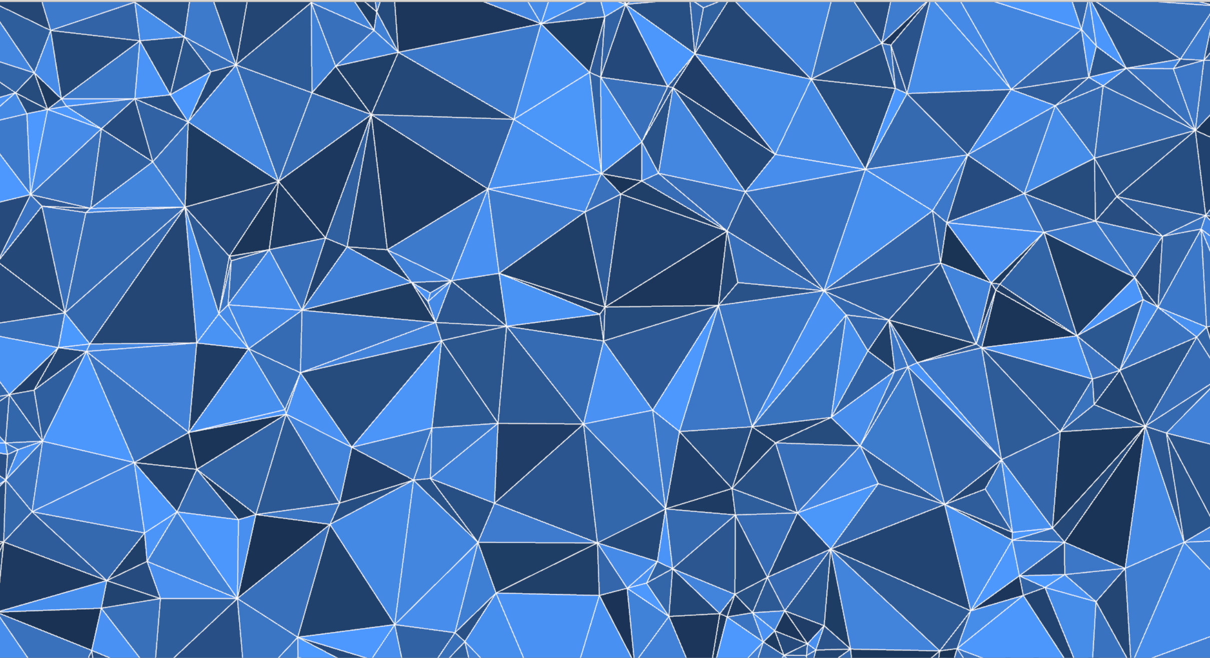
Delaunay Triangulation Mosaic



Algorithmical Geometry: Delaunay Triangulation

Markus Pawellek

January 1, 2022

Outline

Introduction

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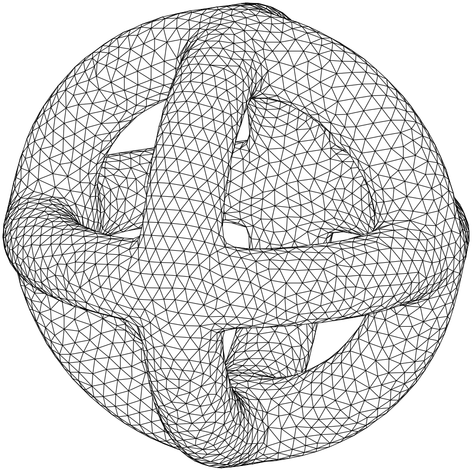

*https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg, December 29, 2021

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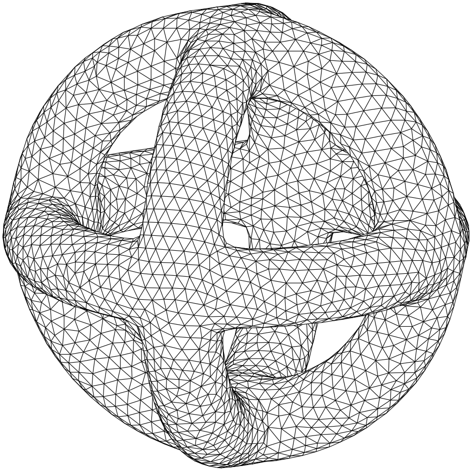

Educational Problems:

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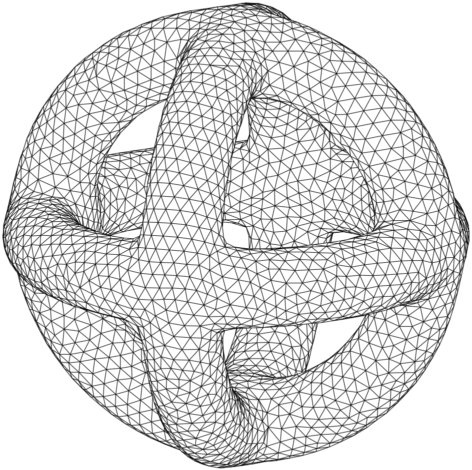

Educational Problems:

- Many Resources

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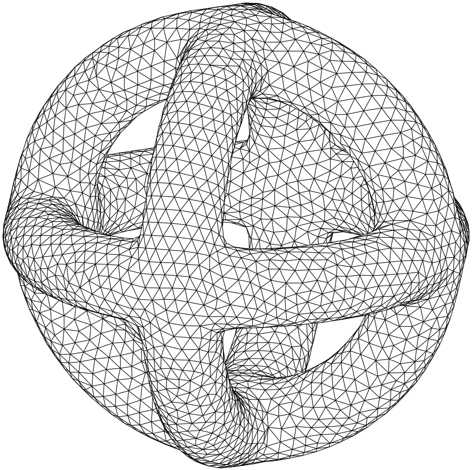

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams

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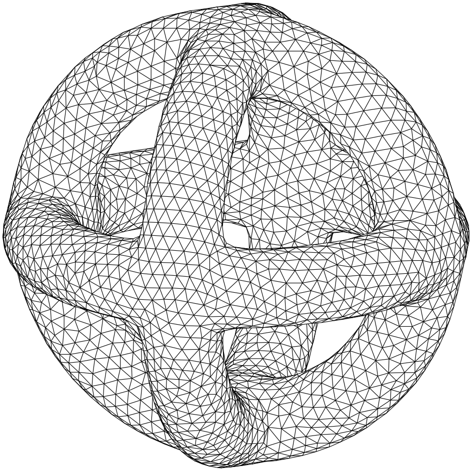

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer

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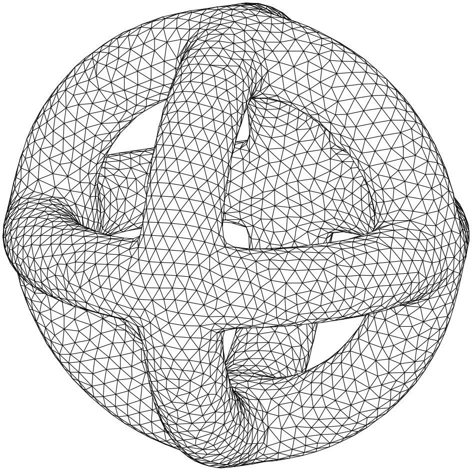

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer
- Varying Data Structures

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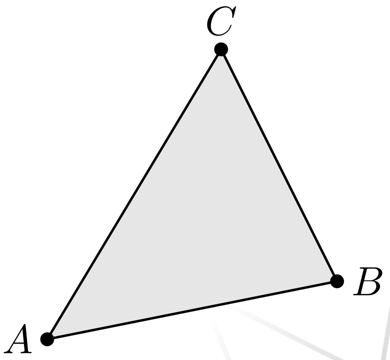
Mathematical Preliminaries

Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.



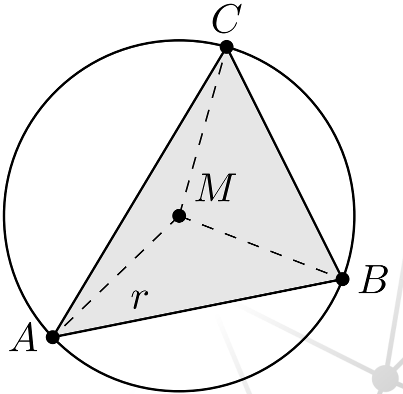
Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.

Circumcircle

Circle that intersects with all vertices of the triangle.

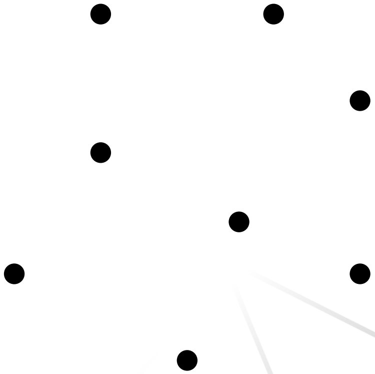


Mathematical Preliminaries: Point Set

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,

affinely span $\mathbb{R}^2$



Mathematical Preliminaries: Triangulation

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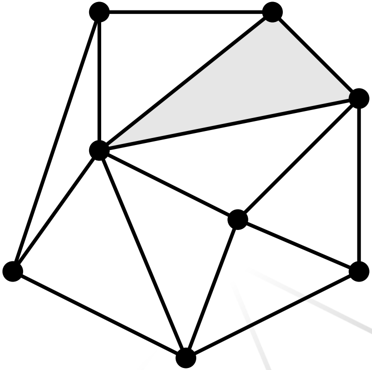
$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$, affinely span $\mathbb{R}^2$

Triangulation

Planar straight-line graph over $\mathcal{V}$ such that its edges form a maximal subset of non-crossing edges.



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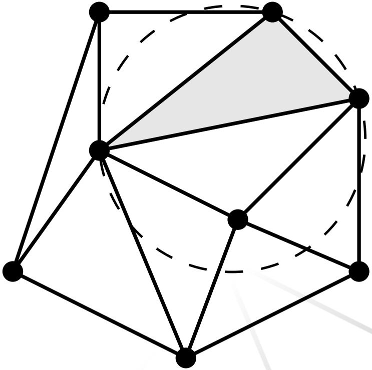
Delaunay Triangulation

Circumcircle of any triangle contains no other points of $\mathcal{V}$.



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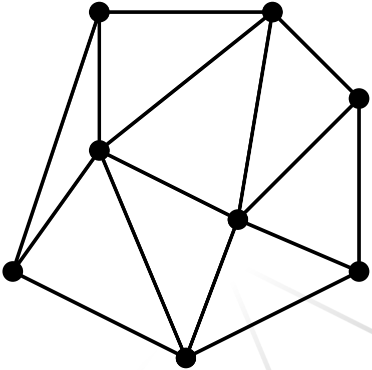

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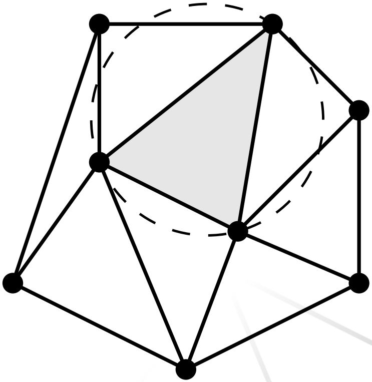
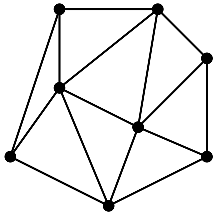
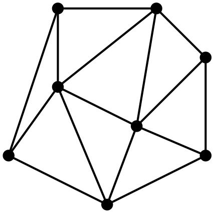


Mathematical Preliminaries: Properties

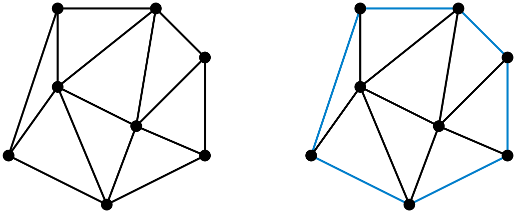


Mathematical Preliminaries: Properties



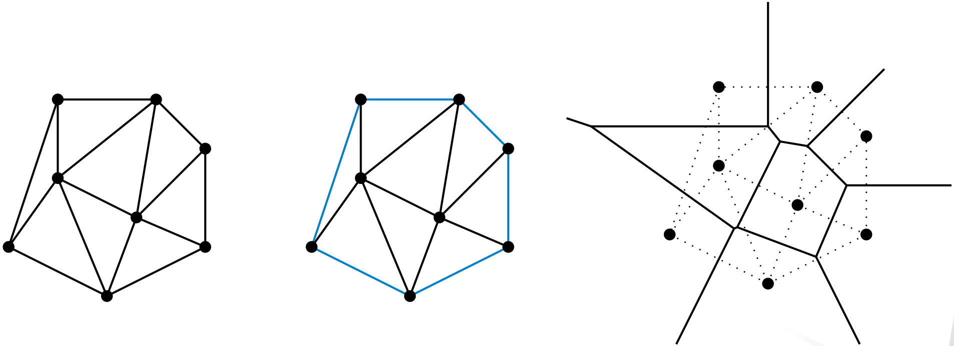
- Optimality: maximization of the minimum angle of all angles

Mathematical Preliminaries: Properties



- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained

Mathematical Preliminaries: Properties



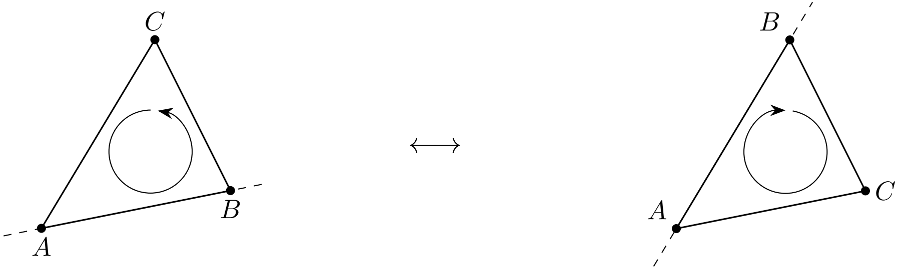
- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained
- Voronoi diagram is the dual

Mathematical Preliminaries: Properties of Delaunay Triangulation

- Duality to Voronoi Diagram (also useful for proofs)
- always exists -> proof
- If no points are cocircular, unique -> proof
- optimality: maximization of the minimum angle of all angles -> proof, reason why delaunay is good
- boundary is convex hull

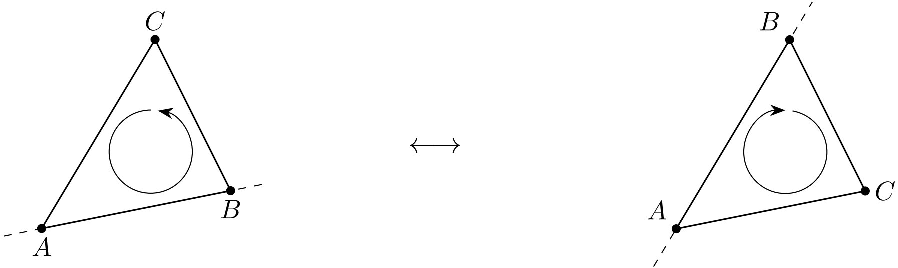
Geometric Primitives

Geometric Primitives: Counterclockwise



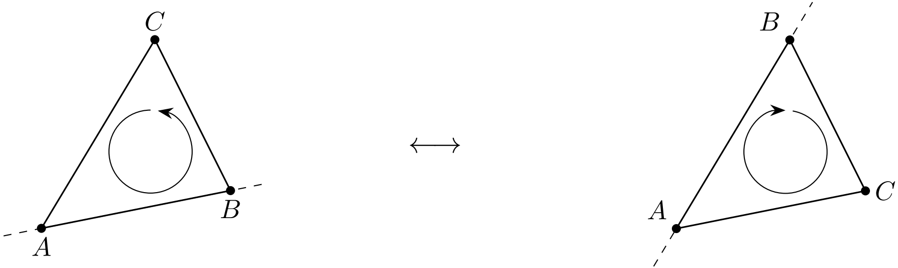
Counterclockwise Order

Geometric Primitives: Counterclockwise



Counterclockwise Order $\iff$ C is left of $\overline{AB}$

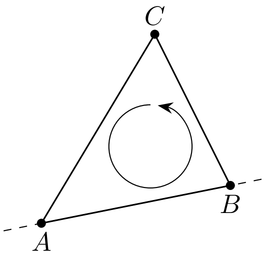
Geometric Primitives: Counterclockwise



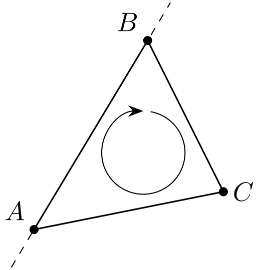
Counterclockwise Order $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

Geometric Primitives: Counterclockwise



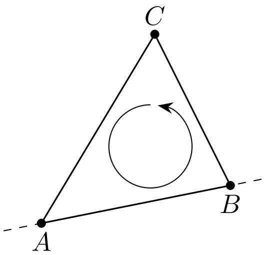
$\longleftrightarrow$



Counterclockwise Order $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

Geometric Primitives: Counterclockwise



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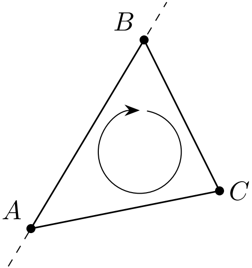

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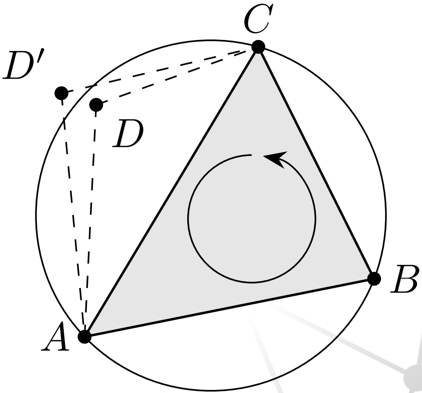
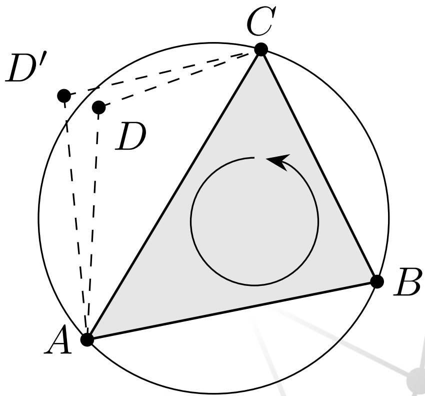
Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

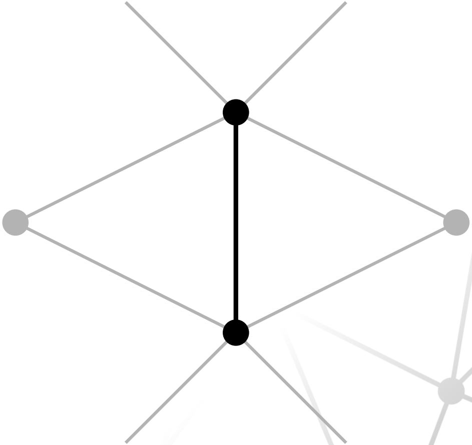
$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



Data Structure

Data Structure: Quad-Edge

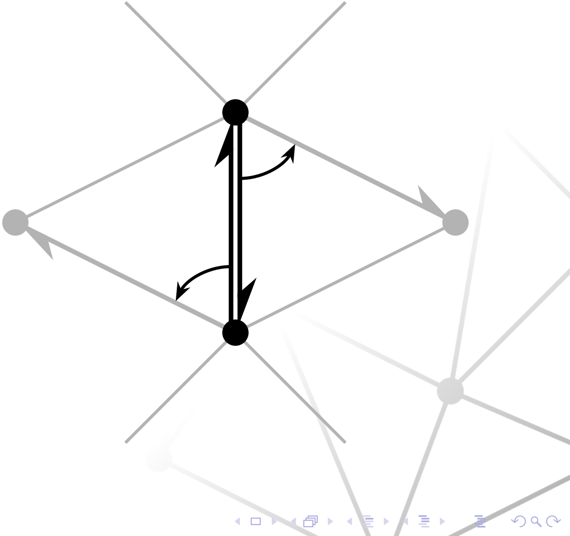
Edge-Based Data Structure:



Data Structure: Quad-Edge

Edge-Based Data Structure:

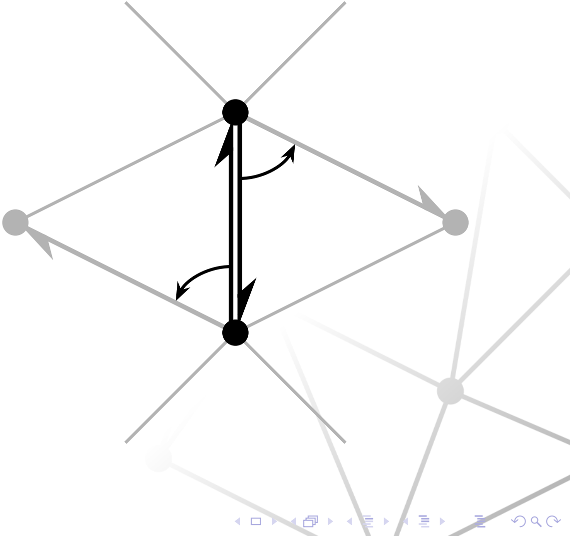
- Directed edges for vertices



Data Structure: Quad-Edge

Edge-Based Data Structure:

- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex



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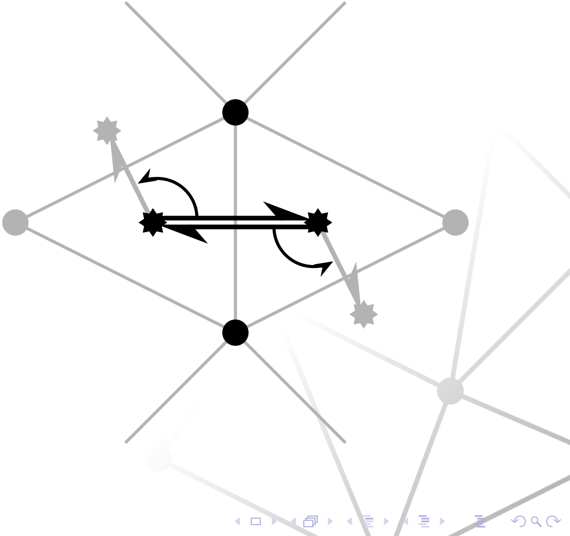
Edge-Based Data Structure:

- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex
- Directed dual edges for adjacent faces



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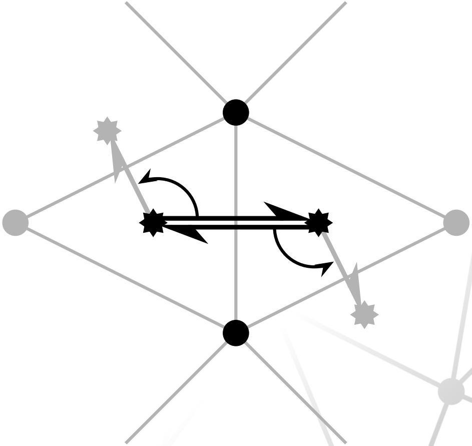

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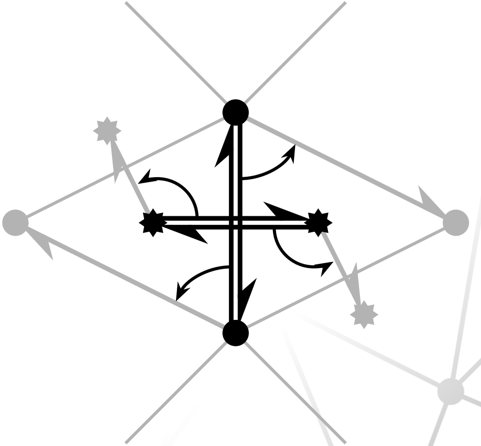
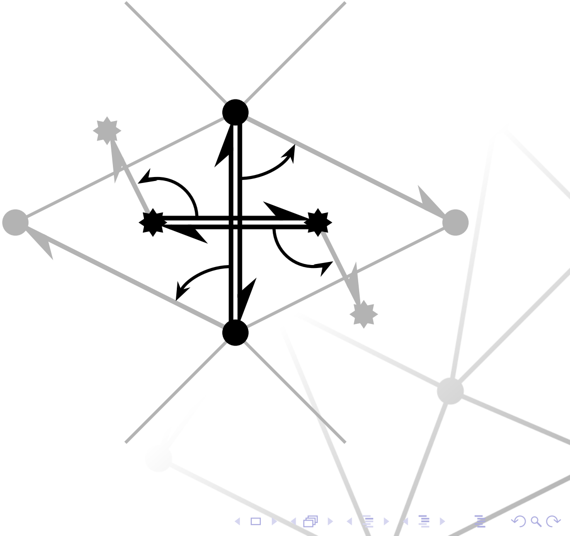


Data Structure: Quad-Edge Implementation

```
struct quad_edge_algebra {
  struct edge {
    size_t next;
    size_t data;
  };

  struct quad_edge {
    edge data[4];
  };

  vector<quad_edge> quad_edges{};
  vector<size_t> free_edges{};
};
```

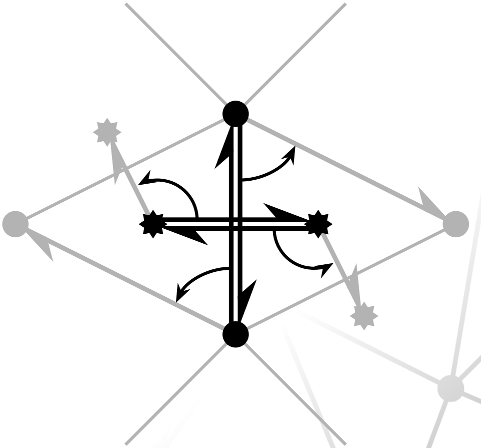


Data Structure: Quad-Edge Implementation

```
struct quad_edge_algebra {
  // ...
  static constexpr size_t edge_type_mask =
    sizeof(quad_edge) / sizeof(edge) - 1;
  static constexpr size_t quad_edge_mask =
    ~edge_type_mask;

  auto rot(size_t e, int n = 1) const -> size_t {
    const size_t t = e + n;
    return (e & quad_edge_mask) |
           (t & edge_type_mask);
  }

  auto onext(size_t e) const -> size_t {
    return ((edge*)quad_edges.data() + e)->next;
  }
};
```



Algorithm

Implementation

Implementation

- Geometric Primitives need exact computation and therefore arbitrary precision

Applications

Conclusions

Thank you for Your Attention!

References

(1) D. T. Lee and B. J. Schachter. "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9 (1980), pp. 219–242. DOI: 10.1007/BF00977785.

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(10) P. Cignoni, C. Montani, and R. Scopigno. "DeWall: A Fast Divide-and-Conquer Delaunay Triangulation Algorithm in E^d". In: *Computer-Aided Design* 30 (1998), pp. 333–341. DOI: 10.1016/S0010-4485(97)00082-1.

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