Wilf, generatingfunctionology

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Abstract

Notes and selected solutions from *Wilf, generatingfunctionology*(third edition). I will generally leave out tedious computations but will refer to the text whenever possible.

Contents

1 Introductory Ideas and Examples

As a motivating example, take the Fibonacci numbers $F_0, F_1, \ldots$ with the recurrence relation $F_{n+1} = F_n + F_{n-1}$ for $n \geq 0$, with $F_0 = 0, F_1 = 1$. There are exact formulas for F_n, but another useful representation is as follows: the n-th Fibonacci number F_n, is the coefficient of x^n in the expansion of

$$F(x) = \frac{x}{1 - x - x^2}$$

as a power series about the origin.

Generating functions are typically used for the following:

- Finding an exact formula for members of a sequence. This is not always possible, depending on the sequence, but it is often a good starting point.
- Finding a recurrence formula. Even though we mostly obtain generating functions from recurrences, we can sometimes use them to generate new recurrence formulas, which can potentially provide new insights.
- Find statistical properties of sequences. These are typically called Moment Generating Functions in statistics and have many use cases.
- Finding asymptotic formulas for sequences. An important example of this is the *Prime Number Theorem.*
- Proving analytic properties of the sequence(convexity, unimodality).
- Proving combinatorial identities.

1.1 Example: Two-Term Recurrences

Example 1.1

Take the recurrence defined by $a_{n+1} = 2a_n + 1$ for $n \geq 0$, $a_0 = 0$. We show that $a_n = 2^n - 1$ using generating functions.

Proof. Define $A(x) = \sum_{n \geq 0} a_n x^n$. By multiplying the recurrence relation by x^n and summing over n, we have

$$\sum_{n \geq 0} a_{n+1} x^n = 2 \sum_{n \geq 0} a_n x^n + \sum_{n \geq 0} x^n = 2A(x) + \frac{1}{1 - x}.$$

Then, note that

$$\sum_{n \geq 0} a_{n+1} x^n = \frac{A(x) - a_0}{x} = \frac{A(x)}{x}.$$

We obtain

$$\frac{A(x)}{x} = 2A(x) + \frac{1}{1 - x} \implies A(x) = \frac{x}{(1 - x)(1 - 2x)}.$$

To find a formula for the coefficients, we can use a partial fractions decomposition and expand the corresponding Taylor series. In this case, we have

$$\begin{aligned}\frac{x}{(1-x)(1-2x)} &= \left(\frac{2x}{1-2x} - \frac{x}{1-x}\right)\\ &= \sum_{n\geq 1}(2x)^n - \sum_{n\geq 1} x^n\\ &= \sum_{n\geq 1}(2^n-1)x^n.\end{aligned}$$

□

We now handle a more challenging two term recurrence.

Example 1.2

Take the recurrence defined by $a_{n+1} = 2a_n + n$ for $n \geq 0$, $a_0 = 1$. Find the generating function and determine a closed formula for the coefficients.

Proof. As before, define $A(x) = \sum_{n\geq 0} a_n x^n$. From the recurrence relation, we have

$$\frac{A(x)-1}{x} = 2A(x) + \sum_{n\geq 0} nx^n.$$

Note that

$$\sum_{n\geq 0} nx^n = \sum_{n\geq 0} xD(x^n) = xD\sum_{n\geq 0} x^n = xD\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$$

□

where D denotes the differentiation operator. We are assuming absolute convergence of the sums in these computations so the exchanging of the sum and the differentiation operator is justified.

Plugging this in and solving for $A(x)$, we obtain the generating function

$$A(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)} = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}.$$

To compute the coefficient of x^n, note that the coefficient of $\frac{-1}{(1-x)^2}$ is $-n-1$ and the coefficient from $\frac{2}{1-2x}$ is 2^{n+1}. From this, we obtain that

$$a_n = 2^{n+1} - n - 1.$$

1.2 The Method of Generating Functions

Definition 1.3. Given a power series $f(x)$, the symbol $[x^n]f(x)$ denotes the coefficient of x^n in the series $f(x)$.

Fact 1.4. $[x^n]\{x^a f(x)\} = [x^{n-a}]f(x)$.

Fact 1.5. $[\beta x^n]f(x) = 1/\beta[x^n]f(x)$ for $\beta \in \mathbb{R}$.

Given a recurrence formula, we have the following steps:

1. Note the set of values that are taken by the free variable(it is generally $n \geq 0$ or $n \geq 1$).
2. Define a generating function, $A(x) = \sum_n a_n x^n$.
3. Multiply both sides of the recurrence by x^n and sum over n.
4. Express both sides of the equation in terms of $A(x)$.
5. Solve for $A(x)$. If an exact formula is needed, expand $A(x)$ in a power series.

1.3 Example: Fibonacci Numbers

We return to the example of Fibonacci numbers, calculating the generating function. Define $F(x) = \sum_{n\geq 0} F_n x^n$.

We have

$$\begin{aligned}\sum_{n\geq 1} F_{n+1}x^n &= \sum_{n\geq 1} F_n x^n + \sum_{n\geq 1} F_{n-1}x^n,\\ \frac{F(x)-x}{x} &= F(x) + xF(x),\\ F(x) &= \frac{x}{1-x-x^2}.\end{aligned}$$

We can write $1 - x - x^2 = (1 - xr_+)(1 - xr_-)$ with $r_\pm = \frac{1\pm\sqrt{5}}{2}$. It follows that

$$\begin{aligned}\frac{x}{1-x-x^2} &= \frac{1}{r_+ - r_-}\left(\frac{1}{1-xr_+} - \frac{1}{1-xr_-}\right)\\ &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}(r_+^n - r_-^n)x^n.\end{aligned}$$

1.4 Two Independent Variables

Let n and k be integers with $0 \leq k \leq n$. How many ways can we choose a subset of k objects from $\{1, 2, \ldots, n\}$? We know that this is $\binom{n}{k}$, but we derive this using generating functions.

Proof. Let $f(n, k)$ be the answer to the question. In the possible collection of subsets, we can divide them into two piles: subsets containing n and subsets not containing n. In the first case, there are $f(n-1, k-1)$ possible subjects and the latter case there are $f(n-1, k)$ subsets. From this, we obtain the recurrence

$$f(n, k) = f(n-1, k) + f(n-1, k-1).$$

Note the obvious initial condition $f(n,0)=1$. Define the generating function

$$B_n(x) = \sum_{k\geq 0} f(n,k)x^k.$$

Using the recurrence relation, we obtain

$$B_n(x) - 1 = (B_{n-1}(x) - 1) + xB_{n-1}(x), \quad B_0(x) = 1$$

which gives

$$B_n(x) = (1+x)B_{n-1}(x).$$

This is an easy recurrence to solve: namely $B_n(x) = (1+x)^n$. We obtain the desired result that $[x^k]B_n(x) = \binom{n}{k}$ by the binomial theorem. We could also use Taylor's formula: $f(n,k)$ will be the k-th derivative of $(1+x)^n$ evaluated at $x=0$ divided by $k!$. $\square$

An important thing to note is that our computation holds for arbitrary $n \in \mathbb{C}$, provided $k \in \mathbb{N}$. From this, we obtain a general formula for binomial coefficients, given by

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

1.5 Exponential Generating Functions

The choice of x^n was somewhat arbitrary for our power series. In some cases, other choices can be helpful.

Definition 1.6 (Exponential Generating Function)**.** The EGF of a sequence $\{a_n\}_{n\geq 0}$ is given by

$$\sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

We call our vanilla generating function an "Ordinary Generating Function"(OGF).

A useful operation in the case of EGFs is the $x(D)\log$. This goes as follows:

1. Take the logarithm of both sides of the equation.
2. Differentiate both sides and multiply through by x.
3. Clear the fractions.
4. For each n, find the coefficients of x^n on both sides and equate them.