Ch 1 : Cauchy's Equation

1.1 Additive Cauchy

Def. Let $D \subset \mathbb{R}$ be closed under $+$. $f: D \to \mathbb{R}$ is additive if

$$f(x+y) = f(x) + f(y) \ \forall x, y \in D.$$

Prop. 1.1. Let D be a set of reals s.t. $0 \in D$, $x+y \in D$, $\frac{x}{n} \in D$ $\forall x, y \in D$ $n \in \mathbb{N}$. If $f: D \to \mathbb{R}$ is additive, then

$$f\left(\sum_{k=1}^{n} r_k x_k\right) = \sum_{k=1}^{n} r_k f(x_k)$$

for $x_k \in D$, $r_k \in \mathbb{Q}$.

Proof. $\cancel{f(0) = 2f(0) \Rightarrow}$ $y=0$: $f(x) = f(x) + f(0) \Rightarrow f(0) = 0$

$y = -x$ $\quad 0 = f(0) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$.

By induction on n, we can show

$$f\left(\sum_{k=1}^{n} x_k\right) = \sum_{k=1}^{n} f(x_k).$$

In particular, $f(nx) = nf(x)$ $\forall x \in D$, $n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}^*$, $r = m/n$, $x \in D$. $mx \in D$, $\frac{mx}{n} \in D \Rightarrow rx \in D$

$\Rightarrow nf(rx) = f(nrx) = f(mx) = mf(x) \Rightarrow rf(x) = f(rx)$.

This holds for general $r \in \mathbb{Q}$ since $f(-y) = -f(y)$, $f(0) = 0$.

$$\cancel{\Rightarrow f\left(\sum_{k=1}^{n} r_k x_k\right) = \sum_{k=1}^{n} f(r_k f(x_k)) = \sum_{k=1}^{n} r_k f(x_k)}.$$

$$\Rightarrow f\left(\sum_{k=1}^{n} r_k x_k\right) = \sum_{k=1}^{n} f(r_k x_k) = \sum_{k=1}^{n} r_k f(x_k). \quad \blacksquare$$

Ex. 1. If $f: \mathbb{Q} \to \mathbb{R}$ additive, $f(x) = ax$ where $a = f(1) \in \mathbb{R}$.

Ex 2. If $f: \mathbb{Q}[\sqrt{2}] \to \mathbb{R}$ additive, $f(x) = ax + b\bar{x}$, $x = p + q\sqrt{2}$ $\bar{x} = p - q\sqrt{2}$.

Theorem. Let $f: \mathbb{R} \to \mathbb{R}$ be additive. TFAE:

(i) f has the form $f(x) = ax$ for $a \in \mathbb{R}$

(ii) f is bounded above

(iii) f is increasing on an interval

(iv) f is continuous at a point.

Cor 1.1 Let $f:\mathbb{R}\to\mathbb{R}$ be additive and multiplicative. Then either $f\equiv 0$ or $f(x)\equiv x$.

Proof. $f(x^2) = f(x)^2 \geq 0$, so f is additive and bounded below on $(0,\infty)$.

$\Rightarrow f(x) = ax$ for $a\in\mathbb{R}$.

Then $axy = f(xy) = f(x)f(y) = a^2xy \quad \forall x,y \Rightarrow a=0$ or $a=1$. ∎

Theorem 1.2. Let $f:\mathbb{R}\to\mathbb{R}$ be additive and not $f(x)=ax$. Then any square in the plane contains a point from graph(f).

Theorem 1.3 (Hamel) $\exists H\subset\mathbb{R}$ s.t. (1) $1\in H$, (2) if $h_1,h_2,\dots,h_n\in H$, $r_1,r_2,\dots,r_n\in\mathbb{Q}$ w/ $\sum h_i r_i = 0$, then $r_1=r_2=\dots=r_n=0$, (3) $\forall x\in\mathbb{R}$, $\exists r_1,r_2,\dots,r_n$ $h_1,h_2,\dots,h_n\in H$ w/ $x = r_1h_1+\dots+r_nh_n$, i.e. $\mathbb{R}$ has a basis as a $\mathbb{Q}$-vector space.

Theorem 1.4. Let H be a Hamel basis and $s:H\to\mathbb{R}$ arbitrary. Then $\exists!\ f:\mathbb{R}\to\mathbb{R}$ additive s.t. $f|_H = s$.

Proof. Let $H=\{h_\alpha\}$. Let $x\in\mathbb{R}$. $x = \sum_{i=1}^n r_{\alpha_i}h_{\alpha_i}$, $r_{\alpha_i}\in\mathbb{Q}$.

Set $f(x) = \sum r_{\alpha_i}s(h_{\alpha_i})$.

1.2 Log Cauchy

Def (logarithmic Cauchy) $\quad f(xy) = f(x)+f(y) \quad \forall x,y\in(0,\infty)$.

Theorem 1.6. A solution $f:(0,\infty)\to\mathbb{R}$ of log Cauchy is of the form $f(x) = g(\log x)$, where $g:\mathbb{R}\to\mathbb{R}$ is additive.

Proof. $\forall x\in(0,\infty)$ $\exists!\ u\in\mathbb{R}$ w/ $x=e^u$. Set $g(u)=f(e^u)$.

Then $g(u+v) = f(e^{u+v}) = f(e^u)+f(e^v) = g(u)+g(v)$. ∎

Take log Cauchy on $D\to\mathbb{R}$: If $D=[0,\infty)$, $f\equiv 0$ on D since

$f(0) = f(x)+f(0) \Rightarrow f(x)=0$.

Theorem. Any solution $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{0\}$ of log Cauchy is of the form $f(x) = g(\log|x|)$ where $g:\mathbb{R}\to\mathbb{R}$ is additive.

Proof. $x=y=1$: $f(1)=0$. $x=y=-1 \Rightarrow f(-1)=0$.

$f(x) = f(|x|\cdot\operatorname{sgn} x) = f(|x|)+f(\operatorname{sgn} x) = f(|x|) = g(\log|x|)$

by Thm 1.6.

1.3 Exponential Cauchy

Def. $f(x+y) = f(x)\, f(y)$ $\quad D \to \mathbb{R}$

Theorem 1.9. Let $f: D \to \mathbb{R}$ be exp Cauchy, $D = \mathbb{R}$ or $(0,\infty)$. Then $f \equiv 0$ or $f(x) = e^{g(x)}$ for ~~g~~ $g: D \to \mathbb{R}$ additive.

Proof. if $f(x_0) = 0$ for some $x_0 \in D$:

if $D = \mathbb{R}$, $f(y) f(x_0) = f(y + x_0) = 0 \Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}$

if $D = (0,\infty)$, $f(x) = 0 \quad \forall x \geq x_0$.

$\exists k$ w/ $kx > x_0$. OTOH, $f(nx) = f(x)^n \Rightarrow f(x)^k = f(kx) = 0 \Rightarrow f \equiv 0$.

if $f(x) \neq 0 \quad \forall x \in D$:

$f(x) = \left(f\left(\frac{x}{2}\right)\right)^2 > 0 \quad \forall x \in \mathbb{R}$. Set $f(x) = e^{g(x)}$. Then

$$e^{g(x) + g(y)} = e^{g(x)} e^{g(y)} = f(x) f(y) = f(x+y) = e^{g(x+y)}.$$

$\Rightarrow g(x) + g(y) = g(x+y)$. ∎

1.4 Multiplicative Cauchy

Let D be closed under $(\cdot)$.

Multiplicative: $f(xy) = f(x) f(y)$.

Theorem 1.11. If $f: (0,\infty) \to \mathbb{R}$ multiplicative, then $f \equiv 0$ or $f = e^{g(\log x)}$, $g: \mathbb{R} \to \mathbb{R}$ additive.

Proof. Let $f(x_0) = 0$ for $x_0 \in (0,\infty)$. Then

$$f(x) = f\left(\frac{x}{x_0}\right) f(x_0) = 0 \quad \forall x \in (0,\infty).$$

If $f(x) \neq 0 \quad \forall x \in (0,\infty)$, set $f(x) = e^{g(\log x)}$.

$$e^{g(\log xy)} = f(xy) = f(x)\, f(y) = e^{g(\log x)} e^{g(\log y)} = e^{g(\log x) + g(\log y)}.$$

Theorem 1.12. if $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$ multiplicative,

$f \equiv 0$, $f \equiv 1$ or $f(x) = \begin{cases} 0 & x = 0 \\ e^{g(\log x)} & x > 0 \end{cases}$

Theorem 1.13. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ multiplicative

$f \equiv 0$, $f(x) = e^{g(\log |x|)}$, $f(x) = \operatorname{sgn} x \cdot e^{g(\log |x|)}$.