additional fact. We can then strengthen the results (1), (2) and (3) without any difficulty by replacing (1) by the following stronger fact:

(1′) *Let $S(i)$ be the complement in $W(i)$ of the set of those points of $W(i)$ which are simple points of $W(i)$ and at which all the $V_j(i)$ $(1 \leqq j \leqq \beta)$ are normally flat along $W(i)$. Then, B_i is a non-singular irreducible subscheme of $W(i)$ which is contained in $S(i)$.* (Note that $S(i)$ is a closed subset of $W(i)$ which is nowhere dense (Th. I, § 1, Ch. II).) In fact, by Corollary 4 of Proposition 1, (§ 1, Ch. II), B_i of (1) is either contained in the singular locus of $W(i)$ or in its complement in $W(i)$. Moreover, by Corollary of Theorem 3, (§ 3, Ch. II), for each j $(1 \leqq j \leqq \beta)$, $V_j(i)$ is normally flat along $W(i)$ either at all the points of B_i or at no points of B_i. Thus, B_i is either contained in $S(i)$ or in the complement of $S(i)$ in $W(i)$. This result enables us to eliminate those monoidal transformation with B_i not contained in $S(i)$ from the above succession without affecting (2) and (3) of its properties. To be precise, we construct a new succession of monoidal transformations $\{\bar{f}_i: \bar{X}(i+1) \to \bar{X}(i)\}$ where $0 \leqq i < r$ and $\bar{X}(0) = X$, such that there exists a morphism g_i: $X(i) \to \bar{X}(i)$ for every i having the following properties:

(a) the center $\bar{B}_i$ of $\bar{f}_i$ is either the image of B_i or the empty subscheme of $\bar{X}(i)$;

(b) if $\bar{V}_j(i+1) \to \bar{V}_j(i)$ (resp. $\bar{W}(i+1) \to \bar{W}(i)$) is the succession of induced monoidal transformations for $\bar{V}_j(0) = V_j$ (resp. $\bar{W}(0) = W$), then $\bar{B}_i$, $\bar{W}(i)$ and $\bar{V}_j(i)$ for all j have the properties (1′), (2) and (3); and finally

(c) if $\bar{S}(i)$ denotes the complement in $\bar{W}(i)$ of the set of those points of $\bar{W}(i)$ which are simple points of $\bar{W}(i)$ and at which all the $\bar{V}_j(i)$ $(1 \leqq j \leqq \beta)$ are normally flat along $\bar{W}(i)$, then there exists an open neighborhood of $\bar{S}(i)$ in $\bar{X}(i)$ in which g_i^{-1} exists as a morphism.

The new succession may be constructed without any difficulty.

Now, we want to generalize the above result to the case in which there is given no ambient algebraic scheme X, non-singular and irreducible. Namely we assert the following theorem for all non-negative integers n:

MAIN THEOREM I(n). *Let V_0 be an arbitrary algebraic scheme, V_j $(1 \leqq j \leqq \beta)$ a finite number of subschemes of V_0, and W a reduced subscheme of V_0 of dimension n which is contained in $\bigcap_{j=1}^{\beta} V_j$. Then there exists a finite succession of monoidal transformations $\{f_i: V_0(i+1) \to V_0(i)\}$ with centers B_i in $V_0(i)$ for $0 \leqq i < r$, where $V_0(0) = V_0$, such that if $\{V_j(i+1) \to V_j(i)\}$ and $\{W(i+1) \to W(i)\}$ are the monoidal transformations induced by f_i, where $V_j(0) = V_j$ and $W(0) = W$, then*

(1) B_i is a non-singular irreducible subscheme of $W(i)$;

(2) B_i does not contain any simple point of $W(i)$ at which all the $V_j(i)$ for $0 \leqq j \leqq \beta$ are normally flat along $W(i)$;

(3) all the $V_j(i)$ $(0 \leqq j \leqq \beta)$ and $W(i)$ are normally flat along B_i; and finally

(4) $W(r)$ is non-singular and all the $V_j(r)$ for $0 \leqq j \leqq \beta$ are normally flat along $W(r)$.

The proof is accomplished by induction on n. If $n = 0$, then we take the empty succession which obviously will do. Assume that the theorem is proved for all $n' < n$ where n is any fixed positive integer. We shall show that if f_i: $V_0(i+1) \to V_0(i)$ for $0 \leqq i < r$ is any succession of monoidal transformations with centers B_i in $V_0(i)$, where $V_0(0) = V_0$, which satisfy the conditions (1), (2) and (3) for all i; and if $S(r)$ denotes the closed subset of $W(r)$ complementary to the set of those simple points of $W(r)$ at which all the $V_j(r)$ for $0 \leqq j \leqq \beta$ are normally flat along $W(r)$, then we can find a prolongation of the succession, say f_i: $V_0(i+1) \to V_0(i)$ for $0 \leqq i \leqq r'$ for some r' $(>r)$, such that the image of $S(r')$ in V_0 is different from (but always contained in) that of $S(r)$, provided $S(r)$ is not empty. We shall denote by $\bar{S}(r)$ the image of $S(r)$ in V_0, which is a closed subset of V_0 (contained in W) because all the morphisms f_i are proper. We first remark that the above assertion is equivalent to that of the theorem. Now, take a generic point x of $\bar{S}(r)$, assuming that $S(r)$ is not empty. Let $\hat{V}_0 = \mathrm{Spec}(\hat{O}_{V_0,x})$, where $\hat{O}_{V_0,x}$ denotes the completion of $O_{V_0,x}$ (the local ring of V_0 at x). We have a canonical morphism of $\hat{V}_0$ to V_0. We shall denote by $\hat{V}_j(i)$ (resp. $\hat{W}(i)$) the product of V_0-schemes $V_j(i)$ (resp. $W(i)$) and $\hat{V}_0$. Let c_i denote the canonical morphism of $\hat{V}_0(i)$ to $V_0(i)$. We see that c_i is a flat morphism. Hence, in view of the property (ii) of the class $\mathscr{B}$ (in the beginning of this chapter), we have $\hat{S}(r) = c_r^{-1}(S(r))$ where $\hat{S}(r)$ denotes the complement in $\hat{W}(r)$ of the set of those simple points of $\hat{W}(r)$ at which all the $\hat{V}_j(r)$ for $0 \leqq j \leqq \beta$ are normally flat along $\hat{W}(r)$. Moreover, we see that $\hat{V}_j(i) = c_i^{-1}(V_j(i))$ for all j and i and that, if $\hat{B}_i = c_i^{-1}(B_i)$ ($=$ the product of V_0-schemes B_i and $\hat{V}_0$), then the canonical morphisms $\hat{V}_j(i+1) \to \hat{V}_j(i)$ and $\hat{W}(i+1) \to \hat{W}(i)$ may be identified with the monoidal transformations with center $\hat{B}_i$ for every i. Obviously, $\hat{B}_i$, $\hat{W}(i)$ and the $\hat{V}_j(i)$ for all j satisfy the conditions (1), (2) and (3). We shall denote by $\hat{f}_i$ the monoidal transformation $\hat{V}_0(i+1) \to \hat{V}_0(i)$ with center $\hat{B}_i$ for each i. For the same reason as above, we see that any prolongation of the succession $\{f_i\}$ satisfying (1), (2) and (3) induces the same kind of prolongation of $\{\hat{f}_i\}$. Since $\hat{V}_0$ can be imbedded in a non-singular algebraic scheme,[37] we can apply the previous result to $\hat{V}_j(r)$ $(0 \leqq j \leqq \beta)$ and $\hat{W}(r)$ so that there exists a prolongation of $\{\hat{f}_i\}$ satisfying (4), in addition to (1), (2) and (3). Therefore, we have only to show that any prolongation of $\{\hat{f}_i\}$ satisfying (1), (2) and (3) can be induced (possibly when we modify it by adding a suitable number of monoidal transformations with *empty centers* before each of the given monoidal transformations) by a prolongation of $\{f_i\}$ satisfying (1), (2) and (3). It suffices to consider the case in which the prolongation of $\{\hat{f}_i\}$ is obtained by adding a single monoidal transformation, say $\hat{g}$: $\hat{V}_0(r+1) \to \hat{V}_0(r)$. Let $\hat{C}$ be the center of $\hat{g}$. Then $\hat{C}$ is a reduced irreducible subscheme of $\hat{V}_0(r)$, which is contained in $\hat{S}(r)$. Since the point x is a generic point of $\bar{S}(r)$, the image of $\hat{S}(r)$ in $\hat{V}_0$ is the unique closed point. Therefore, we can find a reduced irreducible subscheme C of $V_0(r)$ such that $\hat{C} = c_r^{-1}(C)$. In fact, we take the smallest subscheme C of $V_0(r)$ such that $\hat{C} = c_r^{-1}(C)$. We have $C \subseteqq S(r)$. Hence $\dim(C) = n' < n$. We apply *theorem* I(n') to $V_0(r)$, $V_1(r)$, $\cdots$, $V_\beta(r)$, $W(r)$, and C. Since $\hat{C}$ is non-singular and all the $\hat{V}_j(r)$ and $\hat{W}(r)$ are normally flat along $\hat{C}$, any succession of monoidal transformations which appears in the application of *theorem* I(n') induces in $\hat{V}_0(r)$ a succession of monoidal transformations

[37] Every complete local ring in the class $\mathscr{B}$ is a homomorphic image of a formal power series ring over a field of characteristic zero. Hence, $\hat{V}_0$ can be embedded in a prime spectrum of such a formal power series ring, which is non-singular.

with *empty centers*. Let $\{f_i, 0 \leqq i < r' - 1\}$ be the prolongation of the $\{f_i, 0 \leqq i < r\}$ obtained by adding a succession in the above application of *theorem* I(n'), and let $C(i + 1) \to C(i)$ for $r \leqq i < r' - 1$ be the succession of monoidal transformations induced by f_i, where $C(r) = C$. Let $f_{r'-1}$: $V_0(r') \to V_0(r' - 1)$ be the monoidal transformation with center $C(r' - 1)$. We can easily see that $\{f_i, 0 \leqq i < r'\}$ is a prolongation of the $\{f_i, 0 \leqq i < r\}$ satisfying (1), (2) and (3), and that it induces the prolongation of the $\{\hat{f}_i\}$ by $\hat{g}$ except for the $(r' - r - 1)$ monoidal transformations with empty centers which should precede $\hat{g}$. *Main theorem* I(n) is now established.

Let X be a non-singular algebraic scheme. Let J be a coherent sheaf of non-zero ideals on X. Let B be an irreducible non-singular subscheme of X. If $\nu = \nu(J_x)$ for a generic point x of B, and if $f: X' \to X$ is the monoidal transformation of X with center B, then the sheaf of ideals $f^{-1}(J)$ is divisible by the ν^{th} power of the sheaf of ideals of $f^{-1}(B)$ on X'. The quotient obtained by this division is called *the weak transform of J by the monoidal transformation f*. This is the kind of transform of a coherent sheaf of ideals that has appeared in the definition of the transform of a resolution datum of type $\mathfrak{R}_{\text{II}}^N$. (See Definition 7, § 1.) The following theorem can be deduced from the *fundamental theorem* II$_2^N$.

MAIN THEOREM II(N). *Let X be a non-singular algebraic scheme of dimension N. Let J be a coherent sheaf of non-zero ideals on X, and E a reduced subscheme of X which is everywhere of codimension one and has only normal crossings. Then there exists a finite succession of monoidal transformations $\{f_i: X_{i+1} \to X_i\}$ with centers B_i in X_i, where $0 \leqq i < r$ and $X_0 = X$, which has the following properties*:

(1) *B_i is a non-singular irreducible subscheme of X_i;*

(2) *if J_{i+1} is the weak transform of J_i by the monoidal transformations f_i for all i, where $J_0 = J$, and if d_i is the maximal integer attained by $\nu(J_{iy})$ for the points y of X_i, then d_i is a positive integer and $\nu(J_{iz}) = d_i$ for all the points z of B_i;*

(3) *if E_{i+1} denotes the reduced subscheme* $\mathrm{red}(f_i^{-1}(E_i \cup B_i))$ *of X_{i+1} for all i, where $E_0 = E$, then E_i has only normal crossings with B_i; and*

(4) *E_r has only normal crossings and J_r is equal to O_{X_r}.*

As is easily seen, the proof of the *main theorem* can be reduced to the case in which X is irreducible. Assuming that X is irreducible, we apply the *fundamental theorem* II$_2^N$ to the resolution datum on X:

$$\mathfrak{R}_{\text{II}}^N = \left(\begin{matrix} E_1, & \cdots, & E_\alpha \\ a_1, & \cdots, & a_\alpha \end{matrix}\middle| J, b\right),$$

where $b = 1$, the E_i denote the irreducible components of E, and all the integers a_i are zero. If $\{f_i\colon X_{i+1} \to X_i\}$ for $0 \leqq i < r'$ is a succession of monoidal transformations which is obtained by applying the *fundamental theorem* to the above $\mathfrak{R}_{\mathrm{II}}^N$, then we take the smallest integer r such that O_{X_r} is the sheaf of ideals attached to the transform of $\mathfrak{R}_{\mathrm{II}}^N$ by the succession $\{f_i, 0 \leqq i < r\}$. Then this succession of r monoidal transformations has the properties stated in *main theorem* II(N).

4. Comments on fundamental theorems and on their proofs

We have shown in § 2 the implication diagram of induction among the *fundamental theorems* $\mathrm{I}_1^{N,n}$, $\mathrm{I}_2^{N\ n}$, II_1^N and II_2^N, which will be established in Chapter IV after the local theory of singularities developed in Chs. II and III. According to the implication diagram we actually establish, the resolution of singularities of an algebraic scheme of dimension 1 imbedded in a non-singular algebraic scheme of dimension N comes out only after the same is proved for all the algebraic schemes imbedded in a non-singular scheme of dimension $N-1$. This sounds absurd when N is large. In fact, we know a simple direct proof of a resolution of singularities of an algebraic scheme of dimension 1 by means of monoidal transformations with points as their centers, and which does not depend upon an immersion of the scheme into a non-singular algebraic scheme. At the same time, we know that any resolution of singularities of an algebraic scheme of dimension 2 is highly non-trivial, at least in comparison with the 1-dimensional case, even if it is imbedded in a non-singular algebraic scheme of dimension 3. For that matter, we remark that a simple modification of the proof of the implication (A), given in Ch. IV, gives us the following implication:

$$
\begin{array}{ll}
 & \text{Th. } \mathrm{I}_1^{N',n'} \ \textit{with} \ \left\{\begin{array}{l} n' < N' < N \\ n' < n \end{array}\right. \\
 & \text{Th. } \mathrm{I}_2^{N,n''} \ \textit{with} \quad n'' < n \\
\textit{and} & \\
 & \text{Th. } \mathrm{II}_2^{N''} \ \textit{with} \quad N'' \leqq n
\end{array}
\left.\vphantom{\begin{array}{l} a\\a\\a\\a\\a\\a \end{array}}\right\} \Longrightarrow \text{Th. } \mathrm{I}_1^{N,n}\ ,
$$

and also that a simple modification of the proof of the implication (B), given in Chapter IV, gives us the following implication:

$$
\begin{array}{ll}
 & \text{Th. } \mathrm{I}_2^{N,n''} \ \textit{with} \ \left\{\begin{array}{l} n' < N' < N \\ n' < n \end{array}\right. \\
 & \text{Th. } \mathrm{I}_2^{N'\ n'} \ \textit{with} \quad n'' < n \\
\textit{and} & \\
 & \text{Th. } \mathrm{II}_2^{N''} \ \textit{with} \quad N'' \leqq n
\end{array}
\left.\vphantom{\begin{array}{l} a\\a\\a\\a\\a\\a \end{array}}\right\} \Longrightarrow \text{Th. } \mathrm{I}_2^{N\ n}\ .
$$

The modification of the proofs in Chapter IV consists in changing the way of choosing a regular frame $(\mathbf{T}; z)$ is $\hat{\mathbf{S}}$ in [a] and [b] and accordingly the other objects in [c]–[i]. (Refer to [a]–[i] in the paragraphs between Lemma (AB, 3) and Lemma (AB, 4), § 3, Ch. IV.) To be precise, we choose a regular frame $(\mathbf{T}; z_1, \cdots, z_\tau)$ of $\hat{\mathbf{S}}$ such that τ is at least equal to the codimension of W in X at the point x_0 and such that:

[a*] If $\mathfrak{R} = (\mathfrak{R}_{\mathrm{I}}^{N\ n}, F)$, then z_1 is an element of $\mathbf{S}$ which generates the prime ideal of F in $\mathbf{S}$ and $(\mathbf{T}\{z_1\}; z_2, \cdots, z_\tau)$ is $\hat{\mathbf{J}}_0$-stable.

[b*] If $\mathfrak{R} = (\mathfrak{R}_{\mathrm{I}}^{N\ n}, U)$, then $(\mathbf{T}; z_1, \cdots, z_\tau)$ is $\hat{\mathbf{J}}_0$-stable.

[c*] We take a $\hat{\mathbf{J}}_j$-stable standard base of $\hat{\mathbf{J}}_j$,

$$(g_{j1}, \cdots, g_{jm_j})$$

for $0 \leqq j \leqq \beta$, and we write

$$g_{jk} = \textstyle\sum_{A \in Z_0^\tau} g_{jkA} z^A$$

where $z = (z_1, \cdots, z_\tau)$ and $g_{jkA} \in \mathbf{T}$ for all (j, k, A).

[d*] (the same as [d])

$$b = \textstyle\prod_{j=0}^{\beta} \left(\prod_{k=1}^{m_j} (\nu_{jk}!)\right) \qquad \text{with } \nu_{jk} = \nu^{(k)}(\mathbf{J}_j)\ .$$

[e*] $\bar{\mathbf{J}} = (g_{jkA}^{b/|A|})\bar{\mathbf{T}}$, with $\bar{\mathbf{T}} = \hat{\mathbf{S}}/(z_1, \cdots, z_\tau)\hat{\mathbf{S}}$,

which denotes the ideal in $\bar{\mathbf{T}}$ generated by the images in $\bar{\mathbf{T}}$ of the $(b/|A|)$-powers of g_{jkA} for $0 \leqq j \leqq \beta$, $1 \leqq k \leqq m_j$ and $A \in Z_0^\tau$ with $|A| < \nu_{jk}$.

The other objects on [f]–[i] should be defined in the same way as in Chapter IV. The existence of the above regular frame $(\mathbf{T}; z_1, \cdots, z_\tau)$ can be deduced from Theorem 9, (§ 10, Ch. III). Here we must use the fact that if $\mathbf{J}$ is an ideal in a regular local ring $\mathbf{R}$ which is different from $\mathbf{R}$ and if $(z_1, \cdots, z_\tau)$ is a regular system of τ-parameters of $\mathbf{R}$ for $\mathbf{J}$, then $\dim(\mathbf{R}) \leqq \tau + \dim(\mathbf{R}/\mathbf{J})$. (This follows from the fact that if $\mathbf{M}$ is the maximal ideal of $\mathbf{R}$, then $\dim(\mathrm{gr}_{\mathbf{M}}(\mathbf{R}/\mathbf{J})) = \dim(\mathbf{R}/\mathbf{J})$, which is well-known.) The rest of the proofs in § 3 of Chapter IV must be modified according to the above selections of $(\mathbf{T}; z_1, \cdots, z_\tau)$ and $\bar{\mathbf{J}}$, but the reasoning is essentially the same as that given there. The details in the modified proofs will be left to the reader.

We remark also that the reader who finds himself familiar with the arguments in Ch. IV will easily find an inductive proof of the following theorem:

THEOREM III^N. *Let X be a non-singular irreducible algebraic scheme of dimension $N \geqq 0$. Let $\{\mathfrak{R}_i (1 \leqq i \leqq r), \mathfrak{R}'_j (1 \leqq j \leqq s)\}$ be a finite number of resolution data on X which have the following forms*:

$$\mathfrak{R}_i = (E;\ V_{i1}, \cdots, V_{i\beta};\ W_i) \qquad \textit{for } 1 \leqq i \leqq r,$$

and

$$\mathfrak{R}'_j = \left(\begin{matrix} E_1, \cdots, E_\alpha \\ a_{j1}, \cdots, a_{j\alpha} \end{matrix} \middle| J_j, b_j\right) \qquad \text{for } 1 \leqq j \leqq s,$$

where $E = \bigcup_{k=1}^{\alpha} E_k$ *and* $\nu(j) = \nu(\mathfrak{R}'_j) > 0$ *for all* j $(1 \leqq j \leqq s)$. *Here we do not exclude the case in which the principal subscheme* W_i *of* $\mathfrak{R}_i$ *is equal to* X *for some* i. *Then there exists a finite succession of monoidal transformations* $f\colon \tilde{X} \to X$, *which is permissible for all the data* $\mathfrak{R}_i$ *and* $\mathfrak{R}'_j$, *such that (either* $\tilde{X}$ *is empty or)* $\bigcap_{j=1}^{s} S_{\nu(j)}(f^*(\mathfrak{R}'_j))$ *has no points in the intersection of the principal subschemes of the* $f^*(\mathfrak{R}_i)$ *for* $1 \leqq i \leqq r$.

Theorem III^N may be proved as follows: Let $N > 0$, and assume *theorems* III$^{N''}$ for all $N'' < N$. Then, first of all, by means of *theorems* $\mathrm{I}_2^{N',n'}$ for $n' < N' \leqq N$ we can prove that there exists a finite succession of monoidal transformations $f\colon \tilde{X} \to X$, which is permissible for all the $\mathfrak{R}_i$ and $\mathfrak{R}'_j$, such that if S denotes the set of points of X contained in both $f(\bigcap_{j=1}^{s} S_{\nu(j)}(f^*(\mathfrak{R}'_j)))$ and $f(\bigcap_{i=1}^{r} \tilde{W}_i)$, where $\tilde{W}_i$ denotes the principal subscheme of $f^*(\mathfrak{R}_i)$, then

(1) S consists of a finite number of points at which X has dimension N. (cf. the *localization theorems* and their proofs in § 1, Ch. IV). For the proof of *theorem* III^N, it suffices to show that the above f can be extended and replaced in such a way that, for every point y of $\tilde{X}$ at which $\tilde{X}$ has dimension N and such that $x = f(y) \in S$,

(2) we have either

(a) $\nu_y^*(\tilde{V}_{i1}, \cdots, \tilde{V}_{i\beta}, \tilde{W}_i/\tilde{X}) < \nu_x^*(V_{i1}, \cdots, V_{i\beta}, W_i/X)$ for at least one i, or

(b) $\nu_y^*(\tilde{V}_{i1}, \cdots, \tilde{V}_{i\beta}, \tilde{W}_i/\tilde{X}) = \nu_x^*(V_{i1}, \cdots, V_{i\beta}, W_i/X)$ and
$\tau_y^*(\tilde{V}_{i1}, \cdots, \tilde{V}_{i\beta}, \tilde{W}_i/\tilde{X}) > \tau_x^*(V_{i1}, \cdots, V_{i\beta}, W_i/X)$

for at least one i, where

$$f^*(\mathfrak{R}_i) = (\tilde{E};\ \tilde{V}_{i1}, \cdots, \tilde{V}_{i\beta};\ \tilde{W}_i)\ ,$$

or

(c) $\nu((\tilde{J}_j)_y) < \nu((J_j)_x)$ for at least one j, where

$$f^*(\mathfrak{R}'_j) = \left(\begin{matrix} \tilde{E}_1, \cdots, \tilde{E}_{\tilde{\alpha}} \\ a_{j1}, \cdots, a_{j\tilde{\alpha}} \end{matrix} \middle| \tilde{J}_j, b_j\right)$$

(cf. the arguments in §3, Ch; IV). To find such an extension of f, we may assume in (1) that S consists of a single point x_0. We then prove that, if there exists at least one i such that $W_i \neq X$ and W_i has only normal crossings with E at the point x_0, then we can find an extension of f having the property (2) by means of *theorems* $\mathrm{II}_2^{N'}$ for $N' \leqq \dim_{x_0}(W)$ (cf. the proofs of *theorem* A*, § 3, Ch. IV, and of *theorem* C, § 3, Ch. IV). By means of this result, we can extend and replace the succession f with property (1) in the general case, so that

(3) for any m such that $1 \leqq m \leqq \alpha$, $\tilde{E}_m$ does not contain any point which is a common point of $\bigcap_{j=1}^{s} S_{\nu(j)}(f^*(\mathfrak{N}'_j))$ and $\bigcap_{j=1}^{r} \tilde{W}_i$.

After having (1) and (3), an extension of f with the property (2) can be obtained by means of *theorems* $\mathrm{II}_2^{N'}$ foy $N' < N$ and *theorems* $\mathrm{I}_2^{N',n'}$ for $n' < N' \leqq N$ (cf. the proofs of *theorem* B*, § 3, Ch. IV, and of *theorem* D*, § 4, Ch. IV). A detailed proof is left to the reader.

The following theorem can be deduced easily from the above *theorem* III^N (cf. the deduction of the *main theorem* II(N) from the *fundamental theorem* II_2^N).

MAIN THEOREM III(N, n). *Let X, J and E be the same as in the main theorem* II(N) *of* § 3, *this chapter. Let W be a reduced subscheme of dimension n of X. Then there exists a finite succession of monoidal transformations $\{f_i\colon X_{i+1} \to X_i\}$ with centers B_i in X_i, where $0 < i < r$ and $X = X_0$, which has the following properties*:

(a) *If W_i is the strict transform of W in X_i by the succession of monoidal transformations, then B_i is a non-singular irreducible subscheme of W_i along which W_i is normally flat,*

(b) *if J_i is the weak transform of J in X_i by the succession of monoidal transformations, then $\nu(J_{iz}) = d_i$ for all the points z of B_i, where d_i is the maximal integer attained by $\nu(J_{iy})$ with a point y of W_i,*

(c) *the same as* (3) *of* II(N), § 3, *and*

(d) *W_r is empty.*[38]

The details of the proof will be left to the reader.

CHAPTER II. NORMAL FLATNESS ALONG SUBSCHEMES

1. Flatness of the associated graded algebra

Let $\mathbf{A}$ be a noetherian ring and $\mathbf{I}$ an ideal in $\mathbf{A}$. Then we associate with $\mathbf{I}$ the following graded $\mathbf{A}/\mathbf{I}$-algebra of finite type:

$$\mathrm{gr}_{\mathbf{I}}(\mathbf{A}) = \sum_{m=0}^{\infty} \mathbf{I}^m/\mathbf{I}^{m+1} .$$

The homogeneous part of degree m of $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$ is the $\mathbf{A}/\mathbf{I}$-module of finite type $\mathbf{I}^m/\mathbf{I}^{m+1}$ and will be denoted by $\mathrm{gr}_{\mathbf{I}}^m(\mathbf{A})$. If M is an $\mathbf{A}$-module, then we associate the following graded $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$-module:

$$\mathrm{gr}_{\mathbf{I}}(M) = \sum_{m=0}^{\infty} \mathbf{I}^m M/\mathbf{I}^{m+1} M .$$

If M is an $\mathbf{A}$-module of finite type, then $\mathrm{gr}_{\mathbf{I}}(M)$ is a $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$-module of finite type. The homogeneous part of degree m of $\mathrm{gr}_{\mathbf{I}}(M)$, that is $\mathbf{I}^m M/\mathbf{I}^{m+1} M$, will be denoted by $\mathrm{gr}_{\mathbf{I}}^m(M)$. If M is an $\mathbf{A}$-module of finite

38 This assertion (d) is equivalent to saying that $B_{r-1} \supseteq W_{r-1}$ (hence, by (a), $B_{r-1} = W_{r-1}$).

type, then $\mathrm{gr}_{\mathbf{I}}^m(M)$ is an $\mathbf{A}/\mathbf{I}$-module of finite type.

Let V be an algebraic scheme (in the sense specified in the beginning of Ch. I), and W a subscheme of V. We then have the coherent sheaf of ideals $P_W(V)$ on V defining the subscheme W of V. We associate with W the following quasi-coherent sheaf of graded algebras on the scheme W:

$$\mathrm{gr}_W(V) = \sum_{m=0}^{\infty} P_W(V)^m / P_W(V)^{m+1} \qquad \text{(restricted to } W) .$$

The homogeneous part of degree m of $\mathrm{gr}_W(V)$, denote by $\mathrm{gr}_W^m(V)$, is the coherent sheaf of modules $P_W(V)^m/P_W(V)^{m+1}$ on W. If $\mathbf{O}$ is the local ring of V at a point of W and if $\mathbf{P}$ is the ideal in $\mathbf{O}$ which defines the subscheme W, then $\mathrm{gr}_{\mathbf{P}}(\mathbf{O})$ (viewed as a graded $(\mathbf{O}/\mathbf{P})$-algebra) is the stalk of the sheaf $\mathrm{gr}_W(V)$ on W at the point.

If $\mathbf{A}$ is an affine ring of V, i.e., the affine scheme $\mathrm{Spec}(\mathbf{A})$ is identified with an open subset U (with the induced structure of algebraic scheme) of V, and if $\mathbf{I}$ is the ideal in $\mathbf{A}$ which defines the subscheme W of V, then the restriction of $\mathrm{gr}_W(V)$ to U (i.e., its restriction to $W \cap U$) is generated by $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$. This comes from the fact that if $\mathbf{P}$ is a prime ideal in $\mathbf{A}$ then $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})_{\mathbf{P}}$ (which denotes the localization of $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$ viewed as an $\mathbf{A}$-module with respect to $\mathbf{P}$) is canonically isomorphic to $\mathrm{gr}_{\mathbf{IA_P}}(\mathbf{A_P})$.

We shall first recall some basic results on the flatness of modules from the seminar note by A. Grothendieck: *Séminaire de Géométrie Algébrique* de l'Institut des Hautes Études Scientifiques, No. IV (1960). Throughout this paper all the local rings with which we are concerned will be assumed to be noetherian.

LEMMA 1. *Let $\mathbf{O}$ be a local ring and M an $\mathbf{O}$-module of finite type. Then the following conditions are equivalent to each other*:

(i) *M is a free $\mathbf{O}$-module*

(ii) *M is a projective $\mathbf{O}$-module*

(iii) *M is a flat $\mathbf{O}$-module.*

(See Corollaire 4.3 to Proposition 4.1 of the above cited seminar note by Grothendieck.)

Let $\mathbf{A}$ be a noetherian ring and $\mathbf{I}$ an ideal in $\mathbf{A}$. If M is an $\mathbf{A}$-module, then we have the canonical homomorphisms

$$M \otimes_{\mathbf{A}} \mathbf{I}^m/\mathbf{I}^{m+1} \longrightarrow \mathbf{I}^m M/\mathbf{I}^{m+1}M$$

for all non-negative integers m which are obtained by the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & M \otimes_{\mathbf{A}} \mathbf{I}^{m+1} & \longrightarrow & M \otimes_{\mathbf{A}} \mathbf{I}^m & \longrightarrow & M \otimes_{\mathbf{A}} \mathbf{I}^m/\mathbf{I}^{m+1} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \mathbf{I}^{m+1}M & \longrightarrow & \mathbf{I}^m M & \longrightarrow & \mathbf{I}^m M/\mathbf{I}^{m+1}M & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are exact and the first two vertical arrows are induced by the operation of $\mathbf{A}$ on the $\mathbf{A}$-module M. The above canonical homomorphisms give us a canonical homomorphism of graded $\operatorname{gr}_{\mathbf{I}}(\mathbf{A})$-modules:

$$\varphi\colon \operatorname{gr}^0_{\mathbf{I}}(M) \otimes_{\mathbf{A}/\mathbf{I}} \operatorname{gr}_{\mathbf{I}}(\mathbf{A}) \longrightarrow \operatorname{gr}_{\mathbf{I}}(M)\ .$$

LEMMA 2. *Let* $\mathbf{A} \to \mathbf{B}$ *be a homomorphism of noetherian rings,* $\mathbf{I}$ *an ideal in* $\mathbf{A}$ *such that* $\mathbf{IB}$ *is contained in the Jacobson radical of* $\mathbf{B}$ (i.e., *in the intersection of all the maximal ideals of* $\mathbf{B}$), *and* M *a* $\mathbf{B}$-*module of finite type. Then the following conditions are equivalent to each other*:

(i) M *is a flat* $\mathbf{A}$-*module,*

(ii) $M \otimes_{\mathbf{A}} \mathbf{A}/\mathbf{I}$ *is a flat* $\mathbf{A}/\mathbf{I}$-*module and* $\operatorname{Tor}^{\mathbf{A}}_1(M, \mathbf{A}/\mathbf{I}) = 0$,

(iii) $M \otimes_{\mathbf{A}} \mathbf{A}/\mathbf{I}$ *is a flat* $\mathbf{A}/\mathbf{I}$-*module, and the canonical homomorphism*

$$\varphi\colon \operatorname{gr}^0_{\mathbf{I}}(M) \otimes_{\mathbf{A}/\mathbf{I}} \operatorname{gr}_{\mathbf{I}}(\mathbf{A}) \longrightarrow \operatorname{gr}_{\mathbf{I}}(M)$$

is an isomorphism. (See Théorème 5.6 of the above cited seminar note by Grothendiek.)

COROLLARY. *The three conditions of the above lemma remain equivalent to each other if we replace the assumptions on* $\mathbf{IB}$ *and* M *by the following ones*: $\mathbf{B}$ *is a graded* $\mathbf{A}$-*algebra of finite type which is generated by homogeneous elements of positive degrees as an* $\mathbf{A}$-*algebra,* $\mathbf{I}$ *is contained in the Jacobson radical of* $\mathbf{A}$, *and* M *is a graded* $\mathbf{B}$-*module of finite type.*

PROOF. The $\mathbf{A}$-algebra $\mathbf{B}$ is a direct sum of its homogeneous parts $\mathbf{B}_n$ for all the non-negative integers n, and each of the $\mathbf{B}_n$ is an $\mathbf{A}$-module of finite type. Since M is a graded $\mathbf{B}$-module of finite type, it is a homomorphic image of a direct sum of a finite number of copies of $\mathbf{B}$. Therefore, it is clear that all the homogeneous parts of M_n of M are $\mathbf{A}$-modules of finite type. Then the assertion of the corollary can be verified by applying Lemma 2 to ($\mathbf{A}$ =) $\mathbf{A}$, ($\mathbf{B}$ =) $\mathbf{A}$, and (M =) M_n for every integer n. In fact, first of all by the definition of flatness, the $\mathbf{A}$-module M is flat if and only if all the direct summands M_n of M are flat $\mathbf{A}$-modules. Secondly, the functor $\operatorname{Tor}^{\mathbf{A}}_1$ commutes with direct summation of modules, and finally $\operatorname{gr}^0_{\mathbf{I}}(M)$ (resp. $\operatorname{gr}_{\mathbf{I}}(M)$) is a direct sum of the modules $\operatorname{gr}^0_{\mathbf{I}}(M_n)$ (resp. $\operatorname{gr}_{\mathbf{I}}(M_n)$) for all integers n, so that the homomorphism φ is the direct sum of the canonical homomorphisms

$$\varphi_n\colon \operatorname{gr}^0_{\mathbf{I}}(M_n) \otimes_{\mathbf{A}/\mathbf{I}} \operatorname{gr}_{\mathbf{I}}(\mathbf{A}) \longrightarrow \operatorname{gr}_{\mathbf{I}}(M_n)$$

for all integers n. q.e.d.

Let $\mathbf{O}$ be a local ring. Let $\mathbf{P}$ and $\mathbf{m}$ be prime ideals in $\mathbf{O}$ such that

(1) $\mathbf{P} \subset \mathbf{m}$, and

(2) $\mathbf{O}/\mathbf{P}$ and $\mathbf{O}/\mathbf{m}$ are regular local rings.

Let $\bar{\mathbf{O}} = \mathbf{O}/\mathbf{P}$ and $\mathbf{K} = \mathbf{O}/\mathbf{m}$. Let $\bar{\mathbf{m}} = \mathbf{m}/\mathbf{P}$, which is a prime ideal in $\bar{\mathbf{O}}$ such that $\mathbf{K} = \bar{\mathbf{O}}/\bar{\mathbf{m}}$. The injections $\mathbf{P}^n \rightarrow \mathbf{m}^m$ for all integers m induce a homomorphism of graded $\mathbf{O}$-algebras $\mathrm{gr}_{\mathbf{P}}(\mathbf{O}) \rightarrow \mathrm{gr}_{\mathbf{m}}(\mathbf{O})$, whose kernel contains $\mathbf{m}\,\mathrm{gr}_{\mathbf{P}}(\mathbf{O})$. Thus we get a homomorphism of graded $\mathbf{K}$-algebras:

$$\begin{array}{l} \alpha\colon\ \mathrm{gr}^0_{\bar{\mathbf{m}}}\,\mathrm{gr})\ (\mathbf{O})) \longrightarrow \mathrm{gr}_{\mathbf{m}}(\mathbf{O}) \\ \qquad\quad \Vert \\ \mathrm{gr}_{\mathbf{P}}(\mathbf{O})/\bar{\mathbf{m}}\,\mathrm{gr}_{\mathbf{P}}(\mathbf{O})\ . \end{array}$$

On the other hand, the natural homomorphisms of $\mathbf{O}$-modules: $\mathbf{m}^m \longrightarrow \bar{\mathbf{m}}^m$ for all non-negative integers m induce a homomorphism of graded $\mathbf{K}$-algebras:

$$\beta\colon \mathrm{gr}_{\mathbf{m}}(\mathbf{O}) \longrightarrow \mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})\ .$$

Let us choose a system of elements $(x_1, \cdots, x_r)$ of $\mathbf{m}$ which induces a minimal base of the prime ideal $\bar{\mathbf{m}}$ in $\bar{\mathbf{O}}$, say $(\bar{x}_1, \cdots, \bar{x}_r)$ where $\bar{x}_i$ is the residue class of x_i modulo $\mathbf{P}$ for $1 \leqq i \leqq r$ ($r = \dim \bar{\mathbf{O}}_{\bar{\mathbf{m}}}$). Since $\mathbf{K} = \bar{\mathbf{O}}/\bar{\mathbf{m}}$ and $\bar{\mathbf{O}}$ are both regular, $(\bar{x}_1, \cdots, \bar{x}_r)$ can be extended to a regular system of parameters of $\bar{\mathbf{O}}$.[39] Therefore, if $\bar{X}_i$ denotes the class of $\bar{x}_i$ in $\bar{\mathbf{m}}/\bar{\mathbf{m}}^2$, then $\mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})$ can be identified with the polynomial ring $\mathbf{K}[\bar{X}] = \mathbf{K}[\bar{X}_1, \cdots, \bar{X}_r]$ of r variables $\bar{X}_1, \cdots, \bar{X}_r$. Let X_i denote the class of x_i in $\mathbf{m}/\mathbf{m}^2$ for $1 \leqq i \leqq r$, and T the $\mathbf{K}$-subalgebra of $\mathrm{gr}_{\mathbf{m}}(\mathbf{O})$ generated by $X_1, \cdots, X_r$. Then β induces an isomorphism of T onto $\mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) = \mathbf{K}[\bar{X}]$. Hence taking the inverse of this induced isomorphism, we obtain a monomorphism of $\mathbf{K}$-algebras:

$$\beta^*\colon \mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) \longrightarrow \mathrm{gr}_{\mathbf{m}}(\mathbf{O})\ .$$

Thus we obtain a homomorphism of $\mathbf{K}$-algebras:

$$\psi = \alpha \otimes \beta^*\colon\ \mathrm{gr}^0_{\bar{\mathbf{m}}}\big(\mathrm{gr}_{\mathbf{P}}(\mathbf{O})\big) \otimes_{\mathbf{K}} \mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) \longrightarrow \mathrm{gr}_{\mathbf{m}}(\mathbf{O})\ .$$

Moreover, $\mathrm{gr}^0_{\bar{\mathbf{m}}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{O}))$ and $\mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})$ are graded $\mathbf{K}$-algebras, and hence the tensor product $\mathrm{gr}^0_{\bar{\mathbf{m}}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{O})) \otimes_{\mathbf{K}} \mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})$ is a doubly graded $\mathbf{K}$-algebra. In terms of total degrees, we can view $\mathrm{gr}^0_{\bar{\mathbf{m}}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{O})) \otimes_{\mathbf{K}} \mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})$ as a simply graded $\mathbf{K}$-algebra. In this manner, we easily see that ψ is a homomorphism of graded $\mathbf{K}$-algebras. It should be noted that ψ is not canonical

[39] Let $\mathbf{A}$ be a regular local ring, $\mathbf{M}$ the maximal ideal of $\mathbf{A}$, and $\mathbf{N}$ a prime ideal in $\mathbf{A}$ such that $\mathbf{A}/\mathbf{N}$ is regular. Then a system of elements of $\mathbf{N}$ generates $\mathbf{N}$ if and only if their images in $\mathbf{M}/\mathbf{M}^2$ span the $\mathbf{A}/\mathbf{M}$-submodule $(\mathbf{N} + \mathbf{M}^2)/\mathbf{M}^2$.

and depends upon the choice of the system $(x_1, \cdots, x_r)$.

PROPOSITION 1. *Let* $\mathbf{O}$ *be a local ring and* $\mathbf{P}$, $\mathbf{m}$ *prime ideals in* $\mathbf{O}$ *such that* $\mathbf{m}$ *contains* $\mathbf{P}$ *and* $\mathbf{O}/\mathbf{P}$, $\mathbf{O}/\mathbf{m}$ *are regular. Let* $\bar{\mathbf{O}} = \mathbf{O}/\mathbf{P}$ *and* $\bar{\mathbf{m}} = \mathbf{m}/\mathbf{P}$. *Let* $\mathbf{K} = \mathbf{O}/\mathbf{m} = \bar{\mathbf{O}}/\bar{\mathbf{m}}$. *If* $\mathrm{gr}_{\mathbf{P}}(\mathbf{O})$ *is flat over* $\bar{\mathbf{O}}$, *then*

(1) *the homomorphism of graded* $\mathbf{K}$*-algebras, obtained as above,*

$$\psi\colon \mathrm{gr}^0_{\mathbf{m}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{O})) \otimes_{\mathbf{K}} \mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) \longrightarrow \mathrm{gr}_{\mathbf{m}}(\mathbf{O})$$

is an isomorphism.

(2) $\mathrm{gr}_{\mathbf{m}}(\mathbf{O})$ *is flat over* $\mathbf{K}$.

PROOF. We shall use the same notation as in Proposition 1. Let $\mathbf{F}$ be the ideal in $\mathbf{O}$ generated by the chosen system of elements $x_1, \cdots, x_r$. For every non-negative integer q, we have the following commutative diagram of canonical $\mathbf{O}$-homomorphisms:

$$\begin{array}{ccc} \mathbf{m}^q \supset \mathbf{F}^q & \longrightarrow & (\mathbf{F}^q + \mathbf{P})/\mathbf{P} = \bar{\mathbf{m}}^q \\ \downarrow & & \downarrow \\ \mathrm{gr}^q_{\mathbf{m}}(\mathbf{O}) & \overset{\beta^*}{\underset{\beta}{\rightleftarrows}} & \mathrm{gr}^q_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) \end{array}$$

where $\supset$ is the inclusion mapping and the other arrows except β^* are natural homomorphisms.

The above diagram, combined with the natural commutative diagram for each non-negative integer p:

$$\begin{array}{ccc} & \mathbf{P}^p & \\ \swarrow & & \searrow \\ \mathrm{gr}^p_{\mathbf{m}}(\mathbf{O}) & \xleftarrow[\alpha]{} & \mathrm{gr}^0_{\bar{\mathbf{m}}}(\mathrm{gr}^p_{\mathbf{P}}(\mathbf{O})) \end{array},$$

produces the following commutative diagram for each non-negative integer n:

$$\begin{array}{ccc} & \sum_{p+q=n} \mathbf{P}^p \otimes_{\mathbf{O}} \mathbf{F}^q & \\ {\scriptstyle\nu}\swarrow & & \searrow{\scriptstyle\mu} \\ \mathrm{gr}^n_{\mathbf{m}}(\mathbf{O}) & \xleftarrow{\psi} & \sum_{p+q=n} \mathrm{gr}^0_{\bar{\mathbf{m}}}(\mathrm{gr}^p_{\mathbf{P}}(\mathbf{O})) \otimes_{\mathbf{K}} \mathrm{gr}^q_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) \end{array}$$

where ν is obtained by first sending $\sum_{p+q=n} \mathbf{P}^p \otimes_{\mathbf{O}} \mathbf{F}^q$ into $\mathbf{m}^n$.

As is easily seen, ν and μ are both surjective. Therefore, to prove that ψ is an isomorphism, it suffices that the kernel of ν is equal to that of μ. Let h be an arbitrary element of the kernel of ν. We shall prove that $\mu(h) = 0$. Let us choose a system of elements $(f_i^{(p)})$ of $\mathbf{P}^p$ whose residue classes mod $\mathbf{P}^{p+1}$ form a free base of $\mathbf{P}^p/\mathbf{P}^{p+1}$ as $\bar{\mathbf{O}}$-module. Then it is easy to show that $\mathbf{P}^p = (f_1^{(p)}, f_2^{(p)}, \cdots)\mathbf{O}$. Therefore we can write

$$h = \sum_{p+q=n} \left(\sum_i f_i^{(p)} \otimes g_i^{(q)}\right)$$

where $g_i^{(q)} \in \mathbf{F}^q$. If $g_i^{(q)} \in \mathbf{m}^{q+1}$, then $\nu(f_i^{(p)} \otimes g_i^{(q)}) = \mu(f_i^{(p)} \otimes g_i^{(q)}) = 0$. Therefore, to prove $\mu(h) = 0$, we may omit all those terms $f_i^{(p)} \otimes g_i^{(q)}$ with $g_i^{(q)} \in \mathbf{m}^{q+1}$ from the expression of h. Thus we assume that for every (q, i) either $g_i^{(q)} = 0$ or $g_i^{(q)} \notin \mathbf{m}^{q+1}$. Since $\nu(h) = 0$, the image $\tilde{h}$ of h in $\mathbf{m}^n$ must be in $\mathbf{m}^{n+1} = (\mathbf{P}, \mathbf{F})^{n+1}$. Therefore $\tilde{h} \in \sum_{p+q'=n+1} \mathbf{P}^p\mathbf{F}^{q'}$. Let us take the maximal integer p_1 such that $\tilde{h} \in \sum_{\left(\substack{p+q'=n+1 \\ p \geqq p_1}\right)} \mathbf{P}^p\mathbf{F}^{q'}$, and write $\tilde{h} = \sum_{\left(\substack{p+q'=n+1 \\ p \geqq p_1}\right)} \left(\sum_i f_i^{(p)}\lambda_i^{(q')}\right)$, with the above $f_i^{(p)}$ and $\lambda_i^{(q')} \in \mathbf{F}^{q'}$.

We claim that there exists at least one i such that $\lambda_i^{(q_1')} \notin \mathbf{P}$, where $q_1' = n - p_1 + 1$, unless $p_1 = n + 1$. In fact, if $\lambda_i^{(q_1')} \in \mathbf{P}$ then we must have $\lambda_i^{(q_1')} \in \mathbf{m}^{q_1'+1}$ because the generators $x_1, \cdots, x_r$ of $\mathbf{F}$ form a regular system of parameters of $\bar{\mathbf{O}}_{\bar{\mathbf{m}}}$. It then follows that $\lambda_i^{(q_1')} \in \sum_{\left(\substack{q_1'+1=u+v \\ u \geqq 0}\right)} \mathbf{P}^u\mathbf{F}^v \subset \sum_{\left(\substack{q_1'=s+t \\ s>0}\right)} \mathbf{P}^s\mathbf{F}^t + \mathbf{F}^{q_1'+1}$. By the same reason, we must have $\lambda_i^{(q_1')} \in \sum_{\left(\substack{q_1'=s+t \\ s>0}\right)} \mathbf{P}^s\mathbf{F}^t + \mathbf{F}^{q_1'+2}$. Repeating this argument, we have $\lambda^{(q_1')} \in \sum_{\left(\substack{q_1'=s+t \\ s>0}\right)} \mathbf{P}^s\mathbf{F}^t + \mathbf{F}^{q_1'+m}$ for all positive integer m. Hence $\lambda^{(q_1')} \in \sum_{\left(\substack{q_1'=s+t \\ s>0}\right)} \mathbf{P}^s\mathbf{F}^t$, and $f_i^{(p_1)}\lambda_i^{(q_1')} \in \sum_{\left(\substack{p+q'=n+1 \\ p \geqq p_1+1}\right)} \mathbf{P}^p\mathbf{F}^{q'}$. Thus we conclude that, unless $p_1 = n + 1$ and $q_1' = 0$, we have at least one i with $\lambda_i^{(q_1')} \notin \mathbf{P}$.

Now let δ be the smallest integer such that one of the $\lambda_i^{(n-\delta+1)}$ and $g_i^{(n-\delta)}$ is not zero. If $\delta = n + 1$, then all the $g_i^{(q)} = 0$, hence, $\mu(h) = 0$ is clear. Suppose $\delta \leqq n$. Then we should have

$$\sum_i f^{(\delta)} g_i^{(n-\delta)} \equiv \sum_i f_i^{(\delta)}\lambda_i^{(n-\delta+1)} \mod \mathbf{P}^{\delta+1} .$$

Since $(f_i^{(\delta)})$ form a free base of the $\bar{\mathbf{O}}$-module $\mathbf{P}^\delta/\mathbf{P}^{\delta+1}$,

$$g_i^{(n-\delta)} \equiv \lambda_i^{(n-\delta+1)} \mod \mathbf{P}, \qquad \text{for all } i .$$

Since $g_i^{(n-\delta)} \in \mathbf{F}^{n-\delta}$ and $\lambda_i^{(n-\delta+1)} \in \mathbf{F}^{n-\delta+1}$, the assumption on the $g_i^{(q)}$ implies that $g_i^{(n-\delta)} = 0$ for all i, so that $\delta = p_1$ and $\lambda_i^{(n-\delta+1)} \in \mathbf{P}$ for all i. This contradicts what was shown above.

The assertion (1) of the proposition is now verified. The second assertion (2) follows from (1). In fact, $\mathrm{gr}^0_{\bar{\mathbf{m}}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{O}))$ is obviously flat over $\mathbf{K}$ and so is $\mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})$ because $\bar{\mathbf{O}}$ and $\bar{\mathbf{O}}/\bar{\mathbf{m}}$ are assumed to be regular. q.e.d.

COROLLARY 1. *Let V be an algebraic scheme and W a non-singular subscheme of V. Suppose V be normally flat along W. Then, for every non-singular subscheme W_1 of W, V is normally flat along W_1.*

PROOF. Let x be any point of W_1. Let $\mathbf{O}$ be the local ring of V at x, $\mathbf{P}$ the prime ideal in $\mathbf{O}$ defining the subscheme W, and $\mathbf{m}$ the prime ideal in $\mathbf{O}$ defining the subscheme W_1. Then by assumption, $\mathrm{gr}_{\mathbf{P}}(\mathbf{O})$ is flat over $\bar{\mathbf{O}} = \mathbf{O}/\mathbf{P}$. Hence by proposition 1, $\mathrm{gr}_{\mathbf{m}}(\mathbf{O})$ is flat over $\mathbf{K} = \mathbf{O}/\mathbf{m}$. Thus V is normally flat along W_1. q.e.d.

COROLLARY 2. *Let $\mathbf{O}$ be a local ring with the maximal ideal $\mathbf{m}$. Let $\mathbf{P}$ be a prime ideal in $\mathbf{O}$ such that $\bar{\mathbf{O}} = \mathbf{O}/\mathbf{P}$ is regular and that $\mathrm{gr}_{\mathbf{P}}(\mathbf{O})$ is flat over $\bar{\mathbf{O}}$. Let $r = \dim \bar{\mathbf{O}}$. Then the following are true:*

(i) *Let $F(n)$ be the function of non-negative integer n defined by*

$$F(n) = \sum_{i=0}^{n} \dim_{\mathbf{O}_{\mathbf{P}}/\mathbf{P}\mathbf{O}_{\mathbf{P}}} \left(\mathrm{gr}^{i}_{\mathbf{P}\mathbf{O}_{\mathbf{P}}}(\mathbf{O}_{\mathbf{P}})\right) .$$

Then we have

$$\sum_{p+q=n} F(p)\binom{q+r-1}{r-1} = \sum_{i=0}^{n} \dim_{\mathbf{K}}\left(\mathrm{gr}^{i}_{\mathbf{m}}(\mathbf{O})\right)$$

where $\mathbf{K} = \mathbf{O}/\mathbf{m}$.

(ii) *The local rings $\mathbf{O}$ and $\mathbf{O}_{\mathbf{P}}$ have the same multiplicity.*

PROOF. In general, the localization $\mathrm{gr}_{\mathbf{P}}(\mathbf{O})_{\mathbf{P}}$ of $\mathrm{gr}_{\mathbf{P}}(\mathbf{O})$ with respect to $\mathbf{P}$ is canonically isomorphic to $\mathrm{gr}_{\mathbf{P}\mathbf{O}_{\mathbf{P}}}(\mathbf{O}_{\mathbf{P}})$. Since $\mathrm{gr}^{p}_{\mathbf{P}}(\mathbf{O})$ is a free $\bar{\mathbf{O}}$-module for every non-negative integer p, we have

$$\dim_{\mathbf{O}_{\mathbf{P}}/\mathbf{O}\mathbf{P}_{\mathbf{P}}} \left(\mathrm{gr}^{p}_{\mathbf{P}\mathbf{O}_{\mathbf{P}}}(\mathbf{O}_{\mathbf{P}})\right) = \dim_{\mathbf{K}}\left(\mathrm{gr}^{0}_{\bar{\mathbf{m}}}\left(\mathrm{gr}^{p}_{\mathbf{P}}(\mathbf{O})\right)\right)$$

where $\bar{\mathbf{m}} = \mathbf{m}/\mathbf{P}$. By Proposition 1, we have an isomorphism of $\mathbf{K}$-modules

$$\sum_{p+q=i} \mathrm{gr}^{0}_{\bar{\mathbf{m}}}\left(\mathrm{gr}^{p}_{\mathbf{P}}(\mathbf{O})\right) \otimes_{\mathbf{K}} \mathrm{gr}^{q}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}}) \longrightarrow \mathrm{gr}^{i}_{\mathbf{m}}(\mathbf{O})$$

for every non-negative integer i. We know that $\mathrm{gr}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})$ is a polynomial ring of r variables, and hence $\dim_{\mathbf{K}}(\mathrm{gr}^{q}_{\bar{\mathbf{m}}}(\bar{\mathbf{O}})) = \binom{q+r-1}{r-1}$ for all non-negative integer q. Thus we conclude

$$\sum_{p+q=n} F(p)\binom{q+r-1}{r-1} = \sum_{i=0}^{n} \dim_{\mathbf{K}}\left(\mathrm{gr}^{i}_{\mathbf{m}}(\mathbf{O})\right)$$

for all non-negative integer n. Let $G(n) = \sum_{p+q=n} F(p)\binom{q+r-1}{r-1}$. Let $F^0(n)$ be the polynomial such that $F^0(n) = F(n)$ for all sufficiently large integer n. We can write

$$F^0(n) = a_0\binom{n+s}{n} + a_1\binom{n+s-1}{s-1} + \cdots + a_s$$

where $s = \dim \mathbf{O}_{\mathbf{P}}$ and $a_0, \cdots, a_s$ are rational numbers. The multiplicity of the local ring $\mathbf{O}_{\mathbf{P}}$ is equal to a_0. Let $G^0(n)$ be the polynomial such that $G^0(n) = G(n)$ for all sufficiently large n. Then, as is easily seen,

$$G^0(n) - \sum_{p+q=n} F^0(p)\binom{q+r-1}{r-1}$$

is equal to a polynomial of degree $\leqq r-1$ and to the constant zero if $\mathrm{r}=0$. By using the formula

$$\sum_{p+q=n} \binom{p+s}{s}\binom{q+r-1}{r-1} = \binom{n+s+r}{s+r}$$

we get

$$\sum_{p+q=n} F^0(p)\binom{p+q-1}{r-1}$$
$$= a_0\binom{n+s+r}{s+\mathrm{r}} + a_1\binom{n+s+r-1}{s+r-1} + \cdots + a_s\binom{n+r}{r}.$$

Therefore the multiplicity of $\mathbf{O}$ is equal to a_0, which is the multiplicity of $\mathbf{O}_{\mathrm{P}}$. q.e.d.

COROLLARY 3. *Let V be an algebraic scheme and W a non-singular irreducible subscheme of V. Suppose V is normally flat along W. Then V has the same multiplicity at every point of W.*

REMARK. Let V and W be the same as in Corollary 3. Let x be a point of W, $\mathbf{O}$ the local ring of V at x and $\mathbf{m}$ the maximal ideal of $\mathbf{O}$. Let $F_x(n)$ denote the function of non-negative integers n defined by

$$F_x(n) = \sum_{i=0}^{n} \dim_{\mathbf{O}/\mathbf{m}} (\mathbf{m}^i/\mathbf{m}^{i+1}) .$$

Let us call this function F_x *the Samuel characteristic function of V at x*. Let us write F_W for $F_{\bar{x}}$ with the generic point $\bar{x}$ of W. F_W is then uniquely determined by the pair (W, V). Now, Corollary 2 implies that for every point x of W, we have

$$F_x(n) = \sum_{p+q=n} F_W(p)\binom{q+r-1}{r-1}$$

for all non-negative integers n, where r denotes the dimension of W at x. This fact is certainly stronger than the equi-multiplicity of V along W as is asserted is Corollary 3. For instance, let us suppose that the generic point of W is a simple point of V. It is not true in general that the equi-multiplicity of V along W implies that every point of W is a simple point of V. As a matter of fact, an algebraic scheme can have a multiple point at which the multiplicity of the scheme is equal to one. However, the normal flatness of V along W implies that every point of W is a simple point of V. In fact, by the assumption on the generic point of W, we have $F_W(p) = \binom{p+s}{s}$ where s denotes the codimension of W in V, i.e., the dimension of V at the generic point of W. Therefore, by the normal flatness, we have $F_x(n) = \binom{n+s+r}{s+r}$ for every point of W, where r denotes the dimension of W at x. This implies immediately that the local ring of V at x is regular, i.e., x is a simple point of V. Thus

COROLLARY 4. *Let V and W be the same as in Corollary 3. If V is normally flat along W, and W contains at least one simple point of V, then every point of W is a simple point of V.*

LEMMA 3. *Let $\mathbf{A}$ be a noetherian integral domain, $\mathbf{S}$ an $\mathbf{A}$-algebra of finite type and M an $\mathbf{S}$-module of finite type. Then there exists a non-zero element f of $\mathbf{A}$ such that M_f is a free $\mathbf{A}_f$-module, where M_f and $\mathbf{A}_f$ denote the localizations of M and $\mathbf{A}$ respectively with respect to the multiplicatively closed subset of $\mathbf{A}$ consisting of the powers of f.* (See Lemma 6.7 of the above cited seminar note by Grothendieck.)

LEMMA 4. *Let $\mathbf{A}$ be a noetherian ring, $\mathbf{B}$ a graded $\mathbf{A}$-algebra of finite type and M a graded $\mathbf{B}$-module of finite type. Assume that $\mathbf{B}$ is generated by homogeneous elements of positive degrees as an $\mathbf{A}$-algebra. Let $\mathbf{P}$ be a prime ideal in $\mathbf{A}$ such that $M_{\mathbf{P}}$ is a flat $\mathbf{A}_{\mathbf{P}}$-module. Then there exists an element $f \in \mathbf{A} - \mathbf{P}$ such that*

(a) *$(M/\mathbf{P}M)_f$ is a flat $(\mathbf{A}/\mathbf{P})_f$-module,*

(b) $\mathrm{Tor}_1^{\mathbf{A}}(M, \mathbf{A}/\mathbf{P})_f = 0$.

(See Lemma 6.8 of the above cited seminar note by Grothendieck.)

PROOF. We may assume that $\mathbf{B}$ is a polynomial ring over $\mathbf{A}$, suitably graded. Since $M_{\mathbf{P}}$ is a flat $\mathbf{A}_{\mathbf{P}}$-module, we have

(a′) $(M/\mathbf{P}M)_{\mathbf{P}}$ is a flat $(\mathbf{A}/\mathbf{P})_{\mathbf{P}}$-module (which is trivial because $(\mathbf{A}/\mathbf{P})_{\mathbf{P}}$ is a field), and

(b′) $\mathrm{Tor}_1^{\mathbf{A}}(M, \mathbf{A}/\mathbf{P})_{\mathbf{P}} = 0$.

(See corollary to Lemma 2.) By Lemma 3, (a′) implies (a). Since M is a graded $\mathbf{B}$-module of finite type, $\mathrm{Tor}_1^{\mathbf{A}}(M, \mathbf{A}/\mathbf{P})$ is also a graded $\mathbf{B}$-module of finite type. Therefore it is clear that (b′) implies (b). q.e.d.

THEOREM 1. (Krull-Seidenberg-Grothendieck).[40] *Let $\mathbf{A}$ be a noetherian ring, $\mathbf{B}$ a graded $\mathbf{A}$-algebra of finite type, and M a graded $\mathbf{B}$-module of finite type. We assume that $\mathbf{B}$ is generated by homogeneous elements of positive degrees as an $\mathbf{A}$-algebra. Then the set of those prime ideals $\mathbf{P}$ in $\mathbf{A}$ such that $M_{\mathbf{P}}$ is a flat $\mathbf{A}_{\mathbf{P}}$-module is an open subset of* Spec(A).

PROOF. Let U be the set of those prime ideals $\mathbf{P}$ in $\mathbf{A}$ such that $M_{\mathbf{P}}$ is a flat $\mathbf{A}_{\mathbf{P}}$-module. In order to prove that U is open in Spec(A), it is necessary and sufficient that

[40] This is a special case of a more general theorem of Grothendieck, [7], which says that if $\mathbf{B}$ is an $\mathbf{A}$-algebra of finite type (without gradation) and M is a $\mathbf{B}$-module of finite type, then the set of those points x of Spec ($\mathbf{B}$) for which M_x is flat over the local ring of Spec ($\mathbf{A}$) at the point corresponding to x is an open subset of Spec ($\mathbf{B}$).

(i) if $\mathbf{P} \in U$ and $\mathbf{F}$ is a prime ideal contained in $\mathbf{P}$, then $\mathbf{F} \in U$, and

(ii) if $\mathbf{P} \in U$, then there exists a non-empty open subset of Spec $(\mathbf{A}/\mathbf{P})$ whose image in Spec $(\mathbf{A})$ is contained in U. First of all, (i) is clear. In fact, for each non-negative integer n, if $(M_n)_{\mathbf{P}}$ is a free $\mathbf{A}_{\mathbf{P}}$-module, then $(M_n)_{\mathbf{F}}$ is a free $\mathbf{A}_{\mathbf{F}}$-module where M_n denotes the homogeneous part of degree n of M. Then (ii) follows Lemma 4 and corollary to Lemma 2. q.e.d.

COROLLARY. *Let V be an algebraic scheme and W a reduced subscheme of V. Then there exists an open dense subset U of W such that $x \in U$ if and only if V is normally flat along W at x.*

PROOF. The assertion is of local nature. Hence we may assume that V is an affine scheme, say $V = \text{Spec}(\bar{\mathbf{A}})$ with a noetherian ring $\bar{\mathbf{A}}$. Let $\mathbf{I}$ be the ideal of W in $\bar{\mathbf{A}}$. Let $\mathbf{A} = \bar{\mathbf{A}}/\mathbf{I}$ and $\mathbf{B} = \text{gr}_{\mathbf{I}}(\bar{\mathbf{A}})$. Applying Theorem 1 to these $\mathbf{A}$ and $\mathbf{B}$ ($M = \mathbf{B}$), we obtain an open subset U of W which has the property that $x \in U$ if and only only if V is normally flat along W at x. It is easy to see that U is dense in W, i.e., that U contains all the generic points of W. q.e.d.

REMARK. In the later applications of the corollary, we shall be interested only in the subset of W of those simple points of W at which V is normally flat along W. The openness of this subset of W is all that will be needed later. This openness can also be proved by means of the generalized theorem of Krull-Seidenberg, Hironaka [8].

2. The associated graded ideal

Let $\mathbf{A}$ be a noetherian ring, and $\mathbf{I}$ an ideal in $\mathbf{A}$, say $\mathbf{I} \neq \mathbf{A}$. For each element $f \in \mathbf{A}$, we define the symbol $\nu_{\mathbf{I}}(f)$ as follows: $\nu_{\mathbf{I}}(f)$ is the largest integer $\nu \geqq 0$ such that $f \in \mathbf{I}^\nu$, if it exists, and $\nu_{\mathbf{I}}(f) = \infty$ if $f \in \mathbf{I}^\nu$ for all positive integers ν. If $\mathbf{A}$ is a local ring, then $\nu_{\mathbf{I}}(f) = \infty$ implies that $f = 0$. If $\nu = \nu_{\mathbf{I}}(f)$ is a non-negative integer, then the class of f in $\mathbf{I}^\nu/\mathbf{I}^{\nu+1}$ will be called *the initial form of f in* $\text{gr}_{\mathbf{I}}(\mathbf{A})$. For the sake of convenience, we define the initial form of f with $\nu_{\mathbf{I}}(f) = \infty$ to be the zero element of $\text{gr}_{\mathbf{I}}(\mathbf{A})$.

If J is an ideal in $\mathbf{A}$, *then we shall denote by* $\text{gr}_{\mathbf{I}}(\mathbf{J}, \mathbf{A})$ *the graded ideal in* $\text{gr}_{\mathbf{I}}(\mathbf{A})$ *generated by the initial forms of element of* $\mathbf{J}$. We can write

$$\text{gr}_{\mathbf{I}}(\mathbf{J}, \mathbf{A}) = \sum_{n=0}^{\infty} \big((\mathbf{J} \cap \mathbf{I}^n) + \mathbf{I}^{n+1}\big)/\mathbf{I}^{n+1} .$$

We denote by $\text{gr}_{\mathbf{I}}^n(\mathbf{J}, \mathbf{A})$ *the homogeneous part of degree n of the graded ideal* $\text{gr}_{\mathbf{I}}(\mathbf{J}, \mathbf{A})$.

Let $\bar{\mathbf{A}} = \mathbf{A}/\mathbf{J}$ and $\bar{\mathbf{I}} = \mathbf{I}\bar{\mathbf{A}} = (\mathbf{I} + \mathbf{J})/\mathbf{J}$. In view of the canonical isomorphism $\bar{\mathbf{I}}^n = (\mathbf{I}^n + \mathbf{J})/\mathbf{J} \approx \mathbf{I}^n/\mathbf{I}^n \cap \mathbf{J}$, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbf{I}^{n+1} \cap \mathbf{J} \arrow[r] \arrow[d] & \mathbf{I}^{n+1} \arrow[r] \arrow[d] & \bar{\mathbf{I}}^{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{I}^{n} \cap \mathbf{J} \arrow[r] \arrow[d] & \mathbf{I}^{n} \arrow[r] \arrow[d] & \bar{\mathbf{I}}^{n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{gr}^n_{\mathbf{I}}(\mathbf{J}, \mathbf{A}) \arrow[r] \arrow[d] & \mathrm{gr}^n_{\mathbf{I}}(\mathbf{A}) \arrow[r] \arrow[d] & \mathrm{gr}^n_{\bar{\mathbf{I}}}(\bar{\mathbf{A}}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the vertical sequences and the first two horizontal sequences are exact.

The exactness of the last horizontal sequence follows, namely:

LEMMA 5. *Notation being as above, we have a canonical exact sequence of graded* $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$*-modules*:

$$0 \longrightarrow \mathrm{gr}_{\mathbf{I}}(\mathbf{J}, \mathbf{A}) \longrightarrow \mathrm{gr}_{\mathbf{I}}(\mathbf{A}) \longrightarrow \mathrm{gr}_{\bar{\mathbf{I}}}(\bar{\mathbf{A}}) \longrightarrow 0 .$$

LEMMA 6. *Notation being as above, if* $\mathbf{A}$ *is a local ring, and if* $(f_1, \cdots, f_m)$ *is a system of elements of* $\mathbf{J}$ *such that the initial forms of the* f_i *in* $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$ *generate the ideal* $\mathrm{gr}_{\mathbf{I}}(\mathbf{J}, \mathbf{A})$, *then the* f_i *generate the ideal* $\mathbf{J}$.

PROOF. Let $\mathbf{J}_0 = (f_1, \cdots, f_m)\mathbf{A}$. Then we first show that $\mathbf{J} \subseteqq \mathbf{J}_0 + \mathbf{I}^n$ for every positive integer n. It is clear for $n = 1$. Inductively, if $g \in \mathbf{J}$ and $\nu_{\mathbf{I}}(g) = n$ then the initial form of g in $\mathrm{gr}_{\mathbf{I}}(\mathbf{A})$ is in $\mathrm{gr}^n_{\mathbf{I}}(\mathbf{J}, \mathbf{A})$ which is generated by the initial forms of $f_1, \cdots, f_m$. Hence there exists a linear combination $g_0 = \sum_i h_i f_i$ with $h_i \in \mathbf{A}$ such that $\nu_{\mathbf{I}}(g - g_0) > n$. In this manner, $\mathbf{J} \subseteqq \mathbf{J}_0 + \mathbf{I}^n$ implies $\mathbf{J} \subseteqq \mathbf{J}_0 + \mathbf{I}^{n+1}$. Now, since $\mathbf{A}$ is a local ring

$$\mathbf{J} \subseteqq \bigcap_n (\mathbf{J}_0 + \mathbf{I}^{n+1}) = \mathbf{J}_0 .$$

But obviously $\mathbf{J} \supseteqq \mathbf{J}_0$, hence $\mathbf{J} = \mathbf{J}_0$. q.e.d.

LEMMA 7. *Let* $\mathbf{R}$ *be a regular local ring,* $\mathbf{J}$ *an ideal in* $\mathbf{R}$, *and* $\mathbf{P}, \mathbf{Q}$ *prime ideals such that* $\mathbf{Q} \supset \mathbf{P} \supseteqq \mathbf{J}$, *and that* $\mathbf{R}/\mathbf{P}$ *and* $\mathbf{R}/\mathbf{Q}$ *are both regular. Let* $\bar{\mathbf{R}} = \mathbf{R}/\mathbf{P}$ *and* $\bar{\mathbf{Q}} = \mathbf{Q}/\mathbf{P}$. *Suppose we have a system of elements* $(f_1, \cdots, f_m)$ *of* $\mathbf{J}$ *such that*

(i) $\nu_{\mathbf{Q}}(f_j) = \nu_{\mathbf{P}}(f_j)$ *for* $1 \leqq j \leqq m$,

(ii) *if* φ_j *denotes the initial form of* f_j *in* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$, *then* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) = (\varphi_1, \cdots, \varphi_m)\,\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$.

Let Φ_j *be the initial form of* f_j *in* $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$ *and* $\mathbf{H} = (\Phi_1, \cdots, \Phi_m)\,\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$. *Then we have*

$$\bar{\mathbf{Q}}^n\,\mathrm{gr}_{\mathbf{P}}(\mathbf{R}) \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) = \bar{\mathbf{Q}}^n\mathbf{H}$$

for all non-negative integers n. *In particular, we have* $\mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) = (\Phi_1, \cdots, \Phi_m)\,\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$.

PROOF. Let us choose a regular system of parameters $(z_1, \cdots, z_r, y_1, \cdots, y_s, x_1, \cdots, x_t)$ of $\mathbf{R}$ such that $\mathbf{P} = (z_1, \cdots, z_r)\mathbf{R}$ and $\mathbf{Q} = (z_1, \cdots, z_r, y_1, \cdots, y_s)\mathbf{R}$, where $t = \dim \mathbf{R}/\mathbf{Q}$, $s + t = \dim \mathbf{R}/\mathbf{P}$, and $r + s + t = \dim \mathbf{R}$.[41] We can identify $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$ with the polynomial ring $\bar{\mathbf{R}}[Z] = \bar{\mathbf{R}}[Z_1, \cdots, Z_r]$ of r variables over $\bar{\mathbf{R}}$ in such a way that Z_i corresponds to the class of z_i in $\mathrm{gr}^1_{\mathbf{P}}(\mathbf{R})$. Similarly if $\mathbf{K}$ denotes the regular local ring $\mathbf{R}/\mathbf{Q} = \bar{\mathbf{R}}/\bar{\mathbf{Q}}$, then we can identify $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ with the polynomial ring $\mathbf{K}[Z, Y] = \mathbf{K}[Z_1, \cdots, Z_r, Y_1, \cdots, Y_s]$ of $r + s$ variables over $\mathbf{K}$ in such a way that Z_i corresponds to the class of z_i in $\mathrm{gr}^1_{\mathbf{Q}}(\mathbf{R})$ and Y_j does to the class of y_j in $\mathrm{gr}^1_{\mathbf{Q}}(\mathbf{R})$. First of all, we note that by (i), the φ_j are all forms only in $Z_1, \cdots, Z_r$. We shall prove that for every integer n $\geqq 0$,

$$\bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z] \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) \subseteqq \bar{\mathbf{Q}}^n\mathbf{H} + \bar{\mathbf{Q}}^{n+1}\bar{\mathbf{R}}[Z]\ .$$

Let Ψ be any element of $\bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z] \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R})$. We want to show that $\Psi \in \bar{\mathbf{Q}}^n\mathbf{H} + \bar{\mathbf{Q}}^{n+1}\bar{\mathbf{R}}[Z]$. We may assume that n is the largest integer such that $\Psi \in \bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z]$. Obviously, we may assume thae Ψ is homogeneous. Let $\nu = \deg \Psi$. We have an element $g \in \mathbf{J}$ such that Ψ is the initial form of g in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R}) = \bar{\mathbf{R}}[Z]$. Let $\mu = \nu_{\mathbf{Q}}(g)$. Then we have

$$\nu + n \geqq \mu \geqq \nu\ .$$

Let ψ be the initial form of g in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$ which is a form of degree μ. We write

$$\psi = \sum\nolimits_\alpha T_\alpha(Y)\,\xi_\alpha(Z)$$

where $T_\alpha(Y)$ are distinct monomials in $Y_1, \cdots, Y_s$ of degree d_α, and $\xi_\alpha(Z)$ are forms only in $Z_1, \cdots, Z_r$ of degree $\mu - d_\alpha$. Since $\varphi_1, \cdots, \varphi_m$ are forms only in $Z_1, \cdots, Z_r$, (ii) implies that $\xi_\alpha(Z) \in (\varphi_1, \cdots, \varphi_m)\mathbf{K}[Z]$ for all α so that we can write each non-zero $\xi_\alpha(Z)$ in the form

$$\xi_\alpha(Z) = \sum\nolimits_\beta \zeta_{\alpha\,\beta}(Z)\,\varphi_\beta(Z)$$

where $\zeta_{\alpha\,\beta}(Z)$ are forms only in $Z_1, \cdots, Z_r$ of degree $\mu - d_\alpha - \nu_\beta \geqq 0$.

[41] cf. footnote[39] of this chapter.

($\nu_\beta = \deg \varphi_\beta = \nu_{\mathbf{Q}}(f_\beta)$.) Let us take any element $h_{\alpha,\beta} \in (z_1, \cdots, z^r)^{\mu-d_\alpha-\nu_\beta}\mathbf{R}$ whose initial form in $\mathbf{K}[Z, Y]$ ($= \mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$) is equal to $\zeta_{\alpha\,\beta}(Z)$. Let $T_\alpha(y)$ denote the monomial in $y_1, \cdots, y_s$ which is obtained by replacing Y_i by y_i in $T_\alpha(Y)$. Let

$$h = \sum_\alpha \left(T_\alpha(y) \sum_\beta h_{\alpha,\beta} f_\beta\right) .$$

Then, as is easily seen, $\nu_{\mathbf{Q}}(h) = \mu$, and the initial form of h in $\mathbf{K}[Z, Y]$ is equal to ψ. Hence $\nu_{\mathbf{Q}}(g - h) > \mu$. Now, let us consider the case in which $\nu + n > \mu$. Since Ψ is the initial form of g in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, we have $g \in \mathbf{Q}^n\mathbf{P}^\nu + \mathbf{P}^{\nu+1}$. Therefore $\nu + n > \mu$ implies that $\psi \in (Z_1, \cdots, Z_r)^{\nu+1}\mathbf{K}[Z, Y]$. Hence every non-zero $\xi_\alpha(Z)$ must have degree $\mu - d_\alpha \geqq \nu + 1$. Therefore we have $\nu_{\mathbf{P}}(h) \geqq \nu + 1$. This means that g and $g - h$ have the same initial form Ψ in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, while $\nu_{\mathbf{Q}}(g - h) > \mu = \nu_{\mathbf{Q}}(g)$. Thus we could choose g in such a way that $\nu + n = \mu$. Let us then consider the case in which $\nu + n = \mu$. Clearly $\psi \in (Z_1, \cdots, Z_r)^\nu\mathbf{K}[Z, Y]$ and hence $\nu_{\mathbf{P}}(h) \geqq \nu$. If $\bar{y}_i$ denotes the residue class of y_i in $\bar{\mathbf{R}} = \mathbf{R}/\mathbf{P}$ and $\bar{x}_j$ that of x_j, then $(\bar{y}_1, \cdots, \bar{y}_s, \bar{x}_1, \cdots, \bar{x}_t)$ is a regular system of parameters of $\bar{\mathbf{R}}$ and $\mathbf{Q} = (\bar{y}_1, \cdots, \bar{y}_s)\mathbf{R}$. We can identify $\mathrm{gr}_{\bar{\mathbf{Q}}}(\bar{\mathbf{R}})$ with the polynomial ring $\mathbf{K}[Y_1, \cdots, Y_s]$ in such a way that Y_j corresponds to the class of $\bar{y}_j$ in $\mathrm{gr}^1_{\bar{\mathbf{Q}}}(\bar{\mathbf{R}})$. Let ψ_0 be the form in $\mathbf{K}[Z, Y]$ obtained by replacing the coefficients in $\bar{\mathbf{Q}}$ of Ψ by their images in $\mathrm{gr}^n_{\bar{\mathbf{Q}}}(\bar{\mathbf{R}})$. Then we can see that ψ_0 is the partial sum of the form ψ consisting of those terms (monomials in Z and Y) which are exactly of degree ν in $Z_1, \cdots, Z_r$. Since n is the largest integer such that $\Psi \in \bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z]$, ψ_0 is different from zero. Hence we must have $\nu_{\mathbf{P}}(h) = \nu$ and, if Ψ^* is the initial form of h in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, we must have

$$\Psi - \Psi^* \in \bar{\mathbf{Q}}^{n+1}\bar{\mathbf{R}}[Z] ,$$

because we know that ψ is the initial form of h as well as that of g. Moreover, in view of the expression

$$h = \sum_\alpha \left(T_\alpha(y) \sum_\beta h_{\alpha,\beta} f_\beta\right) ,$$

Ψ^* must be of the form

$$\sum\nolimits_\alpha^* \left(T_\alpha(\bar{y}) \sum_\beta \zeta^*_{\alpha,\beta}(Z)\, \Phi_\beta(Z)\right)$$

where $T_\alpha(\bar{y})$ is the monomial obtained by replacing the y_j by their residue classes $\bar{y}_j$ in $\bar{\mathbf{R}}$ in the above monomials $T_\alpha(Y)$, $\zeta^*_{\alpha,\beta}(Z)$ is the image of $h_{\alpha,\beta}$ in $\mathrm{gr}_{\mathbf{P}}^{\mu-d_\alpha-\nu_\beta}(\mathbf{R})$, and the summation $\sum^*_\alpha$ extends only to those α with $\mu - d_\alpha = \nu$, i.e., $d_\alpha = n$. It is clear that Ψ^* is contained in $\bar{\mathbf{Q}}^n\mathbf{H}$. Thus we conclude that

$$\bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z] \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) \subset \bar{\mathbf{Q}}^n\mathbf{H} + \bar{\mathbf{Q}}^{n+1}\bar{\mathbf{R}}[Z]$$

for every non-negative integer n. This implies immediately that

$$\bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z] \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) \subset \bar{\mathbf{Q}}^n\mathbf{H} + \bar{\mathbf{Q}}^{n+e}\bar{\mathbf{R}}[Z]$$

for every non-negative integer n and all positive integers e. Applying Nakayama's lemma to each homegeneous part of $\mathrm{gr}_{\mathbf{P}}(\mathbf{R}) = \bar{\mathbf{R}}[Z]$, which is an $\bar{\mathbf{R}}$-module of finite type, we obtain

$$\bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z] \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) \subseteqq \bar{\mathbf{Q}}^n\mathbf{H} .$$

Hence

$$\bar{\mathbf{Q}}^n\bar{\mathbf{R}}[Z] \cap \mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}) = \bar{\mathbf{Q}}^n\mathbf{H} . \qquad \text{q.e.d.}$$

Let $\mathbf{R}$ be a regular local ring and $\mathbf{J}$ an ideal in $\mathbf{R}$. Let $\mathbf{P}$ and $\mathbf{Q}$ be prime ideals in $\mathbf{R}$ such that $\mathbf{Q} \supset \mathbf{P} \supseteqq \mathbf{J}$. Assume that $\mathbf{R}/\mathbf{P}$ and $\mathbf{R}/\mathbf{Q}$ are both regular. Let $\mathbf{O} = \mathbf{R}/\mathbf{J}$, $P = \mathbf{P}/\mathbf{J}$, and $Q = \mathbf{Q}/\mathbf{J}$. We have the following commutative diagram of canonical homomorphisms of graded $\mathbf{O}/Q\,(=\mathbf{R}/\mathbf{Q})$-algebras:

$$\begin{array}{ccc} \mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) & \xrightarrow{\alpha} & \mathrm{gr}_{\mathbf{Q}}(\mathbf{R}) \\ \downarrow & & \downarrow \\ \mathrm{gr}^0_{Q}(\mathrm{gr}_{P}(\mathbf{O})) & \xrightarrow{\alpha'} & \mathrm{gr}_{Q}(\mathbf{O}) . \end{array}$$

Let $\bar{\mathbf{O}} = \mathbf{O}/P = \mathbf{R}/\mathbf{P}$ and $\bar{Q} = Q/P = \mathbf{Q}/\mathbf{P}$. Let us choose a system of elements $(y_1, \cdots, y_s)$ in $\mathbf{R}$ such that the residue classes $(\bar{y}_1, \cdots, \bar{y}_s)$ of them in $\bar{\mathbf{O}}$ form a minimal base of $\bar{Q}$ ($s = \dim \bar{\mathbf{O}}_{\bar{Q}}$). Let y'_j denote the residue class of y_j in $\mathbf{O}$. Obviously $\bar{y}_j$ is the residue class of y'_j in $\bar{\mathbf{O}}$. The systems $(y_1, \cdots, y_s)$ and $(y'_1, \cdots, y'_s)$ determine the following commutative diagram of homomorphisms of $\bar{\mathbf{O}}/\bar{Q}$ $(= \mathbf{O}/Q)$-algebras:

$$\begin{array}{ccc} \mathrm{gr}_{\bar{Q}}(\bar{\mathbf{O}}) & \xrightarrow{\beta^*} & \mathrm{gr}_{\mathbf{Q}}(\mathbf{R}) \\ & \searrow^{\beta^{*\prime}} & \downarrow \\ & & \mathrm{gr}_{Q}(\mathbf{O}) \end{array}$$

where the class of $\bar{y}_j$ in $\mathrm{gr}^1_{\bar{Q}}(\bar{\mathbf{O}})$ is mapped to the class of y_j in $\mathrm{gr}^1_{\mathbf{Q}}(\mathbf{R})$ by β^*, and to the class of y'_j in $\mathrm{gr}^1_{Q}(\mathbf{O})$ by $\beta^{*\prime}$ (cf. the paragraph preceding Proposition 1, § 1). By virtue of Lemma 5, the homomorphism α maps the ideal $\mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}))$ into the ideal $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$. Therefore combining the two diagrams above, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{J},\mathbf{R})) \otimes_{\bar{\mathbf{O}}/\bar{Q}} \mathrm{gr}_{\bar{Q}}(\bar{\mathbf{O}}) \arrow[r, "\psi''"] \arrow[d] & \mathrm{gr}_{\mathbf{Q}}(\mathbf{J},\mathbf{R}) \arrow[d] & \\
0 \arrow[r] & \mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) \otimes_{\bar{\mathbf{O}}/\bar{Q}} \mathrm{gr}_{\bar{Q}}(\bar{\mathbf{O}}) \arrow[r, "\psi", "\approx"'] \arrow[d] & \mathrm{gr}_{\mathbf{Q}}(\mathbf{R}) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{gr}^0_{Q}(\mathrm{gr}_{P}(\mathbf{O})) \otimes_{\bar{\mathbf{O}}/\bar{Q}} \mathrm{gr}_{\bar{Q}}(\bar{\mathbf{O}}) \arrow[r, "\psi'"] \arrow[d] & \mathrm{gr}_{Q}(\mathbf{O}) \arrow[r] \arrow[d] & 0\,. \\
 & 0 & 0 &
\end{tikzcd}$$

(with a 0 arrow into $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J},\mathbf{R})$ from above)

Here the two vertical sequences are exact by Lemma 5, and by the right exactness of tensor product. Moreover, it is easily seen that ψ is isomorphic and that ψ' is surjective. Therefore, it follows from the diagram that *ψ' is isomorphic if and only if ψ'' is surjective.*

Let γ be the canonical homomorphism of graded $\bar{\mathbf{O}}$-algebras:

$$\gamma\colon \mathrm{gr}_{\mathbf{P}}(\mathbf{R}) \longrightarrow \mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$$

(which induces the above homomorphism α). The ideal $\mathrm{gr}_{\mathbf{P}}(\mathbf{J},\mathbf{R})$ is mapped into the ideal $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J},\mathbf{R})$ by γ. As is easily seen, the image of ψ'' is a graded ideal in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ which is generated by the image of $\mathrm{gr}_{\mathbf{P}}(\mathbf{J},\mathbf{R})$ by γ. Therefore, *ψ' is isomorphic if and only if the image of* $\mathrm{gr}_{\mathbf{P}}(\mathbf{J},\mathbf{R})$ *by γ in* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ *generates the ideal* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J},\mathbf{R})$. We shall study this last condition more closely. Let f be any element of $\mathbf{J}$. Let $\nu = \nu_{\mathbf{P}}(f)$ and $\mu = \nu_{\mathbf{Q}}(f)$. It is clear that $\nu \leqq \mu$. If $\nu < \mu$, then the initial form Φ of f in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$ is such that $\deg\Phi = \nu$ and $\Phi \in \mathbf{Q}^{\mu-\nu}\,\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$. The image of Φ in $\mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R}))$ is then zero, hence, $\gamma(\Phi) = 0$. If $\nu = \mu$, then the image $\gamma(\Phi)$ of Φ is equal to the initial form of f in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$. (In fact, γ is induced by the inclusion of $\mathbf{P}^\nu$ into $\mathbf{Q}^\nu$). Thus the following lemma is verified:

LEMMA 8. *Notation and assumptions being as above,*

$$\psi'\colon\ \mathrm{gr}^0_{Q}(\mathrm{gr}_{P}(\mathbf{O})) \otimes_{\bar{\mathbf{O}}/\bar{Q}} \mathrm{gr}_{\bar{Q}}(\bar{\mathbf{O}}) \longrightarrow \mathrm{gr}_{Q}(\mathbf{O})$$

is an isomorphism of graded $\bar{\mathbf{O}}/\bar{Q}$ $(= \mathbf{O}/Q)$-algebras if and only if there exists a system of elements $(f_1, \cdots, f_m)$ of $\mathbf{J}$ such that

(1) $\nu_{\mathbf{Q}}(f_j) = \nu_{\mathbf{P}}(f_j)$ *for* $1 \leqq j \leqq m$, *and*

(2) *if φ_j denotes the initial form of f_j in* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$, *then* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J},\mathbf{R}) = (\varphi_j, \cdots, \varphi_m)\,\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$.

The following theorem includes Proposition 1 as well as its converse statement.

THEOREM 2.[42] *Let* $\mathbf{R}$ *be a regular local ring and* $\mathbf{J}$ *an ideal in* $\mathbf{R}$. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be prime ideals in* $\mathbf{R}$ *such that* $\mathbf{Q} \supseteqq \mathbf{P} \supseteqq \mathbf{J}$ *and that* $\mathbf{R}/\mathbf{P}$ and $\mathbf{R}/\mathbf{Q}$ *are both regular. Let* $\mathbf{O} = \mathbf{R}/\mathbf{J}$, $Q = \mathbf{Q}/\mathbf{J}$, *and* $P = \mathbf{P}/\mathbf{J}$. *Then the following conditions are equivalent to each other*:

(i) $\mathrm{gr}_P(\mathbf{O})$ *is flat over* $\mathbf{O}/P$.

(ii) $\mathrm{gr}_Q(\mathbf{O})$ *is flat over* $\mathbf{O}/Q$ *and the above homomorphism of graded* $\mathbf{O}/Q$*-algebras*:

$$\psi': \mathrm{gr}_Q^0\left(\mathrm{gr}_P(\mathbf{O})\right) \otimes_{\mathbf{O}/Q} \mathrm{gr}_{Q/P}(\mathbf{O}/P) \longrightarrow \mathrm{gr}_Q(\mathbf{O})$$

is isomorphic.

(iii) $\mathrm{gr}_Q(\mathbf{O})$ *is flat over* $\mathbf{O}/Q$, *and there exists a system of elements* $(f_1, \cdots, f_m)$ *of* $\mathbf{J}$ *such that*

(1) $\nu_{\mathbf{Q}}(f_j) = \nu_{\mathbf{P}}(f_j)$ *for* $1 \leqq j \leqq m$, *and*

(2) *if* φ_j *denotes the initial form of* f_j *in* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ *for* $1 \leqq j \leqq m$, *then* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) = (\varphi_1, \cdots, \varphi_m)\, \mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$.

(iv) $\mathrm{gr}_Q^0(\mathrm{gr}_P(\mathbf{O}))$ *is flat over* $\mathbf{O}/Q$, *and the canonical homomorphism of graded* $\mathbf{O}/Q$*-algebras*:

$$\varphi: \mathrm{gr}_Q^0\left(\mathrm{gr}_P(\mathbf{O})\right) \otimes_{\mathbf{O}/Q} \mathrm{gr}_{Q/P}(\mathbf{O}/P) \longrightarrow \mathrm{gr}_Q\left(\mathrm{gr}_P(\mathbf{O})\right)$$

is isomorphic.

PROOF. By Proposition 1 of § 1, (i) $\Rightarrow$ (ii). By Lemma 8, (ii) $\Leftrightarrow$ (iii). By corollary to Lemma 2 of § 1, (iv) $\Leftrightarrow$ (i). Thus it suffices to prove that (ii) and (iii) together imply (iv). First of all, the isomorphism ψ' of (ii) induces isomorphisms of $\mathbf{O}/Q$-modules :

$$\sum\nolimits_{p+q=n} \mathrm{gr}_Q^0\left(\mathrm{gr}_P^p(\mathbf{O})\right) \otimes_{\mathbf{O}/Q} \mathrm{gr}_{Q/P}^q(\mathbf{O}/P) \approx \mathrm{gr}_Q^n(\mathbf{O})$$

for all non-negative integers n, where $\sum$ signifies the direct sum. By the assumption that $\mathrm{gr}_Q(\mathbf{O})$ is flat over $\mathbf{O}/Q$, $\mathrm{gr}_Q^n(\mathbf{O})$ must be a free $(\mathbf{O}/Q)$-module, and hence the direct summands $\mathrm{gr}_Q^0(\mathrm{gr}_P^p(\mathbf{O})) \otimes_{\mathbf{O}/Q} \mathrm{gr}_{Q/P}^q(\mathbf{O}/P)$ are such also. (See Lemma 1 of § 1.) Note that $\mathrm{gr}_{Q/P}^q(\mathbf{O}/P)$ are free $(\mathbf{O}/Q)$-modules. Therefore $\mathrm{gr}_Q^0(\mathrm{gr}_P(\mathbf{O}))$ is flat over $\mathbf{O}/Q$.

We want to prove that φ is an isomorphism. We have the following commutative diagram of canonical homomorphisms of doubly graded $\mathbf{O}/Q$-algebras:

[42] The assumption that $\mathbf{O}$ is given as a homomorphic image of a regular local ring $\mathbf{R}$ is only necessary for (iii). The equivalence among the other three conditions holds without this assumption. (cf. Proposition 1, § 1.) In fact, we can easily reduce the proof of the equivalence (without the assumption) to the case in which $\mathbf{O}$ is complete, hence, it is a homomorphic image of a formal power series ring over a discrete valuation ring of rank one (or, rank zero).

$$\begin{tikzcd}
\mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{J},\mathbf{R}))\otimes_{\mathbf{O}/Q}\mathrm{gr}_{Q/P}(\mathbf{O}/P) \arrow[r, "\psi'"] \arrow[d] & \mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{J},\mathbf{R})) \arrow[d, "\delta"] \\
\mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R}))\otimes_{\mathbf{O}/Q}\mathrm{gr}_{Q/P}(\mathbf{O}/P) \arrow[r, "\approx"] \arrow[d] & \mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) \arrow[d, "\varepsilon"] \\
\mathrm{gr}^0_{Q}(\mathrm{gr}_{P}(\mathbf{O}))\otimes_{\mathbf{O}/Q}\mathrm{gr}_{Q/P}(\mathbf{O}/P) \arrow[r, "\psi"] \arrow[d] & \mathrm{gr}_{Q}(\mathrm{gr}_{P}(\mathbf{O})) \arrow[d] \\
0 & 0
\end{tikzcd}$$

where the left vertical sequence is exact, the middle horizontal arrow is isomorphic, and the other horizontal arrows φ and φ' are surjective. It is easy to see that if the right vertical sequence is proved exact, then φ is an isomorphism. To show that the right vertical sequence is exact, we consider the following commutative diagram for each of their homogeneous parts with double grades (p, q):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bar{Q}^{p+1}\mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R}) \arrow[r, "\delta''"] \arrow[d] & \bar{Q}^{p+1}\mathrm{gr}^q_{\mathbf{P}}(\mathbf{R}) \arrow[r, "\varepsilon''"] \arrow[d] & \bar{Q}^{p+1}\mathrm{gr}^q_{P}(\mathbf{O}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{Q}^{p}\mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R}) \arrow[r, "\delta'"] \arrow[d] & \bar{Q}^{p}\mathrm{gr}^q_{\mathbf{P}}(\mathbf{R}) \arrow[r, "\varepsilon'"] \arrow[d] & \bar{Q}^{p}\mathrm{gr}^q_{P}(\mathbf{O}) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{gr}^p_{\mathbf{Q}}(\mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R})) \arrow[r, "\delta"] \arrow[d] & \mathrm{gr}^p_{\mathbf{Q}}(\mathrm{gr}^q_{\mathbf{P}}(\mathbf{R})) \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{gr}^p_{Q}(\mathrm{gr}^q_{P}(\mathbf{O})) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the vertical sequences are all exact, δ' and δ'' are injective, ε' and ε'' are surjective. It is clear that ε is surjective. In order to prove $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(\delta)$ for this diagram, it suffices that $\mathrm{Ker}(\varepsilon') = \mathrm{Im}(\delta')$. The homomorphism ε' is induced by the canonical homomorphism of $\mathrm{gr}^q_{\mathbf{P}}(\mathbf{R})$ onto $\mathrm{gr}^q_{P}(\mathbf{O})$ whose kernel is equal to $\mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R})$, by Lemma 5. Hence $\mathrm{Ker}(\varepsilon') = \bar{Q}^p\,\mathrm{gr}^q_{\mathbf{P}}(\mathbf{R}) \cap \mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R})$. By virtue of Lemma 7, the assumption (iii) implies that $\bar{Q}^p\,\mathrm{gr}^q_{\mathbf{P}}(\mathbf{R}) \cap \mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R}) = \bar{Q}^p\,\mathrm{gr}^q_{\mathbf{P}}(\mathbf{J},\mathbf{R})$, which is $\mathrm{Im}(\delta')$. q.e.d.

COROLLARY 1. *The notation and assumptions being as in Theorem* 2, *assume that* $\mathbf{Q}$ *is the maximal ideal of* $\mathbf{R}$. *Then the following conditions are equivalent to each other*:

(i) $\mathrm{gr}_P(\mathbf{O})$ *is flat over* $\mathbf{O}/P$.

(ii) *The homomorphism of graded* $\mathbf{O}/Q$-*algebras*:

$$\psi': \mathrm{gr}_Q^0(\mathrm{gr}_P(\mathbf{O})) \otimes_{\mathbf{O}/Q} \mathrm{gr}_{Q/P}(\mathbf{O}/P) \longrightarrow \mathrm{gr}_Q(\mathbf{O})$$

is isomorphic.

(iii) *There exists a system of element* $(f_1, \cdots, f_m)$ *of* $\mathbf{J}$ *such that*

(1) $\nu_{\mathrm{Q}}(f_j) = \nu_{\mathrm{P}}(f_j)$ *for* $1 \leqq j \leqq m$, *and*

(2) *the initial forms of the* f_j *in* $\mathrm{gr}_{\mathrm{Q}}(\mathbf{R})$ *generate the ideal* $\mathrm{gr}_{\mathrm{Q}}(\mathbf{J}, \mathbf{R})$.

(iv) *The canonical homomorphism of graded* $\mathbf{O}/Q$-*algebras*:

$$\varphi: \mathrm{gr}_Q^0(\mathrm{gr}_P(\mathbf{O})) \otimes_{\mathbf{O}/Q} \mathrm{gr}_{Q/P}(\mathbf{O}/P) \longrightarrow \mathrm{gr}_Q(\mathrm{gr}_P(\mathbf{O}))$$

is isomorphic.

PROOF. The assertion follows Theorem 2 because every module is flat over a field. q.e.d.

COROLLARY 2. *Let V be an algebraic scheme which is imbedded in a non-singular algebraic scheme X. Let us suppose that the sheaf of ideals of V on X is invertible,* i.e., *locally everywhere generated by a single non-zero element. Let W be a non-singular irreducible subscheme of V. Then V is normally flat along W if and only if V is equi-multiple along W,* i.e., *V has a constant multiplicity at the points of W.*

PROOF. Let x be an arbitrary point of W and $\mathbf{O}$ the local ring of V at x. Then, by the assumption, we have a regular local ring $\mathbf{R}$ and an element f of $\mathbf{R}$ such that $\mathbf{O} = \mathbf{R}/(f)\mathbf{R}$. Let $\mathbf{P}$ be the prime ideal in $\mathbf{R}$ such that $\bar{\mathbf{P}} = \mathbf{P}/(f)\mathbf{R}$ is the ideal of W in $\mathbf{O}$. Let $\mathbf{M}$ be the maximal ideal of $\mathbf{R}$. By Corollary 1 of Theorem 2, $\mathrm{gr}_{\bar{\mathbf{P}}}(\mathbf{O})$ is flat over $\mathbf{O}/\bar{\mathbf{P}}$ (i.e., V is normally flat along W at x) if and only if $\nu_{\mathbf{M}}(f) = \nu_{\mathbf{P}}(f)$. This proves the assertion of Corollary 2 because $\nu_{\mathbf{M}}(f)$ (resp. $\nu_{\mathbf{P}}(f)$) is equal to the multiplicity of V at x (resp. at the generic point of W). Here it should be remarked that $\nu_{\mathbf{P}}(f) = \nu_{\mathbf{PR_P}}(f)$ because of the regularity of $\mathbf{R}/\mathbf{P}$. q.e.d.

3. Normal flatness along an ascending chain of subschemes

Let $\mathbf{A}$ be a noetherian ring and M an $\mathbf{A}$-module. We say that M is *faithfully flat* over $\mathbf{A}$ (or, *a faithfully flat* $\mathbf{A}$-*module*) if one (hence all) of the following conditions (equivalent to each other) is satisfied:

(i) M is flat over $\mathbf{A}$ and, for any $\mathbf{A}$-module N, $M \otimes_{\mathbf{A}} N = 0$ implies $N = 0$.

(ii) M is flat over $\mathbf{A}$ and $M \otimes_{\mathbf{A}} \mathbf{A}/\mathbf{m} \neq 0$ for every maximal ideal $\mathbf{m}$ of $\mathbf{A}$.

(iii) For any sequence of homomorphisms of $\mathbf{A}$-modules: $N' \to N \to N''$, it is exact if and only if $N' \otimes_{\mathbf{A}} M \to N \otimes_{\mathbf{A}} M \to N'' \otimes_{\mathbf{A}} M$ is exact. (See Proposition 2.2 of the Seminar note No. IV by A. Grothendieck at l'Institut des Hautes Études Scientifiques.) It follows immediately that: Let $\mathbf{A} \to \mathbf{B}$ be a local homomorphism of local rings and M a $\mathbf{B}$-module of finite type. Then M is faithfully flat over $\mathbf{A}$ if and only if it is flat over $\mathbf{A}$ and different from zero.

LEMMA 9. *Let $\mathbf{A} \to \mathbf{A}'$ be a homomorphism of noetherian rings. Let $\mathbf{I}$ be an ideal in $\mathbf{A}$ and $\mathbf{I}' = \mathbf{I}\mathbf{A}'$. Suppose $\mathbf{A}'$ be faithfully flat over $\mathbf{A}$. Then, for an $\mathbf{A}$-module M of finite type, $\mathrm{gr}_{\mathbf{I}}(M)$ is flat over $\mathbf{A}/\mathbf{I}$ if and only if $\mathrm{gr}_{\mathbf{I}'}(M \otimes_{\mathbf{A}} \mathbf{A}')$ is flat over $\mathbf{A}'/\mathbf{I}'$.*

PROOF. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbf{I}^{n+1}M) \otimes_{\mathbf{A}} \mathbf{A}' & \longrightarrow & (\mathbf{I}^{n}M) \otimes_{\mathbf{A}} \mathbf{A}' & \longrightarrow & \mathrm{gr}^n_{\mathbf{I}}(M) \otimes_{\mathbf{A}} \mathbf{A}' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_{n+1}} & & \downarrow{\scriptstyle \lambda_n} & & \downarrow{\scriptstyle \varepsilon_n} & & \\
0 & \longrightarrow & \mathbf{I}^{n+1}(M \otimes_{\mathbf{A}} \mathbf{A}') & \longrightarrow & \mathbf{I}^{n}(M \otimes_{\mathbf{A}} \mathbf{A}') & \longrightarrow & \mathrm{gr}^n_{\mathbf{I}'}(M \otimes_{\mathbf{A}} \mathbf{A}') & \longrightarrow & 0
\end{array}$$

Since $\mathbf{A}'$ is flat over $\mathbf{A}$, the first horizontal sequence is exact. The second sequence is clearly exact. (Note that $\mathbf{I}^n(M \otimes_{\mathbf{A}} \mathbf{A}') = \mathbf{I}'^n(M \otimes_{\mathbf{A}} \mathbf{A}')$ for all n.) It is clear that λ_n is surjective for all n, hence, so is ε_n. Therefore the above diagram shows that, if λ_n is bijective, then λ_{n+1} and ε_n are bijective. Since λ_0 is bijective, λ_n and ε_n can be proved bijective by induction on n. Thus we have an isomorphism of $\mathbf{A}'$-modules:

$$\varepsilon_n\colon\ \mathrm{gr}^n_{\mathbf{I}}(M) \otimes_{\mathbf{A}} \mathbf{A}' \longrightarrow \mathrm{gr}^n_{\mathbf{I}'}(M \otimes_{\mathbf{A}} \mathbf{A}')$$

for every non-negative integer n. Now, since $\mathbf{A}'$ is faithfully flat over $\mathbf{A}$, so is $\mathbf{A}'/\mathbf{I}'$ over $\mathbf{A}/\mathbf{I}$. Hence $\mathrm{gr}^n_{\mathbf{I}}(M)$ is flat over $\mathbf{A}/\mathbf{I}$ if and only if $\mathrm{gr}^n_{\mathbf{I}}(M) \otimes_{\mathbf{A}/\mathbf{I}} \mathbf{A}'/\mathbf{I}'$ $(= \mathrm{gr}^n_{\mathbf{I}}(\mathbf{A}) \otimes_{\mathbf{A}} \mathbf{A}')$ is flat over $\mathbf{A}'/\mathbf{I}'$. In view of the isomorphism ε_n, $\mathrm{gr}^n_{\mathbf{I}}(M)$ is flat over $\mathbf{A}/\mathbf{I}$ if and only if $\mathrm{gr}^n_{\mathbf{I}'}(M \otimes_{\mathbf{A}} \mathbf{A}')$ is flat over $\mathbf{A}'/\mathbf{I}'$. q.e.d.

COROLLARY. *Let $\mathbf{O}$ be a local ring. Let P be an ideal in $\mathbf{O}$ and Q a prime ideal in $\mathbf{O}$ such that $Q \supset P$ and $\mathbf{O}/Q$ is regular. Let $\hat{\mathbf{O}}$ be the completion of $\mathbf{O}$, $\hat{P} = P\hat{\mathbf{O}}$ and $\hat{Q} = Q\hat{\mathbf{O}}$. Then $\hat{Q}$ is a prime ideal in $\hat{\mathbf{O}}$, and $\mathrm{gr}_{P\mathbf{O}_Q}(\mathbf{O}_Q)$ is flat over $\mathbf{O}_Q/P\mathbf{O}_Q$ if and only if $\mathrm{gr}_{\hat{P}\hat{\mathbf{O}}_{\hat{Q}}}(\hat{\mathbf{O}}_{\hat{Q}})$ is flat over $\hat{\mathbf{O}}_{\hat{Q}}/\hat{P}\hat{\mathbf{O}}_{\hat{Q}}$.*

PROOF. Since $\hat{\mathbf{O}}/\hat{Q}$ is isomorphic to the completion of $\mathbf{O}/Q$ which is regular, $\hat{\mathbf{O}}/\hat{Q}$ is regular. In particular, $\hat{Q}$ is a prime ideal in $\hat{\mathbf{O}}$. We know that $\hat{\mathbf{O}}$ is flat over $\mathbf{O}$, hence, $\hat{\mathbf{O}}_{\hat{Q}}$ is flat over $\mathbf{O}_Q$. But $\mathbf{O}_Q \to \hat{\mathbf{O}}_{\hat{Q}}$ is a local homomorphism and therefore $\hat{\mathbf{O}}_{\hat{Q}}$ is faithfully flat over $\mathbf{O}_Q$. Now the assertion is clear by Lemma 9. q.e.d.

Let $\mathbf{R}$ be a regular local ring, and $\mathbf{J}$ an ideal in $\mathbf{R}$. Let $\mathbf{P}$ and $\mathbf{Q}$ be prime ideals in $\mathbf{R}$ such that $\mathbf{Q} \supset \mathbf{P} \supseteqq \mathbf{J}$ and that $\mathbf{R}/\mathbf{P}$ and $\mathbf{R}/\mathbf{Q}$ are both regular. Then we have the following canonical isomorphism of doubly graded $\mathbf{R}/\mathbf{Q}$-algebras:

$$\varphi\colon \mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) \otimes_{\mathbf{R}/\mathbf{Q}} \mathrm{gr}_{\mathbf{Q}/\mathbf{P}}(\mathbf{R}/\mathbf{P}) \xrightarrow{\approx} \mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R}))\,.$$

Moreover if we choose a system of elements $(y_1, \cdots, y_s)$ of $\mathbf{R}$ such that their residue classes $(\bar{y}_1, \cdots, \bar{y}_s)$ in $\mathbf{R}/\mathbf{P}$ form a minimal base of $\mathbf{Q}/\mathbf{P}$, then it determines an isomorphism of simply graded $\mathbf{R}/\mathbf{Q}$-algebras:

$$\psi\colon \mathrm{gr}^0_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) \otimes_{\mathbf{R}/\mathbf{Q}} \mathrm{gr}_{\mathbf{Q}/\mathbf{P}}(\mathbf{R}/\mathbf{P}) \xrightarrow{\approx} \mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$$

such that the class of $\bar{y}_j$ in $\mathrm{gr}^1_{\mathbf{Q}/\mathbf{P}}(\mathbf{R}/\mathbf{P})$ is mapped to that of y_j in $\mathrm{gr}^1_{\mathbf{Q}}(\mathbf{R})$. By means of these two isomorphisms, we get an isomorphism of simply graded $\mathbf{R}/\mathbf{Q}$-algebras:

$$\lambda\colon \mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) \xrightarrow{\approx} \mathrm{gr}_{\mathbf{Q}}(\mathbf{R})\,.$$

Obviously, this isomorphism λ depends upon the choice of the system $(y_1, \cdots, y_s)$.

Let Φ be a form in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, say $\deg \Phi = \nu$. Then by $\nu_{\mathbf{Q}/\mathbf{P}}(\Phi)$ we shall denote the maximal integer n such that $\Phi \in \mathbf{Q}^n \mathrm{gr}^\nu_{\mathbf{P}}(\mathbf{R}) = (\mathbf{Q}/\mathbf{P})^n \mathrm{gr}^\nu_{\mathbf{P}}(\mathbf{R})$. Let $n = \nu_{\mathbf{Q}/\mathbf{P}}(\Phi)$. Then the class Φ_0 of Φ in $\mathrm{gr}^n_{\mathbf{Q}}(\mathrm{gr}^\nu_{\mathbf{P}}(\mathbf{R})) = \mathbf{Q}^n \mathrm{gr}^\nu_{\mathbf{P}}(\mathbf{R})/\mathbf{Q}^{n+1}\mathrm{gr}^\nu_{\mathbf{P}}(\mathbf{R})$ is a form of double grade (ν, n). We shall call Φ_0 *the initial form of* Φ *in* $\mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R}))$. If g is an element of $\mathbf{R}$, and if Φ is the initial form of g in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, then Φ_0 will be called *the initial form of g in* $\mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R}))$.

LEMMA 10. *Notation and assumptions being as above, let* $\mathbf{O} = \mathbf{R}/\mathbf{J}$, $P = \mathbf{P}/\mathbf{J}$, *and* $Q = \mathbf{Q}/\mathbf{J}$. *Let us fix an isomorphism* $\lambda\colon \mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) \to \mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ *as above. Suppose* $\mathrm{gr}_{P\mathbf{O}_Q}(\mathbf{O}_Q)$ *be flat over* $\mathbf{O}_Q/P\mathbf{O}_Q$. *Then for every element* $g \in \mathbf{J}$, *the λ-image of the initial form of g in* $\mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R}))$ *is contained in* $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}}$.

PROOF. Since gr commutes with localization, we may assume that $\mathbf{Q}$ is the maximal ideal of $\mathbf{R}$. Then, by the assumption that $\mathrm{gr}_P(\mathbf{O})$ is flat over $\mathbf{O}/P$, λ induces an isomorphism of graded $\mathbf{O}/Q$ $(= \mathbf{R}/\mathbf{Q})$-algebras:

$$\lambda'\colon \mathrm{gr}_Q(\mathrm{gr}_P(\mathbf{O})) \longrightarrow \mathrm{gr}_Q(\mathbf{O})\,,$$

and we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathrm{gr}_{\mathbf{Q}}((\mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R})) & \xrightarrow{\delta} & \mathrm{gr}_{\mathbf{Q}}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) & \xrightarrow{\varepsilon} & \mathrm{gr}_Q(\mathrm{gr}_P(\mathbf{O})) & \longrightarrow & 0 \\
 & & & & \wr\Big\downarrow\lambda & & \wr\Big\downarrow\lambda' & & \\
0 & \longrightarrow & \mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) & \longrightarrow & \mathrm{gr}_{\mathbf{Q}}(\mathbf{R}) & \longrightarrow & \mathrm{gr}_Q(\mathbf{O}) & \longrightarrow & 0
\end{array}$$

where the second horizontal sequence is clearly exact. Moreover, by Lemma 7,[43] if Φ is a form in $\mathrm{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R})$, and if $\Phi \in \mathbf{Q}^n \mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, then

[43] The flatness of $\mathrm{gr}_P(\mathbf{O})$ over $\mathbf{O}/P$ implies the existence of $(f_1, \cdots, f_m)$ in the assumption of Lemma 7. (cf. Cor. 1 of Th. 2, § 2.)

$\Phi \in \mathbf{Q}^n \operatorname{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R})$. Hence, if g is an element of $\mathbf{J}$, the initial form of g in $\operatorname{gr}_{\mathbf{Q}}(\operatorname{gr}_{\mathbf{P}}(\mathbf{R}))$ is in the image of δ. It is clear from the diagram that

$$\lambda\delta(\operatorname{gr}_{\mathbf{Q}}(\operatorname{gr}_{\mathbf{P}}(\mathbf{J}, \mathbf{R}))) \subseteqq \operatorname{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) . \qquad \text{q.e.d.}$$

LEMMA 11. *Let* $\mathbf{R}$, $\mathbf{J}$, $\mathbf{Q}$, $\mathbf{O}$ *and* Q *be the same as in Lemma* 10. *If* $\operatorname{gr}_{\mathbf{Q}}(\mathbf{O})$ *is flat over* $\mathbf{O}/Q$, *then we have*

$$\operatorname{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}} \cap \operatorname{gr}_{\mathbf{Q}}(\mathbf{R}) = \operatorname{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) .$$

PROOF. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) & \longrightarrow & \operatorname{gr}_{\mathbf{Q}}(\mathbf{R}) & \longrightarrow & \operatorname{gr}_{Q}(\mathbf{O}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}} & \longrightarrow & \operatorname{gr}_{\mathbf{Q}}(\mathbf{R})_{\mathbf{Q}} & \longrightarrow & \operatorname{gr}_{Q}(\mathbf{O})_{Q} & \longrightarrow & 0 \end{array}$$

where horizontal sequences are exact and the first two vertical arrows are injections. Since $\operatorname{gr}_{Q}(\mathbf{O})$ is flat over $\mathbf{O}/Q$, the last vertical arrow is also injective. The assertion in the lemma follows. q.e.d.

THEOREM 3. *Let* $\mathbf{O}$ *be a local ring. Let* P *and* Q *be prime ideals in* $\mathbf{O}$ *such that* $Q \supset P$ *and that* $\mathbf{O}/P$ *and* $\mathbf{O}/Q$ *are both regular. Then the following two conditions are equivalent to each other*:

(i) $\operatorname{gr}_{\mathbf{P}}(\mathbf{O})$ *is flat over* $\mathbf{O}/P$

(ii) $\operatorname{gr}_{\mathbf{Q}}(\mathbf{O})$ *is flat over* $\mathbf{O}/Q$ *and* $\operatorname{gr}_{P\mathbf{O}_Q}(\mathbf{O}_Q)$ *is flat over* $\mathbf{O}_Q/P\mathbf{O}_Q$.

PROOF. (i) $\Rightarrow$ (ii) is clear from Proposition 1 and the fact that $\operatorname{gr}_{P\mathbf{O}_Q}(\mathbf{O}_Q) = \operatorname{gr}_P(\mathbf{O})_Q$. We shall prove (ii) $\Rightarrow$ (i). By Corollary of Lemma 9, we may assume that $\mathbf{O}$ is complete. By Cohen's structure theorem of a complete local ring, $\mathbf{O}$ is a quotient ring $\mathbf{R}/\mathbf{J}$ of a complete regular local ring $\mathbf{R}$ by an ideal $\mathbf{J}$ in $\mathbf{R}$. We shall fix such $\mathbf{R}$ and $\mathbf{J}$. Then we have prime ideals $\mathbf{P}$ and $\mathbf{Q}$ in $\mathbf{R}$ such that $P = \mathbf{P}/\mathbf{J}$ and $Q = \mathbf{Q}/\mathbf{J}$. We take and fix a system of elements $(z_1, \cdots, z_r, y_1, \cdots, y_s)$ of $\mathbf{R}$ such that $\mathbf{P} = (z_1, \cdots, z_r)\mathbf{R}$ and $\mathbf{Q} = (z_1, \cdots, z_r, y_1, \cdots, y_s)\mathbf{R}$ where $r = \dim \mathbf{R}_{\mathbf{P}}$ and $r + s = \dim \mathbf{R}_{\mathbf{Q}}$. We also take and fix the following identifications:

$$\operatorname{gr}_{\mathbf{P}}(\mathbf{R}) = \mathbf{R}'[Z_1, \cdots, Z_r]$$

where $\mathbf{R}' = \mathbf{R}/\mathbf{P} = \mathbf{O}/P$ and Z_j corresponds to the class of z_j $(1 \leqq j \leqq r)$ in $\operatorname{gr}^1_{\mathbf{P}}(\mathbf{R})$,

$$\operatorname{gr}_{\mathbf{Q}}(\mathbf{R}) = \mathbf{R}''[Z_1, \cdots, Z_r, Y_1, \cdots, Y_s]$$

where $\mathbf{R}'' = \mathbf{R}/\mathbf{Q} = \mathbf{O}/Q$ and Z_i corresponds to the class of z_i $(1 \leqq i \leqq r)$ in $\operatorname{gr}^1_{\mathbf{Q}}(\mathbf{R})$, while Y_j corresponds to that of y_j $(1 \leqq j \leqq s)$ in $\operatorname{gr}^1_{\mathbf{Q}}(\mathbf{R})$, and

$$\begin{aligned} \operatorname{gr}_{\mathbf{Q}}(\operatorname{gr}_{\mathbf{P}}(\mathbf{R})) &= \operatorname{gr}^0_{\mathbf{Q}}(\operatorname{gr}_{\mathbf{P}}(\mathbf{R})) \otimes_{\mathbf{R}''} \operatorname{gr}_{\mathbf{Q}'}(\mathbf{R}') \\ &= \mathbf{R}''[Z_1, \cdots, Z_r, Y_1, \cdots, Y_s] \end{aligned}$$

where $\mathbf{Q}' = \mathbf{Q}/\mathbf{P} = Q/P$ and, if y'_j denotes the residue class of y_j in $\mathbf{R}'$, then Y_j corresponds to the class of y'_j in $\mathrm{gr}^1_{\mathbf{Q}'}(\mathbf{R}')$. (Recall the isomorphisms φ, ψ and λ in the paragraph preceding Lemma 10.) Let $\mathbf{K}$ denote the field of quotients of $\mathbf{R}''$. Then we get in a natural way the identification $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})_{\mathbf{Q}} = \mathrm{gr}_{\mathbf{QR_Q}}(\mathbf{R_Q}) = \mathbf{K}[Z_1, \cdots, Z_r, Y_1, \cdots, Y_s]$. Applying Corollary 1 of Theorem 2, § 2, to $\{\mathbf{R_Q}, \mathbf{JR_Q}, \mathbf{PR_Q}, \text{ and } \mathbf{O}_Q = \mathbf{R_Q}/\mathbf{JR_Q}\}$, we get a system of elements h_j of $\mathbf{JR_Q}$ such that $\nu_{\mathbf{PR_Q}}(h_j) = \nu_{\mathbf{QR_Q}}(h_j)$ for all j and the initial forms of h_j in $\mathrm{gr}_{\mathbf{QR_Q}}(\mathbf{R_Q}) = \mathbf{K}[Z, Y]$ generate the ideal $\mathrm{gr}_{\mathbf{QR_Q}}(\mathbf{JR_Q}, \mathbf{R_Q}) = \mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}}$. The initial forms of such h_j in $\mathbf{K}[Z, Y]$ are obviously forms only in $Z_1, \cdots, Z_r$. Thus the ideal $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}}$ in $\mathbf{K}[Z, Y]$ is generated by forms only in $Z_1, \cdots, Z_r$. This implies that, if ψ is any element of $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$, and if ψ is written in the form $\sum_i \psi_i N_i$ with $\psi_i \in \mathbf{R}''[Z]$ and with distinct monomials N_i in the Y_j $(1 \leqq j \leqq s)$, then all the ψ_i belong to $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}} = \mathbf{K}[Z]$, because $\mathbf{K}[Z, Y]$ is a free $\mathbf{K}[Z]$-module generated by all the distinct monomials in the Y_j $(1 \leqq j \leqq s)$. By Lemma 11, $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}} \cap \mathbf{R}''[Z, Y] = \mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$, and therefore $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}} \cap \mathbf{R}''[Z] = \mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) \cap \mathbf{R}''[Z]$. It follows that all the ψ_i belong to $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) \cap \mathbf{R}''[Z]$. Thus we conclude that $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$ is generated by forms only in $Z_1, \cdots, Z_r$. We take a system of elements $(\varphi_1, \cdots, \varphi_m)$ in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$ such that

(a) all the φ_j $(1 \leqq j \leqq m)$ in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ $(= \mathbf{R}''[Z, Y])$ are forms only in $Z_1, \cdots, Z_r$,

(b) $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R}) = (\varphi_1, \cdots, \varphi_m)\, \mathbf{R}''[Z, Y]$, and

(c) if $\nu_j = \deg \varphi_j$, then $\nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_m$.

We claim that for each j there exists an element $f_j \in \mathbf{J}$ such that φ_j is the initial form of f_j in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$, and $\nu_{\mathbf{Q}}(f_j) = \nu_{\mathbf{P}}(f_j)$. If this is done, then the conclusion (i) follows the equivalence of (i) and (iii) of Theorem 2. Now let i be the largest integer $(\geqq 1)$ such that there exists $f_j \in \mathbf{J}$ such that $\nu_{\mathbf{P}}(f_j) = \nu_{\mathbf{Q}}(f_j)$, and that $\varphi_j =$ the initial form of f_j in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ for every $j < i$. Suppose $i \leqq m$. This will lead to a contradiction. Now let d be the largest integer such that there exists $g \in \mathbf{J}$ with $\nu_{\mathbf{P}}(g) = d$ and with φ_i as its initial form in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$. We must have $d < \nu_i$. Let Ψ be the initial form in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$ of an element $g \in \mathbf{J}$ with the above property. Let μ be the largest integer such that $\Psi \in \mathbf{Q}^{\mu}\,\mathrm{gr}^d_{\mathbf{P}}(\mathbf{R})$. Since the initial form φ_i of g in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ is a form only in $Z_1, \cdots, Z_r$ of degree ν_i, we have $g \in \mathbf{P}^{\nu_i} + \mathbf{Q}^{\nu_i+1}$, and hence $d + \mu > \nu_i$. We claim that there exists $g_1 \in \mathbf{J}$ such that:

(1) $\varphi_i =$ the initial form of g_1 in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$;

(2) $\nu_{\mathbf{P}}(g_1) = d$, and if Ψ_1 is the initial form of g_1 in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, then $\Psi_1 \in \mathbf{Q}^{\mu_1}\,\mathrm{gr}^d_{\mathbf{P}}(\mathbf{R})$ with $\mu_1 > \mu$, and

(3) $g - g_1 \in \mathbf{Q}^{d+\mu}$.

To prove this, let ψ be the initial form of Ψ in $\mathrm{gr}_{\mathbf{Q}'}(\mathrm{gr}_{\mathbf{P}}(\mathbf{R})) = \mathbf{R}''[Z, Y]$, which is homogeneous in $Z_1, \cdots, Z_r$ of degree d and in $Y_1, \cdots, Y_s$ of degree μ. Let us write

$$\psi = \sum_\alpha T_\alpha(Y)\, \psi_\alpha(Z)$$

where $T_\alpha(Y)$ are distinct polynomials in $Y_1, \cdots, Y_s$ of degree μ, and $\psi_\alpha(Z)$ are forms only in $Z_1, \cdots, Z_s$ of degree d. By Lemma 10 we have $\psi \in \mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})_{\mathbf{Q}}$, and therefore, by Lemma 11, we have $\psi \in \mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$. Since $\mathrm{gr}_{\mathbf{Q}}(\mathbf{J}, \mathbf{R})$ is generated by $\varphi_1, \cdots, \varphi_m$ which are forms only in $Z_1, \cdots, Z_r$, we must have

$$\psi_\alpha \in (\varphi_1, \cdots, \varphi_m)\mathbf{R}''[Z] \qquad \text{for all } \alpha .$$

But $\deg \psi_\alpha = d < \nu_i \leqq \deg \varphi_j$ for $j \geqq i$, and therefore we must have

$$\psi_\alpha \in (\varphi_1, \cdots, \varphi_{i-1})\mathbf{R}''[Z] \qquad \text{for all } \alpha .$$

Let us write

$$\psi_\alpha = \sum_\beta \lambda_{\alpha,\beta}\varphi_\beta \qquad (\beta < i)$$

where $\lambda_{\alpha,\beta}$ is a form in $\mathbf{R}''[Z]$ of degree $d - \nu_\beta$ for all α and β. Let us choose $h_{\alpha,\beta} \in (z_1, \cdots, z_r)^{d-\nu_\beta}\mathbf{R}$ whose initial form in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R}) = \mathbf{R}''[Z, Y]$ is equal to $\lambda_{\alpha,\beta}$. Let:

$$h = \sum_\alpha \left(T_\alpha(y) \sum_\beta h_{\alpha,\beta} f_\beta\right)$$

where $T_\alpha(y)$ denotes the monomial in $y_1, \cdots, y_s$ which has y_j in the place of Y_j of $T_\alpha(Y)$. Let $g_1 = g - h$. Clearly $\nu_{\mathbf{Q}}(h) \geqq \mu + d > \nu_i$, and therefore the initial form of g_1 in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ is equal to that of g which is φ_i. Also it is clear that $\nu_{\mathbf{P}}(h) \geqq d$ because $f_\beta \in \mathbf{P}^{\nu_\beta}$ for $\beta < i$. Hence $\nu_{\mathbf{P}}(g_1) \geqq d$. But, by the assumption on the integer d, we must have $\nu_{\mathbf{P}}(g_1) = d$. Let Ψ_1 be the initial form of g_1 in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$. Then it is clear from the above expression of h, that we must have $\Psi_1 \in \mathbf{Q}^{\mu+1}\mathrm{gr}^d_{\mathbf{P}}(\mathbf{R})$. It is also clear that $g - g_1 = h \in \mathbf{Q}^{d+\mu}$. Thus we have obtained g_1 with the properties (1), (2), and (3). If μ_1 is the maximal integer such that $\Psi_1 \in \mathbf{Q}^{\mu_1}\mathrm{gp}^d_{\mathbf{P}}(\mathbf{R})$, then by applying the same process to g_1 as it was done to g, we get $g_2 \in \mathbf{J}$ such that

(1_1) $\varphi_i =$ the initial form of g_2 in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$.

(2_1) $\nu_{\mathbf{P}}(g_2) = d$ and, if Ψ_2 is the initial form of g_2 in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, then $\Psi_2 \in \mathbf{Q}^{\mu_2}\mathrm{gr}^d_{\mathbf{P}}(\mathbf{R})$ with $\mu_2 > \mu_1$, and

(3_1) $g_1 - g_2 \in \mathbf{Q}^{d+\mu_1}$.

By repeating the process, we get a convergent sequence of elements $\{g_k\}$ in $\mathbf{J}$ such that $\varphi_i =$ the initial form of g_k in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$, $\nu_{\mathbf{P}}(g_k) = d$ and, if Ψ_k is the initial form of g_k in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$, then $\Psi_k \in \mathbf{Q}^{\mu_k}\mathrm{gr}^d_{\mathbf{P}}(\mathbf{R})$ where the sequence μ_k is monotone increasing. By the completeness of $\mathbf{R}$, we have $f_i = \lim_{k \to \infty} g_k$ which is in $\mathbf{J}$. Clearly φ_i must be the initial form of f_i in $\mathrm{gr}_{\mathbf{Q}}(\mathbf{R})$ and $\nu_{\mathbf{P}}(f_i) \geqq d$. But, by the assumption on d, we must have $\nu_{\mathbf{P}}(f_i) = d$ and, on the other hand, since $\Psi_i \in \mathbf{Q}^{\mu_k} \mathrm{gr}_{\mathbf{P}}^d(\mathbf{R})$ and $\mu_{k+1} > \mu_k$ for all k, the initial form of f_i in $\mathrm{gr}_{\mathbf{P}}(\mathbf{R})$ must be in $\bigcap_k \mathbf{Q}^{\mu_k} \mathrm{gr}_{\mathbf{P}}^d(\mathbf{R}) = (0)$. This is a contradiction. q.e.d.

COROLLARY. *Let V be an algebraic scheme, W a subscheme of V, and D a non-singular irreducible subscheme of W. Suppose that both V and W are normally flat along D, and that there exists at least one simple point of W which is contained in D and at which V is normally flat along W. Then W has no multiple points in D and V is normally flat along W at every point of D.*

PROOF. It follows from Corollary 4 of Proposition 1, § 1, that D contains no multiple points of W. By the assumption that V is normally flat along W at some point of D, V is normally flat along W at the generic point of D. (See Theorem 1, § 1.) Since D is non-singular and W has no multiple points in D, it is sufficient, for the normal flatness of V along W at every point of D, that V is normally flat along D. (See Theorem 3.) q.e.d.