


Algorithmical Geometry: Computation of Delaunay Triangulations Using a Divide-and-Conquer Algorithm

Markus Pawellek

January 21, 2022



Outline

Related Work

Mathematical Preliminaries

Geometric Primitives

Quad-Edge Data Structure

Algorithm

Implementation Notes

Conclusions



Related Work

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Educational Problems:

Related Work

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- ▶ Many Resources

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- ▶ Duality to Voronoi Diagrams

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- ▶ Multiple Algorithm Types:
Incremental, Sweepline, Divide-and-Conquer

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- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams
- ▶ Multiple Algorithm Types:
Incremental, Sweepline, Divide-and-Conquer
- ▶ Varying Data Structures

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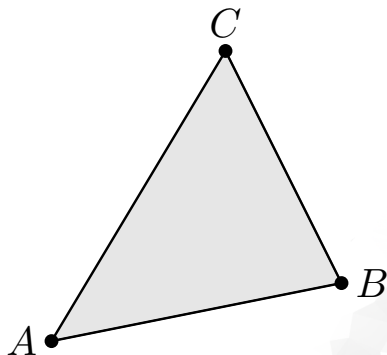


Mathematical Preliminaries

Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent
define vertices of a triangle.



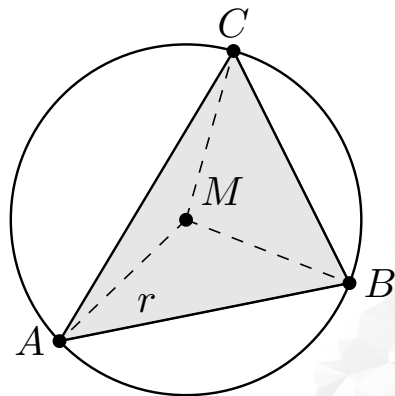
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Triangle

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Circumcircle

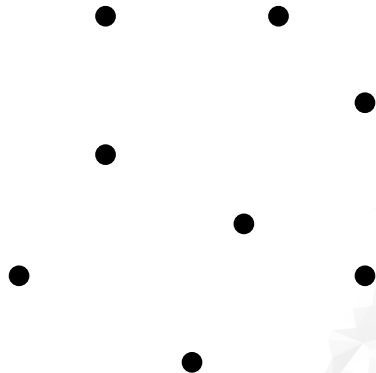
Circle that intersects with
all vertices of the triangle.



Mathematical Preliminaries: Point Set

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
affinely span \mathbb{R}^2



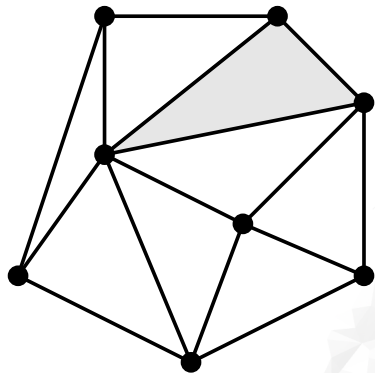
Mathematical Preliminaries: Triangulation

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Triangulation

Planar straight-line graph over \mathcal{V}
such that its edges form a maximal
subset of non-crossing edges.



Mathematical Preliminaries: Delaunay Triangulation

Point Set

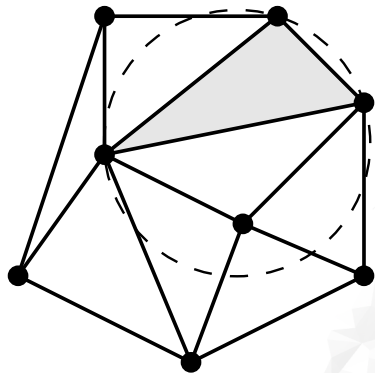
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Delaunay Triangulation

Circumcircle of any triangle
contains no other points of \mathcal{V} .



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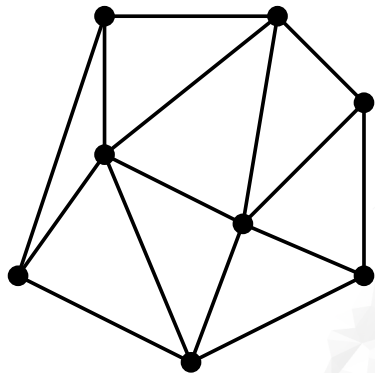
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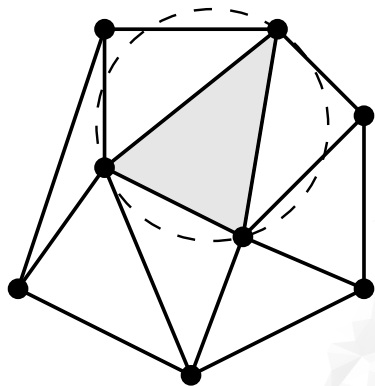
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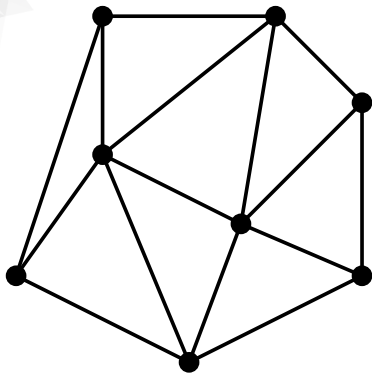
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Delaunay Triangulation

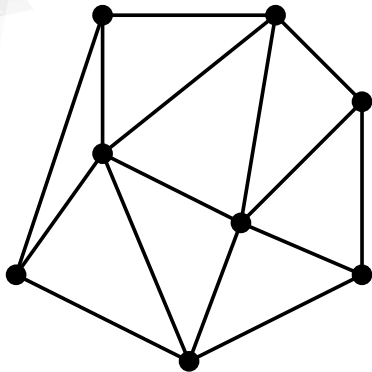
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Mathematical Preliminaries: Properties

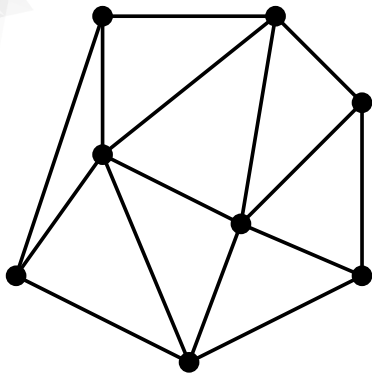


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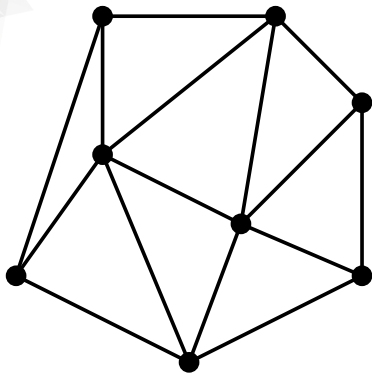
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Mathematical Preliminaries: Properties



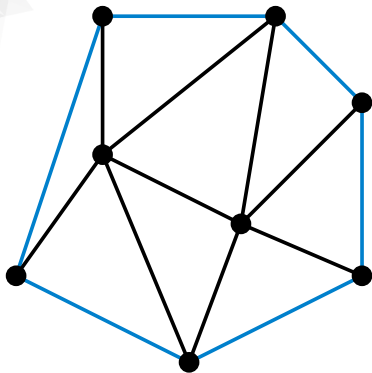
- ▶ Existence is guaranteed
- ▶ Unique if there are no four points that are cocircular

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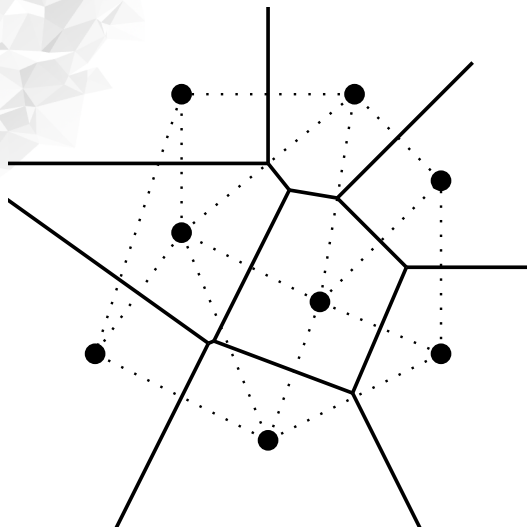
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- ▶ Optimality: maximization of the minimum angle of all angles

Mathematical Preliminaries: Properties



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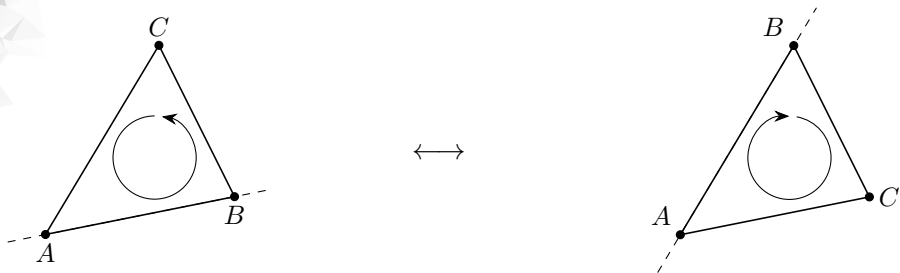


- ▶ Existence is guaranteed
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- ▶ Convex hull is contained
- ▶ Dual of Voronoi diagram

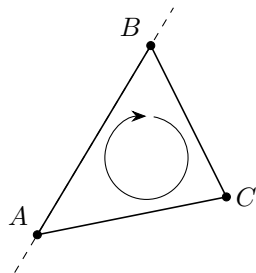
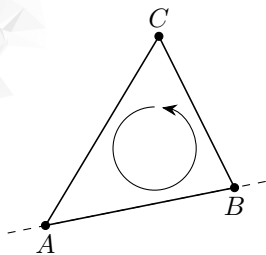


Geometric Primitives

Geometric Primitives: Triangle Orientation



Geometric Primitives: Triangle Orientation

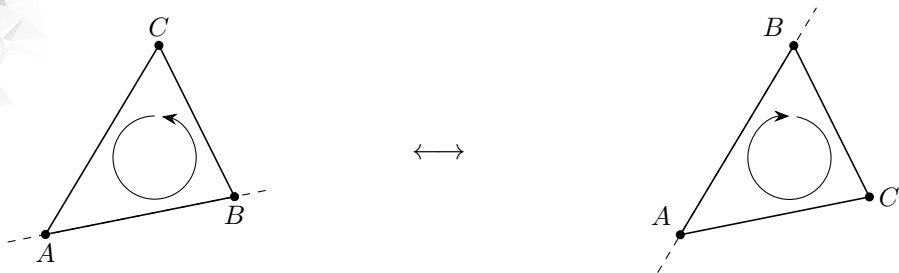


Counterclockwise Orientation



C is left of \overline{AB}

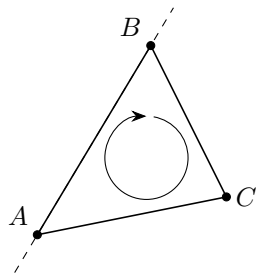
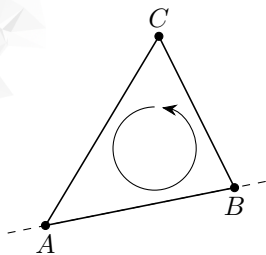
Geometric Primitives: Triangle Orientation



Counterclockwise Orientation $\iff C$ is left of \overline{AB}

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

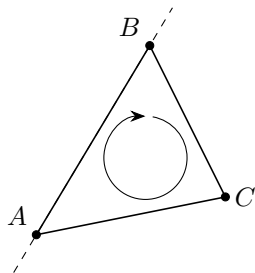
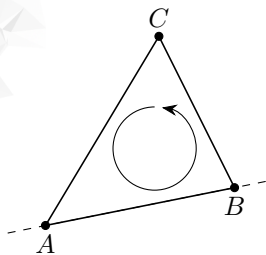
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$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

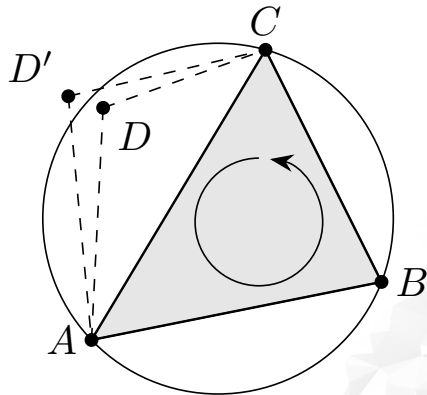
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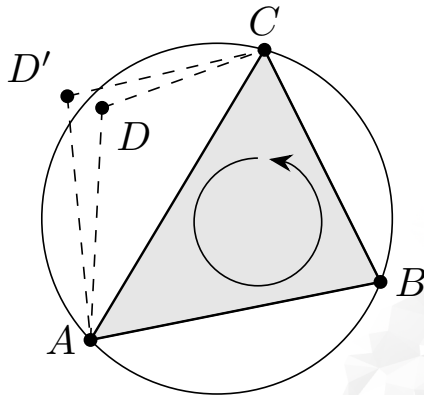
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Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

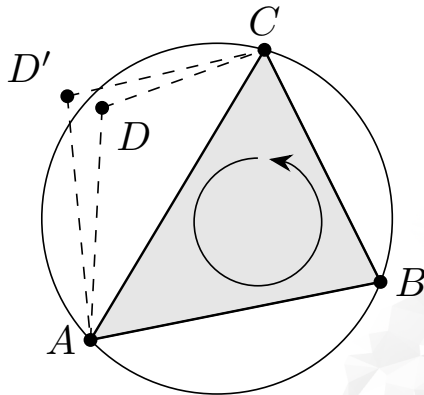
$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



Geometric Primitives: Inside Circumcircle

$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$

$$= \left\langle x, \text{adj} \begin{pmatrix} u & v \end{pmatrix}^T \begin{pmatrix} \|u\|^2 \\ \|v\|^2 \end{pmatrix} \right\rangle \\ - \det \begin{pmatrix} u & v \end{pmatrix} \|x\|^2$$



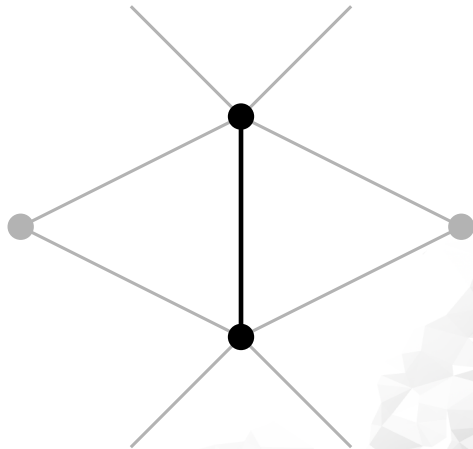
$$u := B - A, \quad v := C - A, \quad x := D - A$$



Quad-Edge Data Structure

Quad-Edge Data Structure: Scheme

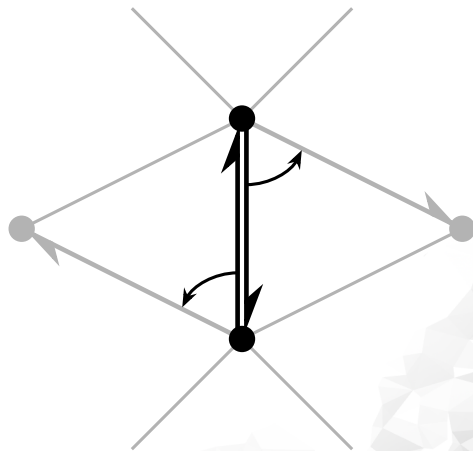
Edge-Based List-Like Data Structure
for Storing Neighbor Information:



Quad-Edge Data Structure: Scheme

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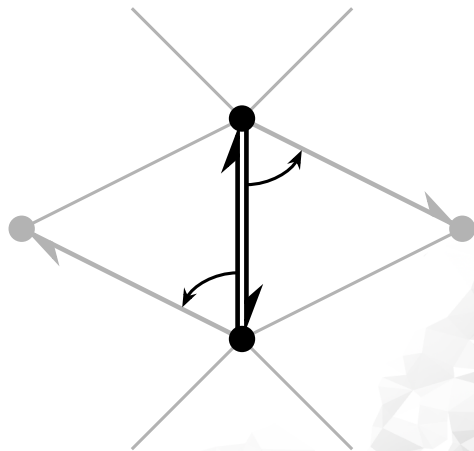
- ▶ Directed edges for vertices



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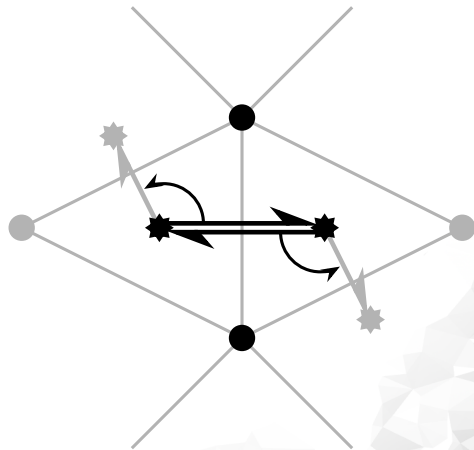
- ▶ Directed edges for vertices
- ▶ Pointer to ccw. next directed edge with same origin vertex



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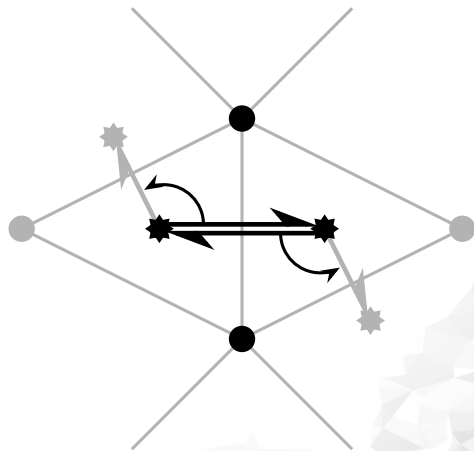
- ▶ Directed edges for vertices
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- ▶ Directed dual edges for adjacent faces



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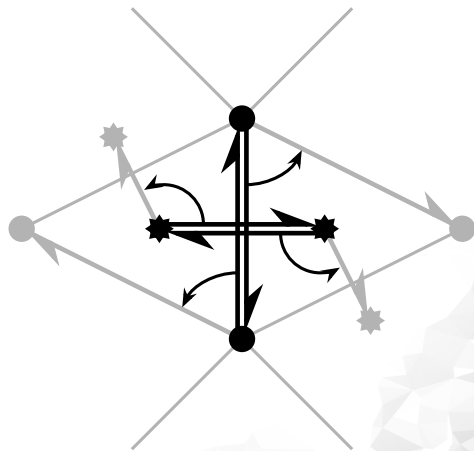
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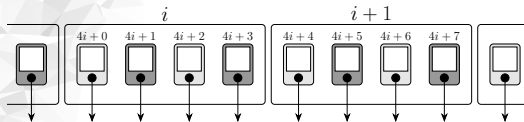
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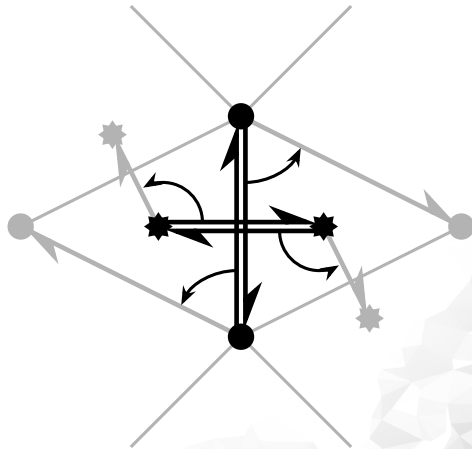
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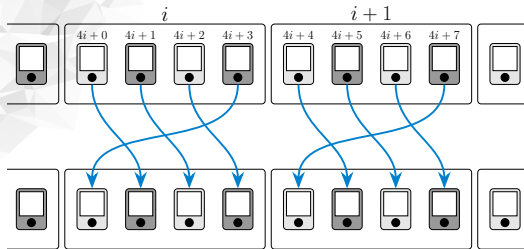
Quad-Edge Data Structure: Implementation



```
struct edge {  
    size_t next;  
    size_t data;  
};  
  
struct quad_edge {  
    edge data[4];  
};  
  
vector<vertex>    vertices{};  
vector<quad_edge> quad_edges{};  
vector<size_t>    free_edges{};
```

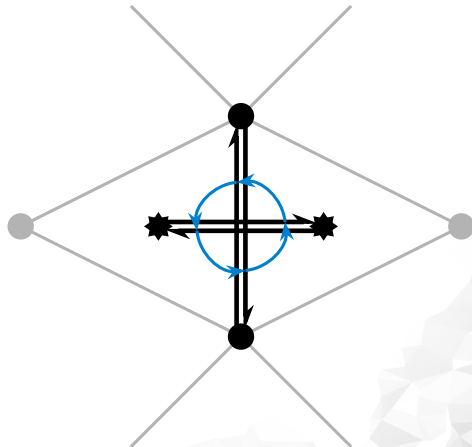


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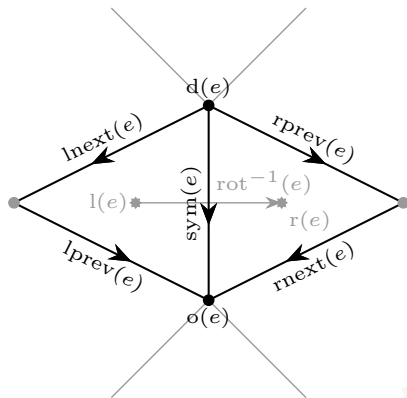
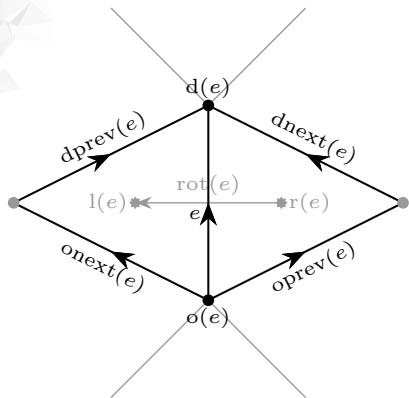


$\text{rot}: \mathbb{N}_0 \rightarrow \mathbb{N}_0$

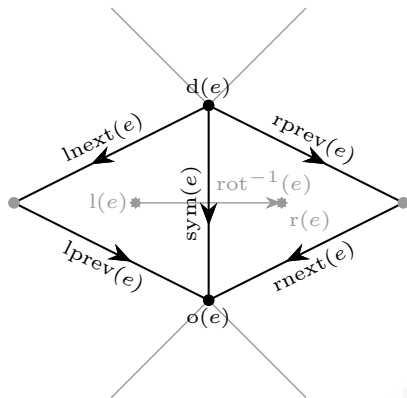
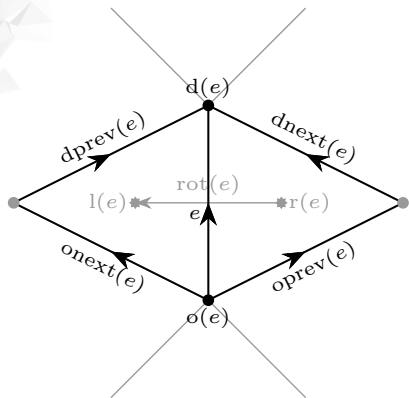
$$\text{rot}(x) = 4 \cdot \left\lfloor \frac{x}{4} \right\rfloor + (x + 1 \mod 4)$$



Quad-Edge Data Structure: Edge Functions and Operators

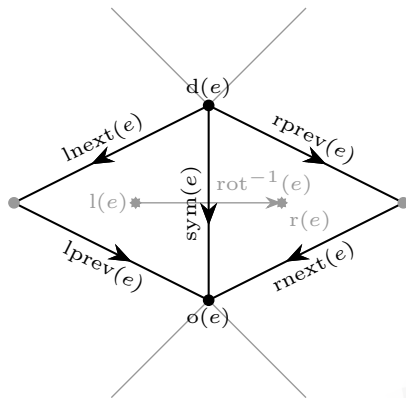
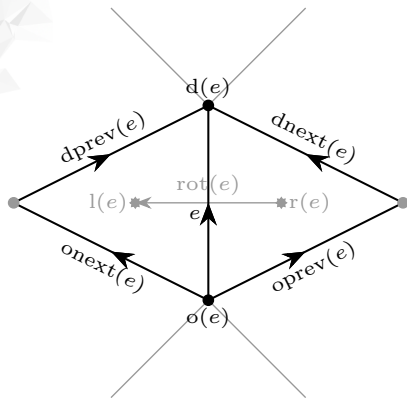


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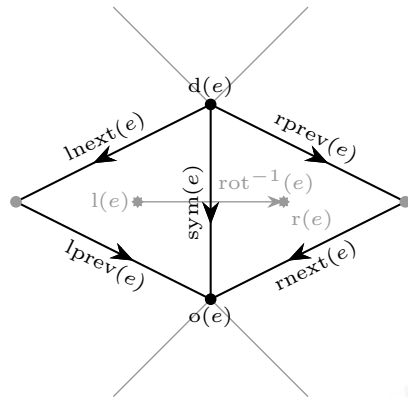
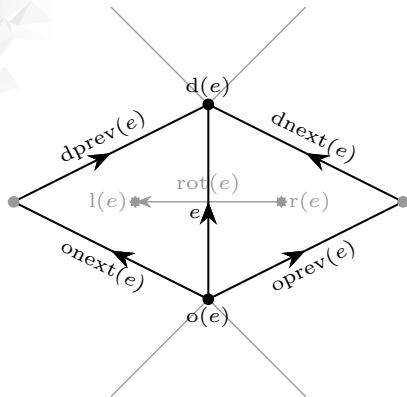
- Create a new edge

Quad-Edge Data Structure: Edge Functions and Operators



- ▶ Create a new edge
- ▶ Delete existing edge

Quad-Edge Data Structure: Edge Functions and Operators



- ▶ Create a new edge
- ▶ Delete existing edge
- ▶ Connect points by a new edge



Algorithm

Algorithm: Overview

Algorithm: Overview

Triangulation Algorithm

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

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Subroutine: Triangulate

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Algorithm: Overview

Triangulation Algorithm

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1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.

Algorithm: Overview

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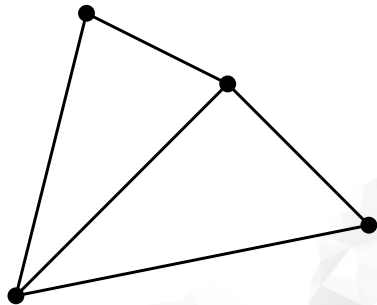
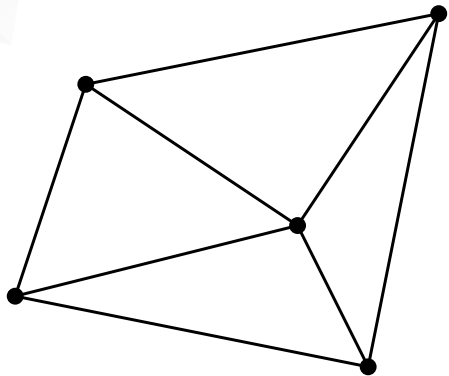
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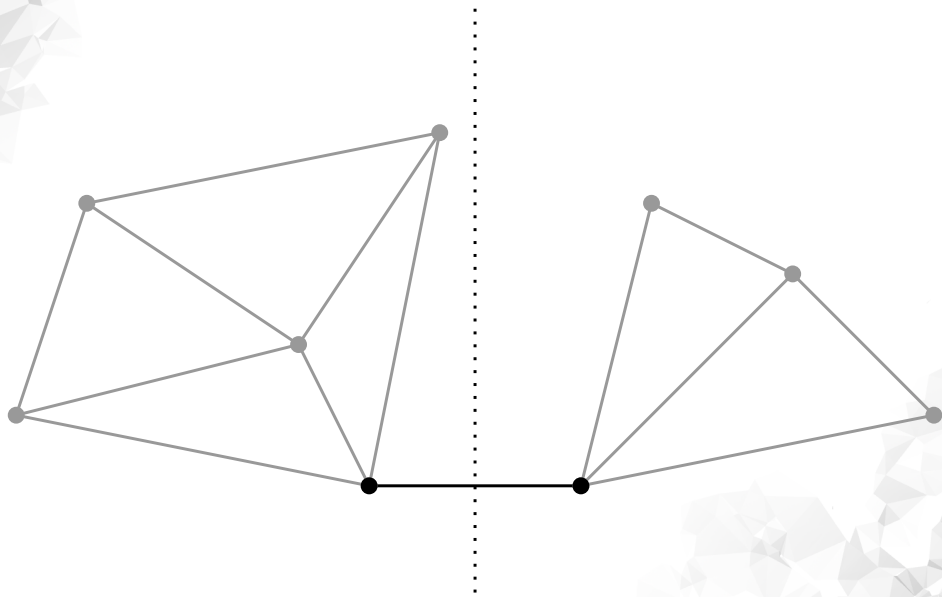
Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.
3. Triangulate left and right half.
4. Merge left and right triangulations.

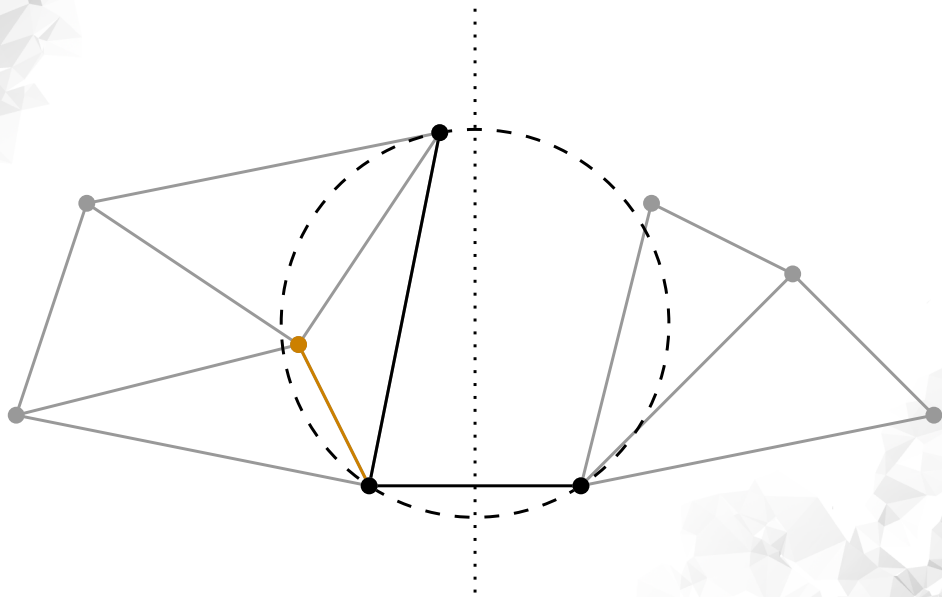
Algorithm: Merge Triangulations Example



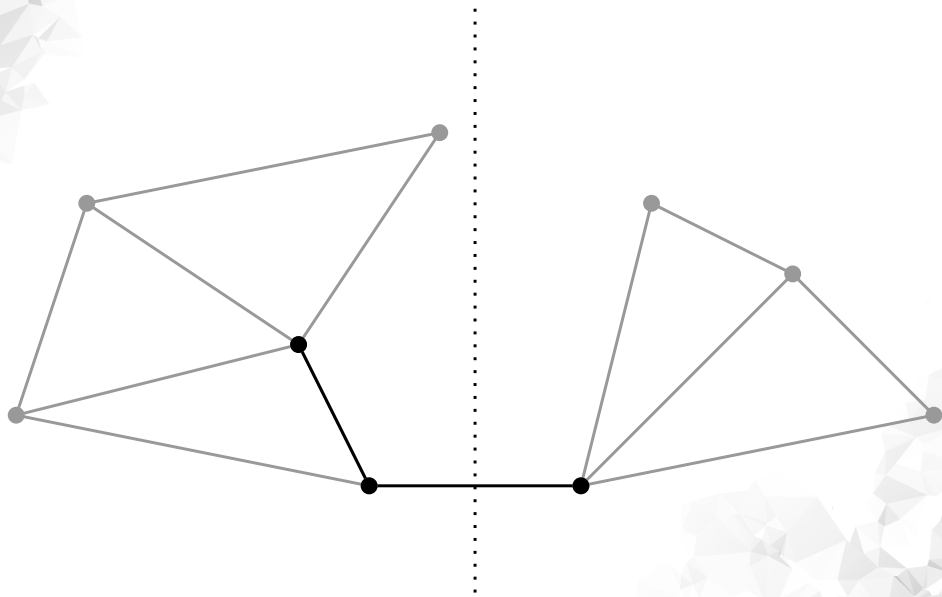
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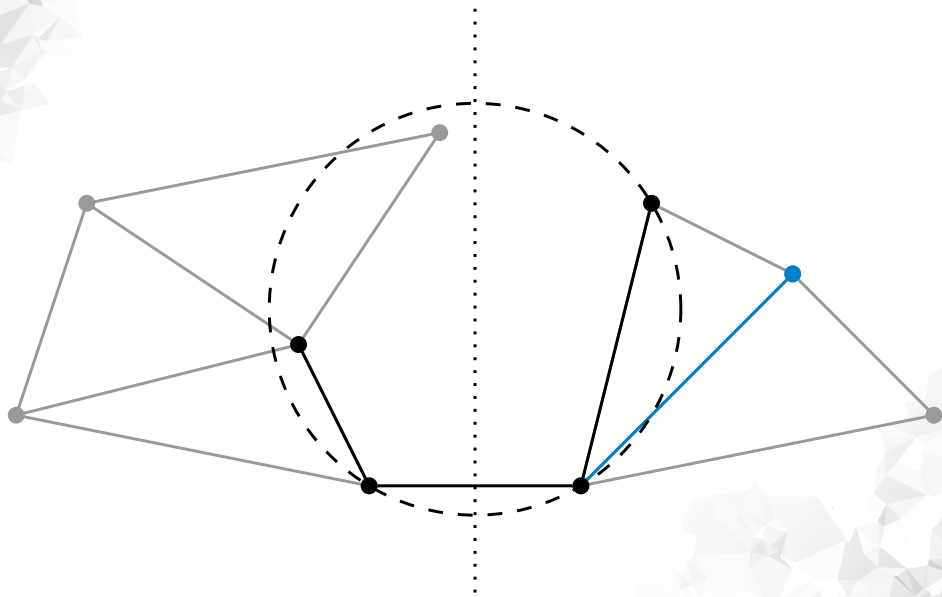
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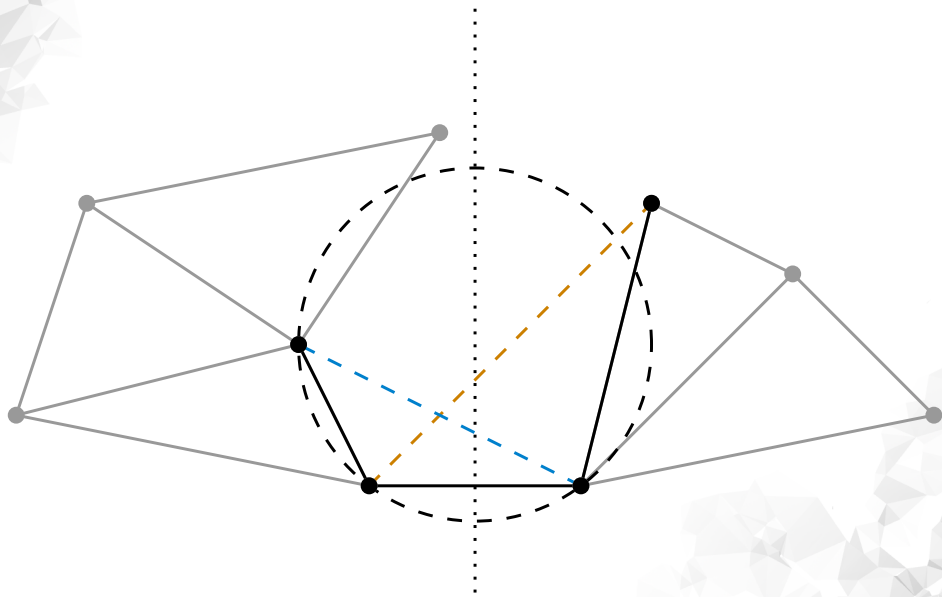
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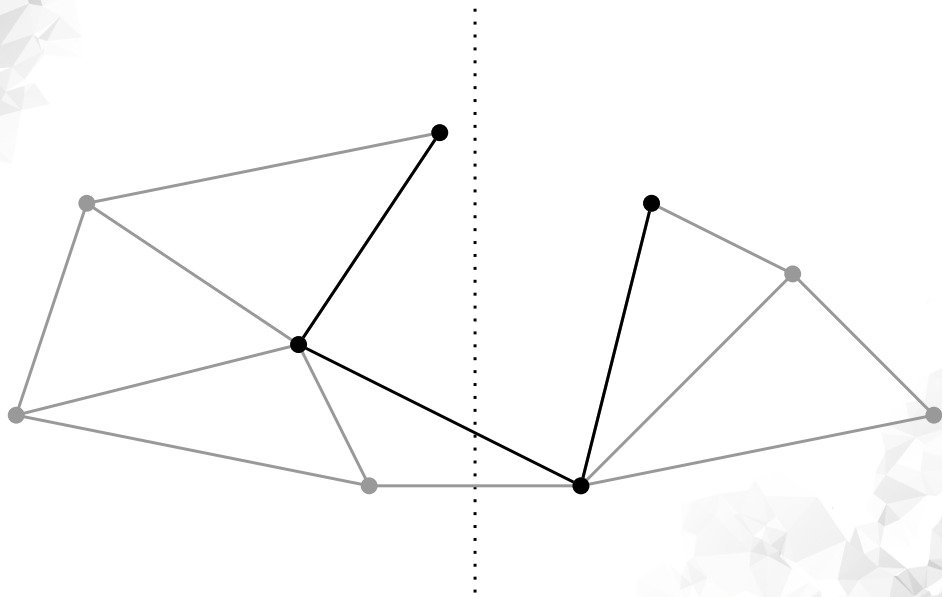
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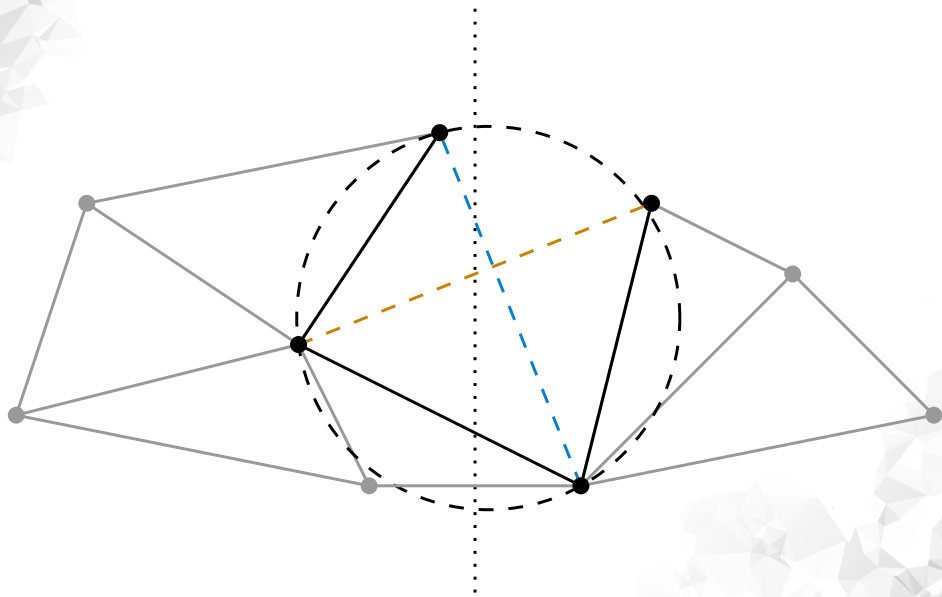
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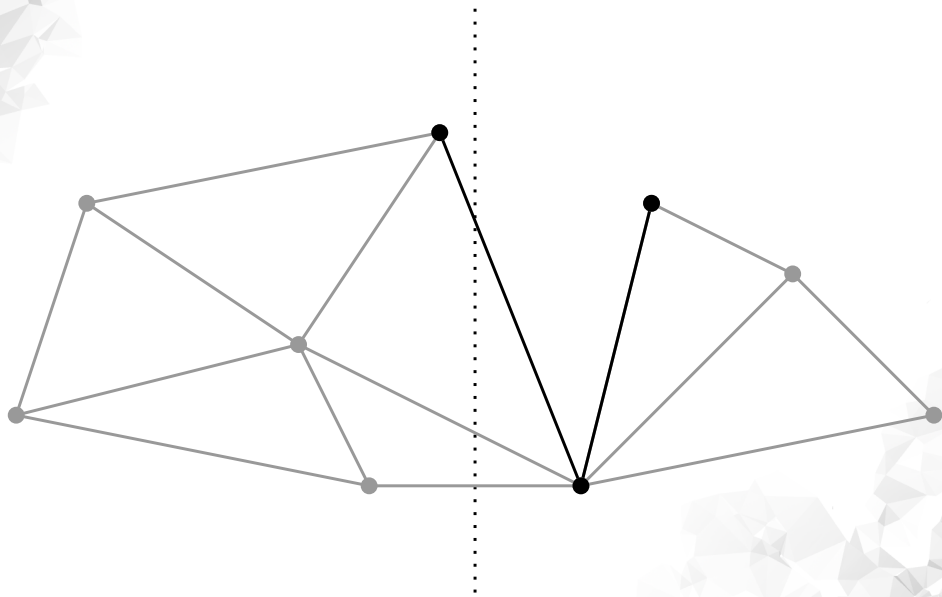
Algorithm: Merge Triangulations Example



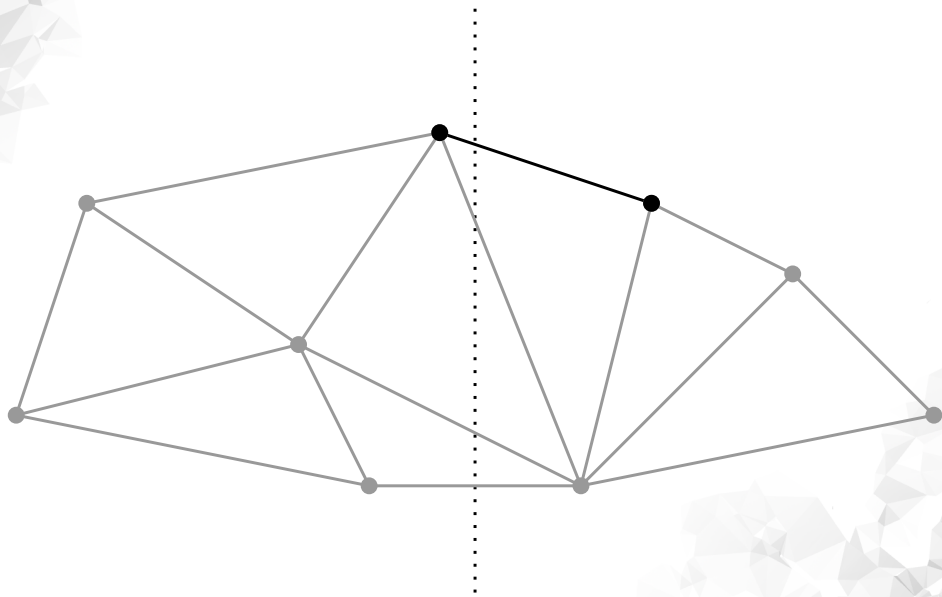
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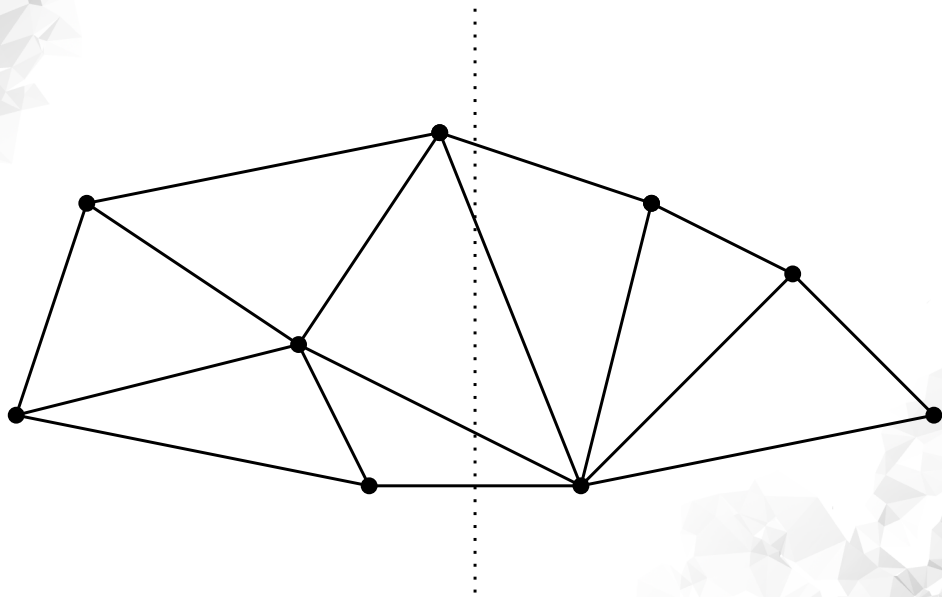
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- ▶ Algorithm is correct



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Thank you for Your Attention!

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