Math 218a Lecture Notes, Fall 2020

Probability Theory

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§1.1 Introduction

Consider a **random experiment** - this involves a state space Ω and some "probability" on it. The outcome of an experiment would be $\omega \in \Omega$.

Example 1.1 (Fair Coin Toss)
$\Omega = \{0, 1\}, P(0) = 1/2, P(1) = 1/2$ models a fair coin toss. The outcomes are $\omega \in \Omega, \omega = 0$ or $\omega = 1$.

Example 1.2 (Continuous State Space)
$\Omega = [0, 1]$, X is the outcome of a random experiment. Suppose X is uniformly distributed random variable. $P(X \in [0, \frac{1}{2}]) = 1/2$. Take $A = \mathbb{Q} \cap [0, 1]$. $P(x \in A) = 0$, since A has no "volume". Similarly, taking $A_1 = \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$, then $P(x \in A_1) = 1 - P(x \in A) = 1$. Finally, take $E \subset [0, 1]$. $P(x \in E) =$"volume" of E.

The issue: we need to define some notion of volume. Some properties we would like are the following:

- Translation Invariance
- Countable Additivity: $A_1, A_2, \ldots$ disjoint with $A = \bigcup A_i$, then $P(A) = \sum_{i=1}^{\infty} P(A_i)$.

§1.2 Nonmeasurable Sets

Take $I = [-1, 2]$, and define $x \sim y$ iff $x - y \in \mathbb{Q}$. [Exercise: check that $\sim$ is an equivalence relation.] This decomposes I into equivalence classes $I/\sim$. Note that the equivalence classes are countable, since any class is $x + A, A \subset Q$.

For each equivalence class B, pick $x_B \in B \cap [0, 1]$. Define $E = \{x_B\}$ over all the equivalence classes. Note that x_B is a representative of B in E, so $B = \{x_b + q : x_b + q \in I, q \in \mathbb{Q}\}$.

Now, consider the set $[0, 1] \subset \bigcup_{q \in [-1,1]} E + q \subset [-1, 2]$. Equality doesn't hold, because there can be B s. t. x_b is close to 0. Then $E + (\mathbb{Q} \cap [-1, 1])$ will only recover elements of B near 1 and will not go up to 2.

Proposition 1.3
We claim that $E + q$ are disjoint for different values of q.

Proof. Suppose $E + q_1 \cap E + q_2 \neq \emptyset$ for some q_1, q_2. Then, there exists $x, y \in E$ such that $x + q_1 = y + q_2$. This implies that $x - y = q_2 - q_1 \in \mathbb{Q}$, so $x \sim y$, but by definition, there is exactly one member of each equivalence class in E. $\square$

The big question: What is $P(E)$? Suppose $P(E) > 0$. Then $\bigcup_{q \in [-1,1]} E + Q \subset [-1, 2]$ and $P(E + q_1) = P(E + q_2) = P(E)$ for all q_1, q_2. Furthermore, by countable additivity,

$$1 \geq P(\bigcup_{q \in [-1,1] E+q}) = \sum_{q \in [-1,1]} P(E + q) = \infty \cdot P(E).$$

This would imply that $P(E) = 0$. However,

$$[0,1] \subseteq \bigcup_{q \in [-1,1]} E + q \Rightarrow P([0,1]) = 1/3 \leq \sum_{q \in [-1,1]} P(E+q) = 0.$$

Hence, $P(E)$ cannot be defined.

The issue is the step where we pick x_B, since we need to pick x_B from uncountably many points, which assumes the axiom of choice. It was proved by Robert M. Solovay that all models of set theory excluding the axiom of choice have the property that all sets are Lebesgue measurable.

Our goal is thus to come up with a general framework where things can be consistently defined for a large class of sets.

§1.3 Measure Theory Beginnings

For the definitions, we take Ω to be the state space.

Definition 1.4 (Sigma-Algebra)**.** Suppose Σ follows the following properties:

1. $\emptyset \in \Sigma$
2. $A \in \Sigma \Rightarrow A^c \in \Sigma$
3. $A_1, A_2, \cdots \in \Sigma$, then $\bigcup A_i \in \Sigma$

Note that 2 and 3 imply 1 since $(A \cup A^c)^c = \emptyset$. Then Σ is a sigma-algebra.

Note that we also have countable intersections(this is an easy exercise).