Complex Analysis Notes
STEIN

# 1. Preliminaries to Complex Analysis

## 1.1.1 Identities

- $|z+w| \le |z| + |w|$
- $||z| - |w|| \le |z-w|$
- $|Re(z)| \le |z|$, $|Im(z)| \le |z|$
- $Re(z) = \frac{z+\bar{z}}{2}$ $Im(z) = \frac{z-\bar{z}}{2i}$
- $|z|^2 = z\bar{z}$, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

## 1.1.2. Convergence

Def. $\{z_i\} \to w \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - w| = 0, \quad w = \lim_{n\to\infty} z_n$$

Exercise. $\{z_n\} \to w \iff Re(z_n) \to Re(w)$, $Im(z_n) \to Im(w)$

Proof.

$|Re(z_n) - Re(w)| = |Re(z_n - w)| \le |z_n - w| \to 0$

$|Im(z_n) - Im(w)| = |Im(z_n - w)| \le |z_n - w| \to 0$

$|z_n - w| = |Re(z_n) - Re(w) + Im(z_n) - Im(w)| \le |Re(z_n) - Re(w)| + |Im(z_n) - Im(w)| \to 0.$ ∎

Def. $\{z_n\}$ is Cauchy if $|z_n - z_m| \to 0$ as $n, m \to \infty$.

Theorem. $\mathbb{C}$ is complete.

## 1.1.3. Sets in Complex Plane

- $z_0 \in \mathbb{C}$, $r > 0$, $D_r(z_0)$ is the open disk centered at $z_0$, radius $r$.

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

$$\overline{D_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \le r\}$$

$$\partial \overline{D_r(z_0)} = C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

$$\mathbb{D} = D_1(0).$$

- $\Omega^\circ = \Omega \setminus \partial\Omega$.
- $\Omega$ is bounded if $\exists M > 0$ w/ $|z| < M$ for $z \in \Omega$.

$$diam(\Omega) = \sup_{z, w \in \Omega} |z - w|$$

- Theorem. $\Omega \subset \mathbb{C}$ is compact iff every sequence $\{z_n\} \subset \Omega$ has a subsequence converging to a point in $\Omega$.

· Theorem. $\Omega$ is compact if every open covering has a finite subcovering.

Prop. If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a sequence of non-empty compact sets in $\mathbb{C}$ w/ $\operatorname{diam}(\Omega_n) \longrightarrow 0$, there exists a unique point $w \in \mathbb{C}$ w/ $w \in \Omega_n$ for all $n$.

Proof. Choose $z_n \in$ each $\Omega_n$. $\operatorname{diam}(\Omega_n) \to 0$ says $\{z_n\}$ is Cauchy, so has a limit $w$. $\Omega_n$ is compact, so $w \in \Omega_n$ $\forall n$. If $w'$ also satisfies above w/ $|w-w'|>0$, then $\operatorname{diam}(\Omega_n) \not\to 0$. ∎

1.2 Functions in Complex Space

Def. $f$ is continuous at $z_0$ if $\forall \varepsilon>0$ $\exists \delta>0$ s.t. for $z \in \Omega$ w/ $|z-z_0|<\delta$, $|f(z)-f(z_0)|<\varepsilon$.

Equivalently, for $\{z_n\} \to z_0$, $f(z_n) \longrightarrow f(z_0)$.

· If $f$ is continuous, $z \longmapsto |f(z)|$ is continuous

Theorem. Continuous function on a compact set is bounded, attains min, max.

1.2.2 Holomorphic Functions.

Def. Let $\Omega$ be open of $f$ on $\Omega$. $f$ is <u>holomorphic</u> at $z_0 \in \Omega$ if

$$\frac{f(z_0+h)-f(z_0)}{h}$$

converges ~~when~~ $h \to 0$. We define

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Def. $f$ is holomorphic on $\Omega$ if holomorphic for each $z_0 \in \Omega$.

Def. " closed $C \subseteq \mathbb{C}$ if $f$ is holomorphic on some open $\Omega \supset C$.

Def. $f$ is entire if holomorphic on all of $\mathbb{C}$.

Ex. $f(z)=\bar{z}$ is not holomorphic

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\bar{h}}{h}$$

which has no limit.

Theorem. $f$ is holomorphic at $z_0 \in \Omega \Leftrightarrow \exists a \in \mathbb{C}$ w/

$$f(z_0+h)-f(z_0)-ah = h\,\psi(h)$$

where $\psi$ is defined for small $h$, $\lim_{h\to 0}\psi(h)=0$.

Complex Functions as Mappings

Warning: Real derivative does not imply holomorphic. See $f(z) = \bar{z}$.

$F(x,y) = (u(x,y), v(x,y))$ is diff. at $P_0 = (x_0, x_0)$ if $\exists\, J \in \mathrm{Hom}(\mathbb{R}^2; \mathbb{R}^2)$ w/

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \longrightarrow 0 \quad \text{as } |H| \longrightarrow 0, \quad H \in \mathbb{R}^2$$

Equivalently,

$$F(P_0 + H) - F(P_0) = J(H) + |H|\, \Psi(H), \qquad |\Psi(H)| \xrightarrow{|H| \to 0} 0 .$$

If $F$ is diff. we have

$$J = J_F(x,y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}.$$

Suppose $h$ is real. $z = x + iy$, $z_0 = x_0 + iy_0$, $f(z) = f(x,y)$

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1}$$

$$= \frac{\partial f}{\partial x}(z_0).$$

For purely imaginary $h$, $h = ih_2$,

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2}$$

$$= \frac{1}{i} \frac{\partial f}{\partial y}(z_0).$$

Hence, if $f$ is holomorphic,

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

W/ $f = u + iv$, $1/i = -i$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Define $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Prop 2.3. If $f$ is holomorphic at $z_0$, then
$$\frac{\partial f}{\partial \bar z}(z_0) = 0 \qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

If we rewrite $F(x,y) = f(x)$, then $F$ is differentiable in the sense of reals and $\det J_F(x_0,y_0) = |f'(z_0)|^2$.

Proof. Taking Re, Im, C-R gives $\partial f/\partial \bar z = 0$. Moreover
$$f'(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) = \frac{\partial f}{\partial z}(z_0).$$
~~and C-R gives ∂~~

and C-R gives $\partial f/\partial z = 2\partial u/\partial z$.

To prove $F$ is differentiable, STS if $H = (h_1, h_2)$ $h = h_1 + ih_2$, then
$$J_F(x_0,y_0)(H) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) = f'(z_0)h.$$

Thm. Suppose $f = u + iv$ on $\Omega$ open. If $u, v$ are continuously differentiable and satisfy C-R on $\Omega$, then $f$ is holomorphic on $\Omega$ and $f'(z) = \partial f/\partial z$.

Proof.
$$u(x+h_1, y+h_2) - u(x,y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h).$$
$$v(x+h_1, y+h_2) - v(x,y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h),$$
where $\psi_i(h) \xrightarrow{h\to 0} 0$, $h = h_1 + ih_2$. By C-R,
$$f(z+h) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi(h)$$
w/ $\psi(h) = \psi_1(h) + \psi_2(h)$, as $|h| \to 0$. Hence $f$ is holomorphic and
$$f'(z) = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

1.2.3 Power Series

- For $z \in \mathbb{C}$,
$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{converges absolutely.}$$

$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!} \quad \text{so} \quad \sum |z|^n/n! = e^{|z|} < \infty.$$

Theorem. Given $\sum_{n=0}^{\infty} a_n z^n$, $\exists\ 0 \le R \le \infty$ s.t.
(i) if $|z| < R$, the series converges absolutely
(ii) if $|z| > R$, the series diverges.
Moreover,
$$1/R = \limsup |a_n|^{1/n}.$$
Proof. Let $L := 1/R$. Suppose $L \neq 0, \infty$ (these case are easy).
If $|z| < R$, choose $\varepsilon > 0$ s.t.
$$(L+\varepsilon)|z| = r < 1.$$
$|a_n|^{1/n} \le L+\varepsilon$ $\forall$ large $n$, so
$$|a_n||z|^n \le \{(L+\varepsilon)|z|\}^n = r^n.$$
So it converges by comparison w/ the geometric series. ∎

- $$\cos z = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

Theorem. The $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on its disk of convergence.
Furthermore
$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1},$$
and $f'$ has the same radius of convergence.
Proof.
$$\limsup |n a_n|^{1/n} = \limsup |a|^{1/n} \quad \text{since } \lim_{n\to\infty} n^{1/n} = 1.$$

Let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Let $R$ be the radius of convergence of $f$, suppose $|z_0| < r < R$. Let
$$f(z) = S_N(z) + E_N(z)$$
$$S_N(z) = \sum_{n=0}^{N} a_n z^n \qquad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

If $h$ is s.t. $|z_0+h|<r$,

$$\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right)$$
$$+ (S_N'(z_0)-g(z_0)) + \left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right).$$

$$\Rightarrow \left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| \le \sum_{N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right|$$
$$\le \sum_{N+1}^{\infty} |a_n|\, n r^{n-1}.$$

Assuming $|z_0|<r$
$|z_0+h|<r$
$(a^n-b^n) = (a-b)(a^{n-1}+\dots+b^{n-1})$

The RHS is tail of geometric, since $g$ converges absolutely on the $|z|<R$.
Hence given $\varepsilon>0$, $\exists N_1$ w/ $N>N_1 \Rightarrow$

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| < \varepsilon.$$

Since $\lim_{N\to\infty} S_N'(z_0) = g(z_0)$ $\exists N_2$ w/ $N>N_2 \Rightarrow |S_N'(z_0)-g(z_0)|<\varepsilon$.

For $N>\max(N_1,N_2)$, $\exists\delta>0$ s.t. $|h|<\delta \Rightarrow$

$$\left|\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right| < \varepsilon$$

$$\Rightarrow \left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| < 3\varepsilon.$$

Def. $f$ on open $\Omega$ is analytic at $z_0\in\Omega$ if $\exists$ power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \forall z \in D_r(z_0)$$

w/ positive radius of convergence.

1.3 Integration Along Curves

- $z(t)$ maps $[a,b] \subset \mathbb{R}$ to $\mathbb{C}$, is "parameterized curve".

Smooth if $z'(t)$ exists and is continuous on $[a,b]$, $z'(t) \neq 0$ for $t \in [a,b]$.

$$z'(a) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{z(a+h) - z(a)}{h} \qquad z'(b) = \lim_{\substack{h \to 0 \\ h < 0}} \frac{z(b+h) - z(b)}{h}.$$

- Two parameterizations are equivalent if there is a continuously differentiable bijection between them. $s \mapsto t(s)$ $[c,d] \to [a,b]$ w/ $t'(s) > 0$, $\tilde{z}(s) = z(t(s))$
- The set of equivalent parameterizations determines a smooth curve $\gamma \subset \mathbb{C}$.
- Positive orientation is CCW.
- Given $\gamma$ in $\mathbb{C}$ param by $z : [a,b] \to \mathbb{C}$ and $f$ continuous on $\gamma$, the integral of $f$ along $\gamma$ is

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

If $\gamma$ is piecewise smooth,

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))\,z'(t)\,dt$$

$$\text{length}(\gamma) = \int_a^b |z'(t)|\,dt.$$

- Properties
  - $\alpha, \beta \in \mathbb{C}$,
  $$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int g(z)\,dz$$
  - if $\gamma^-$ is negatively oriented $\gamma$,
  $$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$
  -
  $$\left| \int_\gamma f(z)\,dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma).$$

Def. A primitive for $f$ on $\Omega$ is a function $F$ that is holomorphic on $\Omega$ s.t. $F'(z) = f(z)$ for all $z \in \Omega$.

• Thm. If $f$ has primitive on $\Omega$, $\gamma$ is a curve on $w_1$, $w_2$, then

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1).$$

Corollary. If $\gamma$ is a closed curve, $f$ is continuous w/ a primitive

$$\int_\gamma f(z)\,dz = 0.$$

Corollary. If $f$ is holomorphic in $\Omega$ and $f' = 0$, $f$ is constant.