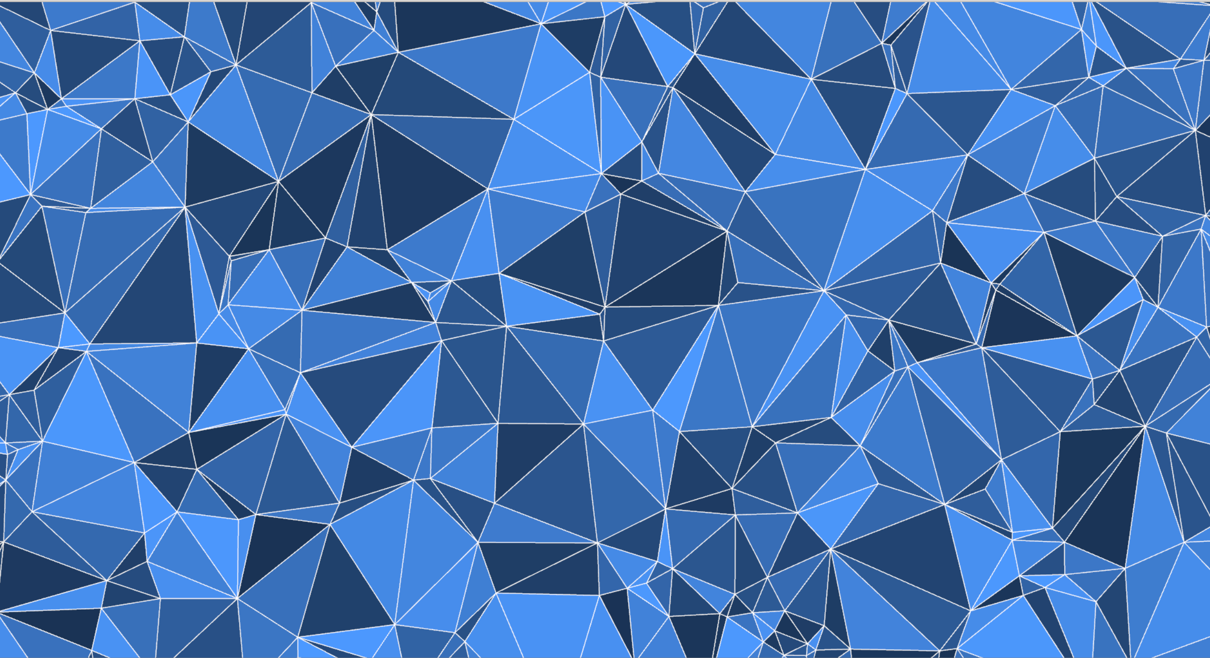
Delaunay Triangulation Mosaic



# Algorithmical Geometry: Delaunay Triangulation

Markus Pawellek

December 30, 2021

# Outline

# Introduction

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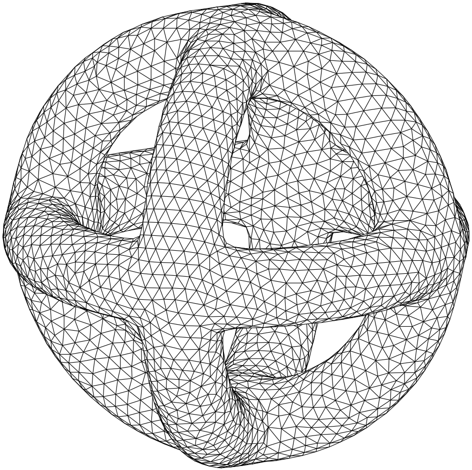

*https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg, December 29, 2021

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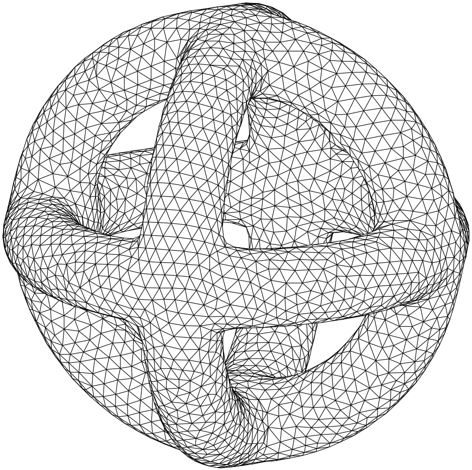

Educational Problems:

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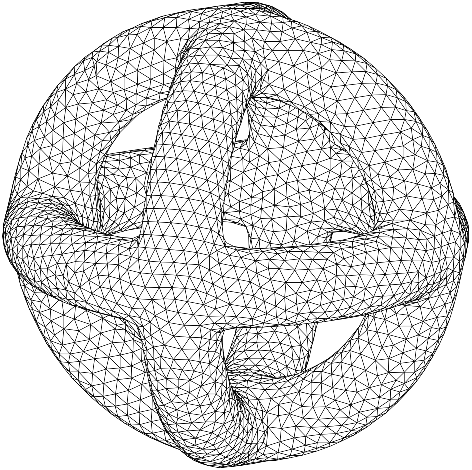

Educational Problems:

- Many Resources

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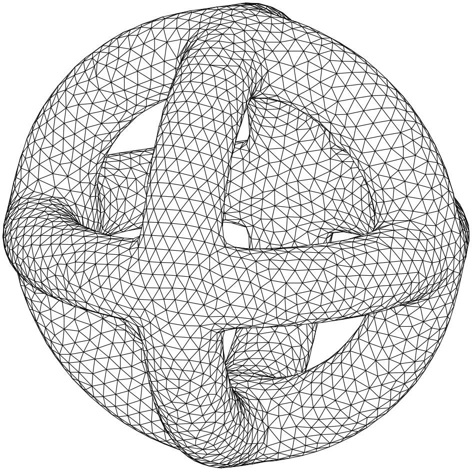

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams

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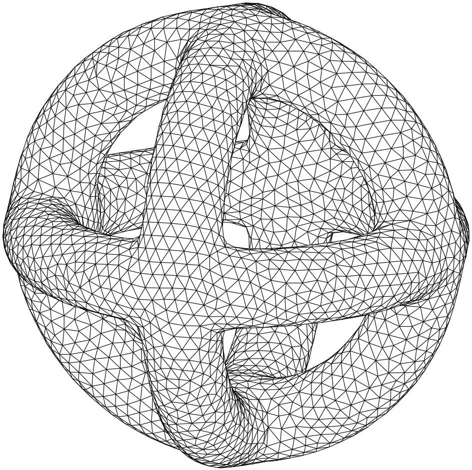

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer

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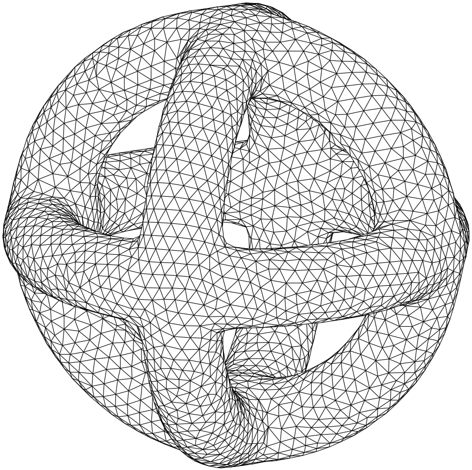

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer
- Varying Data Structures

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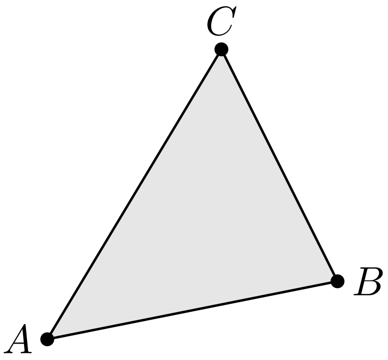
# Mathematical Preliminaries

# Mathematical Preliminaries: Triangle and Circumcircle

**Triangle**

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.



# Mathematical Preliminaries: Triangle and Circumcircle

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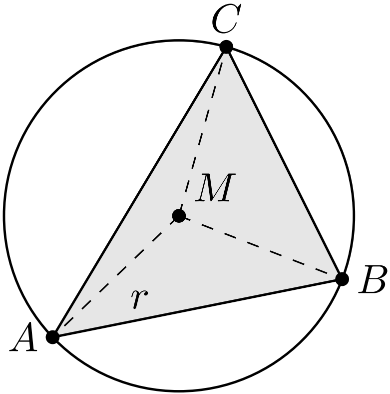
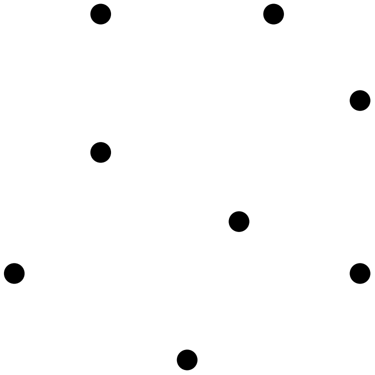
**Circumcircle**

Circle that intersects with all vertices of the triangle.



**Point Set**

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
affinely span $\mathbb{R}^2$



# Mathematical Preliminaries: Triangulation

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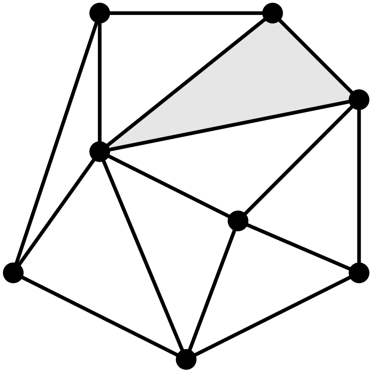
**Triangulation**

Planar straight-line graph over $\mathcal{V}$ such that its edges form a maximal subset of non-crossing edges.



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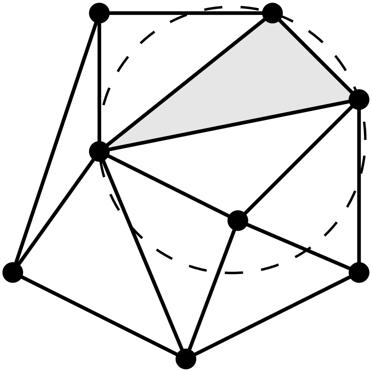
**Delaunay Triangulation**

Circumcircle of any triangle contains no other points of $\mathcal{V}$.



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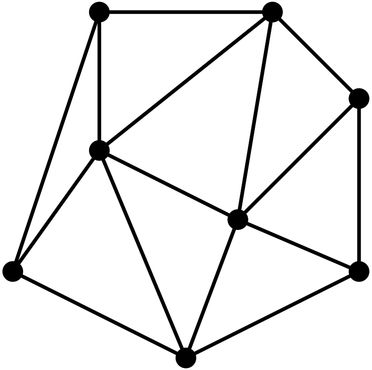

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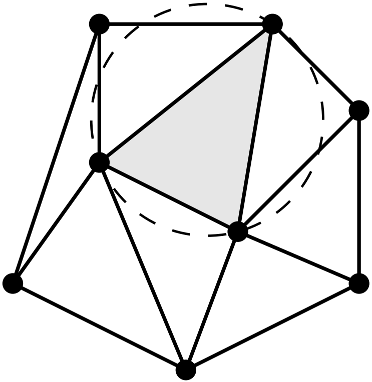
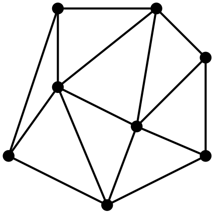


# Mathematical Preliminaries: Properties



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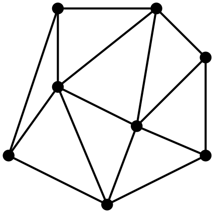
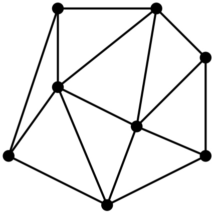
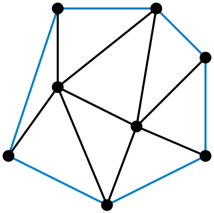


- Optimality: maximization of the minimum angle of all angles

# Mathematical Preliminaries: Properties





- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained

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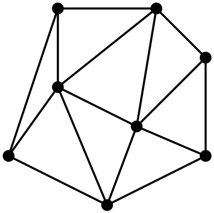
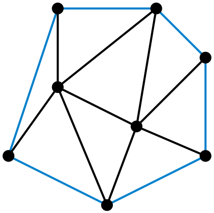
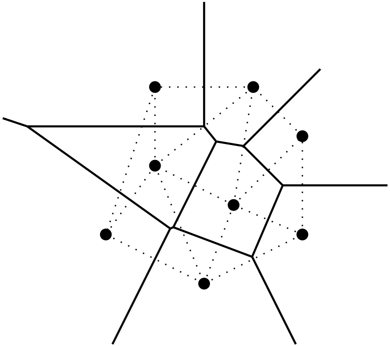




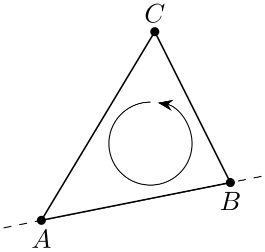
- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained
- Voronoi diagram is the dual

# Mathematical Preliminaries: Properties of Delaunay Triangulation

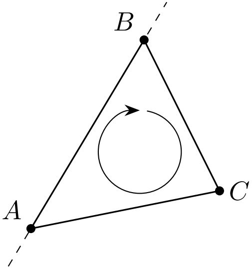
- Duality to Voronoi Diagram (also useful for proofs)
- always exists -> proof
- If no points are cocircular, unique -> proof
- optimality: maximization of the minimum angle of all angles -> proof, reason why delaunay is good
- boundary is convex hull

# Geometric Primitives

# Geometric Primitives: Counterclockwise

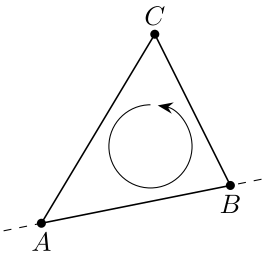



$\longleftrightarrow$




Counterclockwise Order

# Geometric Primitives: Counterclockwise

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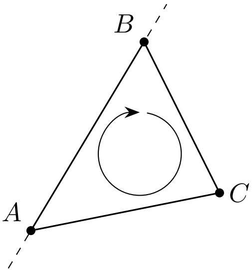
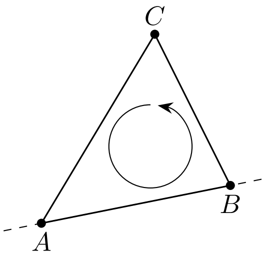



Counterclockwise Order $\iff$ $C$ is left of $\overline{AB}$

# Geometric Primitives: Counterclockwise

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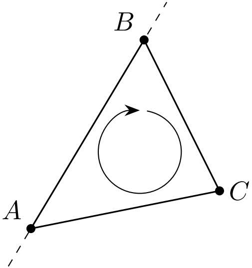
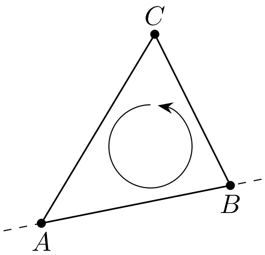



Counterclockwise Order $\iff$ $C$ is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

# Geometric Primitives: Counterclockwise

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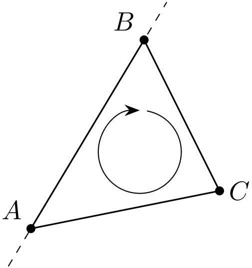


Counterclockwise Order $\iff$ $C$ is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

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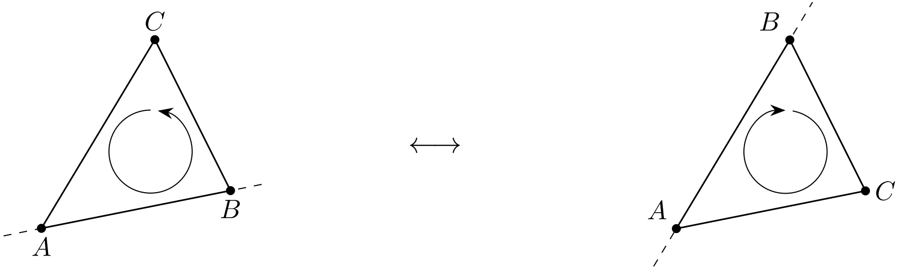


Counterclockwise Order $\iff$ $C$ is left of $\overline{AB}$

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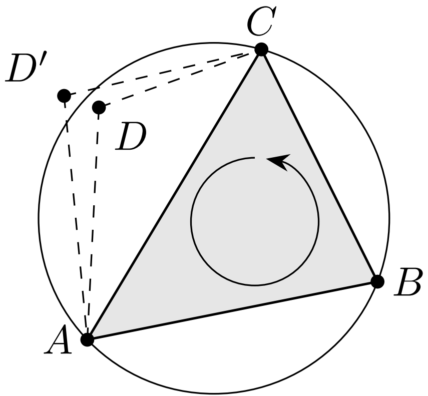
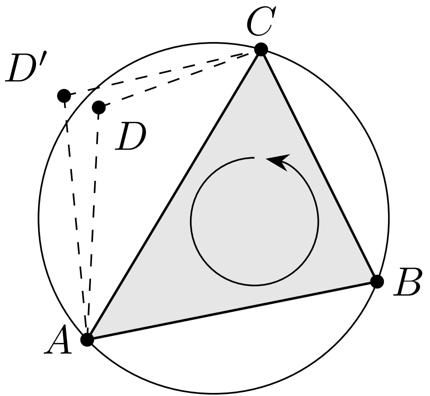
# Geometric Primitives: Inside Circumcircle



# Geometric Primitives: Inside Circumcircle

$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



# Data Structures

# Algorithm

# Implementation

# Implementation

- Geometric Primitives need exact computation and therefore arbitrary precision

# Applications

# Conclusions

Thank you for Your Attention!

# References


[1] D. T. Lee and B. J. Schachter. "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9 (1980), pp. 219–242. DOI: 10.1007/BF00977785.

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