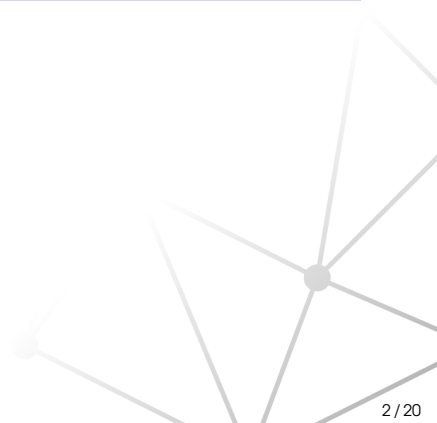


Algorithmical Geometry: Delaunay Triangulation

Markus Pawellek

January 16, 2022



Outline

Introduction

Mathematical Preliminaries

Geometric Primitives

Data Structure

Algorithm

Implementation

Applications

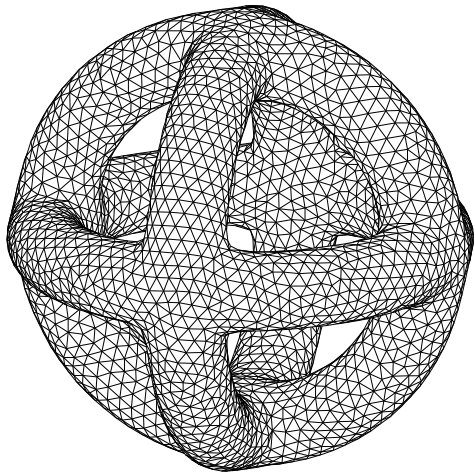
Conclusions



Introduction

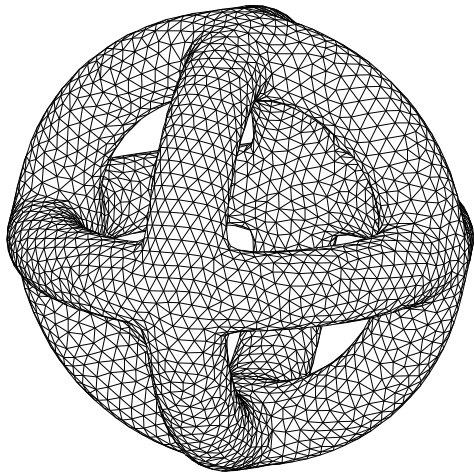


Introduction



*<https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg>, December 29, 2021

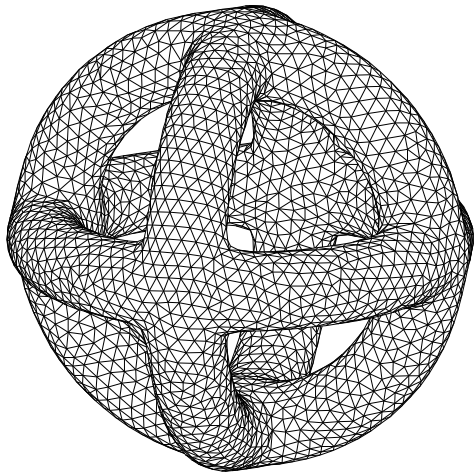
Introduction



Educational Problems:

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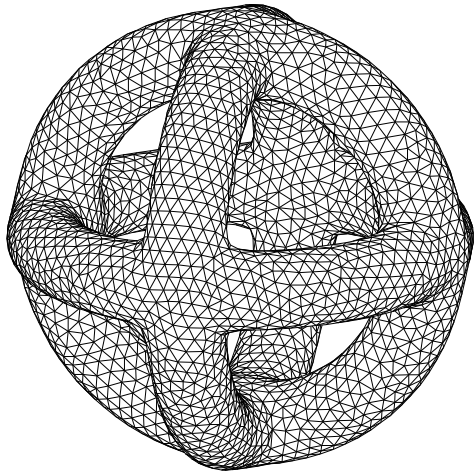


Educational Problems:

- ▶ Many Resources

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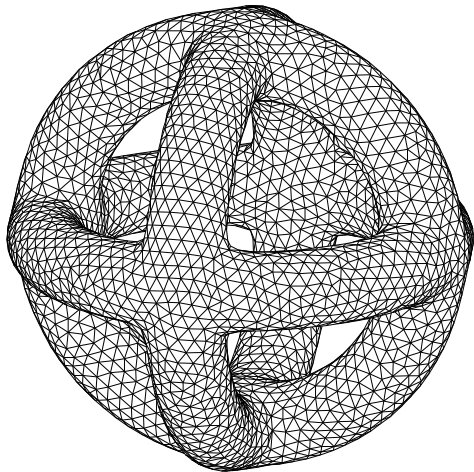


Educational Problems:

- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams

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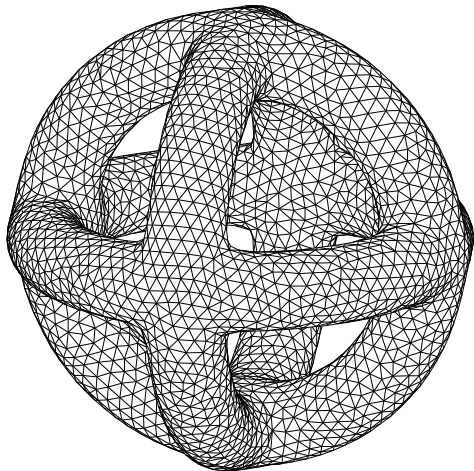


Educational Problems:

- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams
- ▶ Multiple Algorithm Types:
Incremental, Sweepline,
Divide-and-Conquer

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Educational Problems:

- ▶ Many Resources
- ▶ Duality to Voronoi Diagrams
- ▶ Multiple Algorithm Types:
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Divide-and-Conquer
- ▶ Varying Data Structures

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Introduction: Previous Work



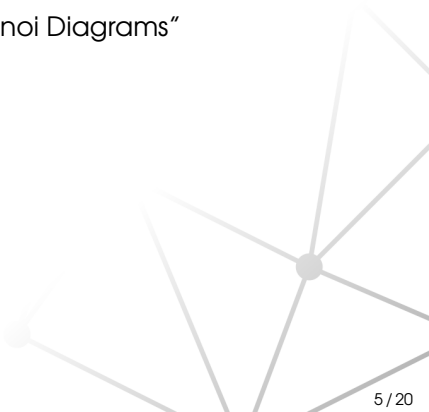
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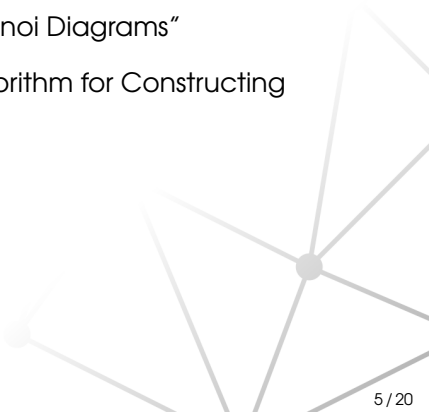
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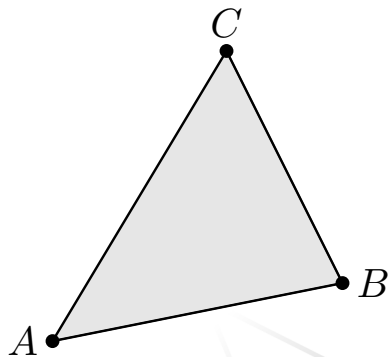
Mathematical Preliminaries



Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent
define vertices of a triangle.



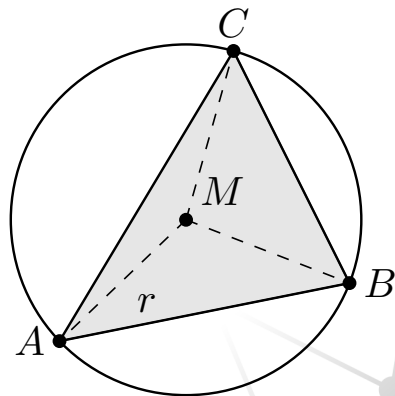
Mathematical Preliminaries: Triangle and Circumcircle

Triangle

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Circumcircle

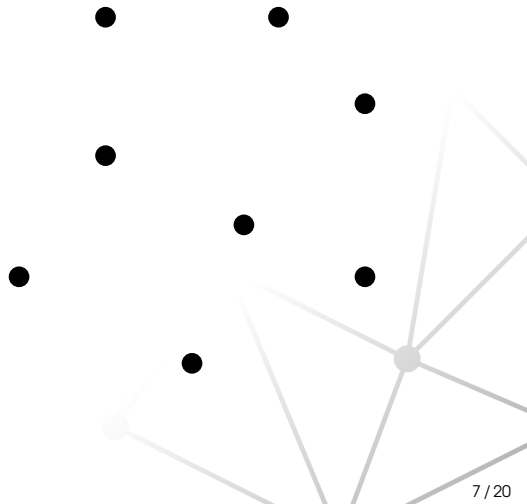
Circle that intersects with
all vertices of the triangle.



Mathematical Preliminaries: Point Set

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
affinely span \mathbb{R}^2



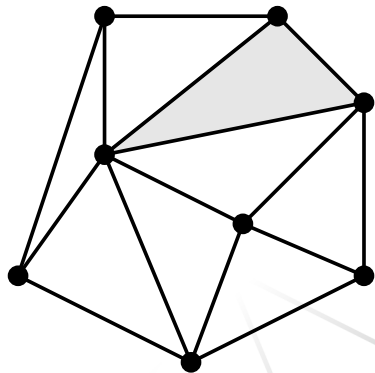
Mathematical Preliminaries: Triangulation

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
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Triangulation

Planar straight-line graph over \mathcal{V}
such that its edges form a maximal
subset of non-crossing edges.



Mathematical Preliminaries: Delaunay Triangulation

Point Set

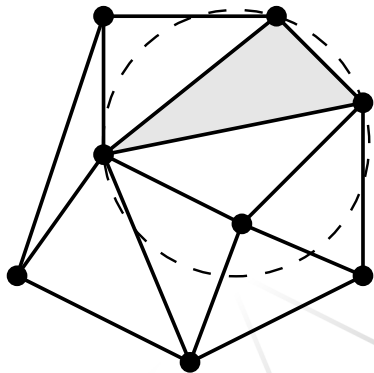
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Delaunay Triangulation

Circumcircle of any triangle
contains no other points of \mathcal{V} .



Mathematical Preliminaries: Delaunay Triangulation

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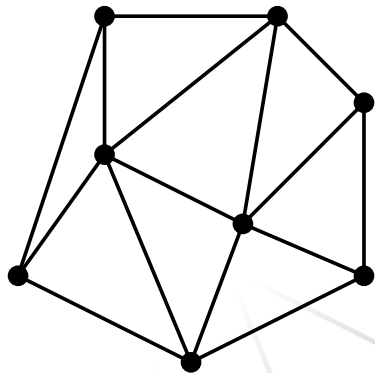
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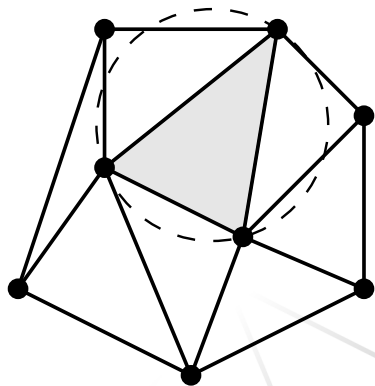
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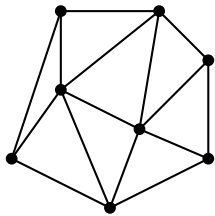
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Delaunay Triangulation

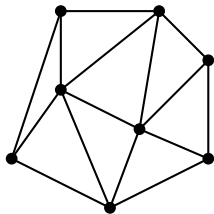
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Mathematical Preliminaries: Properties

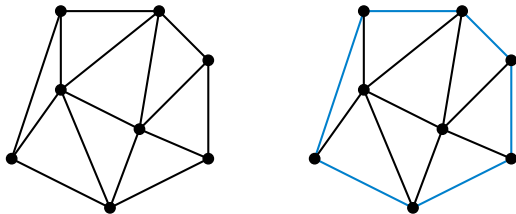


Mathematical Preliminaries: Properties



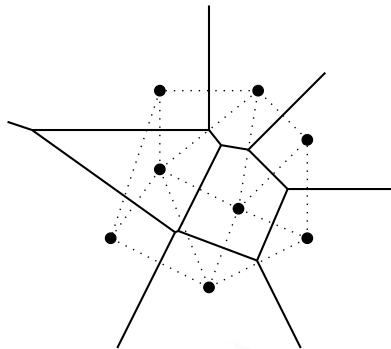
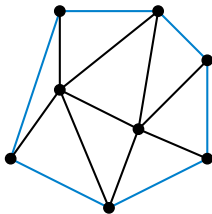
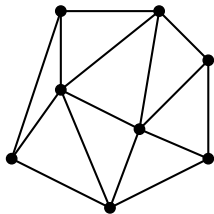
- Optimality: maximization of the minimum angle of all angles

Mathematical Preliminaries: Properties



- ▶ Optimality: maximization of the minimum angle of all angles
- ▶ Convex hull is contained

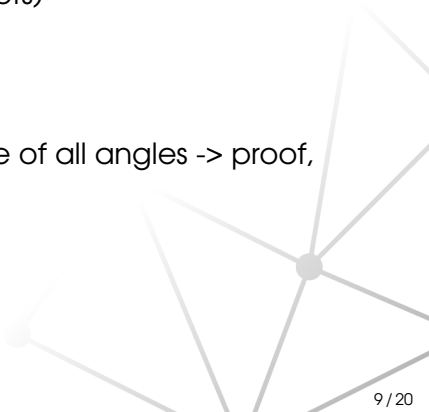
Mathematical Preliminaries: Properties



- ▶ Optimality: maximization of the minimum angle of all angles
- ▶ Convex hull is contained
- ▶ Voronoi diagram is the dual

Mathematical Preliminaries: Properties of Delaunay Triangulation

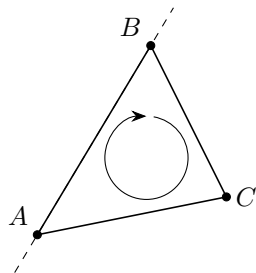
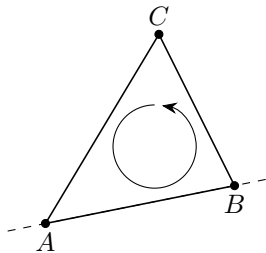
- ▶ Duality to Voronoi Diagram (also useful for proofs)
- ▶ always exists -> proof
- ▶ If no points are cocircular, unique -> proof
- ▶ optimality: maximization of the minimum angle of all angles -> proof, reason why delaunay is good
- ▶ boundary is convex hull



Geometric Primitives

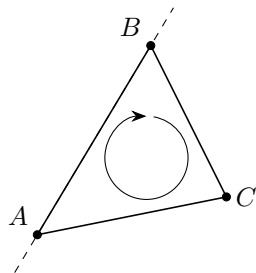
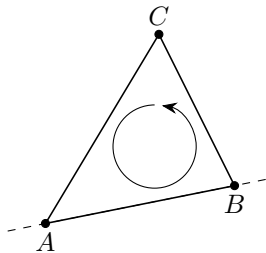


Geometric Primitives: Counterclockwise



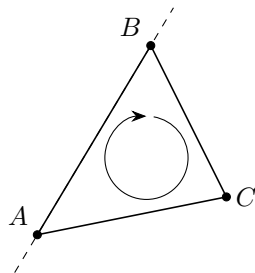
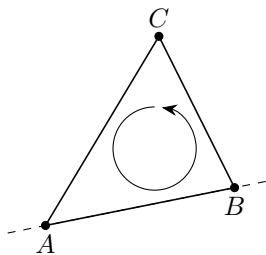
Counterclockwise Order

Geometric Primitives: Counterclockwise



Counterclockwise Order \iff C is left of \overline{AB}

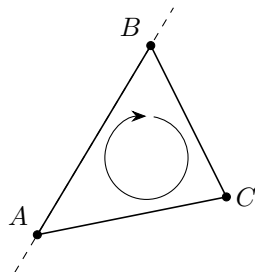
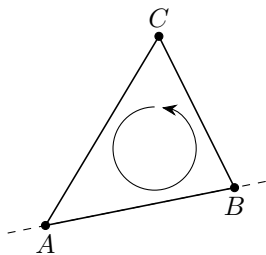
Geometric Primitives: Counterclockwise



Counterclockwise Order \iff C is left of \overline{AB}

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

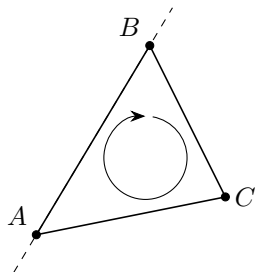
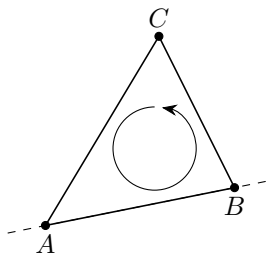
Geometric Primitives: Counterclockwise



Counterclockwise Order \iff C is left of \overline{AB}

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

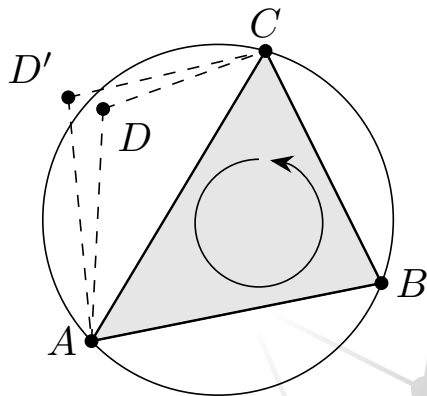
Geometric Primitives: Counterclockwise



Counterclockwise Order \iff C is left of \overline{AB}

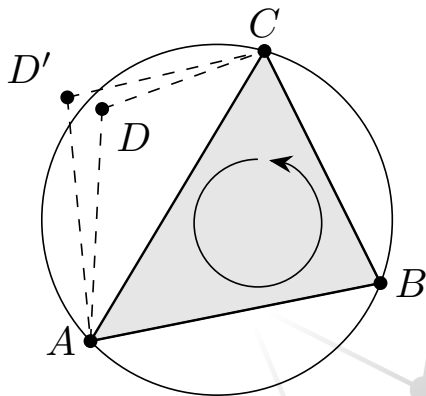
$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix} = \det \begin{pmatrix} B - A & C - A \end{pmatrix}$$

Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$

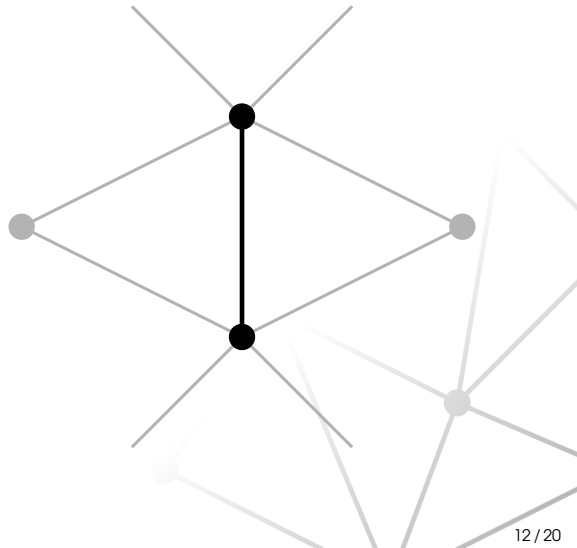


Data Structure



Data Structure: Quad-Edge

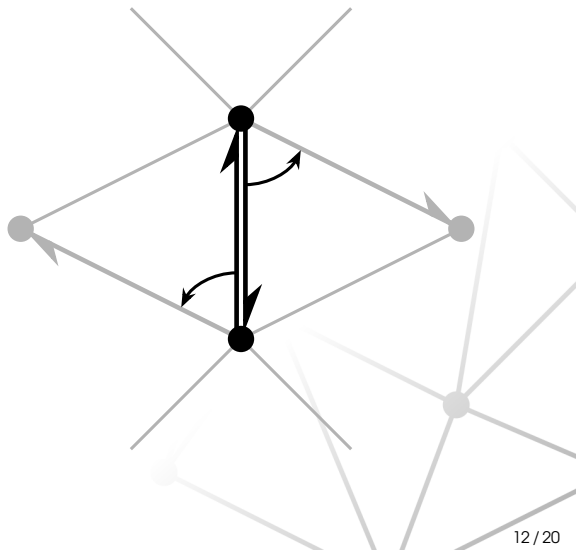
Edge-Based List-Like Data Structure:



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

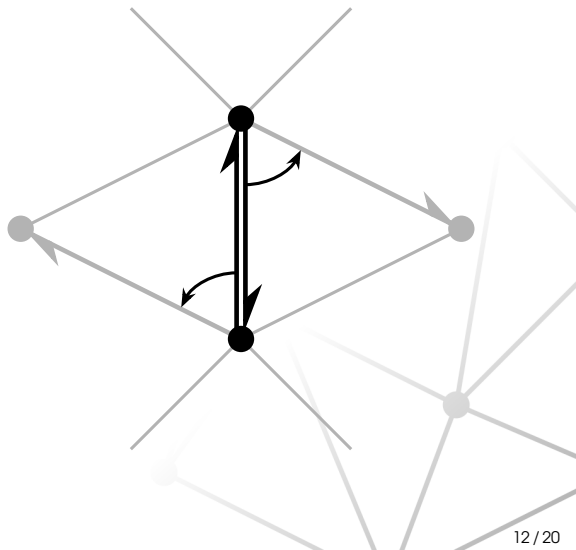
- ▶ Directed edges for vertices



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

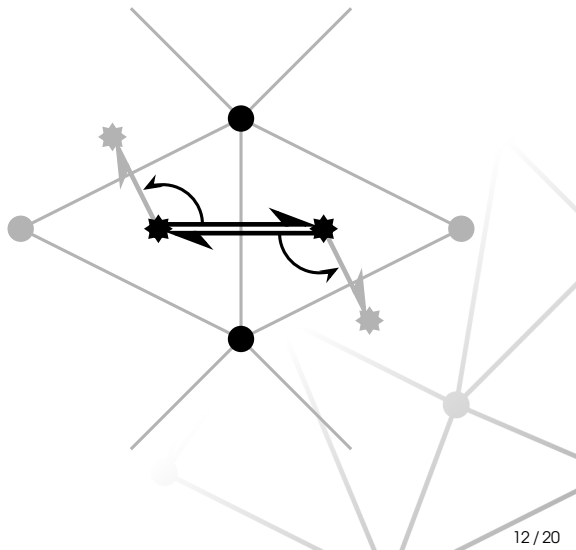
- ▶ Directed edges for vertices
- ▶ Pointer to ccw. next directed edge with same origin vertex



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

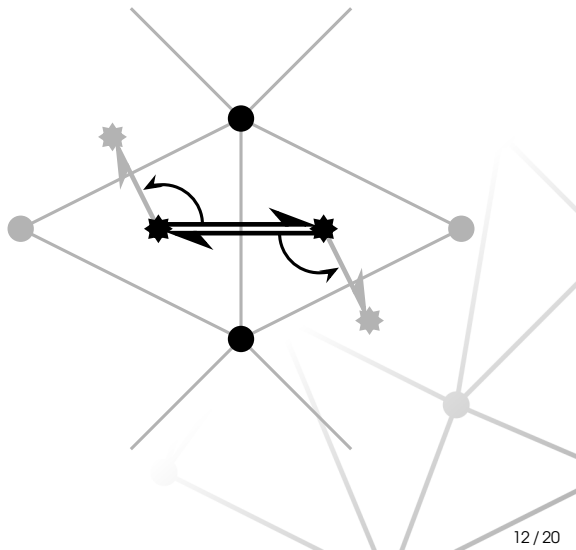
- ▶ Directed edges for vertices
- ▶ Pointer to ccw. next directed edge with same origin vertex
- ▶ Directed dual edges for adjacent faces



Data Structure: Quad-Edge

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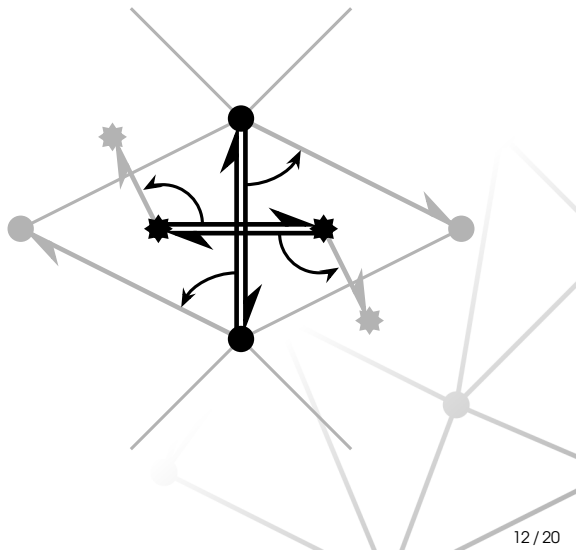
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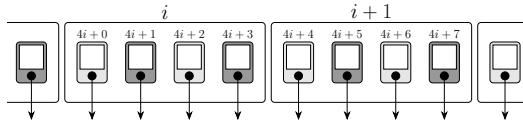
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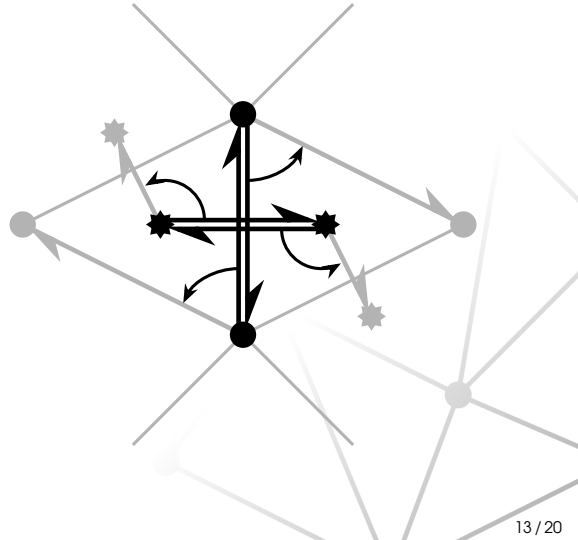
- ▶ Directed edges for vertices
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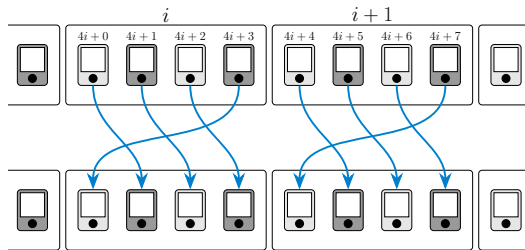
Data Structure: Quad-Edge Implementation



```
struct edge {  
    size_t next;  
    size_t data;  
};  
  
struct quad_edge {  
    edge data[4];  
};  
  
vector<vertex>    vertices{};  
vector<quad_edge> quad_edges{};  
vector<size_t>    free_edges{};
```

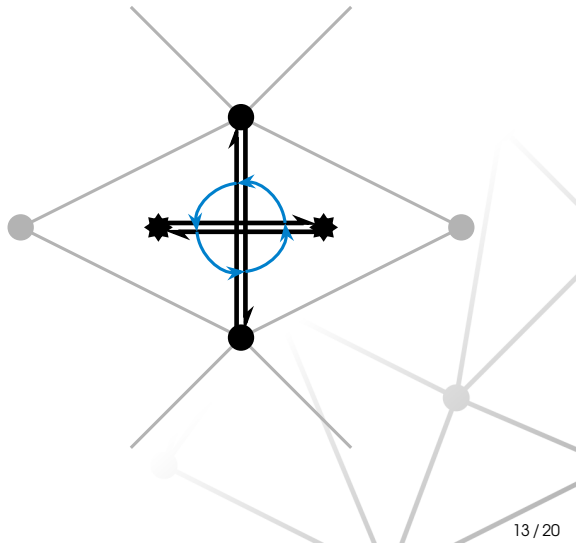


Data Structure: Quad-Edge Implementation

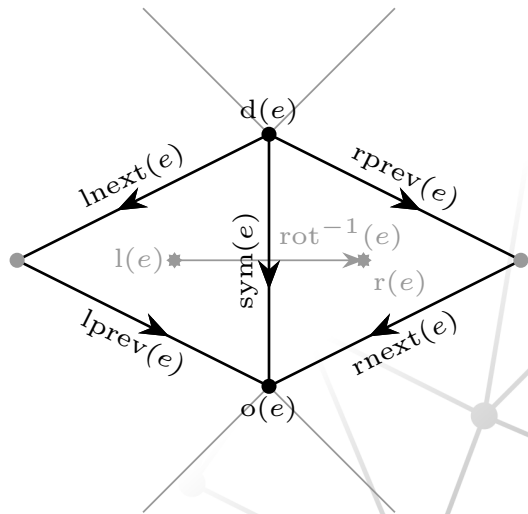
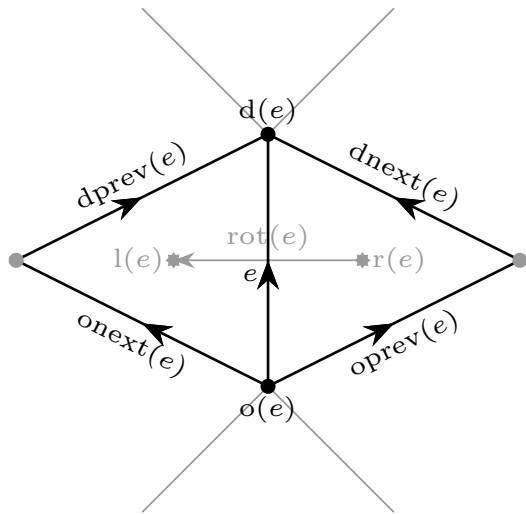


$$\text{rot}: \mathbb{N}_0 \rightarrow \mathbb{N}_0$$

$$\text{rot}(x) = 4 \cdot \left\lfloor \frac{x}{4} \right\rfloor + (x + 1 \bmod 4)$$



Data Structure: Quad-Edge Functions



Data Structure: Quad-Edge Operations

- ▶ edge functions
- ▶ create edge
- ▶ splice
- ▶ connect
- ▶ delete edge



Algorithm



Algorithm: Overview



Algorithm: Overview

Triangulation Algorithm



Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.



Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.



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Subroutine: Triangulate

Algorithm: Overview

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1. Sort the given point set by increasing x coordinate.
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1. If point count is smaller than four, make edge or triangle and return.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
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1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
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2. Split point set into left and right half.
3. Triangulate left and right half.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.
3. Triangulate left and right half.
4. Merge left and right triangulations.

Algorithm: Merge Triangulations



Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:
 - 3.1 Remove invalid edges adjacent to and above baseline.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:
 - 3.1 Remove invalid edges adjacent to and above baseline.
 - 3.2 Insert cross edge above baseline.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:
 - 3.1 Remove invalid edges adjacent to and above baseline.
 - 3.2 Insert cross edge above baseline.
 - 3.3 Make this cross edge the new baseline.

Algorithm: Correctness



Algorithm: Complexity



Implementation



- ▶ Geometric Primitives need exact computation and therefore arbitrary precision

Applications



Conclusions



Thank you for Your Attention!



References

- (1) D. T. Lee and B. J. Schachter. "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9 (1980), pp. 219–242. DOI: 10.1007/BF00977785.
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