

Algorithmical Geometry: Delaunay Triangulation

Markus Pawellek

December 30, 2021

Outline

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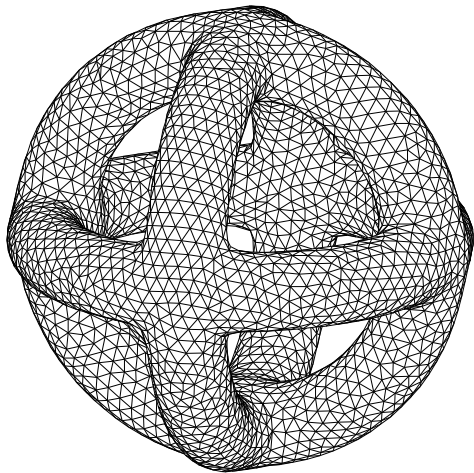
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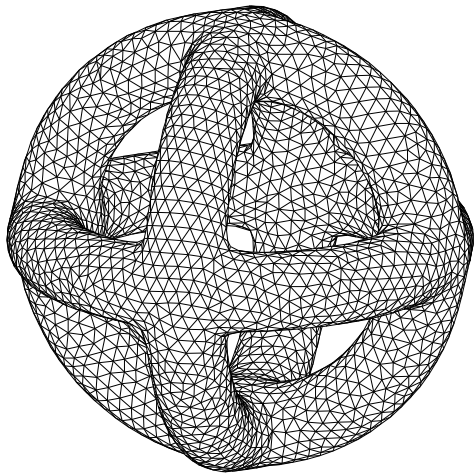
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*<https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg>, December 29, 2021

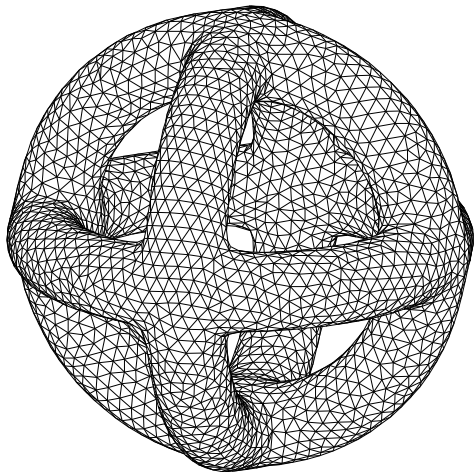
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Educational Problems:

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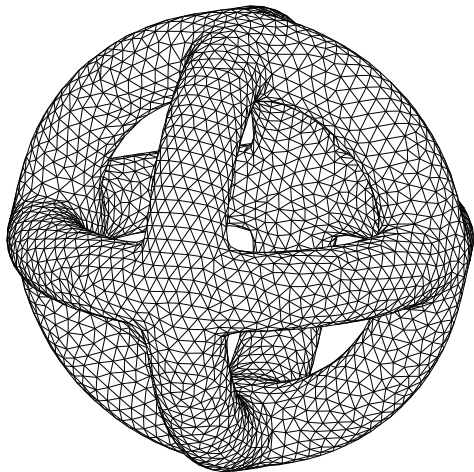
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Educational Problems:

- ▶ Many Resources

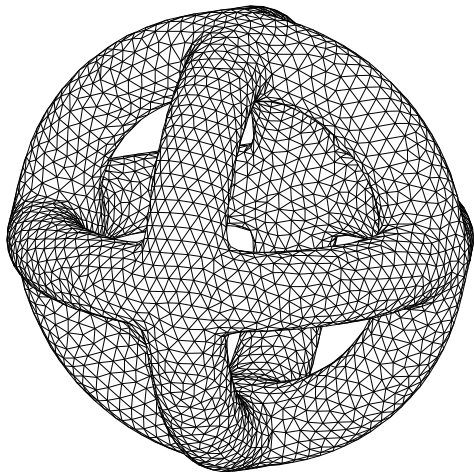
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Educational Problems:

- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams

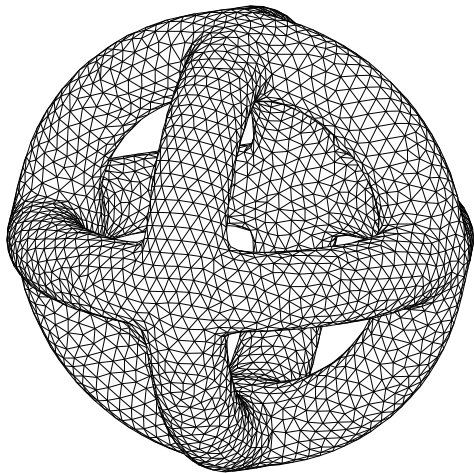
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Educational Problems:

- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams
- ▶ Multiple Algorithm Types:
Incremental, Sweepline,
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Educational Problems:

- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams
- ▶ Multiple Algorithm Types:
Incremental, Sweepline,
Divide-and-Conquer
- ▶ Varying Data Structures

Introduction: Previous Work

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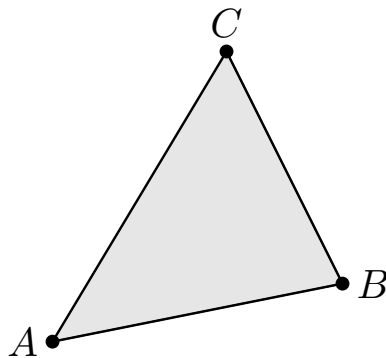
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Mathematical Preliminaries

Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent
define vertices of a triangle.



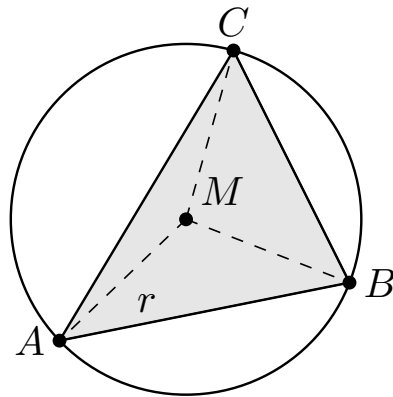
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Circumcircle

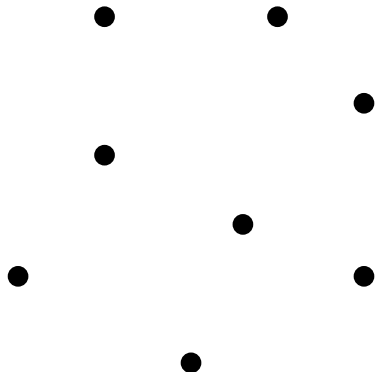
Circle that intersects with
all vertices of the triangle.



Mathematical Preliminaries: Point Set

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
affinely span \mathbb{R}^2



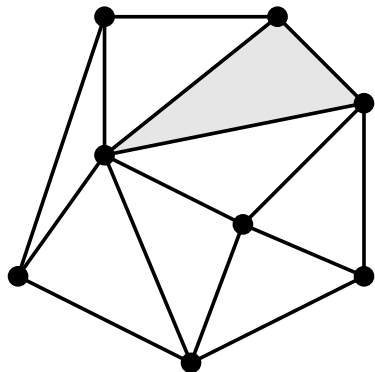
Mathematical Preliminaries: Triangulation

Point Set

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Triangulation

Planar straight-line graph over \mathcal{V}
such that its edges form a maximal
subset of non-crossing edges.



Mathematical Preliminaries: Delaunay Triangulation

Point Set

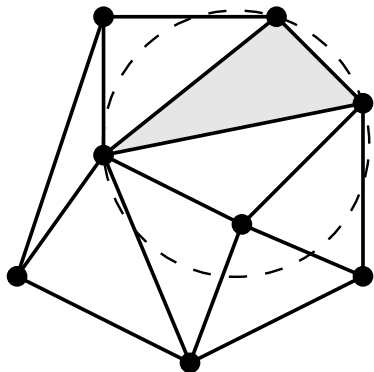
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Delaunay Triangulation

Circumcircle of any triangle
contains no other points of \mathcal{V} .



Mathematical Preliminaries: Delaunay Triangulation

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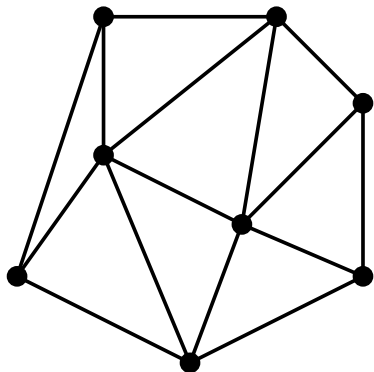
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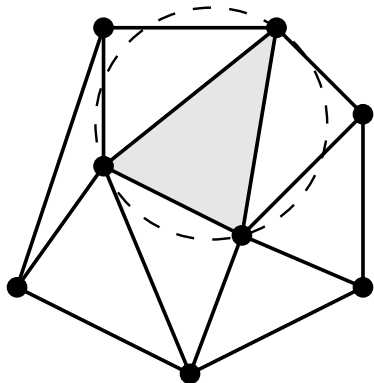
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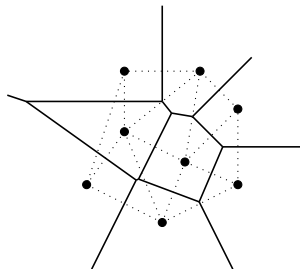
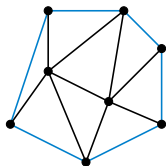
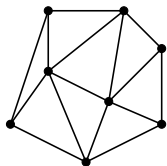
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Mathematical Preliminaries: Voronoi Duality

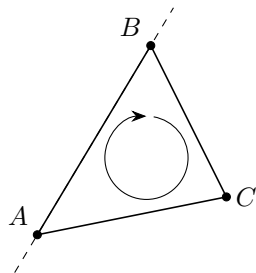
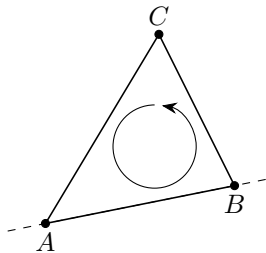


Mathematical Preliminaries: Properties of Delaunay Triangulation

- ▶ Duality to Voronoi Diagram (also useful for proofs)
- ▶ always exists -> proof
- ▶ If no points are cocircular, unique -> proof
- ▶ optimality: maximization of the minimum angle of all angles -> proof, reason why delaunay is good
- ▶ boundary is convex hull

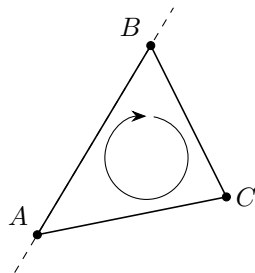
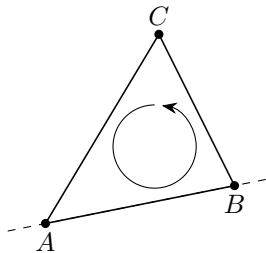
Geometric Primitives

Geometric Primitives: Counterclockwise



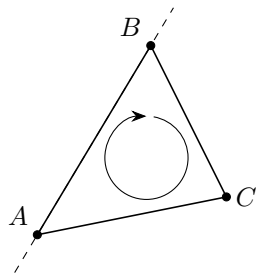
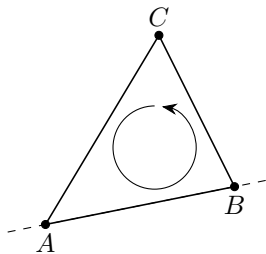
Counterclockwise Order

Geometric Primitives: Counterclockwise



Counterclockwise Order \iff C is left of \overline{AB}

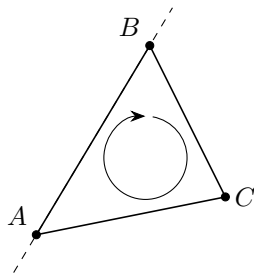
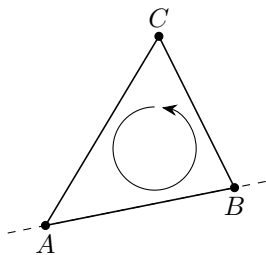
Geometric Primitives: Counterclockwise



Counterclockwise Order \iff C is left of \overline{AB}

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

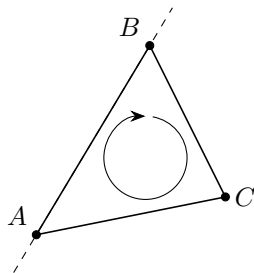
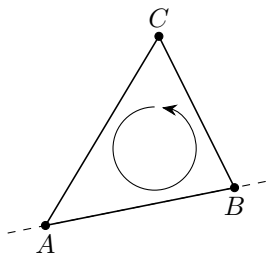
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$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

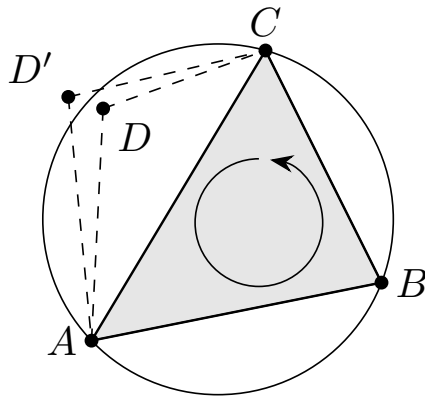
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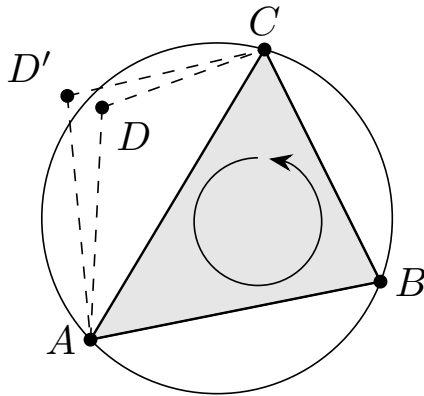
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Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



Data Structures

Algorithm

Implementation

Implementation

- ▶ Geometric Primitives need exact computation and therefore arbitrary precision

Applications

Conclusions

Thank you for Your Attention!

References

- (1) D. T. Lee and B. J. Schachter. "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9 (1980), pp. 219–242. DOI: 10.1007/BF00977785.
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