

Math 205: Complex Variables

Professor: Dan-Virgil Voiculescu, Spring 2021

Scribe: Vishal Raman

Contents

1	January 20th, 2021	3
1.1	Intro to Riemann Mapping Theorem	3
1.2	Cauchy's Integral Formula	3
1.3	Schwarz Lemma	3
1.4	Maximum Principles	4
2	January 25th, 2021	5
2.1	Uniform Convergence	5
3	January 27th, 2021	7
3.1	Uniform Convergence, continued	7
3.2	Metric Convergence	7
3.3	Riemann Sphere	7
4	February 1st, 2021	8
4.1	Connectivity	8
4.2	Fractional Linear Transformations	8
4.3	Fractional Linear Transformations, Unit Disk	9
5	February 3rd, 2021	10
5.1	Remark on the Zeta Function	10
5.2	Fractional Linear Transformations, continued	10
5.3	Automorphisms of the Half Plane	10
5.4	The Cross Ratio	11
6	February 8th, 2021	12
6.1	Mappings of Circles and Lines	12
6.2	Revisiting the Schwarz Lemma	12
6.3	Functions on Simply Connected Regions	13
6.4	Injective Functions	13
7	February 10th, 2021	14
7.1	Riemann Mapping Theorem	14

§1 January 20th, 2021

§1.1 Intro to Riemann Mapping Theorem

Our first goal is to prove a fundamental theorem of Riemann on conformal mappings. We start with several preparations, including some detours. The theorem essentially says that lots of open sets in \mathbb{C} are holomorphically isomorphic, given that they satisfy some simple topological conditions.

§1.2 Cauchy's Integral Formula

Recall Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz$$

where Γ is a simple closed curve, piecewise differentiable, $z_0 \in \text{Int}(\Gamma)$, and $f : \Omega \rightarrow \mathbb{C}$ is a holomorphic function, with Ω open, $\Omega \supset \Gamma \cup \text{Int}(\Gamma)$.

If Γ is the circle $|z - z_0| = R$, we parameterize with $z = Re^{i\theta} + z_0$ with $\theta \in [0, 2\pi)$. This gives

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta,$$

which represents the average of f on the circle.

It follows that

$$|f(z_0)| \leq \max_{\partial B_R(z_0)} |f(z)|,$$

with equality if and only if f is constant.

If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic for Ω connected, open and $z_0 \in \Omega$, then

$$|f(z_0)| \leq \sup_{z \in \Omega} |f(z)|$$

with equality if and only if f is constant.

§1.3 Schwarz Lemma

Theorem 1 (Schwarz Lemma)

For $f : B_1(0) \rightarrow \mathbb{C}$ holomorphic with $|f(z)| \leq 1$ for all z and $f(0) = 0$. Then

$$|f(z)| \leq |z|, |f'(0)| \leq 1.$$

If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = cz$ for some $|c| = 1$.

Proof. Define a function

$$g(z) = \begin{cases} f(z)/z, & \text{if } 0 < |z| \leq 1 \\ f'(0), & \text{if } z = 0 \end{cases}.$$

Note that $g(z)$ is continuous since at zero,

$$\lim_{z \rightarrow 0} \frac{f(z)}{z} = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Hence, $|g(z)| \leq C < \infty$ using the Weierstrass Extreme Value theorem. If $0 < \epsilon < |w| < r < 1$, note that taking a Keyhole Contour, we have

$$g(w) = \frac{1}{2\pi i} \left(\int_{|z|=r} - \int_{|z|=\epsilon} \right) \frac{g(z)}{z-w} dz.$$

Note that

$$\left| \int_{|z|=\epsilon} \frac{g(z)}{z-w} dz \right| \leq (2\pi\epsilon) \cdot C \frac{1}{|w|-\epsilon} \xrightarrow{\epsilon \rightarrow 0} 0.$$

It follows that

$$g(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(z)}{z-w} dz$$

for $0 < |w| < r$. The right side is holomorphic in w if $|w| < r$, so it follows that

$$g(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(z)}{z-w} dz$$

is holomorphic in $|z| < 1$.

This can also be proved by taking a Taylor series about the origin. Since there is no constant term, we can divide by z to still have a convergent Taylor series.

If $r < 1$,

$$\sup_{|z| \leq r} |g(z)| = \sup_{|z|=r} |g(z)| \leq \sup_{|z|=r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

If we let $r \uparrow 1$, then we get $\sup_{|z| < 1} |g(z)| \leq 1$. It follows that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

If $|f(z_0)| = |z_0|$ for some $0 < |z_0| < 1$ then $|g(z_0)| = 1$ and g is constant by the maximum principle so $g(z) = c$, $f(z) = cz$. If $|f'(0)| = 1$, then $|g(0)| = 1$ so g is constant and $f = cz$. \square

§1.4 Maximum Principles

In the above proof, we used the maximum principle. Some other versions we will use are the following:

If $K \subset \mathbb{C}$ compact and $f : K \rightarrow \mathbb{C}$ continuous, and the restriction of f to the interior of K is holomorphic, then

$$\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|.$$

If Ω is open and connected, $f : \Omega \rightarrow \mathbb{C}$, $z_0 \in \Omega$, and $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$, then f is constant. Applying this to e^f and using that $|e^f| = e^{\operatorname{Re} f}$, we find that

$$\operatorname{Re} f(z_0) = \sup_{z \in \Omega} \operatorname{Re} f(z),$$

implies that f is constant. We have the same result for $\operatorname{Im} f$ by replacing f with $-if$.

§2 January 25th, 2021

§2.1 Uniform Convergence

Remark 2.1. They sometimes call open connected sets "regions".

Definition 2.2 (Uniform Convergence). Let $\Omega \subset \mathbb{C}$ be open. Let $f_n : \Omega \rightarrow \mathbb{C}$ be holomorphic and $f : \Omega \rightarrow \mathbb{C}$ a function so that $\lim_{n \rightarrow \infty} \sup_{z \in K} |f(z) - f_n(z)| = 0$ for all $K \subset \Omega$ compact (also denoted $K \subset\subset \Omega$).

Remark 2.3. Recall from real analysis that f is a continuous function.

Some further remarks:

- It suffices to check the result for a sequence of compact subsets K_m so that $\bigcup_m K_m^\circ = \Omega$, then it suffices to check those. If $K \subset\subset \Omega$, then K is compact and covered by the union of the subsets so there exists a finite subcovering, and uniform convergence on the subcovering implies uniform convergence on K .
- It is often convenient to introduce $\|g\|_K = \sup_{z \in K} |g(z)|$. Uniform convergence can be restated as $\|f_n - f\|_K \rightarrow 0$ for all $K \subset\subset \Omega$.
- If $\|f_n - f\|_K \rightarrow 0$ for all $K \subset\subset \Omega$, then f is also holomorphic. It follows by passing to the limit in the Cauchy Integral formula. Namely, take $\{z : |z - z_0| \leq R\} \subset \Omega$ and consider the points in $|z_0 - \zeta| < R$.

$$\begin{aligned} \left| f_n(\zeta) - \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z-\zeta} dz \right| &= \left| \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f_n(z)}{z-\zeta} dz - \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z-\zeta} dz \right| \\ &\leq \frac{1}{2\pi} \frac{1}{R - |z_0 - \zeta|} \cdot (2\pi R) \|f_n - f\|_{|z-z_0|=R} \rightarrow 0. \end{aligned}$$

So it follows that

$$f(\zeta) = \lim_{n \rightarrow \infty} f_n(\zeta) = \frac{1}{2\pi i} \int_{|z-z_0|} \frac{f(z)}{z-\zeta} dz.$$

It follows that f continuous on $|z - z_0| = R$ is holomorphic in $\zeta \in \{|z - z_0| < R\}$, so it follows that f is holomorphic.

- We can similarly show that

$$f_n^{(j)}(\zeta) = \frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f_n(z)}{(z-\zeta)^{n+1}} dz$$

$$\text{and } \|f_n^{(j)} - f^{(j)}\|_K \rightarrow 0.$$

From the last item, we have the following theorem.

Theorem 2

If $f_n \rightarrow f$ on compact subsets of Ω , then if f_n is holomorphic we find that f is holomorphic and $f_n^{(j)} \rightarrow f^{(j)}$ uniformly on compact subsets of Ω .

Theorem 3 (Hurwitz)

Let Ω be a region, $f : \Omega \rightarrow \mathbb{C}$ and $f_n : \Omega \rightarrow \mathbb{C}$ holomorphic with $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$ and $\|f_n - f\|_K \rightarrow 0$ for all compact subsets. Then either $f \equiv 0$ or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

Proof. If f is not identically zero on ω , then since f is holomorphic, its zeros are isolated. If $z_0 \in \Omega$, $f(z_0) = 0$, then there is $\epsilon > 0$ so that when $0 < |z - z_0| < \epsilon$, $f(z) \neq 0$.

Since $f(z) \neq 0$ for $|z - z_0| = \epsilon/2$, by the Weierstrass theorem applied to $|f|$ on $|z - z_0| = \epsilon$, we have $|f(z)| \geq m > 0$ on $\{|z - z_0| = \epsilon/2\} = \Gamma$. If $\|f_n - f\|_\Gamma \leq m/2$ for $n \geq N$, then

$$|f_n(z)| \geq |f(z)| - m/2 \geq m - m/2 = m/2$$

for $z \in \Gamma$. Hence, it follows that $\|1/f_n - 1/f\|_\Gamma \rightarrow 0$ (we leave this as an exercise).

Since $\|f'_n - f'\|_\Gamma \rightarrow 0$, we find that $\|f'_n/f_n - f'/f\| \rightarrow 0$ (another exercise) and hence

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'_n}{f_n} dz \rightarrow \frac{1}{2\pi i} \int_\Gamma \frac{f'}{f} dz.$$

The integrand of the left hand side is $(\log f_n)'$, whose integral is 0, and the right side is the order of the zero of f at z_0 by the argument principle. It follows that the order of z_0 as a possible zero is 0, so $f(z_0) \neq 0$. \square

Theorem 4

For $\Omega \subset \mathbb{C}$ open, \mathcal{F} a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω .

Proof. We first show 2 implies 1. If $\sup_{f \in \mathcal{F}} \|f\|_K = \infty$, then we can find for each $n \in \mathbb{N}$ $f_n \in \mathcal{F}$ so that $\|f_n\|_K \geq n$. If we abstract a convergence subsequence, then $\|f_{n_j} - f\|_K \leq C < \infty$ and $\|f_{n_j}\|_K \leq \|f\|_K + C$, while $\|f_{n_j}\|_K \rightarrow \infty$, a contradiction. \square

§3 January 27th, 2021

§3.1 Uniform Convergence, continued

Theorem 5

For $\Omega \subset \mathbb{C}$ open, \mathcal{F} a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω .

I missed the beginning of the class, but I will add the proof of the theorem once notes are posted.

§3.2 Metric Convergence

One can put a metric on holomorphic functions so that convergence in the metric is uniform convergence on compact sets. For $f : \Omega \rightarrow \mathbb{C}$, but $K_n \Subset \Omega$ so that $\bigcup_n K_n^\circ = \Omega$ and take

$$d(f, g) = \sum_{n=1}^{\infty} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}} 2^{-n}.$$

§3.3 Riemann Sphere

On the set $\mathbb{C} \cup \{\infty\}$, we consider the topology which makes it the Alexandroff(one-point) compactification of \mathbb{C} . If $z \in \mathbb{C}$, a neighborhood is one that contains a neighborhood in \mathbb{C} and a neighborhood of ∞ is of the form $\{\infty\} \cup (\mathbb{C} \setminus K)$ for $K \Subset \mathbb{C}$.

Let $U_+ = \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ and $U_- = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$. Note that the union of the two sets covers the Riemann Sphere. Define $\psi_+ : U_+ \rightarrow \mathbb{C}$ by $\psi_+(z) = z$ and $\psi_- : U_- \rightarrow \mathbb{C}$ is given by $\psi_-(w) = 1/w$ if $w \in \mathbb{C} \setminus \{\infty\}$ and 0 if $w = \infty$. Notice that these two functions are bijections.

If $V \subset \mathbb{C} \cup \{\infty\}$ is open, a function $f : V \rightarrow \mathbb{C}$ is holomorphic if

$$f|_{V \cup U_{\pm}} \circ (\psi_{\pm}|_{V \cup U_{\pm}})^{-1} : \psi_{\pm}(V \cup U_{\pm}) \rightarrow \mathbb{C}$$

is holomorphic. In this way, we know what holomorphic functions are on open sets of $\mathbb{C} \cup \{\infty\}$.

More generally, we can describe a Riemann surface in the following way - Let X be a topological space. Take $\{(U_{\alpha}, z_{\alpha})\}_{\alpha \in I}$ where $U_{\alpha} \subset X$ is open, and $\bigcup_{\alpha \in I} U_{\alpha} = X$ and $z_{\alpha} : U_{\alpha} \rightarrow \mathbb{C}$ is continuous, $z_{\alpha}(U_{\alpha})$ is open and z_{α} is a homeomorphism. The key requirement is that the maps $z_{\alpha} \circ z_{\beta}^{-1} : z_{\beta}(U_{\alpha} \cap U_{\beta}) \rightarrow z_{\alpha}(U_{\alpha} \cap U_{\beta})$ are holomorphic.

Then, if $U \subset X$ is open, $f : U \rightarrow \mathbb{C}$ is holomorphic if for all $\alpha \in I$,

$$f|_{U \cap U_{\alpha}} \circ (z_{\alpha}|_{U \cap U_{\alpha}})^{-1}$$

is holomorphic. Two such atlases give the same Riemann surface if put together, we get an atlas.

§4 February 1st, 2021

§4.1 Connectivity

Definition 4.1. $\Omega \subset \mathbb{C}$ open is connected if $\Omega = \Omega_1 \cup \Omega_2$ open with $\Omega_1 \cap \Omega_2 = \emptyset$ implies that one of the two is empty. For open sets, this is equivalent to arcwise connected.

Definition 4.2. A set is arcwise connected if for every $z_1, z_2 \in \Omega$, there is a path $\varphi : [0, 1] \rightarrow \Omega$ which is continuous and $\varphi(0) = z_1, \varphi(1) = z_2$.

Definition 4.3. Ω is simply connected if for $z_0 \in \Omega$, $\Gamma : [0, 1] \rightarrow \Omega$ continuous and $\Gamma(0) = \Gamma(1) = z_0$, then there is $G : [0, 1] \times [0, 1] \rightarrow \Omega$ continuous with $G(t, 0) = \Gamma(t)$ for $t \in [0, 1]$ and $G(t, 1) = z_0$, for $t \in [0, 1]$.

Simply connected corresponds to the idea of being able to continuously deform the set to a point for each point.

In $\mathbb{R}^2 \cong \mathbb{C}$, Ω -open simply connected is equivalent to $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ is connected in $\mathbb{C} \cup \{\infty\}$. That is, if $F = \mathbb{C} \cup \{\infty\} \setminus \Omega$, which is closed in $\mathbb{C} \cup \{\infty\}$, with $F \cap V_1 \cap V_2 = \emptyset$, then at least one of the $F \cap V_k = \emptyset$. If $0 \in \Omega$, then Ω is simply connected if and only if $\{0\} \cup \{1/z : z \in \mathbb{C} \setminus \Omega\}$ is connected (this is a local representation).

- Take $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^m \{tz_j : t \in [1, \infty)\}$ for $z_1, \dots, z_n \in \mathbb{C} \setminus \{0\}$.
- $\mathbb{C} \setminus$ spirals.

Theorem 6 (Riemann Mapping Theorem)

If $\Omega \subset \mathbb{C}$ open, connected, simply connected, $\emptyset \neq \Omega \neq \mathbb{C}$, then Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic isomorphisms.

§4.2 Fractional Linear Transformations

Recall that if $f \in \text{Aut}(\mathbb{D})$ then $f(z) = \frac{az+b}{cz+d}$, which was proved using the Schwarz lemma. We view the fractional linear maps from a different context.

We define a map $p : \mathbb{C}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \rightarrow \mathbb{C} \cup \{\infty\}$ given by

$$p\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0 \end{cases}.$$

Then $p(\xi) = p(\eta)$ if and only if $\xi = \lambda\eta$ for $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

There is a larger group acting on $\mathbb{C}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ given by $GL(2, \mathbb{C})$ the invertible 2×2 matrices in the natural way so that

$$A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \frac{A_{11}p(\xi) + A_{12}}{A_{21}p(\xi) + A_{22}}.$$

Define $T_g : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ given by

$$T_g z = \frac{az + b}{cz + d},$$

with $T_g(\infty) = \frac{a}{c}$. We have the action $T_g p(\xi) = p(g\xi)$ for $g \in GL(2, \mathbb{C})$.

This gives

$$\begin{aligned} T_{g_1} \circ T_{g_2} &= T_{g_1 g_2}, \\ (T_g)^{-1} &= T_{g^{-1}}. \end{aligned}$$

We can also ask about the fixed point:

$$T_g p(\xi) = p(\xi) \leftrightarrow p(\xi) = p(g\xi) \Leftrightarrow g\xi = \lambda\xi, \lambda \in C^\times$$

It follows that the fixed points of T_g correspond to the eigenvectors of $GL(2, \mathbb{C})$.

§4.3 Fractional Linear Transformations, Unit Disk

If we have $\xi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, then $p(\xi) \in \mathbb{D}$ if and only if $|z_1| < |z_2|$ if and only if $z_1 \bar{z}_1 - z_2 \bar{z}_2 < 0$.
If we let

$$J = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix},$$

we consider the sesquilinear form $\langle J \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \rangle$, where it is linear in the first coordinate and conjugate linear in the second coordinate. Note that

$$\left\langle J \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle = \xi_1 \bar{\eta}_1 - \xi_2 \bar{\eta}_2.$$

When does $g \in GL(2, \mathbb{C})$ preserve $\langle J\xi, \xi \rangle$?

This means that

$$\langle Jg\xi, g\xi \rangle = \langle J\xi, \xi \rangle$$

for all $\xi \in C^2 \setminus \{0\}$. Then,

$$\langle g^* Jg\xi, \xi \rangle = \langle J\xi, \xi \rangle$$

so it follows that $g^* Jg = J$. (We prove this by transforming ξ in polar coordinates, $\xi = x + i^k y$, and considering $k = 0, 1, 2, 3$. These four equations allow us to determine the equality). Note that $U(1, 1) = \{g : g^* Jg = J\}$ forms a group structure where J has eigenvalues ± 1 for this reason, we denote $U(1, 1) \subset GL_2(\mathbb{C})$.

We claim the following: $T_g \in \text{Aut}(\mathbb{D}) \Leftrightarrow g \in C^\times \cdot U(1, 1)$.

§5 February 3rd, 2021

§5.1 Remark on the Zeta Function

Theorem 5.1 (S.M. Voronin 1975)

For $D = \{\frac{1}{2} < \operatorname{Re}(z) < 1\}$, $f : D \rightarrow \mathbb{C} \setminus \{0\}$. If $K \subset\subset D$ and $\epsilon > 0$, then there exists $t \in \mathbb{R}$ such that

$$\|f(\cdot) - \zeta(\cdot + it)\|_K < \epsilon.$$

This theorem essentially says that if I slide around the zeta function in the strip D , I can uniformly approximate pretty much any function I want.

§5.2 Fractional Linear Transformations, continued

Note that $\operatorname{Ker}(g \mapsto T_g) = \mathbb{C}^\times I_2$. We define $SL(2; \mathbb{C}) = \{g \in GL(2; \mathbb{C}) : \det g = 1\}$, the special linear group.

Theorem 5.2

For $g \in SL(2; \mathbb{C})$, $T_g \in \operatorname{Aut}(\mathbb{D})$ if and only if $g \in U(1, 1)$.

Proof. We start with the forward direction. From the first homework, we showed that $f \in \operatorname{Aut}(\mathbb{D})$ implies that $f(z) = T_g z$ where g is the composition of a rotation g_1 and $g_2 = \begin{pmatrix} 1 & z_0 \\ \bar{z}_0 & 1 \end{pmatrix}$ for $z_0 \in \mathbb{D}$. It suffices to check that $g_1, g_2 \in U(1, 1) \times \mathbb{C}^\times I_2$. This is easy to check.

Now, we show the converse. If $g \in U(1, 1)$, then $g^{-1} \in U(1, 1)$. If $z \in \mathbb{D}$, then $z = p(\xi)$, $\langle J\xi, \xi \rangle < 0$. We have $T_g z = p(g\xi)$ and $\langle J\xi, \xi \rangle < 0$ implies that $\langle g^* J g \xi, \xi \rangle < 0$, which implies that $\langle J g \xi, g \xi \rangle < 0$, which shows that $T_g z = p(g\xi) \in \mathbb{D}$. Hence $T_g \mathbb{D} \subset \mathbb{D}$. The same argument holds for $T_g^{-1} \mathbb{D} \subset \mathbb{D}$ so we have $T_g \mathbb{D} = \mathbb{D}$ exactly, so $T_g = \operatorname{Aut}(\mathbb{D})$. \square

§5.3 Automorphisms of the Half Plane

There is a conformal map from $\mathbb{H}_+ \rightarrow \mathbb{D}$ given by $f : z \mapsto \frac{z-i}{z+i}$. This corresponds to

$$f = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Note that

$$f^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Now, $\operatorname{Aut}(\mathbb{H}_+) = \{(T_f)^{-1} T_g T_f | T_g \in \operatorname{Aut}(\mathbb{D})\} = \{T_{f^{-1} g f} | g \in SU(1, 1)\}$. It follows that $\operatorname{Aut}(\mathbb{H}_+) = \{T_h | f h f^{-1} \in SU(1, 1)\}$ (assuming $h \in SL(2, \mathbb{C})$, $f h f^{-1} \in SL(2, \mathbb{C})$). It follows that $(f h f^{-1})^* J (f h f^{-1}) = J$, so $h^*(f^* J f) h = f^* J f$. We can compute

$$f^* J f = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}.$$

It follows that

$$h^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h = I_2.$$

If we check the computation, we find that $a, b, c, d \in \mathbb{R}$, so it follows that $h \in SL(2, \mathbb{R})$.

§5.4 The Cross Ratio

Note that T_g is completely determined by $T_g 0, T_g 1, T_g \infty$. Suppose $T_g 0 = T_h 0, T_g 1 = T_h 1, T_g \infty = T_h \infty$. If we let $r = g^{-1}h$, we have $T_r 0 = 0, T_r 1 = 1, T_r \infty = \infty$, so it follows that $r \in C^\times I_2$ (carry out the matrix multiplication for an arbitrary matrix).

if we look at g^{-1} instead of g , we find that T_g is completely determined by $a, b, c \in C \cup \infty$ so that $Ta = 1, Tb = 0, Tc = \infty$. Given, a, b, c , such a T_g is the map

$$z \mapsto \frac{z - b}{z - c} : \frac{a - b}{a - c}.$$

We denote the RHS by (z, a, b, c) , which is a fractional linear map taking a, b, c to $1, 0, \infty$. This is called the cross ratio of z, a, b, c .

Theorem 5.3

If T_g is a fractional linear transformation and z_1, z_2, z_3, z_4 are distinct points in $\mathbb{C} \cup \infty$, then

$$(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4).$$

Remark 5.4. The above theorem shows that cross ratios are invariant under fractional linear transformations.

§6 February 8th, 2021

§6.1 Mappings of Circles and Lines

Lemma 6.1

For $g \in GL_2(\mathbb{C})$, $\{w \in \mathbb{C} \cup \{\infty\} : T_g w \in \mathbb{R} \cup \{\infty\}\}$ is a circle or a straight line with a point at infinity.

Proof.

$$\frac{aw + b}{cw + d} = \frac{\overline{aw + b}}{\overline{cw + d}},$$

Then $(a\bar{c} - c\bar{a})|w|^2 + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - d\bar{b} = 0$. If $a\bar{c} - c\bar{a} = 0$, then we have a straight line. If $a\bar{c} - c\bar{a} \neq 0$, we have

$$\left| w + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a} \right| = \left| \frac{ad - bc}{\bar{a}c - \bar{c}a} \right|,$$

a circle. □

§6.2 Revisiting the Schwarz Lemma

Recall we have $f \in \text{Aut}(\mathbb{D})$, with $f(0) = 0$. We will use the fractional linear transformations so that $0 \in \mathbb{D}$ no longer has a special role.

Given $f : \mathbb{D} \rightarrow \mathbb{D}$ holomorphic with $z_0 \in \mathbb{D}$. Take an automorphism mapping $0 \rightarrow z_0$ given by $\frac{\cdot + z_0}{1 + \bar{z}_0(\cdot)}$. Then, applying f and applying $(\frac{\cdot + f(z_0)}{1 + \bar{f}(z_0)(\cdot)})^{-1}$, which sends $f(z_0) \rightarrow 0$. These are all holomorphic, so it follows that the composition is a holomorphism from $\mathbb{D} \rightarrow \mathbb{D}$ mapping $0 \rightarrow 0$. Now, we can apply the Schwarz Lemma as usual: For the derivatives, we use the chain rule:

$$\left(\frac{\cdot + z_0}{1 + \bar{z}_0(\cdot)} \right)' \Big|_{z=0} = 1 - |a|^2.$$

Composing the derivatives along the composition, we find the derivative evaluated at 0 which we require to be ≤ 1 .

It follows that

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} \leq \frac{1}{1 - |z_0|^2}.$$

Moreover, by the Schwarz Lemma, we have equality if and only if $f \in \text{Aut}(\mathbb{D})$. if we put $w = f(z)$, then $dw = f'dz$ and the inequality is

$$\frac{|dw|}{1 - |w|^2} \leq \frac{dz}{1 - |z|^2}.$$

This can be interpreted as having on \mathbb{D} the Riemannian metric

$$\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

and $f : \mathbb{D} \rightarrow \mathbb{D}$, contracting the metric.

§6.3 Functions on Simply Connected Regions

Recall the following properties of holomorphic functions in simply connected regions:

- For $f : \Omega \rightarrow \mathbb{C}$ holomorphic, then there is $F : \Omega \rightarrow \mathbb{C}$ holomorphic so that $F' = f$.
- $f : \Omega \rightarrow \mathbb{C} \setminus \{0\}$, then there exists $g : \Omega \rightarrow \mathbb{C}$ holomorphic so that $e^g = f$.
- $f : \Omega \rightarrow \mathbb{C} \setminus \{0\}$ holomorphic, then there exists $g : \Omega \rightarrow \mathbb{C}$ so that $h^n = f$.
- $f : \Omega \rightarrow \mathbb{C}$ holomorphic and non-constant, Ω a region, then $f(V)$ is open if $V \subset \Omega$, V is open.

§6.4 Injective Functions

Let $f : \Omega \rightarrow G$ be a holomorphic function with Ω open and connected. If f is injective, then $f'(z) \neq 0$. If so, then $f(z) - f(z_0) = u(z)^n$ if $0 = f'(z_0) = \dots, f^{(n-1)}(z_0)$ and $f^{(n)}(z_0) \neq 0$, with $u(z_0) = 0$. Then $u(\{|z - z_0| < \epsilon\})$ is open for some $\epsilon > 0$ so it contains $\{|\zeta| < \delta\}$ for some $\delta > 0$. It follows that $U(z_k) = \frac{\delta}{z} e^{2\pi i k/n}$ for $1 \leq k \leq n$ and $f(z_1) = \dots = f(z_n)$. We could also use the argument principle to show that $f'(z) \neq 0$.

Then, $f(\Omega)$ is open and f has local inverses: for each $z \in \Omega$, there is a neighborhood V_z , where f is a holomorphic isomorphism in the region. It follows that $f : \Omega \rightarrow G$ is holomorphic, injective, then $f|f(\Omega) : \Omega \rightarrow f(\Omega)$ is a holomorphic isomorphism.

If Ω is an open region so that $f : \Omega \rightarrow \mathbb{D}$ is a holomorphic isomorphism, then if fix $z_0 \in \Omega$, we have $g \in \text{Iso}(\Omega, \mathbb{D}) \rightarrow (g(z_0), \frac{g'(z_0)}{|g'(z_0)|}) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

§7 February 10th, 2021

Lemma 7.1

If Ω is an open region so that $f : \Omega \rightarrow \mathbb{D}$ is a holomorphic isomorphism, then if fix $z_0 \in \Omega$, we have $g \in \text{Iso}(\Omega, \mathbb{D}) \rightarrow (g(z_0), \frac{g'(z_0)}{|g'(z_0)|}) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

Proof. We provide a sketch of the proof. Replace f with

$$\left(\frac{\cdot - f(z_0)}{1 - \overline{f(z_0)} \cdot} \right) \circ f$$

so that $f(z_0) = 0$. Then, $\text{Iso}(\Omega, \mathbb{D}) \ni g \rightarrow g \circ f^{-1} \in \text{Aut}(\mathbb{D})$ is a bijection and

$$\left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|} \right) = \left((g \circ f^{-1})(0), \frac{(g \circ f^{-1})'(0)}{|(g \circ f^{-1})'(0)|} \frac{f'(z_0)}{|f'(z_0)|} \right)$$

so the proof reduces to the case where $\Omega = \mathbb{D}$ and $z_0 = 0$. It is easy to show that the map is onto and 1-1. \square

§7.1 Riemann Mapping Theorem

Theorem 7 (Riemann Mapping Theorem)

Suppose Ω is simply connected and $\Omega \neq \mathbb{C}$. Then, there exists $f : \Omega \rightarrow \mathbb{D}$ a holomorphic isomorphism.

Remark 7.2. There is no holomorphic isomorphism from $\mathbb{D} \rightarrow \mathbb{C}$ because of Liouville's Theorem.

Proof. (Kobe) Let $z_0 \in \Omega$ and $\mathcal{F} = \{f : \Omega \rightarrow \mathbb{D} : f \text{ injective}, f(z_0) = 0, f'(z_0) > 0\}$. The steps are as follows:

- $\mathcal{F} \neq \emptyset$.

Proof. If $\Omega \neq \mathbb{C}$, there is a point $a \in \mathbb{C} \setminus \Omega$. If Ω is simply connected, there exists $h : \Omega \rightarrow \mathbb{C}$ holomorphic with $h^2(z) = z - a$. Then $h(\Omega)$ is open and there exists r such that $B_r(h(z_0)) \subset h(\Omega)$. Then $h^2(\cdot) = \cdot - a$ is injective, so h is injective. Then $-B(h(z_0), r) \cap h(\Omega) = \emptyset$. Otherwise, there are z_1, z_2 with $h(z_1) = -h(z_2) \neq 0$. Then, we have $z_1 \neq z_2$ and $h(z_1) = -h(z_2)$ which implies that $h^2(z_1) = h^2(z_2)$.

Hence, $|h(z) - h(z_0)| \geq r$ for all $z \in \Omega$. It we take $p = r/2 > 0$, then we have $|h(z) + h(z_0)| \geq p$. Then, we find $c \in \mathbb{C}^\times$ so that

$$c \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathbb{D}.$$

Rotating by a sufficient $\theta \in \mathbb{R}$, we have

$$z \mapsto ce^{i\theta} \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathcal{F}$$

\square

- Show there is f which maximizes $f'(z_0)$ in \mathcal{F} .

Proof. Let $g_n \in \mathcal{F}$ so that $\lim_{n \rightarrow \infty} g'_n(z_0) = \sup_{f \in \mathcal{F}} f'(z_0)$. Since $\|g_n\|_{\Omega} \leq 1$, $n \in \mathbb{N}$, we can pass to a subsequence so that $g_n \rightarrow g$ uniformly on compact subsets of Ω for some holomorphic $g : \Omega \rightarrow \mathbb{C}$ and $g'_n \rightarrow g'$ uniformly on compact sets in Ω . Hence $\lim_{n \rightarrow \infty} g'_n(z_0) = g'(z_0)$ and $\sup_{f \in \mathcal{F}} f'(z_0) = g'(z_0) < \infty$ and $g'(z_0) > 0$.

We still need to show g is injective. Let $z_1 \neq z_2$, $z_1, z_2 \in \Omega$, $g(z_1) = g(z_2)$. Then in $\Omega \setminus \{z_1\}$, $g_n(\cdot) - g_n(z_1) \neq 0$ for all points in $\Omega \setminus \{z_1\}$. By the Hurwitz theorem, $g(\cdot) - g(z_1)$ is either 0 or never vanishes. But $g(\cdot)$ is not a constant function since $g'(z_0) > 0$, so we have $g(\cdot) - g(z_1)$ never vanishes on $\Omega \setminus \{z_1\}$, so $g(z_2) \neq g(z_1)$, a contradiction.

Moreover, $\|g\|_{\Omega} \leq 1$ gives that $g(\Omega) \subset \overline{\mathbb{D}}$, but by the maximum principle, we have $g(\Omega) \subset \mathbb{D}$.

□

- If $f'(z_0)$ maximal, then f is an isomorphism.

Proof. It suffices to show that $g(\Omega) = \mathbb{D}$. Suppose there is $w_0 \in \mathbb{D} \setminus g(\Omega)$. We perform several modifications of g .

First, let $F(z) = \sqrt{\frac{g(z) - w_0}{1 - \overline{w_0}g(z)}}$. This is well-defined since Ω is simply connected. Note that $F(\Omega) \subset \mathbb{D}$ and F is injective with $0 \notin F(\Omega)$.

Second, we make z_0 go to 0. Define $G(z) = \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}$. Then, G is injective from $\Omega \rightarrow \mathbb{D}$ and $G(z_0) = 0$.

We now show that $G'(z_0) > g'(z_0)$, a contradiction. We will show that $g = k \circ G$, where $k : \mathbb{D} \rightarrow \mathbb{D}$, holomorphic. The inverse of G is a fractional linear transformation given by $\begin{pmatrix} 1 & F(z_0) \\ \overline{F(z_0)} & 1 \end{pmatrix}$.

From F to g , we take the $T_w \circ (z \mapsto z^2)$, where w is the corresponding matrix from the initial FLT. So we have $k = T_w \circ (z \mapsto z^2) \circ T_h$. Note that $k(\mathbb{D}) \subset \mathbb{D}$ and $k(0) = \frac{F(z_0)^2 + w_0}{1 + \overline{w_0}F(z_0)^2}$, so since we have $F(z_0)^2 = -w_0$, we get $k(0) = 0$.

Since $k \notin \text{Aut}(\mathbb{D})$, so we must have $|k'(0)| < 1$ by the Schwarz Lemma. It follows that

$$|G'(z_0)| > |k'(0)||G'(z_0)| = |(k \circ G)'(z_0)| = |g'(z_0)|,$$

a contradiction.

□

□