Math 214: Differentiable Manifolds

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Contents

§1 January 19th, 2021

§1.1 Topology Review

Definition 1.1 (Topological Space)**.** $(X, O_X \subset \mathcal{P}(X))$, where $A \in O_x$ are the open sets which satisfy the following:

1. $\emptyset, X \in O_X$.
2. $A, B \in O_X$ implies $A \cap B \in O_X$
3. $A_i \in O_X$, $i \in I$, then $\bigcup_{i\in I} A_i \in O_X$.

We say that $A \subset X$ is closed if $X \setminus A$ is open. $U \subset X$ is a neighborhood of $p \in X$ if $\exists A$ such that $p \in A \subset U$.

Example 1.2

Take a metric space (X, d). The topology is generated as follows: $A \subset X$ is open if $\forall p \in A$, $\exists r > 0$ such that $B_r(p) \subset A$.

Definition 1.3. $\mathcal{B} \subset \mathcal{P}(X)$ is called a **basis** for the topology on X if for every subset $A \subset X$, A is open if and only if A is a union of elements of $\mathcal{B}$.

Example 1.4

For a Euclidean space, $\mathcal{B} = \{B_r(x) \subset \mathbb{R}^n : r \in \mathbb{Q}, r > 0, x \in \mathbb{Q}^n\}$ is a basis for the topology. Note that this basis is countable, so $\mathbb{R}^n$ is 2nd countable.

Let (X, O_X), (Y, O_Y) be topological spaces.

Definition 1.5. A function $\varphi : X \to Y$ is continuous if for any open subset $B \subset Y$, $\varphi^{-1}(B) \subset X$ is open.

Definition 1.6. $\varphi : X \to Y$ is a homeomorphism if it is a continuous bijection whose inverse is continuous.

Definition 1.7. Let $Y \subset X$ a topological space. We set $O_Y = \{A \cap Y : A \in O_X\}$.

Example 1.8

The subspace topology is the coarsest topology so that the inclusion map $Y \to X$ is continuous(also called the initial topology).

Example 1.9

$\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ has the same topology as $\mathbb{R}$. In other words, it is clear that $\mathbb{R} \approx \mathbb{R} \times \{0\}$, where the approximate sign indicates a homeomorphism.

Theorem 1

(Topological Invariance of Dimension) If we take $\mathbb{R}^m, \mathbb{R}^n$ with open subsets $U \subset \mathbb{R}^m$ and $V \subset R^n$. If we have $\varphi : U \to V$ a homeomorphism, then we must have $m = n$.

The proof is beyond the scope of the class, but uses homology groups.

Definition 1.10. Given a topological space X, X is called locally Euclidean (of dimension n) at $p \in X$ if there is an open neighborhood about $p \in U \subset X$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

Lemma 1.11

The n is uniquely determined by p.

Proof. Assume that X was locally Euclidean at p of dimensions n_1, n_2. There are neighborhoods $p \in U_i \subset X$ and homeomorphisms $\varphi_i : U_i \to \widehat{U}_i \subset \mathbb{R}^{n_i}$. Consider the image of $U_1 \cap U_2$ under both homeomorphisms. If we take $\varphi_2 \circ \varphi_1^{-1} : \varphi(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$, a homeomorphism, so it follows that $n_1 = n_2$ by Topological Invariance of Dimension. $\square$

Definition 1.12. A space X is **Hausdorff** if for any $p, q \in X$, $p \neq q$ there exists open subsets U, V with $p \in U$, $q \in V$ so that $U \cap V = \emptyset$.

Exercise 1.13. For any $p, q \in X$, if there is a separating continuous function $f : X \to \mathbb{R}$ such that $f(p) \neq f(q)$, then X is Hausdorff.

Definition 1.14. $K \subset X$ is compact if every open cover of K has a finite subcover.

Some useful facts, a subspace of a Hausdorff space is Hausdorff, Hausdorff + Compact implies Closed, $\varphi : X \to Y$ continuous, K is compact, then $\varphi(K)$ is compact. We can use these to show that for $\varphi : X \to Y$ with X compact, Y Hausdorff with φ continuous, bijective, then φ is a homeomorphism.

§1.2 Smooth Manifolds

Definition 1.15. A topological space M is called an n-dimensional **topological manifold** if M satisfies the following:

- M is locally Euclidean at any point,
- M is Hausdorff,
- M is second countable.

Example 1.16 (Manifold - Hausdorff)

Suppose we drop the Hausdorff condition. Take $X = (\mathbb{R} \times \{0, 1\})\backslash \sim$, where $(x, 0) \sim (x, 1)$ if $x < 0$. Consider the quotient map $\pi : \mathbb{R} \times \{0, 1\} \to X$. Call $A \subset X$ open iff $\pi^{-1}(A)$ is open. Each branch of the line are open subsets, each homeomorphic to $\mathbb{R}$.

Example 1.17 (Manifold - Second Countable)

Take an uncountable subset S equipped with the discrete topology. Set $X = S \times \mathbb{R}$.

A more interesting example called the "long line" is as follows:

Lemma 1.18

There is an uncountable, well-ordered set S such that S has a maximal element $\Omega \in S$ and for all $\alpha \in S$, $\alpha \neq \Omega$, the set $\{x \in S | x < \alpha\}$ is countable.

Now, set $X = (-\infty, 0) \cup S \times [0, 1)$ under the lexicographic ordering. This turns out to be Hausdorff and locally Euclidean but not second countable.

Exercise 1.19. If M is 0-dimensional topological manifold , then M is a finite or countable set equipped with the discrete topology.

Exercise 1.20. If M^n is a top. manifold and $M' \subset M^n$ is open, then M' is an n-dimensional top. manifold.