Algebra Problems

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A collection of algebra problems and solutions sorted in roughly increasing difficulty.

§1 Easy Problems

Problem 1.1 (IMO 2000/2)**.** Let $A, B, C \in \mathbb{R}^+$ with $ABC = 1$. Prove that

$$\left(A - 1 + \frac{1}{B}\right)\left(B - 1 + \frac{1}{C}\right)\left(C - 1 + \frac{1}{A}\right) \leq 1.$$

Proof. Apply the substitution $A = \frac{x}{y}, B = \frac{y}{z}, C = \frac{z}{x}$. Then, we have

$$\begin{aligned}\prod_{\text{cyc}}\left(A - 1 + \frac{1}{B}\right) &= \prod_{\text{cyc}} \frac{x+z-y}{y} \\ &= \frac{(x+z-y)(y+x-z)(z+y-x)}{xyz}.\end{aligned}$$

Thus, it suffices to show

$$(x+z-y)(y+x-z)(z+y-x) \leq xyz.$$

Let $m = x+z-y, n = y+x-z, p = z+y-x$. The above is equivalent to

$$mnp \leq \frac{(m+n)(n+p)(p+m)}{8},$$

which follows from AM-GM. $\square$

Problem 1.2 (IMO 2001/4)**.** Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be integers. For each permutation $a \in S_n$, let

$$S(a) = \sum_{i=1}^{n} k_i a(i).$$

Show that there exists two permutations $b, c \in S_n$ such that $n!$ divides $S(b) - S(c)$.

Proof. It suffices to show that there exist two permutations with the same remainder modulo $n!$ upon applying S. For sake of contradiction, suppose $S(a)$ is distinct modulo $n!$ for all permutations. Then,

$$\begin{aligned}\sum_{i=1}^{n!} i &\equiv \sum_{\sigma\in S_n} S(\sigma) \pmod{n!}\\ &= \sum_{\sigma\in S_n}\sum_{i=1}^{n} k_i\sigma(i)\\ &= \sum_{i=1}^{n} k_i \sum_{\sigma\in S_n}\sigma(i)\\ &= \sum_{i=1}^{n} k_i(n-1)!\sum_{i=1}^{n} i\\ &= (n-1)!\frac{n(n+1)}{2}\sum_{i=1}^{n} k_i\\ &= n!\left(\frac{n+1}{2}\right)\sum_{i=1}^{n} k_i \equiv 0 \pmod{n!}.\end{aligned}$$

However, $\sum_{i=1}^{n!} i = \frac{n!(n!+1)}{2}$ is not divisible by $n!$, as $\frac{n!+1}{2}$ is not an integer for $n \geq 2$. □

Problem 1.3 (IMO 2007/1)**.** Real numbers $a_1, \ldots, a_n$ are given. For each $i \in [1, n] \cap \mathbb{Z}$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}.$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

1. Prove that, for real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,
$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}.$$
2. Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that the equality holds in (1).

Proof. First, note that

$$d = \max_{1\leq i\leq j\leq n}(a_i - a_j).$$

Suppose a_i, a_j are the maximal indexes such that $d = a_i - a_j$. Note that $d \geq 0$, since $d_i \geq a_i - a_i = 0$.

$$\begin{aligned}|a_i - x_i| + |x_j - a_j| &> |a_i - a_j + x_j - x_i|\\ &= |d + (x_j - x_i)|\\ &= d + (x_j - x_i) \geq d,\end{aligned}$$

where we used the fact that $x_j \geq x_i$ so $d + (x_j - x_i) \geq 0$. Hence, one of $|a_i - x_i|, |a_j - x_j|$ must be at least $d/2$, so it follows that

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}.$$

For the equality case, let

$$x_k = \begin{cases} \min(x_{k+1}, a_k) & \text{if } k < i \\ \frac{a_i+a_j}{2} & \text{if } i \le k \le j \\ \max(x_{k-1}, a_k) & \text{if } k > j \end{cases}.$$

Then

$$|x_k - a_k| \le \left|\frac{a_i + a_j}{2} - a_k\right| \le \left|\frac{a_i - a_j}{2}\right| = \frac{d}{2},$$

as desired. $\square$