

Math 214 Study Guide

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§1 Smooth Manifolds

§1.1 Definitions

- Topological Manifold
- Locally Euclidean
- Coordinate Charts
- Projective Spaces
- Smooth Atlas
- Smooth Structure
- Smooth Manifold
- Manifolds with Boundary

§2 Smooth Maps

§2.1 Definitions

- Smooth function
- Coordinate representation
- Smooth Map
- Diffeomorphism
- Partitions of Unity

§2.2 Theorems

- 2-3: If $f : M \rightarrow \mathbb{R}^k$ is smooth, $f \circ \varphi^{-1}$ is smooth for every smooth chart (U, φ)
- 2-5: Equivalent Characterizations of Smoothness
- 2-8: Gluing Lemma for Smooth Maps
- 2-25: Existence of Smooth Bump Functions
- 2-26: Extension Lemma for Smooth Functions
- 2-28: Existence of Smooth Exhaustion Functions
- 2-29: Level Sets of Smooth Functions

§3 Tangent Vectors

§3.1 Definitions

- Geometric Tangent Space
- Derivations
- Tangent Vectors
- Differentials
- Computations in Coordinates
- Differential in Coordinates
- Pushforward
- Change of Coordinates
- Tangent Bundle
- Velocity Vectors of Curves

§3.2 Theorems

- 3-4: Properties of Tangent Vectors
- 3-6: Properties of Differentials
- 3-23: Correspondence of Velocity Vectors and Tangent Vectors

§4 Submersions, Immersions, and Embeddings

§4.1 Definitions

- Rank of Smooth Map
- Smooth Submersion
- Smooth Immersion
- Smooth Embedding
- Local Section
- Smooth Covering Map

§4.2 Theorems

- 4-5: Inverse Function Theorem
- 4-6: Properties of Local Diffeomorphisms
- 4-12: Rank Theorem
- 4-14: Global Rank Theorem
- 4-26: Local Section Theorem
- 4-28: Smooth submersion is an open map/quotient map

§5 Submanifolds

§5.1 Definitions

- Embedded Submanifold
- Properly Embedded
- k -dimensional Slice Chart
- Regular/Critical Points
- Regular/Critical Value
- Regular Level Set
- Defining Map
- Local Defining Function
- Immersed Submanifold

§5.2 Theorems

- 5-8: Local Slice Criterion
- 5-14: Regular Level Set Theorem
- 5-37: Tangent Space to Submanifold
- 5-38: Tangent Space of Embedded Submanifold is the Kernel of Local Defining Map

§6 Sard's Theorem

§6.1 Definitions

- Measure Zero Sets
- Normal Bundle
- Tubular Neighborhood
- Smooth Homotopy
- Transversality

§6.2 Theorems

- 6-10: Sard's Theorem
- 6-15: Whitney Embedding Theorem
- 6-18: Whitney Immersion Theorem
- 6-19: Strong Whitney Embedding Theorem
- 6-20: Strong Whitney Immersion Theorem
- 6-21: Whitney Approximation Theorem for Functions

- 6-24: Tubular Neighborhood Theorem
- 6-26: Whitney Approximation Theorem
- 6-35: Parametric Transversality Theorem
- 6-36: Transversality Homotopy Theorem

§7 Lie Groups

§7.1 Definitions

- Lie Group
- Left/Right Translation
- Lie Group Homomorphism
- Lie Subgroup
- Lie Group Actions
- Equivariant Maps
- Orbit Map

§7.2 Theorems

- 7-7: Existence/Uniqueness of Universal Covering Group
- 7-21: Lie subgroup closed if and only if embedded
- 7-25: Equivariant Rank Theorem: Equivariant implies Constant Rank
- 7-26: Properties of Orbit Map

§8 Vector Fields

§8.1 Definitions

- Vector Fields
- Rough Vector Field
- Support of Vector Field
- Local and Global Frames
- Parallelizable
- F -related vector fields
- Pushforward of Vector Fields
- Lie Bracket
- Lie Algebra
- Left-invariant
- Lie Algebra homomorphism

§8.2 Theorems

8-1: Smoothness Criterion for Vector Fields

8-7: Given $p \in M, v \in T_p M$, exists $X \in \mathfrak{X}(M)$ with $X_p = v$.

8-13: Gram-Schmidt for Frames

8-14: Alternate Smoothness Criterion (Vector Field as Derivations of $C^\infty(M)$)

8-27: Lie Bracket Computation

8-31: Pushforward of Lie Bracket

§9 Integral Curves and Flows

§9.1 Definitions

- Integral Curve
- Flows
- Singular/Regular Points
- Lie Derivative
- Commuting Vector Fields
- Commuting Flows

§9.2 Theorems

9-2: Existence/Uniqueness

9-3: Rescaling Lemma

9-4: Translation Lemma

9-12: Fundamental Theorem on Flows

9-20: Flowout Theorem

9-38/39: Properties of Lie Derivative

9-42: Equivalent Conditions for Commuting Vector Fields

9-44: Vector fields commute if and only if flows commute

§10 Vector Bundles

§10.1 Definitions

- Vector Bundles
- Local Trivialization
- Sections of Vector Bundles

- Zero Section
- Local and Global Frames
- Bundle Homomorphisms
- Subbundles

§10.2 Theorems

10-4: Tangent Bundle as a Vector Bundle

10-15: Completion of Local Frames

§11 Cotangent Bundle

§11.1 Definitions

- Dual Basis
- Cotangent Space
- Covariant/Contravariant Vectors
- Cotangent Bundle
- Covector Field
- Component Functions of Covector Field
- Local and Global Coframes
- Differential
- Pullback of Covector Field
- Line Integral
- Exact Covector Field
- Conservative Covector Field
- Closed Covector Field

§11.2 Theorems

11-11: Smoothness Criteria for Covector Fields

11-18: Differential is a smooth covector field

11-20: Properties of Differential

11-23: Derivative of a Function Along a Curve

11-42: Smooth covector field is conservative if and only if it is exact

11-44: Exact implies closed

11-46: Pullback takes closed to closed, exact to exact

11-49: Poincaré Lemma

§12 Tensors

§12.1 Definitions

- Multilinear Map
- Tensor Products
- Covariant Tensors
- Contravariant Tensors
- Mixed Tensors
- Symmetric Tensor
- Symmetrization
- Symmetric Product
- Alternating Tensor
- Tensor Bundles
- Component Functions of Tensor Bundle
- Pullback of Tensor Fields
- Lie Derivative of Tensor Field

§12.2 Theorems

- 12-4: Basis for Multilinear Functions
- 12-9: Associativity of Tensor Product Spaces
- 12-25: Properties of Tensor Pullbacks
- 12-26: Coordinate Representation of Tensor Pullback
- 12-32: Properties of Lie Derivative on Tensor Fields
- 12-34: Differential Commutes with Lie Derivative

§13 Riemannian Metrics

§13.1 Definitions

- Riemannian Metric
- Euclidean Metric
- Product Metric
- Pullback Metric
- Local Isometry
- Flat Metric

- Riemannian Submanifold
- Surface of Revolution
- Riemannian Length
- Riemannian Distance
- Musical Isomorphisms
- Gradient of Vector Field

§13.2 Theorems

13-3: Existence of Riemannian Metrics

13-9: Pullback Metric Criterion

13-19: Flatness Criterion for Surfaces of Revolution

13-29: Riemannian Manifolds as Metric Spaces

§14 Differential Forms

§14.1 Definitions

- Alternating Covariant Tensors
- Elementary Covectors
- Wedge Product
- Interior Multiplication
- Differential Forms
- Exterior Derivative
- Lie Derivatives of Differential Forms

§14.2 Theorems

14-8: Basis for $\Lambda^k(V^*)$

14-11: Properties of Wedge Product

14-20: Pullback Formula for Top-Degree Forms

14-23: Properties of Exterior Derivative

14-35: Cartan's Magic Formula

14-36: Lie Derivative Commutes with d .

§15 Orientations

§15.1 Definitions

- Oriented Vector Space
- Positively/Negatively Oriented
- Pointwise Orientation
- Positively/Negatively Oriented Frame
- Continuous pointwise orientation
- Orientable Manifold
- Orientation Form
- Oriented Coordinate Chart
- Orientation-preserving Action
- Generalized covering maps
- Orientation Covering/Double Cover

§15.2 Theorems

- 15-5: Orientation Determined by an n -Form
- 15-11: Orientations of Codimension-0 Submanifolds
- 15-15: Pullback Orientation
- 15-24: Induced Orientation on a Boundary
- 15-35: If $\pi : E \rightarrow M$ is a smooth covering map, M is orientable, then E is orientable
- 15-40: Properties of Orientation Covering
- 15-41: Orientation Covering Theorem

§16 Integration on Manifolds

§16.1 Definitions

- Integral of Differential Form
- Integration on Manifolds

§16.2 Theorems

- 16-1: Transformation Theorem
- 16-6: Properties of Integrals of Forms
- 16-8: Integration over Parameterizations
- 16-11: Stokes' Theorem

§17 De Rham Cohomology

§17.1 Definitions

- Cycles, Boundaries, de Rham Cohomology Group
- Homotopy Invariants
- Homotopy Operator/Cochain Homotopy
- Degree of Smooth Map

§17.2 Theorems

17-2: Induced Cohomology Maps

17-13: Cohomology of Contractible Manifolds

17-14: Poincaré Lemma

17-15: Local Exactness of Closed Forms

17-20: Mayer-Vietoris Sequence

17-35: Degree of Smooth Map