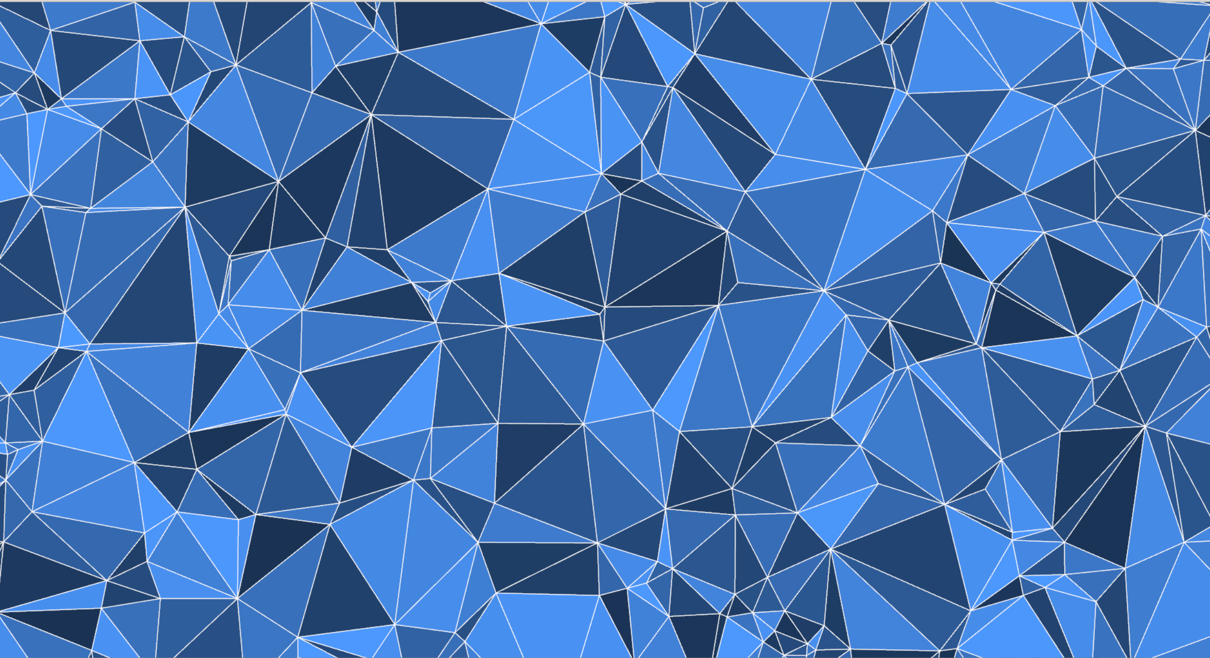
Delaunay Triangulation Mosaic



Algorithmical Geometry: Delaunay Triangulation

Markus Pawellek

January 16, 2022

Outline

Introduction

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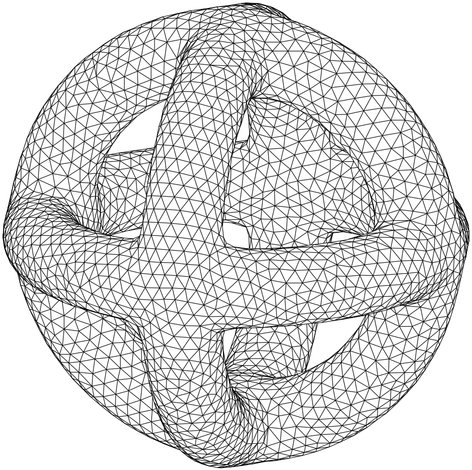

*https://upload.wikimedia.org/wikipedia/commons/b/b8/Approx-3tori.svg, December 29, 2021

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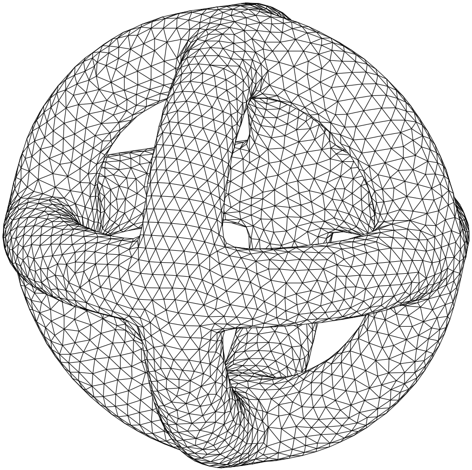

Educational Problems:

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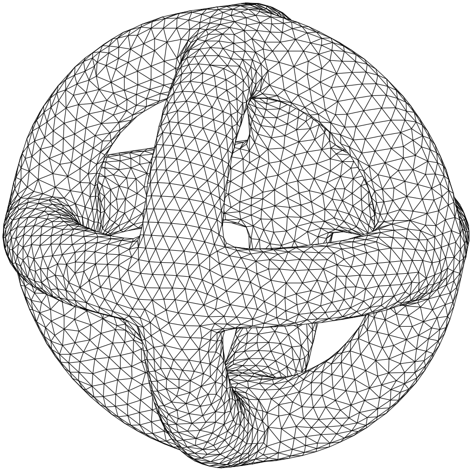

Educational Problems:

- Many Resources

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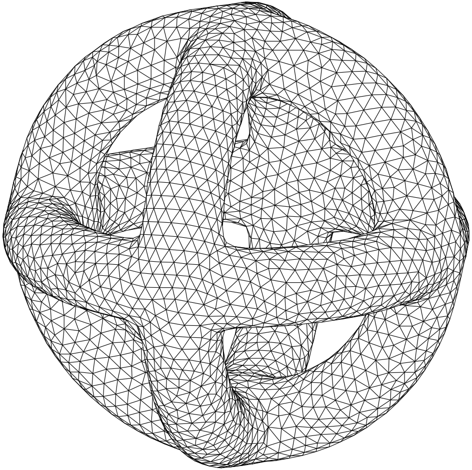

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams

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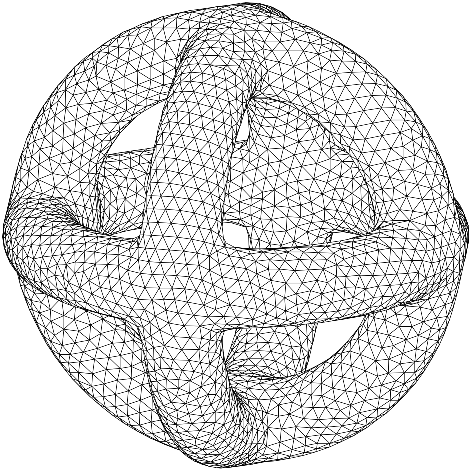

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer

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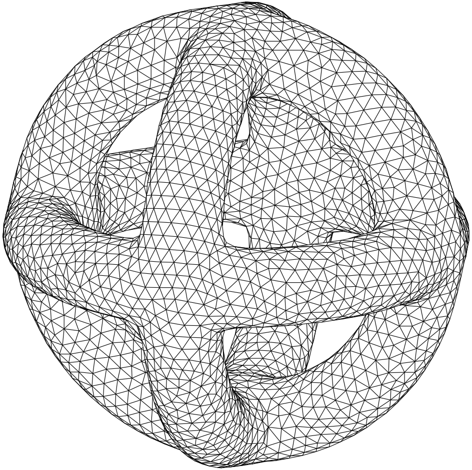

Educational Problems:

- Many Resources
- Duality to Voronoi Diagrams
- Multiple Algorithm Types: Incremental, Sweepline, Divide-and-Conquer
- Varying Data Structures

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Introduction: Previous Work

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1980 Lee and Schachter, “Two Algorithms for Constructing a Delaunay Triangulation”

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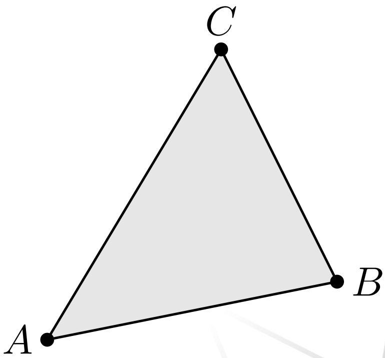
Mathematical Preliminaries

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.



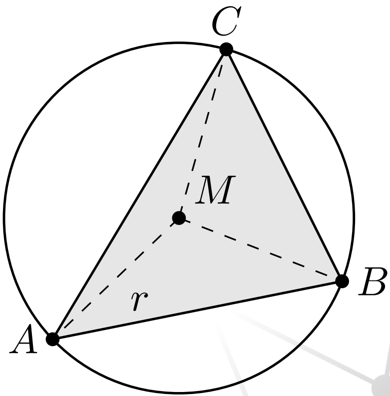
Mathematical Preliminaries: Triangle and Circumcircle

Triangle

$A, B, C \in \mathbb{R}^2$ affinely independent define vertices of a triangle.

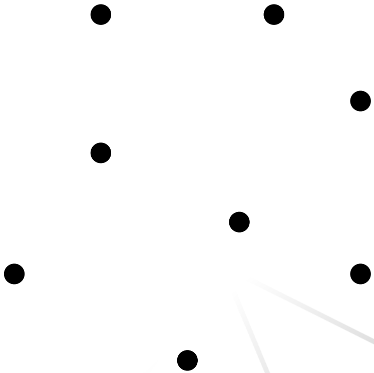
Circumcircle

Circle that intersects with all vertices of the triangle.



Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
affinely span $\mathbb{R}^2$



Mathematical Preliminaries: Triangulation

Point Set

$\mathcal{V} \subset \mathbb{R}^2$ finite, $\#\mathcal{V} \geq 3$,
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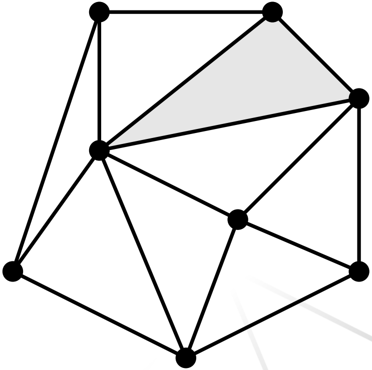
Triangulation

Planar straight-line graph over $\mathcal{V}$ such that its edges form a maximal subset of non-crossing edges.



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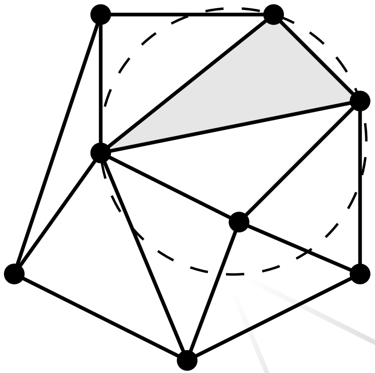
Delaunay Triangulation

Circumcircle of any triangle contains no other points of $\mathcal{V}$.



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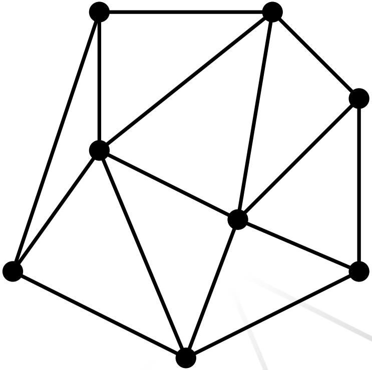

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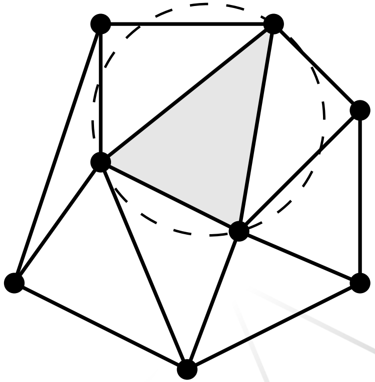
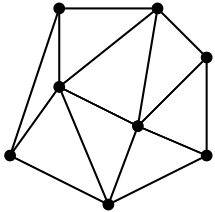
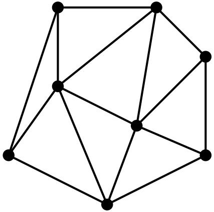


Mathematical Preliminaries: Properties

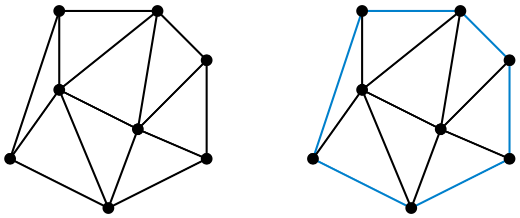


Mathematical Preliminaries: Properties



- Optimality: maximization of the minimum angle of all angles

Mathematical Preliminaries: Properties



- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained

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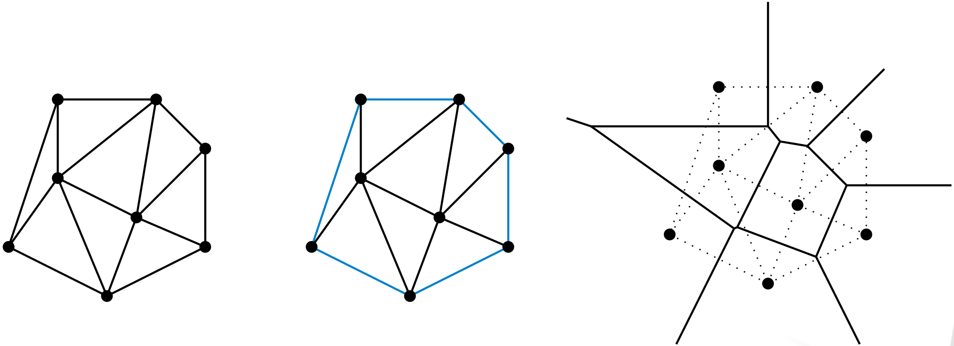


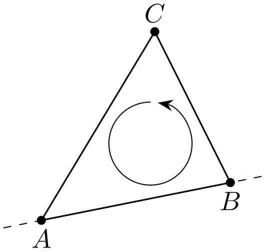
- Optimality: maximization of the minimum angle of all angles
- Convex hull is contained
- Voronoi diagram is the dual

Mathematical Preliminaries: Properties of Delaunay Triangulation

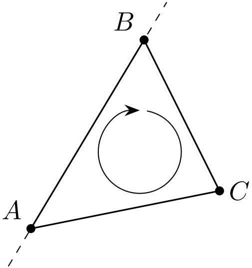
- Duality to Voronoi Diagram (also useful for proofs)
- always exists -> proof
- If no points are cocircular, unique -> proof
- optimality: maximization of the minimum angle of all angles -> proof, reason why delaunay is good
- boundary is convex hull

Geometric Primitives

Geometric Primitives: Counterclockwise

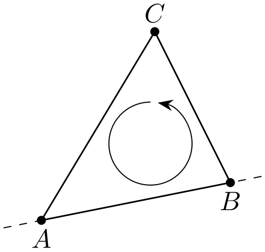


$\longleftrightarrow$



Counterclockwise Order

Geometric Primitives: Counterclockwise



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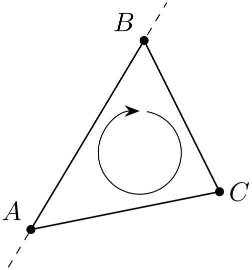
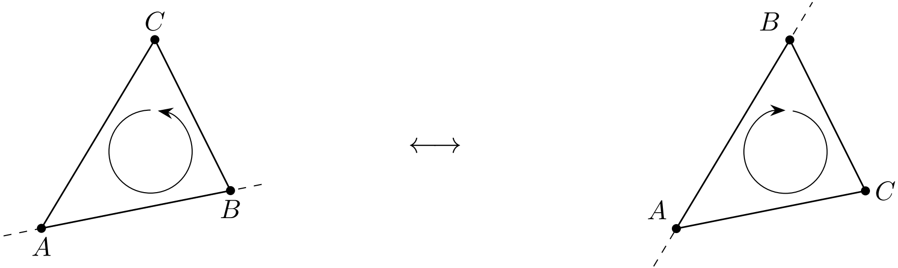


Counterclockwise Order $\iff$ C is left of $\overline{AB}$

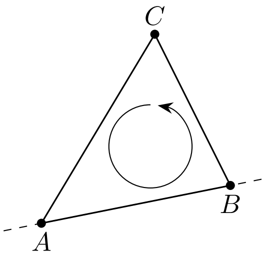
Geometric Primitives: Counterclockwise



Counterclockwise Order $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix}$$

Geometric Primitives: Counterclockwise



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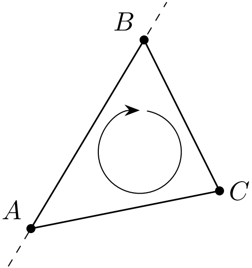
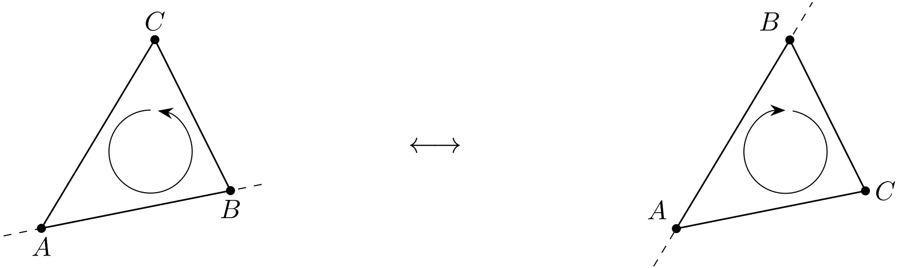


Counterclockwise Order $\iff$ C is left of $\overline{AB}$

$$0 < \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

Geometric Primitives: Counterclockwise



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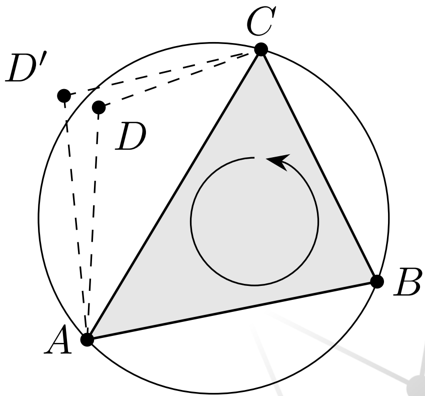
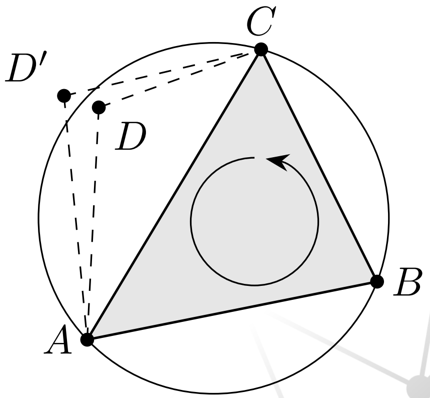
Geometric Primitives: Inside Circumcircle



Geometric Primitives: Inside Circumcircle

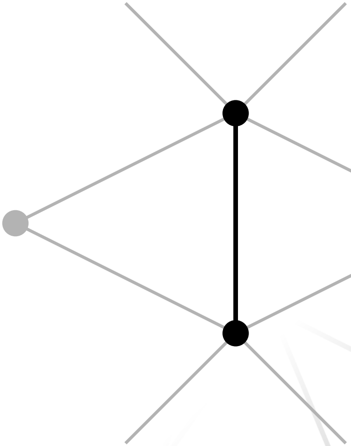
$$0 < \begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix}$$



Data Structure

Data Structure: Quad-Edge

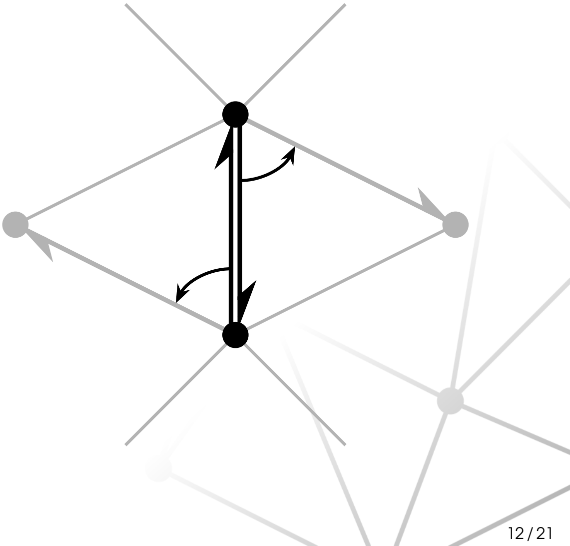
Edge-Based List-Like Data Structure:



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

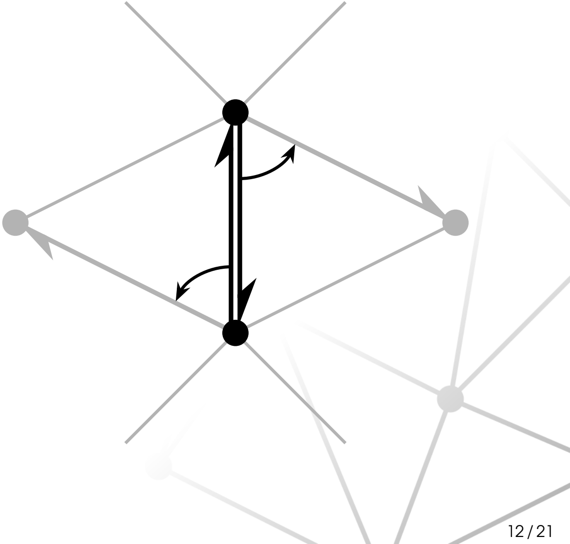
- Directed edges for vertices



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

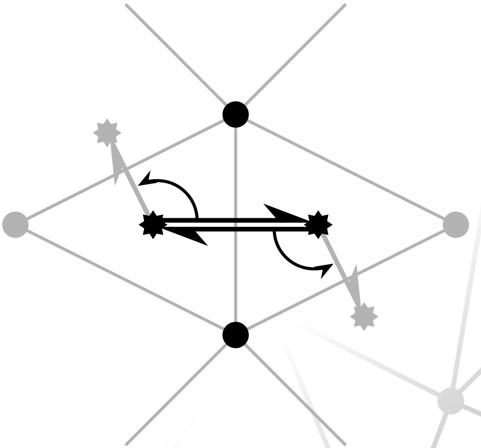
- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex



Data Structure: Quad-Edge

Edge-Based List-Like Data Structure:

- Directed edges for vertices
- Pointer to ccw. next directed edge with same origin vertex
- Directed dual edges for adjacent faces



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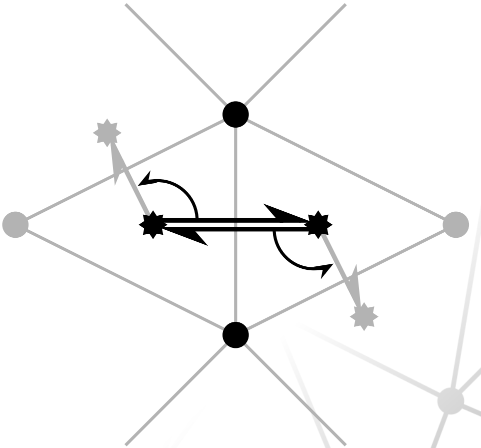

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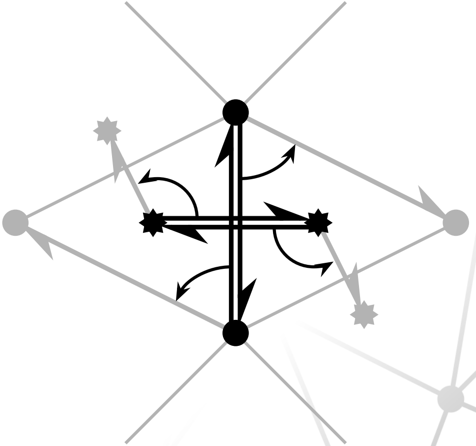
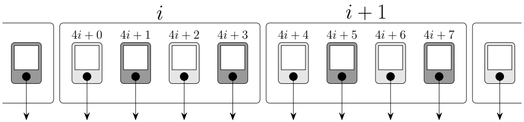


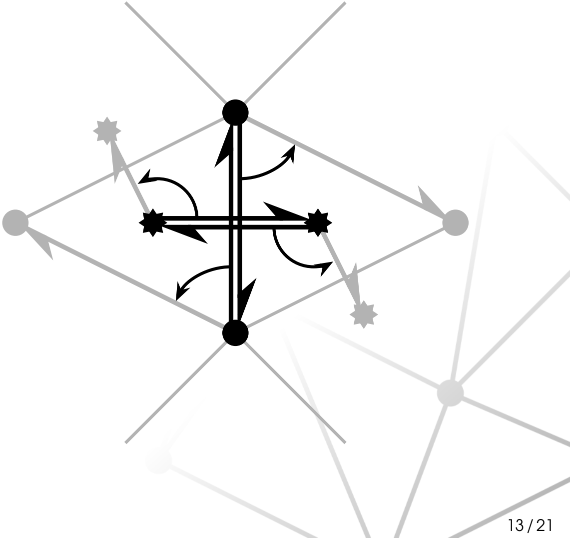
Data Structure: Quad-Edge Implementation




```
struct edge {
  size_t next;
  size_t data;
};

struct quad_edge {
  edge data[4];
};

vector<vertex>    vertices{};
vector<quad_edge> quad_edges{};
vector<size_t>    free_edges{};
```



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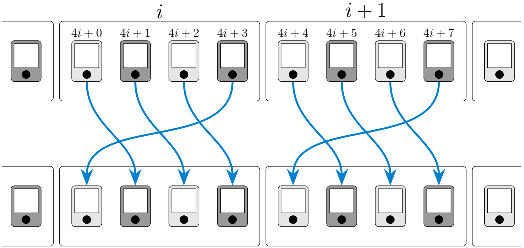
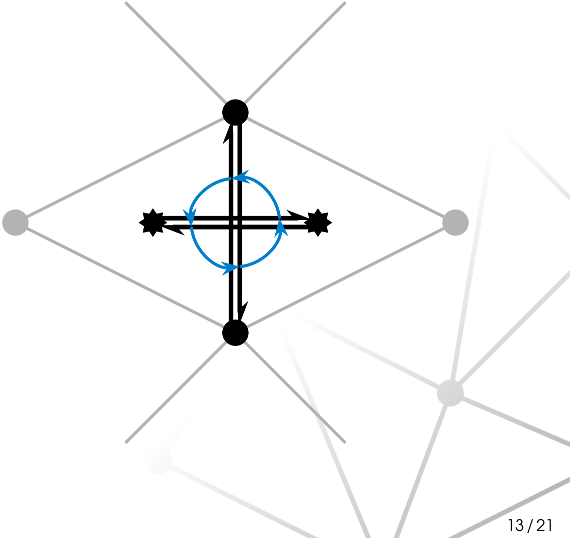
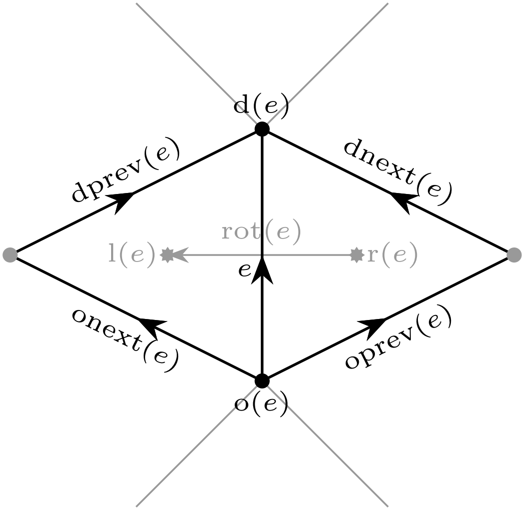


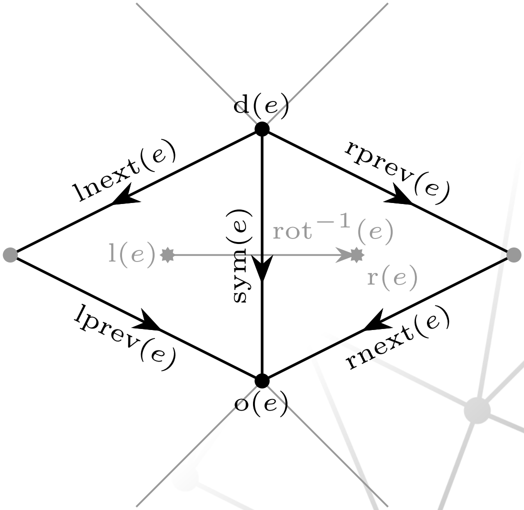
$$\mathrm{rot}\colon \mathbb{N}_0 \to \mathbb{N}_0$$

$$\mathrm{rot}(x) = 4 \cdot \left\lfloor \frac{x}{4} \right\rfloor + (x + 1 \mod 4)$$



Data Structure: Quad-Edge Functions





Data Structure: Quad-Edge Operations

- edge functions
- create edge
- splice
- connect
- delete edge

Algorithm

Algorithm: Overview

Algorithm: Overview

Triangulation Algorithm

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

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1. Sort the given point set by increasing x coordinate.
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Subroutine: Triangulate

Algorithm: Overview

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1. Sort the given point set by increasing x coordinate.
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Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.
3. Triangulate left and right half.

Algorithm: Overview

Triangulation Algorithm

1. Sort the given point set by increasing x coordinate.
2. Triangulate sorted point set.

Subroutine: Triangulate

1. If point count is smaller than four, make edge or triangle and return.
2. Split point set into left and right half.
3. Triangulate left and right half.
4. Merge left and right triangulations.

Algorithm: Merge Triangulations

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:
   3.1 Remove invalid edges adjacent to and above baseline.

Algorithm: Merge Triangulations

Subroutine: Merge Triangulations

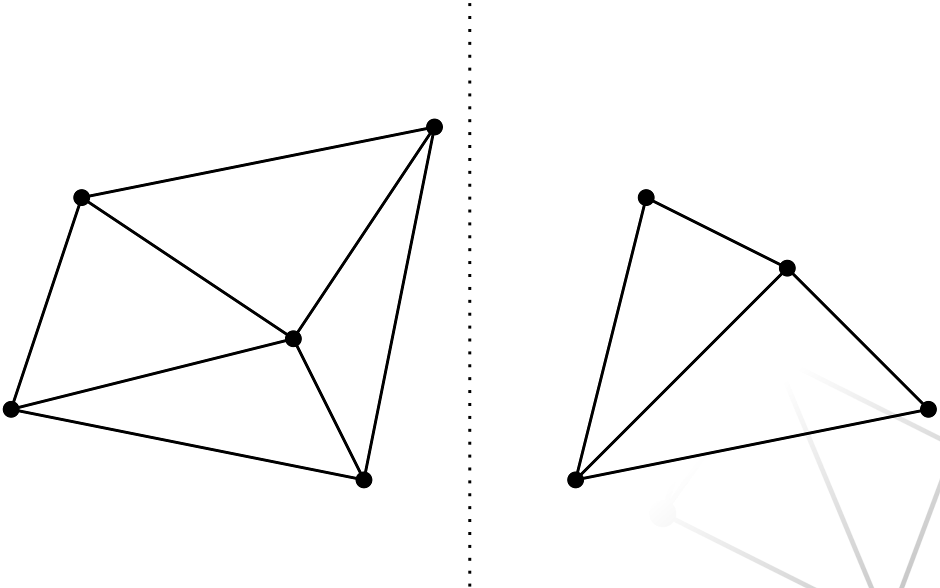
1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:
   3.1 Remove invalid edges adjacent to and above baseline.
   3.2 Insert cross edge above baseline.

Algorithm: Merge Triangulations

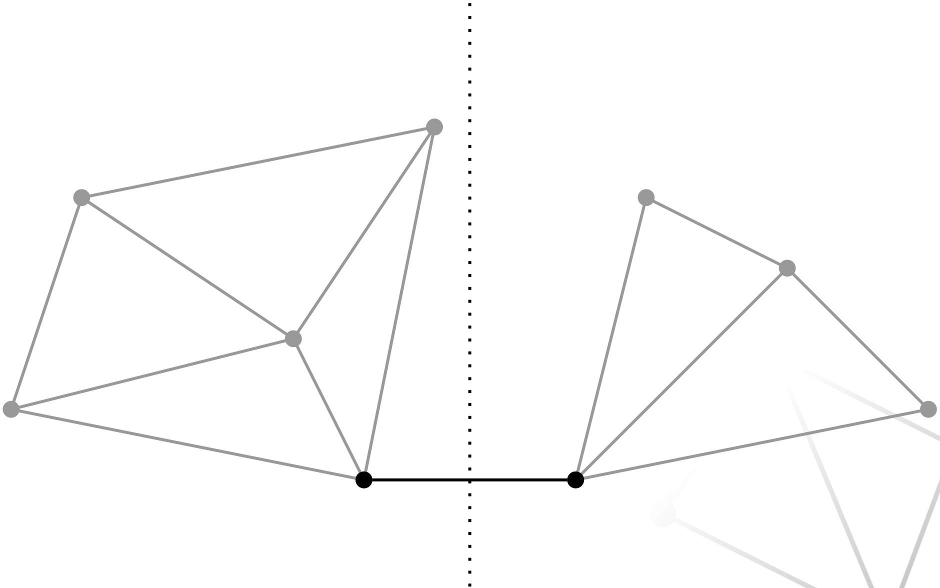
Subroutine: Merge Triangulations

1. Compute and add lower common tangent.
2. Use lower common tangent as baseline.
3. Loop until baseline becomes upper common tangent:
   3.1 Remove invalid edges adjacent to and above baseline.
   3.2 Insert cross edge above baseline.
   3.3 Make this cross edge the new baseline.

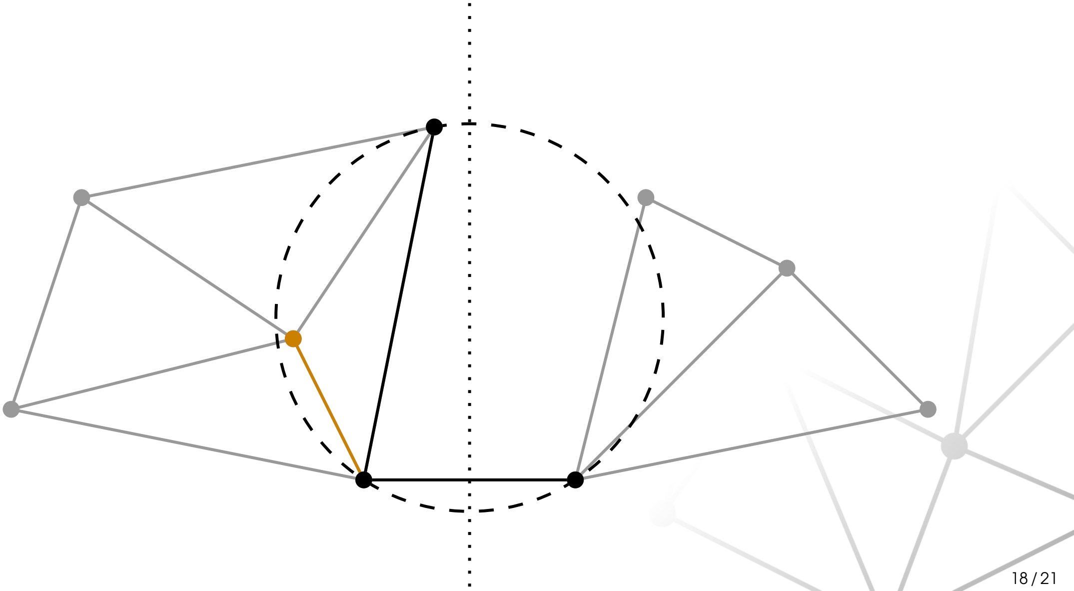
Algorithm: Merge Triangulations Example



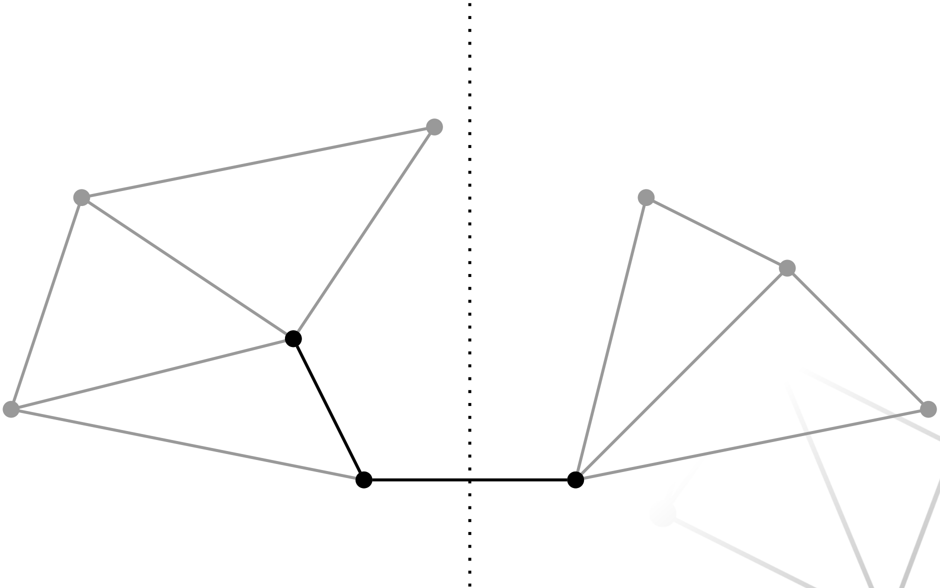
Algorithm: Merge Triangulations Example



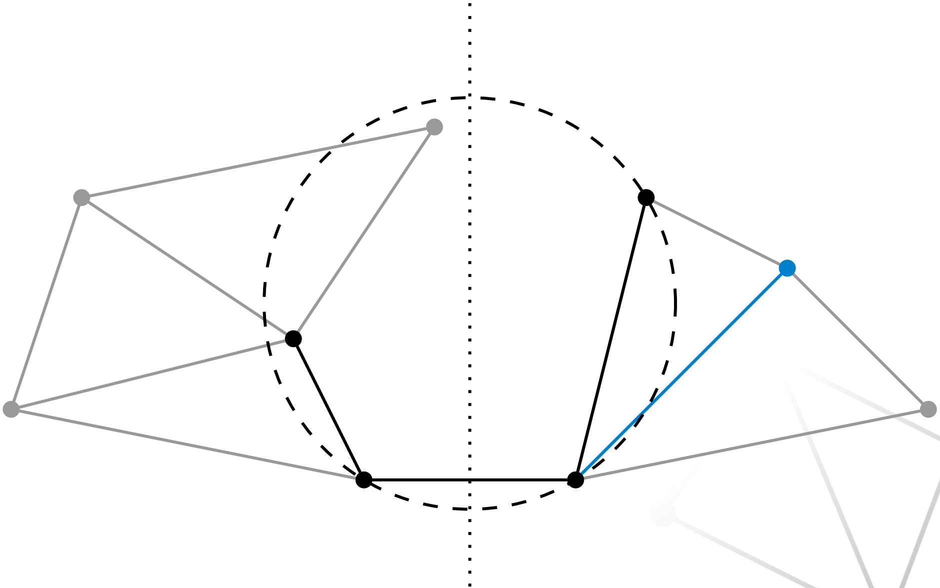
Algorithm: Merge Triangulations Example



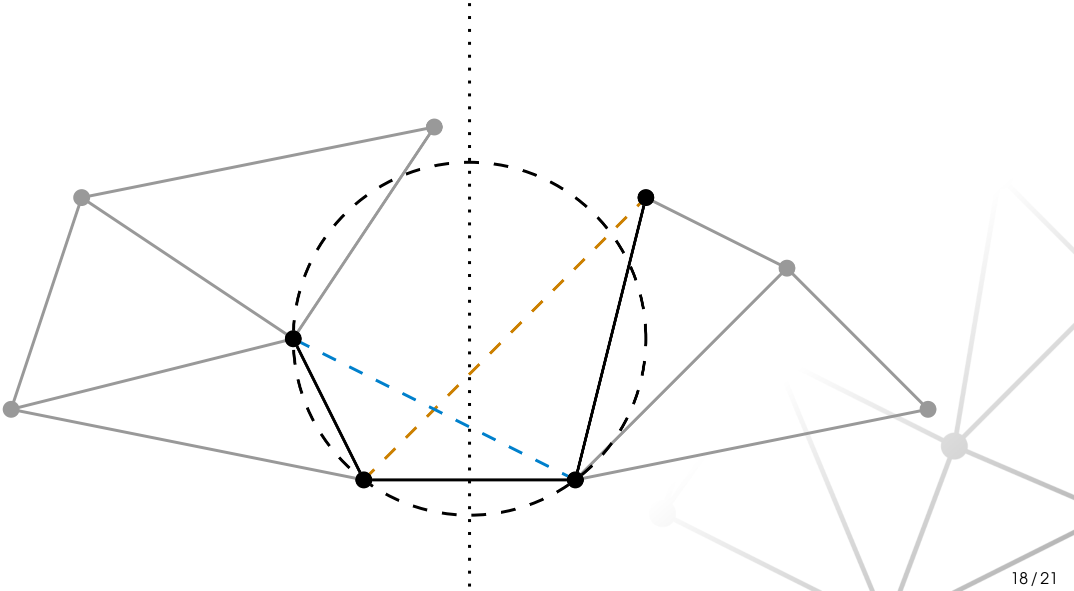
Algorithm: Merge Triangulations Example



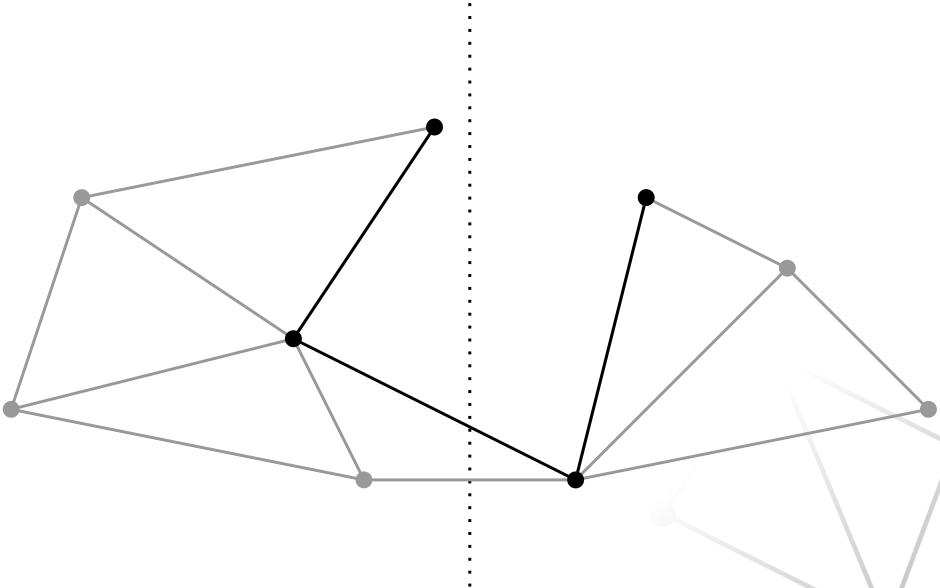
Algorithm: Merge Triangulations Example



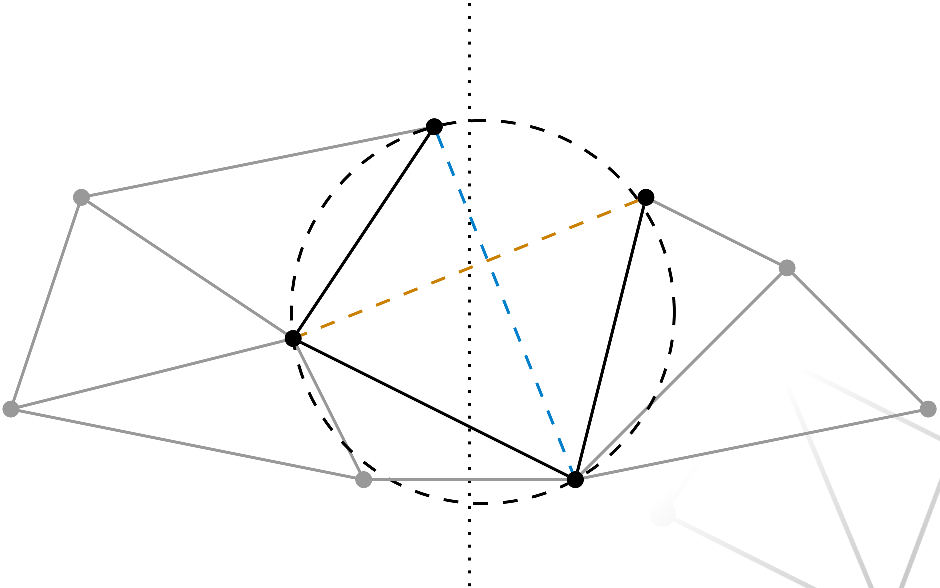
Algorithm: Merge Triangulations Example



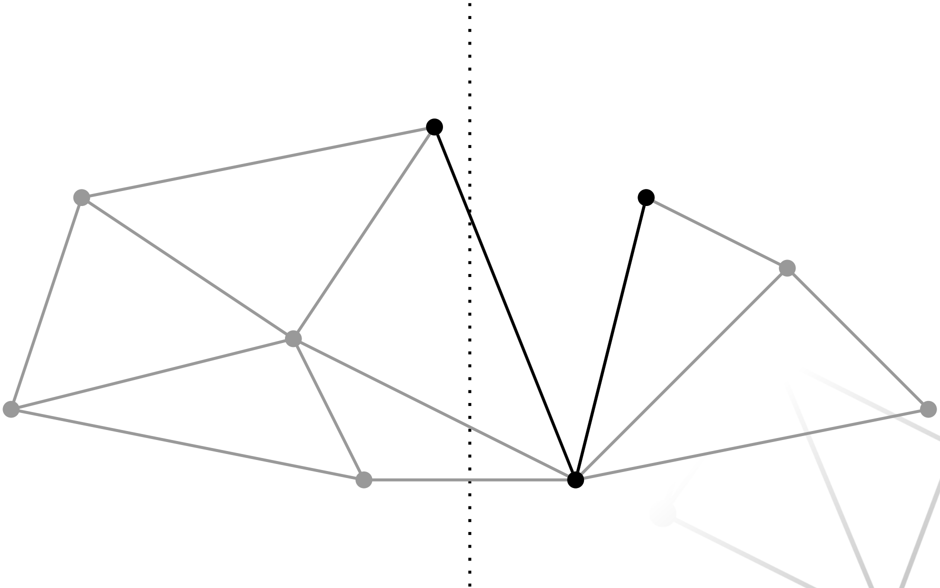
Algorithm: Merge Triangulations Example



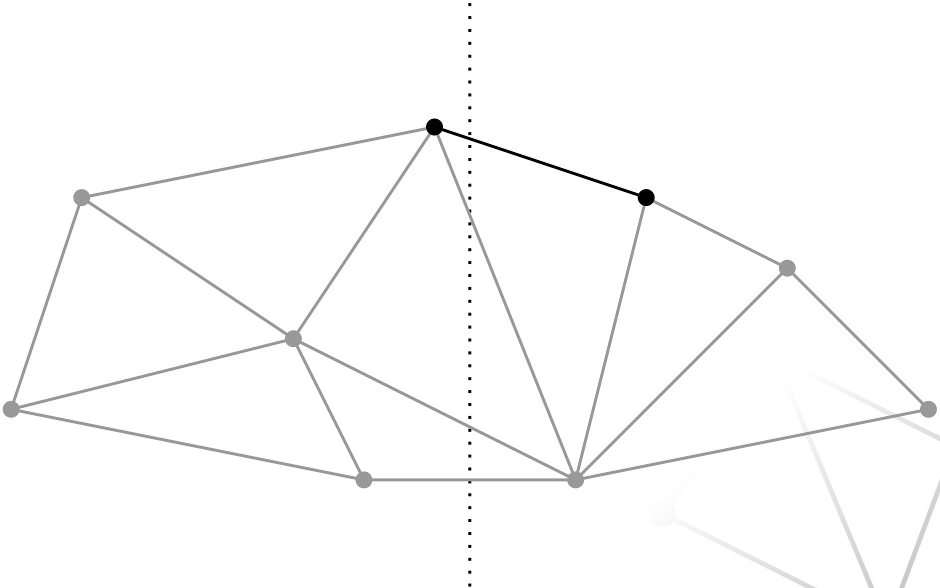
Algorithm: Merge Triangulations Example



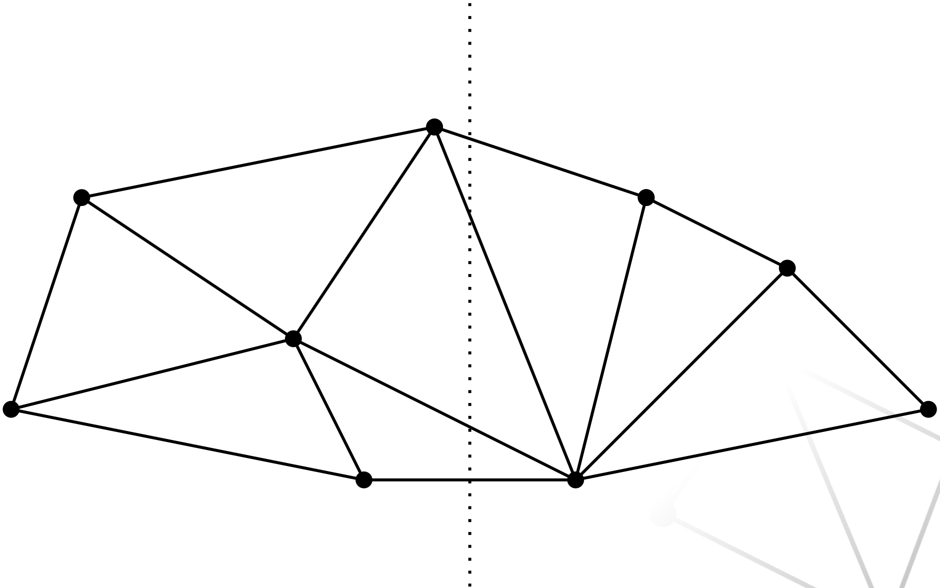
Algorithm: Merge Triangulations Example



Algorithm: Merge Triangulations Example



Algorithm: Merge Triangulations Example



Algorithm: Correctness

Algorithm: Complexity

Implementation

Implementation

- Geometric Primitives need exact computation and therefore arbitrary precision

Applications

Conclusions

Thank you for Your Attention!

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