Notes on Free Probability

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§1 Free Products

We first define the notion of a free product for objects. They can all be boiled down to universal products in the categorical sense.

Definition 1.1 (Free Product of Groups)**.** let $(G_i)_{i\in I}$ denote a family of groups. The *group free product*, denoted $*_{i\in I}G_i$ is the unique group G up to isomorphsim with homomorphisms $\psi_i : G_i \to G$ so that given any group H and $\phi_i : G_i \to H$, there exists a unique homomorphism $\Phi = *_{i\in I}\phi_i : G \to H$ making the diagram commute:

$$\begin{tikzcd} G_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & G \arrow[dl, "\Phi"] \\ H & \end{tikzcd}$$

The free product can also be constructed as

$$H = \{g_1g_2\dots g_n : g_j \in G_{i_j} \setminus \{e\}, i_1 \neq i_2 \neq \cdots \neq i_n\} \cup \{\emptyset\},$$

with multiplication defined by concatenation followed by reduction.

Definition 1.2 (Free Product of Groups)**.** let $(A_i)_{i\in I}$ denote a family of unital algebras. The *unital algebra free product*, denoted $*_{i\in I}A_i$ is the unique unital algebra A up to isomorphsim with homomorphisms $\psi_i : A_i \to A$ so that given any group B and $\phi_i : A_i \to B$, there exists a unique homomorphism $\Phi : *_{i\in I}A \to B$ making the diagram commute:

$$\begin{tikzcd} A_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & A \arrow[dl, "\Phi"] \\ B & \end{tikzcd}$$

As a vector space, we construct the unital algebra free product by first taking a basis formed by

$$B = \{a_1a_2\dots a_n : a_j \in A_{i_j} : i_1 \neq i_2 \neq \dots i_n\},$$

then taking the quotient with the subspace generated by relations

$$a_1\dots a_{j_1}(\lambda a_j^{(0)} + \mu a_j^{(1)})a_{j+1}\dots a_n, a_j = 1.$$

We also have the following proposition:

Proposition 1.3. *If $A_i = \mathbb{C}1 \oplus V_i$, then*

$$*_{i\in I}A_i \cong \mathbb{C}1 \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{i_1\neq i_2\neq\cdots\neq i_n} V_{i_1}\otimes\cdots\otimes V_{i_n}\right).$$

References: [1] Free Random Variables, D. V. Voiculescu, J. H. Dykema, A. Nica