Midterm Exam

I hereby swear that the work done on this assignment is my own and I have not given nor received aid that is inappropriate for this assignment.

Problem I

Suppose that X and Y are finite CW-complexes with Euler characteristics $\chi(X)$ and $\chi(Y)$. Show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Proof. Note that $\chi(X) = \sum_{i=0}^{n}(-1)^i c_i^X$ and $\chi(Y) = \sum_{j=0}^{m}(-1)^j c_j^Y$, where c_k^X and c_k^Y denote the number of k-cells for X and Y respectively.

Furthermore, each k-cell of $X \times Y$ is given from the product of a ℓ-cell from X and an $k-\ell$-cell from Y(where we can set $c_\ell^X, c_\ell^Y = 0$ if it has no ℓ-cells). Hence, it follows that

$$c_k^{X\times Y} = \sum_{\ell=0}^{k} c_\ell^X c_{k-\ell}^Y.$$

Thus,

$$\begin{aligned}
\chi(X) \cdot \chi(Y) &= \left(\sum_{i=0}^{n}(-1)^i c_i^X\right)\left(\sum_{j=0}^{m}(-1)^j c_j^Y\right) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{i+j} c_i^X c_j^Y \\
&= \sum_{i+j=0}^{n+m}(-1)^{i+j}\sum_{\ell=0}^{i+j} c_\ell^X c_{i+j-\ell}^Y \\
&= \sum_{k=0}^{n+m}(-1)^k \sum_{\ell=0}^{k} c_\ell^X c_{k-\ell}^Y \\
&= \sum_{k=0}^{n+m}(-1)^k c_k^{X\times Y} \\
&= \chi(X \times Y).
\end{aligned}$$

$\square$

Problem II

Suppose that X is a finite CW-complex and $n > 1$. Show that $H_i(X \times \mathbb{S}^n; \mathbb{F}) = H_i(X; \mathbb{F}) \oplus H_{i-n}(X; \mathbb{F})$.

Proof. First, we claim that $H_i(X \times \mathbb{S}^n) \cong H_i(X) \oplus H_i(X \times \mathbb{S}^n, X \times \{pt\})$. Define $r : X \times \mathbb{S}^n \to X \times \{pt\}$ by $(x, a) \mapsto (x, pt)$. Note that this is a retraction since $r \circ i(x, pt) = r(x, pt) = (x, pt)$, where $i : X \times \{pt\} \hookrightarrow X \times \mathbb{S}^n$ is the inclusion map. Furthermore, note that

$$H(r) \circ H(i) = H(r \circ i) = H(\mathrm{id}_{X \times \mathbb{S}^n}) = id_{H(X \times \mathbb{S}^n)},$$

which implies that $H(i)$ is injective. It follows that the exact sequence

$$0 \to X \times \{pt\} \hookrightarrow X \times \mathbb{S}^n \to X \times \mathbb{S}^n / X \times \{pt\} \to 0$$

induces the short exact sequence

$$0 \to H_i(X \times \{pt\}) \to H_i(X \times \mathbb{S}^n) \to H_i(X \times \mathbb{S}^n, X \times \{pt\}) \to 0,$$

which implies that $H_i(X \times \mathbb{S}^n) \cong H_i(X \times \{pt\}) \oplus H_i(X \times \mathbb{S}^n, X \times \{pt\})$.

Next, we show that $H_i(X \times \mathbb{S}^n, X \times \{pt\}) \cong H_{i-1}(X \times \mathbb{S}^{n-1}, X \times \{pt\})$. Decompose $\mathbb{S}^n = \tilde{A} \cup \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are the upper and lower hemispheres respectively. We replace $\tilde{A}$ and $\tilde{B}$ with A and B where the hemisphere is slightly thickened at the equator by a factor $\epsilon > 0$. Note that $\mathbb{S}^n = A \cup B = \operatorname{int} A \cup \operatorname{int} B$. Note that A, B are homeomorphic to $\mathbb{D}^n$ and $A \cap B$ is homeomorphic to $\mathbb{S}^{n-1} \times (-\epsilon, \epsilon)$, which is homotopy equivalent to $\mathbb{S}^{n-1}$. From Mayer-Vietoris, we have the sequence

$$\cdots \to H_i(X \times \mathbb{D}^n, X \times \{pt\}) \oplus H_i(X \times \mathbb{D}^n, X \times pt) \to H_i(X \times \mathbb{S}^n, X \times \{pt\})$$

$$\xrightarrow{\delta_{i-1}} H_{i-1}(X \times \mathbb{S}^{n-1}, X \times \{pt\}) \to H_{i-1}(X \times \mathbb{D}^n, X \times \{pt\}) \oplus H_{i-1}(X \times \mathbb{D}^n, X \times pt) \to \dots$$

Since $\mathbb{D}^n$ is homotopy equivalent to a point, it follows from exactness that $H_i(X \times \mathbb{S}^{n-1}, X \times \{pt\}) \cong H_{i-1}(X \times \mathbb{S}^{n-1}, X \times \{pt\})$. By iterating this n times, we obtain $H_i(X \times \mathbb{S}^n, X \times \{pt\}) \cong H_{i-n}(X \times \mathbb{S}^0, X \times \{pt\}) \cong H_{i-n}(X)$, since S^0 consists of two points and $H_{i-n}(X \times \{pt\}) \cong H_{i-n}(X)$. $\square$

Problem III

Let X be the topological space we get by identifying opposite points on the equator of $\mathbb{S}^2$. What is $H_*(X;\mathbb{F})$?

Proof. We give a CW-decomposition of X consisting of a point attached to S^1, and attaching the northern and southern hemispheres to S^1. Then $C_0(X) = \mathbb{F}$ since it is generated by a point, $C_1(X) = \mathbb{F}$ since it is generated by the equator, and $C_2(X) = \mathbb{F}^2$ since it is generated by the two hemispheres. Note that the gluing maps for the hemispheres are of degree 2 and -2 respectively since under the quotient, when going around the boundary of each hemisphere we wind twice around S^1, and the two maps go in opposite directions.

This gives the sequence:

$$0 \to \mathbb{F}^2 \xrightarrow{d_2} \mathbb{F} \xrightarrow{d_1} \mathbb{F} \to 0.$$

Note that $d_1 = 0$ since when we go around S^1, we meet the point from both sides. If we denote e_1^2, e_2^2 to be the gluing maps for the hemispheres and e as the gluing map for S^1, from the Cellular Boundary Formula, we have

$$d_2(e_1^2) = 2e^1, d_2(e_2^2) = -2e^1.$$

It follows that $\operatorname{im} d_2$ is generated by $2e^1$, which is isomorphic to $2\mathbb{F}$.

$$0 = d_2(ae_1^2 + be_2^2) = 2ae^1 - 2be^1,$$

which happens when $a = b$. Thus, $\ker d_2$ is generated by $e_1^2 + e_2^2$, which is isomorphic to $\mathbb{F}$. Thus, $H_2(X) = \mathbb{F}$, $H_1(X) = \mathbb{F}/2\mathbb{F}$, and $H_0(X) = \mathbb{F}$, $H_i(X) = 0$ for $i > 2$. Therefore, $H_*(X) = \mathbb{F}_{(2)} \oplus \mathbb{F}/2\mathbb{F} \oplus F_{(0)}$.

□

Problem IV

Let X be the topological space we get from the full triangle Δ^2 by identifying its three vertices. Compute $H_*(X;\mathbb{F})$.

Proof. We give two arguments. First note that Δ^2 is homeomorphic to the closed disc $\mathbb{D}^2$, which is homotopic to a point. Since homology is preserved under homotopy equivalence, it follows that $H_*(X;\mathbb{F}) = \mathbb{F}_{(0)}$.

We can also compute this explicitly. We take a triangle with vertices x, y, z, edges $u = [xy]$, $v = [yz]$, $w = [zx]$, and face $T = [xyz]$. Note that $C_0(X) = \mathbb{F}^3$ since it is generated by x, y, z, $C_1(X) = \mathbb{F}^3$ since it is generated by u, v, w and $C_2(X) = \mathbb{F}$ since it is generated by T. This gives the chain complex:

$$0 \to \mathbb{F} \xrightarrow{\partial_2} \mathbb{F}^3 \xrightarrow{\partial_1} \mathbb{F}^3 \to 0.$$

Note that $\partial_1 u = y - x$, $\partial_1 v = z - y$, $\partial_1 w = x - z$. Furthermore, $\partial_2 T = v + w + u$. Note that

$$0 = \partial_1(au + bv + cw) = a(y - x) + b(z - y) + c(x - z) = x(-a + c) + y(a - b) + z(b - c),$$

which happens whenever $a = b = c$. This implies that $\ker \partial_1$ is generated by $u + v + w$, which is isomorphic to $\mathbb{F}$. Furthermore, note that the image of ∂_2 is generated by $u+v+w$, so $H_1(X) = 0$.

Then, the image of ∂_1 is generated by $x - y, y - z, z - x$ and the kernel of ∂_0 is generated by x, y, z so it follows that $H_0(X) = \mathbb{F}$. For $i > 1$, it is clear that $H_i(X;\mathbb{F}) = 0$, so it follows that $H_*(X;\mathbb{F}) = \mathbb{F}_{(0)}$, as desired. $\square$

Problem V

Show that chain homotopy of chain maps is an equivalence relation.

Proof. Suppose $f, g, h : C \to D$ are chain maps.

- Reflexive: Note that $f - f = 0$, so if we take the zero map $0 : C \to D$, then $0 = \partial_D \circ 0 - 0 \circ \partial_C = f - f$.
- Symmetric: Suppose f is chain homotopic to g. There exists a homomorphism $\varphi : C \to D$ so that $f - g = \partial_D \circ \varphi - \varphi \circ \partial_C$. Then, note that $g - f = \partial_D \circ (-\varphi) - (-\varphi) \circ \partial_C$, so it follows that g is chain homotopic to f.
- Transitive: Suppose that f is chain homotopic to g and g is chain homotopic to h. There exist homomorphisms $\varphi, \psi : C \to D$ such that $f - g = \partial_D \circ \varphi - \varphi \circ \partial_C$ and $g - h = \partial_D \circ \psi - \psi \circ \partial_C$. Then, note that

$$\begin{aligned} f - h &= (f - g) + (g - h) \\ &= \partial_D \circ \varphi - \varphi \circ \partial_C + \partial_D \circ \psi - \psi \circ \partial_C \\ &= \partial_D \circ (\varphi + \psi) - (\varphi + \psi) \circ \partial_C. \end{aligned}$$

□

Problem VI

Suppose that X is a finite CW-complex and $A, B \subset X$ are subcomplexes with the property that $X = A \cup B$. Show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Proof. It suffices to show that $c_n^{A \cup B} = c_n^A + c_n^B - c_n^{A \cap B}$. This is precisely the principle of inclusion-exclusion: for finite sets C, D, $|C \cup D| = |C| + |D| - |C \cap D|$. A short proof of this is as follows. In order to count the elements of $C \cup D$, we count the number of elements in C once and the number of elements in D once. However, the elements in $C \cap D$ are counted twice, so we subtract this from our count so that every element is counted exactly once. The result follows from setting $C = X_n^A$ and $D = X_n^B$, the respective n-skeletons. $\square$