Applications of Fourier Analysis

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We present several theorems and applications of Fourier Analysis in different sub-fields.

§1 Kronecker's Theorem

We begin with the statement of the theorem.

Theorem 1.1 (Kronecker)
Suppose that $k \geq 1$ and $\{x_1, x_2, \ldots, x_k, 2\pi\}$ is linearly independent over $\mathbb{Q}$. Then for any $\epsilon > 0$ and any $z_1, \ldots, z_k \in S^1 \subset \mathbb{C}$, there exists $\ell \in \mathbb{Z}$ so that for all $j \leq k$,

$$|e^{i\ell x_j} - z_j| < \epsilon.$$

Before proving the theorem, we require 2 key lemmas.

Lemma 1.2
If $f \in L^2(\mathbb{T}^d)$ and $\widehat{f} \in \ell^1(\mathbb{Z}^d)$, then the Fourier series of f converges uniformly to a continuous function, which equals f almost everywhere.

Proof. Let $S_N f(x) = \sum_{|n| \leq N} \widehat{f}(n) e^{in \cdot x}$, for $x \in \mathbb{T}^d$. We show that

$$\|S_N f(x) - f\|_\infty \xrightarrow{N \to \infty} 0.$$

$\square$