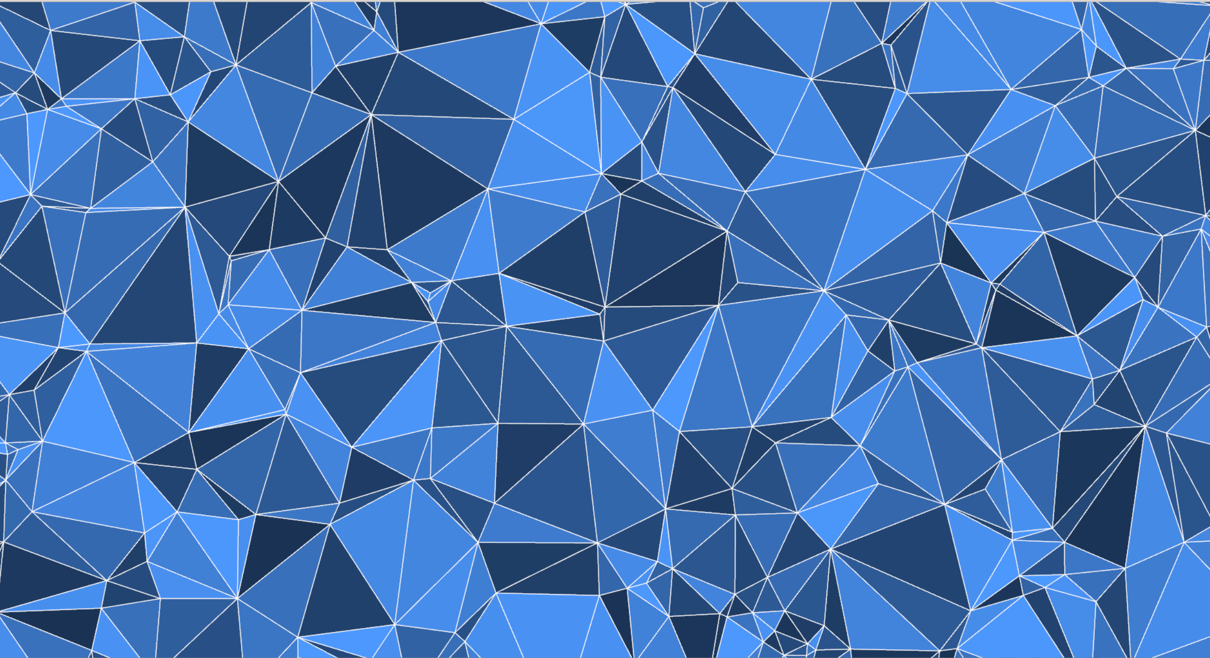
Delaunay Triangulation Mosaic



Algorithmical Geometry: Computation of Delaunay Triangulations Using a Divide-and-Conquer Algorithm

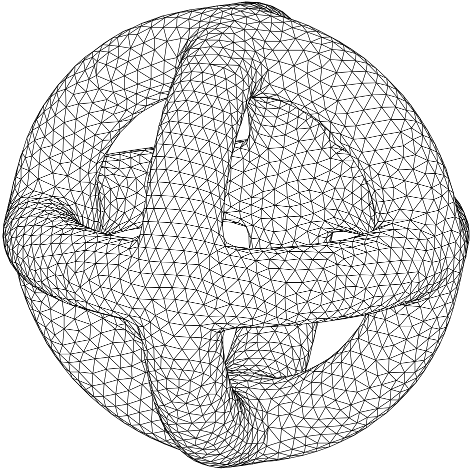
Markus Pawellek

January 17, 2022

Outline

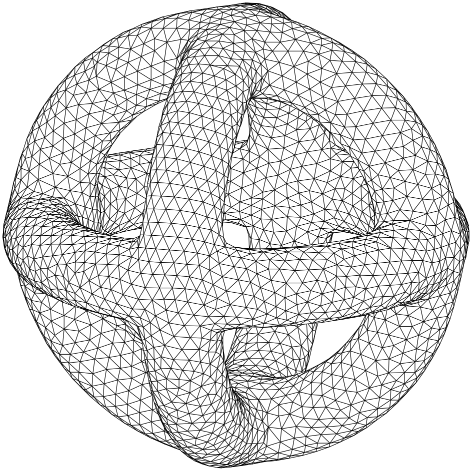
Related Work

Related Work



*https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg, December 29, 2021

Related Work



Educational Problems:

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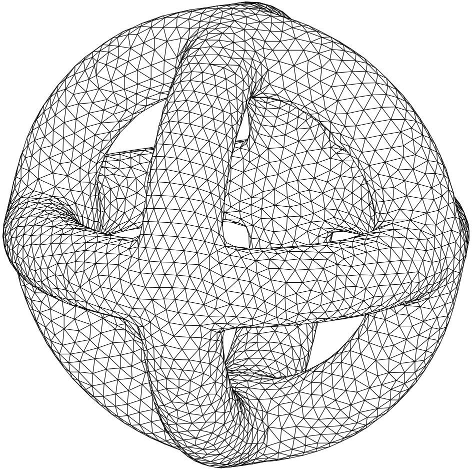

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Educational Problems:

- Many Resources

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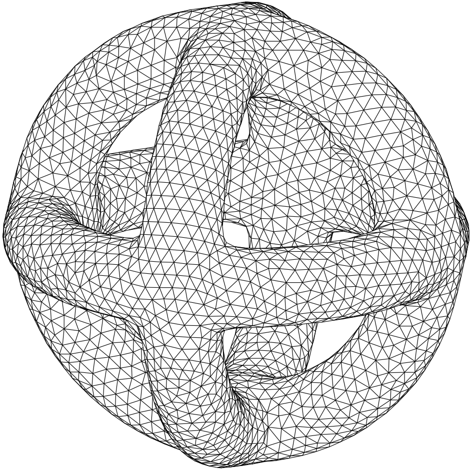

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams

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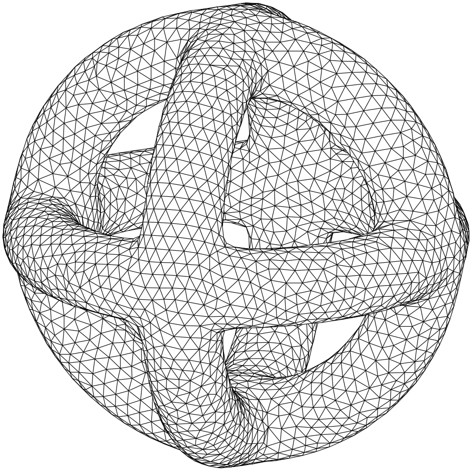

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer

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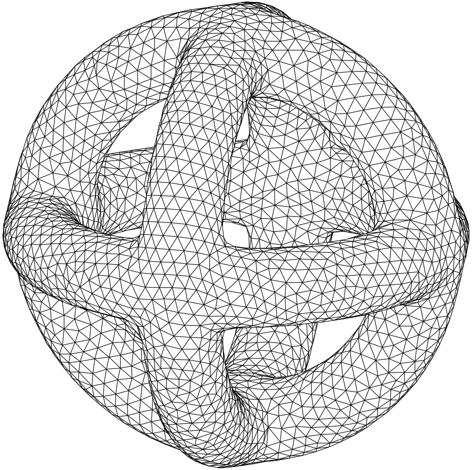

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types:
  Incremental, Sweepline,
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- Varying Data Structures

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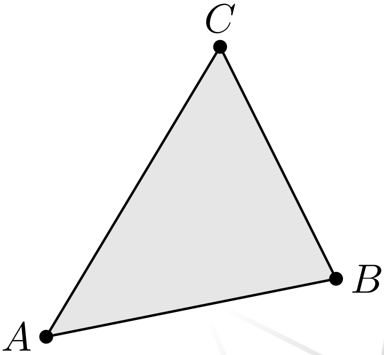
Mathematical Preliminaries

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.



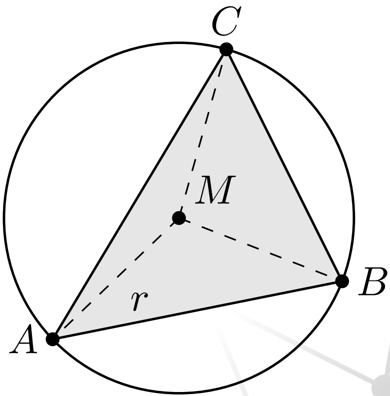
Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent
define vertices of a triangle.

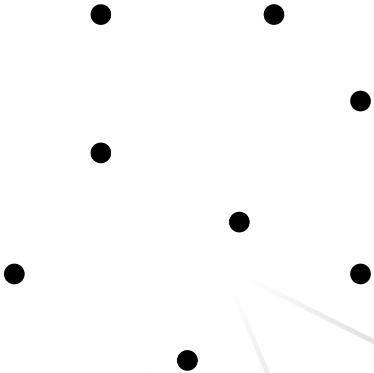
Circumcircle

Circle that intersects with
all vertices of the triangle.



Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
affinely span $\mathbb{R}^2$



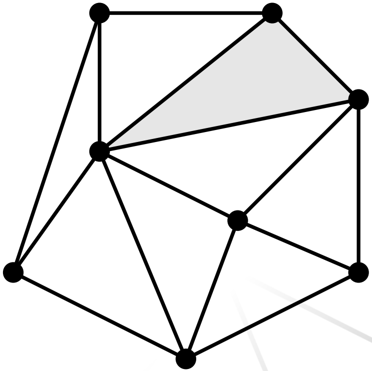
Mathematical Preliminaries: Triangulation

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$, affinely span $\mathbb{R}^2$

Triangulation

Planar straight-line graph over $\mathcal{V}$ such that its edges form a maximal subset of non-crossing edges.



Mathematical Preliminaries: Delaunay Triangulation

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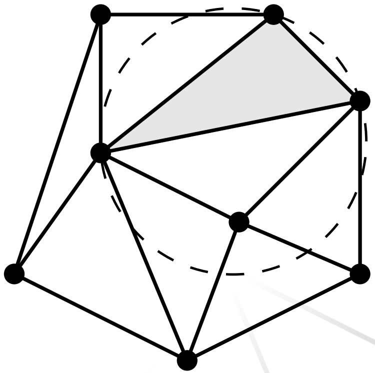
Delaunay Triangulation

Circumcircle of any triangle contains no other points of $\mathcal{V}$.



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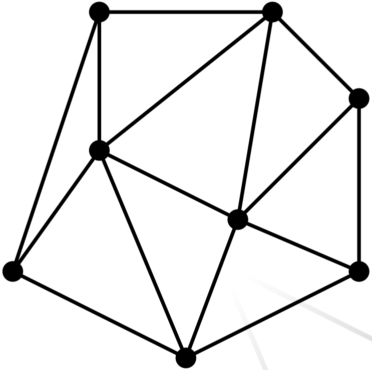

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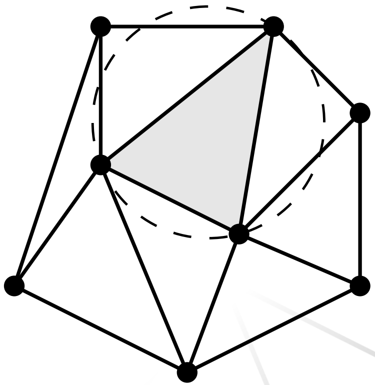
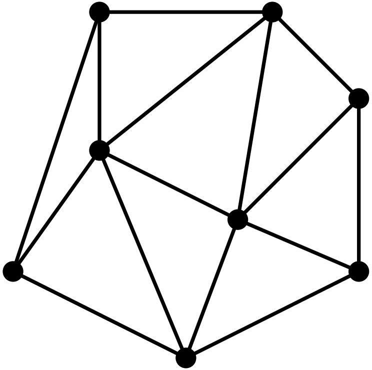


Mathematical Preliminaries: Properties



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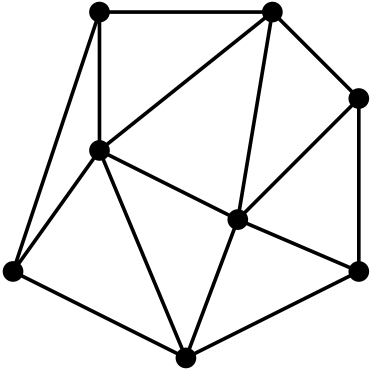
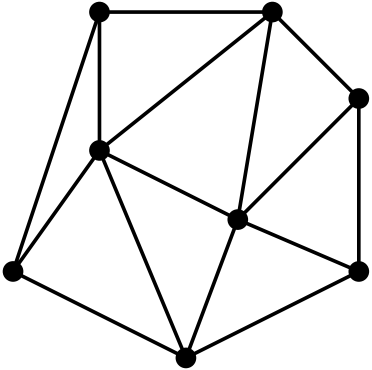


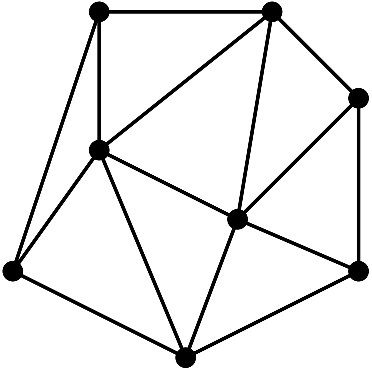
- Existence is guaranteed

Mathematical Preliminaries: Properties



- Existence is guaranteed
- Unique if there are no four points that are cocircular

Mathematical Preliminaries: Properties



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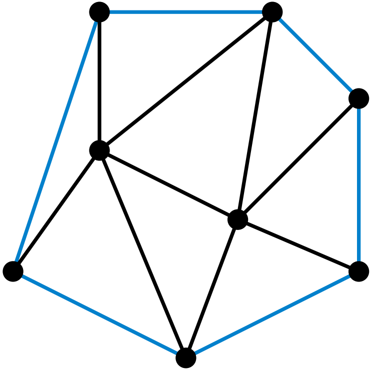

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- Convex hull is contained

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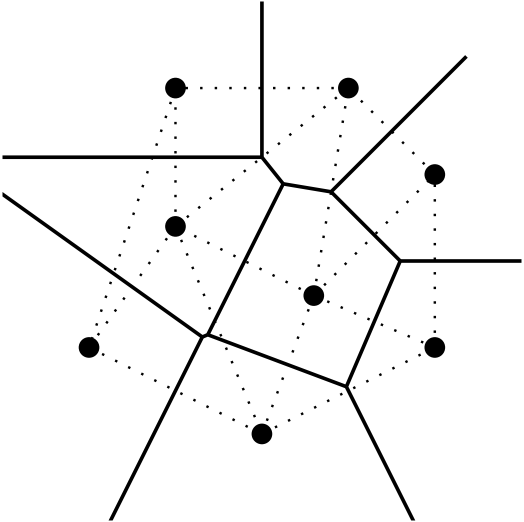
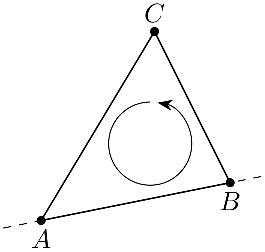


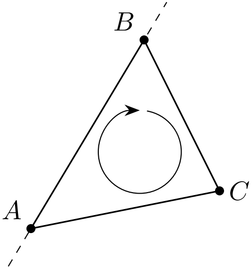
- Existence is guaranteed
- Unique if there are no four points that are cocircular
- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained
- Dual of Voronoi diagram

Geometric Primitives

Geometric Primitives: Triangle Orientation



$\longleftrightarrow$



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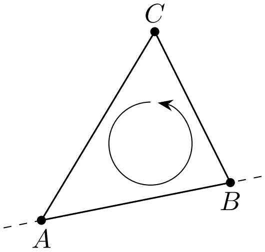

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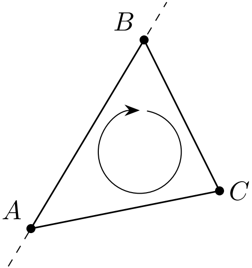
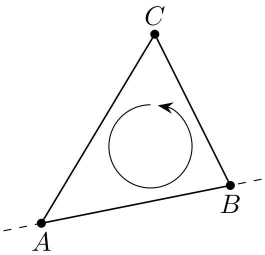


Counterclockwise Orientation $\iff$ C is left of $\overline{AB}$

Geometric Primitives: Triangle Orientation



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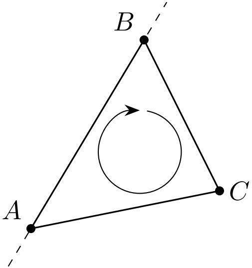
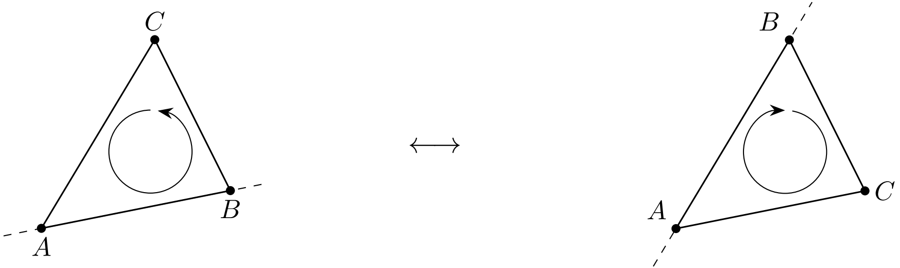


Counterclockwise Orientation $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

Geometric Primitives: Triangle Orientation



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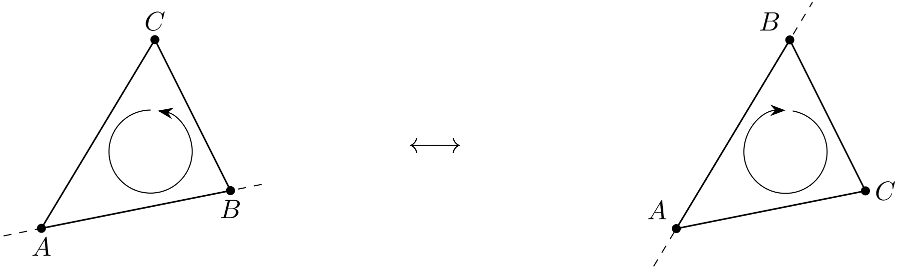
$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

Geometric Primitives: Triangle Orientation



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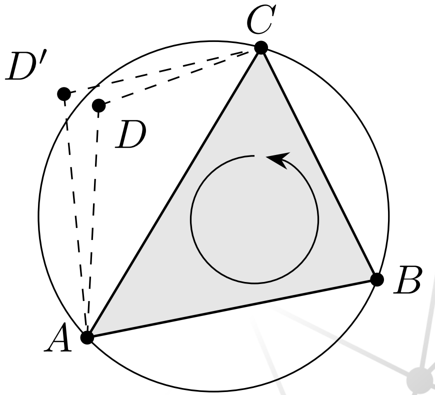
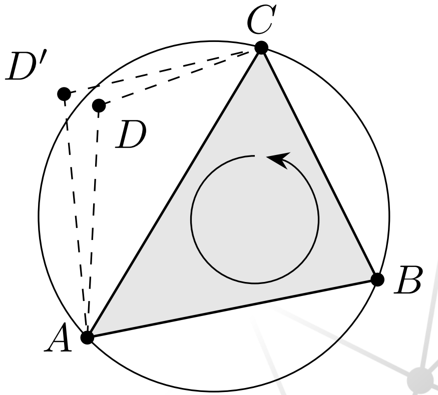
Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



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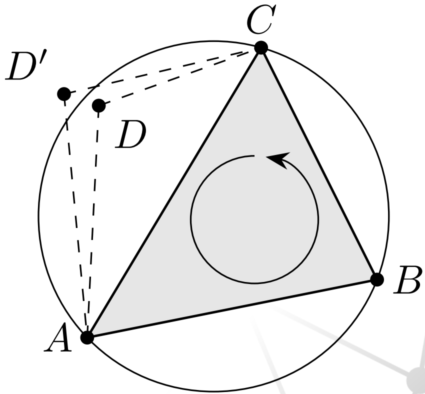
$$= \left\langle x, \operatorname{adj} \begin{pmatrix} u & v \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \|u\|^2 \\ \|v\|^2 \end{pmatrix} \right\rangle - \det \begin{pmatrix} u & v \end{pmatrix} \|x\|^2$$

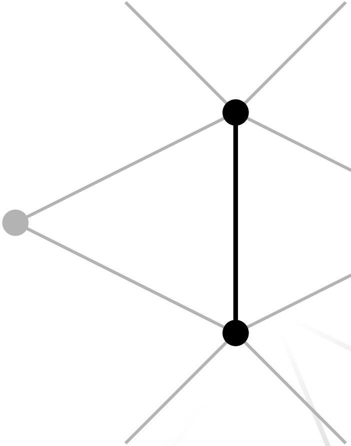
$$u := B - A, \quad v := C - A, \quad x := D - A$$



Quad-Edge Data Structure

Quad-Edge Data Structure: Scheme

Edge-Based List-Like Data Structure
for Storing Neighbor Information:



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Edge-Based List-Like Data Structure
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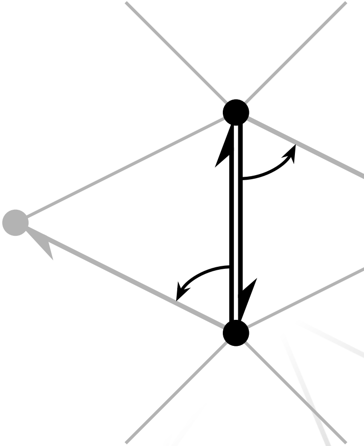
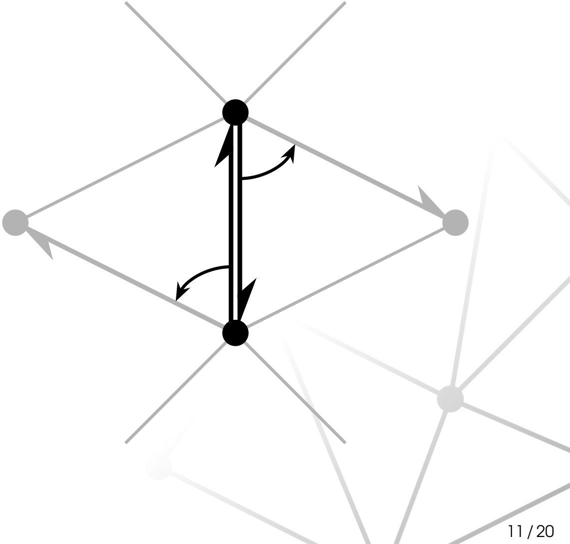
- Directed edges for vertices



Quad-Edge Data Structure: Scheme

Edge-Based List-Like Data Structure for Storing Neighbor Information:

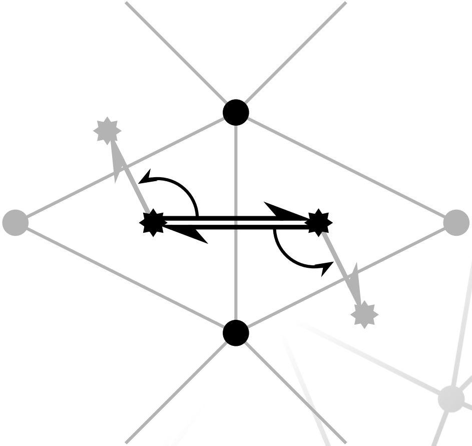
- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex



Quad-Edge Data Structure: Scheme

Edge-Based List-Like Data Structure for Storing Neighbor Information:

- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex
- Directed dual edges for adjacent faces



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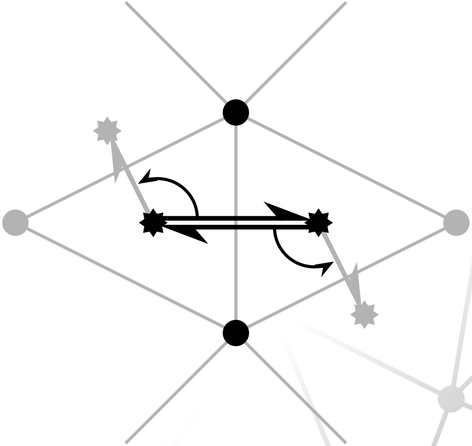

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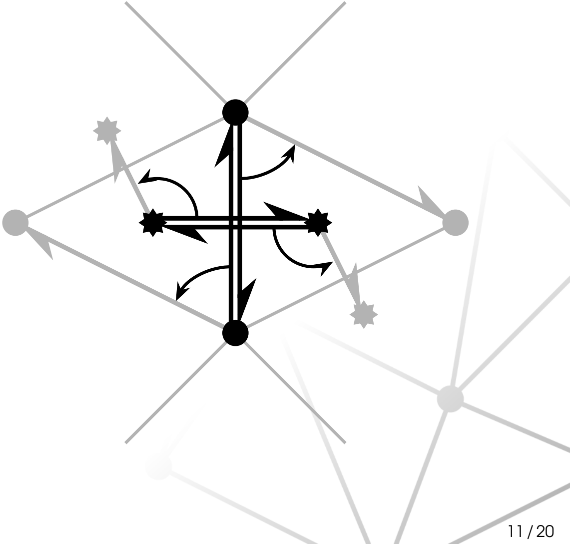
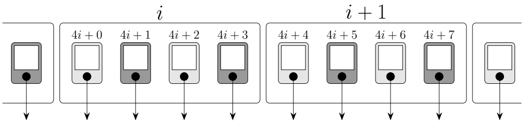


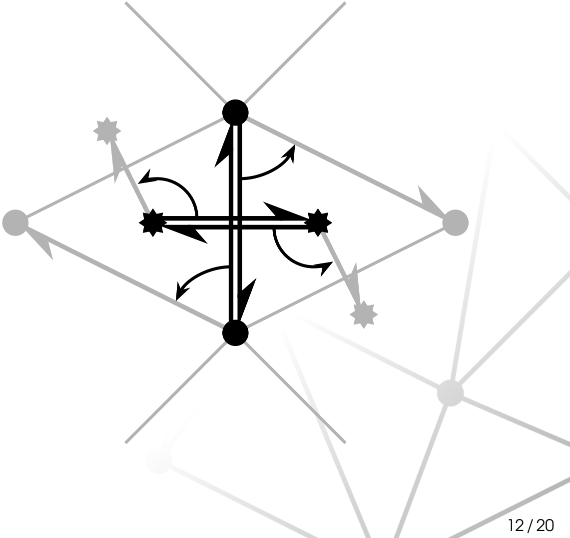
Quad-Edge Data Structure: Implementation




```
struct edge {
  size_t next;
  size_t data;
};

struct quad_edge {
  edge data[4];
};

vector<vertex>    vertices{};
vector<quad_edge> quad_edges{};
vector<size_t>    free_edges{};
```



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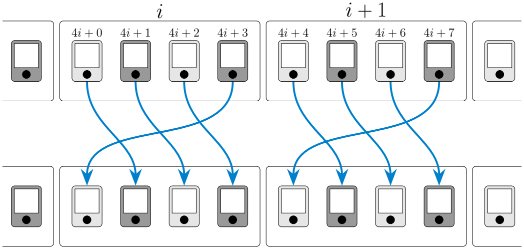
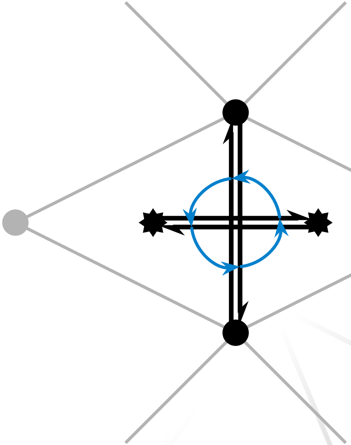
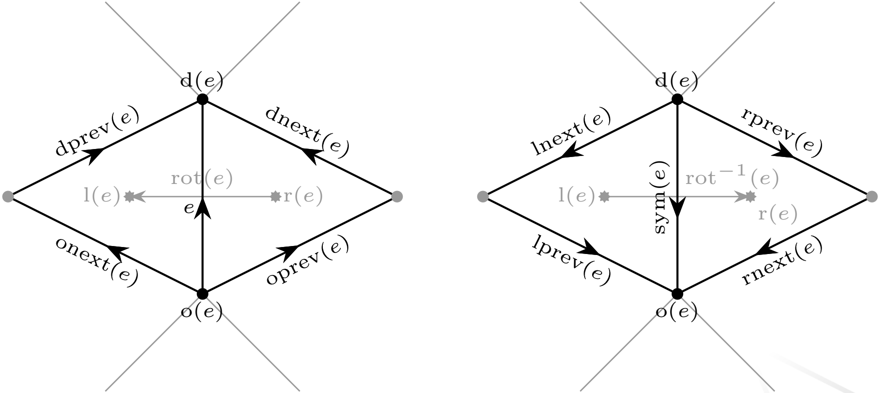


$$\mathrm{rot}\colon \mathbb{N}_0 \to \mathbb{N}_0$$

$$\mathrm{rot}(x) = 4 \cdot \left\lfloor \frac{x}{4} \right\rfloor + (x+1 \mod 4)$$



Quad-Edge Data Structure: Edge Functions and Operators



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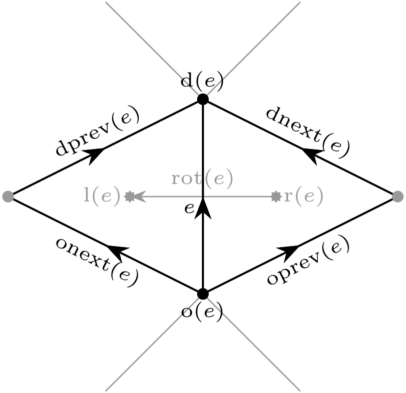
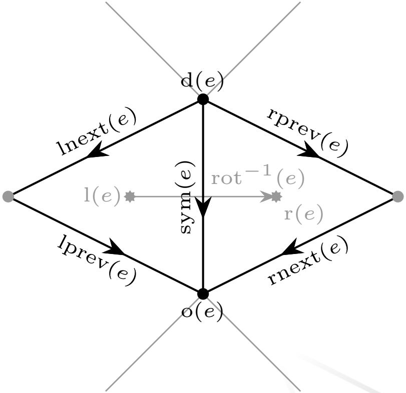
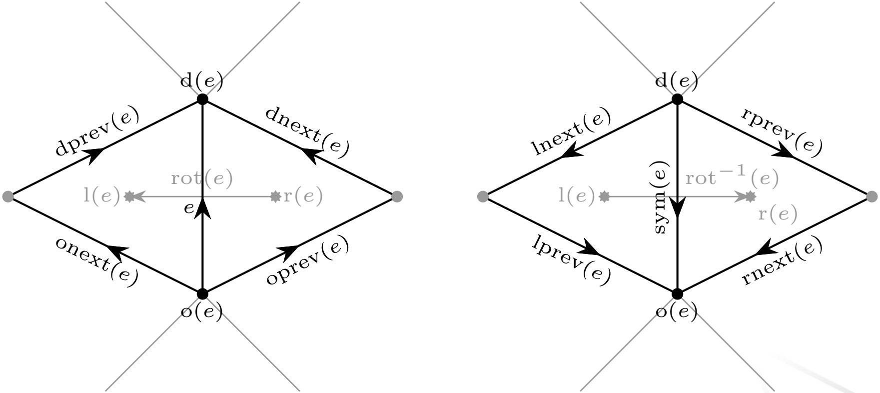




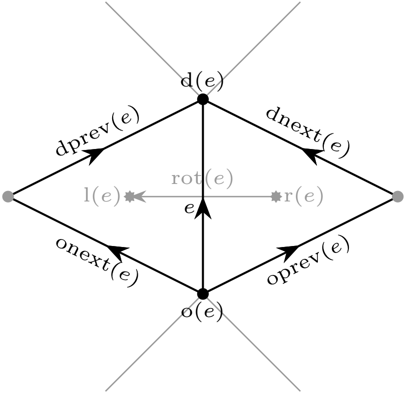
- Create a new edge

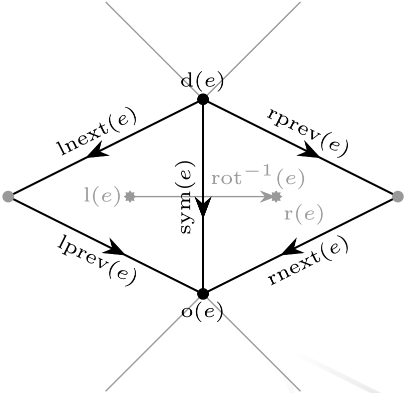
Quad-Edge Data Structure: Edge Functions and Operators



- Create a new edge
- Delete existing edge

Quad-Edge Data Structure: Edge Functions and Operators





- Create a new edge
- Delete existing edge
- Connect points by a new edge

Algorithm

Algorithm: Overview

Algorithm: Overview

Triangulation Algorithm

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.

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2. Triangulate sorted point set.

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Subroutine: Triangulate

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1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.

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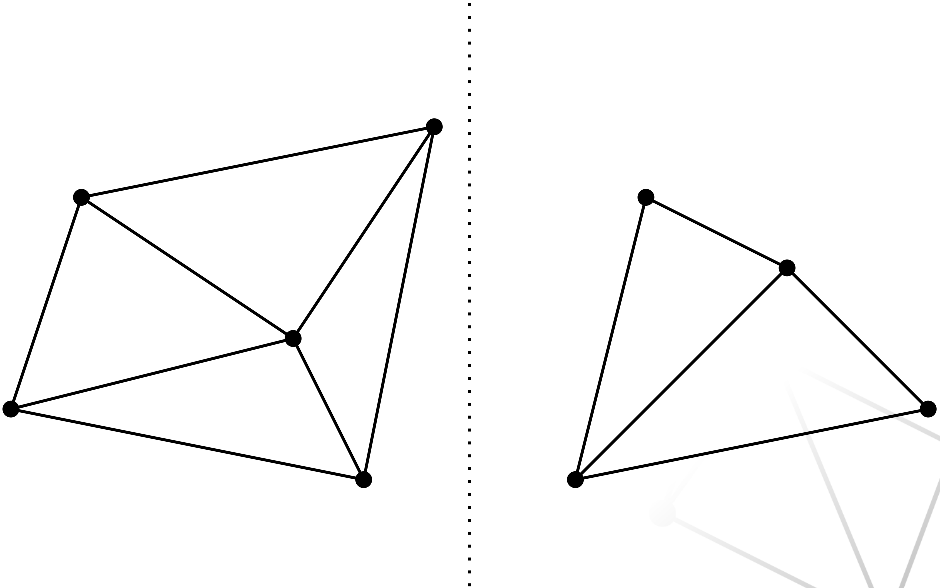
Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.
3. Triangulate left and right half.
4. Merge left and right triangulations.

Algorithm: Merge Triangulations Example



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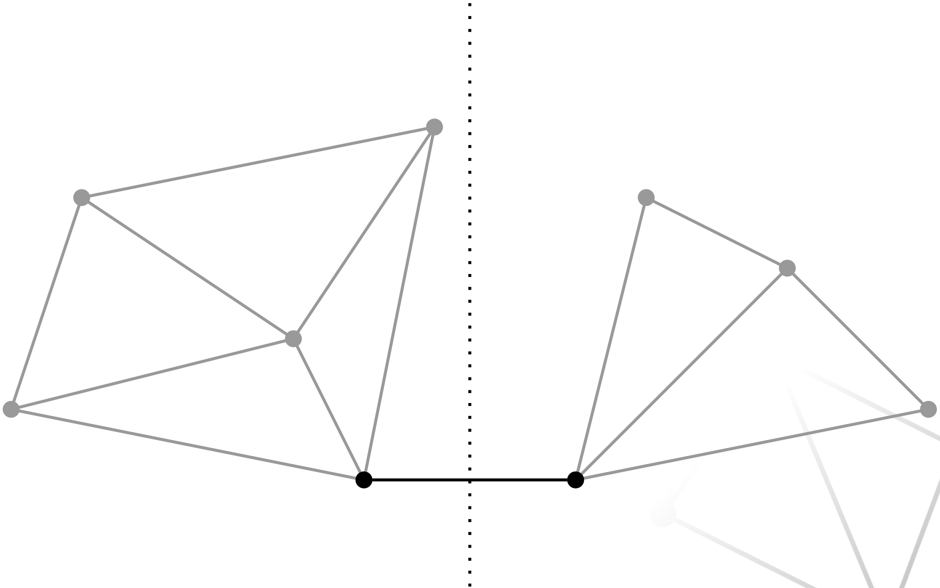

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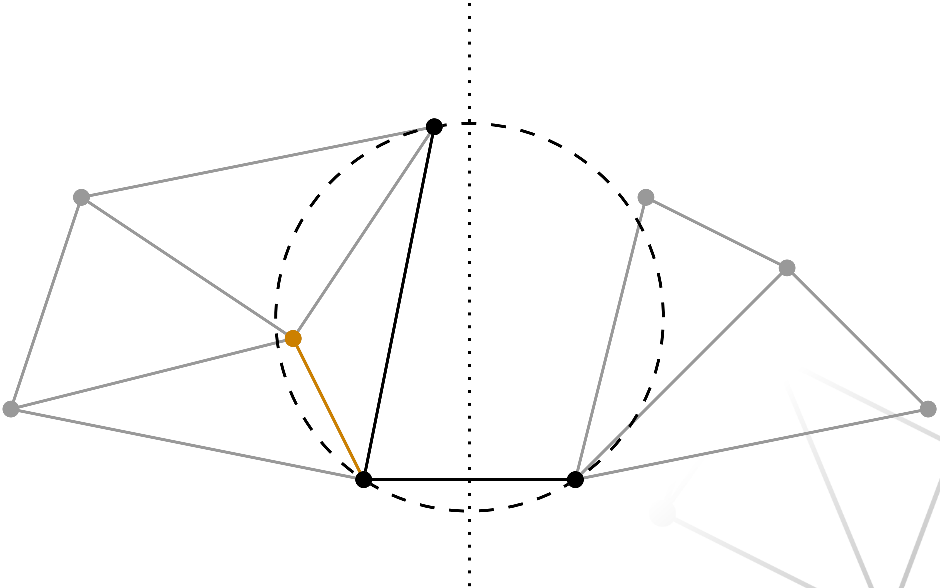

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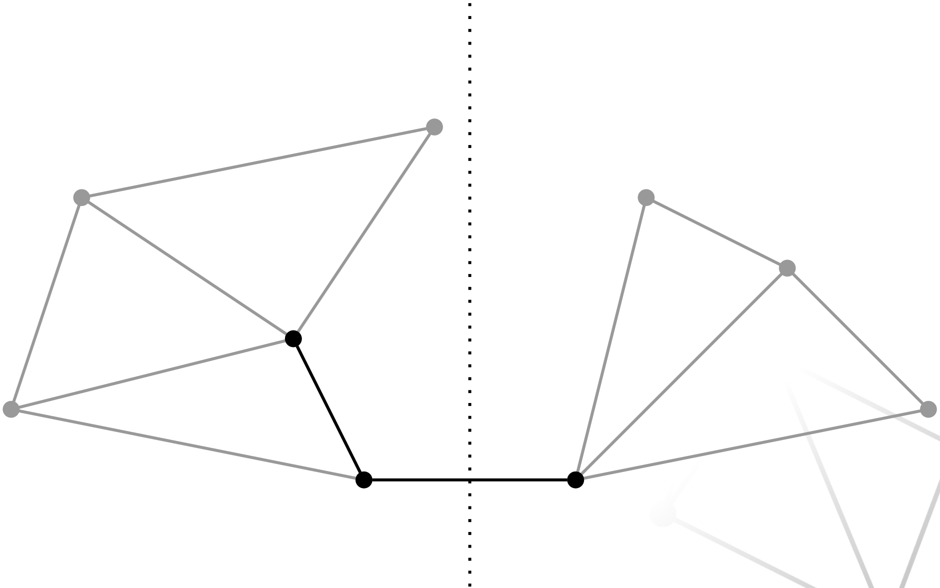

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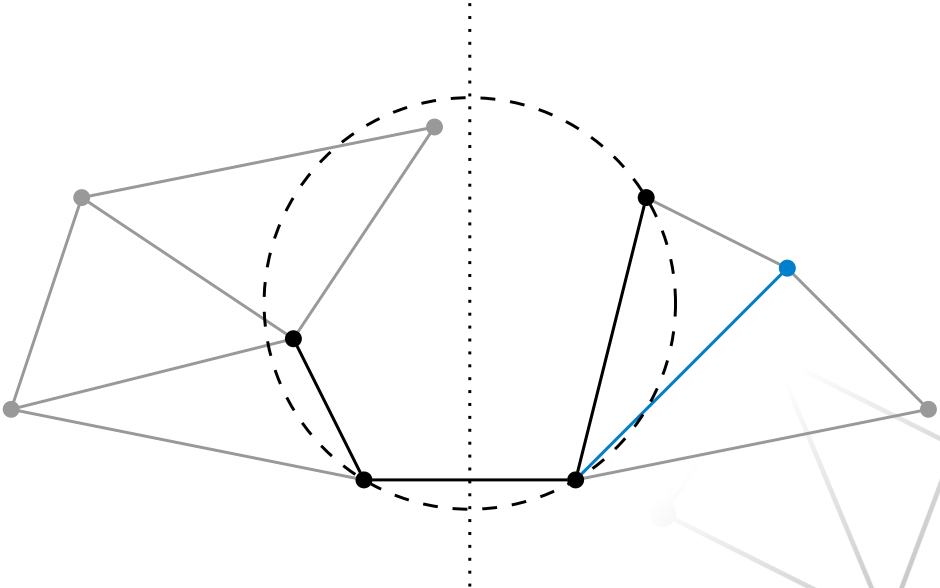

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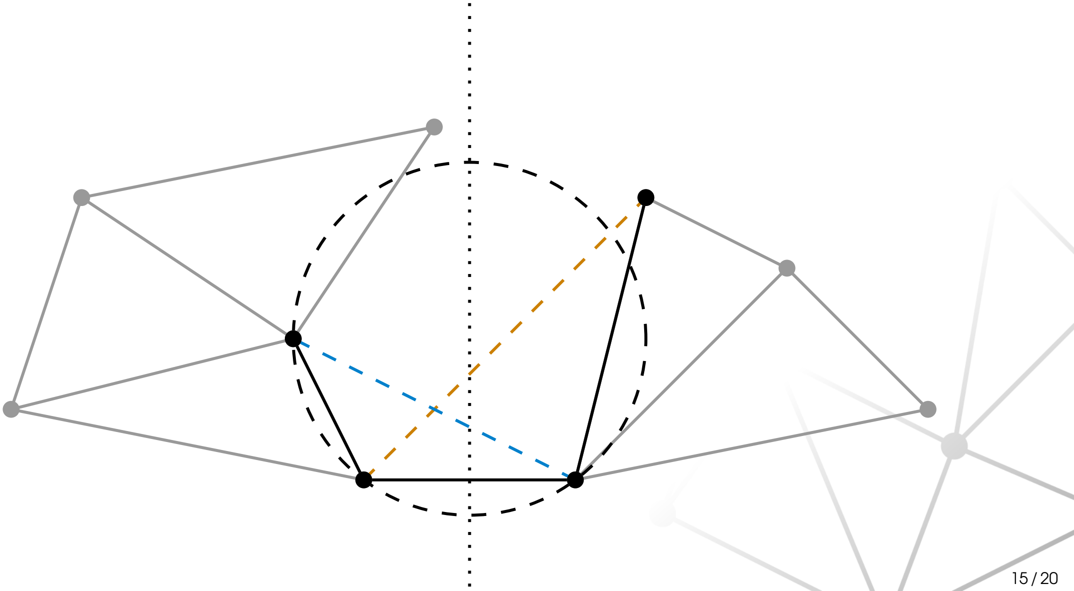

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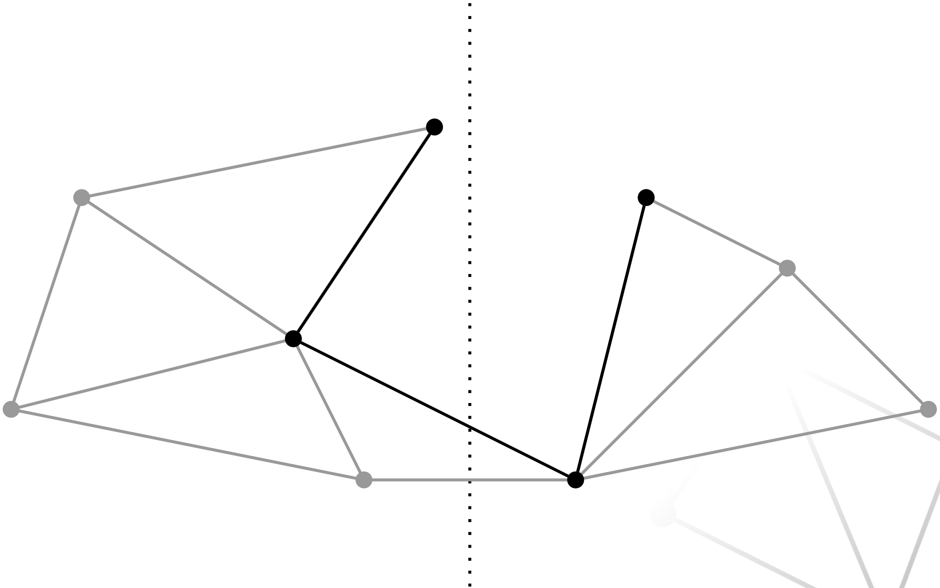

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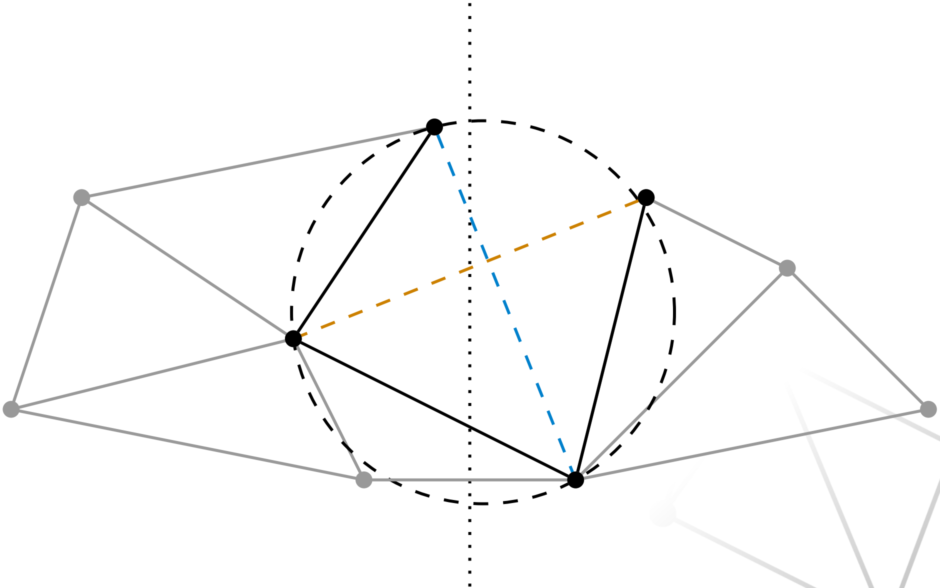

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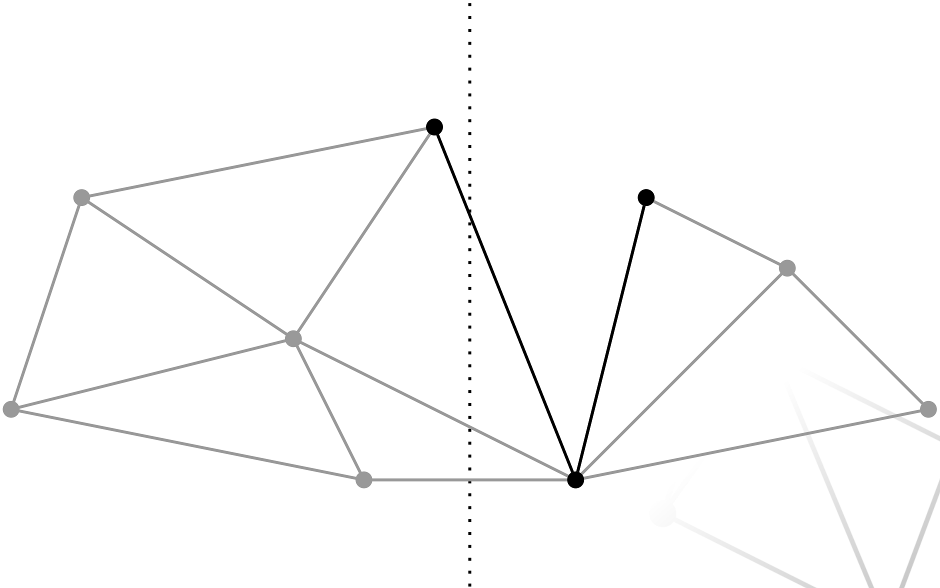

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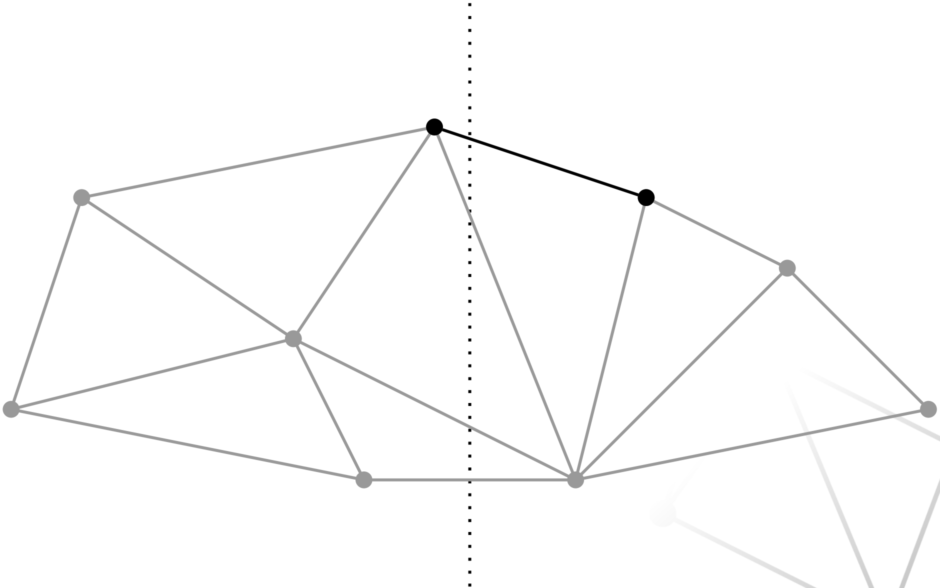

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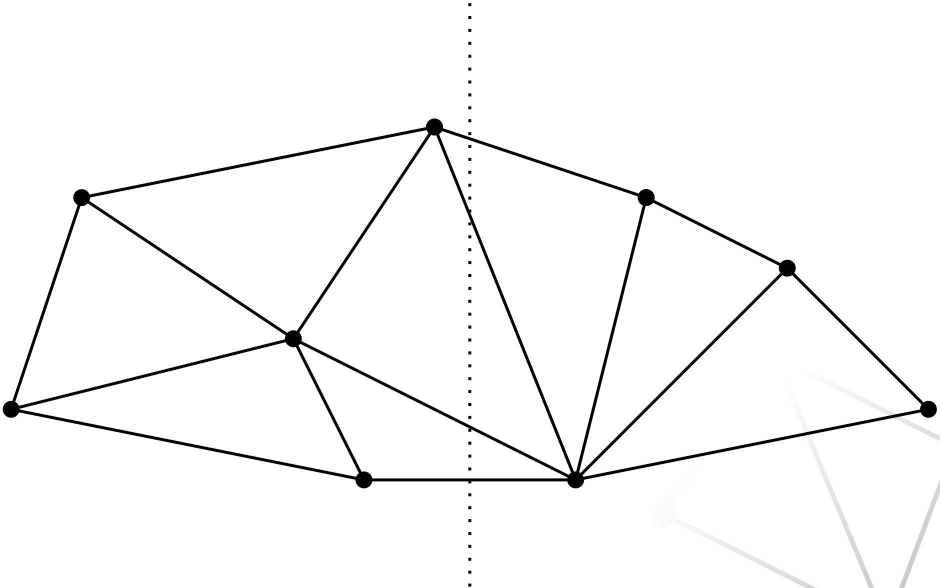

Algorithm: Merge Triangulations

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Subroutine: Merge Triangulations

- linear complexity by using Euler's formula for planar graphs
- computation of lower common tangent
- circle test for adjacent edges
- circle test for cross edge

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1. Compute and add lower common tangent.

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Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.

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Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:

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Algorithm: Merge Triangulations

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    3.1 Remove invalid edges adjacent to and above baseline.

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   3.1 Remove invalid edges adjacent to and above baseline.
   3.2 Insert cross edge above baseline.
   3.3 Make this cross edge the new baseline.

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Algorithm: Complexity

- Use master theorem
- Merge step is linear in point count

$$t(n) = 2t\left(\frac{n}{2}\right) + \mathcal{O}(n)$$

$$\mathcal{O}(n \log n)$$

Subroutine: Triangulate

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Algorithm: Correctness

Proof by induction:

Show that for two given Delaunay triangulations $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}(\mathcal{R})$ separated by a vertical line the merge subroutine generates a Delaunay triangulation $\mathcal{T}(\mathcal{L} \cup \mathcal{R})$.

- For two Delaunay triangulations separated by a vertical line, it is enough to remove inner edges and insert cross edges.
- Common tangents are elements of the Delaunay triangulation.
- Removed edges are indeed not Delaunay.
- Insertion of cross edges generates new Delaunay triangle.
- There are no other edges that have to be removed.
- Theorem: Algorithm is correct.

Implementation Notes

Implementation Notes

- Geometric Primitives need exact computation and therefore arbitrary precision
- still no robust split, use Dwyer instead (no sorting, parallelization)
- triangular data structure increases speed but algorithm is more complicated
- Divide-and-conquer variant seems to be most powerful and robust

Applications

Conclusions

Conclusions

Summary:

- Delaunay triangulation can be generated by given divide-and-conquer algorithm in $\mathcal{O}(n \log n)$
- Data structure needs to store neighbor information

Future Work:

- Use triangular data structure instead of quad-edge data structure
- Use Dwyer's approach to make algorithm more robust
- Parallelization

Thank you for Your Attention!

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