Jordan Curve Theorem

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We used the Jordan Curve Theorem in order to present the Caratheodory Extension theorem for conformal maps. The following is a proof of the theorem as a corollary of a more general theorem using Homology and the Mayer-Vietoris Theorem. Any mistakes and typos are my own - kindly direct them to my inbox.

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§1 Statement of Theorem

Theorem 1.1

Let $n, k, i \in \mathbb{N}$ be arbitrary proved that $k \leq n-1$.

- Suppose $h : \mathbb{D}^k \to \mathbb{S}^n$ is a topological embedding. Then

$$\widetilde{H}_i(\mathbb{S}^n \setminus h(\mathbb{D}^k); \mathbb{Z}) = 0.$$

- If $h : \mathbb{S}^k \to \mathbb{S}^n$ is a topological embedding, then

$$\widetilde{H}_i(\mathbb{S}^n \setminus h(\mathbb{S}^k); \mathbb{Z}) = \widetilde{H}_i(\mathbb{S}^{n-k-1}; \mathbb{Z}).$$

Corollary 1.2 (Jordan Curve Theorem)

Taking $n = 2$, $k = 1$ in the above theorem, $\widetilde{H}_0(\mathbb{S}^2 \setminus h(\mathbb{S}^1)) = \widetilde{H}_0(\mathbb{S}^0; \mathbb{Z}) \cong \mathbb{Z}$, so $H_0(\mathbb{S}^2 \setminus h(\mathbb{S}^1)) = \mathbb{Z}^2$, which implies that $\mathbb{S}^2 \setminus h(\mathbb{S}^1)$ has two path-connected components.

§2 Proof of the Theorem

Proof. We proceed by induction on k: the $k = 0$ case is clear. Now, consider $\mathbb{D}^k \cong [0,1]^k$ and setting $I = [0,1]$, define $A_+ = \mathbb{S}^n \setminus (I^{k-1} \times [0,1/2])$ and $A_- = \mathbb{S}^n \setminus (I^{k-1} \times [1/2,1])$. It is easy to see that $A_+ \cup A_- = \mathbb{S}^n \setminus h(I^k)$ and $A_+ \cup A_- = \mathbb{S}^n \setminus (I^{k-1})$. By induction, we know that the homologies of $A_+ \cup A_-$ are zero.

By Mayer-Vietoris, we have the sequence We have the sequence

$$\cdots \to \widetilde{H}_{i+1}(A_+ \cup A_-) \to \widetilde{H}_i(A_+ \cap A_-) \to H_i(A_+) \oplus H_i(A_-) \to \widetilde{H}_i(A_+ \cup A_-) \to \ldots$$

Since $\widetilde{H}_{i+1}(A_+ \cup A_-) = \widetilde{H}_i(A_+ \cup A_-) = 0$, it follows that $\widetilde{H}_i(\mathbb{S}^n \setminus h(I^k)) \cong \widetilde{H}_i(A_+) \oplus \widetilde{H}_i(A_-)$. It suffices to check that one of these is zero. Suppose that $\widetilde{H_i}(A_+ \cap A_i) \neq 0$. Then, one of $\widetilde{H}_i(A_+)$ or $\widetilde{H}_i(A_-)$ is nonzero. Now, suppose $\alpha \in \widetilde{H}_i(\mathbb{S}^n \setminus h(I^k))$ is not a boundary. Then it is not a boundary in A_+ or A_-. We use the same principle for further subdivisions of the interval $[0,1]$, as in the proof of the Mayer-Vietoris Theorem. By iteration, we obtain a nested sequence of intervals

$$I_1 \supset I_2 \supset \cdots \supset I_j \supset \ldots$$

and it follows that there exists $p \in \bigcup I_j$ with α not a boundary in $\mathbb{S}^n \setminus h(I^{k-1} \times I_j)$. However, we must have α as a boundary in $\mathbb{S}^n \setminus h(I^{k-1} \times \{p\})$, a contradiction. So we must have α as a boundary in some finite step. This concludes the proof of the first statement.

Now, we prove the second statement. We again proceed by induction on k. If $k = 0$ then $\widetilde{H}_i(\mathbb{S}^n \setminus \mathbb{S}^0) \cong \widetilde{H}_i(\mathbb{R}^n \setminus \{p\}) \cong \widetilde{H}_i(\mathbb{S}^{n-1})$. Decompose $\mathbb{S}^k = \mathbb{D}^k_+ \cup D^k_-$ as the ϵ-neighborhood of the upper and lower hemisphere. We denote the decomposition of the subsets as $B_+ = \mathbb{S}^n \setminus h(\mathbb{D}^k_+)$ and $B_- = \mathbb{S}^n \setminus h(\mathbb{D}^k_-)$.

By the first part, we know that $\widetilde{H}_i(B_\pm) = 0$. By Mayer-Vietoris, we know that $B_+ \cap B_- \sim \mathbb{S}^n \setminus h(\mathbb{S}^{n-1})$($\sim$ denotes homotopy equivalence), so we obtain

$$\widetilde{H}_i(\mathbb{S}^n \setminus h(\mathbb{S}^k)) \cong \widetilde{H}_{i+1}(\mathbb{S}^n \setminus h(\mathbb{S}^{k-1})) = H_{i+1}(\mathbb{S}^{n-k+1-1}) \cong \widetilde{H}_{i+1}(\mathbb{S}^{n-k}) \cong \widetilde{H}_i(\mathbb{S}^{n-k-1}).$$

$\square$