

# Olympiad Geometry

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I first present my notes from the Art of Problem Solving(AoPS) Olympiad Geometry courses. This includes problems, solutions and expository material.

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## §1 Week 1: Fundamentals of Geometry

### §1.1 Similar Triangles

The first fundamental tool at our disposal is similar triangles, which give us relationships between the lengths and angles of segments.

**Definition 1.1.** Two triangles  $\triangle ABC$ ,  $\triangle DEF$  are similar (denoted  $\triangle ABC \sim \triangle DEF$ ) if  $\angle A = \angle D$ ,  $\angle B = \angle E$ , and  $\angle C = \angle F$ . If the above relations hold, then we also have

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

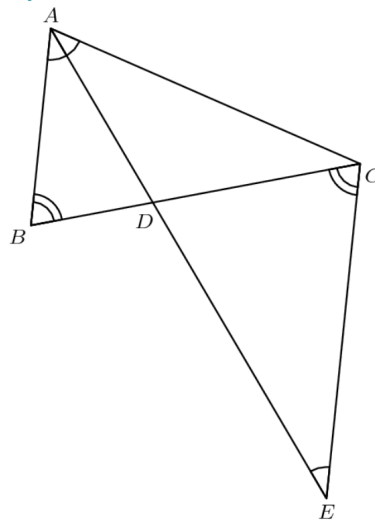
Similar triangles can be useful if a problem involves ratios or products of lengths. Another use (though rare) is that we show triangles are similar by showing  $AB/DE = AC/DF = BC/EF$  and deduce the angles are equal. We could also show that pair of sides have equal ratio and the included angle is equal:  $AB/DE = AC/DF$  and  $\angle BAC = \angle EDF$ , then  $\triangle ABC \sim \triangle DEF$ .

We begin by present some applications.

#### Theorem 1 (Angle Bisector)

Take  $\triangle ABC$ . If  $D \in BC$  so that  $AD$  bisects  $\angle BAC$ , then  $AB/BD = AC/CD$ .

*Proof.* Draw a line through  $C$  parallel to  $AB$  and mark  $E$  as the intersection of the parallel line through  $C$  and the extension of  $AD$ .



Then  $\angle ABC = \angle ECD$  and  $\angle DEC = \angle DAC$  so it follows that  $\triangle ABC \sim \triangle ECD$ . Thus,  $AB/BD = EC/CD$ . Finally,  $\angle CED = \angle DAC$  so it follows that  $\triangle ACE$  is isosceles and  $AC = EC$  so we find that

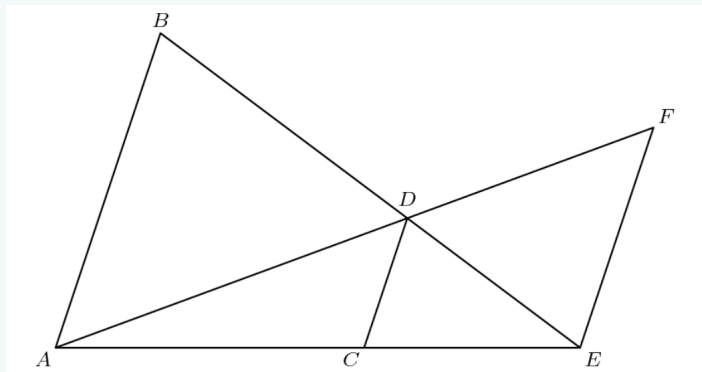
$$\frac{AB}{BD} = \frac{AC}{CD},$$

as desired. □

**Remark 1.2.** We also could prove this using the Law of Sines or the ratio of areas of the two triangles.

## Problem 1

Given that  $AB \parallel CD \parallel EF$ , prove that  $\frac{1}{AB} + \frac{1}{EF} = \frac{1}{CD}$  in the following diagram:



*Proof.* Multiplying through by  $CD$ , we get that

$$CD/AB + CD/EF = 1.$$

Note that  $\triangle ACD \sim \triangle AEF$  and  $\triangle ECD \sim \triangle EAB$  so it follows that  $CD/AB = CE/AE$  and  $CD/EF = CA/AE$ .

Finally,

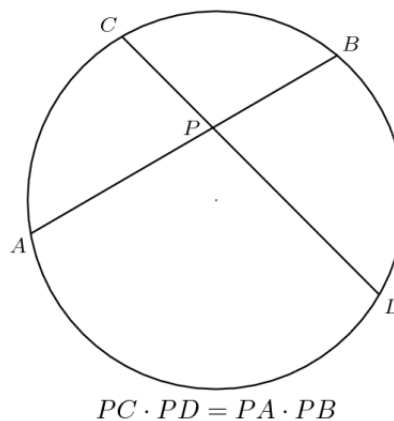
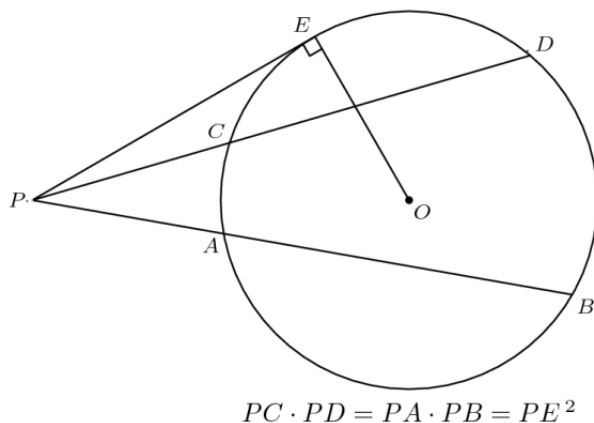
$$\frac{CD}{AB} + \frac{CD}{EF} = \frac{CE}{AE} + \frac{CA}{AE} = \frac{AE}{AE} = 1.$$

☐

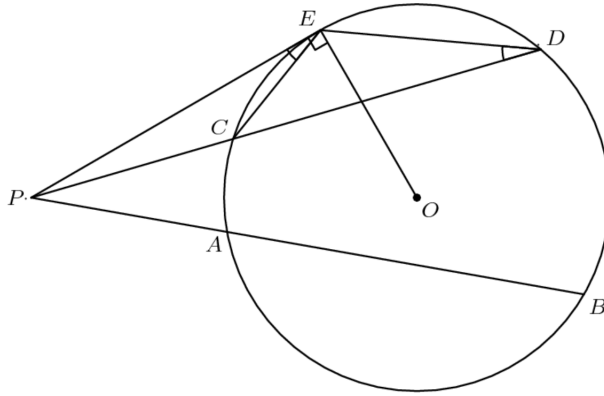
## §1.2 Power of a Point

## Theorem 2 (Power of a Point)

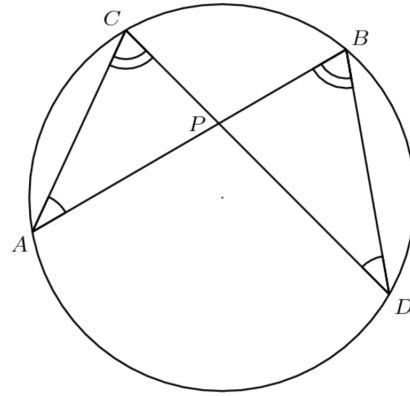
Take a point  $P$  and circle  $O$ . For any line that passes through  $P$  and intersects  $O$  at two points  $X$  and  $Y$ , the product  $(PX)(PY)$  is constant. We call this product the **power of point**  $P$  with respect to circle  $O$ .



The Power of a Point Theorem follows from similar triangles:



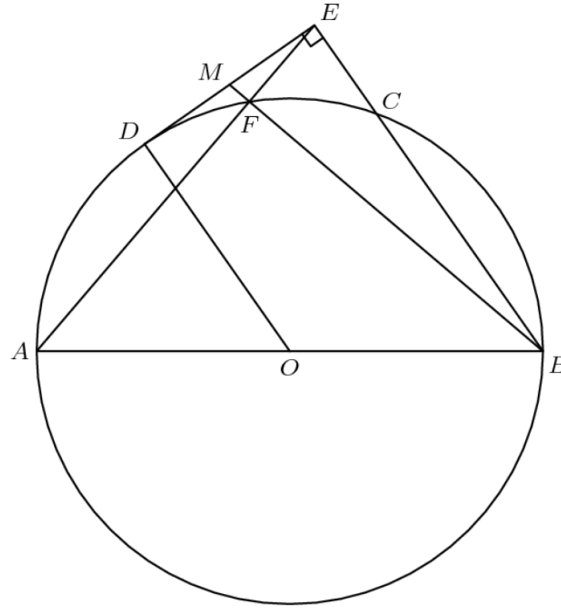
$$\triangle PCE \sim \triangle PED$$



$$\triangle PCA \sim \triangle PBD$$

### Problem 2

$AB$  is a diameter of circle  $O$ . Points  $C$  and  $D$  are on the circle such that  $D$  bisects arc  $AC$ . Point  $E$  is on the extension of  $BC$  that that  $BE$  is perpendicular to  $DE$ .  $F$  is the intersection of  $AE$  and circle  $O$ . Prove that the extension of  $BF$  bisects segment  $DE$  at  $M$ .



*Proof.* We first claim that  $OD \parallel EB$ . This is because

$$\angle AOD = \text{arc}(AD) = \text{arc}(AC)/2 = \angle ABE.$$

It follows that  $ED$  is tangent to the circle, since  $\angle ODE$  is a right angle. Furthermore,  $\angle AFB$  is a right angle since  $AB$  is the diameter of the circle. Now, note that  $\text{Pow}_O(M) = MD^2 = MF \cdot FB$ . It suffices to show that  $EF^2 = MF \cdot FB$ . This follows from the fact that  $MFE \sim EFB$ , so it follows that

$$\frac{EF}{FB} = \frac{ME}{FE} \implies EF^2 = ME \cdot FB.$$

□

### §1.3 Cyclic Quadrilaterals

**Definition 1.3.** A quadrilateral is called **cyclic** if a circle can be drawn that passes through all four vertices.

There are 4 equivalent methods to showing a quadrilateral  $ABCD$  is cyclic, namely:

- Showing  $\angle ABD = \angle ACD$  (or any of the other pairs of similarly defined angles).
- Showing a pair of opposite angles sum to 180 degrees.
- The converse of the Power of a Point: if  $P$  is the intersection of lines  $AB$  and  $CD$  and

$$PA \cdot PB = PC \cdot PD$$

or

$$QC \cdot QD = QB \cdot QA,$$

then  $A, B, C, D$  are all on a circle.

- The equality condition of **Ptolemy's Inequality**: In a quadrilateral  $ABCD$ ,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

with equality if and only if  $ABCD$  is cyclic.

I omit the basic examples but present some of the interesting ones:

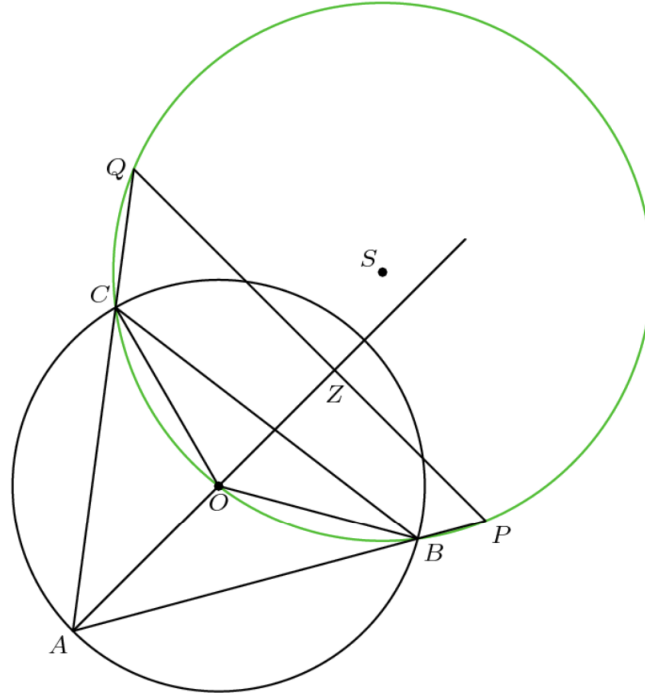
#### Proposition 1.4

A chord  $ST$  of constant length slides around a semicircle with diameter  $AB$ .  $M$  is the midpoint of  $ST$  and  $P$  is the foot of the perpendicular from  $S$  to  $AB$ . Prove that the angle  $SPM$  is constant for all positions of  $ST$ .

*Proof.* If  $SM = MT$ , then it follows  $M$  is the perpendicular bisector of  $\triangle OST$ . Thus,  $OPSM$  is cyclic and  $\angle SPM = \angle SOM$ . Finally, the length of  $SM$  is constant, so it follows that the arc between intersection of the extension of  $OM$  and the circle and  $S$  is constant. Thus,  $\angle SPM$  is constant, as desired.  $\square$

#### Proposition 1.5

$ABC$  is an acute triangle with  $O$  as its circumcenter. Let  $S$  be the circle through  $C, O, B$ . The lines  $AB$  and  $AC$  meet circle  $S$  again at  $P$  and  $Q$ , respectively. Show that  $AO$  and  $PQ$  are perpendicular.



*Proof.* It suffices to show that  $\angle AZP$  is right, where  $Z = AO \cap PQ$ . This reduces to showing that  $\angle ZPA + \angle ZAP = 90$ . Since  $PBCQ$  is cyclic, note that

$$\angle ZPA = 180 - \angle BCQ = \angle ACB,$$

so it suffices to show that  $\angle ACB + \angle OAB = 90$ . Mark  $D$  as the intersection of  $AO$  with the original circle. Then,

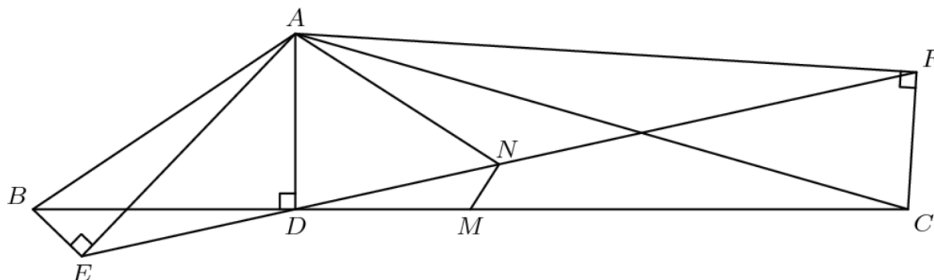
$$\angle ACB + \angle OAB = \frac{\text{arc}(AB) + \text{arc}(BD)}{2} = \frac{\text{arc}(AD)}{2} = 90.$$

□

## §1.4 Problems

### Problem 3

Let  $ABC$  be a triangle and  $D$  be the foot of the altitude from  $A$ . Let  $E$  and  $F$  be on a line passing through  $D$  such that  $AE$  is perpendicular to  $BC$ ,  $AF$  is perpendicular to  $CF$ , and  $E$  and  $F$  are different from  $D$ . Let  $M$  and  $N$  be the midpoints of the line segments  $BC$  and  $EF$ , respectively. Prove that  $AN$  is perpendicular to  $NM$ .



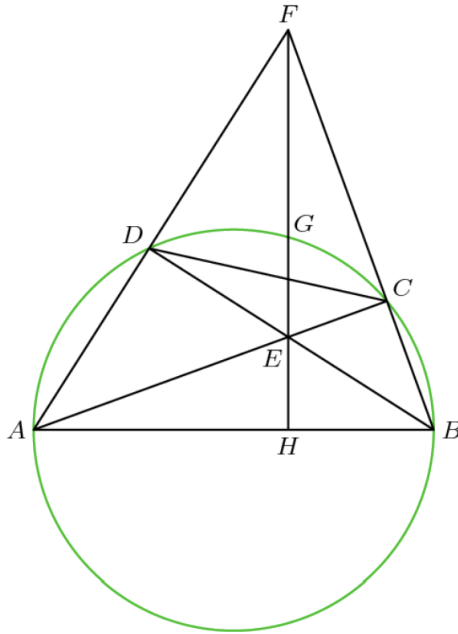
*Proof.* Note that  $ABED$  and  $AFCD$  are cyclic quadrilaterals. It follows that  $ABC \sim AEF$  since  $\angle ABD = \angle AED$  and  $\angle AFD = \angle ACD$ . Similarly, we can show that  $ABM \sim AEN$  since

$$\frac{AB}{AE} = \frac{BC}{EF} = \frac{2BN}{2EM} = \frac{BN}{EM}.$$

Therefore,  $\angle AND = \angle AMD$  and it follows that  $ANMD$  is cyclic. Therefore  $\angle ANM = 180 - \angle AD = 90$ , as desired.  $\square$

#### Problem 4

Let  $ABCD$  be a convex quadrilateral inscribed in a semicircle with diameter  $AB$ . The lines  $AC$  and  $BD$  intersect at  $E$  and the lines  $AD$  and  $BC$  meet at  $F$ . The line  $EF$  meets the semicircle at  $G$  and  $AB$  at  $H$ . Prove that  $E$  is the midpoint of  $GH$  if and only if  $G$  is the midpoint of the line segment  $FH$ .



*Proof.* Note that  $\angle ADB = \angle ACB = 90$ . It follows that  $E$  is the orthocenter of  $FAB$  and  $\angle FAH = 90$ . We obtain many similar triangles, with one notable one being  $\triangle AEH \sim FBH$  which gives the relation

$$HE \cdot HF = HA \cdot HB.$$

However, note that

$$\text{Pow}(H) = HG^2 = HA \cdot HB,$$

so it follows that

$$\frac{HG}{HF} = \frac{HE}{HG},$$

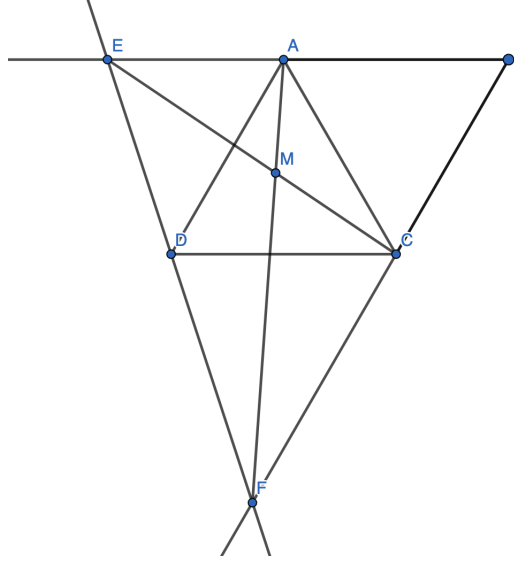
which proves the result.  $\square$

## §2 Homework 1: Fundamentals of Geometry

### §2.1 Problem 1

#### Problem 5

Let  $ABCD$  be a quadrilateral such that all sides have equal length and  $\angle ABC = 60^\circ$ . Let  $k$  be a line through  $D$  and not intersecting the quadrilateral. Let  $E$  and  $F$  be the intersection of  $k$  with lines  $AB$  and  $BC$  respectively. Let  $M$  be the point of intersection of  $CE$  and  $AF$ . Prove that  $CA^2 = CM \cdot CE$ .



*Proof.* It suffices to show that  $\triangle MCA \sim \triangle ACE$ . We already have that  $\angle MCA = \angle ACE$  so we finish by showing that  $\angle CAM = \angle CEA$ .

We first claim that  $AD \parallel CB$  and  $AB \parallel DC$ . Note that  $AB = BC$  and  $\angle ABC = 60^\circ$  so it follows that  $\triangle ABC$  is equilateral. Hence  $AB = CB = CA$ . But note that  $AD = DC = AB = CA$ , so it follows that  $\triangle ADC$  is also equilateral. Hence  $\angle DAB = 120^\circ$  and  $\angle ADC = \angle ABC = 60^\circ$  showing that  $AD \parallel CB$  and  $AB \parallel DC$ .

Note that  $\angle EAC = \angle ACE = 120^\circ$ , so it suffices to show that  $\frac{EA}{AC} = \frac{AC}{CF}$ , since it follows that  $\triangle EAC \sim \triangle ACF$  and  $\angle CAM = \angle CEA$ . Furthermore, we have that  $\triangle DCF \sim \triangle EAD$  since  $\angle EAD = \angle DCA$  and  $\angle AED = \angle CDF$ . It follows that

$$\frac{EA}{AC} = \frac{DA}{FC} = \frac{AC}{FC},$$

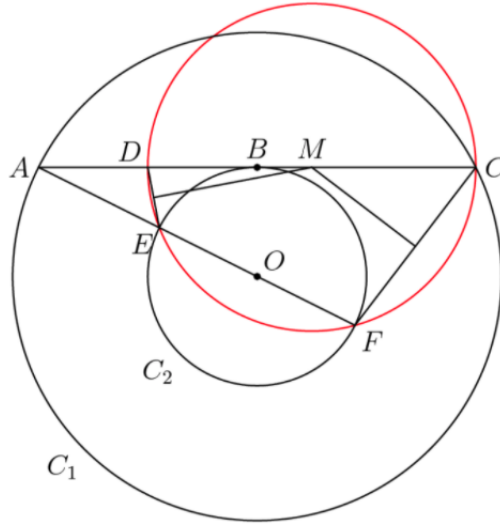
since  $DA = AC$ , which completes the proof.  $\square$



## §2.2 Problem 2

### Problem 6

Let  $C_1$  and  $C_2$  be concentric circles with  $C_2$  inside  $C_1$ . Let  $A$  and  $C$  be on  $C_1$  such that  $AC$  is tangent to  $C_2$  at  $B$ . Let  $D$  be the midpoint of  $AB$ . A line passing through  $A$  meets  $C_2$  at  $E$  and  $F$  such that the perpendicular bisectors of  $DE$  and  $CF$  meet at a point  $M$  on a segment  $DC$ . Find the ratio  $AM/MC$ .



*Proof.* Note that  $\text{Pow}_{C_2}(A) = AB^2 = AE \cdot AF$ . Furthermore, since  $AD = \frac{1}{2}AB$  and  $AC = 2AB$ , it follows that

$$AD \cdot AC = \frac{1}{2}AB \cdot 2AB = AB^2 = AE \cdot AF.$$

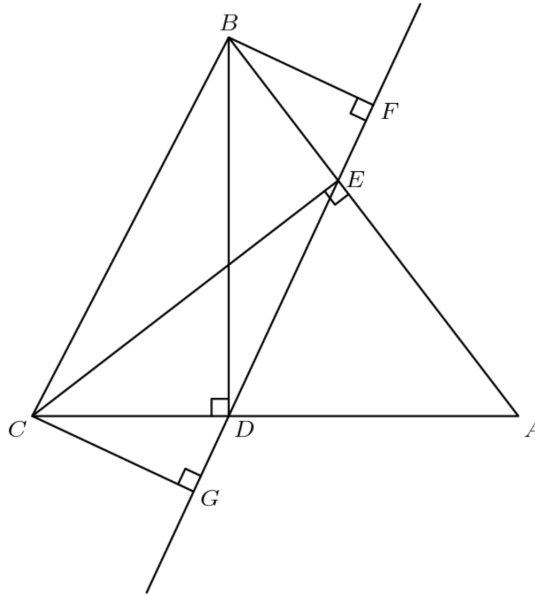
Hence,  $DCFE$  is a cyclic quadrilateral. Furthermore, Since  $M$  is the perpendicular bisector of the chords  $DE$  and  $CF$ , it follows that  $M$  is the center of the corresponding circle. Hence  $M$  is the midpoint of  $DC$ . It follows that  $AM = \frac{5}{8}AC$  and  $MC = \frac{3}{8}AC$  so  $AM/MC = 5/3$ .  $\square$

## §3 Week 2: Fundamentals, Continued

### §3.1 Warm-up Problem

#### Problem 7

$\triangle ABC$  is acute;  $BD$  and  $CE$  are altitudes. Points  $F$  and  $G$  are the feet of perpendiculars  $BF$  and  $CG$  to line  $DE$ . Prove that  $EF = DG$ .



We present two proofs for the problem, though there are many. The first uses basic facts about cyclic quadrilaterals and similar triangles.

*Proof.* Note that  $BEDC$  is a cyclic quadrilateral. Note that  $\angle BCD = \angle BEF = 180 - \angle BED$ . Hence,  $\triangle BEF \sim \triangle BCD$ . Similarly,  $\triangle CGD \sim \triangle CEB$ . Therefore,

$$\frac{EF}{CD} = \frac{BE}{BC} = \frac{DG}{CD},$$

so it follows that  $EF = DG$ . □

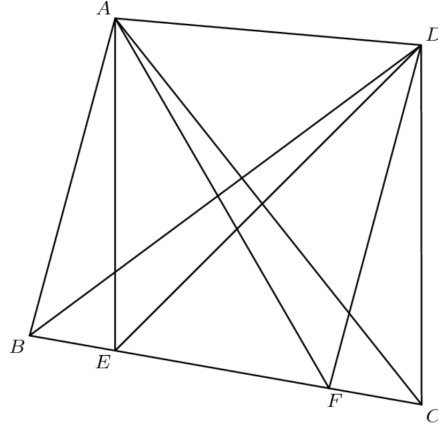
The second proof uses properties of projections.

*Proof.* The midpoint of  $BC$  is the circumcenter of circle  $BCDE$ , so it projects to the midpoint of  $DE$ . On the other hand, the midpoint of  $BC$  projects to the midpoint of  $FG$ , since  $BFGC$  is a trapezoid. It follows that  $DE$  and  $GF$  have the same midpoint, so  $DG = EF$ . □

### §3.2 Russia

#### Problem 8 (Russia)

Points  $E$  and  $F$  are on side  $BC$  of a convex quadrilateral  $ABCD$  with  $BE < BF$ . Given that  $\angle BAE = \angle CDF$  and  $\angle EAF = \angle FDE$ , prove that  $\angle FAC = \angle EDB$ .



*Proof.* Note that  $\angle EAF = \angle FDE$  implies that  $AEFD$  is cyclic. It suffices to show that  $ABCD$  is cyclic. Note that  $\angle ADC = \angle ADF + \angle FDC$ , so we have

$$\angle ABC + \angle ADC = \angle ABC + \angle ADF + \angle FDC.$$

Then,  $\angle ABC = \angle AEF - \angle BAE$ , so it follows that

$$\begin{aligned} \angle ABC + \angle ADC &= \angle ABC + \angle ADF + \angle FDC \\ &= \angle AEF - \angle BAE + \angle ADF + \angle FDC \\ &= \angle AEF + \angle ADF \\ &= 180, \end{aligned}$$

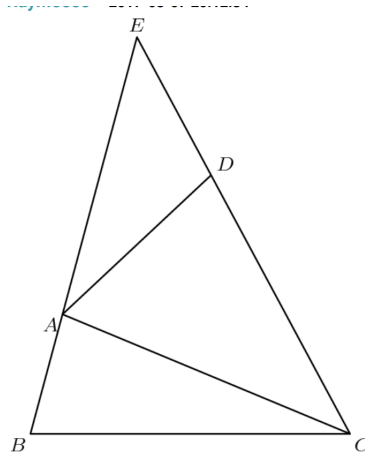
which shows that  $ABCD$  is cyclic, as desired.  $\square$

### §3.3 Bulgaria

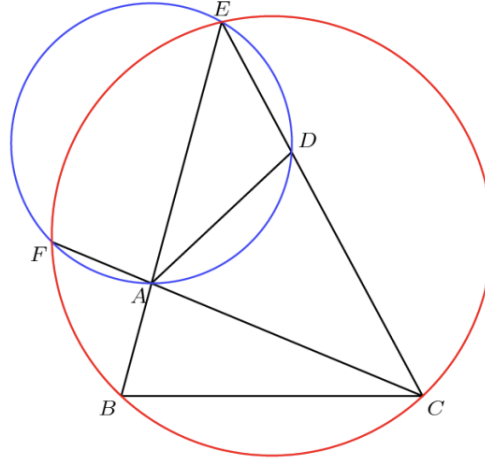
#### Problem 9 (Bulgaria)

A convex quadrilateral  $ABCD$  is given for which  $\angle ABC + \angle BCD < 180$ .  $AB$  and  $CD$  extended meet at  $E$ . Prove that  $\angle ABC = \angle ADC$  if and only if  $AC^2 = CD \cdot CE - AB \cdot AE$ .

**Remark 3.1.** After drawing the diagram for the problem, one should check that it corresponds to the solution in the problem. One can enter a trap proceeding without checking for this problem specifically.



*Proof.* Let  $\omega_1$  be the circumcircle of  $ADE$  and  $\omega_2$  be the circumcircle of  $EBC$ . Note that  $\text{Pow}_{\omega_1}(C) = CD \cdot CE$  and  $\text{Pow}_{\omega_2}(A) = AB \cdot AE$ . Extend  $CA$  to  $\omega_2$  and label the intersection  $F$ .



Assuming that  $AC^2 = CD \cdot CE - AB \cdot AE = CA \cdot CF - AB \cdot AE$ , it follows that

$$AC(CF - AC) = AC \cdot AF = AB \cdot AE,$$

so from the converse of the Power of a Point, it follows that  $F \in \omega_2$ .

Finally,

$$\angle ABC = \angle AFE = 180 - \angle ADE = \angle ADC.$$

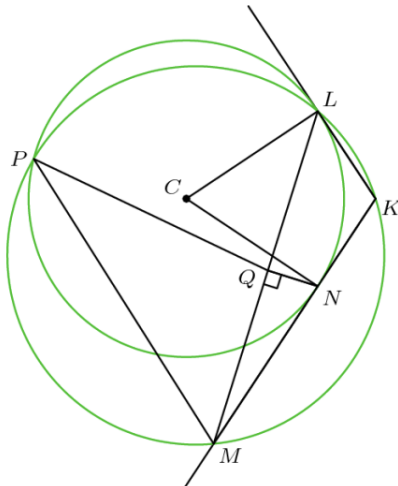
We can go back and show that each of the steps are reversible, but this is left as an exercise.  $\square$

### §3.4 Iran

**Warning:** This is a very difficult problem.

#### Problem 10 (Iran)

Point  $K$  is outside circle  $C$  and points  $L$  and  $N$  are on  $C$  such that  $KL$  and  $KN$  are tangent to  $C$ . Let  $M$  be on ray  $KN$  beyond  $N$ , and let  $P$  be the second intersection of the circumcircle of  $KLM$  and  $C$ . Let  $Q$  be the foot of the perpendicular from  $N$  to  $ML$ . Prove that  $\angle MPQ = 2\angle KML$ .



*Proof.* Let  $S$  be the intersection of  $QM$  and circle  $C$ . We show that  $PS$  bisects  $P$ . Let  $T$  be the intersection of  $PM$  and the circumcircle of  $PNL$ .  $\square$