Riemannian Geometry

with an introduction to Optimization on Manifolds

written by Vishal Raman

We present detailed expository notes on Riemannian Geometry mainly following the treatment from *Lee, Riemannian Manifolds*, *Do Carmo, Riemannian Geometry*. We finish with an introduction to optimization algorithms on smooth manifolds, following the treatment from Boumal. Any typos or mistakes are my own - please redirect them to my email.

Contents

§1 Review of Smooth Manifolds

§1.1 Tensors on a Vector Space

Let V be a finite-dimensional vector space. Recall the dual space V^*, the set of covectors on V. We denote the natural pairing $V^* \times V \to \mathbb{R}$ by the notation $(\omega, X) \mapsto \omega(X)$ for $\omega \in V^*, X \in V$.

Definition 1.1 (Covariant Tensor)**.** A covariant k-tensor on V is a multilinear map

$$F : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

The space of covariant k-tensors on V is denoted $T^k(V)$.

Definition 1.2 (Contravariant Tensor)**.** A contravariant k-tensor on V is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \to \mathbb{R}.$$

The space of contravariant k-tensors on V is denoted $T_k(V)$.

Definition 1.3 (Mixed Tensor)**.** A mixed $\binom{k}{l}$-tensor on V is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ times}} \times \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

The space of mixed $\binom{k}{l}$-tensors on V is denoted $T^k_l(V)$.

Some identifications:

- $T_0^k(V) = T^k(V)$, $T_0^k(V) = T_0^k(V)$,.
- $T^1(V) = V^*$, $T_1(V) = V^{**} = V$.
- $T^0(V) = \mathbb{R}$.
- $T_1^1(V) = \mathrm{End}(V)$, the space of linear endomorphisms of V.

The last identification is a consequence of the following lemma:

Lemma 1.4. *Let V be a finite-dimensional vector space. There is a natural isomorphism between $T_{l+1}^k(V)$ and the space of multilinear maps*

$$\underbrace{V^* \times \cdots \times V^*}_{l} \times \underbrace{V \times \cdots \times V}_{k} \to V.$$

Definition 1.5 (Tensor Product)**.** If $F \in T_l^k(V)$ and $G \in T_q^p(V)$, the tensor $F \otimes G \in T_{l+q}^{k+p}(V)$ is defined by

$$F \otimes G(\omega^1, \ldots, \omega^{l+q}, X_1, \ldots, X_{k+p}) = F(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k) G(\omega^{l+1}, \ldots, \omega^{l+q}, X_{k+1}, \ldots, X_{k+p}).$$

If $(E_1, \ldots, E_n)$ is a basis for V and $(\varphi^1, \ldots, \varphi^n)$ denotes the corresponding dual basis for V^*(defined by $\varphi^i(E_j) = \delta_{ij}$), a basis for $T_l^k(V)$ is given by the set of tensors of the form

$$E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}.$$

Definition 1.6 (Trace)**.** Using Lemma 1.4, we can define the *trace operator* given by $\mathrm{tr} : T_{l+1}^{k+1}(V) \to T_l^k(V)$ where $(\mathrm{tr}\, F)(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l)$ is the trace of the tensor

$$F(\omega^1, \ldots, \omega^k, \cdot, v_1, \ldots, v_l, \cdot) \in T_1^1(V).$$

§1.2 Vector Bundles and Vector Fields

§1.3 Tensor Fields

§1.4 Lie Theory

§2 Riemannian Metrics

Definition 2.1 (Riemannian Metric)**.** Let M be a smooth manifold. A *Riemannian metric* on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ whose value g_p at each $p \in M$ is an inner product on T_pM; i. e., g is a symmetric 2-tensor field that is positive definite in the sense that $g_p(v, v) \geq 0$ for each $p \in M$ and each $v \in T_pM$, with equality if and only if $v = 0$.

Definition 2.2 (Riemannian Manifold)**.** A *Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric on M.

Proposition 2.3. *Every smooth manifold admits a Riemannian metric.*

Proof. Let M^n be a smooth manifold with a corresponding covering of smooth charts $(U_\alpha, \varphi_\alpha)$. In each of the coordinate domains, there is a Riemannian metric $g_\alpha = \varphi_\alpha^* \overline{g}$, where $\overline{g} = \delta_{ij} dx^i dx^j$ is the Euclidean metric on $\mathbb{R}^n$. Now, if we choose $\{\psi_\alpha\}$ to be a smooth partition of unity subordinate to $\{U_\alpha\}$, then, we can define $g = \sum_\alpha \varphi_\alpha g_\alpha$, where each term is interpreted to be zero outside the support of φ_α.

By local finiteness, there are only finitely many terms in a neighborhood of each point, so this defines a smooth tensor field. It is also symmetric by construction. Finally, if $v \in T_pM$ is nonzero,

$$g_p(v,v) = \sum_{\alpha} \psi_\alpha(p) g_\alpha|_p(v,v) > 0$$

since $g_\alpha|_p(v,v) > 0$ and at least one of the $\psi_\alpha(p) > 0$. $\square$

We can similarly define a Riemannian manifold with boundary when M is a smooth manifold with boundary.

We will use the notation $\langle v, w \rangle_g = g_p(v,w)$ since g_p is an inner product on T_pM. This motivates the notion of angles, lengths, and orthogonality.

§2.1 Isometries

Suppose (M,g) and $(\tilde{M},\tilde{g})$ are Riemannian manifolds.

Definition 2.4 (Isometry)**.** An *isometry* from (M,g) to $(\tilde{M},\tilde{g})$ is a diffeomorphism $\varphi : M \to \tilde{M}$ such that $\varphi^*\tilde{g} = g$. Equivalently, this is equivalent to the requirement that φ is a bijection and $d\varphi_p : T_pM \to T_{\varphi(p)}\tilde{M}$ is a linear isometry.

We denote the $\mathrm{Iso}(M,g)$ as the isometry group of (M,g) under composition.

List of Definitions and Theorems