Linear Algebra

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July 14, 2021

Abstract

A collection of problems and solutions from topics in linear algebra in the olympiad setting. I include some expositions when possible. Any typos or mistakes are my own - kindly direct them to my inbox.

Contents

1 Problems

Problem 1.1. Let $A \in M_n(\mathbb{R})$ be skew-symmetric. Show that $\det(A) \geq 0$.

Proof. If n is odd, note that

$$\det(A) = \det(A^\intercal) = \det(-A) = (-1)^n \det(A) = -\det(A).$$

It follows that $\det(A) = 0$.

Otherwise, suppose n is even and let $p(\lambda) = \det(A - I_n\lambda)$. If $\lambda \neq 0$ is an eigenvalue, note that $p(\lambda) = 0$ by the Cayley-Hamilton Theorem. Moreover,

$$p(-\lambda) = \det(A + I_n\lambda) = \det(A^\intercal + I_n^\intercal\lambda) = \det(-A + I_n\lambda) = 0.$$

Moreover, let v be an eigenvector with corresponding eigenvalue λ. Note that

$$\langle Av, v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2,$$

$$\langle Av, v\rangle = \langle v, A^\intercal v\rangle = \langle v, -Av\rangle = -\bar{\lambda}\langle v, v\rangle = -\bar{\lambda}\|v\|^2.$$

It follows that $\lambda = -\bar{\lambda}$, which implies that $\lambda = ri$ for $r \in \mathbb{R}$. Hence,

$$\det(A) = \prod_{j=1}^{n/2} (i\lambda_j)(-i\lambda_j) = \prod_{j=1}^{n} \lambda_j^2 \geq 0.$$

□

Problem 1.2. Let $A \in M_n(\mathbb{R})$ with $A^3 = A + I_n$. Show that $\det(A) > 0$.

Proof. Let $p(x) = x^3 - x - 1$. Note that $p(0) = -1$, $p(2) = 5$, so the polynomial has a root in the interval $(0, 2)$ by the intermediate value theorem. Furthermore, $p'(x) = 3x^2 - 1$ so the polynomial has critical points at $\pm\frac{1}{\sqrt{3}}$. It is easy to see that at both of these values, $p(x) < 0$ so it follows that the other roots of $p(x)$ are conjugate complex numbers. Let the roots be $\lambda_1, \lambda_2, \lambda_3$ with λ_1 being the positive real root and λ_2, λ_3 the conjugate complex ones. If A satisfies $A^3 = A + I_n$, then we must have the eigenvalues of A are λ_1, λ_2 and λ_3, with multiplicity $\alpha_1, \alpha_2, \alpha_3$ respectively. Since λ_2, λ_3 are complex conjugates, we must have $\alpha_2 = \alpha_3$, so it follows that

$$\det(A) = \lambda_1^{\alpha_1}(\lambda_2\lambda_3)^{\alpha_2} = \lambda_1^{\alpha_1}|\lambda_2|^{\alpha_2} > 0.$$

□

Problem 1.3. If $A, B \in M_n(\mathbb{R})$ such that $AB = BA$, then $\det(A^2 + B^2) \geq 0$.

Proof.

$$\det(A^2 + B^2) = \det(A + iB)\det(A - iB) = \det(A + iB)\overline{\det(A + iB)} = |\det(A + iB)|^2 \geq 0.$$

□

Problem 1.4. Let $A, B \in M_2(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + B^2) = 0$. Show that $\det(A) = \det(B)$.

Proof. Let $p_{A,B}(\lambda) = \det(A + \lambda B) = \det(B)\lambda^2 + (\operatorname{tr} A + \operatorname{tr} B - \operatorname{tr}(AB))\lambda + \det(A)$. By Problem 1.3, we have $\det(A + iB)$ and $\det(A - iB) = 0$, which implies that $p_{A,B}(\lambda) = c(\lambda - i)(\lambda + i) = c(\lambda^2 + 1)$. It follows that $c = \det B = \det A$. $\square$

Problem 1.5. Let $A \in M_2(\mathbb{R})$ with $\det A = -1$. Show that $\det(A^2 + I_2) \geq 4$. When does equality hold?

Proof. First, note the identity

$$\det(X + Y) + \det(X - Y) = 2(\det X + \det Y).$$

This follows from writing $p(z) = \det(X + zY) = \det(Y)z^2 + (\operatorname{tr} X + \operatorname{tr} Y - \operatorname{tr}(XY))z + \det(X)$ and taking

$$p(1) + p(-1) = \det(X + Y) + \det(X - Y) = 2\det Y + 2\det X.$$

Then, taking $X = A^2 + I$ and $Y = 2A$, we have

$$0 \leq \det(A + I)^2 + \det(A - I)^2 = 2(\det(A^2 + I) + \det(2A)) = 2(\det(A^2 + I) - 4).$$

It follows that $\det(A^2 + I) \geq 4$ as desired. We have equality when the eigenvalues of A are 1 and -1. $\square$

Problem 1.6. Let $A, B \in M_3(\mathbb{C})$ with $\det(A) = \det(B) = 1$. Show that $\det(A + \sqrt{2}B) \neq 0$.

Proof. $\square$