Math 202 Review, Fall 2019

Topology and Analysis

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Contents

1 Lecture I

1.1 Metrics and Norms

Definition 1.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$, such that

1. $d(x, x) = 0$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$,
4. $d(x, y) = 0 \Rightarrow x = y$ (Note that dropping this condition gives a **semi-metric**).

Definition 1.2. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** is a function $|| \ || : V \to \mathbb{R}^+$ such that

1. $||v|| = 0$ iff $v = 0$ (drop this condition for **semi-norms**),
2. $||\alpha v|| = |\alpha| ||v||$,
3. $||v|| + ||w|| \geq ||v + w||$

A norm induces a metric by $d(v, w) = ||v - w||$.

1.2 Convergence

1.3 Complete Metric Spaces

Definition 1.3. A metric space X is **complete** if every cuachy sequences converges to some point of X.

1.4 Completions of Metric Spaces

Definition 1.4. Let (X, d) be a metric space. A **completion** of X is a complete metric space $(\tilde{X}, \tilde{d})$ together with a function $j : X \to \tilde{X}$ such that

1. j is an isometry: $\tilde{d}(j(x), j(y)) = d(x, y)$,
2. $j(X)$ is dense in X.

Note that every metric space has a unique completion, up to isometry.

2 Lecture II

2.1 General Algorithm for Completing Metric Spaces(Using Cauchy Sequences)

Theorem 2.1. An algorithm for completing a metric space.

Proof. Let (X, d) be a metric space and let $\mathrm{CS}(X, d)$ be the set of Cauchy sequences in (X, d)

(Step 1: Define $\bar{d}$ for our completed space.) Let $\{x_n\}, \{y_n\} \in \mathrm{CS}(X, d)$. We begin by proving a lemma:

Lemma 1. $\{d(x_n, y_n)\}$ is a Cauchy sequence.

Proof. By the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) \Rightarrow d(x, y) - d(x, z) \leq d(y, z).$$

Note that we can switch y and z without changing the inequality, which gives

$$|d(x, y) - d(x, z)| \leq d(y, z).$$

Now, we have

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &= |\,(d(x_n, y_n) - d(x_n, y_m)) + (d(x_n, y_m) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \\ &\leq d(y_n, y_m) + d(x_n, x_m) \end{aligned}$$

Since $\{x_n\}, \{y_n\}$ are Cauchy, this can be made arbitrarily small, which completes the proof. $\square$

Define $\bar{d}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty}\{d(x_n, y_n)\}$. It's trivial to verify that this is a semi-metric.

(Step 2: Define an equivalence relation on $\mathrm{CS}(X, d)$) Define an equivalence relation on X by $x \sim y$ if $d(x, y) = 0$.

Define $\hat{d}$ on $X/\sim$ by $\hat{d}([x], [y]) = d(x, y)$. Note that if $x' \in [x], y' \in [y]$, then

$$d(x', y') \leq d(x', x) + d(x, y) + d(y, y') = d(x, y).$$

Taking the reverse inequality gives $d(x', y') = d(x, y)$. Now, let $\tilde{d}$ be the corresponding metric on $\mathrm{CS}(X, d)$. $\mathrm{CS}(X, d)/\sim$ is the set of equivalence classes of cauchy sequences.

(Step 3: Define the isometry) Embed (X, d) in $\mathrm{CS}(X, d)/\sim$ by $x \mapsto \{x_n\}$, converging to x. Define $\varphi(x) = \{x_n = x\}$. Note that

$$\tilde{d}(\varphi(x), \varphi(y)) = \lim\{d(x_n, y_n)\} = \bar{d}(x, y).$$

(Step 4: show that $\varphi(CS(X, d)/\sim)$ is dense in $CS(X, d)/\sim$) Let $\{x_n\}$ be any cauchy sequence. Given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for $n, m \geq N$, $d(x_m, x_n) < \epsilon$.

Consider $\varphi(x_N)$. We have $\tilde{d}(\{x_n\}, \varphi(x_n)) = \lim\{d(x_n, x_N)\}$.

(Step 5: Show that $(CS(X, d)/\sim, \hat{d})$ is complete) . For each m, let $S^{=}\{x_n^m\}_{n=0}^{\infty} \in CS(X, d)$. Assume the sequence $\{S^m\}$ is a cauchy sequence. For each k, we find $x_k \in X$ s.t. $\tilde{d}(\varphi(x_k), S_m) < \frac{1}{k}$(since cauchy). Then $S = \{x_k\}$ is a cauchy sequence, so $\tilde{d}(S^m, S) \to 0$, as desired. $\square$

2.2 General Notion of Continuity

Definition 2.1. Let $(X, d_x), (Y, d_y)$ be metric spaces. Let $f : X \to Y$, $x_0 \in X$. We say f is **continuous** at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$.

Theorem 2.2. A function f is continuous if and only if the preimage of an open set is open.

3 Lecture III

3.1 Topology

Definition 3.1. Let X be a set. By a **topology** for X, we mean a collection, $\mathscr{T}$ of subsets of X, such that:

1. Arbitrary unions of elements in $\mathscr{T}$ are in $\mathscr{T}$.
2. Finite intersections of elements of $\mathscr{T}$ are in $\mathscr{T}$.
3. X and $\varnothing$ are closed.

Definition 3.2. A closed set is the compliment of an open set.

Note 3.1. Note that:

1. Arbitrary intersections of closed sets are closed.
2. Finite unions of closed sets are closed.
3. X and $\varnothing$ are closed.

Definition 3.3. Let $A \subseteq X$, the **closure** of A is the smallest closed set that contains A; that is, the intersection of all closed sets that contain A.

Definition 3.4. The **interior** of A is the biggest open set contained in A, or equivalently, the union of all open sets contained in A.

Definition 3.5. Let C be a closed set, and let $A \subseteq C$, we say that A is **dense** in C if $\bar{A} = C$.

3.2 Basis, Sub-base

Definition 3.6. Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X, the smallest topology containing the intersection of topologies that contain $\mathcal{S}$ is said to be the topology generated by $\mathcal{S}$, and $\mathcal{S}$ is said to be a **subbase** for that topology.

Note 3.2. If $\mathcal{C}$ is a collection of topologies for X, then

$$\bigcap_{\mathscr{T} \in \mathscr{C}} \mathscr{T}$$

is a topology in X.

Definition 3.7. Let X be a set, and let D be a collection of subsets of X. D is a topology for X called the **discrete topology** for X. It is given by a metric:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

D is the biggest topology in X.

Definition 3.8. The smallest topology in X is $\{\varnothing, X\}$. This topology is called the indiscrete topology.

Note 3.3. If $\mathscr{T}_1 \subseteq \mathscr{T}_2$ are topologies on X, $\mathscr{T}_1$ is coarser, smaller, weaker, and $\mathscr{T}_2$ is larger, stronger, and finer. We generally require that $\bigcup \mathscr{S} = X$.

Definition 3.9. A collection of subsets of X is a base for a topology if the set of all arbitrary unions of elements of $\mathcal{S}$ is a topology.

Note 3.4. For $\mathcal{S}$ to be a base, it must have the property that if $A, B \in \mathcal{S}$, then $A \cap B$ is the union of elements of $\mathcal{S}$.

Note 3.5. If $\mathcal{S}$ is any collection of subsets of X, then the collection of all finite intersections must be a topology.

3.3 Properties of Pre-Image

Definition 3.10. A function between topological spaces is said to be continuous if the inverse image of every open set is also open.

Note 3.6. Let Y be a set and $\mathscr{S} = \{A_\alpha\}$, let X be a set, and $f : X \to Y$ be a function. Then,

1. $f^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f^{-1}(A_\alpha)$
2. $f^{-1}(\bigcap_\alpha A_\alpha) = \bigcap_\alpha f(A_\alpha)$
3. If $A, B \subseteq Y$, then $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$.

3.4 Constructing New Topologies

Example 3.1. Let X be a set and let $(X_\alpha, \mathscr{T}_\alpha)$ be a collection of topological spaces. Let there be a quasifunction $f_\alpha : X_\alpha \to X$. Let $\mathscr{T}$ be the strongest topology such that all of the f_α's are continuous. Given α_0, f_α. If $A \subseteq X$, then if A is to be open, we must have that $f_{\alpha_0}(A) \in \mathscr{T}_{\alpha_0}$. Now, let $\mathscr{S}_{\alpha_0} = \{A \subseteq : f_{\alpha_0}^{-1}(A) \in \mathscr{T}_{\alpha_0}\}$ is a topology for X; in fact, it is the strongest topology making f_{α_0} continuous. The strongest topology making all of the f_α continuous is the intersection of the $\mathscr{S}_\alpha$.

Example 3.2. Let (X, τ) be a topological space, let Y be a set. Then, $f : X \to Y$, $\{A \subseteq Y : f^{-1}(A) \in \tau_X\}$ is the strongest topology making f continuous. Usually, we want f to be onto Y.

4 Lecture IV

4.1 Product, Quotient, Relative Topologies

Definition 4.1. $\mathcal{S}$ is a **base** for a topology on X if the union of the sets in $\mathcal{S}$ is X.

Definition 4.2. Let $(X, \mathscr{T})$ be a topological space, Y a set, and $f : X \to Y$ surjective. The **quotient topology** on Y is the set $\mathscr{T}_y = \{A \subseteq Y : f^{-1}(A) \in \mathscr{T}\}$.

Example T. he conditional topology

Definition 4.3. The **relative topology** on a set X is the set $\{X \cap O : O \in \mathscr{T}_Y\}$ for $(Y, \mathscr{T}_Y)$, a topological space.

Definition 4.4. Let $(X_1, \mathscr{T}_1), (X_2, \mathscr{T}_2)$ be given. The set $\mathcal{S} = \{\mathscr{O} \times \mathscr{U} : \mathscr{O} \in \mathscr{T}_1, \mathscr{U} \in \mathscr{T}_2\}$ form a sub-base for the **product topology**.

For infinite product topology, we let all but finitely many of the terms be the whole set.

4.2 Separation Axioms

Note T. he Separation Axioms:

1. T_2(Hausdorff)
2. T_1: Given $x, y, x \neq y$, there exists $\mathscr{O}_x$ with $x \in \mathscr{O}_x$, $y \notin \mathscr{O}_x$ and there exists a similar $\mathscr{O}_y$.
3. T_0: Given $x, y, x \neq y$, there exists $\mathscr{O}$ such that only one of x or y is in $\mathscr{O}$.

4.3 Urysohn's Lemma

5 Lecture V

5.1 Homeomorphisms

5.2 Properties of Normal Topological Spaces

5.3 Boundedness

5.4 Categories

6 Lecture VI

6.1 Tietze Extension Theorem

Theorem (Tietze Extension Theorem). Let $(X, \mathscr{T})$ be a normal topological space, and let $A \to$ be continuous. Then there is $\tilde{f} : X \to \mathbb{R}$, continuous that extends f, if $\tilde{f}|_A = f$. If $f : A \to [a, b], a, b \in \mathbb{R}$ then can arrange that $\tilde{f} : X \to [a, b]$.

6.2 Compactness

Definition L. et X be a set, $\mathcal{C}$ a collection of subsets of X. We say that $\mathcal{C}$ is a covering of X if

$$\bigcup\{A \in \mathcal{C}\} = X$$

. If $B \subseteq X$, $\mathcal{C}$ is a collection of subsets of X, we say that $\mathcal{C}$ covers $\mathcal{B}$ if $\mathcal{B} \subseteq \bigcup\{A \in \mathcal{C}\}$. If $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ is a subcover of $\mathcal{C}$ if $\mathcal{D}$ also is a c.

Let $(X, \mathscr{T})$ be a topological space. We say that it is compact if every open cover of X has a finite subcover.

Theorem I. f $(X, \mathscr{T})$ is compact and $A \subseteq X$, then the following are equivalent.

1. A is compact for the relative topology
2. If $\mathcal{C} \subseteq \mathscr{T}$ is a cover of A, then A has a finite subcover of $\mathcal{O}$.

Theorem I. f $(X, \mathscr{T})$ is compact and $A \subseteq X$ is closed then A is compact for the relative topology.

6.3 Compact + Hausdorff

Theorem L. et $(X, \mathscr{T})$ be Hausdorff. Let $A \subseteq X$ be compact for the relative topology, then A is closed.

Theorem L. et $(X, \mathscr{T})$ be compact and Hausdorff. For any closed subset A of X and any pf (?) $y \in X$, $y \notin A$, there are open sets u, v, disjoint, with $A \subseteq u$, $y \in V$.

Definition . (X, T)$isregularforall$A $\subseteq X$ closed and all $y \in X$, $y \notin A$.

Theorem E. very compact Hausdorff space is normal.

7 Lecture VII

7.1 Compact + Continuity

7.2 Axiom of Choice/Zorn's Lemma

7.3 Tychonoff's Theorem

8 Lecture VIII

8.1 Completing the Proof of Tychonoff's Theorem

8.2 Tychonoff's Theorem implies Axiom of Choice

8.3 Compactness for Metric Spaces

Definition A. subset A of a metric space (X, d) is said to be totally bounded if for any $\epsilon > 0$, it call be covered by a finite number of ϵ-balls.

Theorem A. ny subset of a compact subset of a metric space is totally bounded.

Theorem I. f A is totally bounded subset of a metric space, then $\bar{A}$ is totally bounded.

Theorem A. metric that is not complete can be compact.

Theorem I. f X is complete, if $A \subset X$ is totally bounded, then $\bar{A}$ is compact.

Theorem L. et (X, d) be a complete metric space. Then, if (X, d) is totally bounded then it is compact.

Corollary L. et (X, d) be a complete metric space, let $A \subseteq X$, with A totally bounded. Then $\bar{A}$ is compact.

Corollary . [a,b] $\subseteq \mathbb{R}$, the first is compact. Any closed bounded subset of $\mathbb{R}^n$ is compact.

8.4 Arzela - Ascoli

Theorem (. Core of the Arzeli-Ascoli Theorem) Let $(X, \mathscr{T})$ be compact. Let $F \subseteq C(X, M)$. If F is equicontinuous and pointwise totally bounded, then F is totally bounded for d_∞.

Theorem (. Arzela-Ascoli): Let $(X, \mathscr{T})$ be a complete metric space. Then, $F \subseteq C(X, M)$ is compact in d_∞ if it is closed and equicontinuous and pointwise totally bounded.

9 Lecture IX

9.1 Locally Compact Hausdorff(LCH)

Definition L. ocally compact spaces. A topological space $(X, \mathscr{T})$ is locally compact if for each $x \in X$, there is a $\mathcal{O} \in \mathscr{T}, x \in \mathcal{O}, \bar{\mathcal{O}}$ is compact.

Lemma L. et $C \subseteq X$ be compact. Then there is open $\mathcal{O}$ with $C \subseteq \mathcal{O}$, $\mathcal{O}$ compact.

Theorem L. et $(X, \mathscr{T})$ be a LCH. Let $C = X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is open $\mathcal{U}$, $C \subseteq \mathcal{U}$, $\mathcal{U}$ compact, $\mathcal{U} \subseteq \mathcal{O}$.

Theorem L. et $(X, \mathscr{T})$ be LCH. Let $C \subseteq X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is a continuous $f : X \to [0, 1]$ with $f(x) = 1$, for $x \in C$ and $f(x) = 0$ for $x \notin \mathcal{O}$.

Definition F. or $(X, \mathscr{T})$ LCH, let $C_c(X)$ be the set of continuous $-$valued functions on X "of compact support", i.e. there is a compact set outside of which $f \equiv 0$. $C_c(X)$ is an algebra for pointwise operations. $e, f, g \in C_c(X)$, then $f + g$, fg, $rf(r \in \mathbb{R}) \in C_c(X)$.

9.2 Measure Theory

9.3 Rings, Sigma-Rings, Algebras, Sigma-Algebras

10 Lecture X

10.1 Pre-rings

10.2 Half-open Measure

10.3 Premeasures

11 Lecture XI

11.1 Completion of the Half-open Measure

11.2 Countable Subadditivity

11.3 Outer Measures