$$\frac{2^k z^{2^k}}{1+z^{2^k}} = \frac{(1+2+2^2+\dots+2^{k-1})z^{2^k}}{1+z^{2^k}} + \frac{z^{2^k}}{1+z^{2^k}}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{2^k z^{2^k}}{1+z^{2^k}} = \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k-1} \frac{2^\ell z^{2^k}}{1+z^{2^k}}\right) + \frac{z^{2^k}}{1+z^{2^k}}$$

Chapter 2 : Cauchy's Theorem

2.1 Goursat's Theorem

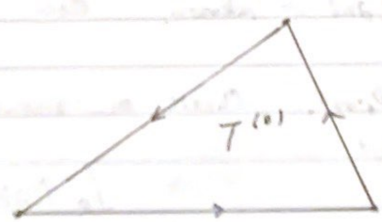
Theorem (Goursat): If Ω is open in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then

$$\int_T f(z)\,dz = 0,$$

whenever f is holomorphic in Ω.

Proof. We call $T^{(0)}$ the original triangle, $d^{(0)}$ and $p^{(0)}$ denoting the diameter and perimeter of $T^{(0)}$ respectively. WLOG we let $T^{(0)}$ has positive orientation. Bisect each side of $T^{(0)}$ and connect midpoints, creating $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}$ and $T_4^{(1)}$ similar to the original triangle.

$$\int_{T^{(0)}} f(z)\,dz = \sum_{j=1}^{4} \int_{T_j^{(1)}} f(z)\,dz$$



For some j, we have

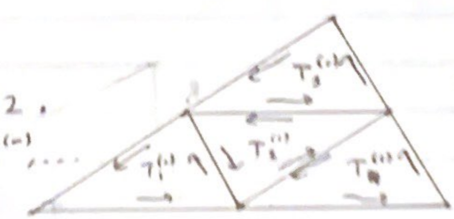
$$\left|\int_{T^{(0)}} f(z)\,dz\right| \le 4\left|\int_{T_j^{(1)}} f(z)\,dz\right|.$$

Note that $d^{(1)} = d^{(0)}/2$, $p^{(1)} = p^{(0)}/2$.

Repeating the process, we have $T^{(0)}, T^{(1)}, \dots, T^{(n)}, \dots$

w/ $\left|\int_{T^{(0)}} f(z)\,dz\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,dz\right|$

$d^{(n)} = 2^{-n} d^{(0)}$, $p^{(n)} = 2^{-n} p^{(0)}$.



If we denote $\mathcal{T}^{(n)}$ the solid triangle w/ boundary $T^{(n)}$, we have nested compact sets

$$\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \dots \supset \mathcal{T}^{(n)} \supset \dots$$

w/ diameter going to 0. Hence, $\exists z_0 \in \mathcal{T}^{(n)}$ $\forall n \in \mathbb{N}$.

Since f is holomorphic at z_0, we have

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)$$

w/ $\psi(z) \longrightarrow 0$ as $z \longrightarrow z_0$.

Since $f(z_0)$, $f'(z_0)(z-z_0)$ have primitives, we integrate to find

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} \psi(z)(z-z_0)\,dz.$$

Since $z_0 \in \mathcal{T}^{(n)}$, and $z \in \partial \mathcal{T}^{(n)}$, we have $|z-z_0| \leq d^{(n)}$.
It follows that

$$\left|\int_{T^{(n)}} f(z)\,dz\right| \leq \varepsilon_n d^{(n)} p^{(n)},$$

$\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore,

$$\left|\int_{T^{(n)}} f(z)\,dz\right| \leq \varepsilon_n 4^{-n} d^{(0)} p^{(0)}$$

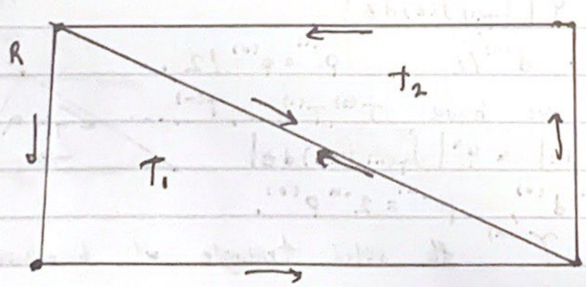
$$\Rightarrow \left|\int_{T^{(0)}} f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \leq \varepsilon_n d^{(0)} p^{(0)} \xrightarrow{n\to\infty} 0.$$

Corollary. If f is holomorphic in an open set Ω containing a rectangle R and its interior, then

$$\int_R f(z)\,dz = 0.$$

Proof. Choose an orientation as in the proof of Goursat's and note that

$$\int_R f(z)\,dz = \int_{T_1} f(z)\,dz + \int_{T_2} f(z)\,dz$$

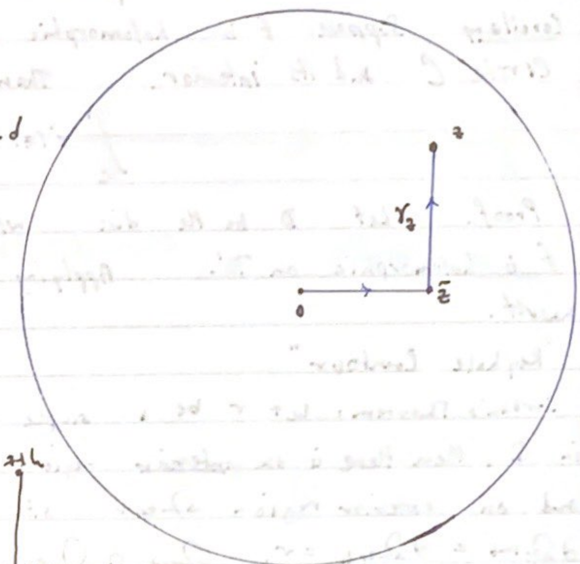


2.2 Local Existence of Primitive and Cauchy's Theorem on a Disc

Theorem 2. A holomorphic function in an open disc has a primitive in that disc.

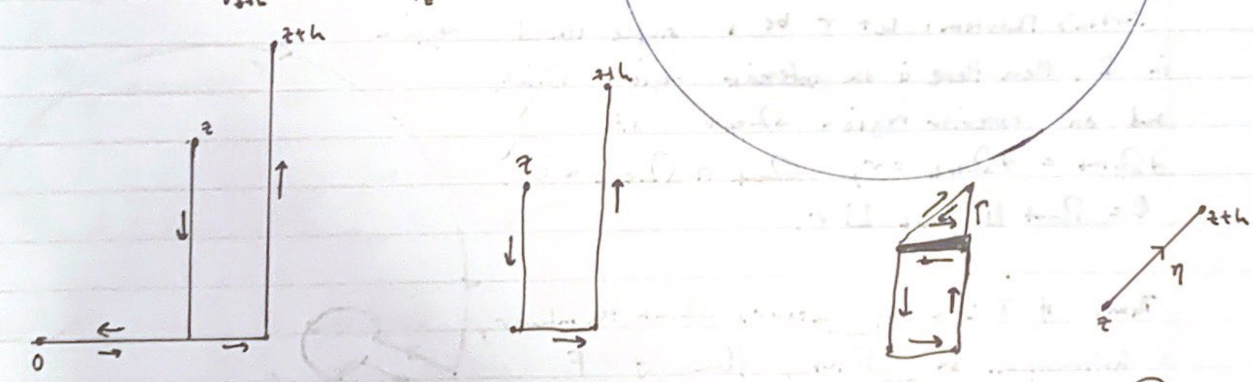
Proof. WLOG, disc D centered at origin. Given $z \in D$, consider the piecewise-smooth curve joining 0 to z by moving horizontally to $\tilde{z} = \mathrm{Re}(z)$ then vertically from $\tilde{z}$ to z. Define the curve as γ_z.

Define $F(z) = \int_{\gamma_z} f(w)\, dw$.



We claim F is holomorphic in D and $F'(z) = f(z)$. Fix $z \in D$ and let $h \in \mathbb{C}$ be so small s.t. $z+h$ belongs to the disc.

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\, dw$$



(A) (B) (C) (D)

$$\Rightarrow \quad F(z+h) - F(z) = \int_\eta f(w)\, dw$$

Since f is continuous at z, we have

$$f(w) = f(z) + \psi(w) \quad \text{where} \quad \psi(w) \xrightarrow{w \to z} 0$$

$$F(z+h) - F(z) = \int_\eta f(z)\, dw + \int_\eta \psi(w)\, dw = f(z) \int_\eta dw + \int_\eta \psi(w)\, dw.$$

$$\Rightarrow \quad \left| \int_\eta \psi(w)\, dw \right| \le \sup_{w \in \eta} |\psi(w)| \, |h|$$

$$\Rightarrow \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

So F is a primitive for f on the disc.

<u>Theorem</u> (Cauchy on Disc) If f is holomorphic in a disc, then

$$\int_\gamma f(z)\,dz = 0.$$

Proof. Since f has a primitive, apply the theorem from Ch. 1. ∎

<u>Corollary</u> Suppose f is holomorphic in an open set containing the circle C and its interior. Then

$$\int_C f(z)\,dz = 0.$$

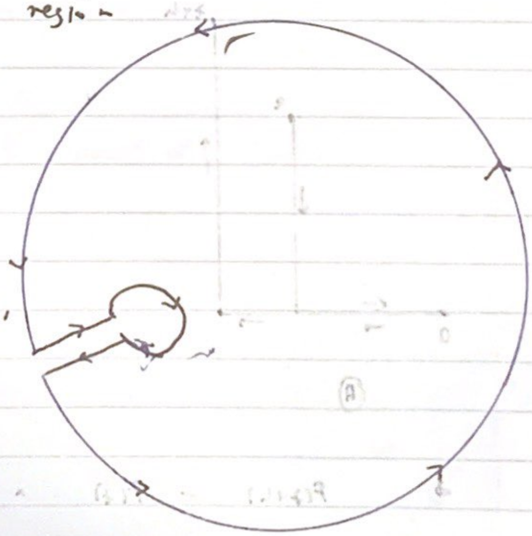
Proof. Let D be the disc w/ $\partial D = C$. $\exists D' \supset D$ s.t. f is holomorphic on D'. Applying Cauchy's Theorem on D' gives the result. ∎

"Keyhole Contour"

Jordan's Theorem: Let γ be a simple closed region in $\mathbb{C}$, then there is an <u>interior</u> region Ω_{int} and an exterior region Ω_{ext} s.t.

$\partial\Omega_{int} = \partial\Omega_{ext} = \gamma$, $\Omega_{int} \cap \Omega_{ext} = \emptyset$.

$\mathbb{C} = \Omega_{int} \sqcup \Omega_{ext} \sqcup \gamma$.

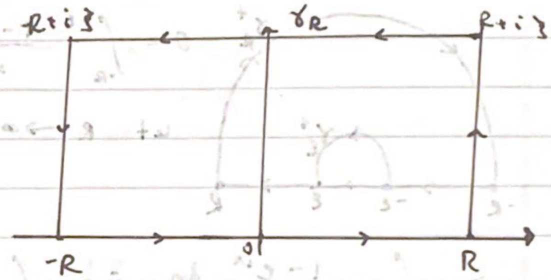


Thm. If γ is a toy contour, Ω_{int} the interior, f holomorphic on $\overline{\Omega_{int}}$, then $\exists$ F holomorphic on Ω_{int} w/ $F' = f$ on $\overline{\Omega_{int}}$.

Example 1. If $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx.$$

Proof. For $\xi = 0$, this is well-known Gaussian Integral.
For $\xi > 0$, consider $f(z) = e^{-\pi z^2}$, which is entire and holomorphic in the interior of γ_R.



By Cauchy's Theorem,

$$\int_{\gamma_R} f(z)\,dz = 0.$$

Note that $\int_{-R}^{R} e^{-\pi z^2} dz \xrightarrow{R\to\infty} 1$

$$I(R) = \int_0^{\xi} f(R+iy)\, i\, dy = \int_0^{\xi} e^{-\pi(R^2 + 2iRy - y^2)}\, i\, dy \xrightarrow{R\to\infty} 0$$

since $|I(R)| \le Ce^{-\pi R^2}$

and

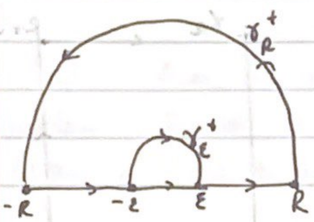
$$\int_R^{-R} e^{-\pi(x+i\xi)^2} dx = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

$$\Rightarrow \quad 0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx,$$ which gives the result.

Then $\xi < 0$ case is identical.

Example 2. $$\int_0^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$

Proof. Let $f(z) = (1-e^{iz})/z^2$, and integrate over the upper semi circles



By Cauchy,

$$0 = \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx$$

Let $R \longrightarrow \infty$, note that

$$\left| \frac{1-e^{iz}}{z^2} \right| \le \frac{2}{|z|^2} \quad \text{so} \quad \gamma_R^+ \text{ part} \longrightarrow 0.$$

$$\Rightarrow \int_{|x| \ge \varepsilon} \frac{1-e^{ix}}{x^2}\,dx = -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz.$$

Next,

$$f(z) = \frac{-iz}{z^2} + E(z),$$

w/ $E(z)$ is bounded as $z \longrightarrow 0$, and $z = \varepsilon e^{i\theta}$, $dz = i\varepsilon e^{i\theta} d\theta$ on γ_ε^+.

$$\Rightarrow \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2} \longrightarrow \int_{\pi}^{0} (-i\,i)\,d\theta = -\pi \quad \text{as } \varepsilon \longrightarrow 0$$

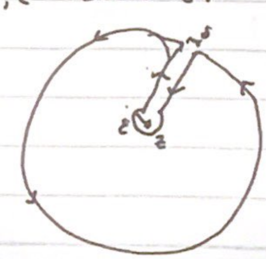
$$\Rightarrow \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi.$$

2.4 Cauchy's Integral Formulas

Theorem. If f is holomorphic in an open set that contains the closure of a disc D, if $C = \partial D^+$, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \text{for } z \in D.$$

Proof. Fix $z \in D$ and consider the keyhole $T_{\delta,\varepsilon}$ about z. $F(\zeta) = f(\zeta)/(\zeta - z)$ is holomorphic away from $\zeta = z$, so



$$\int_{T_{\delta,\varepsilon}} F(\zeta)\,d\zeta = 0.$$

If we let $\delta \longrightarrow 0$, we use the continuity of F to see that the integrals over the corridor cancel, leaving a circle C and a small circle C_ε centered at z of radius ε oriented $\ominus$ly.

Then, note that

$$F(\zeta) = \underbrace{\frac{f(\zeta) - f(z)}{\zeta - z}}_{①} + \underbrace{\frac{f(z)}{\zeta - z}}_{②}.$$

Since f is holomorphic, $① \xrightarrow{\varepsilon \to 0} 0$ since it is bounded over C_ε.

Finally,

$$\int_{C_\varepsilon} \frac{f(z)}{\zeta - z} d\zeta = f(z) \int_{C_\varepsilon} \frac{d\zeta}{\zeta - z}$$
$$= -f(z) \int_0^{2\pi} \frac{\varepsilon i e^{-it}}{\varepsilon e^{-it}} dt$$
$$= -f(z)\, 2\pi i$$

$$\Rightarrow \quad 0 = \int_C \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i f(z). \quad \blacksquare$$

Corollary. If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{for } z \in C^{\circ}.$$

Proof. Follows from induction on n.

Cauchy's Inequality: If f is holomorphic in an open set that contains $\overline{D}$ for a disc $D(z_0, R)$, then

$$|f^{(n)}(z_0)| \leq \frac{n! \, \|f\|_C}{R^n}.$$

$$\|f\|_C = \sup_{z \in C} |f(z)|.$$

Proof.

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| = \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} Rie^{i\theta} d\theta \right| \leq \frac{n!}{2\pi} \frac{\|f\|_C}{R^n} 2\pi$$

Theorem. If f is holomorphic in an open set Ω, $D(z_0)$ disc w/ $\bar{D} \subset \Omega$, then f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Proof. Fix $z \in D$. By CIF,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta - z},$$

where $C = \partial D^+$. Then

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0}\left(\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right).$$

Since $\zeta \in C$, $z \in D$ fixed, $\exists r$ w/ $0 < r < 1$,

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < r.$$

$$\Rightarrow \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right) \cdot (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

(Liouville's Theorem)

Corollary. If f is entire and bounded, then f is constant.

Proof. NTS $f' = 0$ since $\mathbb{C}$ is connected.

For $z_0 \in \mathbb{C}$, $R > 0$,

$$|f'(z_0)| \leq \frac{B}{R} \xrightarrow{R \to \infty} 0.$$

where $B = \|f\|_\infty$.

Fundamental Theorem of Algebra. Every non-constant polynomial $P(z) = a_n z^n + \dots + a_0$ w/ complex coefficients has a root in $\mathbb{C}$.

Proof. If P has no roots, $1/P(z)$ is bounded, holomorphic.

$$\frac{P(z)}{z^n} = a_n + \underbrace{\left(\frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right)}_{①} \quad \text{for } z \neq 0$$

Since ① $\longrightarrow 0$ as $|z| \longrightarrow \infty$, $\exists R>0$ s.t. if $c = |a_n|/2$,

$$|P(z)| \geq c|z|^n, \quad |z| > R.$$

Since it has no roots in $|z| \leq R$, it is bounded below everywhere. By Liouville's Theorem, $1/P$ is constant, contradiction. ∎

Theorem $f : \Omega \longrightarrow \mathbb{C}$ holomorphic. If $\exists \{z_i\} \subset \Omega$ w/ $z_i \longrightarrow z_0 \in \Omega$ and $f(z_n) = 0$, $f(z) = 0 \ \forall z \in \Omega$.

Proof. Fix $\overline{D_r(z_0)} \subset \Omega$.

$$f(z) = \sum_{n \geq 0}^{\infty} a_n \cdot (z - z_0)^n \quad \forall z \in D_r(z_0).$$

Claim. $a_n = 0 \ \forall n$.

If not let a_m be the first nonzero.

$$f(z) = a_m (z - z_0)^m (1 + g(z - z_0))$$

where $g(z-z_0) \xrightarrow{z \to z_0} 0$. Take $z = z_k \neq z_0$

$a(z_k - z_0)^m \neq 0$, $1 + g(z_k - z_0) \neq 0$

but $f(z_k) = 0$ ↯.

Let $U :=$ interior of points where $f(z) = 0$. Then U is open and non-empty. U is closed since $z_n \in U$, $z_n \to z$, $f(z) = 0$ by continuity, f vanishes in a neighborhood of z as above.
$\Rightarrow$ $V = U^c$ are open, disjoint, $\Omega = U \cup V$ $\Rightarrow$ $z_0 \in \Omega \cap U$
$\Rightarrow \Omega = U$, since Ω is connected.

Def. Given f, F in Ω, Ω' w/ $\Omega \subset \Omega'$. If $F|_{\Omega} = f$, then F is an <u>analytic continuation</u> of f.

2.5 More Applications

Morera's Theorem

Thm. Suppose f is continuous in D open s.t. $\forall T \subset D$ triangle,

$$\int f(z)\,dz = 0.$$

Then, f is holomorphic.

Proof. By the proof of Cauchy, f has F in D w/ $F' = f$. By the regularity theorem, F is indefinitely differentiable, so f is holomorphic. ∎

Sequences of Functions

Thm. If $\{f_n\}_{n=1}^{\infty}$ are holomorphic converging uniformly to f in every compact subset of Ω, then f is holomorphic in Ω.

Proof. Take $\underset{\Delta}{T} \subset \underset{\text{Disc}}{D} \subset \overline{D} \subset \Omega$. Since f_n is holomorphic

$$\int_T f_n(z)\,dz = 0 \qquad \forall n.$$

By assumption, $f_n \longrightarrow f$ uniformly in $\overline{D}$ so f is continuous and

$$\int_T f_n(z)\,dz \longrightarrow \int_T f(z)\,dz = 0.$$

By Morera's Theorem, f is holomorphic on D, and hence Ω. ∎

Corollary. $\{f_n'\} \longrightarrow f'$ uniformly over every compact subset of Ω.

Proof. Let $\Omega_\delta = \{z \in \Omega : \overline{D_\delta}(z) \subset \Omega\}$. S.T.S.

$$\sup_{z \in \Omega_\delta} |F'(z)| \le \frac{1}{\delta} \sup_{z \in \Omega} |F(z)| \qquad \forall F \text{ holomorphic.}$$

$$F'(z) = \frac{1}{2\pi i} \int_{\partial D_\delta(z)} \frac{F(\zeta)}{(\zeta - z)^2}\,d\zeta$$

$$\Rightarrow |F'(z)| \le \frac{1}{2\pi} \int_{C_\delta(z)} \frac{|F(\zeta)|}{|\zeta - z|^2}\,|d\zeta|$$

$$\le \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \, \frac{1}{\delta^2}\, 2\pi\delta$$

$$= \frac{1}{\delta} \sup_{\zeta \in \Omega} |F(\zeta)|.$$

Now, take $F = f_n - f$. ∎

Holomorphic Functions as Integrals.

Thm. Let $F(z,s)$ be defined for $(z,s) \in \Omega \times [0,1]$ where $\Omega \overset{open}{\subseteq} \mathbb{C}$.

If (i) $F(z,s)$ is holomorphic in z for each s

(ii) F is continuous on $\Omega \times [0,1]$

Then $$f(z) := \int_0^1 F(z,s)\, ds$$ is holomorphic.

Proof. For $n \geq 1$,
$$f_n(z) := \frac{1}{n} \sum_{k=1}^{n} F(z, k/n).$$
By (i), f_n is holomorphic. f_n continuous on compact set
$\Rightarrow f_n$ uniformly continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\sup_{z \in D} | F(z,s_1) - F(z,s_2) | < \varepsilon \qquad \text{whenever} \quad |s_1 - s_2| < \delta.$$

If $n > 1/\delta$, $z \in D$,
$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} F(z, k/n) - F(z,s)\, ds \right| \\ &\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} | F(z, k/n) - F(z,s) |\, ds \\ &\leq \sum_{k=1}^{n} \varepsilon/n \\ &< \varepsilon. \end{aligned}$$
■

Schwarz Reflection Principle

~~Let Ω be open~~ Let $\Omega \overset{open}{\subset} \mathbb{C}$ symmetric WRT real line; $z \in \Omega$ iff $\bar{z} \in \Omega$. Ω^+, Ω^- as $+\mathrm{Im}$, $-\mathrm{Im}$ respectively.

$I = \Omega \cap \mathbb{R} \Rightarrow \Omega = \Omega^+ \cup \Omega^- \cup I$.

Thm. If f^+, f^- holomorphic on Ω^+, Ω^-, resp. and extend continuously to I w/ $f^+(x) = f^-(x)$ on I, then
$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(z) = f^-(z) & z \in I \\ f^-(z) & z \in \Omega^- \end{cases}$$ is holomorphic on Ω.

Theorem. If f holomorphic on Ω^+ extending continuously to I, f is real valued on I, then $\exists F$ holomorphic on Ω w/ $F = f$ on Ω^+.

Proof. $F(z) = \overline{f(\bar{z})}$ on Ω^-. ■

Runge Approximation

Theorem. Any holomorphic function in a neighborhood of a compact set K can be approximated uniformly on K by rational functions whose singularities are in K^c. If K^c is connected, any function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials.

Proof.

Lemma 1. If f holomorphic on $\Omega \overset{open}{\subseteq} \mathbb{C}$, $K \overset{compact}{\subset} \Omega$, $\exists\, \gamma_1, \gamma_2, \ldots, \gamma_N$ in $\Omega \setminus K$ w/

$$f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad \forall z \in K.$$

Sketch. Drop a square lattice covering K. ∘

Lemma 2. For line segment $\gamma \subset \Omega \setminus K$, $\exists$ a sequence of rational functions w/ singularities on γ that approximate $\int_\gamma f(\zeta)/(\zeta - z)\, d\zeta$ uniformly on K.

Proof.
$$\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t)\, dt$$

$\gamma \cap K = \emptyset$, so $F(z,t)$ is jointly continuous on $K \times [0,1]$ and K is compact so, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\| F(z, t_1) - F(z, t_2) \|_K < \varepsilon \qquad \text{whenever } |t_1 - t_2| < \delta.$$

Then, we approximate $\int_0^1 F(z,t)\,dt$ w/ Riemann sums as before. ∎

Lemma 3. If K^c is connected, $z_0 \notin K$, then $1/(z - z_0)$ can be approximated uniformly by polynomials.

Proof.

Choose z_1 outside $K \subset D \overset{open}{\subset} \Omega$.

$$\frac{1}{z - z_1} = -\frac{1}{z_1} \frac{1}{1 - z/z_1} = \sum_{n=1}^{\infty} -\frac{z^n}{z_1^{n+1}}.$$

Let γ be a curve on K^c s.t. $\gamma(0) = z_0$, $\gamma(1) = z_1$.

Let $\rho := \frac{1}{2} d(K, \gamma) > 0$. Choose $\{w_1, \ldots, w_\ell\}$ on γ w/ $w_0 = z_0$, $w_\ell = z_1$, $|w_j - w_{j+1}| < \rho \quad \forall\ 0 \le j < \ell$.

If w on γ and w' w/ $|w-w'|<\rho$, $1/(z-w)$ can be approximated uniformly by $1/(z-w')$.

$$\frac{1}{z-w} = \frac{1}{z-w'} \frac{1}{1-\frac{w-w'}{z-w'}}$$
$$= \sum_{n\geq 0}^{\infty} \frac{(w-w')^n}{(z-w')^{n+1}}.$$

Then, we travel from z_0 to z_1 through $\{w_j\}$ to approximate $1/(z-z_0)$ by polynomials in $1/(z-z_1)$.