ALGORITHMEN UND DATENSTRUKTUREN

ÜBUNG 13: DAS PROZESSPROBLEM

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Algebraisches Pfadproblem

ABSTRAKTION: ALGEBRAISCHES PFADPROBLEM

- **bisher:** kürzeste Wege
  - ▷ Summation $+$ entlang der Pfade
  - ▷ Minimum min über alle Pfade

- **jetzt:** Verallgemeinerung
  - ▷ Pfadoperation $\odot$ entlang der Pfade
  - ▷ Akkumulationsoperation $\oplus$

- **Ergebnis:** allgemeine algebraische Struktur — Semiring $(S, \oplus, \odot, \mathbf{0}, \mathbf{1})$

	Werte S	$\oplus$	$\odot$	$\mathbf{0}$	$\mathbf{1}$
kürzeste Wegeproblem	$\mathbb{R}_{\geq 0}^{\infty}$	min	$+$	∞	0
Kapazitätsproblem	$\mathbb{N}_{\infty}$	max	min	0	∞
Erreichbarkeitsproblem	$\{\text{true}, \text{false}\}$	$\vee$	$\wedge$	false	true
Zuverlässigkeitsproblem	$[0, 1]$	max	$\cdot$	0	1
Prozessproblem	$\mathcal{P}(\Sigma^*)$	$\cup$	$\circ$	$\varnothing$	$\{\varepsilon\}$

- modifizierte Adjazenzmatrix

$$mA_G = \begin{cases} A_G(u,v) & \text{wenn } u \neq v \\ A_G(u,v) \oplus \mathbf{1} & \text{wenn } u = v \end{cases}$$
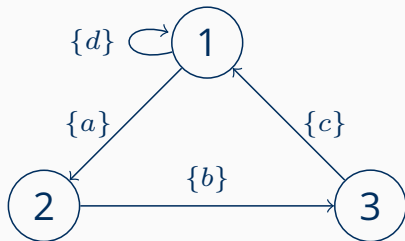
- **Initialisierung**: $D_G^{(0)} = mA_G$
- **Rekursion**:

$$\begin{aligned} & D_G^{(k+1)}(u,v) \\ = & D_G^{(k)}(u,v) \oplus \left(D_G^{(k)}(u,k+1) \odot (D_G^{(k)}(k+1,k+1))^* \odot D_G^{(k)}(k+1,v) \right) \end{aligned}$$
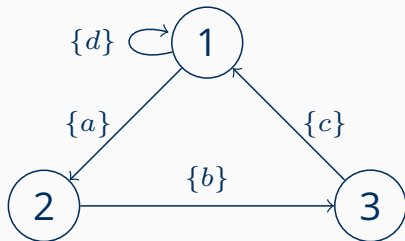
- vgl. dazu Floyd-Warshall:

$$\begin{aligned} & D_G^{(k+1)}(u,v) \\ = & \min \left\{ D_G^{(k)}(u,v), D_G^{(k)}(u,k+1) + D_G^{(k)}(k+1,v) \right\} \end{aligned}$$

Teil (a) :
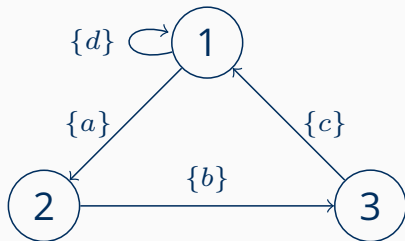$$(S, \oplus, \odot, \mathbf{0}, \mathbf{1}) =$$
$$(\mathcal{P}(\Sigma^*), \cup, \circ, \varnothing, \{\varepsilon\})$$

Teil (a) :
$$(S, \oplus, \odot, \mathbf{0}, \mathbf{1}) =$$
$$(\mathcal{P}(\Sigma^*), \cup, \circ, \varnothing, \{\varepsilon\})$$

Update-Formel: $D_G^{(k+1)}(u, v)$

$$= D_G^{(k)}(u, v) \oplus \left(D_G^{(k)}(u, k+1) \odot (D_G^{(k)}(k+1, k+1))^* \odot D_G^{(k)}(k+1, v) \right)$$

$$= D_G^{(k)}(u, v) \cup \left(D_G^{(k)}(u, k+1) \circ (D_G^{(k)}(k+1, k+1))^* \circ D_G^{(k)}(k+1, v) \right)$$

$$= \quad \text{alt} \quad \cup \left(\quad \text{Zeile} \quad \circ \quad (\text{Diagonale})^* \quad \circ \quad \text{Spalte} \quad \right)$$

Teil (b) : $mA_G = D_G^{(0)} = \begin{pmatrix} \{\varepsilon, d\} & \{a\} & \varnothing \\ \varnothing & \{\varepsilon\} & \{b\} \\ \{c\} & \varnothing & \{\varepsilon\} \end{pmatrix}$

Teil (b) : $mA_G = D_G^{(0)} = \begin{pmatrix} \{\varepsilon, d\} & \{a\} & \varnothing \\ \varnothing & \{\varepsilon\} & \{b\} \\ \{c\} & \varnothing & \{\varepsilon\} \end{pmatrix}$

Teil (c) : $D_G^{(1)} = \begin{pmatrix} \{d\}^* & \{d\}^* \{a\} & \varnothing \\ \varnothing & \{\varepsilon\} & \{b\} \\ \{c\} \{d\}^* & \{c\} \{d\}^* \{a\} & \{\varepsilon\} \end{pmatrix}$

Teil (d) :

$$D_G^{(2)}(3,3) = D_G^{(1)}(3,3) \cup \left(D_G^{(1)}(3,2) \circ (D_G^{(1)}(2,2))^* \circ D_G^{(1)}(2,3) \right)$$

$$= \{\varepsilon\} \cup \left(\{c\} \{d\}^* \{a\} \circ \{\varepsilon\}^* \circ \{b\} \right)$$
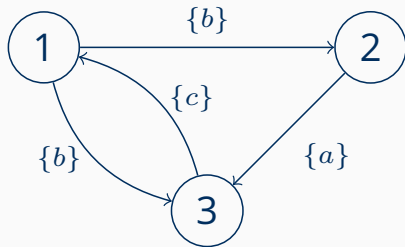
$$= \{\varepsilon\} \cup \left(\{c\} \{d\}^* \{ab\} \right)$$

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$$D_G^{(2)}(3,3) = D_G^{(1)}(3,3) \cup \left(D_G^{(1)}(3,2) \circ (D_G^{(1)}(2,2))^* \circ D_G^{(1)}(2,3) \right)$$
$$= \{\varepsilon\} \cup \left(\{c\}\{d\}^*\{a\} \circ \{\varepsilon\}^* \circ \{b\} \right)$$
$$= \{\varepsilon\} \cup \left(\{c\}\{d\}^*\{ab\} \right)$$

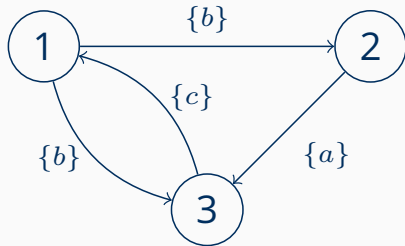$$D_G^{(3)}(3,3) = D_G^{(2)}(3,3) \cup \left(D_G^{(2)}(3,3) \circ (D_G^{(2)}(3,3))^* \circ D_G^{(2)}(3,3) \right)$$
$$= D_G^{(2)}(3,3) \cup \left(D_G^{(2)}(3,3) \right)^*$$
$$= \left(D_G^2(3,3) \right)^*$$
$$= \left(\{\varepsilon\} \cup \{c\}\{d\}^*\{ab\} \right)^*$$
$$= \left(\{c\}\{d\}^*\{ab\} \right)^*$$

Übungsblatt 13

Teil (a) :

$$mA_G = \begin{pmatrix} \{\varepsilon\} & \{b\} & \{b\} \\ \varnothing & \{\varepsilon\} & \{a\} \\ \{c\} & \varnothing & \{\varepsilon\} \end{pmatrix}$$

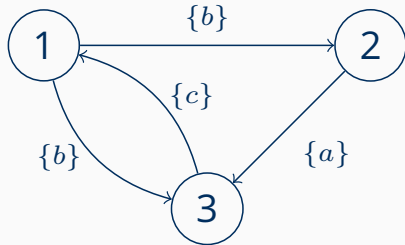**Teil (a)** :

$$mA_G = \begin{pmatrix} \{\varepsilon\} & \{b\} & \{b\} \\ \varnothing & \{\varepsilon\} & \{a\} \\ \{c\} & \varnothing & \{\varepsilon\} \end{pmatrix}$$

Teil (b):

$$D_G^{(1)} = \begin{pmatrix} \{\varepsilon\} & \{b\} & \{b\} \\ \varnothing & \{\varepsilon\} & \{a\} \\ \{c\} & \{cb\} & \{cb, \varepsilon\} \end{pmatrix} \qquad D_G^{(2)} = \begin{pmatrix} \{\varepsilon\} & \{b\} & \{b, ba\} \\ \varnothing & \{\varepsilon\} & \{a\} \\ \{c\} & \{cb\} & \{cb, cba, \varepsilon\} \end{pmatrix}$$

Teil (c):

$$
\begin{aligned}
D_G(3,1) &= D_G^{(3)}(3,1) \\
&= D_G^{(2)}(3,1) \cup \left\{ \; D_G^{(2)}(3,3) \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,1) \right\} \\
&= \quad \{c\} \quad \cup \left\{ \; \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \quad \{c\} \quad \right\} \\
&= \{cb, cba\}^* \cdot \{c\}
\end{aligned}
$$

Teil (c):

$$D_G(3,1) = D_G^{(3)}(3,1)$$

$$= D_G^{(2)}(3,1) \cup \left\{ \; D_G^{(2)}(3,3) \; \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,1) \; \right\}$$

$$= \quad \{c\} \quad \cup \left\{ \; \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \quad \{c\} \quad \right\}$$

$$= \{cb, cba\}^* \cdot \{c\}$$

$$D_G(3,2) = D_G^{(2)}(3,2) \cup \left\{ \; D_G^{(2)}(3,3) \; \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,2) \; \right\}$$

$$= \quad \{cb\} \quad \cup \left\{ \; \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \quad \{cb\} \quad \right\}$$

$$= \{cb, cba\}^* \cdot \{cb\}$$

Teil (c):

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$$= D_G^{(2)}(3,1) \cup \left\{ \; D_G^{(2)}(3,3) \; \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,1) \; \right\}$$
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$$D_G(3,2) = D_G^{(2)}(3,2) \cup \left\{ \quad D_G^{(2)}(3,3) \; \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,2) \right\}$$
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$$D_G(3,3) = D_G^{(2)}(3,3) \cup \left\{ D_G^{(2)}(3,3) \; \cdot (D_G^{(2)}(3,3))^* \cdot \; D_G^{(2)}(3,3) \; \right\}$$
$$= \{cb, cba, \varepsilon\} \cup \left\{ \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \{cb, cba, \varepsilon\} \right\}$$
$$= \{cb, cba\}^*$$

Teil (c):

$$D_G(3,1) = D_G^{(3)}(3,1)$$
$$= D_G^{(2)}(3,1) \cup \left\{ \ D_G^{(2)}(3,3) \ \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,1) \ \right\}$$
$$= \ \{c\} \ \cup \left\{ \ \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \ \{c\} \ \right\}$$
$$= \{cb, cba\}^* \cdot \{c\}$$

$$D_G(3,2) = D_G^{(2)}(3,2) \cup \left\{ \ D_G^{(2)}(3,3) \ \cdot (D_G^{(2)}(3,3))^* \cdot D_G^{(2)}(3,2) \ \right\}$$
$$= \ \{cb\} \ \cup \left\{ \ \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \ \{cb\} \ \right\}$$
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$$D_G(3,3) = D_G^{(2)}(3,3) \cup \left\{ \ D_G^{(2)}(3,3) \ \cdot (D_G^{(2)}(3,3))^* \cdot \ D_G^{(2)}(3,3) \ \right\}$$
$$= \{cb, cba, \varepsilon\} \cup \left\{ \{cb, cba, \varepsilon\} \cdot \{cb, cba, \varepsilon\}^* \cdot \{cb, cba, \varepsilon\} \right\}$$
$$= \{cb, cba\}^*$$

Teil (d): $D_G(3,3) = \{cb, cba, ba\}^* \rightsquigarrow (3, 3, \{cb, cba, ba\}^*)$