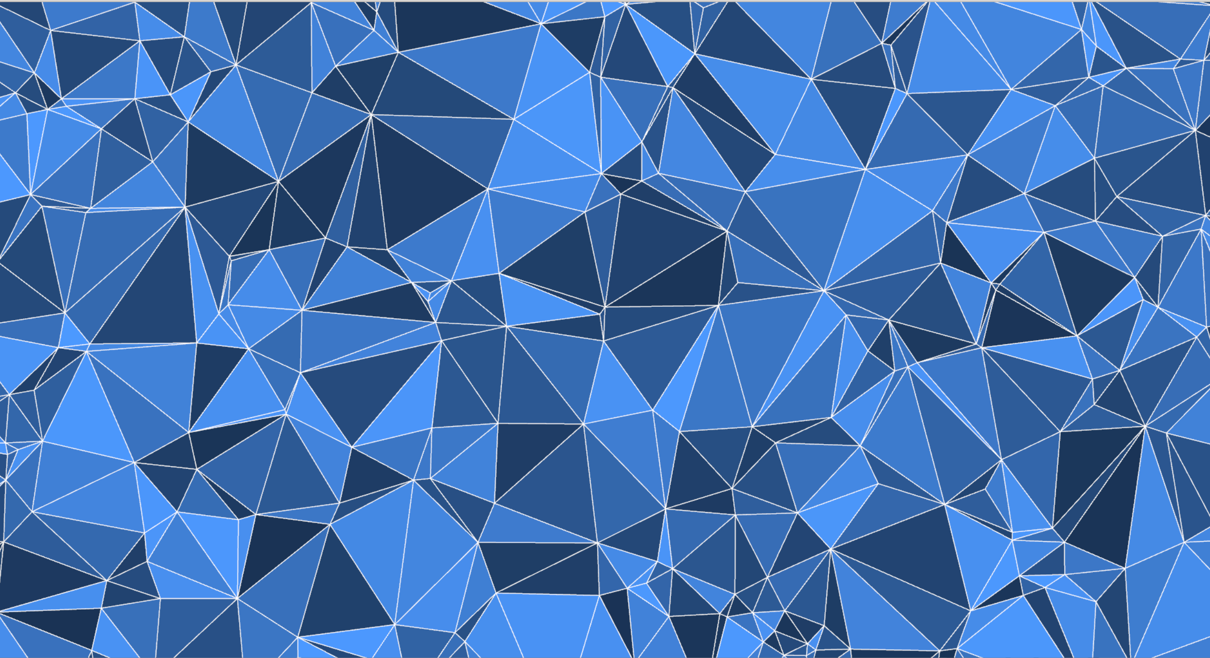
Delaunay Triangulation Mosaic



Algorithmical Geometry: Delaunay Triangulation

Markus Pawellek

January 3, 2022

Outline

Introduction

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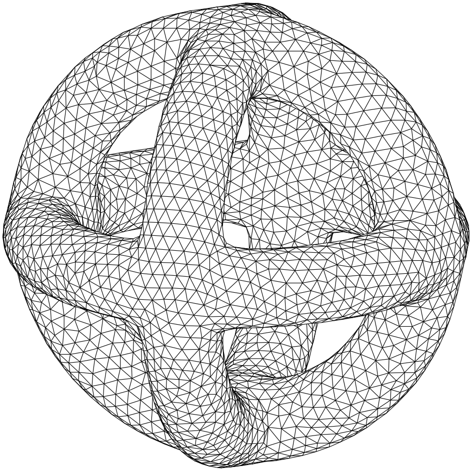

*https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg, December 29, 2021

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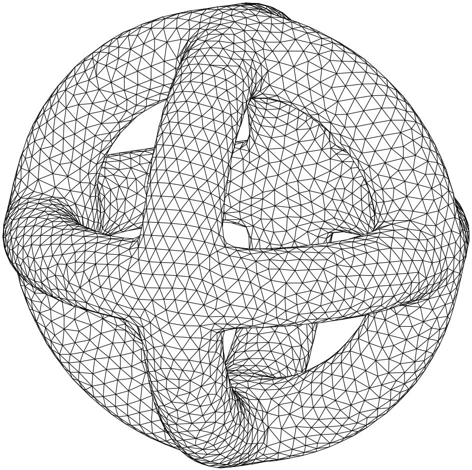

Educational Problems:

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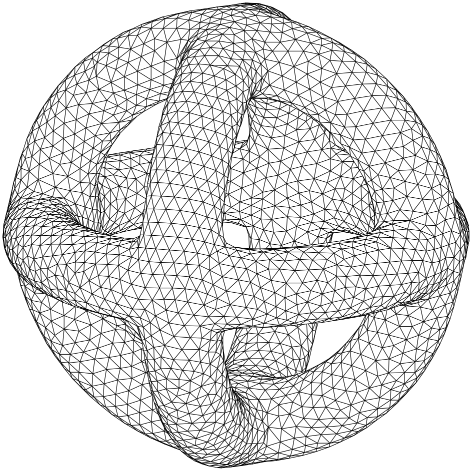

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- Many Resources

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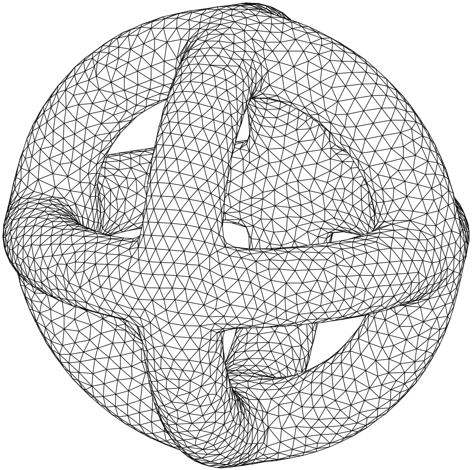

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams

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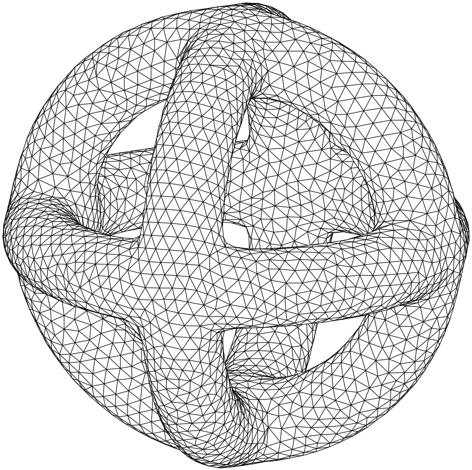

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer

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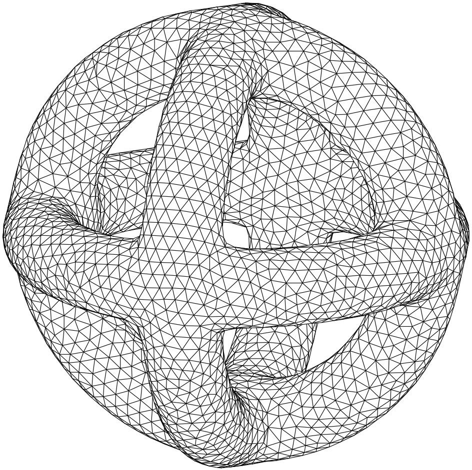

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer
- Varying Data Structures

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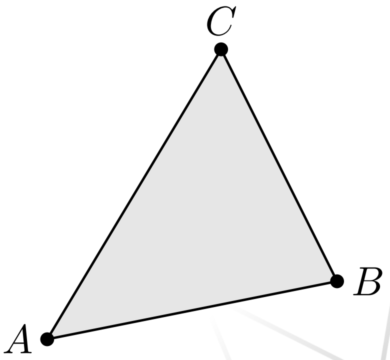
Mathematical Preliminaries

Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.



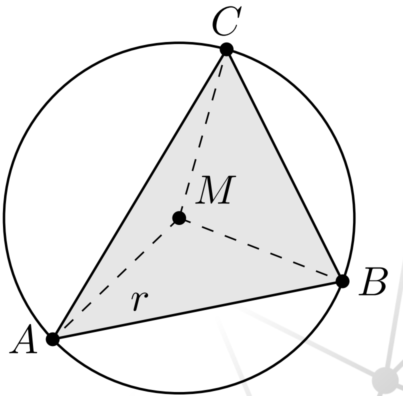
Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.

Circumcircle

Circle that intersects with all vertices of the triangle.

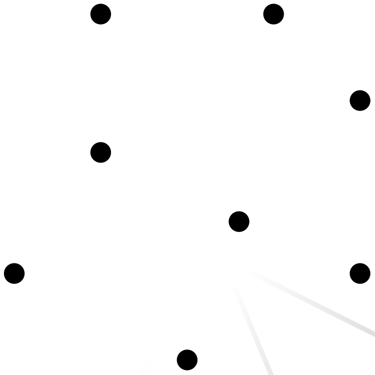


Mathematical Preliminaries: Point Set

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,

affinely span $\mathbb{R}^2$



Mathematical Preliminaries: Triangulation

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
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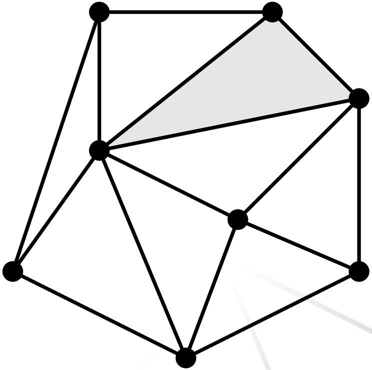
Triangulation

Planar straight-line graph over $\mathcal{V}$ such that its edges form a maximal subset of non-crossing edges.



Mathematical Preliminaries: Delaunay Triangulation

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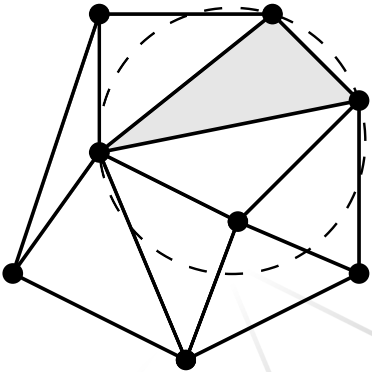
Delaunay Triangulation

Circumcircle of any triangle contains no other points of $\mathcal{V}$.



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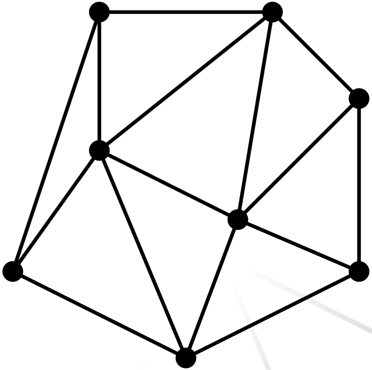

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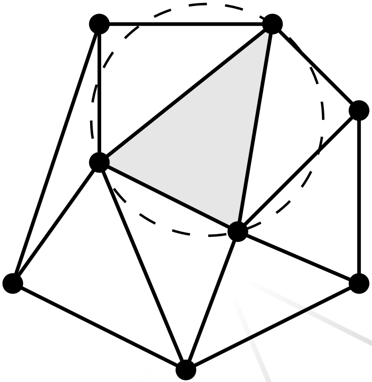
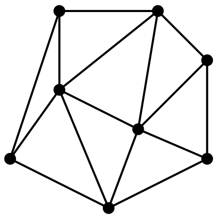


Mathematical Preliminaries: Properties



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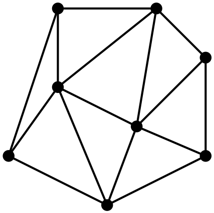
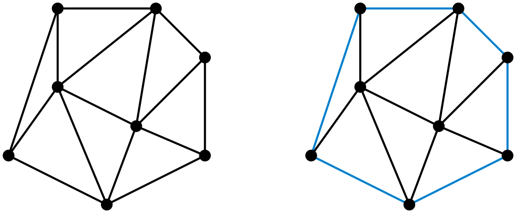


- Optimality: maximization of the minimum angle of all angles

Mathematical Preliminaries: Properties



- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained

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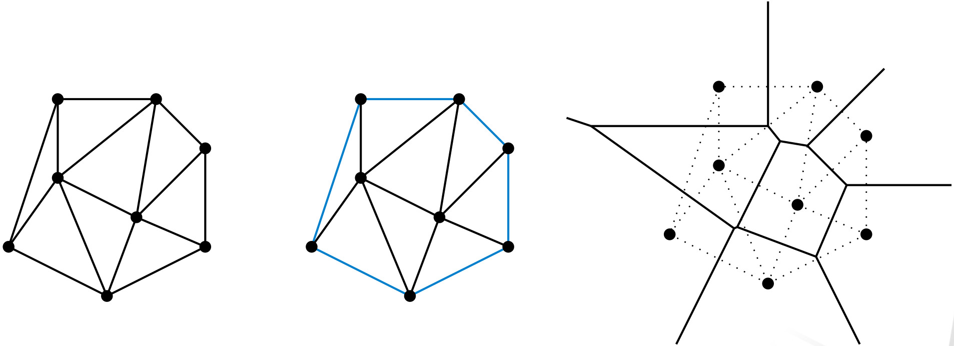


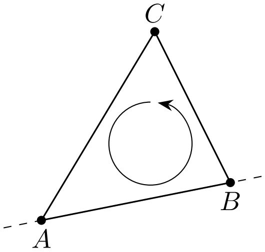
- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained
- Voronoi diagram is the dual

Mathematical Preliminaries: Properties of Delaunay Triangulation

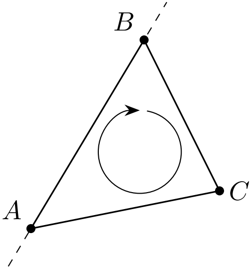
- Duality to Voronoi Diagram (also useful for proofs)
- always exists -> proof
- If no points are cocircular, unique -> proof
- optimality: maximization of the minimum angle of all angles -> proof, reason why delaunay is good
- boundary is convex hull

Geometric Primitives

Geometric Primitives: Counterclockwise

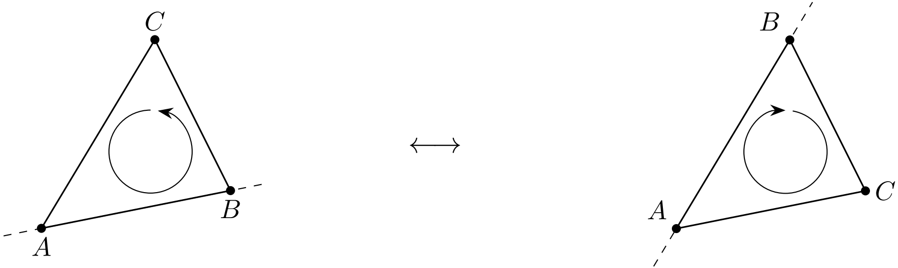


$\longleftrightarrow$



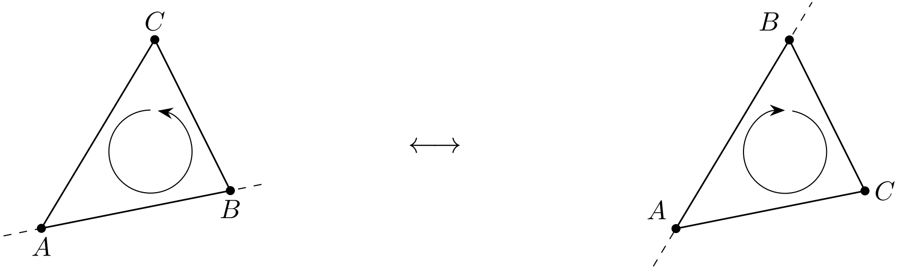
Counterclockwise Order

Geometric Primitives: Counterclockwise



Counterclockwise Order $\iff$ C is left of $\overline{AB}$

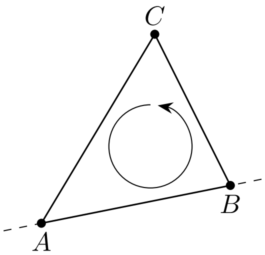
Geometric Primitives: Counterclockwise



Counterclockwise Order $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

Geometric Primitives: Counterclockwise



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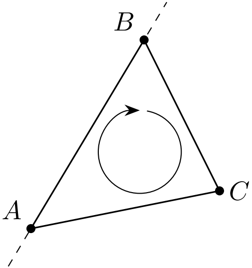

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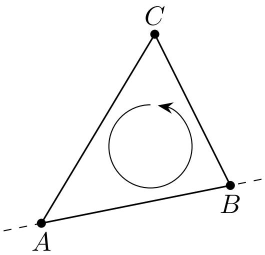
$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

Geometric Primitives: Counterclockwise



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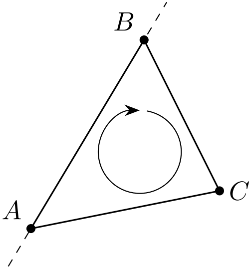

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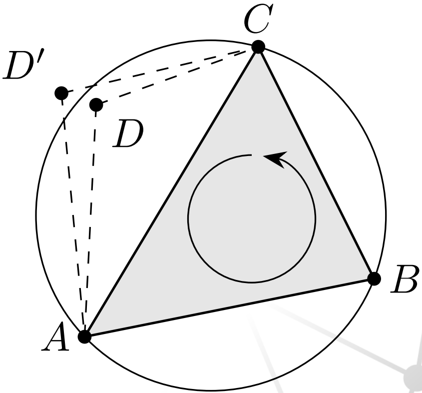
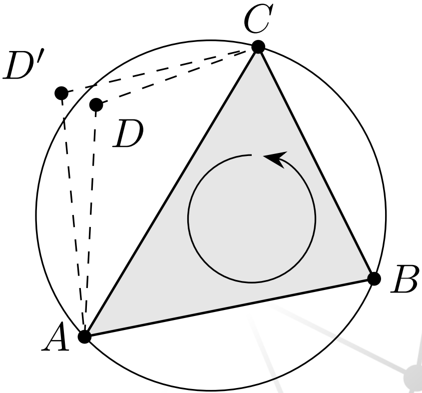
Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

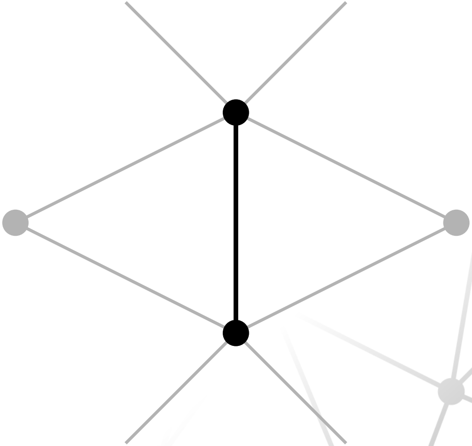
$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



Data Structure

Data Structure: Quad-Edge

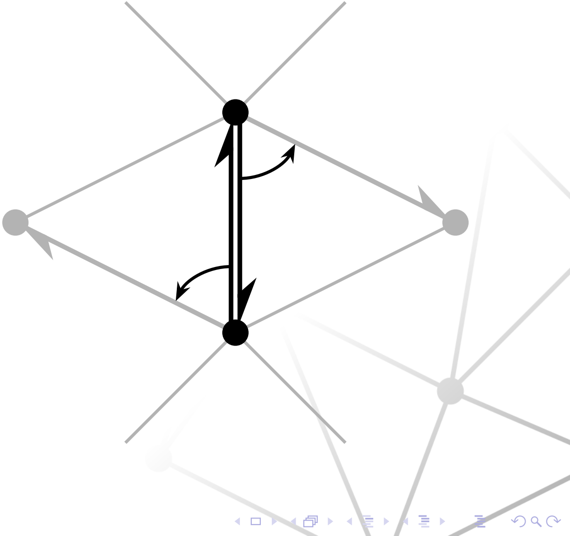
Edge-Based List-Like Data Structure:



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

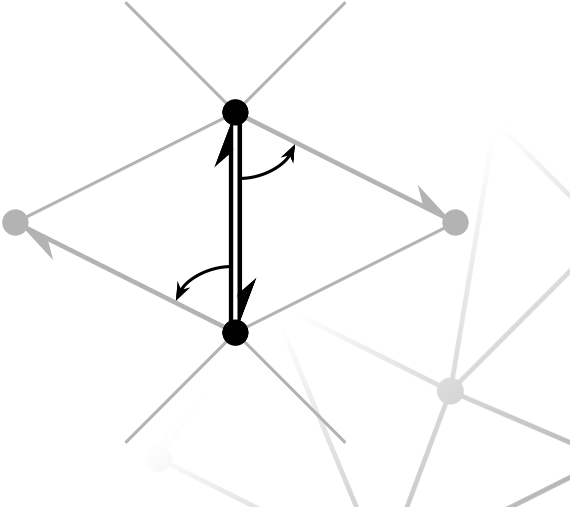
- Directed edges for vertices



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

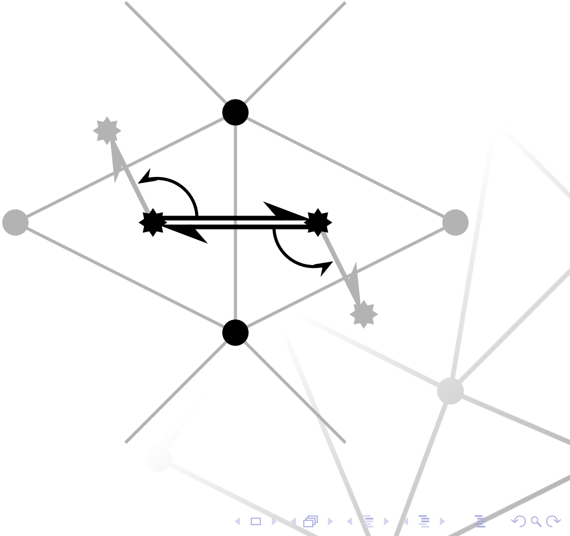
- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex
- Directed dual edges for adjacent faces



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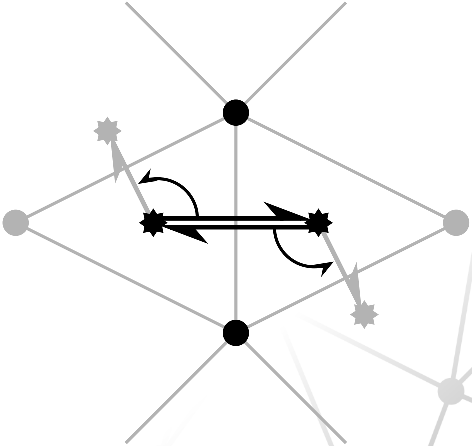

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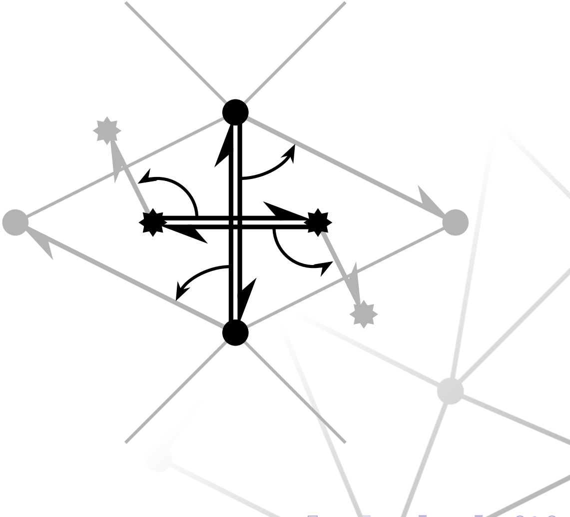
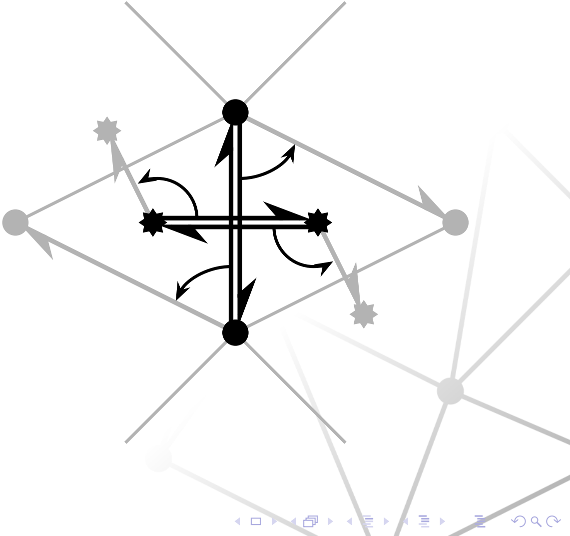


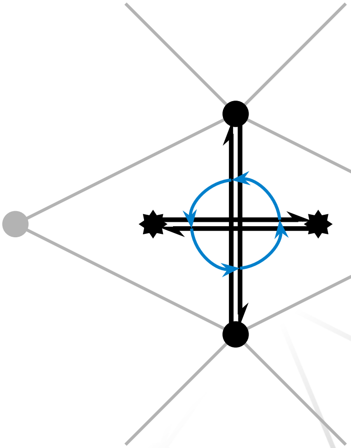
Data Structure: Quad-Edge Implementation

```
struct quad_edge_algebra {
  struct edge {
    size_t next;
    size_t data;
  };

  struct quad_edge {
    edge data[4];
  };

  vector<quad_edge> quad_edges{};
  vector<size_t> free_edges{};
};
```

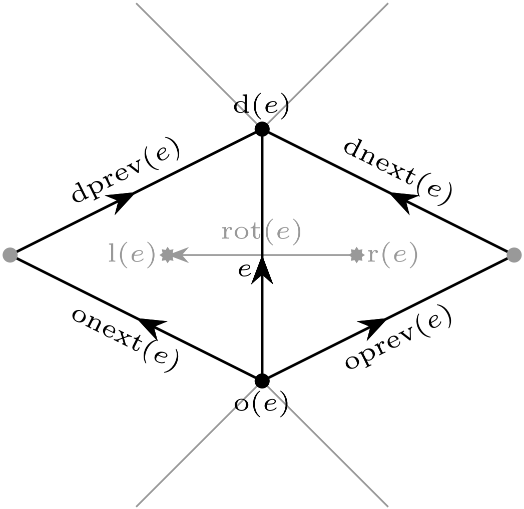


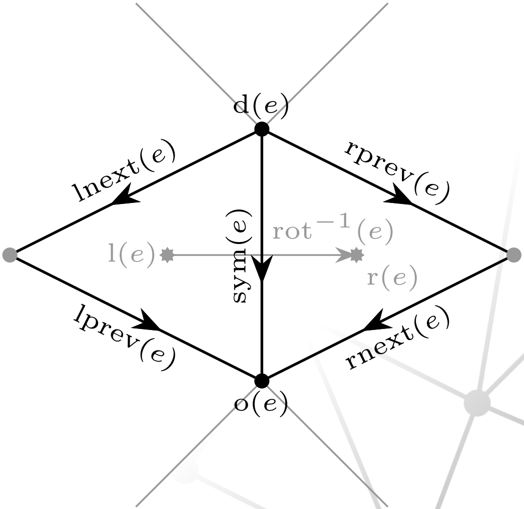
Data Structure: Quad-Edge Implementation

```
struct quad_edge_algebra {
  // ...
  static constexpr size_t edge_type_mask =
    sizeof(quad_edge) / sizeof(edge) - 1;
  static constexpr size_t quad_edge_mask =
    ~edge_type_mask;

  auto rot(size_t e, int n = 1) const -> size_t {
    const size_t t = e + n;
    return (e & quad_edge_mask) |
           (t & edge_type_mask);
  }

  auto onext(size_t e) const -> size_t {
    return ((edge*)quad_edges.data() + e)->next;
  }
};
```



Data Structure: Quad-Edge Functions





Data Structure: Quad-Edge Operations

- edge functions
- create edge
- splice
- connect
- delete edge

Algorithm

Algorithm: Overview

1. Sort points by increasing x coordinate

1. If number of points is less than four, create an edge or a triangle.
2. Separate points into left and right half
3. Compute the lower common tangent and make it a cross edge
4. Merge Loop to insert crossing edges until upper tangent is reached
5. Return left and right convex hull edge

1. Remove edges from left partition that fail circle test
2. Remove edges from right partition that fail circle test
3. Check for upper tangent
4. Insert crossing edge by using circle test

Implementation

Implementation

- Geometric Primitives need exact computation and therefore arbitrary precision

Applications

Conclusions

Thank you for Your Attention!

References

(1) D. T. Lee and B. J. Schachter. "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9 (1980), pp. 219–242. DOI: 10.1007/BF00977785.

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(10) P. Cignoni, C. Montani, and R. Scopigno. "DeWall: A Fast Divide-and-Conquer Delaunay Triangulation Algorithm in E^d". In: *Computer-Aided Design* 30 (1998), pp. 333–341. DOI: 10.1016/S0010-4485(97)00082-1.

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