Math 205: Complex Variables

Professor: Dan-Virgil Voiculescu, Spring 2021
Scribe: Vishal Raman

Contents

§1 January 20th, 2021

§1.1 Intro to Riemann Mapping Theorem

Our first goal is to proof a fundamental theorem of Riemann on conformal mappings. We start with several preparations, including some detours. The theorem essentially says that lots of open sets in $\mathbb{C}$ are holomorphically isomorhpic, given that they satisfy some simple topological conditions.

§1.2 Cauchy's Integral Formula

Recall Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz$$

where Γ is a simple closed curve, piecewise differentiable, $z_0 \in \text{Int}(\Gamma)$, and $f : \Omega \to \mathbb{C}$ is a holomorphic function, with Ω is open, $\Omega \supset \Gamma \cup \text{Int}(\Gamma)$.

If Γ is the circle $|z - z_0| = R$, we parameterize with $z = Re^{i\theta} + z_0$ with $\theta \in [0, 2\pi)$. This gives

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta,$$

which represents the average of f on the circle.

It follows that

$$|f(z_0)| \leq \max_{\partial B_R(z_0)} |f(z)|,$$

with equality if and only if f is constant.

If $f : \Omega \to \mathbb{C}$ is holomorphic for Ω connected, open and $z_0 \in \Omega$, then

$$|f(z_0)| \leq \sup_{z\in\Omega} |f(z)|$$

with equality if and only if f is constant.

§1.3 Schwarz Lemma

Theorem 1 (Schwarz Lemma)
For $f : B_1(0) \to \mathbb{C}$ holomorphic with $|f(z)| \leq 1$ for all z and $f(0) = 0$. Then

$$|f(z)| \leq |z|, |f'(0)| \leq 1.$$

If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = cz$ for some $|c| = 1$.

Proof. Define a function

$$g(z) = \begin{cases} f(z)/z, \text{if } 0 \leq |z| \leq 1 \\ f'(0), \text{if } z = 0 \end{cases}.$$

Note that $g(z)$ is continuous since at zero,

$$\lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Hence, $|g(z)| \leq C < \infty$ using the Weierstrass Extreme Value theorem. If $0 < \epsilon < |w| < r < 1$, note that taking a Keyhole Contour, we have

$$g(w) = \frac{1}{2\pi i}\left(\int_{|z|=r} - \int_{|z|=\epsilon}\right) \frac{g(z)}{z-w}\, dz.$$

Note that

$$\left|\int_{|z|=\epsilon} \frac{g(z)}{z-w}\, dz\right| \leq (2\pi\epsilon)\cdot C\frac{1}{|w|-\epsilon} \xrightarrow{\epsilon\to 0} 0.$$

It follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)}{z-w}\, dz$$

for $0 < |w| < r$. The right side is holomorphic in w if $|w| < r$, so it follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)}{z-w}\, dz$$

is holomorphic in $|z| < 1$.

This can also be proved by taking a Taylor series about the origin. Since there is no constant term, we can divide by z to still have a convergent Taylor series.

If $r < 1$,

$$\sup_{|z|\leq r} |g(z)| = \sup_{|z|=r} |g(z)| \leq \sup_{|z|=r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

If we let $r \uparrow 1$, then we get $\sup_{|z|<1} |g(z)| \leq 1$. It follows that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

If $|f(z_0)| = z_0$ for some $0 < |z_0| < 1$ then $|g(z_0)| = 1$ and g is constant by the maximum principle so $g(z) = c$, $f(z) = cz$. If $|f'(0)| = 1$, then $|g(0)| = 1$ so g is constant and $f = cz$. $\square$

§1.4 Maximum Principles

In the above proof, we used the maximum principle. Some other versions we will use are the following:

If $K \subset \mathbb{C}$ compact and $f : K \to \mathbb{C}$ continuous, and the restriction of f to the interior of K is holomorphic, then

$$\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|.$$

If Ω is open and connected, $f : \Omega \to \mathbb{C}$, $z_0 \in \Omega$, and $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$, then f is constant. Applying this to e^f and using that $|e^f| = e^{\mathrm{Re}\, f}$, we find that

$$\mathrm{Re}\, f(z_0) = \sup_{z\in\Omega} \mathrm{Re}\, f(z),$$

implies that f is constant. We have the same result for $\mathrm{Im}\, f$ by replacing f with $-if$.

§1.5 Homework

Show the Automorphisms of the unit disk are fractional linear transformations. Hint: Compose f with special Automorphisms of the unit disk and move special points to zero.