Iterations and Recurrence Relations

For $f: D \to D$, $D \subseteq \mathbb{R}$, denote $f^{(n)}$ as the n-th iterate of f.
If f is injective, we also have $f^{(-n)}$, the n-th iterate of f^{-1}.
It is easy to check $f^{(m+n)}(x) = f^{(m)}(f^{(n)}(x))$ and for $x \in D$,
$O(x) = \{ f^{(0)}(x), f^{(1)}(x), \dots \}$ is called the orbit of x.

(IMO Longlist 1982) Determine all $f: \mathbb{Z} \to \mathbb{R}$ s.t.

$$f(n)f(m) = f(n+m) + f(n-m) \quad \forall n, m \in \mathbb{Z}$$

if (a) $f(1) = \frac{5}{2}$ (b) $f(1) = \sqrt{3}$.

Proof. $m=0, n=1$: $f(1)f(0) = 2f(1) \Rightarrow f(0) = 2$.

$n=0$: $f(m) = f(-m)$.

Set $m=1$:

$$a_{m+1} - f(1)a_n + a_{n-1} = 0, \quad a_n = f(n)$$

(a)

$$x^2 - \frac{5}{2}x + 1 = 0, \quad x_1 = 2, \; x_2 = \frac{1}{2}.$$

$$a_n = A \cdot 2^n + B \cdot 2^{-n}.$$

$a_0 = f(0) = 2$, $a_1 = f(1) = \frac{5}{2} \Rightarrow A = B = 1$.

Since $f(-n) = f(n)$, we have

$$f(n) = 2^n + 2^{-n}, \quad \forall n \in \mathbb{N}.$$

(b)

$$x^2 - \sqrt{3}x + 1 = 0$$

So $x_{1,2} = \cos\left(\frac{\pi}{6}\right) \pm i \sin\left(\frac{\pi}{6}\right) = e^{\pm i\pi/6}$.

$$\Rightarrow f(n) = a_n = A\cos\left(\frac{n\pi}{6}\right) + B\sin\left(\frac{n\pi}{6}\right)$$

$f(0) = 2$, $f(1) = \sqrt{3} \Rightarrow A = 2, B = 0$

$$\Rightarrow f(n) = 2\cos\left(\frac{n\pi}{6}\right).$$

Ex. Determine all $f : [0,\infty) \longrightarrow \mathbb{R}$ s.t. $f(0) = 0$ and

$$f(x) = 1 + 5 f\left(\left\lfloor \tfrac{x}{2} \right\rfloor\right) - 6 f\left(\left\lfloor \tfrac{x}{4} \right\rfloor\right) \qquad \forall x > 0.$$

Proof. $0 \le x < 2$, $\quad \lfloor \frac{x}{2} \rfloor = \lfloor \frac{x}{4} \rfloor = 0$.

$$f(x) = 1 + 5f(0) - 6f(0) = 1 \quad \Rightarrow f(1) = 1.$$

$2 \le x < 4$

$$f(x) = 1 + 5f(1) - 6f(0) = 6$$

$\Rightarrow f(x) = a_n$ for $x \in [2^n, 2^{n+1})$,

$a_0 = 1$, $a_1 = 6$, $\quad a_n = 1 + 5a_{n-1} - 6a_{n-2} = 0 \quad$ for $n \ge 2$.

Set $a_n = \frac{1}{2} + b_n$, $\quad b_0 = \frac{1}{2}$, $b_1 = \frac{11}{2}$, $\quad b_n - 5b_{n-1} + 6b_{n-2} = 0 \quad$ for $n \ge 2$.

$$x^2 - 5x + 6 = 0, \qquad x_1 = 2,\ x_2 = 3.$$

$$\Rightarrow b_n = \tfrac{1}{2}\left(3^{n+2} - 2^{n+3}\right)$$

$$a_n = \tfrac{1}{2}\left(1 + 3^{n+2} - 2^{n+3}\right) \quad \text{for } n \ge 0.$$

$$\Rightarrow \quad f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \in (0,2) \\ \frac{1}{2}\left(1 + 3^{n+2} - 2^{n+3}\right) & x \in [2^n, 2^{n+1}),\ n \ge 1 \end{cases}.$$

Ex. Find $f : \mathbb{Z} \to \mathbb{Z}$ w/ $\quad f(m+n) + f(mn-1) = f(m)f(n) \quad \forall\, m,n \in \mathbb{Z}$

Proof. If $f = c$ is constant, $\quad 2c = c^2 \Rightarrow c = 0, 2$.

If f is not constant, $m = 0 \Rightarrow f(n)(1 - f(0)) = -f(-1)$, $\quad$ possible only for $f(0) = 1$, $f(-1) = 0$.

Setting $m = -1$, $\quad f(n-1) + f(-n-1) = 0$. $\quad m = 1$, $\quad f(n+1) + f(n-1) = f(1)f(n)$.

$\Rightarrow \quad x^2 - f(1)x + 1 = 0$. $\quad$ If $f(1) = 0$,

$$f(n-1) + f(n+1) = 0 \quad \Rightarrow \quad f(n+2) = -f(n)$$

and $\quad f(2k) = (-1)^k f(0) = (-1)^k$. $\quad f(2k+1) = (-1)^k f(1) = 0$.

If $f(1) = -1$, $\quad f(n) = (n-1) \pmod 3 \ -1 \quad \forall n$ by induction on n.

If $f(1) = 2$, $\quad f(n-1) - 2f(n) + f(n-1) = 0$ and $f(n) = n+1$.

If $f(1) = 1$, $\quad f(n+1) + f(n-1) = f(n) \quad \Rightarrow f(n+2) + f(n) = f(n+1)$

$\Rightarrow f(n+2) = -f(n-1)$.

Thus, $f(8) = -f(5) = f(2) = -f(-1) = 0$ and $-1 = f(8) + f(6) = f^2(3) = 1$, $\lightning$

We continue these cases.

Ex. Find all $f: \mathbb{N} \to \mathbb{N}$ s.t. $f^{(3)}(n) + f^{(2)}(n) + n = 3f(n)$ $\forall n$.

Proof.

Fix n s.t. $a_0 = n$, $a_{k+1} = f(a_k)$, $k \geq 0$.

$$\Rightarrow a_{k+3} + a_{k+2} - 3a_{k+1} + a_k = 0.$$
$$x^3 + x^2 - 3x + 1 = 0.$$

$$\Rightarrow a_k = c_0 + c_1(-1+\sqrt{2})^k + c_2(-1-\sqrt{2})^k.$$

If $c_2 > 0$, $a_{2k+1} \longrightarrow -\infty$, similar w/ $c_2 < 0 \Rightarrow c_2 = 0$.

$a_0, a_1, a_2 \in \mathbb{Q} \Rightarrow c_1 = 0 \Rightarrow a_1 = a_0$, $f(n) = n$ for all n. ∎

Ex. Let $a, b \in (0, 1/2)$, $f: \mathbb{R} \to \mathbb{R}$ be a continuous function w/
$$f^{(2)}(x) = af(x) + bx.$$

Show $\exists c \in \mathbb{R}$ w/ $f(x) = cx$.

Proof. Note f is injective + continuous $\Rightarrow$ monotone.

Then, f is unbounded since bx is unbounded, so f is onto.

Define $x_{n+1} = f(x_n)$, $x_0 \in \mathbb{R}$ and $x_{n-1} = f^{-1}(x_n)$.

Then $x_{n+2} = ax_{n+1} + bx_n$.

Let t_1, t_2 be roots of $x^2 - ax - b = 0$. Then $t_1 > 0 > t_2$,
$1 > |t_2| > |t_2|$

and $\exists c_1, c_2 \in \mathbb{R}$ w/ $x_n = c_1 t_1^n + c_2 t_2^n$ for $n \in \mathbb{Z}$.

If f is increasing, $c_2 > 0$, then $0 < x_n < x_{n+2}$ and $0 < x_{n+3} < x_{n+1}$ for odd $n < 0$, so $f(x_n) > f(x_{n+2})$ but $x_n < x_{n+2}$ ↯

It follows that $c_2 < 0$ is impossible so $c_2 = 0 \Rightarrow x_0 = c_1$, $x_1 = c_1 t_1 = t_1 x_0$.
$\Rightarrow f(x) = t_1 x$.

Similarly follows if f is decreasing.