Strong convexity and Lipschitz continuity of gradients

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Firmly non-expansive operators

Let T be an operator (like a gradient) that is necessarily single-valued. Then it is **firmly non-expansive** if (Defn 4.1 [BC11])

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \le \|x - y\|^2 \tag{1}$$

Proposition 1 (Prop 4.2 [BC11])**.** *The following are equivalent:*

1. T *is firmly non-expansive*
2. $I - T$ *is firmly non-expansive*
3. $2T - I$ *is non-expansive*
4. $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle$
5. $0 \le \langle Tx - Ty, (I - T)x - (I - T)y\rangle$

We say T is β **cocoercive** (or β-inverse **strongly monotone**) if βT is firmly nonexpansive (Defn 4.4). N.B. This is an unusual use of β: you might think β^{-1} is more natural, so do not confuse it!

Exercise 4.12 [BC11]: if T_1 and T_2 are both firmly nonexpansive, then $T_1 - T_2$ is nonexpansive. N.B. while nonexpansiveness is closed under composition, firm nonexpansiveness is not!

Fact: the projector onto a closed convex set is firmly nonexpansive (Prop 4.8 [BC11]). More generally, all proximity operators are firmly nonexpansive (Prop 12.27 [BC11]). Even more generally, all resolvents $T = (A+I)^{-1}$ are firmly nonexpansive iff A is monotone, and T has full domain iff A is maximally monotone.

Fact (corollary 16.16 [BC11], Baillon-Haddad theorem). Let f be Frechet differentiable convex. Then ∇f is β-Lipschitz continuous *if and only if* ∇f is β^{-1} cocoercive; in particular, ∇f is nonexpansive iff it is firmly nonexpansive.

Monotone Operators

A set-valued operator A is **monotone** if $\langle x - y, Ax - Ay\rangle \ge 0$ (note: I use Ax to denote an arbitrary element of the set Ax), and **strongly monotone** if $A - \beta I$ is monotone, i.e., $\langle x - y, Ax - Ay\rangle \ge \beta\|x - y\|^2$. See defn. 22.1 [BC11]. These notions can be localized to a subset C. Obvious fact: if f is **strongly convex** with constant β, then ∂f is strongly monotone with β. Vandenberghe's notes use "strongly monotone" (with $A = \nabla f$) and "coercive" interchangeable.

Let T be a single-valued operator and $A = T^{-1}$. Then T is β-cocoercive iff A is strongly monotone with constant β.

Defn 11.10 [BC11], coercive: f is coercive of $\lim_{\|x\|\to+\infty} f(x) = +\infty$ and supercoercive if $f(x)/\|x\| \to +\infty$ as well.

References

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