§1 Game Theory Analysis

Formulation: We restrict the problem to the case of classes of linear maps. Let $\mathbf{X} \in \mathbb{R}^D$ be data. For a map $T_L^b : \mathbb{R}^{d\times D} \times \mathbb{R}^{D\times d} \to \mathbb{R}$ defined by $T_L^b(\mathbf{F}, \mathbf{G}) = \Delta R(\mathbf{Z}(\mathbf{F}), \widehat{\mathbf{Z}}(\mathbf{F}, \mathbf{G}))$ where $\mathbf{Z}(\mathbf{F}) = \mathbf{FX}$ and $\widehat{\mathbf{Z}}(\mathbf{F}, \mathbf{G}) = \mathbf{FGFX}$.

Recall

$$\Delta R(\mathbf{Z}, \widehat{\mathbf{Z}}) = R\left(\begin{bmatrix}\mathbf{Z} & \widehat{\mathbf{Z}}\end{bmatrix}\right) - \frac{1}{2}R(\mathbf{Z}) - \frac{1}{2}R(\widehat{\mathbf{Z}})$$

where

$$R(\mathbf{Z}) = \frac{1}{2}\log\det\left(\mathbf{I} + \alpha\mathbf{Z}\mathbf{Z}^*\right).$$

Definition 1.1 (Local Nash Equilibrium for T_L^b)**.** The point $(\mathbf{F}_*, \mathbf{G}_*) \in \mathbb{R}^{d\times D} \times \mathbb{R}^{D\times d}$ is a local Nash Equilibrium for T_L^bif there exists $\delta > 0$ such that for all $(\mathbf{F}, \mathbf{G}) \in \mathbb{R}^{d\times D} \times \mathbb{R}^{D\times d}$ satisfying $\|\mathbf{F} - \mathbf{F}_*\|_F \le \delta$, $\|\mathbf{G} - \mathbf{G}_*\|_F \le \delta$,

$$T_L^b(\mathbf{F}_*, \mathbf{G}) \le T_L^b(\mathbf{F}_*, \mathbf{G}_*) \le T_L^b(\mathbf{F}, \mathbf{G}_*).$$

Proposition 1.2 (First Order Necessary Condition)
If $(\mathbf{F}_*, \mathbf{G}_*)$ is a local Nash Equilibrium, we must have that $\nabla_{\mathbf{F}} T_L^b(\mathbf{F}_*, \mathbf{G}_*) = 0$ and $\nabla_{\mathbf{G}} T_L^b(\mathbf{F}_*, \mathbf{G}_*) = 0$.

Fact 1.3. $\nabla_{\mathbf{Z}} R(\mathbf{Z}) = \alpha(I + \alpha\mathbf{Z}\mathbf{Z}^*)^{-1}\mathbf{Z}$.

Fact 1.4. $\nabla_{\mathbf{F}} \Delta R(\mathbf{FX}, \mathbf{FGFX}) =?$

Fact 1.5. $\nabla_{\mathbf{G}} \Delta R(\mathbf{FX}, \mathbf{FGFX}) =?$

Theorem 1.6 (Characterization of Linear Local Nash Equilibria)
Suppose that $(\mathbf{F}_*, \mathbf{G}_*)$ is a local Nash Equilibrium for T_L^b satisfying $T_L^b(\mathbf{F}_*, \mathbf{G}_*) = 0$ or equivalently, $\mathbf{Z}\mathbf{Z}^* = \widehat{\mathbf{Z}}\widehat{\mathbf{Z}}^*$.