- Topology
    - Definitions:
        - Hausdorff
        - Normal
        - Final topology
        - Initial topology
        - Relative Topology
        - Product Topology
        - Base
    - Subbase
    - Quotient Topology
    - Regular
    - Finite Intersection Property
    - Totally Bounded
    - Equicontinuous
    - Pointwise totally bounded
    - Locally Compact Hausdorff
    - Nets
    - Convergence(Nets)
    - Theorems/Propositions:
        - Main Theorems:
            - Urysohn's Lemma
            - Tietze Extension Theorem
            - Tychonoff's Theorem
            - Pointwise totally bounded + Equicontinuous implies Totally bounded(Core of Arzela Ascoli)
        - Compactness
            - Compact + Closed implies Compact for Relative Topology
            - Compact + Hausdorff implies Closed
            - Compact + Hausdorff implies Normal
            - Continuous + Compact implies Compact Image
            - Subset of Compact subset of metric space is totally bounded
            - Compact implies Complete
            - Complete + Totally Bounded implies Compact
        - LCH Theorems: Let $(X, \mathcal{T})$ be LCH.

        - If $C \subseteq X$ is compact, there exists $\mathcal{O}$ with $C \subseteq \mathcal{O}$ and $\bar{\mathcal{O}}$ compact.
        - If $C \subseteq X$ is compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$, then there exists U open with $C \subseteq U$, $\bar{U}$ compact
        - If $C \subseteq X$ compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$, then there exists $f : X \to [0, 1]$ continuous with $f|_C = 1, f|_{O^c} = 0$.
    - Extras:
        - Any metric space is normal
- Measure Theory/Integration
    - Definitions:
        - Finite Additivity
        - Ring
        - Algebra
        - σ-ring
        - Countably Additive
        - Pre-ring/Semi-ring
        - Left-continuous
        - Pre-measure
        - Monotone
        - Countably Subadditive
        - Outer Measure
            - Given a family of subsets of $X, \mathcal{F}$, $\mu^*(A) = \inf\{\sum \mu(F_j) : F_j \in F, A \subseteq \bigcup F_j\}$.
        - Hereditary σ-ring
        - Measurable(Carathedory) Sets
        - Complete Measure
            - A measure ν is complete if given any nullset A in $\mathscr{M}(\nu)$ for any $B \subseteq A$ with $\nu(B) = 0$, $B \in \mathscr{M}(\nu)$.
        - Lebesgue-Stietjes Measure
        - σ-finite
            - A measure μ on a $\sigma-$ring S is σ-finite if, for $E \in \mathcal{S}$, there exists $\{F_j\}_1^\infty$ with $\mu(F_j) < \infty$ and $E \subseteq \bigcup_{j=1}^\infty F_j$.
        - Measure Space
        - Simple measurable B-valued function(SMF)
        - Simple $\mu-$integrable function(SIF)
        - S- measurable
        - Almost Uniform Convergence/Cauchy
        - Convergence/Cauchy in Measure

- Separable Range
  - Contains a countable dense set
  - There exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space so that every non-empty open-subset of the space contains at least 1 element of the sequence.
- $\|\cdot\|_1 = \int_X \|f\|_B d\mu$.
- Convergence/Cauchy in Mean
- μ-integrable
- $\mathcal{L}^1(X, \mathcal{S}, \mu, B)$ is the set of $\mu - integrable$ functions.
- $f \in \mathcal{L}^1$, $\int f d\mu = \lim \int f_n d\mu$.
- Carrier, C_f
- Indefinite Integral, $\mu_f(E) = \int_E \chi_E f(x) d\mu$.
- Locally S-measurable
  - $A \subseteq X$ is locally S measurable if $A \cap E \in S$ whenever $E \in S$.
  - X is locally S measurable if $X \setminus E$ is S-measurable.
- Theorems:
  - Measure Theory:
    - $\mu_\alpha([a, b)) = \alpha(b) - \alpha(a)$ on $\mathcal{P} = \{[a, b) : a < b \in \mathbb{R}\}$ is countably additive. [Done]
    - If $\mu : \mathcal{P} \to \mathbb{R}^+$ is finitely additive, $E \supseteq \bigoplus_{k=1}^n F_j$, for $E, F_j \in \mathcal{P}$, then $\mu(E) \geq \sum \mu(F_j)$. [Done]
    - If $\mathcal{P}$ is a semiring and $\mu : \mathcal{P} \to \mathbb{R}^+$ is countably additive, then μ is countably subadditive. [Done]
    - μ^* is monotone, countably sub additive [done]
    - If $\mathcal{P}$ is a semiring, μ a premeasure on $\mathcal{P}$ is countably additive, μ^* is the corresponding outer measure on $\mathcal{H}(\mathcal{P})$. For $E \in \mathcal{P}$, $\mu^*(E) = \mu(E)$. [done]
    - $\mathcal{M}(\nu)$ is a σ-ring, and $\nu|_{\mathcal{M}}$ is countably additive. [done]
    - Let $(\mathcal{P}, \mu)$ be a premeasure. Let $\mathcal{H}(\mathcal{P})$ have outer measure μ^*. Then $\mathcal{P} \subseteq \mathcal{M}(\mu^*)$. [done]
    - If $(\mathcal{P}, \mu)$ is σ-finite, for any σ-ring $\mathcal{S}$ with $\mathcal{S}(\mathcal{P}) \subseteq \mathcal{S} \subseteq \mathcal{M}(\mu^*)$ and any extension μ' of μ, then $\mu'(F) = \mu^*(F)$ for any $F \in \mathcal{S}$. [done]
    - Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $\{E_j\} \subseteq \mathcal{S}$ be increasing. If $E = \bigcup^{\infty} E_j$, then $\mu(E) = \lim \mu(E_j)$. [done]
    - Construction of a non-measurable set[done]
  - Integration:
    - If f, g are SMF, then $f + g$ is SMF. [done]
    - If f, g are SIF, then $f + g$ is SIF. [done]
    - Properties of $\mathscr{M}(X, \mathcal{S}, B)$(S - measurable functions):
      - Closure under addition and scalar multiplication
      - If $f \in \mathscr{M}(X, \mathcal{S}, B)$ and if $h \in \mathscr{M}(X, \mathcal{S}, \mathbb{R}/\mathbb{C})$, then $hf \in \mathscr{M}(X, \mathcal{S}, B)$.

- If $f \in \mathscr{M}(X, \mathcal{S}, B)$, then the range of f is separable.
- If $\{f_n\}$ are a sequence of functions with separable range, then $f_n \to f$ has separable range.

- Let $\{f_n\}$ be a sequence of B-valued functions on X with the property that for any open set $U \subseteq B$, $f^{-1}(U \setminus \{0\}) \in \mathcal{S}$. Then $f_n \to f$ has this property.
  - Corollary 1. If $f \in \mathscr{M}(X, \mathcal{S}, B)$, then for any open $U \subseteq B$, $f^{-1}(U \setminus \{0\}) \in \mathcal{S}$.
  - Corollary 2. If $f : X \to B$ is the pointwise limit of $\{f_n\} \subseteq \mathscr{M}(X, \mathcal{S}, B)$, then $f \in \mathcal{S}$.
- Let $(X, \mathcal{S}, B)$ be given. If $f : X \to B$ has separable range and $f^{-1}(U \setminus \{0\}) \in \mathcal{S}$ for all open $U \subseteq B$, then $f \in \mathscr{M}(X, \mathcal{S}, B)$.
  - Corollary 1. $\mathscr{M}(X, \mathcal{S}, B)$ is "closed" under taking pointwise limits of sequences.
- Egoroff's Theorem: Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of B-valued functions. If $\{f_n\}$ converges almost everywhere on $E \in \mathcal{S}$, where $\mu(E) < \infty$, then for every $\epsilon > 0$, there exists $F \subseteq E$ with $\mu(E \setminus F) < \epsilon$ so that $\{f_n\} \to f$ uniformly on F.
- If $f_n \to f$ almost uniformly on E, then $\{f_n\} \to f$ almost everywhere.
- (B-complete)If $\{f_n\}$ is almost uniformly cauchy on $E \in \mathcal{S}$, then there is a function f defined almost everywhere on E such that on $F \subseteq E$ with $\mu(E \setminus F) = 0$ such that $\{f_n\}$ converges almost uniformly on F.
- If $\{f_n\}$ is a sequence of ISF and is cauchy for $\| \cdot \|_1$, then it is Cauchy in measure.
- The Riesz-Weyl Theorem: Let $\{f_n\} \in \mathscr{M}(\mathcal{S}, X, \mu, B)$ that is cauchy in measure. Then there exists a subsequence that is almost uniformly cauchy.
  - Rapidly Cauchy Sequence
  - $E_j = \{x : \|f_{n_{j+1}} - f_{n_j}\| > 1/2^j\}$.
  - Take $E = \bigcup C_{f_n}$, $F = E \setminus \bigcup_{j=j_0}^{\infty} E_j$.
- If $\{f_n\} \in M \to f$ almost uniformly on E, then $\{f_n\} \to f$ in measure.
- If $\{f_n\} \to f$ in measure and $\{f_n\} \to g$ in measure, then $f = g$ almost everywhere.
- If $\{f_n\}$ is cauchy in measure and a subsequence $\{f_{n_j}\} \to f$ in measure, then $\{f_n\} \to f$ in measure.
- If f_n, g_n are mean cauchy sequences of ISF with $\|f_n - g_n\|_1 \to 0$, and $f_n \to f$ in measure, then $\{g_n\} \to f$ in measure.
- If $\{f_n\}, \{g_n\}$ are mean cauchy sequences of ISF and they both converge to f in meausre, then $\{f_n\}, \{g_n\}$ are equivalent.
- If $\{f_n\}$ is a MCS of ISFs then $\{\int f_n d\mu\}$ is a mean cauchy seuqnece in B.
- $\mathcal{L}^1$ is a vector space.
- $\mathcal{L}^1$ is complete(so, is a Banach space).

- Let $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.
  - C_f is σ-finite.
  - If $f \in \mathcal{L}^1(\dots, \mathbb{R})$, $E \in \mathcal{S}$ with $X_E \leq f$, then $\mu(E) < \infty$.
- If $\{f_n\} \in L^1$ cauchy in mean, then cauchy in measure.
- μ_f is a B-valued measure(finite measure).
- (Strong Absolute Continuity) If $f \in \mathcal{L}^1$, then for any $\epsilon > 0$, there exists $\delta > 0$ with the property that if $\mu(E) < \delta$, $\|\mu_f(E)\| < \epsilon$.
- Lebesgue Dominated Convergence Theorem: Let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ such that $\{f_n\} \to f$ almost everywhere. If there is a $g \in \mathcal{L}^1(X, S, \mu, \mathbb{R})$ with $\|f_n(x)\| \leq g(x)$ for all x, for all n, then $\{f_n\}$ is mean-cauchy.
- Monotone Convergence Theorem: Let $\{f_n\} \in \mathcal{L}^1(X, S, \mu, \mathbb{R})$ that is monotonically increasing. If there is c so that $\int f_n < c$, for all n, then $\{f_n\}$ is mean cauchy.
- If $f \in \mathscr{M}(X, \mathcal{S}, \mu, \mathbb{R})$, $f \geq 0$. If f is not integrable, set $\int f d\mu = \infty$. If there is $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ such that $\|f(x)\|_B \leq g(x)$ almost everywhere, then $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.
  - Force the lebesgue

- Extra:
  - Examples/Counterexamples
    - A mean-cauchy sequence that doesn't converge pointwise for any point on a given measure space.

- L^p spaces
  - Definitions:
    - L^p space
    - $\|\cdot\|_p$
  - Theorems:
    - L^p is a vector space.
    - $\|\cdot\|_p$ is a norm for $1 \leq p < \infty$.
    - Young's Inequality
    - Holder's Inequality
    - Minkowski's Inequality
    - Fatou's Lemma
    - L^p is complete.
    - For a ring R that generates $\mathcal{S}$, $ISF(X, R, \mu, B)$ is dense in $L^p(X, \mathcal{S}, \mu, B)$ for $1 \leq p < \infty$.
    - $C_c(\mathbb{R})$(compact support) is dense in $L^p(X, \mathcal{S}, \mu, B)$ for $1 \leq p < \infty$.
- Product Measures
  - Definitions:
    - $\mathcal{S} \otimes \mathcal{T}$

    - $\mu \otimes \nu$
- Theorems:
    - Tonelli's Theorem
    - Fubini's Theorem