FYSB21 - Mathematical Methods for Vibrations, Waves and Diffusion

Heat Diffusion Lab

Lab Manual

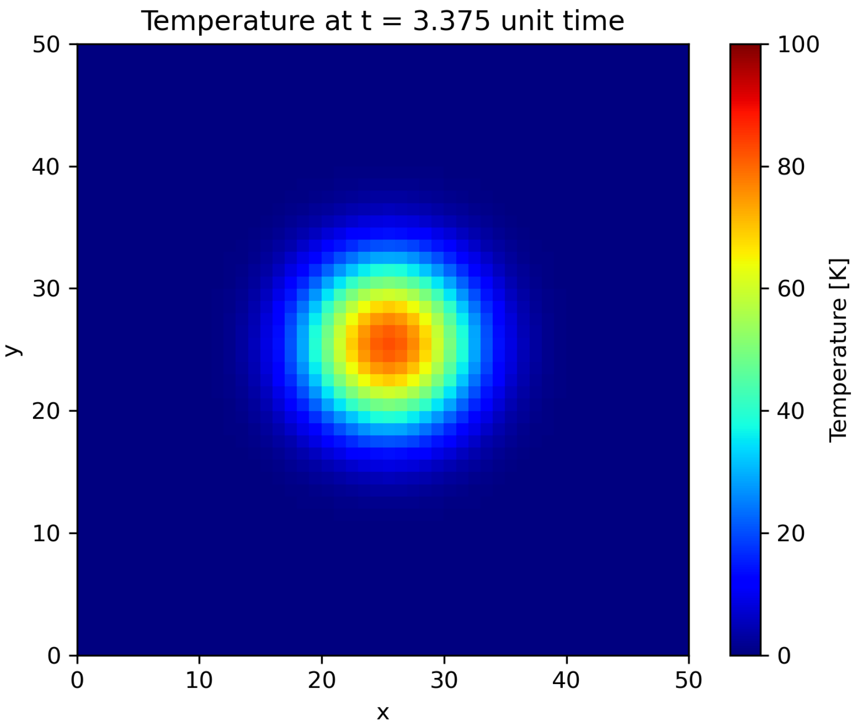

Figure credits: D. Hobbs, B.Prinoth

Lund University

Lab Instructors:
Bibiana Prinoth
bibi@astro.lu.se

David Hobbs
david@astro.lu.se

1 Introduction

The heat equation was first applied by Joseph Fourier in 1822 when modelling the diffusivity behaviour of a quantity such as heat. Describing the heat flow in a homogeneous and isotrpic medium the heat equation is given as:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{1}$$

where α is a positive coefficient called the diffusivity of the medium. $u(x, y, z, t)$ is the temperature of the medium at a given point in space at a given time.

1.1 Solving the Heat Equation with Fourier

Instead of using a discretised version of the heat equation resulting in a very pixelated solution, we aim at solving the heat equation in Fourier space. This allows us to come up with a solution that is fully continuous and analytical, only being limited by the plot and not the mathematics. We can use the following recipe to solve the heat equation with Fourier:

1. Transform the equation into Fourier space.
2. Solve the resulting ordinary differential equation
3. Transform back into real space
4. Evaluate the inverse Fourier integral
5. Evaluate $u(x, y, z, t)$ given the initial conditions of the input function.

Of course you have already done this in the lecture, and this shall only be a reminder on how we arrived at the following equations.

1.2 Equations used in this lab

In the following we summarise the important equations for the scope of this lab. You can use them without deriving them again, but make sure to include the theory behind it in your report (e.g. explain the derivation in your own words, why is Fourier space easier?)

1.2.1 One-dimensional heat equation (heat pulse)

In the one-dimensional case the heat equation becomes

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{+\infty} u_0(y) e^{-\frac{(x-y)^2}{4\alpha t}} dy \tag{2}$$

The initial conditions of the one-dimensional heat equation (heat pulse) are described using a rectangular function given by

$$u_0(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases} \tag{3}$$

Including these initial conditions, the solution for heat equation in one-dimension is found to be

$$u(x,t) = \frac{1}{2}\left[erf\left(\frac{+y_{\text{lim}} - x}{\sqrt{4\alpha t}}\right) - erf\left(\frac{-y_{\text{lim}} - x}{\sqrt{4\alpha t}}\right)\right] \tag{4}$$

where erf is the error-function.

1.3 General two-dimensional case

In the two-dimensional case, the heat equation takes the form

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{5}$$

From the one dimensional example we just replace the scalar x by the vector (x, y) and scale the solution by u_0. We then define similar initial conditions as: $r_{0x} = 17.5$ $r_{0y} = 37.5$ $r_{\text{max}} = 5/2$ and and centre our calculation using $r = \sqrt{(i\Delta x - r_{0x})^2 + (j\Delta y - r_{0y})^2)}$ and in our error function calculation we use $\pm r_{\text{max}} - r$ instead of $\pm y_{\text{lim}} - x$.

The solution then becomes

$$u(x,y,z,t) = \frac{1}{2}u_0\left[erf\left(\frac{+r_{\text{lim}} - r}{\sqrt{4\alpha t}}\right) - erf\left(\frac{-r_{\text{lim}} - r}{\sqrt{4\alpha t}}\right)\right]. \tag{6}$$

2 Tasks

This lab is split into three different tasks, whereas all of them have to be addressed in order to pass. Using the theory learned in the lecture on heat diffusion, you will create three animations for different cases with increasing difficulty:

- Heat diffusion in one dimension (heat pulse)
- Simulation of heat diffusion in two dimensions (to prepare for the final step)
- Simulation of the impact of a comet/asteroid on Earth using the heat equation in two

For all the tasks you have to submit a short video (mp4 or gif) similar to the one shown in the lecture. Task 1 corresponds to 40% of the points, Task 2a&b corresponds to 40%. If you manage to implement the exponential decaying function (see Task 2b), you will receive another 20%.

2.1 Task 1: Heat diffusion in one dimension

Using Eq. (4), implement a code showing the solution of the heat equation in one dimension. The sharpness of the function will diminish over time, slowly approaching an equilibrium solution. As an example, set $\alpha = 1$ and try to reproduce the video shown in the lecture. For your personal solution, you may vary the initial condition or α, but make sure to state your choice in the report.

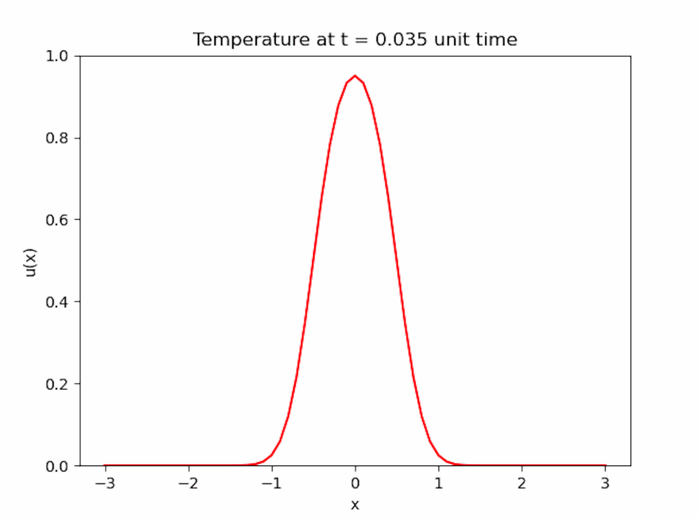

Figure 1: Solution of the heat equation in one dimension at time step $t = 0.035$ unit time. Credits: D. Hobbs.

2.2 Task 2a: Heat diffusion in two dimensions

We implement a similar code to the case in one dimension, but keep in mind that our plane of action is now two-dimensional. Use Eq. (6) to implement a code showing the solution of the heat equation in two dimensions. In order to reproduce the video in the lecture you may use the following initial conditions:

$$
\begin{aligned}
r_{0x} &= 17.5 \\
r_{0y} &= 37.5 \\
r_{\max} &= 2.5 \\
u_0 &= 100 \\
\alpha &= 2
\end{aligned}
$$

You are allowed to use different initial conditions, but make sure to state them in your report and plots.

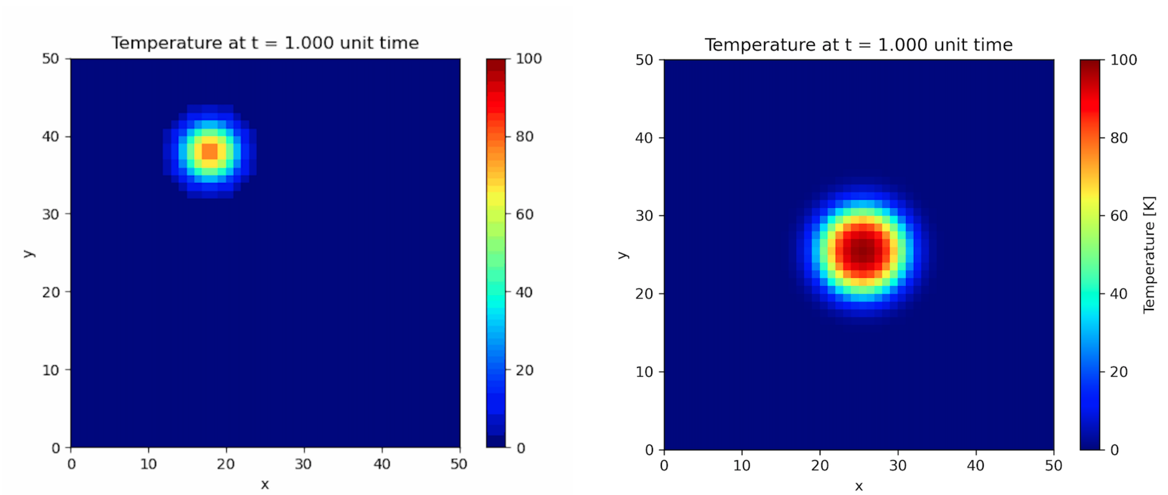

Figure 2: Solution of the heat equation in two dimensions in the general case at time step $t = 1$ unit time. Credits: D. Hobbs.

Note: You don't have to produce these plots, this is just an intermediate step you can / should take when creating the comet impact. The comet impact in 2b is the animation you want.

2.3 Task 2b: Comet impact on Earth

The last task focuses on a more realistic application of the heat equation modelling the impact of a comet on Earth. The layers of the Earth can be modelled as layers with different diffusivity α (trial and error). You also want to set different initial conditions for the different layers (i.e. different temperatures as the core is hotter than the layer above, etc.). In addition you want to use an exponential decaying function to simulate the source of heat decaying after the main event.

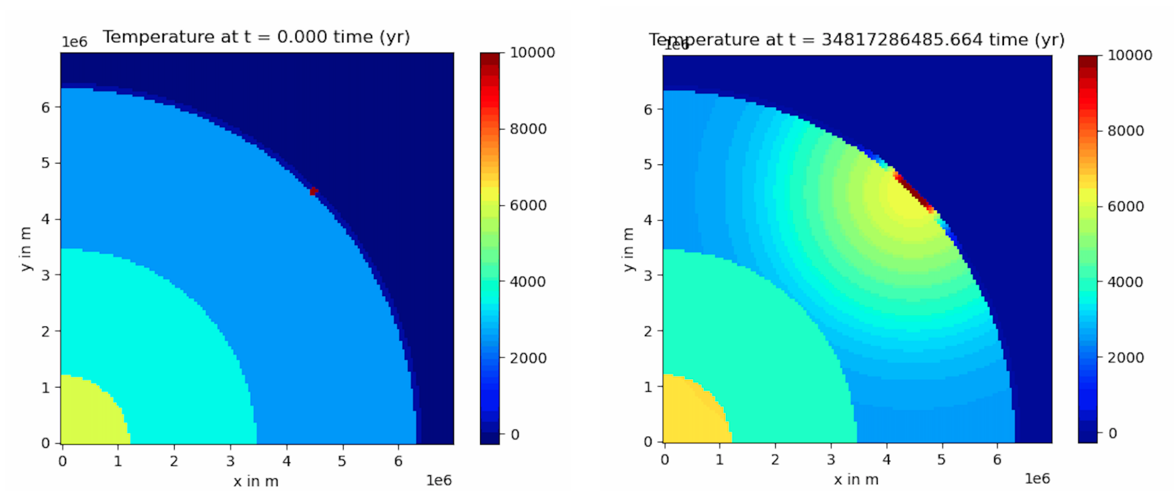

Figure 3: Solution of the heat equation in two dimensions modelling a comet impact on Earth at two time steps. Credits: D. Hobbs.

The lab report

Please typeset your lab report as a PDF document, and email it to the lab instructor. The deadline for handing in the lab report is by the end of the day two weeks after your second lab session. Please follow the general guideline of the report:

- **Title** contains the title and author information
- **Introduction** should contain a short general introduction to the subject, purpose of the lab, the tool, and a brief outline of the lab procedures.
- **Results** should contain the answers to the lab questions and any related plots. Include a plot of one of the frames of your videos into the report. *Do not omit your procedure, reasoning, and/or calculations, where applicable.*
- **Discussion** should contain any extended thought(s) you have related to the lab, besides what have been questioned and answered in the previous sections. You should discuss the approximations made while doing these numerical experiments.
- **Conclusion** briefly summarise the whole lab report.

The introduction, discussion, and conclusion do not have to be long. Longer does not mean it is better; instead, try to be concise but self-contained, and most importantly, coherent.

Moreover, *write the report in your own words.* Do not copy your answers from any other sources (like your classmates or the internet), which is considered plagiarism. Plagiarism is a serious violation of the academic code of conduct.