USAJMO 2010 - Problems and Solutions

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§1 Problems

Problem 1.1 (USAJMO 2010/1)**.** A *permutation* of the set of positive integers $[n] = \{1, 2, ..., n\}$ is a sequence $(a_1, a_2, ..., a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \leq k \leq n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

Problem 1.2 (USAJMO 2010/2)**.** Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, ..., x_{n-1}$ of positive integers with the following three properties:
(a) $x_1 < x_2 < ... < x_{n-1}$;
(b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, ..., n-1$;
(c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

Problem 1.3 (USAJMO 2010/3)**.** Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Problem 1.4 (USAJMO 2010/4)**.** A triangle is called a parabolic triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

Problem 1.5 (USAJMO 2010/5)**.** Two permutations $a_1, a_2, ..., a_{2010}$ and $b_1, b_2, ..., b_{2010}$ of the numbers $1, 2, ..., 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Show that there exist 1006 permutations of the numbers $1, 2, ..., 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

Problem 1.6 (USAJMO 2010/6)**.** Let ABC be a triangle with $\angle A = 90^o$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

§2 Solutions

Problem 1 (USAJMO 2010/1)

A *permutation* of the set of positive integers $[n] = \{1, 2, ..., n\}$ is a sequence $(a_1, a_2, ..., a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \leq k \leq n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

Proof. We claim the smallest such n is $n = 67^2 = 4489$. Call a permutation $(a_1, a_2, ..., a_n)$ *square* if it satisfies the condition ka_k is a perfect square for all $k \in [n]$. We will define the sequence $O = (o_1, o_2, ..., o_n)$ so that $o_k = k$. We will first show that $2010|P(4489)$, and we will then show that $n = 4489$ is the smallest such permutation that satisfies this condition.

Note that O is square as $ko_k = k^2$for all $k \in [n]$. Now, Take any two perfect squares $a^2, b^2 \in [4489]$. Since $a^2 o_{b^2} = a^2 b^2$ and vice versa, are perfect squares, we can obtain another square permutation of n by swapping o_{a^2} and o_{b^2}. Since there are 67 perfect squares in [4489], we can obtain 67! square permutations by permuting the perfect square elements of O. Thus, $67!|P(n)$ and since $2010 = 2 \cdot 3 \cdot 5 \cdot 67|67!$, $2010|P(n)$.

The next largest set of elements that can be swapped are in the form $2k^2 \in [n]$ - if $2c^2, 2d^2 \in [n]$, then $2c^2 o_{2d^2} = 2^2 c^2 d^2$ is a perfect square. However, in order to obtain the multiple of 67 in 2010, $n \geq 2(67)^2 > 67^2$. Therefore $n = 4489$ is the smallest such n so that $P(n)$ is a multiple of 2010. □

The second solution is more elegant, rigorously creating all the permutation groups of $[n]$ through the notion of an equivalence relation(based on "mavropnevma", from Art of Problem Solving).

Proof. Use all the definitions from the first part of the above proof. Define on $[n]$ the relation $k\ l$ if and only if kl is a perfect square(is clearly satisfies the reflexive, symmetric, and transitive properties).

A permutation $(a_1, a_2, ..., a_n)$ is square if and only if it $a_k\ k$. Suppose $[n]$ has p equivalence classes, $C_1, C_2, ..., C_p$. Over each of those equivalence classes, we can permute the elements a_k to keep the square condition. Therefore, we have

$$P(n) = \prod_{n=1}^{p} |C_n|!.$$

For $n = 67^2 = 4489$, $C_1 = 67$, which means $|C_1|! = 67! \mid P(n)$, and $2010 = 2*3*5*67 \mid 67! \mid P(n)$. $n = 4489$ is the smallest such solution as $|C_1|$ is clearly the smallest equivalence class, and $n = 4489$ is the smallest n so that $67 \mid C_1$. Therefore, 4489 is the smallest such n so that $2010 \mid P(n)$.

□

Problem 2 (USAJMO 2010/2)

Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, ..., x_{n-1}$ of positive integers with the following three properties:
(a) $x_1 < x_2 < ... < x_{n-1}$;
(b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, ..., n-1$;
(c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

Proof. Firstly, $x_1 + x_{n-1} = 2n$, so $x_{n-1} = 2n - x_1$. Since $x_{n-2} < x_{n-1}$, we must have $x_1 + x_{n-2} < x_1 + x_{n-1} < 2n$. Thus, there must exist an index k such that $x_1 + x_{n-2} = x_k$. However, $x_{n-2} > x_{n-3} > ... > x_2 > x_1$. Therefore, the only possible index for k to be is $k = n-1$, so $x_1 + x_{n-2} = x_{n-1} = 2n - x_1$, which means $x_{n-2} = 2n - 2x_1$.

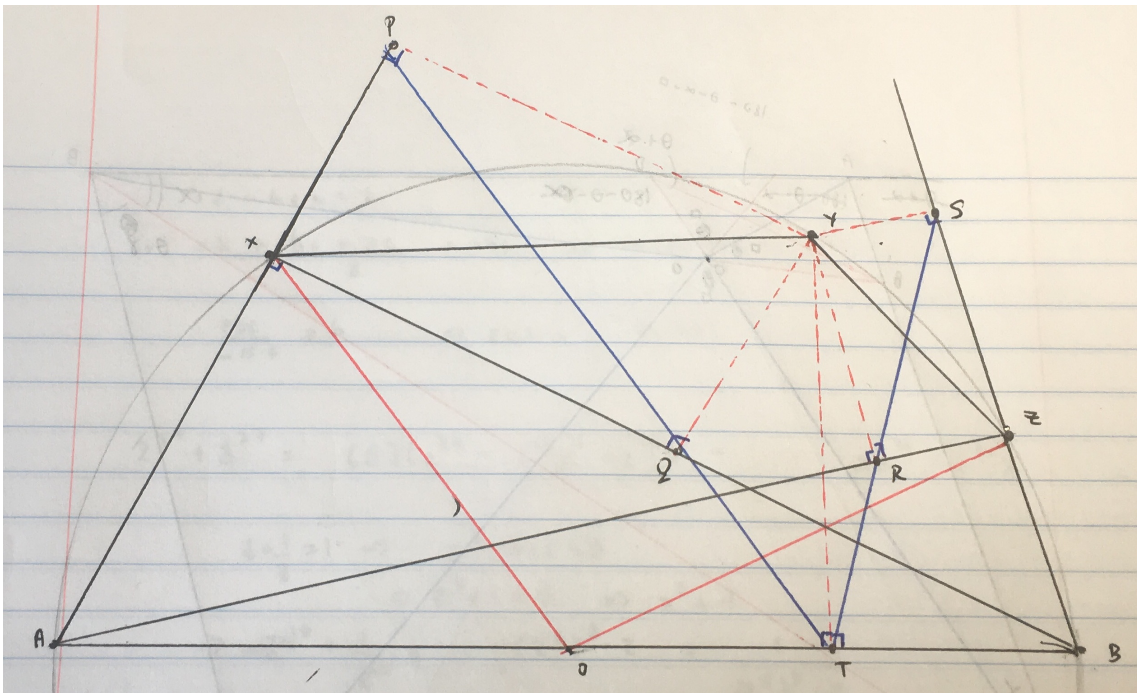
Now, consider $x_1 + x_{n-3}$. Since $x_{n-3} < x_{n-1}$, $x_1 + x_{n-3} < 2n$ which means there exists an index k_1 so that $x_1 + x_{n-3} = x_{k_1}$. k_1 cannot be $n-1$, as this would imply that $x_{n-3} = x_{n-2}$, but x_{n-3} must be less than x_{n-2}. This only leaves $k_1 = n-2$, which means $x_1 + x_{n-3} = x_{n-2}$. It follows that $x_{n-3} = 2n - 3x_1$.

Note that this argument can be repeated for all x_{n-m}, where $1 \leq m \leq n-1$. In particular, $x_{n-m} = 2n - mx_1$. If we take $m = n-1$, then we have $x_{n-(n-1)} = x_1 = 2n-(n-1)x_1$. Solving for x_1 gives $x_1 = 2$. This generates the sequence $2, 4, 6, 8, ..., 2n-2$.

This satisfies the 3 conditions as
(a) $2 < 4 < 6 < ... < 2n-2$;
(b) $x_i + x_{n-i} = 2i + (2n-2i) = 2n$;
(c) If $x_i + x_j < 2n$ for some $i, j < n-1$. then since the sequence contains all the even numbers less than $2n$, there exists an index k so that $x_i + x_j = x_k$. $\square$

Problem 3 (USAJMO 2010/3)

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.



Proof. Let $\measuredangle$ denote directed angles (mod 180^o).

Let T be the altitude from Y onto AB. Note that $PQ \cap SR = T$, since PQ and SR are Simson lines from Y.

Firstly, $\measuredangle YTA = \measuredangle YPA = 90^o$, which implies $YTAP$ is cyclic. Similarly, $\measuredangle YTB = \measuredangle YSB = 90^o$, so $YSBT$ is cyclic.

Then $\measuredangle XOZ = \measuredangle XOY + \measuredangle YOZ = 2(\measuredangle XAZ + \measuredangle YBZ) = 2(\measuredangle PAY + \measuredangle YBS) = 2(\measuredangle PTY + \measuredangle YTS) = 2\measuredangle PTS$, as desired. $\square$

Problem 4 (USAJMO 2010/4)

A triangle is called a parabolic triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

Proof. Let $A(a, b, c)$ denote the area of a parabolic triangle with coordinates (a, a^2), (b, b^2), and (c, c^2). We have

$$A(a, b, c) = \frac{1}{2}\left|\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}\right| = \frac{1}{2}|(a-b)(b-c)(c-a)|.$$

Note that $A(ka, kb, kc) = k^3 A(a, b, c)$. Then, if $A(a, b, c) = (2^n m)^2$, $A(4a, 4b, 4c) = 4^3(2^n m)^2 = (2^{n+3}m)^2$. Therefore, it suffices to show that we have parabolic triangles when $n = 0, 1, 2$.

For the $n = 0$ case, $A(0, 1, -1) = \frac{1}{2}|(0-1)(1-(-1))(-1-0)| = 1 = (2^0 \cdot 1)^2$. For the $n = 1$ case, $A(0, 8, 9) = \frac{1}{2}|(0-8)(8-9)(9-81)| = 36 = (2^1 \cdot 3)^2$. Finally, for the $n = 2$ case, $A(0, 40, 50) = \frac{1}{2}|(0-40)(40-50)(50-0)| = 10000 = (2^2 \cdot 25)^2$.

By induction, there exists an odd number m so that for all $n \in \mathbb{Z}^+$, there exists a parabolic triangle with area $(2^n m)^2$. $\square$

Problem 5 (USAJMO 2010/5)

Two permutations $a_1, a_2, ..., a_{2010}$ and $b_1, b_2, ..., b_{2010}$ of the numbers $1, 2, ..., 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Show that there exist 1006 permutations of the numbers $1, 2, ..., 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

Proof. Consider the 1006 permutations

$$P_1 = (1, 2, 3, ..., 1005, 1006, 1007, 1008, ..., 2009, 2010);$$

$$P_2 = (1006, 1, 2, ..., 1004, 1005, 1007, 1008, ..., 2009, 2010);$$

$$P_3 = (1005, 1006, 1, ..., 1003, 1004, 1007, 1008, ..., 2009, 2010);$$

$$...$$

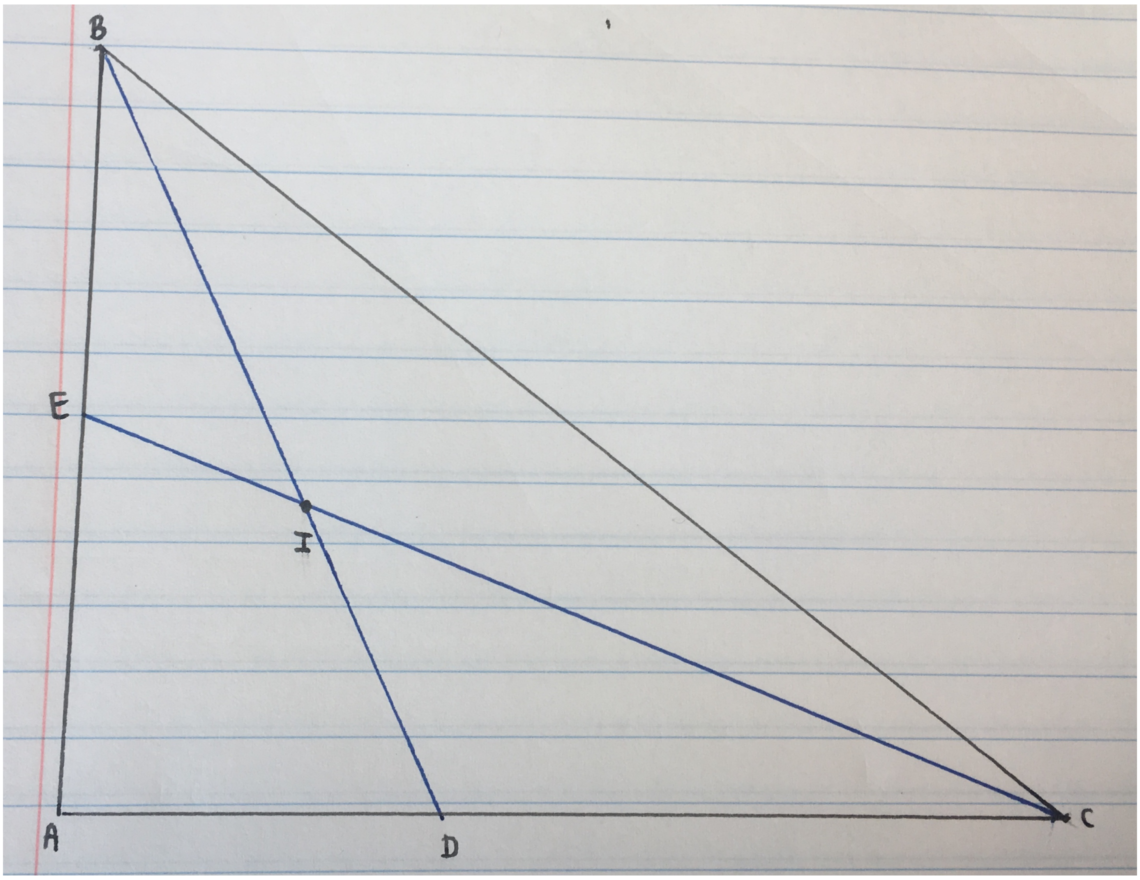
$$P_{1006} = (2, 3, 4, ..., 1006, 1, 1007, 1008, ..., 2009, 2010);$$

formed by cycling through the first 1006 natural numbers and leaving the next 1004 numbers in the same position.

Let $P_i(j)$ denote the j-th element of P_i. Suppose there existed a permutation $X = (x_1, x_2, ..., x_{2010})$ that didn't intersect with any of $P_1, P_2, ..., P_{2010}$. Take $n \in [1, 1006] \cap \mathbb{Z}$. Now, suppose $x_i = n$ for some $i \in [1, 1006] \cap \mathbb{Z}$. If $i = n$, then $P_1(i) = x_i$, which means X and P_i intersect. If $i > n$, then $P_{i-n+1}(i) = x_i$, which means X and P_{i-n+1} intersect. Finally, if $i < n$, then $P_{1007-n+i}(i) = x_i$, which means X and $P_{1007-n+i}$ intersect. Therefore, $x_i \neq n$ for all $i \in [1, 1006] \cap \mathbb{Z}$, so $x_i = n$ for some $i \in [1007, 2010] \cap \mathbb{Z}$. Since n was arbitrarily chosen, this must be true for all $n \in [1, 1006] \cap \mathbb{Z}$. However, $|[1, 1006] \cap \mathbb{Z}| > |[1007, 2010] \cap \mathbb{Z}|$, so by the pigeonhole principle, there exists an $i \in [1007, 2010] \cap \mathbb{Z}$ so that there exists $n, m \in [1, 1006] \cap \mathbb{Z}$ so that $x_i = n$ and $x_i = m$. This is clearly a contradiction, therefore we have shown that there exist 1006 permutations of $1, 2, ..., 2010$ so that any other permutation must intersect at least one of these 1006 permutations. $\square$

Problem 6 (USAJMO 2010/6)

Let ABC be a triangle with $\angle A = 90^o$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.



Proof. Let $\angle ABE = \theta$. Since CI is an angle bisector, $\angle ECB = \theta$, and $\angle A = 90°$, so $\angle B = 90° - 2\theta$. Since BI is an angle bisector, $\angle IBC = 45° - \theta$, which means $\angle BIC = 180° - (45° - \theta) - (\theta) = 135°$. Therefore, by the Law of Cosines,

$$BC^2 = BI^2 + CI^2 - 2(BI)(CI)\cos 135° = BI^2 + CI^2 + (BI)(CI)\sqrt{2},$$

which has no solutions in positive integers. Thus, it is impossible for AB, AC, BI, ID, CI, IE to all have integer lengths. □