Math 212, Lecture Notes

Several Complex Variables

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§1 Lecture 1: 4/26/2021

§1.1 Review of 1D Complex Analysis

Definition 1.1 (Holomorphic)**.** Let $D \subset \mathbb{C}$ be an open connected domain and take $u \in C^1(D)$. The function u is **holomorphic** if $\partial_{\overline{z}} u = 0$ where $\partial_{\overline{z}} = (\partial_x + i\partial_y)$.

We also have the equivalent conditions that

$$u \in \mathrm{Hol}(D) \Leftrightarrow \partial_{\overline{z}} = 0 \Leftrightarrow \lim_{h \to 0} \frac{u(z+h) - u(z)}{h} \text{ exists and is continuous.}$$

Fact 1.2 (Green's Theorem)**.** For $\Omega \subset \mathbb{C}, \partial\Omega \in C^1$, we have

$$\int_{\partial\Omega} u \, dz = \iint_{\Omega} \partial_{\overline{z}} u \, d\overline{z} \wedge dz.$$

Theorem 1.3 (Cauchy-Pompieu Formula)
Let $u \in C^1(\overline{\Omega})$. For all $\zeta \in \Omega$,

$$u(\zeta) = \frac{1}{2\pi i} \left(\int_{\partial\Omega} \frac{u(z)}{z - \zeta} \, dz + \iint_{\Omega} \frac{\partial_{\overline{z}} u(z)}{z - \zeta} \, dz \wedge d\overline{z} \right)$$

Proof. Let $\Omega_\varepsilon = \Omega \setminus \overline{D(\zeta, \epsilon)}$, where $0 < \epsilon << 1$. Applying Green's Theorem to $w(z) = \frac{u(z)}{z-\zeta} \in C^1(\overline{\Omega_\varepsilon})$ and noting that $\partial_{\overline{z}} w = \frac{\partial_{\overline{z}} u(z)}{z-\zeta}$, we have

$$\iint_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} u(z)}{z - \zeta} d\overline{z} \wedge dz = \int_{\partial\Omega} \frac{u(z)}{z - \zeta} \, dz - \int_{\partial D(\zeta, \epsilon)} \frac{u(z)}{z - \zeta} \, dz.$$

The left-hand side converges to $\iint_\Omega \frac{\partial_{\overline{z}} u(z)}{z-\zeta} d\overline{z} \wedge dz$ by the dominated convergence theorem. Parameterizing the disc via polar coordinates, we can write

$$\int_{\partial D(\zeta, \epsilon)} \frac{u(z)}{z - \zeta} \, dz = \int_0^{2\pi} u(\zeta + \epsilon e^{i\theta}) \, d\theta \to 2\pi i u(\zeta).$$

The desired formula follows from rearranging the terms upon taking the limit as $\epsilon \to 0$. □

Remark 1.4. We also have a partial converse: let $\varphi \in C_c^k(\mathbb{C})$ with $k \geq 1$ and $u(z) = \iint \frac{\varphi(z)}{z-\zeta} dz \wedge d\overline{z}$. Then $u \in C^k(\mathbb{C})$ and $\partial_{\overline{z}} u = \varphi$.

Some other notable corollaries that follow from Cauchy's Theorem:

- $u \in \mathrm{Hol}(D) \Rightarrow u \in C^\infty(D)$.
- For all $K \Subset \Omega \Subset D$, k, there exists C such that for all $u \in \mathrm{Hol}(D)$, we have

$$\sup_K |u^{(j)}(z)| \leq C \|u\|_{L^1(\Omega)}.$$

- $u_j \in \mathrm{Hol}(D)$, $u_j \to u$ uniformly on bounded sets, then $u \in \mathrm{Hol}(D)$.