Vector Bundles

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April 13, 2021

These notes correspond to Chapter 10 of Lee, *Smooth Manifolds*. Roughly, we define vector bundles, bundle homomorphisms, subbundles, and fiber bundles, with several examples.

Contents

§1 Vector Bundles

Definition 1.1. Let M be a topological space. A **real vector bundle of rank k over M** is a topological space E together with a surjective continuous map $\pi : E \to M$ satisfying the following conditions:

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

(ii) For each $p \in M$, there exist a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$(called a **local trivialization** of E over U), satisfying the following conditions:

  - $\pi_U \circ \Phi = \pi$, where π_U is the standard projection.
  - for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M, E are smooth manifolds, π is a smooth map, and the local trivializations are diffeomorphisms, then E is called a **smooth vector bundle**. We call any local trivialization that is a diffeormophism onto its image a **smooth local trivialization**. We refer to E as the **total space**, M as the **base space** and π as the **projection**.

Definition 1.2. A rank-1 bundle is called a **line bundle**.

> **Remark 1.3.** We could similarly define complex vector bundles, replacing $\mathbb{R}^k$ with $\mathbb{C}^k$.

Definition 1.4. If there exists a local trivialization of E over all of M, then E is said to be a **trivial bundle**. In this case $E \cong M \times \mathbb{R}^k$.

Definition 1.5. If $E \to M$ is a smooth bundle that admits a smooth global trivialization, then we say that E is **smoothly trivial.**

Example 1.6 (Product Bundles)

Taking $E = M \times \mathbb{R}^k$ with $\pi = \pi_1 : M \times \mathbb{R}^k \to M$ as the projection is called a **product bundle**. Note that the product bundle is trivial and if M is a smooth manifold then $M \times \mathbb{R}^k$ is smoothly trivial.

Example 1.7 (Mobius Bundle)

We start with an equivalent relation on $\mathbb{R}^2$ by $(x, y) \sim (x', y')$ whenever $(x', y') = (x + n, (-1)^n y)$ for $n \in \mathbb{Z}$. Take $E = \mathbb{R}^2/\sim$ and $q : \mathbb{R}^2 \to E$ to be the quotient map. Then, if we take $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ given by $\varepsilon(x) = e^{2\pi i x}$, then we admit a continuous map $\pi : E \to \mathbb{S}^1$ with $\pi \circ q = \varepsilon \circ \pi_1$. E along with π is called the **Mobius bundle**.

Proposition 1.8

Let M be a smooth n-manifold with or without boundary, and let TM be its tangent bundle. With its standard projection map, its natural vector space structure on each fiber, and the standard topology and smooth structure, TM is a smooth vector bundle of rank n over M.

Lemma 1.9

Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \emptyset$. There exists a smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v),$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

Definition 1.10. $\tau : U \cap V \to GL(k, \mathbb{R})$ in the above lemma is called the **transition function** between the local trivializations.

Lemma 1.11

Let M be a smooth manifold with or without boundary, and suppose that for each $p \in M$ we are given a real vector space E_p of some fixed dimension k. Let $E = \bigsqcup_{p \in M} E_p$ and let $\pi : E \to M$ be the map that takes each element of E_p to the point p. Suppose furthermore that we are given the following:

- an open cover $\{U_\alpha\}_{\alpha \in A}$ of M
- for each $\alpha \in A$, a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from $E_p \to \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$.
- for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

Then E has a unique topology and smooth structure making it into a smooth manifold with or without boundary and a smooth rank-k vector bundle over M, with π as projection and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.

§2 Local and Global Sections of Vector Bundles

Definition 2.1. Let $\pi : E \to M$ be a vector bundle. A **section of** E is a section of the map π, that is, a continuous map $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_M$.

Definition 2.2. A **local section of** E is a continuous map $\sigma : U \to E$ defined on some open subset $U \subset M$ and satisfying $\pi \circ \sigma = \mathrm{id}_U$. A **global section** is a local section defined on all of M. Sections defined over smooth manifolds are called **smooth sections**.

Definition 2.3. The **zero section of** E is the global section $\zeta : M \to E$ defined by $\zeta(p) = 0 \in E_p$ for each $p \in M$.

Definition 2.4. The **support** of a section σ is the closure of the set of points where it is nonzero.

Suppose M is a smooth manifold with or without boundary.

- Sections of TM are vector fields on M.
- Given an immersed submanifold $S \subset M$ with or without boundary, a section of the ambient tangent bundle $TM|_s \to S$ is called a **vector field along** S. Note that it is a continuous map $X : S \to TM$ such that $X_p \in T_pM$ for each $p \in S$. This is not quite a vector field on S, which satisfies $X_p \in T_pS$ for each $p \in S$.
- If $E = M \times \mathbb{R}^k$ is a product bundle, there is a natural one-to-one correspondence between sections of E and continuous functions from $M \to \mathbb{R}^k$: a continuous function $F : M \to \mathbb{R}^k$ determines a section $\widetilde{F} : M \to M \times \mathbb{R}^k$ by $\widetilde{F}(x) = (x, F(x))$, and vice versa.

Definition 2.5. If $E \to M$ is a smooth vector bundle, we denote by $\Gamma(E)$ the vector space of smooth global sections under pointwise addition and scalar multiplication. If $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, we obtain $f\sigma \in \Gamma(E)$ defined by $(f\sigma)(p) = f(p)\sigma(p)$.

Lemma 2.6

Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary. Suppose A is a closed subset of M, and $\sigma : A \to E$ is a section of $E|_A$ that is smooth in the sense that σ extends to a smooth local section of E in a neighborhood of each point. For each open $U \subset M$ containing A, there exists a global smooth section $\widetilde{\sigma} \in \Gamma(E)$ such that $\widetilde{\sigma}|_A = \sigma$ and $\operatorname{supp} \widetilde{\sigma} \subset U$.

§3 Local and Global Frames

Definition 3.1. If $U \subset M$ is an open subset, a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be **linearly independent** if the pointwise values form a linear independent k-tuple in E_p for each $p \in U$.

Definition 3.2. A **local frame** for E over U is an ordered k-tuple of linearly independent local sections over E that span E. We similarly have a **global frame** if $E = M$. If $E \to M$ is a smooth vector bundle, the frame is called a **smooth frame**.

Proposition 3.3

Every smooth local frame for a smooth vector bundle is associated with a smooth local triviailization.

Corollary 3.4

A smooth vector bundle is smoothly trivial if and only if it admits a smooth global frame.

Corollary 3.5

Let $\pi : E \to M$ be a smooth vector bundle of rank k, let (V, φ) be a smooth chart on M with coordinate functions (x^i), and suppose there exists a smooth local frame (σ_i) for E over V. Define $\widetilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by

$$\widetilde{\varphi}(v^i \sigma_i(p)) = (x^1(p), \dots, x^n(p), v^1, \dots, v^k).$$

Then $(\pi^{-1}(V), \widetilde{\varphi})$ is a smooth coordinate chart for E.

Proposition 3.6

Proposition 3.7

§4 Bundle Homomorphism

§5 Subbundles

§6 Fiber Bundles