Math 214: Differentiable Manifolds

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Contents

§1 January 19th, 2021

§1.1 Topology Review

Definition 1.1 (Topological Space)**.** $(X, O_X \subset \mathcal{P}(X))$, where $A \in O_x$ are the open sets which satisfy the following:

1. $\emptyset, X \in O_X$.
2. $A, B \in O_X$ implies $A \cap B \in O_X$
3. $A_i \in O_X$, $i \in I$, then $\bigcup_{i \in I} A_i \in O_X$.

We say that $A \subset X$ is closed if $X \setminus A$ is open. $U \subset X$ is a neighborhood of $p \in X$ if $\exists A$ such that $p \in A \subset U$.

Example 1.2

Take a metric space (X, d). The topology is generated as follows: $A \subset X$ is open if $\forall p \in A$, $\exists r > 0$ such that $B_r(p) \subset A$.

Definition 1.3. $\mathcal{B} \subset \mathcal{P}(X)$ is called a **basis** for the topology on X if for every subset $A \subset X$, A is open if and only if A is a union of elements of $\mathcal{B}$.

Example 1.4

For a Euclidean space, $\mathcal{B} = \{B_r(x) \subset \mathbb{R}^n : r \in \mathbb{Q}, r > 0, x \in \mathbb{Q}^n\}$ is a basis for the topology. Note that this basis is countable, so $\mathbb{R}^n$ is 2nd countable.

Let (X, O_X), (Y, O_Y) be topological spaces.

Definition 1.5. A function $\varphi : X \to Y$ is continuous if for any open subset $B \subset Y$, $\varphi^{-1}(B) \subset X$ is open.

Definition 1.6. $\varphi : X \to Y$ is a homeomorphism if it is a continuous bijection whose inverse is continuous.

Definition 1.7. Let $Y \subset X$ a topological space. We set $O_Y = \{A \cap Y : A \in O_X\}$.

Example 1.8

The subspace topology is the coarsest topology so that the inclusion map $Y \to X$ is continuous(also called the initial topology).

Example 1.9

$\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ has the same topology as $\mathbb{R}$. In other words, it is clear that $\mathbb{R} \approx \mathbb{R} \times \{0\}$, where the approximate sign indicates a homeomorphism.

Theorem 1

(Topological Invariance of Dimension) If we take $\mathbb{R}^m, \mathbb{R}^n$ with open subsets $U \subset \mathbb{R}^m$ and $V \subset R^n$. If we have $\varphi : U \to V$ a homeomorphism, then we must have $m = n$.

The proof is beyond the scope of the class, but uses homology groups.

Definition 1.10. Given a topological space X, X is called locally Euclidean (of dimension n) at $p \in X$ if there is an open neighborhood about $p \in U \subset X$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

Lemma 1.11

The n is uniquely determined by p.

Proof. Assume that X was locally Euclidean at p of dimensions n_1, n_2. There are neighborhoods $p \in U_i \subset X$ and homeomorphisms $\varphi_i : U_i \to \widehat{U}_i \subset \mathbb{R}^{n_i}$. Consider the image of $U_1 \cap U_2$ under both homeomorphisms. If we take $\varphi_2 \circ \varphi_1^{-1} : \varphi(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$, a homeomorphism, so it follows that $n_1 = n_2$ by Topological Invariance of Dimension. $\square$

Definition 1.12. A space X is **Hausdorff** if for any $p, q \in X$, $p \neq q$ there exists open subsets U, V with $p \in U$, $q \in V$ so that $U \cap V = \emptyset$.

Exercise 1.13. For any $p, q \in X$, if there is a separating continuous function $f : X \to \mathbb{R}$ such that $f(p) \neq f(q)$, then X is Hausdorff.

Definition 1.14. $K \subset X$ is compact if every open cover of K has a finite subcover.

Some useful facts, a subspace of a Hausdorff space is Hausdorff, Hausdorff + Compact implies Closed, $\varphi : X \to Y$ continuous, K is compact, then $\varphi(K)$ is compact. We can use these to show that for $\varphi : X \to Y$ with X compact, Y Hausdorff with φ continuous, bijective, then φ is a homeomorphism.

§1.2 Smooth Manifolds

Definition 1.15. A topological space M is called an n-dimensional **topological manifold** if M satisfies the following:

- M is locally Euclidean at any point,
- M is Hausdorff,
- M is second countable.

Example 1.16 (Manifold - Hausdorff)

Suppose we drop the Hausdorff condition. Take $X = (\mathbb{R} \times \{0, 1\}) \backslash \sim$, where $(x, 0) \sim (x, 1)$ if $x < 0$. Consider the quotient map $\pi : \mathbb{R} \times \{0, 1\} \to X$. Call $A \subset X$ open iff $\pi^{-1}(A)$ is open. Each branch of the line are open subsets, each homeomorphic to $\mathbb{R}$.

Example 1.17 (Manifold - Second Countable)

Take an uncountable subset S equipped with the discrete topology. Set $X = S \times \mathbb{R}$.

A more interesting example called the "long line" is as follows:

Lemma 1.18

There is an uncountable, well-ordered set S such that S has a maximal element $\Omega \in S$ and for all $\alpha \in S$, $\alpha \neq \Omega$, the set $\{x \in S | x < \alpha\}$ is countable.

Now, set $X = (-\infty, 0) \cup S \times [0, 1)$ under the lexicographic ordering. This turns out to be Hausdorff and locally Euclidean but not second countable.

Exercise 1.19. If M is 0-dimensional topological manifold , then M is a finite or countable set equipped with the discrete topology.

Exercise 1.20. If M^n is a top. manifold and $M' \subset M^n$ is open, then M' is an n-dimensional top. manifold.

Example 1.21

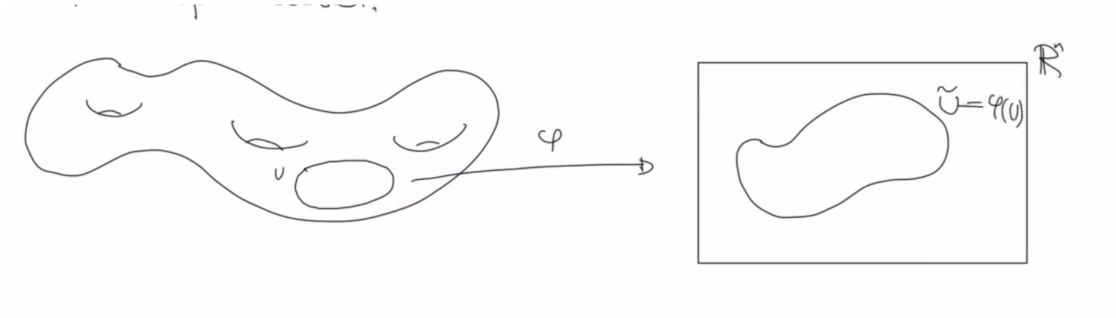
Take $S^1 \subset \mathbb{R}^2$, a circle. This is a 1-dimensional topological manifold.

- It is easy to show that S^1 is Hausdorff and second countable.
- Define $U_i^+ = \{(x_1, x_2) \in S^1 | x_i > 0\}$. We similarly define U_i^-. Then S^1 is the union of all the intervals. We can construct the map $\varphi_i^+ : U_i^+ \to (-1, 1)$ by projecting onto the corresponding axis. This is a homeomorphism.

§2 January 21st, 2021

§2.1 Coordinate Charts

Definition 2.1. A **coordinate chart** on M is a pair (U, φ) where $U \subset M$ is open and $\varphi : U \to \widehat{U}$ is a homeomorphism to an open subset $\widehat{U} \subset \mathbb{R}^n$.



Remark 2.2. We can actually drop the condition that $\widehat{U}$ is open, but the proof of this requires the notion of homology.

We will often write $\varphi(p) = (\varphi^1(p), \varphi^2(p), \ldots, \varphi^n(p))$, which are local coordinates. A way to think about a coordinate chart is just a set of scalar functions, which are the coordinate functions.

Theorem 2

Take $V \subset \mathbb{R}^n$ open, $F : V \to \mathbb{R}^k$ continuous. We claim the graph

$$\Gamma(F) = \{(x, F(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+k}$$

is a manifold.

Proof. Take $(\Gamma(F), \varphi)$, where φ is the projection of the graph onto $\mathbb{R}^n$. It is clear that $\Gamma(F) \cong V$. □

Example 2.3

Take $S^n = \{x \in \mathbb{R}^{n+1:|x|=1} \subset \mathbb{R}^{n+1}\}$. We claim this is a manifold.

Define $U_i^+ = \{(x^1, \ldots, x^{n+1}) : x_i > 0\}$. Similarly define U_i^-. It is clear that M is the union of all the U_i^+'s and U_i^-'s. Note that $U_i^\pm$ is the graph of the map from $B^n(0,1) \to \mathbb{R}$ given by $y \mapsto \pm\sqrt{1 - |y|^2}$. It follows that S^n is a topological manifold.

Example 2.4 (Projective Space)

We define $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where the equivalence relation is defined by $x \sim y$ if $x = \lambda y$ for some $\lambda \neq 0$. We can also view this as a set of lines through the origin. The quotient space is equipped with a projection map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$. We can then use the Quotient topology: $A \subset \mathbb{R}P^n$ is open if $\pi^{-1}(A)$ is open.

We write $[(x_1, \ldots, x^{n+1})] = [x^1 : \cdots : x^{n+1}]$. One should check that $\mathbb{R}P^n$ is Hausdorff and second countable. We show that $\mathbb{R}P^n$ is locally Euclidean.

Define $U_i^* = \{x \in \mathbb{R}^{n+1} \setminus \{0\} : x_i \neq 0\}$ and let $U_i = \pi(U_i^*)$. Note that

$$U_i = \{[x^1 : \cdots : x^{n+1}] : x^i \neq 0\} = \{[\frac{x^1}{x^i} : \cdots : 1 : \frac{x^{n+1}}{x^i}] : x^i \neq 0\}$$

and furthermore

$$U_i = \{[x^1 : \cdots : 1 : \cdots : x^{n+1}]\}.$$

If we define $\varphi_i^* : U_i^* \to \mathbb{R}^n$ given by $(x^1, \ldots, x^{n+1}) \mapsto \left(\frac{x^1}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right)$.

We claim that there exists a continuous map $\varphi_i : U_i \to \mathbb{R}^n$ so that the corresponding commutative diagram commutes: this is just the natural map associated to the quotient.

Furthermore, φ_i is a homeomorphism with inverse $(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}) \mapsto [x^1 : \cdots : 1 : \cdots : x^{n+1}]$.

§2.2 Connectivity

Given a topological space X, we have the following definitions:

Definition 2.5. X is connected if the only subsets that are open and closed are $\emptyset, X$.

Definition 2.6. A space is path-connected if for any $p, q \in X$ there is a continuous path between them.

Theorem 3

If M^n is a topological manifold, M is connected if and only if M is path connected.

Proof. It suffices to show the forward direction. The proof is the same in the case of open subsets of $\mathbb{R}^n$. □

§2.3 Local Compactness and Paracompactness

Proposition 2.7

Given M^n, for all $p \in M$, there exists a compact neighborhood i.e. M is locally compact.

Let X be a topological space.

Definition 2.8. An exhaustion by compact subsets is an increasing sequence of subsets $K_1 \subset K_2 \subset \cdots \subset X$ such that K_i is compact and $K_i \subset \mathrm{Int}(K_{i+1})$ and $\bigcup_i K_i = X$.

> **Remark 2.9.** This also implies that $X = \bigcup_i \text{Int}(K_i)$. If $K^i \subset X$ is some other compact subset, there is some j such that $K^i \subset \text{Int}(K_j)$.

Proposition 2.10

If X is second countable, and locally compact, Hausdorff, then X has an exhaustion by compact subsets.

Proof. First, take $\mathcal{B}$ a countable basis for the topology of X. Take $\mathcal{B}' = \{B \in \mathcal{B} : \overline{B} \text{ compact}\}$, which is still a basis for the topology. Call these sets $\{U_1, U_2, \dots\}$. Choose $K_1 = \overline{U_1}$. For K_2, cover K_1 with possibly several U_i such that $K_1 \subset U_1 \cup \cdots \cup U_{m_2}$ so that $K_2 = \overline{U_1} \cup \cdots \cup \overline{U_{m_2}}$, which is compact. We continue this process to form an exhaustion. □

Definition 2.11. Take $\mathcal{U} \subset \mathcal{P}(X)$. This is a cover of X if $X = \bigcup_{U \in \mathcal{U}} U$. A collection is called locally finite if every $p \in X$ has a neighborhood $p \in W \subset X$ such that W only intersects finitely many $U \in \mathcal{U}$.

Definition 2.12. A collection of subsets $\mathcal{V}$ is called a refinement of some other collection $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.

Definition 2.13. X is called paracompact if every open cover has a locally finite refinement.

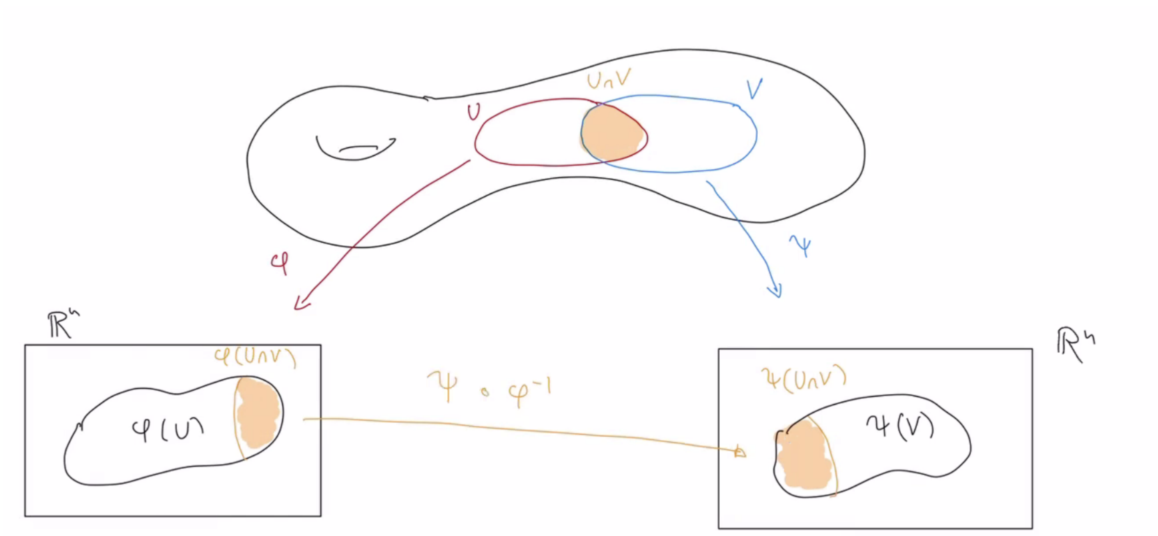
Theorem 4

Every topological manifold is paracompact.

§3 January 26th, 2021

§3.1 Smooth Structures

Definition 3.1. Let M^n be a topological manifold. Two charts $(U, \varphi), (V, \psi)$ of M have a transition map: $\psi \circ \varphi^{-1}$. This map is a homeomorphism.



Definition 3.2. Two charts are smoothly compatible if the transition maps in both directions are smooth.

Definition 3.3. An atlas $\mathcal{A}$ of M is a collection of charts such that the domains of the charts cover M. An atlas $\mathcal{A}$ is smooth if any two charts in $\mathcal{A}$ are smoothly compatible. An atlas $\mathcal{A}$ is called a maximal smooth atlas on M if there is no smooth atlas containing $\mathcal{A}'$ that contains $\mathcal{A}$.

Theorem 3.4

Every smooth atlas $\mathcal{A}$ of M is contained in a unique maximal smooth atlas.

Proof. We first address the existence of a a maximal atlas $\overline{\mathcal{A}}$. We define

$$\overline{\mathcal{A}} = \{(U, \varphi) : (U, \varphi) \text{ is compatible with all } (V, \psi) \in \mathcal{A}\}.$$

$\mathcal{A} \subset \overline{\mathcal{A}}$ is clearly an atlas so it suffices to show that it is smooth.

Take (U_1, φ_1), $(U_2, \varphi_2) \in \overline{\mathcal{A}}$. We check that $\varphi_2 \circ \varphi_1^{-1}(U_1 \cap U_2)$ are smooth. Take some $q \in \varphi_1(U_1 \cap U_2)$ so that $q = \varphi_1(p)$ for $p \in U_1 \cap U_2$. Choose (V, ψ) so that $p \in V$. Then,

$$\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2 \cap V)} = (\varphi_2 \circ \psi^{-1}|_{\psi(U_1 \cap U_2 \cap V)}) \circ (\psi \circ \varphi_1^{-1}|_{\psi(U_1 \cap U_2 \cap V)}).$$

□

Remark 3.5. If $(U_1, \varphi_1) \in \mathcal{A}_1$ is smoothly compatible with any chart $(U_2, \varphi_2) \in \mathcal{A}_2$, then $\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}$.

Definition 3.6. A maximal smooth atlas $\mathcal{A}$ on a topological manifold M is called a smooth structure on M.

Definition 3.7. A smooth manifold is a pair $(M^n, \mathcal{A})$, where M^n is a topological manifold and $\mathcal{A}$ is a smooth structure.

Some exercises:

- If $(U, \varphi) \in \mathcal{A}$ and $U' \subset U$ open, then $(U', \varphi|_{U'}) \in \mathcal{A}$.
- If $(U, \varphi) \in \mathcal{A}$ and a diffeomorphism $\psi : \varphi(Y) \to \psi(\varphi(U)) \subset \mathbb{R}^n$, then $U(\psi \circ \varphi) \in \mathcal{A}$.
- If $\varphi : U \to \mathbb{R}^n$ is injective has the property that for any $p \in U$, there is an open neighborhood $p \in U_p \subset U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

§3.2 Examples of Smooth Structures

- Take $\mathbb{R}^n$. Choose a maximal atlas containing $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$.
- Given $(M^n, \mathcal{A})$ a smooth manifold, $M' \subset M$ open, take $\mathcal{A}' = \{(U, \varphi) \in \mathcal{A} | U \in M'\}$.
- Take a vector space V with dimension n. Take
$$\mathcal{A}' = \{(V, \varphi) : \varphi : V \to \mathbb{R}^n \text{ linear isomorphisms}\}.$$
We take the maximal atlas containing $\mathcal{A}'$.
- Take $M = \mathbb{R}$. Define $\mathcal{A}$ to be the maximal atlas containing $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$. We could also choose $\mathcal{A}^*$ to be the maximal atlas containing $(\mathbb{R}, \varphi)$ where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto x^3$. These two structures are not the same, but the two charts are diffeomorphic.

Does every topological manifold have a smooth structure? For $n \leq 3$, YES and unique up to diffeomorphism. For $n > 4$, NO and if they do exist, they may not be unique. Are there exotic spheres?

§4 January 28th, 2021

§4.1 Construction of Smooth Manifolds

Lemma 4.1

Let M be an uncountable set of points and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$, where $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ are injective. Assume that (1) $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ open for all $\alpha, \beta \in I$ and (2) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth for all $\alpha, \beta \in I$, (3) M is covered by countably many U_α, (4) for all $p, q \in M$, $p \neq q$, there is a $\alpha \in I$ such that $p, q \in U_\alpha$ or $\alpha, \beta \in I$ such that $p \in U_\alpha, q \in U_\beta$.

Then, M has a unique topology and smooth structure such that $(U_\alpha, \varphi_\alpha)$ are smooth charts.

Proof. We define the topology by $A \subset M$ is open whenever $\varphi_\alpha(A \cup U_\alpha) \subset \mathbb{R}^n$ is open for all $\alpha \in I$. We take the smooth structure to be the maximum atlas containing the charts. □

§4.2 Grassmannian Manifolds

Definition 4.2. For $1 \le k \le n$, define $\mathrm{Gr}_k(\mathbb{R}^n) = \{V \subset \mathbb{R}^n | \dim V = k\}$.

Note that $\mathrm{Gr}_1(\mathbb{R}^n) = RP^{n-1}$.

We construct topological and smooth manifolds on $\mathrm{Gr}_k(\mathbb{R}^n)$. Define $I = \{(P, Q), P, Q \subset \mathbb{R}^n, V = P \oplus Q, \dim P = k, \dim Q = n + k\}$.

For a given $(P, Q) = \alpha$ define $U_\alpha = \{V \in \mathrm{Gr}_k(\mathbb{R}^n) | V \cap Q = \{0\}\}$.

Lemma 4.3

For $V \in U_\alpha$, there is a unique linear map $A_{P,Q,V} : P \to Q$ such that $V = \{x + A_{P,Q,V}x \in P \oplus Q | x \in P\}$.

This defines a map $\varphi_\alpha : U_\alpha \to \mathrm{Hom}(P, Q) \cong \mathbb{R}^{kx(n-k)}$.

§4.3 Manifolds with Boundary

We have a manifold M with a boundary ∂M.

We denote $H^n = \{x^n \ge 0\} \subset \mathbb{R}^n$, the upper half space, the most basic example. Note that $\partial H^n = \{x^n = 0\} \cong \mathbb{R}^{n-1}$. The interior Int $H^n = \{x^n > 0\}$.

Definition 4.4. A topological manifold with boundary M^n is a topological space such that is Hausdorff, second countable, and every point $p \in H^n$ has an open neighborhood $p \in U \subset M$ that is homeomorphic to some (relatively) open subset $\widehat{U} \subset H$.

> **Remark 4.5.** For M a topological manifold, we have a topological manifold with boundary, $p \in M$ is interior if it has an open neighborhood homeomorphic to $\widehat{U} \subset \mathbb{R}^n$ and a boundary point, if there is a chart (U, φ) such that $\varphi(p) \in \partial H^n$.

Theorem 4.6 (Boundary Invariance)

$M^n = \int M \cup \partial M$, and ∂M is a topological $(n-1)$ manifold.

Note that the interior and boundary do not correspond to the topological notions of boundary and interior. For example, take $M = \{x^n > 0\} \subset \mathbb{R}^n$, this has a topological boundary in $\mathbb{R}^n$ but no manifold boundary. For any manifold with boundary M, the topological boundary of M is empty and the topological interior within M is M.

If we take $M = S^n \subset \mathbb{R}^{n+1}$, then $\partial M = \emptyset$ but the topological boundary is simply S^n.

> **Remark 4.7.** ∂M is a manifold(without boundary).

§4.4 Smooth Maps

Definition 4.8. $f : M \to \mathbb{R}^m$ is smooth if for every $p \in M$, there is a smooth chart (U, φ), $\widehat{U} = \varphi(U)$ such that $p \in U$ and $\widehat{f} = f \circ \varphi^{-1} : \widehat{U} \to \mathbb{R}^n$.

We denote $C^\infty(M) : \{f : M \to \mathbb{R}^m \text{ smooth}\}$.