Algorithmical Geometry: Computation of Delaunay Triangulations Using a Divide-and-Conquer Algorithm

Markus Pawellek

January 20, 2022

Outline

Related Work

Mathematical Preliminaries

Geometric Primitives

Quad-Edge Data Structure

Algorithm

Implementation Notes

Conclusions

Related Work

Educational Problems:

Related Work

Educational Problems:

- Many Resources

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- Duality to Voronoi Diagrams
- Multiple Algorithm Types:
  Incremental, Sweepline, Divide-and-Conquer

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Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types:
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- Varying Data Structures

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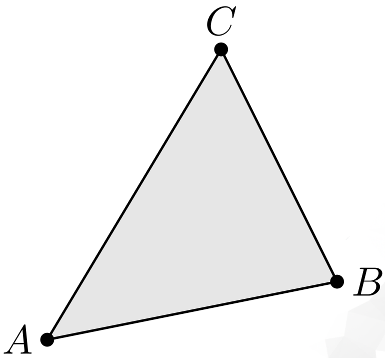
Mathematical Preliminaries

Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.



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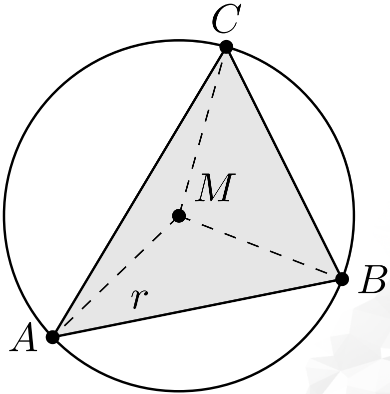
Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.

Circumcircle

Circle that intersects with all vertices of the triangle.

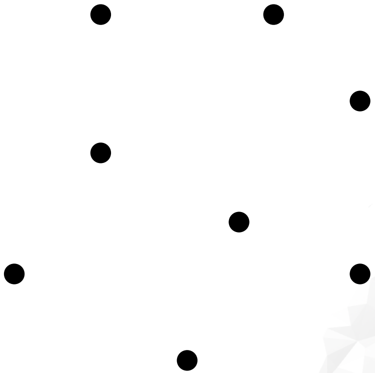


Mathematical Preliminaries: Point Set

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,

affinely span $\mathbb{R}^2$



Mathematical Preliminaries: Triangulation

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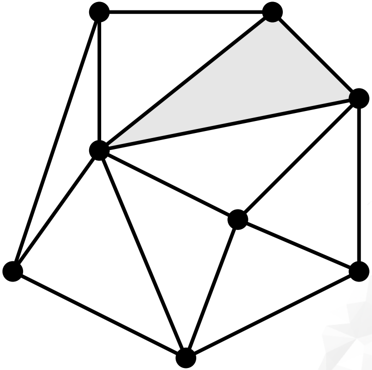
Triangulation

Planar straight-line graph over $\mathcal{V}$ such that its edges form a maximal subset of non-crossing edges.



Mathematical Preliminaries: Delaunay Triangulation

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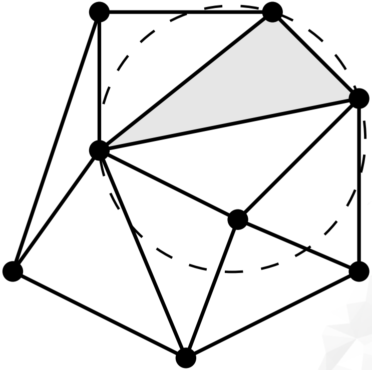
Delaunay Triangulation

Circumcircle of any triangle contains no other points of $\mathcal{V}$.



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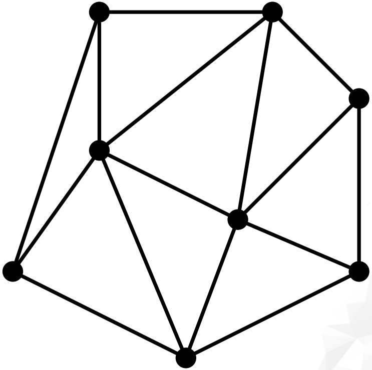

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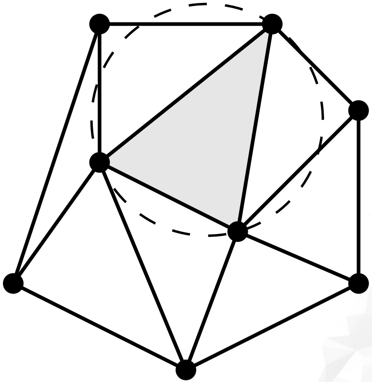
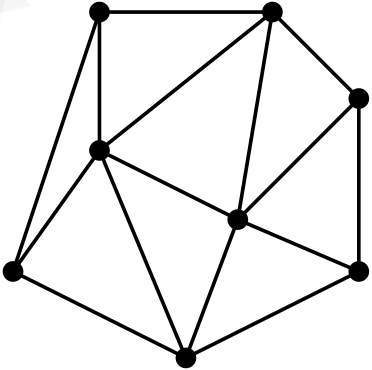


Mathematical Preliminaries: Properties



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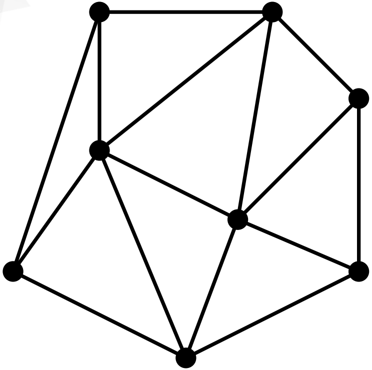

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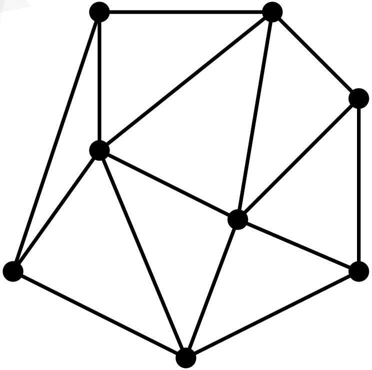

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- Unique if there are no four points that are cocircular

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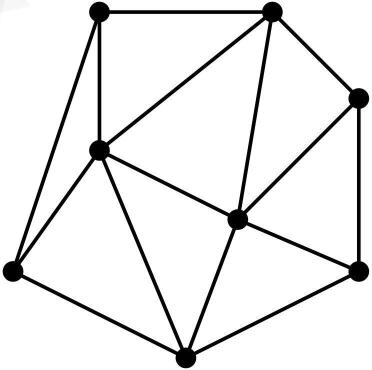

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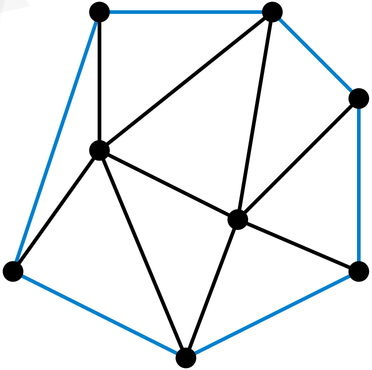

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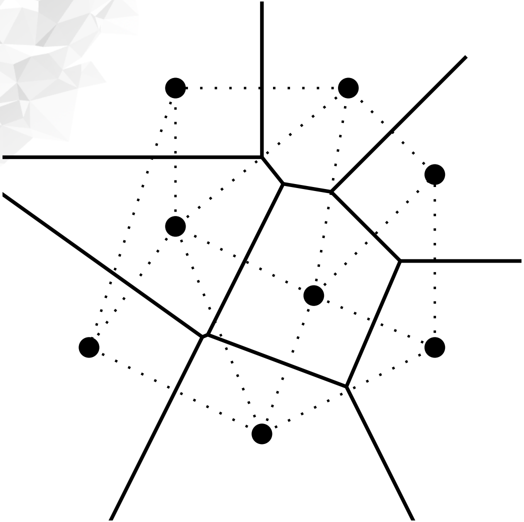
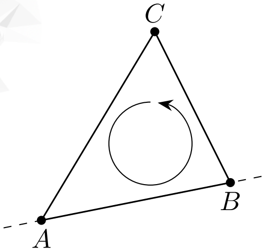


- Existence is guaranteed
- Unique if there are no four points that are cocircular
- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained
- Dual of Voronoi diagram

Geometric Primitives

Geometric Primitives: Triangle Orientation



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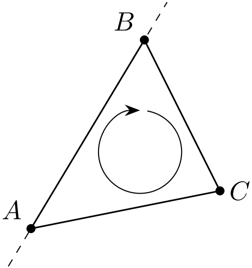

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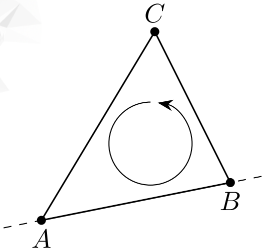

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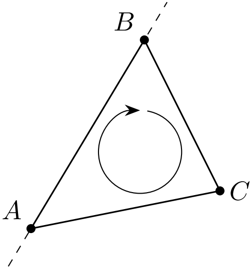
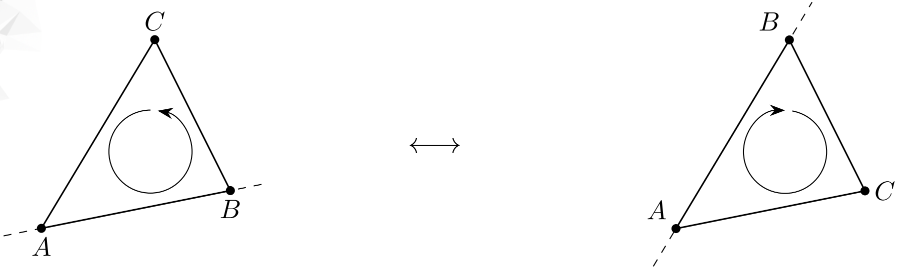


Counterclockwise Orientation $\iff$ C is left of $\overline{AB}$

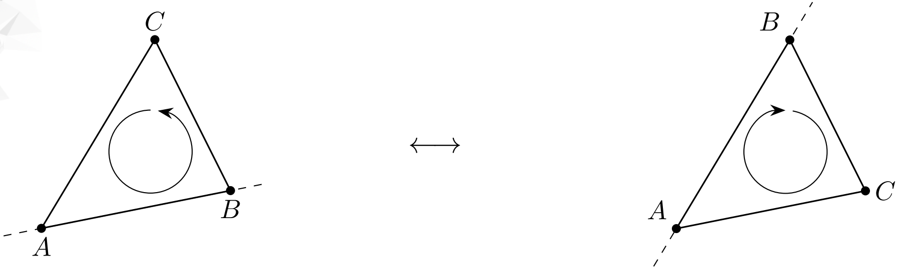
Geometric Primitives: Triangle Orientation



Counterclockwise Orientation $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

Geometric Primitives: Triangle Orientation



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$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

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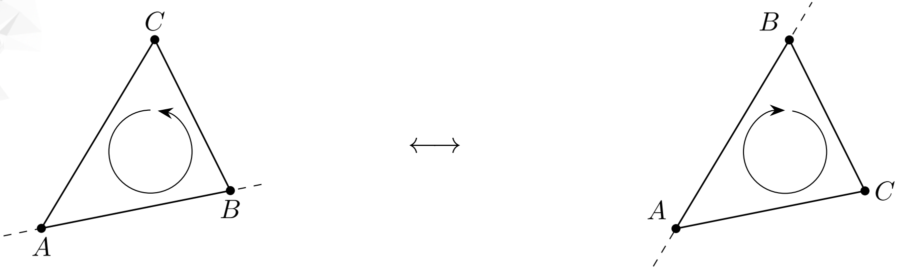

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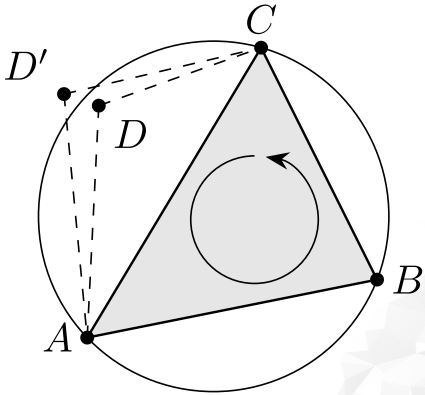
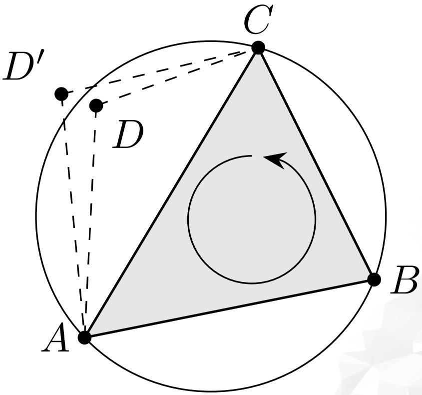
Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



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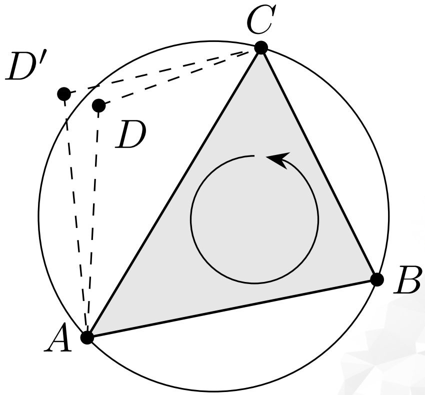
$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$

$$= \left\langle x, \operatorname{adj} \begin{pmatrix} u & v \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \|u\|^2 \\ \|v\|^2 \end{pmatrix} \right\rangle - \det \begin{pmatrix} u & v \end{pmatrix} \|x\|^2$$

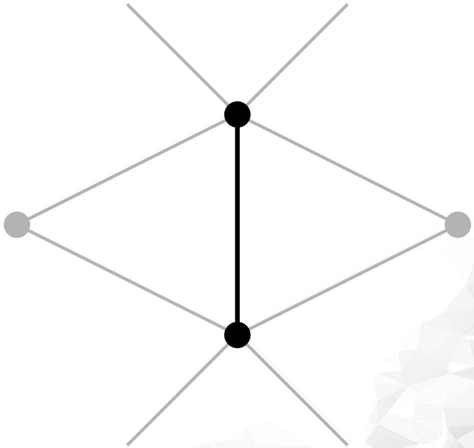
$u := B - A, \quad v := C - A, \quad x := D - A$



Quad-Edge Data Structure

Quad-Edge Data Structure: Scheme

Edge-Based List-Like Data Structure
for Storing Neighbor Information:



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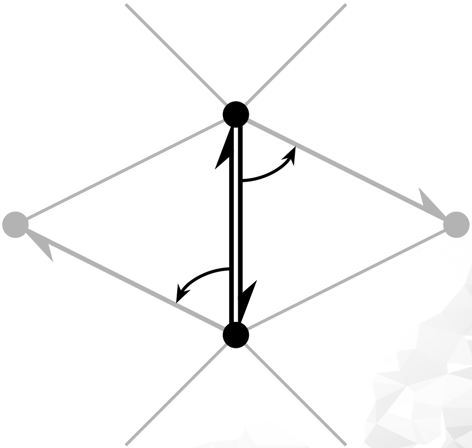
- Directed edges for vertices



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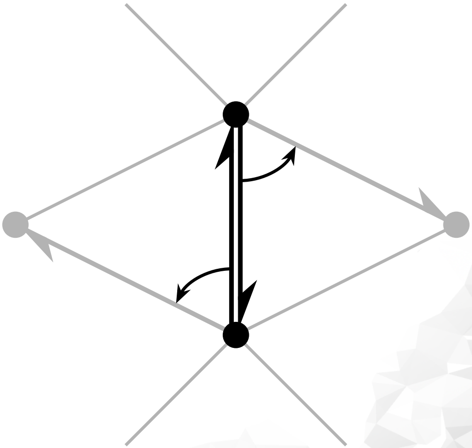
- Pointer to ccw. next directed edge with same origin vertex



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Edge-Based List-Like Data Structure
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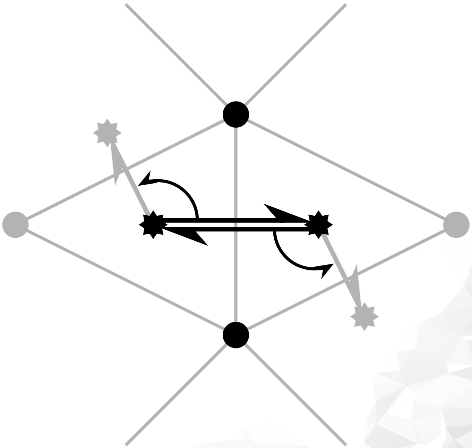
- Directed dual edges for adjacent faces



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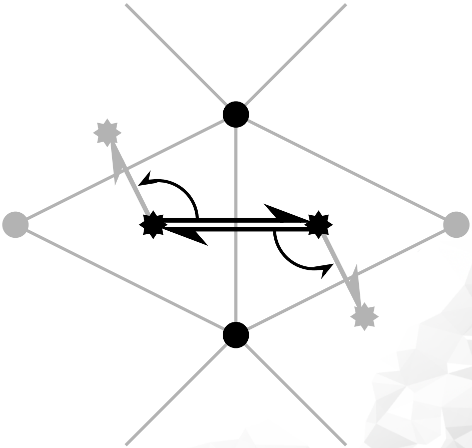

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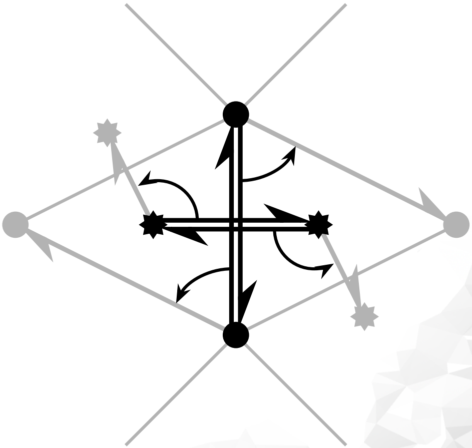
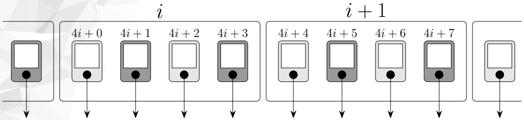


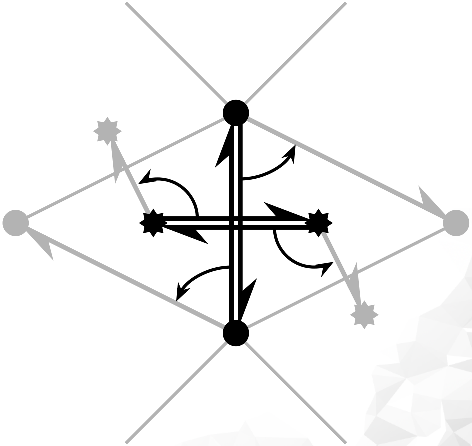
Quad-Edge Data Structure: Implementation




```
struct edge {
  size_t next;
  size_t data;
};

struct quad_edge {
  edge data[4];
};

vector<vertex>    vertices{};
vector<quad_edge> quad_edges{};
vector<size_t>    free_edges{};
```



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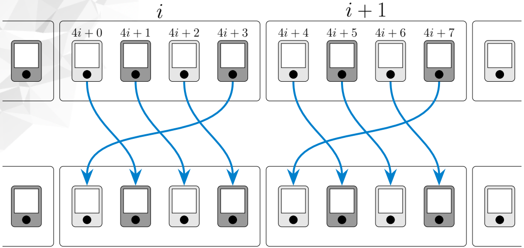
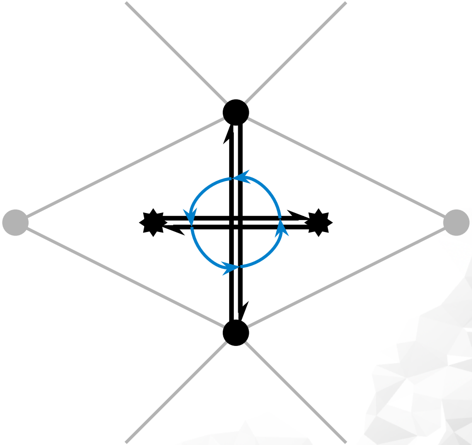
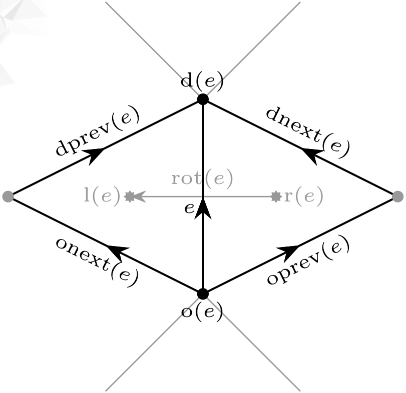


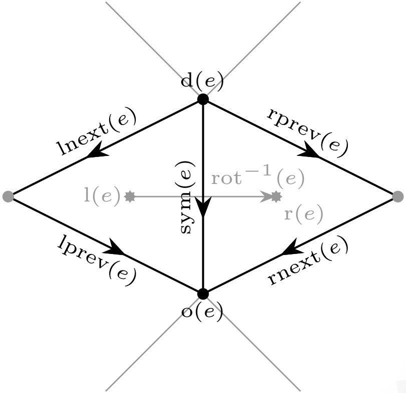
$$\mathrm{rot}\colon \mathbb{N}_0 \to \mathbb{N}_0$$

$$\mathrm{rot}(x) = 4 \cdot \left\lfloor \frac{x}{4} \right\rfloor + (x + 1 \mod 4)$$



Quad-Edge Data Structure: Edge Functions and Operators





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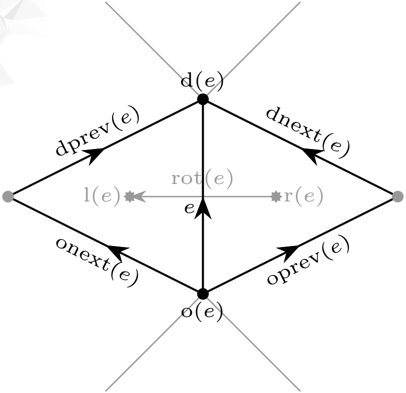
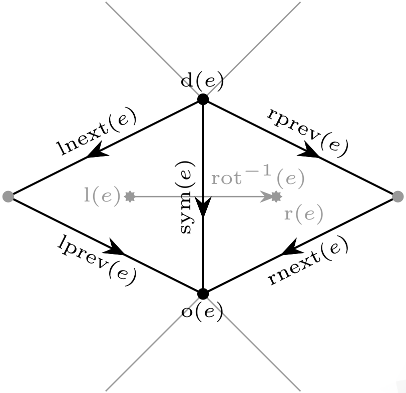
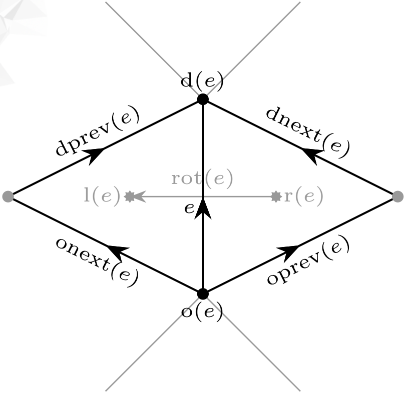
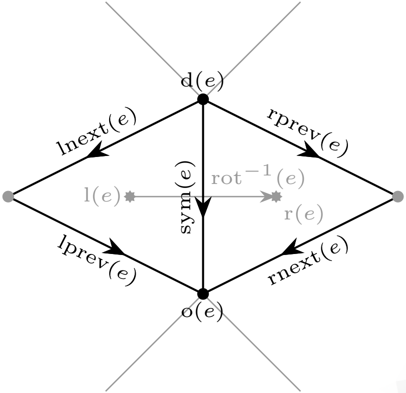




- Create a new edge

Quad-Edge Data Structure: Edge Functions and Operators





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- Delete existing edge

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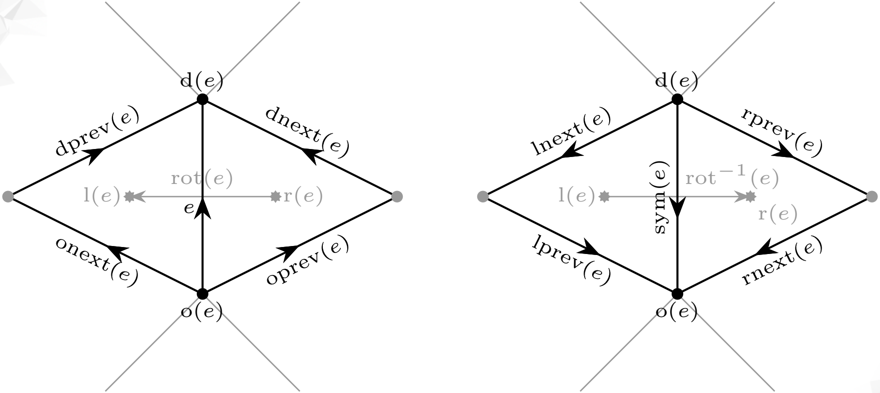

- Create a new edge
- Delete existing edge
- Connect points by a new edge

Algorithm

Algorithm: Overview

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Triangulation Algorithm

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Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.

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Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

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2. Split point set into left and right half.

Algorithm: Overview

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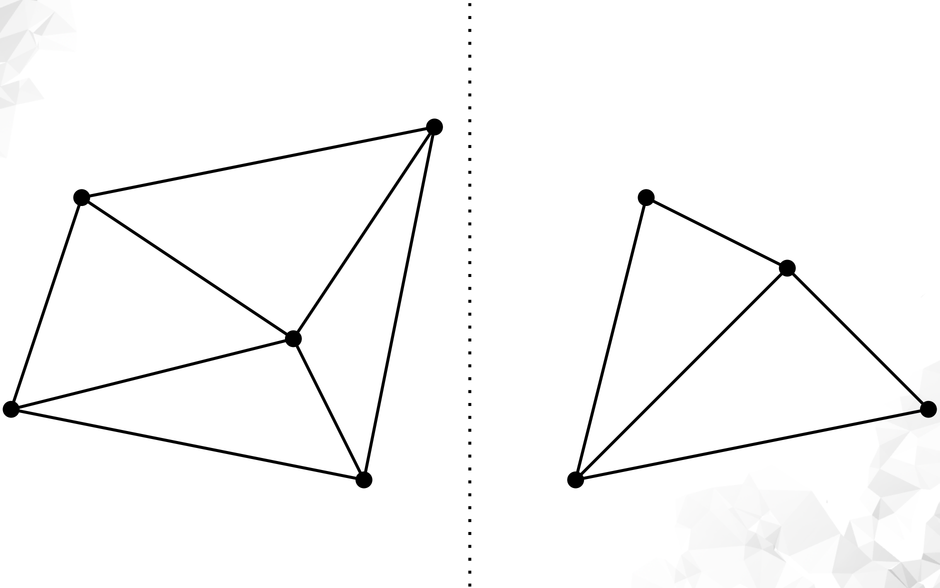
Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.
3. Triangulate left and right half.
4. Merge left and right triangulations.

Algorithm: Merge Triangulations Example



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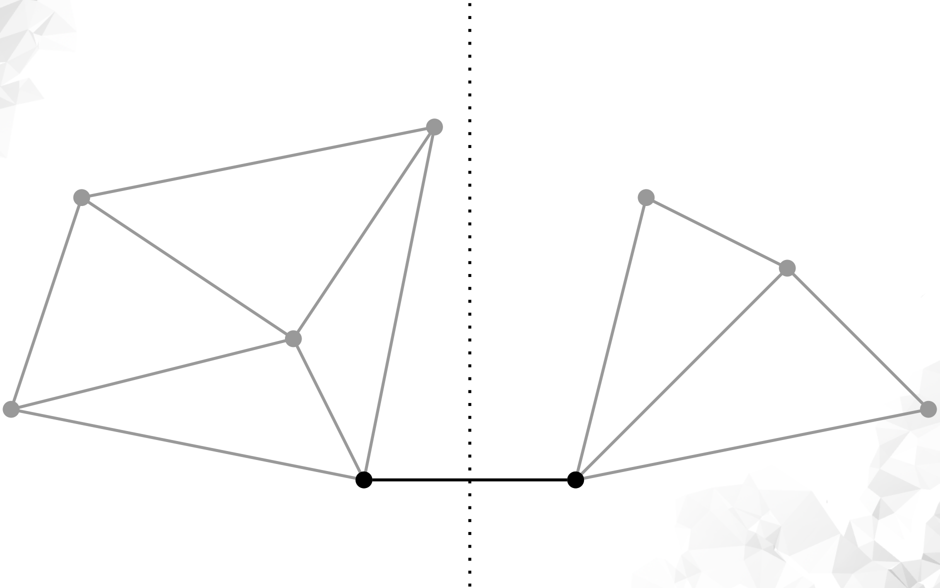

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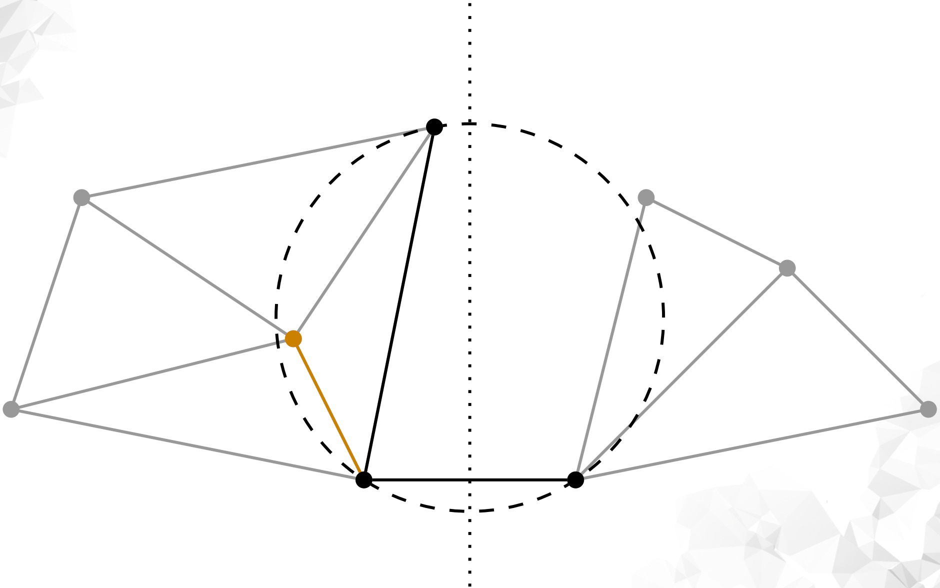

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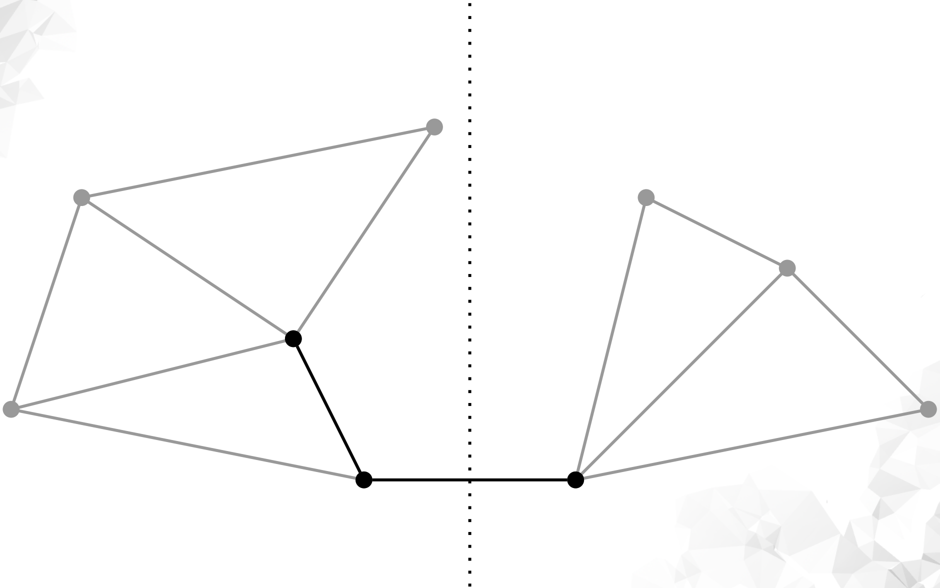

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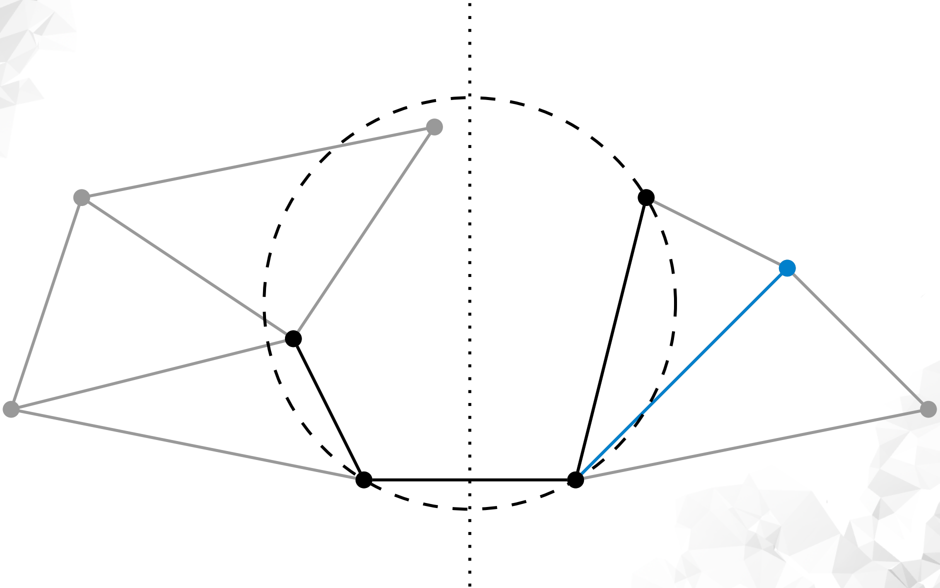

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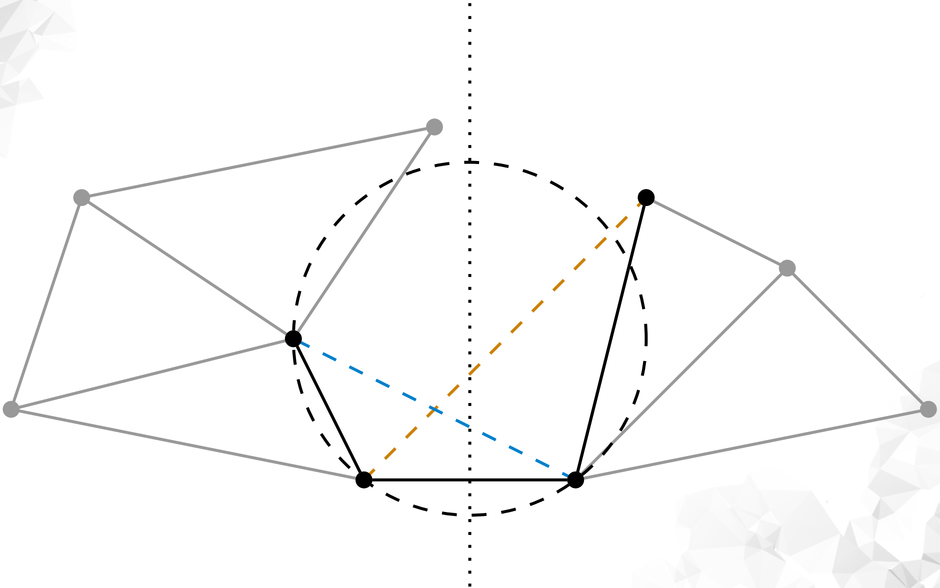

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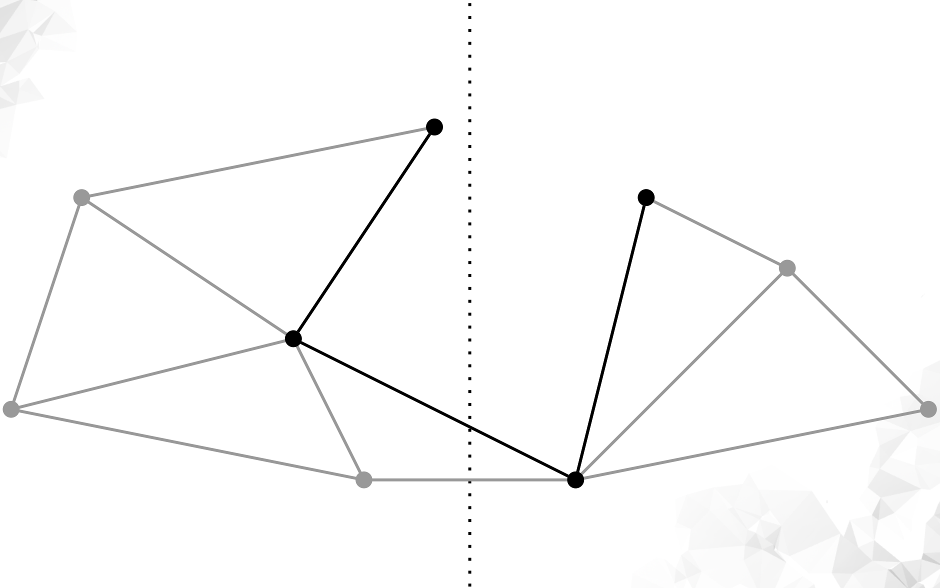

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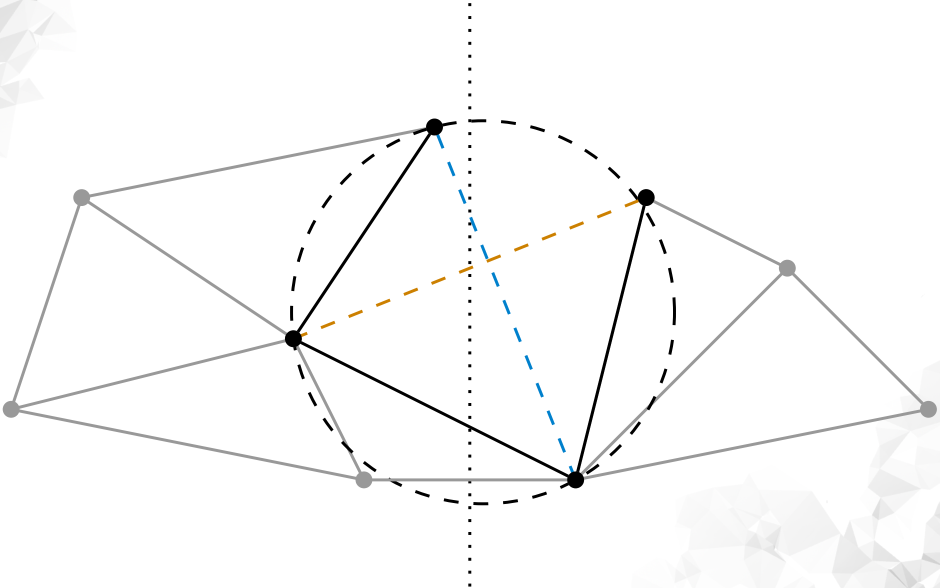

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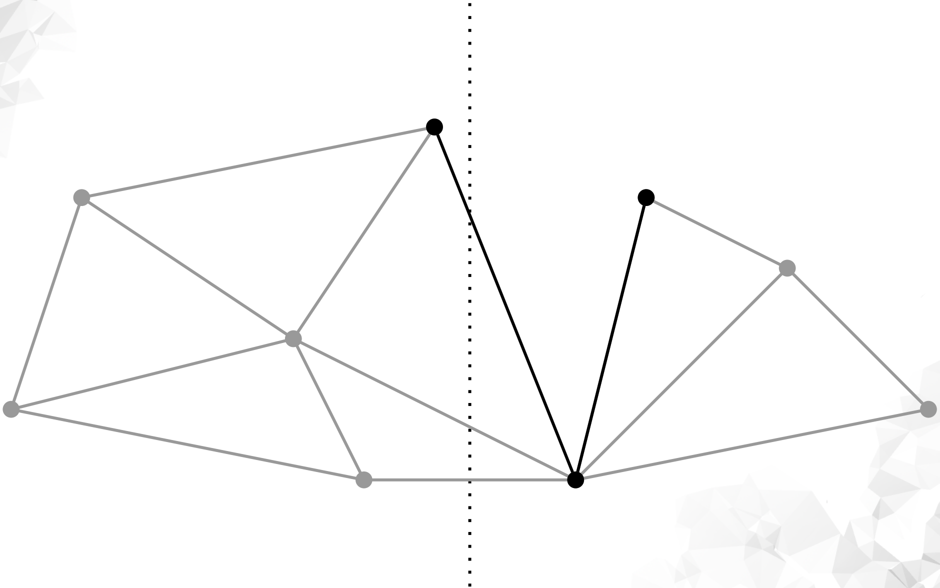

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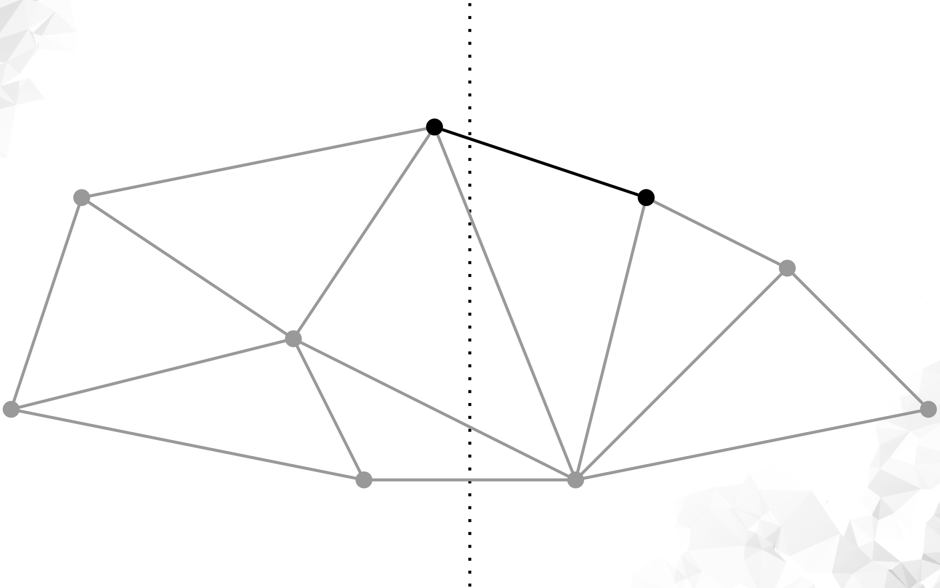

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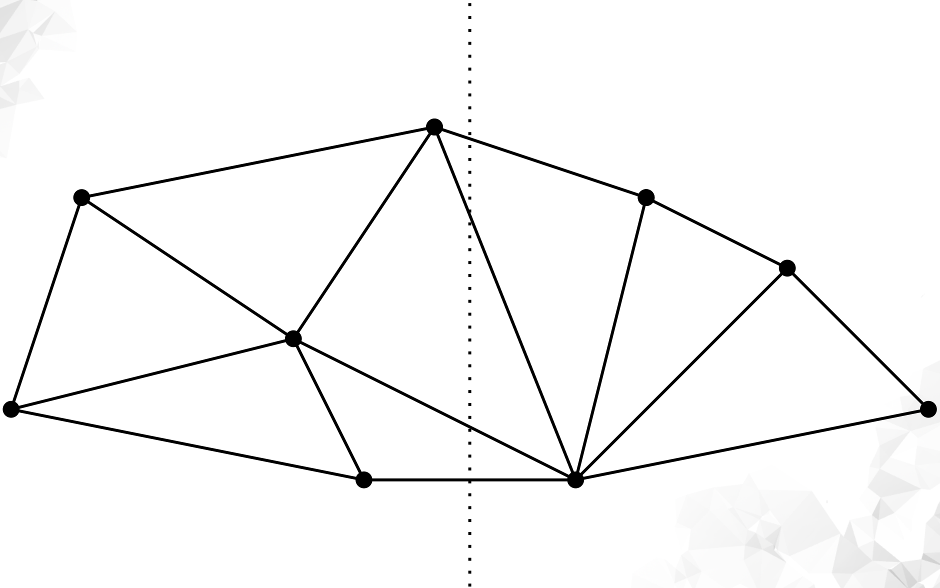

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Thank you for Your Attention!

References

(1) D. T. Lee and B. J. Schachter. "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9 (1980), pp. 219–242. DOI: 10.1007/BF00977785.

(2) Leonidas Guibas and Jorge Stolfi. "Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams". In: *ACM Transactions on Graphics* 4 (April 1985), pp. 74–123. DOI: 10.1145/282918.282923. URL: http://sccg.sk/~samuelcik/dgs/quad_edge.pdf (visited on 11/07/2020).

(3) Rex A. Dwyer. "A Faster Divide-and-Conquer Algorithm for Constructing Delaunay Triangulations". In: *Algorithmica* 2 (November 1987), pp. 137–151. DOI: 10.1007/BF01840356.

(4) Jonathan Richard Shewchuk. "Triangle: Engineering a 2D Quality Mesh Generator and Delaunay Triangulator". In: *Applied Computational Geometry: Towards Geometric Engineering*. Ed. by Ming C. Lin and Dinesh Manocha. Vol. 1148. Lecture Notes in Computer Science. From the First ACM Workshop on Applied Computational Geometry. Springer-Verlag, May 1996, pp. 203–222. URL: https:

(7) D. F. Watson. "Computing the n-Dimensional Delaunay Tessellation with Application to Voronoi Polytopes". In: *The Computer Journal* 24 (1981), pp. 167–172. DOI: 10.1093/comjnl/24.2.167.

(8) A. Bowyer. "Computing Dirichlet Tessellations". In: *The Computer Journal* 24 (1981), pp. 162–166. DOI: 10.1093/comjnl/24.2.162.

(9) Christoph Burnikel. *Delaunay Graphs by Divide and Conquer*. 1998. URL: https://pure.mpg.de/rest/items/item_1819432_4/component/file_2599484/content (visited on 11/07/2020).

(10) P. Cignoni, C. Montani, and R. Scopigno. "DeWall: A Fast Divide-and-Conquer Delaunay Triangulation Algorithm in E^d". In: *Computer-Aided Design* 30 (1998), pp. 333–341. DOI: 10.1016/S0010-4485(97)00082-1.

(11) Jyrki Katajainen and Markku Koppinen. "Constructing Delaunay Triangulations by Merging Buckets in Quad-Tree Order". In: *Fundamenta Informaticae* 11 (April 1988), pp. 275–288.