Math 222a Lecture Notes, Fall 2020
Partial Differential Equations

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§1 September 1st, 2020

§1.1 Introduction

Partial differential equations apply to functions $u : \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$, where u refers to the space dimension. Usually, $n \geq 2(n = 1$ corresponds to ODEs).

We present the following notation:

- $\frac{\partial}{\partial x_i} u = \partial_i u$

- There is also multi-index notation, where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} u$. The size of α is given by $|\alpha| = \sum_{i=1}^n \alpha_i$.

- $C(\mathbb{R}^n)$, continuous functions in $\mathbb{R}^n$.

- $C(\Omega)$, $\Omega \subset \mathbb{R}^n$, continuous functions in Ω.

- $C^1(\mathbb{R}^n), C^1(\Omega)$, continuously differentiable functions.

- $C^k(\mathbb{R}^n), C^k(\Omega)$, k-times differentiable.

- $C^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty C^k(\mathbb{R}^n)$.

We consider an example PDE,

$$F(u, \partial u, \partial^2 u, \dots, \partial^k u) = 0.$$

In the above, $k \geq 1$ and k is the **order** of the equation. We also have the shorthand $F(\partial^{\leq k} u) = 0$.

§1.2 Classification of PDE's

Definition 1.1 (Linear PDE). The PDE is a linear function of its arguments. We can apply multi-index notation, as follows:

$$\sum_{|\alpha| < k} c_\alpha \partial^\alpha u = f(x).$$

If $f(x) = 0$, the PDE is **homogeneous**, otherwise it is **inhomogeneous**.

This can be separated into linear PDEs with constant coefficients, $c_\alpha \in \mathbb{R}, \mathbb{C}$ and variables coefficients, $c_\alpha = c_\alpha(x)$. [In this class, we focus on constant coefficient PDEs, but many of the techniques can be extended to variable coefficient PDEs.]

Definition 1.2 (Nonlinear PDE). We look at a function $F = F(u, \partial u, \dots, \partial^k u)$. The highest order terms are take the *leading role*.

- Semilinear PDE's: F is linear, with constant or variable coefficients in $\partial^k u$:

$$\sum_{|\alpha| = k} c_\alpha(x) \partial^\alpha u = N(\partial^{\leq k-1} u).$$

The LHS is called the principal part, and the RHS is the perturbative role.

- Quasilinear PDE's:

$$\sum_{|\alpha| = k} c_\alpha(\partial^{\leq k-1} u) \partial^\alpha u = N(\partial^{\leq k-1} u).$$

- Fully Nonlinear PDE's: $F(\partial^{\leq k} u) = 0$, with a nonlinear dependence on $\partial^k u$.

Some examples:

- Linear, homogeneous, variable coefficients, order 1:

$$\sum_{k=1}^{u} c_k(x) \partial_k(u) = 0.$$

- Define $\Delta = \partial_1^2 + \cdots + \partial_n^2$, the Laplacian operator. We have a linear, constant coefficients, inhomogeneous, order 2:

$$\Delta u = f.$$

- Semilinear, order 2:

$$\Delta u = u^3.$$

 [Note that translation invariance makes homogeneous vs inhomogeneous not useful for classification in the case of nonlinear PDE's.]

- Harmonic Map Equation:
$$\Delta u = u|\nabla u|^2.$$

 It is still semilinear, but with a stronger nonlienarity.

- Monge Ampere Equation:

$$\mathbb{R}^2, \partial_1^2 u \partial_2^2 u - (\partial_1 \partial_2 u)^2 = 0.$$

 It is a fully nonlinear equation.

§1.3 Initial Value Problems

We have various types of problems:

- (Stationary Problems) With $u : \mathbb{R}^n \to \mathbb{R}$,

$$F(\partial^{\leq k} u) = 0,$$

 might describe an equilibrium configuration of a physical system.

- (Evolution Equations) With $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $u(t,x)$ describes the state at time t. We can think about the order in x or in t.

Definition 1.3 (Initial Value Problem/Cauchy Problem)**.** A PDE with initial conditions.

Example 1.4

Consider the heat equation:
$$\partial_t u = \Delta_x u,$$

$$u(t = 0, x) = u_o(x).$$

The equation is first order in t, but second order in x.

Example 1.5

In $[\mathbb{R} \times \mathbb{R}]$, the vibrating string:

$$\partial_t^2 u = \partial_x^2 u,$$

$$u(t = 0, x) = u_0(x),$$

$$\partial_t u(t = 0, x) = u_1(x).$$

Note that this equation is second order in time, and requires 2 pieces of initial data.

An easier problem: Compute the Taylor series of u at some point $(0, x_0)$. It requires $\partial_t^\alpha \partial_x^\beta u(0, x_0)$.

- This is obvious if we have no time derivative or exactly 1.

- Second order time derivatives come from the equation.

- Third order or higher time derivatives come from differentiating the equation:

$$\partial_t^3 u = \partial_x^2 \partial_t u.$$

§1.4 Boundary Value Problems

We begin with an example.

Example 1.6

Take $\Delta u = f$ in $\Omega \subset \mathbb{R}^3$, which represents equilibrium for temperature in a solid. To solve, we need information about the boundary of Ω. For example,

$$\Delta u = f \in \Omega,$$

$$u = g \in \partial\Omega.$$

§1.5 Fluid Classification

We take $u : \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$, and

$$F(\partial^{\leq k} u) = 0.$$

This is considered to be a **scalar equation**.

We could also take a **system** of equations, where $u : \mathbb{R}^n \to \mathbb{R}^m(\mathbb{C}^m)$, where $u = [u_i]$ a column of equations. These are often more difficult than scalar equation. We should have

$$F(\partial^{\leq k} u) = 0,$$

but $F : \mathbb{R}^{(\cdot)} \to \mathbb{R}^m(\mathbb{C}^m)$.

Example 1.7

A 2-system:

$$\Delta u = v,$$

$$\Delta v = -u.$$

We can often reduce the order of a scalar equation by turning it into a system:

Example 1.8

Consider the vibrating string,
$$\partial_t^2 u = \partial_x^2 u.$$

If we take $v = \partial_t u$, the it suffices to solve the system,

$$\partial_t u = v,$$

$$\partial_t v = \partial_x^2 u.$$

We van reduce it further by saying $u_1 = \partial_x u, u_2 = \partial_t u$ for the system,

$$\partial_t u_1 = \partial_x u_2,$$

$$\partial_t u_2 = \partial_x u_1.$$

§2 September 3rd, 2020

§2.1 Picard-Lindeloff Theorem

Consider the example, $x' = f(x), x(0) = x_0, x : \mathbb{R} \to \mathbb{R}^n$. We ask for existence, uniqueness, continuous dependence on initial data.

Definition 2.1 (Locally Lipschitz). A **Lipschitz** continuous function f is one that satisfies,
$$|f(x) - f(y)| \leq c|x - y|.$$
A function is **Locally Lipschitz** if for each R, there exists $c(R)$ such that

$$|f(x) - f(y)| \leq c(r)|x - y|, x, y \in \text{Ball}(0, R).$$

As examples, $f(x) = x$ is Lipschitz, $f(x) = x^2$ is not Lipschitz, but is locally Lipschitz.

Definition 2.2 (Locally well-posed). For each $x_0 \in \mathbb{R}^n$, there exists $T > 0$(lifespan) and a unique solution $u \in C^1[0, T; \mathbb{R}^n]$ with the property that $u_0 = x_0$ and the solution has a Lipschitz dependence on the data: x_0, y_0 initial data, $T = T(x_0)$. For $T_1 < T$, there exists $\epsilon > 0$ such that if $|y_0 - x_0| \leq \epsilon$ then $T(y) > T_1$ and

$$\sup_{t \in [0, T_1]} |x(t) - y(t)| \leq \tilde{C}|x_0 - y_0|.$$
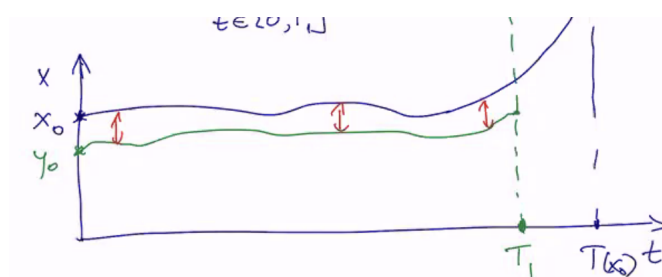
> **Theorem 1** (Picard-Lindelof)
> Assume that f is locally Lipschitz continuous. Then the ODE is locally well-posed.

§2.2 Contraction Principle

We will use the "Contraction principle" - recall the following definitions:

Definition 2.3 (Fixed-point Problem). Let X be a Banach space,let $D \subset X$ be a closed subset of X, and let $F : D \to D$. Question: Can we solve the equation $F(u) = u$ where $u \in D$.

Definition 2.4 (Contraction).

$$\|F(u) - F(v)\|_X \leq L\|u - v\|,$$

where $L < 1$.

If F is a contraction, then it has a unique fixed point. The existence proof follows an iterative construction: start with an arbitrary element $u_0 \in D$ and define $u_{n+1} = F(u_n)$. We would show $\{u_n\}$ is a Cauchy sequence, so it converges.

We now prove the theorem. We have $x' = f(x), x(0) = x_0$, so

$$x(t) = x_0 + \int_0^t f(x(s))ds, t \in [0, T].$$

We choose $X = C[0, T; \mathbb{R}^n]$, $F(x)(t) = x_0 + \int_0^t f(x(s))ds$. Then x solves the ODE in $(0, T)$ if $F(x) = x$.
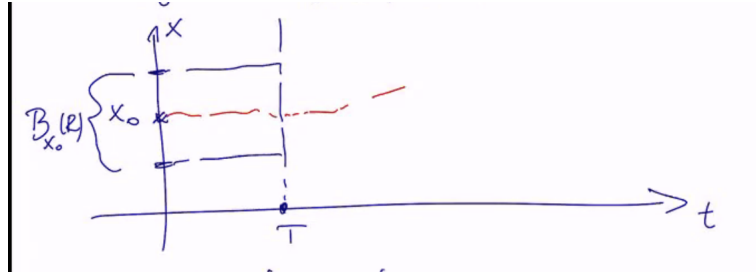
We have to choose R, T. Then

$$D = \{x \in X : \|x - x_0\|_X \leq R\}.$$

Let $R = |x_0|$. Next, we choose T so that $F : D \to D$ is Lipschitz. For $F : D \to D$, we estimate the size of $F(x) - x_0$.

$$
\begin{aligned}
F(x)(t) - x_0| &= \left| \int_0^t f(x(s))ds \right| \\
&\leq \left| \int_0^t f(x_0(s))ds \right| + \left| \int_0^t f(x) - f(x_0)ds \right| \\
&\leq T|f(x_0)| + CT\|x - x_0\|_X
\end{aligned}
$$

Hence,
$$\|F(x) - x_0\| \leq T(|f(x_0)| + CR).$$

Thus, we choose T such that $T(|f(x_0)| + CR) \leq R$.

Now look at differences: For $x, y \in D$,

$$
\begin{aligned}
|F(x)(t) - F(y)(t)| &\leq \int_0^t |f(x(s)) - f(y(s))|ds \\
&\leq TC \sup_{s \in [0,T]} |x(s) - y(s)|
\end{aligned}
$$

thus,
$$\|F(x) - F(y)\|_X \leq CT\|x - y\|_X,$$

so we can choose T so that $CT\|x - y\|_X < 1$.

By the contraction principle, there exists a unique solution $x \in D$.

To prove uniqueness of a solution, we have to show that any solution has to stay in D, up to time T.
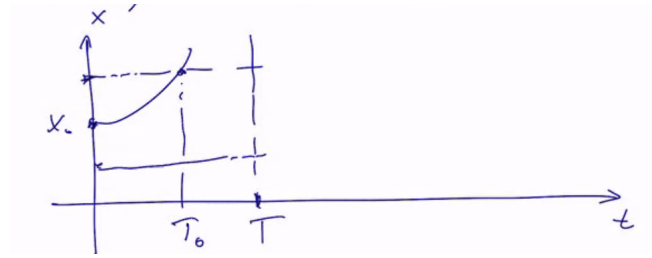
Suppose a solution $\tilde{x}$ leave the ball before time T. We repeat the above computation up to the exit time T_0. Then, $T_0(|f(x_0) + CR|) < T$, since $T_0 < T$. This is a contradiction since T_0 is the exit time.

§2.3 Bootstrap Argument

Consider a bootstrap argument: try to solve an equation and show that the solution x satisfies some bound $\|x\|_T \leq R$. The difficulty is that a priori, we do not know any bound on $\|x\|_T$. The solution: make a bootstrap assumption, $\|x\|_T \leq 2R$ and show that $\|x\|_T \leq R$ under this assumption.
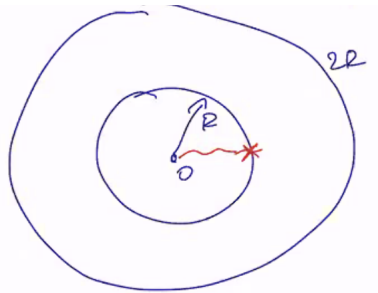
So far, we know uniqueness in $[0, T]$, where $T = T(x_0)$ given by the contraction argument. We now show global uniqueness: Suppose we have a solution x_0 with maximal lifespan $T_{max}(x_0)$. Suppose y is another solution. We look at the maximal T so that $x = y$ in $[0, T)$. We now think of T as the initial time. We $x(T) = y(T)$ from continuity. Then, the solution is unique up to some time $T + T_0$, so $x = y$ in $[T, T + T_0]$, contradicting the maximality of T. This is called a "continuity argument".

Next, we compare two solutions: We have $x(0) = x_0, x : [0, T) \to R^n$. We choose $T_1 < T$. Then $x : [0, T_1] \to \mathbb{R}^n$. We compare x with a "nearby" solution $y(0) = y_0$ close to x_0. We have $\|x\|_{X_{T_1}} \leq R$ since we have continuity on a compact set. We claim the following: if $|y_0 - x_0| < \epsilon$, then x, y stay close. We make a bootstrap assumption $\|y\|_{X_{T_1}} \leq 2R$.

$$\frac{d}{dt}|x - y|^2 = 2(x - y)(f(x) - f(y)) \leq 2C|x - y|^2.$$

This is the *Gronwall Inequality.* It follows that

$$|x - y|^2(t) \leq e^{2ct}|x - y|^2(0) = e^{2ct}|x_0 - y_0|^2.$$

To close the bootstrap:

$$\|y\|_{X_{T_1}} \leq \|x\|_{X_{T_1}} + \|x - y\|_{X_{T_1}} \leq R + e^{cT_1}\|x_0 - y_0\| \leq \frac{3R}{2},$$

which is better than the bootstrap assumption.

§3 September 8th, 2020

Last lecture, we discussed the ordinary differential equation $x' = f(x)$ in R^n with $x(0) = x_0$. We proved the Pircard-Lindelof theorem: if f is locally Lip. then this problem is locally well-posed and the solution has a local Lip. dependence on the initial data. We proved this by the contraction principle, using Picard iterations.

§3.1 Observations regarding Picard-Lindelof

We note the following observations:

1. The result is local, so it can blow up in finite time.

   For example, take $x' = x^2, x(0) = x_0 > 0$. The positive solutions to the ODE are $x(t) = \frac{1}{T-t}, T \geq 0$, where T is the blow up time. In this case, it is $T = \frac{1}{x_0}$.

2. If f is not Lipschitz, then uniqueness might fail.

   Take $x' = \sqrt{x}, x(0) = 0$. An obvious solution is $x = 0$. Other solutions are like $x(t) = ct^2$. We can generate infinitely many solutions from here.
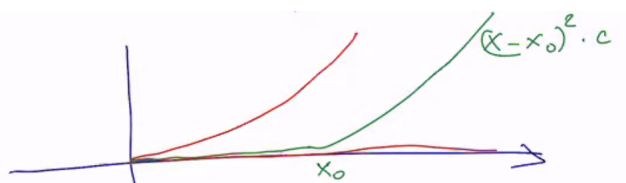
   But solutions might still exist:

   > **Theorem 2** (Peano)
   > If f is continuous, then a local solution exists.

   The proof uses Schauder's fixed point theorem.

3. What if $f \in C^1_{loc}$, the space of differentiable functions on a compact set?

   > **Theorem 3**
   > If $f \in C^1_{loc}$, then the flow map $x_0 \mapsto x(t, x_0) = \Phi(t, x_0)$ is of class C^1.

   Proof. We give a sketch. Take x_0, x_0^h and assume $\frac{d}{dh}x_0^h(0)$ exists and show that $\frac{d}{dh}x^h$ exists. The linearized equation about $h = 0$ is $\dot{y} = Df(x_0)y, y_0 = \frac{d}{dh}x_0^h$. We expect that
   $$x^h(t) = x(t) + hy(t) + o(h).$$
   Let $\tilde{x}^h(t) = x(t) + hy(t)$. We claim that this is an "approximate solution", in the sense that
   $$\dot{\tilde{x}}^h(t) = f(\tilde{x}^h(t)) + o(h).$$
   Furthermore, we have close initial data in the sense that
   $$|x_0^h - \tilde{x}_0^h| \leq o(h).$$
   We repeat the difference bound for one exact and one approximate solution and show that
   $$|x^h(t) - \tilde{x}^h(t)| \leq o(h)$$
   $\square$

This implies that the Flow map is a group of local diffeomorphisms:

$$\Phi(t) \circ \Phi(s) = \Phi(t + s).$$

§3.2 Linearization of an ODE

The leads us to the notion of the linearization of the ODE: If we consider $x_0 \to x_0^h$, a one parameter family of data, assume this is C^1 in h. The corresponding solution $x_0^h \to x^h(t)$ also in C^1 in h.

What can we say about

$$y^h(t) = \frac{d}{dh} x^h(t)?$$

We have $\dot{x}^h = f(x^h), x^h(0) = x_0$. If we differentiate with respect to h, we have $\dot{y}^h = Df(x^h)y^h, y^h(0) = \frac{d}{dh}x_0^h.$, where $Df(x^h)$ is the differential of f, $\left(\frac{\partial f_i}{\partial x_j}\right)_{n \times m}$. This is a linear ODE with variable coefficients.
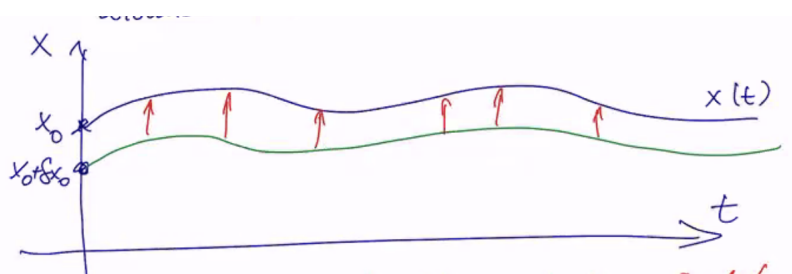
> **Proposition 3.1**
>
> If the linearized equation is well-posed, then we have Lip. dependence of solutions on the initial data.

§3.3 Our First Partial Differential Equation

Our first example is scalar first order equations in $\mathbb{R}^n$,

$$F(x, u, Du) = 0 \in \mathbb{R}^n, y : \mathbb{R}^n \to \mathbb{R}.$$

Today, we look at the case of linear, constant coefficients:

$$\sum a^i \partial_i u = f(x).$$

We will write this as $a^i \partial_i u$ following the Einstein summation convention. Take $A = (a_1, \ldots, a_n)$, so we have $A \cdot Du = f(x)$, with $A \neq 0$. This can be interpreted as a directional derivative of u in the direction A.

$$\frac{d}{dt} u(x(t)) = A \cdot Du(x(t)) = f(x(t)).$$

Note the fundamental theorem of calculus,

$$u(x(t)) = u(x_0) + \int_0^t f(x(t)) dx.$$

Suppose we have a C^1 surface Σ and we are asked to solve a PDE with inidtial data $u = u_0$ on Σ.
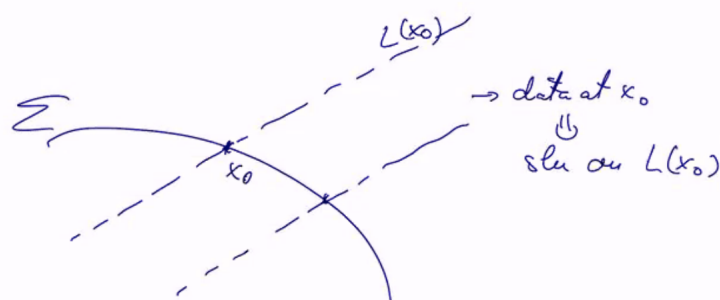
But things can go wrong. If Σ is a circle, we'd could have two intersection points. Furthermore, we could miss the circle entirely and have no solutions. Our solution in this case would be to assume that each line intersects Σ exactly once. However, if solutions are tangent, perturbations of the surface cause problems.

To solve all these issues, we assume that A is always transversal to Σ. This can be written in terms of N, the normal vector to Σ, namely,

$$A \cdot N \neq 0.$$

Definition 3.2 (Noncharacteristic Surface). If $A \cdot N \neq 0$, then we say the surface Σ is noncharacteristic.

We can have solutions that solve the equation at every point but not differentiable everywhere. We learn 2 lessons from this example:

1. We need to enlarge the notion of what is a solution, this leads to the theory of distributions.

2. There are solutions to our PDE with a jump discontinuity along characteristic surfaces. (Γ in the picture)

After applying a change of coordinates, we have a Cauchy problem:

$$u_t + A D_x u = f, u(t = 0) = u_0,$$

where u_t is nonzero, corresponding to the condition that the surface is noncharacteristic.

§4 September 10th, 2020

Last time:

- We began discussing first order scalar equations.
    - Linear, Constant Coefficients,

$$a^j \partial_j u = f.$$

    - We interpret the equation as a directional derivative, so solving the equation reduces to integration along straight lines.
    - For initial data, $u = u_0$ on Σ, for the problem to be well poser, we need Σ to be **noncharacteristic**, namely

$$A \cdot N \neq 0,$$

    where $A = (a_j)$, N is the normal vector to the surface.
    - Our model problem was an evolution in (t, x), where

$$u_t + a^j \partial_j u = f, u(t = 0) = u_0.$$

§4.1 Linear, Variable Coefficients

We have equations of the form
$$a^j(x) \partial_j u = f,$$
the **Transport equation**.
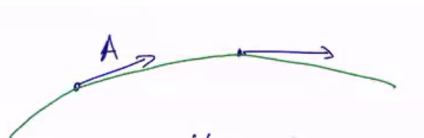
Now, we have integration on curves instead of straight lines. We think about this as ode's along curves γ so that A is tangent to γ at every point:

$$\dot{x} = A(x), x(0) = x_0, x(t) = \Phi(t, x_0).$$

We can rewrite this in the form

$$\frac{d}{dt} u(\Phi(t, x_0)) = f(\Phi(t, x_0)),$$

which reduces to the fundamental theorem of calculus in 2 steps:

1. Solve the ode

2. Integrate

We can also add Cauchy data, $u = u_0$ on Σ. For the problem to be well-posed, we need Σ to be noncharacteristic, $A \cdot N \neq 0$.
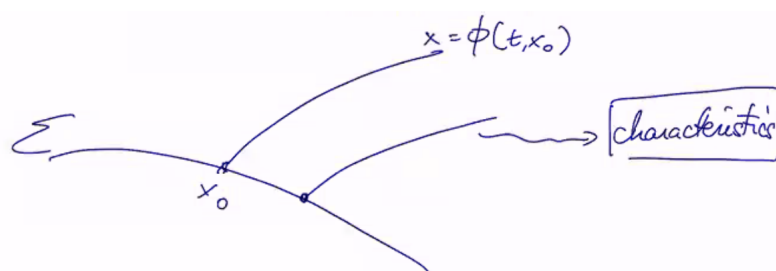
> **Theorem 4**
>
> Assume that $A \in C^1_{loc}$, Σ is a C^1 noncharacteristic surface. Then the problem
>
> $$\begin{cases} A \cdot Du = f \in C \\ u = u_0 \in C \text{ on } \Sigma \end{cases}$$
>
> admits a unique continuous solution. Note that if $(f, u_0) \in C$, then we can get $u \in C$ and if $(f, u_0) \in C^1$, then we can get $u \in C^1$.

Having a local diffeomorphism is equivalent to showing Σ is noncharacteristic. We show this by showing the differential is nonzero.

Take a change of coordinates

$$x = \Phi(x_0, t).$$

In the new coordinates, the equation becomes

$$\frac{\partial u}{\partial t} = f, u_{t=0} = u_0,$$

which reduces completely to the fundamental theorem of calculus.

§4.2 Semilinear Equations

We have equations of the form

$$a^j(x)\partial_j u = f(x, u).$$

We can still interpret this as a directional derivative, with an ODE:

$$\begin{cases} x' = A(x) \\ x(0) = x_0 \end{cases}$$

with a Flow map $(x_0, t) = \varphi(x_0, t)$. We still require our surface Σ to be noncharacteristic.

The equation along characteristics is

$$\begin{cases} \frac{\partial}{\partial t} u(\varphi(x_0, t)) = f(\varphi(x_0, t), u(\varphi(x_0, t))) \\ u(\varphi(x_0, 0)) = u_0(x_0). \end{cases}$$

Then, we solve for the characteristics, and solve the ode along the characteristics.

§4.3 Quasilinear Equations

We have equations of the form

$$\begin{cases} a^j(x, u)\partial_j u = f(x, u) \\ u = u_0 \text{ on } \Sigma \end{cases}$$

A priori, we can no longer draw the vector field, since a given point will depend on the solution u.

Suppose we already have a solution $u \in C^1$. Then, we have a well-defined vector field $A = A(x, u)$. We can also consider the flow of A.
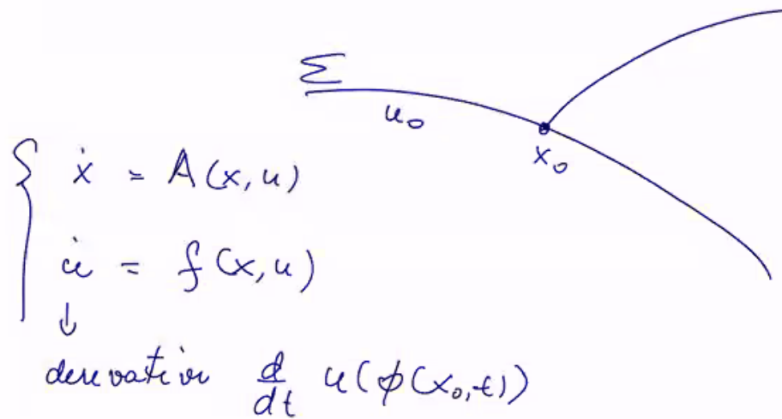
Given initial data $x(0) = x_0, u(0) = u_0(x_0)$, we have a Cauchy problem. From this, we conclude that a good strategy for the problem is the following:

- Given x_0 on Σ, solve the above system for x, u.

$$X = \varphi(t, x_0, u_0), u = U(t, x_0, u_0).$$

- Define the candidate solution u as

$$u(\varphi(t, x_0, u_0)) = U(t, x_0, u_0).$$

Remark: We still want Σ to be noncharacteristic: $A(x_0, u_0(x_0)) \cdot N(x_0) \neq 0$. In this case, we say the problem is noncharacteristic, since it depends on the initial data.

> **Theorem 5**
>
> Let $a^j(x, u), f(x, u)$ be C^1 functions, $u_0 \in C^1(\Sigma)$ and Σ noncharacteristic. Then, the problem
> $$\begin{cases} a^j(x, u)\partial_j u = \delta(x, u) \\ u = u_0 \text{ on } \Sigma \end{cases}$$
> has a unique local solution.

Proof. We outline the steps.

1. Solve the characteristic ode:
$$\begin{cases} \dot{x} = A(x, u) \\ \dot{u} = f(x, u) \\ x(0) = x_0 \in \Sigma \\ u(0) = u_0(x_0) \end{cases}.$$

   This gives us a local diffeomorphism:
$$(x_0, u_0(x_0), t) \to (x, u).$$

2. Define the candidate solution
$$u(\varphi(x_0, u_0(x_0, t))) = U(x_0, u_0(x_0), t).$$

3. Verify that the solution is C^1, which comes from C^1 dependence for ODE's and for the local diffeomorphism.

4. Verify that the solution is unique. [Suppose we have two solutions u_1, u_2 with the same initial data. Then if their characteristic ode's have the same data, they have the same solutions, which implies that the characteristics are the same and $u_1 = u_2$ on characteristics.]

$\square$

The key observation is that solutions given by the theorem are local solutions.

§4.4 Classical model problem: Burgers' Equation

Consider the Burgers Equation: we use coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}$.
$$\begin{cases} u_t + u \cdot u_x = 0 \\ u(t = 0) = u_0(x) \end{cases}$$

$t = 0$

$A = (1, u)$

$N = (1, 0)$

$A \cdot N = 1$

noncharact. by definition

If we choose a point x_0, the characteristic system: denote by s the parameter along characteristics.

$$\begin{cases} \dot{t} = 1, t(0) = 0 \\ \dot{x} = u, x(0) = x_0 \\ \dot{u} = 0, u(0) = u_0(x_0) \end{cases}$$

Our first equation is $t = s$, which tells us that t is the natural parameter along characteristics. The third equation gives us that $u = u_0(x_0)$, which is constant along characteristics. Then, $x(t) = x_0 + tu_0(x_0)$. In particular, characteristics are straight lines.
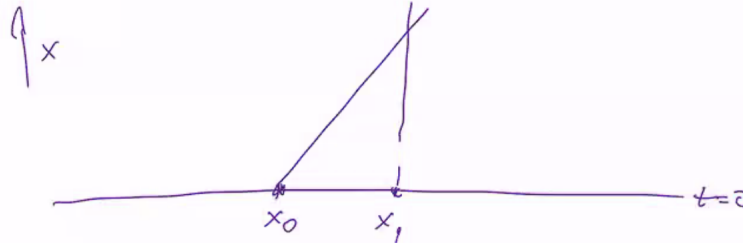
Characteristics can intersect, which loses the C^1 well-posedness. To find the first C^1 blow-up time, we look at the first point where $x_0 \to x$ is no longer a diffeomorphism.

$$\frac{\partial x}{\partial x_0} = 1 + tu_0'(x_0),$$

so we encounter a singularity when $tu_0'(x_0) + 1 = 0 \Rightarrow t = -\frac{1}{u_0'(x_0)}$

§5 September 15th, 2020

§5.1 Fully Nonlinear PDEs

We have an equation $F(x, u, Du) = 0 \in \mathbb{R}^n$, where $Du = (\partial_1 u, \dots, \partial_n u)$. So far, we've been able to interpret the equation as a directional derivative in the direction of u. Naively, it isn't clear how to apply that approach to this system.

First, we compute the linearized equation: We have $u(h)$, a one parameter family of solutions, and take $V = \frac{d}{dh} u(h)|_{h=0}$. The $h = 0$ corresponds to the original solution. We get that

$$F_u(x, u, Du) \cdot v + F_{Du}(x, u, Du) \cdot Dv = 0.$$

We improve the notation by noting that $F : \mathbb{R}^n \times \mathbb{R} \times R^n \to \mathbb{R}$, and we denote $p = Du$ for convenience.

We can rewrite the equation as

$$F_u(x, u, Du)v + F_p(x, u, Du) \cdot Dv = F_u(x, u, Du)v + F_{p_j}(x, u, Du) \cdot \partial_j v = 0.$$

The key observation is that the linearized equation is a transport equation. The vector field is $F_p(x, u, Du)$. Then, we have the ordinary differential equation:

$$\dot{x}(s) = F_p(x, u, Du),$$

which are the integral curves of the vector field. If we have a solution, we can compute it using the chain rule:

$$\dot{u} = F_p(x, u, Du) \cdot Du.$$

We also need $(\dot{Du})$. We can compute this directly with chain rule, or we can simply differentiate the original equation:

$$\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x_j} + \frac{\partial F}{\partial p_j} \cdot \frac{\partial^2 u}{\partial x_j x_k} = 0.$$

Interpreting this as a directional derivative of $\frac{\partial u}{\partial x_j}$, we have

$$F_p \cdot D \frac{\partial u}{\partial x_j} = -F_i \cdot \frac{\partial u}{\partial x_j} - F_x,$$

$$\left(\frac{\dot{\partial u}}{\partial x_j} \right) = -F_u \frac{\partial u}{\partial x_j} - F_x.$$

We think of the three equations as a system of ODEs:

$$\begin{cases} \dot{x} = F_p(x, u, p) \\ \dot{u} = F_p(x, u, Du) \cdot Du \\ \dot{p} = -F_u \cdot p - F_x \end{cases}$$

Lemma 5.1

Suppose u is a C^2 solution to our PDE. Consider the solution to (x, u, p) to the characteristic system above, with initial data $x(0) = x_0, u(0) = u(x_0), p(0) = Du(x_0)$. Then,

$$u(x(s)) = u(s),$$

$$Du(x(s)) = p(s).$$

The goal is to use this property in order to construct solutions.

Add to the ODE an initial solution $u = u_0$ on a surface Σ. Can we solve (locally) the initial value problem?

$$\begin{cases} F(x, u, Du) = 0 \\ u = u_0 \in \Sigma \end{cases}.$$

To compute characteristics starting at x_0, we need $x_0, u(x_0), Du(x_0)$. We have the first two, since $u(x_0) = u_0(x_0)$. However, $Du(x_0)$ is not immediately obvious, since u_0 is only on Σ, in $n - 1$ variables. In other words, we can't compute derivatives in the normal direction to Σ.

In a frame of x_0 where $\Sigma = \{x_n = 0\}$, we are given $\frac{\partial u}{\partial x'} = \frac{\partial u_0}{\partial x'}$, the tangential derivatives, but not $\frac{\partial u}{\partial x_n}$.

We use the equation at x_0:

$$F(x_0, u_0(x_0), Du(x_0)) = 0.$$

We think the above as an equation for $\partial_u u(x_0)$. Either we cannot solve it, in which case there is no solution to the PDE, or we can solve it, with some issues, namely:

- The solution might not be unique, in which case we make a choice that is consistent around x_0.

- The outcome $\partial_n u(x_0) = G(x_0, u_0(x_0), D'u_0(x_0))$, which we want to solve smoothly. Using the "Implicit Function theorem", this happens if $\boxed{F_{p_n}(x_0, u_0(x_0), p) \neq 0}$.

Theorem 6

Consider the PDE:

$$\begin{cases} F(x, u, Du) = 0 \\ u|_\Sigma = u_0 \end{cases}.$$

Let $x_0 \in \Sigma$, and p_0 so that

- $F(x, u_0, p_0) = 0$

- $p_0 \cdot T = Du_0 \cdot T$, where T is tangential

- $F_p(x_0, u_0, p_0) \cdot N \neq 0$.

- $F_p(x_0, u_0, p_0) \cdot N \neq 0$, the noncharacteristic condition.

(Notice that the third and fourth conditions are the same! They come from two different places, the implicit function theorem, and our conditions on initial data.)

Then, there exists a unique C^2 solution of our PDE "near" x_0.

Remark: The solution might be only local because characteristics may intersect.

Remark: We work in C^2 for convenience, but the theorem holds in C^1.

Proof. The proof has 3 key steps:

- The construction of the solution.

- The verification of the equation.

- Verify uniqueness.

- We would also like continuous/Lipschitz dependence on the initial data, but this part is omitted. [It uses the linearized equation]

We begin with the construction. We use the method of characteristics.

$$\begin{cases} \dot{x} = F_p(x, z, p) \\ \dot{z} = F_p(x, z, p) \cdot Du \\ \dot{p} = -F_u \cdot p - F_x \end{cases}$$

Find initial data for this system on $\Sigma : x(0) = x_0, z(0) = u_0(x_0), p(0) = p_0$, which is obtained by solution $F(x_0, u_0(x_0), p_0) = 0$. Recall that p_0 is broken into $p_T = Du_0(x_0)$ tangential, and p_N, normal. We can solve this smoothly with the Implicit Function Theorem by the noncharacteristic condition.

We solve the characteristic system with the initial data, and the outcome is a pair of functions

$$x(s, x_0), z(s, x_0), p(s, x_0),$$

which is defined for x_0 is a neighborhood of the original point, s in a neighborhood of 0.

We want to define one solution u: We set $u(x(s, x_0)) = z(s, x_0)$. Instead, we want u as a function of x,

$$(s, x_0) \to x$$

needs to be invertible, i. e. a local diffeomorphism.

This equivalent to having

$$\det \left[\frac{\partial x(s, x_0)}{\partial(s, x_0)} \right] \neq 0,$$

which is equivalent to the noncharacteristic condition(we showed this before). Now, we have u defined in a neighborhood at x_0.

The next objective is to show that for u, we have

$$\boxed{Du(x(x_0, s)) = p(x_0, s)}.$$

We'll do this next time, but suppose we have this condition. Then to show that u solves the equation, we need that $F(x(x_0, s), u(x_0, s), p(x_0, s)) = 0$. $\qquad\square$

§6 September 17th, 2020

§6.1 Fully Nonlinear PDEs, continued

We complete the proof of the theorem from last time. We have functions $x(t, y), z(t, y), p(t, y)$, the solutions to the characteristic system. They are defined in a neighborhood of x_0. We also saw that the map $(t, y) \mapsto x$ is a local diffeomorphism. Assuming we have this functions, we recall that z corresponds to u, p corresponds to ∇u, so we define the candidate solution u in a neighborhood of x_0 by setting

$$u(x(t, y)) = z(t, y).$$

This equation implies that u is defined in a neighborhood, but more subtlety, it assumes that we already have a solution. We defined p in terms of z, so how do we know that p actually corresponds to ∇u? We know that z has a C^1 dependence on t and y, so p is a C^1 function. Our objective is to show that

$$\nabla u(x(t, y)) = p(t, y).$$

Recall, we have that $F(x_0, u_0(x_0), p_0) = 0$, so we know that on Σ, we have $p = \nabla u$, by the choice of initial data for the characteristic system. We also know that on Σ we have $F(x, u, p) = 0$, $u = z$ by the choice of the normal component of p.

We claim that $F(x, z, p) = 0$ propagates along characteristics.

$$\begin{aligned}
\frac{d}{ds} F(x, z, p) &= F_x \cdot \dot{x} + F_z \cdot \dot{z} + F_p \cdot \dot{p} \\
&= F_x \cdot F_p + F_z F_p \cdot p - F_p (F_z \cdot p - F_x) \\
&= 0.
\end{aligned}$$

To prove the theorem, it remains to show that $p = \nabla u$ along each characteristic. We would like to show that

$$p_j(y, t) = \partial_{x_j} u(x(y, t)).$$

We will instead identify the derivatives of $u(x(y, t))$ with respect to y and t.

- WRT t:

$$\begin{aligned}
\partial_t u(x(y, t)) &= \partial_{x_j} u(x(y, t)) \cdot \partial_t x_j(y, t) \\
&= \partial_{x_j} u(x(y, t)) \cdot F_{p_j}(x(y, t), z(y, t), p(y, t))
\end{aligned}$$

We want this to equal $p_j \cdot F_{p_j}(x, z, p)$. Hence, we need to show that

$$\partial_t z(y, t) = p_j F_{p_j}(x, z, p),$$

by this is given by the characteristic equations.

- WRT y:

$$\partial_{y_k} u(x(y, t)) = (\partial_{x_j} u)(x(y, t)) \cdot \partial_{y_k} x_j(y, t)$$

We would like this to equal $p_j(y, t) \cdot \partial_{y_k} x_j(y, t)$. Recall that $\partial_{y_k} z(y, t) = p_j(y, t) \cdot \partial_{y_k} x_j(y, t)$, which is exactly what we need to show.

Let
$$r_k = \partial_{y_k} z(y,t) - p_j(y,t)\partial_{y_k} x_j(y,t).$$

We want to show that $r_k = 0$ along characteristics. We know that $r_k = 0$ at $t = 0$(on Σ).

We will show that r solves an ode:

$$\dot{r} = G \cdot r, r(0) = 0.$$

If this happens, then we get $r = 0$. Note that the linearization gives us equations for $\partial_{y_k} x, \partial_{y_k} z, \partial_{y_k} p$, with

$$\begin{cases} \left(\frac{\dot{\partial x_j}}{\partial y_k}\right) = F_{p_j x_\ell}\frac{\partial x_\ell}{\partial y_k} + F_{p_j z}\frac{\partial z}{\partial y_k} + F_{p_j p_m}\frac{\partial p_m}{\partial y_k} \\ \left(\frac{\dot{\partial z}}{\partial y_k}\right) = F_{p_j x_\ell}p_j\frac{\partial x_\ell}{\partial y_k} + F_{p_j z}p_j\frac{\partial z}{\partial y_k} + F_{p_j p_m}p_j\frac{\partial p_m}{\partial y_k} + F_{p_j}\frac{\partial p_j}{\partial y_k} \end{cases}$$

We don't actually need $\left(\frac{\dot{\partial p_\ell}}{y_k}\right)$.

Then

$$\begin{aligned} \dot{r}_k &= [\partial_{y_k}\dot{z(y,t)}] - p_j(y,t)[\partial_{y_k}\dot{x_j(y,t)}] \\ &= p_j \partial_{y_k} x_j(y,t) \\ &= \frac{\partial p_j}{\partial y_k}F_{p_j} - (F_z \cdot p_j + F_x)\frac{\partial p_j}{\partial y_k}. \end{aligned}$$

We use the fact that $F(x,z,p) = 0$ and apply $\frac{\partial}{\partial y_k}$:

$$0 = F_{x_j}\frac{\partial x_j}{\partial y_j} + F_z\frac{\partial z}{\partial y_k} + F_{p_j}\frac{\partial p_j}{\partial y_k}$$

Hence

$$\begin{aligned} \dot{r}_k &= -F_{x_j}\frac{\partial x_j}{\partial y_k} - F_z\frac{\partial z}{\partial y_k} + (\mathbb{F}_z \cdot p_j + F_{x_j})\frac{\partial x_j}{\partial y_k} \\ &= -F_z \cdot \left(\frac{\partial z}{\partial y_k} - p_j\frac{\partial x_j}{\partial x_j}\right) \\ &= -F_z \cdot r_k, \end{aligned}$$

which gives $r = 0$.

This gives existence. For uniqueness, note that two solutions will be the same along characteristics and the characteristic equations are satisfied for $(x, u_1, Du_1), (x, u_2, Du_2)$ with the same initial data so they must coincide.

Some concluding remarks:

- The method of characteristics works for scalar first order equations.

- We only get local solutions which last until characteristics start intersecting.

We have two situations where one can go further:

- Hamilton-Jacobi Equations:

$$\begin{cases} u_t + H(x, D_x u) = 0 \\ u(t = 0) = u_0 \end{cases}.$$

The system of characteristics is

$$
\begin{cases}
\dot{x} = H_p \\
\dot{z} = H_p \cdot p \\
\dot{p} = -H_x
\end{cases}.
$$

We can drop the middle equation for a simple pair of equations, which are the equations for Hamiltonian flows.

- Conservation Laws:

$$
\begin{cases}
u_t + F(u)_x = 0 \\
u(0) = u_0
\end{cases}.
$$

Characteristics intersect in this case, which leads to jump discontinuities(shocks). It also extends to the theory of systems.

§7 September 22nd, 2020

§7.1 Introduction to Distribution Theory

We introduce distributions in order to generalize the solutions of equations to functions that are not everywhere differentiable.

Example 7.1

Consider the equation
$$u_t - u_x = 0.$$

In the classic sense, we need u to be differentiable. Recall that $u(t, x) = u_0(t - x)$ solves the equation for $u_0 \in C^1$. However, even if u_0 is not differentiable, we only need it to be differentiable along the directional derivative - we can call that a "generalized solution".

Example 7.2

Let $f(x) = |x|$. Then,
$$f'(x) = \operatorname{sgn} x = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases}.$$
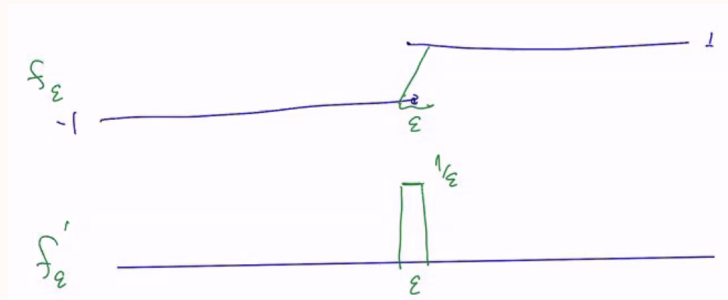
If we approximate sgn x with differentiable functions f_ϵ, then as $\epsilon \to 0$, $f'_\epsilon \to 2\delta_0$, the Dirac mass at $x = 0$, which has integral 2.

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§7.2 Frechet Space

We can define norms on C^k as follows
$$C^0 \Rightarrow \|u\|_{C^0} = \sup |u|, \|u\|_{C^1} = \sup |u| + \sup |\nabla u|, \ldots$$

We define the class of seminorms
$$P(K, m)(u) = \|u\|_{C^m(K)},$$

with K compact and m an integer. The space of seminorms is known as a **Frechet** space, or a **locally convex space**. In a Banach space, the topology is determined by balls $B(0, \epsilon)$(note the use of translation invariance). In the case of a Frechet space, we instead have balls determined by the seminorms:
$$\left(P_{K_1, m_1} + P_{K_2, m_2} + \cdots + P_{K_j, m_j}\right)(u) < \epsilon,$$

where we have finitely many P_{K_i, m_i}.

Convergence in a Frechet space is given by

$$u_j \to u \Leftrightarrow P_{K,m}(u_j - u) \to 0$$

for all K, m. The is the same as saying that

$$\partial^\alpha u_j \to \partial^\alpha u,$$

uniformly on compact sets.

§7.3 Continuous Functions with Compact Support

Definition 7.3. $x_0 \notin \operatorname{supp} u$ if there exists $\epsilon > 0$ such that

$$u = 0 \in B(x_0, \epsilon).$$

Example 7.4

Consider the function

$$u(x) = \begin{cases} 1 - x^2, x \in [-1, 1] \\ 0, \text{ otherwise} \end{cases}.$$

The support of u is given by $[-1, 1]$. Note that support is closed, which follows from the fact that the complement is open.

Fact 7.5. $\operatorname{supp} u$ is closed.

Proof. If $x_0 \notin \operatorname{supp} u$ then $u = 0 \in B(x_0, \epsilon)$ and for $y \notin B(x_0, \epsilon)$, we can find $\delta > 0$ such that $u = 0 \in B(y, \delta)$. Hence, for $u \in C(\mathbb{R}^n)$, $\operatorname{supp} u = \overline{\{x : u(x) \neq 0\}}$. $\qquad\square$

Consider the space $C_0^\infty, C_0^\infty(\mathbb{R}^n) = \mathcal{D}$, the space of continuous functions with compact support. This is a linear space, but it's not clear that the space is nonempty.

Example 7.6

The function

$$f(x) = \begin{cases} e^{-1/x}, x > 0 \\ 0, x \leq 0 \end{cases},$$

is a continuous function with support $[0, \infty)$. It is not clear that the function is infinitely differentiable though. But, we can take $f(x)f(1 - x)$, a function with support in $[0, 1]$.

Fact 7.7. If $O \subset \mathbb{R}^n$ is an open set, then there exists $f \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} f = \overline{O}$. This follows from Urysohn's lemma.

What topology do we put on $\mathcal{D}$?

Definition 7.8 (Convergence in $\mathcal{D}$). We let $u_m \to u$ in $\mathcal{D}$ if

- $\partial^\alpha u_m \to \partial^\alpha u$ uniformly

- There exists a compact set K such that supp $u_m \subseteq K$ for all m.

We cannot put Banach space on $\mathcal{D}$, since the norm cannot be well-defined in the case of C^∞. We also cannot put a Frechet space due to cardinality issues. However, we can say that

$$\mathcal{D} = \bigcup_m \{u \in C^\infty : \text{supp } u \subset B(0, m)\},$$

which is the union of locally convex spaces. This is the **inductive limit topology**. We will refer $\mathcal{D}$ as the space of **test functions**.

§7.4 The Space of Distributions

This is related to the notion of duality in functional analysis.

- If X is a Banach space, we can define X^*, the space of bounded linear functionals on X. More precisely, if we have

$$\|x^*\|_{X^*} = \sup_{\|x\|_X \leq 1} |x^*(x)|.$$

- If X is a Frechet space, X^* is the space of continuous linear functional on X. We can show that for any functional $x^* \in X^*$,

$$|x^*(x)| \leq c \sum_{i=1}^n P_{k_i}(x).$$

Definition 7.9 (Space of Distributions)**.** The space of distributions $\mathcal{D}'$ is the dual of $\mathcal{D}$, the space of test functions. For any $f \in \mathcal{D}'$, $f : \mathcal{D} \to \mathbb{R}(\mathbb{C})$, we require f to be continuous:

$$u_n \to u \in D, f(u_n) \to f(u).$$

Why can we think of distributions as generalized functions? For $u \in C^\infty(\mathbb{R}^n)$, $v \in \mathcal{D}$, we can define the action

$$u(v) = \int_{R^n} u \cdot v dx.$$

This allows us to identify $C^\infty(\mathbb{R}^n)$ with a subset of $\mathcal{D}'$. Note that this still works for $u \in C(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$. Furthermore, $u \in L^1_{loc}(\mathbb{R}^n)$, the functions that are integrable in any compact set.

We can conclude that $\mathcal{D}'$ contains all classical functions in the sense above, but also elements that are not functions.

Example 7.10

For the dirac mass δ_0, we have $\int \delta_0 u = u(0)$, so we can define $\delta_0(u) = u(0)$, and similarly, $\delta_x(u) = u(x)$. This gives us $\delta'_x(u) = -u'(x)$ and

$$\delta_x^{(}\alpha)(u) = (-1)^{|\alpha|} \partial^\alpha u(x).$$

> **Example 7.11**
>
> For a function $f \in L^1_{loc}$, α a multiindex,
>
> $$F(u) = \int_{R^n} f \cdot \partial^\alpha u \, dx.$$
>
> We cannot do $\sum \int f_k \partial^{\alpha_k} u \, dx$, unless the supp $f_k \to \infty$.

What is the topology on $\mathcal{D}'$?

Definition 7.12. $f_n \to f$ in $\mathcal{D}'$ if for every $u \in \mathcal{D}$, we have

$$f_n(u) \to f(u).$$

> **Example 7.13**
>
> If $f_n = \delta_{x_n}$ with $x_n \to y$, $f_n \to f = \delta_y$.

§8 September 24th, 2020

§8.1 The Space of Distributions, continued

Recall, we had $\mathcal{D}$, the space of test functions, a linear topological space. The notion of convergence is given by

$$u_n \to u \text{ if } \begin{cases} \exists K : \text{supp } (u_n) \subset K \\ \partial^\alpha u_n \to \partial^\alpha u \end{cases}.$$

We also defined $\mathcal{D}'$ as the space of distributions, continuous functions $f : \mathcal{D} \to \mathbb{R}$. We have $f_n \to f$ if $f_n(u) \to f(u)$ for all $u \in D$. We call this **convergence in the sense of distributions**.

Example 8.1

If we have $C^0 \supset f_n \to f$ uniformly,

$$f_n(u) = \int f_n \cdot u dx.$$

Then

$$\int f_n \cdot u dx \to \int f \cdot u dx,$$

by uniform convergence.

   This shows that classical convergence implies convergence in $\mathcal{D}'$.

Example 8.2

Suppose $f_n = \sin nx$, bounded functions. This converges pointwise almost nowhere. However,

$$\int f_n \cdot u dx = \int u(x) \sin (nx) dx$$
$$= \int \frac{1}{n} \cos (nx) \partial_x u(x) dx,$$

so

$$\left| \int f_n \cdot u dx \right| \le \frac{1}{n} \int_k C dx \to 0,$$

which implies that $\sin(nx) \to 0$ in $\mathcal{D}'$.

Proposition 8.3

$\partial_x : \mathcal{D} \to \mathcal{D}$ is continuous.

Example 8.4

We defined δ_0, the Dirac mass as some limit of bump functions,

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon}, |x| < \epsilon \\ 0, \text{ else} \end{cases}.$$

We have $\int f_\epsilon dx = 1$. We can show formally that $f_\epsilon \to \delta_0$ in $\mathcal{D}'$.

$$\begin{aligned} f_\epsilon(u) &= \int f_\epsilon \cdot u\, dx \\ &= \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} u\, dx \\ &= \frac{1}{2} \int_{-1}^{1} u(\epsilon x)\, dx \to u(0) = \delta_0(u) \end{aligned}$$

Remark: The same works if we take $f_\epsilon = \frac{1}{\epsilon}\varphi(\epsilon x)$ where

$$\begin{cases} \varphi \in \mathcal{D} \\ \int \varphi = 1 \end{cases}.$$

Example 8.5

Take $\varphi \in \mathcal{D}$, and define $f_n = \varphi(x - n)$.

$$f_n(u) = \int f_n(x) u(x)\, dx \to 0,$$

since the support of f_n and u are disjoint for large n.

§8.2 Properties of Distributions

For partial differential equations, we need to be able to compute $P(x, \partial)f$ for $f \in \mathcal{D}'$,

$$P(x, \partial) = \sum c_\alpha(x) \partial^\alpha.$$

- Multiplication by functions: if we have $\varphi \in C^\infty, f \in \mathcal{D}'$, what is φf?

  Suppose f was a function. Then $(\varphi f)(x) = \varphi(x)f(x)$, so

  $$\varphi f(u) = \int (\varphi f) u\, dx = \int f \cdot (\varphi u)\, dx = f(\varphi u),$$

  Definition 8.6. For $\varphi \in C^\infty, f \in \mathcal{D}', \varphi f(u) = f(\varphi u)$.

  Note that the map $f \to \varphi f$ is continuous in $\mathcal{D}'$.

- Differentiation of distributions: for $f \in C^1$,

  $$\partial_x f(u) = \int (\partial_x f) u\, dx = -\int f(\partial_x u)\, dx = -f(\partial_x u)$$

Definition 8.7. For $f \in \mathcal{D}'$, $\partial_x f(u) = -f(\partial_x u)$.

Observe that $\partial : \mathcal{D}' \to \mathcal{D}'$ is continuous. Furthermore, we have the Leibniz rule,

$$\partial(\varphi f) = \partial \varphi \cdot f + \varphi \cdot \partial f.$$

Example 8.8

$f(x) = |x|$, $\partial_x f = \operatorname{sgn} x$, and

$$\partial_x^2 f(u) = -\partial_x f(\partial_x u) = -\int \partial_x u \cdot \operatorname{sgn} u dx = \int_{-\infty}^{0} \partial_x u dx - \int_{0}^{\infty} \partial_x u dx = 2\delta_0(u).$$

We also have
$$\partial_x \delta_0 = \partial_x \delta_0(u) = -\delta_0(\partial_x u) = -\partial_x u(0) = \delta_0'(u).$$

---

§8.3 The support of distributions

Definition 8.9. Let $f \in \mathcal{D}'$. Then $x_0 \notin \operatorname{supp} f$ if there exists $\epsilon > 0$ such that $f(u) = 0$ whenever $\operatorname{supp} u \in B_\epsilon(x_0)$.

Note that $(\operatorname{supp} f)^c$ is open, so $\operatorname{supp} f$ is closed.

Proposition 8.10

Any closed set is the support of some distribution.

Example 8.11

If we take δ_{x_0}, $\operatorname{supp}(\delta_{x_0}) = \{x_0\}$.

We let $\mathcal{E}' \subset \mathcal{D}'$ as the distributions with compact support.

Proposition 8.12

If $f \in \mathcal{D}'$ has compact support, then f extends to a continuous functional $f : C^\infty \to \mathbb{R}$.

Proof. We have $f : \mathcal{D} \to \mathbb{R}$ with compact support. Let $\varphi \in \mathcal{D}$, so that $f = 1$ in $B(0,1)$. We also define $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$. Then $\varphi_R \in D$ and $\varphi_R = 1 \in B(0,R)$.

Let $R \to \infty$.
$$\lim_{R \to \infty} \varphi_R = 1$$

in $C^\infty = \mathcal{E}$, but not $\mathcal{D}$.

Suppose we take $f \in \mathcal{D}'$, with compact support. Let $c \in \mathcal{E}$. Then

$$v = \lim_{\mathcal{E}} \varphi_R v.$$

Define $f(v) = \lim_{R \to \infty} f(\varphi_R v)$.

Take
$$f(\varphi_{R_1} v) - f(\varphi_{R_2} v) = f((\varphi_{R_1} - \varphi_{R_2})v).$$

Then if $R_1 \subseteq R_2$, supp $(\varphi_{R_1} - \varphi_{R_2}) \subset B(0, R_1)^c$. Since f is compactly supported,

$$f((\varphi_{R_1} - \varphi_{R_2})v) = 0$$

if $R_1 \subset R_2$ are large enough.

Exercise: Show that this definition does not depend on the choice of φ.

Exercise: Show that $f : \mathcal{E} \to \mathbb{R}$ is continuous.

Exercise: Prove the converse: if $f : \mathcal{E} \to \mathbb{R}$ is continuous, then $f \in \mathcal{E}'$.

$\square$

Example 8.13

Consider the PDE $(\partial_t - \partial_x)u = 0$. Formally, any function $u(t, x) = g(t + x)$ should solve this.

Claim: let $g \in L^1_{loc}$. Then $(\partial_t - \partial_x)u = 0$.

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§9 September 29th, 2020

Last time,

- We introduced $\mathcal{D}$, the space of test functions, which are smooth with compact support. Then $\mathcal{E}$ denotes the smooth functions.

- We denoted $\mathcal{D}'$, the space of distributions, and $\mathcal{E}'$, the space of compactly supported distributions.

§9.1 Regularity of Compactly Supported Distributions

We introduced C^k, the set of k-times differentiable functions. We call k the **order of regularity**.

Fix $K \subset R^n$ compact, and consider $\mathcal{D}'(K)$. To measure such distributions, we only need to consider test functions restricted to K, If we consider $\mathcal{D}(2K)$, a locally convex space with seminorms

$$P_\alpha(u) \sup_{2K} |\partial^\alpha u|,$$

$$P_{\leq k}(u) = \sup_{x \in 2K} \sup_{|\alpha| \leq K} |\partial^\alpha u(x)|.$$

If $f \in \mathcal{D}'(K)$, then f must be controlled by finitely many seminorms. But the seminorms are ordered in increasing order, so it suffices to consider a single index k. Namely, there exists k such that $f(u) \leq C P_{\leq k}(u)$.

Definition 9.1. The smallest k for which the above holds gives the order of f, i. e. $-k$.

For arbitrary distributions f:

$$f = \sum \chi_j f = \sum f_j$$

where χ_j is supported in $\{2^{j-1} \leq |x| \leq 2^{j+1}\}$, with $1 = \sum \chi_j$. The supports of f_j are going to infinity. Each f_j has an order k_n, but it is possible for $k_j \to \infty$.

Example 9.2

Take $f \in L_0^1 \subset \mathcal{D}'$. Then $\partial^k f \in \mathcal{D}'$. What is the order of $\partial^k f$?

$$\partial^k f(u) = (-1)^{|k|} \int f \partial^k u \, dx,$$

so

$$|\partial^k f(u)| \leq \int |f| \sup |\partial^k u| \, dx \leq \sup |\partial^k u| \int |f| \, dx.$$

Hence, the order of $\partial^k f$ is greater than or equal to $-k$.

Example 9.3

The Dirac mass, δ_0 has order 0, since $\delta_0(u) = u(x)$ and $|\delta_0(u)| \leq \sup |u|$. Similarly, δ_0' has order -1, δ_0'' has order -2.

> **Theorem 7**
>
> Any distribution f in a compact set admits a finite representation
>
> $$f = \sum_{k \in K} \partial^{\alpha_k} f_k,$$
>
> where $f_k \in L_0^1$ and K is finite.

§9.2 Convolutions

Note that $(\mathcal{E}, +, \cdot)$ is an **algebra**, which is a vector space under addition and is closed under the multiplication operation.

If we have two test functions $u, v \in \mathcal{D}$, define

$$u * v(x) = \int u(y)v(x-y)dy.$$

The integral is well-defined since both functions have compact support.

We have some properties:

- Symmetry, $u * v = v * u$. We see this by making a change of variables, $x - y = z$, $y = x - z$. Then,

$$u * v(x) = \int u(x-z)v(z)dz = v * u(x).$$

- Suppose $u * v(x) \neq 0$. Then, the integrand cannot be identically 0, so there exists $y \in \mathbb{R}^n$ so that $u(y)v(x-y) \neq 0$. Then $y \in \operatorname{supp} u$, $x - y \in \operatorname{supp} v$. Then

$$x = y + x - y \in \operatorname{supp} u + \operatorname{supp} v = \{a + b : a \in \operatorname{supp} u, b \in \operatorname{supp} v\}.$$

  Hence, we have

$$\operatorname{supp} u * v \subset \operatorname{supp} u + \operatorname{supp} v.$$

- Regularity of convolutions:

$$u * v(x) = \int u(y)v(x-y)dy.$$

  If we differentiate under the integral,

$$\partial(u * v)(x) = \int u(y)\partial v(x-y)dy.$$

  Hence

$$\partial(u * v) = u * \partial v = \partial u * v.$$

- The convolution of test functions is a test function: $\mathcal{D} * \mathcal{D} \to \mathcal{D}$.

- Observe that for $u * v$ to be well defined, it is enough that one of them has compact support. Hence $\mathcal{D} * \mathcal{E} \to \mathcal{E}$.

- For rough functions, we can extend this as $u \in \mathcal{D}$, $v \in L_0^1$, where

$$u * v(x) = \int u(x - y)v(y)dy,$$

so

$$|u * v(x)| \leq \sup |u| \int |v|dx = \|u\|_\infty \cdot \|v\|_1.$$

This gives the inequality,

$$\|u * v\|_\infty \leq \|v\|_1 \cdot \|u\|_\infty.$$

In addition, if u is smooth, then $u * v$ is also smooth, so

$$\partial(u * v) = \partial u * v \Rightarrow \mathcal{D} * L_0^1 \to \mathcal{D}.$$

- Associativity: $(u * v) * w = u * (v * w)$.

  Proof.

$$(u * v) * w(x) = \int (u * v)(y)w(x - y)dy$$

$$= \iint u(z)v(y - z)w(x - y)dydz.$$

Similarly,

$$u * (v * w)(x) = \iint u(x - y)v(z)w(y - z)dydz.$$

If we identify the corresponding arguments, the integrals are equal via a change of coordinates. $\square$

- $(\mathcal{D}, +, *)$ is an algebra.

§9.3 Convolution of Distributions

Note that

$$f * u(x) = \int f(y)u(x - y)dy = f(u(x - \cdot)),$$

hence, the RHS is well defined for any distribution f.

Definition 9.4. $f * u(x) = f(u(x - \cdot))$.

Observe that $u(x - \cdot)$ is differentiable in x, so $f * u$ is differentiable and

$$\partial f * u(x) = f(\partial u(x - \cdot))) = f(-\partial_y u(x - \cdot)) = \partial f(u(x - \cdot)),$$

hence,

$$\partial_x(f * u) = f * \partial u = \partial f * u.$$

It follows that $\mathcal{D}' * \mathcal{D} = \mathcal{E}$. We can also show that $\mathcal{E}' * \mathcal{D} = \mathcal{D}$.

Example 9.5

$$\delta_0 * u(x) = \delta_0(u(x - \cdot)) = u(x).$$

In this sense, δ_0 acts like the identity with respect to convolutions.

Similarly,

$$\delta_{x_0} * u(x) = u(x - x_0).$$

---

§10 October 1st, 2020

§10.1 Convolutions of Distributions, continued

- Recall the convolution,

$$u * v(x) = \int u(x - y)v(y)dy,$$

which is a map from $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and $\mathcal{D} \times \mathcal{E} \to \mathcal{E}$.

- We also saw that $\mathcal{D}' \times \mathcal{D} \to \mathcal{E}$. Similarly, $\mathcal{E}' \times \mathcal{D} \to \mathcal{D}$.

- Note that supp $(u * v) \subset$ supp $u +$ supp v, $u * v = v * u$ and $u * (v * w) = (u * v) * w$. This shows that we have a commutative algebra structure on $\mathcal{D}$.

- δ_0 is the identity with respect to convolutions, $\delta_0 * u = \delta_0$.

- Consider $\mathcal{D}' * \mathcal{D} :$
$$f * u(x) = f(u(x - \cdot)).$$

We will continue to fully extend the notion of convolutions to distributions. If we take $f \in \mathcal{D}'$ and $g \in \mathcal{E}'$, we expect $\mathcal{D}' * \mathcal{E}' \to \mathcal{D}'$, since we don't have regularity or compact support for $\mathcal{D}'$. Similarly, we expect $\mathcal{E}' * \mathcal{E}' \to \mathcal{E}'$. If f, g were functions,

$$\begin{aligned}
f * g(\varphi) &= \int f * g(x)\varphi(x)dx \\
&= \iint f(y)g(x - y)\varphi(x)dxdy \\
&= \iint f(y)g(z)\varphi(y + z)dydz \\
&= \int f(y)\left(\int g(z)\varphi(y + z)dx\right)dy \\
&= \int f(y)g_z(\varphi(y + \cdot_z))dz \\
&= f_y(g_z(\varphi(\cdot_y + \cdot_z))).
\end{aligned}$$

Hence $f * g(\varphi) = f_y(g_z(\varphi(\cdot_y + \cdot_z)))$ for functions. We take this as the definition for distributions.

> **Proposition 10.1**
>
> This notion of convolution extends the notion of convolution for functions and has the same properties.
>
> Recall that
> $$\partial(f * g) = \partial f * g = f * \partial g.$$

§10.2 Distributional Solutions to PDEs

Consider constant coefficient partial differential equations:

$$P(\partial) = \sum)|\alpha| \le kc_\alpha \partial^\alpha.$$

We would like to solve the equation $P(\partial)u = f$ in the smooth functions or in distributions: suppose we have $f \in \mathcal{D}'$. Do we have a solution u? This question is split into questions

of existence and uniqueness(and continuous dependence, but this makes more sense to ask for nonlinear problems).

We refine the questions:

- For $f \in \mathcal{D}$, we wish to find a local solution u.

- We would like uniqueness with restricted behavior at infinity, but which requires considering the kernel of $P(\partial)$. Consider exponential solutions, $u(x) = e^{x \cdot y}$. Then

$$\partial_j(e^{x \cdot y}) = y_j e^{x \cdot y},$$

$$\partial_x^\alpha = y^\alpha e^{x \cdot y},$$

    so

$$P(\partial)e^{x \cdot y} = P(y)e^{x \cdot y}.$$

    Hence, $e^{x \cdot y}$ is in the kernel of $P(\partial)$ if and only if $P(y) = 0$. In 1-d, we have finitely many roots, but in higher dimensions we have infinitely many roots. Hence, we consider the restricted setting, with solutions which do not grow exponentially, for example, $y = i\xi$. Then, $u = e^{ix\xi}$ is a solution if $P(i\xi) = 0$.

§10.3 Fundamental Solutions

> **Theorem 8** (Malgrange-Ehrenpreis)
>
> All constant coefficient linear PDE's are solvable.

We will prove the theorem after developing Fourier Anaylsis. It uses the notion of a fundamental solution for constant coefficient linear PDE's. We try to solve the equation $P(\partial)u = \delta_x$. Then, we translate our solution by x, which uses the key property that $P(\partial)$ is invariant with respect to translations.

Definition 10.2. $K \in \mathcal{D}'$ is a fundamental solution for $P(\partial)$ if $P(\partial)K = \delta_0$.

> **Proposition 10.3**
>
> If K is a fundamental solution, then the function
>
> $$u = K * f$$
>
> solves
>
> $$P(\partial)u = f.$$

Proof.
$$P(\partial)u = P(\partial)(K * f) = P(\partial)K * f = \partial_0 * f = f.$$

$\square$

Remark: This is useful for existence, but not for uniqueness.

Example 10.4

Consider $P(\partial) = \partial_x$ in $\mathbb{R}$. We would like to solve $\partial_x K = \partial_0$. Take

$$
K(x) = \begin{cases} 0, x < 0 \\ 1, x > 0 \end{cases} \quad,
$$

the Heaviside Function. More generally, we can take $K + H + c$ for some constant c (which gives all solutions by HW 2.1).

If we want to solve $\partial_x u = f$, one of the solutions is given by $u = f * H$. If f is a nice function, then

$$
u(x) = \int_{\mathbb{R}} H(y)f(x-y)dy = \int_0^\infty f(x-y)dy = \int_{-\infty}^x f(z)dz.
$$

This recovers the fundamental theorem of calculus.

Note that

$$
f * c = \int f(y) \cdot c\,dy = c \int f,
$$

which is a constant for compactly supported f.

In general, K is not unique, even in the class of bounded functions.

Note that supp $H = [0, \infty)$ so supp $f * H \subseteq$ supp $f + [0, \infty)$, which moves the support to the right. Then H is the unique forward solution, and $H - 1$ is a backward solution. Note that

$$
(H-1) * f = \int (H-1)(y)f(x-y)dy = -\int_{-\infty}^0 f(x-y)dy = -\int_x^\infty f(z)dz.
$$

Using H, we solve

$$
\begin{cases} \partial_x u = f \\ u(-\infty) = 0 \end{cases}
$$

and with $H - 1$, we solve

$$
\begin{cases} \partial_x u = f \\ u(\infty) = 0 \end{cases} \quad.
$$

§11 October 6th, 2020

§11.1 Laplacians

We have a PDE $P(\partial)u = f$ and $K \in \mathcal{D}'$ is a fundamental solution for $P(\partial)$ if $P(\partial)u = \delta_0$. Then, $u = K * f$ solves the PDE.

Example 11.1 (1D Laplacian)

Take $P = \partial_x^2$. We want to solve the equation $\partial_x^2 K = \delta_0$. Then, integrating once gives $\partial_x K = H + C$, where H is the heaviside function and C is a constant. Integrating again, we get that
$$K = \begin{cases} 0, x < 0 \\ x, x > 0 \end{cases} + Cx + D.$$

Note that $Cx + D$ solves the homogeneous problem.

We have some special choices of solutions:

- The forward solution:
$$K = \begin{cases} 0, x < 0 \\ x, x > 0 \end{cases}.$$

- The backward solution:
$$K = \begin{cases} -x, x < 0 \\ 0, x > 0 \end{cases}.$$

- The symmetric solution:
$$K = |x|/2.$$

Example 11.2

Let $P = \partial_x^2 - 1$.

First, we solve the homogeneous equation:

$$(\partial_x^2 - 1)K = 0.$$

The characteristic polynomial is $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$. Hence, $u_{1,2} = e^{\pm x}$.

Next, we find the fundamental solution. We solve $PK = \delta_0$. K must solve the homogeneous equation to the left and right of 0.

$$K(x) = \begin{cases} c_1^- u_1 + c_2^- u_2, x < 0 \\ c_1^+ u_1 + c_2^+ u_2, x > 0 \end{cases}$$

Then, $\partial_x K = K'(x) + [K(0)] \cdot \delta_0$. where $K'(x)$ is the formal derivative of the above and $[K(0)]\delta_0$ is the jump of K at 0. Furthermore,

$$\partial_x^2 K = K''(x) + [K'(0)]\delta_0 + [K(0)]\delta_0'.$$

It follows that

$$(\partial_x^2 - 1)K = K''(x) - K + \partial_x^2 K,$$

so

$$(\partial_x^2 - 1)K = [K'(0)]\delta_0 + [K(0)]\delta_0' = \delta_0,$$

hence, $[K(0)] = 0$ and $[K'(0)] = 1$.

By adding multiples of the homogeneous solution, we can ensure that $c_1^+ = 0, c_2^- = 0$.

Hence,

$$K(x) = \begin{cases} c_1^- e^x, x < 0 \\ c_2^+ e^{-x}, x > 0 \end{cases}.$$

Using $[u(0)] = 0$ gives $c_1^- = c_2^+$. The second condition $[K'(0)] = 1$ gives $c_1^- + c_2^+ = 1$, so

$$K(x) = \frac{e^{-|x|}}{2},$$

which decays exponentially at both ends.

Example 11.3

We solve $P = \partial_x^2 + 1$. The solutions to the characteristic equations are $\pm i$ so $e^{\pm ix} = \cos x \pm i \sin x$ are homogeneous solutions. Equivalently, we can take $\cos x$ and $\sin x$, since we take complex linear combinations. We get $K(x) = \frac{1}{2} \sin |x|$ and we can add multiples of the homogeneous solutions. Namely,

- The forward solution is $\sin x \cdot 1\{x > 0\}$.

- The backward solution is $-\sin x \cdot 1\{x < 0\}$.

§12 Wave Equation in 1+1 Dimension

We solve $P = \partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x)$.

We introduce null coordinates: $u = t + x, v = t - x$. Then $t = \frac{u+v}{2}, x = \frac{u-v}{2}$. Hence,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\frac{\partial v}{\partial t} = \partial_u + \partial t.$$

Similarly,

$$\frac{\partial}{\partial x} = \partial_u - \partial_v.$$

This implies that

$$\partial_t + \partial_x = 2\partial u, \partial_t - \partial_x = 2\partial v,$$

so

$$P = (\partial_t + \partial_x)(\partial_t - \partial_x) = 4\partial_u\partial_v.$$

In (u, v) coordinates, we solve

$$4\partial_u\partial_v K = \delta_{0,0} = \delta_{u=0} \cdot \delta_{v=0}.$$

We look for solutions

$$K(u, v) = K_1(u)K_2(v).$$

Then, we solve

$$\partial_u K_1 = \frac{1}{2}\delta_{u=0}, \partial_v(K_2) = \frac{1}{2}\delta_{v=0}.$$

The forward solutions($t > 0$) are $K_1 = \frac{1}{2}H(u), K_2 = \frac{1}{2}H(v)$, so $K = \frac{1}{4}H(u)H(v)$.

We would like to switch this back in the (t, x) coordinates. We have

$$P_{u,v}K = \delta_{0,0},$$

and this corresponds to

$$P_{x,y}K = J\delta_{0,0} = \frac{\delta_{0,0}}{2}.$$

where $J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$.

Hence, the fundamental solution in (x, y) is

$$K = \frac{1}{2} \cdot 1_{t>x,t>-x}.$$

The solutions to the homogeneous equation are as follows:

$$\partial_u\partial_v\varphi = 0,$$

so

$$\partial_v\varphi = c(v)$$

and it follows that

$$\varphi = f(u) + g(v).$$

Going back to x, y coordinates, we find out that

$$\varphi(x, t) = f(t + x) + g(t - x),$$

which are both solutions to transport equations, $\partial_t - \partial_x, \partial_t + \partial_x$.

These solve homogeneous equations for f, g differentiable, f, g bounded and $f, g \in \mathcal{D}'$.

How do we interpret $f(t + x)$ as a distribution in (t, x)? If f is a function, then $\tilde{f} = f(t + x)$, and

$$\tilde{f}(\varphi) = \int \tilde{f}\varphi dt dx = \int f(t+x)\varphi dt dx = \frac{1}{2}\int f(u)\varphi(u, v)du dv = \int f(u)\left(\frac{1}{2}\int \varphi dv\right) du = f(\widehat{\varphi}),$$

where $\widehat{\varphi} = \int \varphi/2$.

§12.1 Back to Theory of Distributions

We would like to approximations of the identity. The idea is to approximate δ_0 with smooth functions. Let $\varphi \in \mathcal{D}$ with $\int \varphi = 1$. Then $\varphi_\epsilon(x) = e^{-n}\varphi(x/\epsilon)$, so the the integral is invariant.

Proposition 12.1

$\varphi_\epsilon \to \delta_0$ is in $\mathcal{D}'$.

Proof.

$$\varphi_\epsilon(u) = \int \varphi_\epsilon(x) \cdot u(x) dx$$

$$= \int \varphi(y) \cdot u(\epsilon y) dy$$

$$\to \int \varphi u(0) dy = u(0).$$

So $\varphi_\epsilon \to \delta_0$ in $\mathcal{D}'$ Hence φ_ϵ is an approximation of the identity. We can also call this a **mollifier**. $\square$

For $f \in \mathcal{D}'$, we define the regularizations as $f_\epsilon = f * \varphi_\epsilon \in \mathcal{E}$.

Note that
$$\lim_{\epsilon \to 0} f_\epsilon = \lim_{\epsilon \to 0} f * \varphi_\epsilon = f * \delta_0 = f$$

in $\mathcal{D}$.

For $f_\epsilon \in \mathcal{E}$, $f \in \mathcal{D}'$ and for $f_\epsilon \in \mathcal{D}$, $f \in \mathcal{E}'$.

Example 12.2

Consider the topology on $C_0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}\}$ with f continuous and $\lim_{x \to \infty} f(x) = 0$.

Then
$$\|f\|_{C_0} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

We have some properties:

- $f_\epsilon \to f$ is in $C_0(\mathbb{R}^n)$.

- $|\partial^\alpha f_\epsilon| \leq c_\alpha \epsilon^{-|\alpha|}$.

We could also consider $f \in L^p$ for $1 \leq p < \infty$. Then $f_\epsilon \to f$ in L^p.

---

§13 October 8th, 2020

§13.1 Homogeneous Functions

Definition 13.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of order m if $f(\lambda x) = \lambda^m f(x)$ for $\lambda \in \mathbb{R}^+$.

Example 13.2 (Homogeneous Functions in $\mathbb{R}$)

If $m \in \mathbb{N}$, the function $f(x) = x^m$ is homogeneous. More generally in $\mathbb{R}$,

$$f(x) = x_+^m = \begin{cases} 0, x < 0 \\ x^m, x > 0 \end{cases}$$

or

$$f(x) = x_-^m = \begin{cases} (-x)^m, x < 0 \\ 0, x > 0 \end{cases}.$$

Linear combinations of these work as well.

Example 13.3 (Homogeneous Functions in $\mathbb{R}^n$)

We consider Polar coordinates: $x = r\Theta$ for $\Theta \in S^n$. Then $f(x) = r^m g(\Theta)$.

These functions are directly distributions if they are locally integrable. The happens in $\mathbb{R}^n$ if $m > -n$. For example, $\frac{1}{|x|^n}$ is not integrable(a borderline case).

Suppose f is m-homogeneous. Then ∂f is $m - 1$ homogeneous.

§13.2 Homogeneous Distributions

Let $u_\lambda = \frac{1}{\lambda^n} u(x/\lambda)$ for $u \in \mathcal{D}$, $\lambda > 0$. In particular, $\int u_\lambda = \int u$.

Consider a homogeneous function f of order m. Then,

$$\int f(\lambda x) u(x) dx = \lambda^m \int f(x) u(x) dx.$$

Furthermore,

$$\int f(\lambda x) u(x) dx = \frac{1}{\lambda} \int f(y) u(y/\lambda) dy.$$

Therefore,

$$f(u_\lambda) = \lambda^m f(u).$$

Definition 13.4. A function f is homogeneous of order m if $f(u_\lambda) = \lambda^m f(u)$.

Proposition 13.5

If f is m homogeneous, then ∂f is $m - 1$ homogeneous.

> **Example 13.6**
>
> Note that
> $$\delta_0(u_\lambda) = u_\lambda(0) = \frac{1}{\lambda^n} u(0) = \frac{1}{\lambda^n} \delta_0(u).$$
>
> Hence δ_0 is homogeneous of order $-n$. Similarly, $\delta_0^{(\alpha)}$ is of order $-n - |\alpha|$.

> **Example 13.7**
>
> We classify all homogeneous distributions of order -1 in $\mathbb{R}$. One example is δ_0, but this is only supported at 0. If we assign the value at $x > 0$ and $x < 0$ independently, we have to consider $1/x$ and $1/|x|$.
>
> If we have $1/x$,
> $$\int \frac{u(x)}{x} dx, u \in \mathcal{D}$$
> will be easy if $u(0) = 0$, but if $u(0) \neq 0$, we have logarithmic divergence. However,
> $$\mathrm{pv} \int \frac{u(x)}{x} dx = \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus [-\epsilon, \epsilon]} \frac{u(x)}{x} dx.$$
>
> Now, the integral exists. In particular,
> $$\lim_{\epsilon \to 0} \int_{[1,-1] \setminus [-\epsilon,\epsilon]} \frac{u(x)}{x} dx = \lim_{\epsilon \to 0} \int (u(x) - u(0))/x dx + \int u(0)/x dx,$$
>
> where the right term is 0 and the left term is well defined.
>
> The function $1/|x|$ turns out to not be homogeneous of order -1.
>
> Hence, the space of homogenous distributions of order -1 is spanned by δ_0, p. v.$\frac{1}{x}$.

> **Proposition 13.8**
>
> There exists a well-defined homogeneous distribution of order -1, which is odd and equals $1/x$ away from 0. We will call this
> $$f = \mathrm{pv} \, \frac{1}{x}.$$

What about for distributions of order $m < -1$. We take $\partial_x x_+^m = x_+^{m-1}$ and $\partial_x x_-^{m-1} = -m x_-^{m-1}$ as definitions. For $m = 0$, we get the homogeneous distributions of order -1.

For homogeneous distributions of order $-n$ in $\mathbb{R}^n$, we have δ_0 and $f = \mathrm{pv}|x|^{-n} g(\Theta)$, where $\int_{S^n} g = 0$.

> **Example 13.9** (Calderon-Zygmond Operators)
>
> Let $P(\partial)u = f$. Let K be a fundamental solution. The $u = K * f$ and $f \to K * f$ is our solution operator.
>
> In 1D, for convolutions of order -1, we have δ_0 and pv $1/x$, which gives $f \to$ pv $1/x * f = Hf$, the Hilbert transform of f. Note that $H^2 = -1$.
>
> We have similar operators in higher dimensions.

§14 October 13th, 2020

§14.1 Introducing the Fourier Transform

We use the notation $D_j = \frac{1}{i}\partial_j$. Then, we consider constant coefficients of the form $P(D)$. Denote $f_\xi(x) = e^{ix\xi}$. Then $D_j f_\xi(x) = \xi_j f_\xi(x)$. Hence, we can think of $f_\xi(x)$ as a joint eigenfunction for all constant coefficient linear PDOs.

Consider an arbitrary function $u(x)$. We then think of $u(x)$ of a linear combination of $e^{ix\xi}$'s. With $\xi \in \mathbb{R}^n$, we represent

$$u(x) = \int f(\xi)e^{ix\xi}d\xi.$$

In $\mathbb{R}^n$, suppose $u \in \mathbb{R}^n$ and $\{e_j\}$ an orthonormal frame. Then, we can represent

$$u = \sum c_j e_j,$$

where $c_j = \langle u, e_j \rangle \in \mathbb{R}$.

On the space of functions,

$$\langle u, v \rangle = \int_{\mathbb{R}^n} u(x)\overline{v}(x)dx.$$

This is also the inner product in $L^2(\mathbb{R}^n)$.

Definition 14.1. The Fourier Transform is

$$u(x) \mapsto \widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int u(x)e^{-ix\xi}dx.$$

The Fourier Inversion Formula is given by

$$u(x) = \frac{1}{(2\pi)^{k/2}} \int \widehat{u}(\xi)e^{ix\xi}d\xi.$$

§14.2 Properties of the Fourier Transform

We have

$$\begin{aligned}
\widehat{\partial_j u}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int \partial_j u(x)e^{-ix\xi}dx \\
&= \frac{i\xi_j}{(2\pi)^{n/2}} \int u(x)e^{-ix\xi}dx \\
&= i\xi_j \widehat{u}(\xi) \\
\Rightarrow \widehat{D_j u}(\xi) &= \xi_j \widehat{u}(\xi) \\
\Rightarrow \widehat{P(D)u}(\xi) &= P(\xi)\widehat{u}(\xi)
\end{aligned}$$

Hence, we have a dictionary $\partial_{x_j} \to i\xi_j$, $D_{x_j} \to \xi_j$, $P(D) \to P(\xi)$.

We also have

$$\begin{aligned}
\widehat{x_j u}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int x_j u(x)e^{-ix\xi}dx \\
&= \frac{i}{(2\pi)^{n/2}} \int u(x)\partial_{\xi_j} e^{-ix\xi}dx \\
&= i\partial_{\xi_j} \frac{1}{(2\pi)^{n/2}} \int u(x)e^{ix\xi}dx \\
&= i\partial_{\xi_j} \widehat{u}(\xi).
\end{aligned}$$

§14.3 Schwartz Space

We will consider a space $\mathcal{S}$ with $\mathcal{D} \subset \mathcal{S}\mathcal{E}$. This is the **Schwartz space**. This is a locally convex space, with seminorms

$$P_{\alpha\beta}(u) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)|.$$

Functions in the Schwartz space are "rapidly decreasing", in that they decrease faster than any polynomial decay.

> **Example 14.2**
> Consider $e^{-x^2} \in \mathcal{S}$. Note that $\frac{1}{1+x^2} \notin \mathcal{S}$. If $u \in \mathcal{S}$, then $|u(x)| \leq \frac{c_N}{(1+|x|)^N}$.

> **Proposition 14.3**
> The Fourier transform takes $\mathcal{S} \to \mathcal{S}$.

Proof. Note that $\partial_x : \mathcal{S} \to \mathcal{S}$ and $x : \mathcal{S} \to \mathcal{S}$. We consider the multiplication operator $u \to x \cdot u$. Take $u \in \mathcal{S}$. We want to show that $\widehat{u} \in \mathcal{S}$. We want to show that the function $\xi^\alpha \partial_\xi^\beta \widehat{u}$ is bounded for all multi-indexes α, β.

$$\xi^\alpha \partial_\xi^\beta \widehat{u} = i^{|\alpha|-|\beta|} \widehat{\partial_x^\alpha x^\beta u} \in \mathcal{S}$$

so we have an inequality

$$|\partial_x^\alpha x^\beta u| \leq \frac{c_{N,n}}{(1+|x|)^N}.$$

Then

$$|\xi^\alpha \partial_\xi^\beta \widehat{u}(\xi)| = \left| \frac{1}{(2\pi)^{n/2}} \int e^{ix\xi} \partial_x^\alpha x^\beta u(x) dx \right|$$
$$\leq \int |\partial_x^\alpha x^\beta u(x)| dx$$
$$\leq c_n \int \frac{1}{1+|x|^N} dx < \infty$$

for large N. Hence $P(\alpha\beta)(\widehat{u}) < \infty$ which implies that $\widehat{u} \in \mathcal{S}$. $\qquad \square$

> **Corollary 14.4**
> The inverse Fourier Transform also maps $\mathcal{S} \to \mathcal{S}$.

§14.4 Fourier Inversion

Observe that for $u \in L^1$,

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int u(x) e^{-ix\xi} dx,$$

Note that

$$|\widehat{u}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \int |u(x)| dx = \frac{\|u\|_1}{(2\pi)^{n/2}}.$$

Hence,

$$\|\widehat{u}\|_\infty \leq \frac{\|u\|_1}{(2\pi)^{n/2}}.$$

By the Lebesgue Dominated Convergence theorem, $\widehat{u} \in C_{\to 0}(\mathbb{R}^n)$.

Exercise 14.5. $\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{Id}$ on $\mathcal{S}$.

§15 October 15th, 2020

§15.1 Fourier Transform in L^2

Suppose we take a function $u \in \mathcal{S}$. We can define $\widehat{u} \in \mathcal{S}$. Similarly, if we look at $f \in \mathcal{S}$ we can consider $\check{f} \in \mathcal{S}$.

Consider

$$\int \widehat{u}(\xi)\overline{f}(\xi)d\xi = \frac{1}{(2\pi)^{n/2}} \iint u(x)e^{-ix\xi}\overline{f}(\xi)d\xi dx$$

$$= \frac{1}{(2\pi)^{n/2}} \iint u(x)\overline{e^{ix\xi}f(\xi)}d\xi dx$$

$$= \int u(x)\overline{\check{f}}dx.$$

Consider the special case where $f = \widehat{u}, \check{f} = u$. Then we have

$$\int |\widehat{u}(\xi)|^2 d\xi = \int |u(x)|^2 dx,$$

so in particular,

$$\|\widehat{u}\|_2 = \|u\|_2.$$

This is known as the **Plancherel Formula.** It tells us that the map

$$u \mapsto \mathcal{F}u$$

is an isometry with resepct to the L^2 norm. We use this to extend $\mathcal{F}: L^2 \to L^2$. Namely, suppose $u \in L^2(\mathbb{R}^n)$. Then, there exists a sequence $u_m \in \mathcal{S}$ (or $\mathcal{D}$) so that $u_m \to u \in L^2$. Then

$$\|\widehat{u_m} - \widehat{u_k}\|_2 = \|u_m - u_k\|_2 \to 0,$$

hence $\widehat{u}_m$ is Cauchy in L^2, which implies convergence. We denote

$$\widehat{u} = \lim_{n \to \infty} \widehat{u_m} \in L^2.$$

Then

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x)e^{-ix\xi}dx$$

$$= \lim_{R \to \infty} \int_{B(0,R)} u(x)e^{-ix\xi}dx$$

$$= \lim_{R \to \infty} \int_{\mathbb{R}^n} 1(x/R)u(x)e^{-ix\xi}dx$$

for a cutoff function $1(x/R)$.

§15.2 Fourier Transform of Distributions

If we omit the conjugates in the above, we have

$$\int \widehat{u}(\xi)f(\xi)d\xi = \int u(x)\widehat{f}(x)dx$$

If we interpret $u, \widehat{u}$ as distributions, then

$$\widehat{u}(f) = u(\widehat{f}).$$

We would like to use this to define the Fourier Transform of distributions, but note that we cannot have f, $\widehat{f}$ be test functions in general.

Recall that

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E},$$
$$\mathcal{D}' \supset \mathcal{S}' \supset \mathcal{E}'.$$

What is $\mathcal{S}'$? This is the space of continuous linear functionals on $\mathcal{S}$. We call $\mathcal{S}'$ the space of **Tempered Distributions** or **Temperate Distributions**.

If $f \in \mathcal{D}'$ then $f \in \mathcal{S}'$ if and only if

$$|f(u)| \leq \sum_{\text{finite}} P_{\alpha\beta}(u),$$

where

$$P_{\alpha\beta}(u) = \sup |x^{\alpha} \partial^{\beta} u(x)|.$$

Suppose $f \in L^1_{loc}$, then for $u \in \mathcal{D}$,

$$f(u) = \int f u \, dx.$$

Example 15.1

Heuristically, f should only have at most polynomial growth at infinity. So $x^n \in \mathcal{S}'$. On the other hand, $e^{x^2} \notin \mathcal{S}'$.

If we had something like $e^{x^2} e^{i e^{x^4}} \in \mathcal{S}'$, since the rapid oscillation of the second term trumps the first.

Definition 15.2. Let $u \in \mathcal{S}'$. Define

$$\widehat{u}(f) = u(\widehat{f}).$$

Now,

$$\mathcal{F} : \mathcal{S}' \to \mathcal{S}'.$$

§15.3 Fourier Transform in L^p

Note that $\mathcal{F} : L^1 \to L^{\infty}$ and

$$\|\widehat{u}\|_{\infty} \leq \frac{1}{(2\pi)^{n/2}} \|u\|_1,$$

and $\|\widehat{u}\|_2 = \|u\|_2$, so through interprelation, we can get

$$\mathcal{F} : L^p \to L^q,$$

where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$\|u\|_q \leq (2\pi)^{n/2(1/2 - 1/p)} \|u\|_p.$$

Similarly, the inverse Fourier transform maps $L^p \to L^q$. Finally, $\mathcal{F} : L^p \to L^q$ for $p > 2$, is false.

§15.4 Fourier Transforms and Convolutions

It is easy to show that $u, v \in \mathcal{S}$ implies that $u * v \in \mathcal{S}$.

$$\widehat{u * v}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\xi} u * v(x) dx$$
$$= \frac{1}{(2\pi)^{n/2}} \iint e^{-ix\xi} u(y) v(x - y) dx dy$$
$$= \frac{1}{(2\pi)^{n/2}} \iint e^{-iy\xi} u(u) e^{-i(x-y)\xi} v(x - y) dx dy$$
$$= (2\pi)^{n/2} \widehat{u}(\xi) \widehat{v}(\xi)$$

Then

$$(\mathcal{S}, +, \cdot) \leftrightarrow (\mathcal{S}, +, *)$$

with maps $\mathcal{F}, \mathcal{F}^{-1}$.

§16 October 20th, 2020

§16.1 Fourier Transforms for Compactly Supported Distributions

Recall the Fourier transform, defined by

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\xi} u(x) dx.$$

This holds for $u \in \mathcal{S}$ and $u \in L^1$. For $u \in \mathcal{E}'$, we can reinterpret this as

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} u(e^{-ix\xi}),$$

in the sense of distributions.

Observation: if $u \in \mathcal{E}'$, then $\widehat{u}$ is a smooth function. In other words, the regularity of u corresponds to the decay of $\widehat{u}$, by the Fourier Transform, and vice versa. Recall $\widehat{u}(\varphi) = u(\widehat{\varphi})$ for $\varphi \in \mathcal{S}$, $u \in \mathcal{S}'$.

§16.2 Fourier Transform of Homogeneous Distributions

A homogeneous function is a function satisfying $f(\lambda x) = \lambda^a f(x)$, where a is the degree of homogeneity. A homogeneous distribution is a distribution where

$$f(\varphi_\lambda) = \lambda^{-a} f(\varphi),$$

where

$$\varphi_\lambda(x) = \lambda^n \varphi(\lambda x),$$

so that $\int \varphi_\lambda = \int \varphi$.

Initially, we were working with $\varphi \in \mathcal{D}$, but $f \in \mathcal{S}'$, so this also holds for $\varphi \in \mathcal{S}$. By definition, we have that

$$\widehat{f}(\varphi_\lambda) = f(\widehat{\varphi_\lambda}).$$

Finally,

$$\begin{aligned}
\widehat{\varphi_\lambda}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int e^{-ix\xi} \lambda^n \varphi(\lambda x) dx \\
&= \frac{1}{(2\pi)^{n/2}} \int e^{-y(\xi/\lambda)} \varphi(y) dy \\
&= \widehat{\varphi}(\xi/\lambda).
\end{aligned}$$

Suppose f is homogeneous of degree a, then

$$\widehat{f}(\varphi_\lambda) = f(\widehat{\varphi_\lambda}) = \lambda^n f(\lambda^{-n} \widehat{\varphi}(\xi/\lambda)) = \lambda^{a+n} f(\widehat{\varphi}) = \lambda^{a+n} \widehat{f}(\varphi).$$

Proposition 16.1

If f is homogeneous of degree a, then $\widehat{f}$ is homogeneous of degree $-a - n$.

Example 16.2

Some examples of Fourier transforms:

- $\mathcal{F}\delta_0 = \frac{1}{(2\pi)^{n/2}}\delta_0(e^{-ix\xi}) = \frac{1}{(2\pi)^{n/2}}$.

- $\mathcal{F}^{-1}\delta_0 = \frac{1}{(2\pi)^{n/2}}$. Hence, $\mathcal{F}(1) = (2\pi)^{n/2}\delta_0$.

- $\mathcal{F}(x_j) = \mathcal{F}(x_j \cdot 1) = -i\partial_{\xi_j}\mathcal{F}(1) = -i(2\pi)^{n/2}\partial_{\xi_j}\delta_0$.

- In 1-D, $\mathcal{F}(x^n) = (-i)^n(2\pi)^{1/2}\delta_0^{(n)}$.

- We have
$$\mathcal{F}(\delta_{x_0})(\xi) = \frac{1}{(2\pi)^{n/2}}\delta_{x_0}(e^{-ix\xi}) = \frac{e^{-ix_0\xi}}{(2\pi)^{n/2}}$$
  and
$$\mathcal{F}^{-1}(\delta_{\xi_0})(x) = \frac{1}{(2\pi)^{n/2}}\delta_{\xi_0}(e^{ix\xi}) = \frac{e^{ix_0x}}{(2\pi)^{n/2}}.$$

- For $f \in \mathcal{S}'$, If $\xi_0 \in \mathbb{R}$, then
$$[fe^{ix\xi_0}](\varphi) = f(e^{ix\xi_0}\varphi)$$
  and $e^{ix\xi_0}\varphi \in \mathcal{S}$, so $fe^{ix\xi_0} \in \mathcal{S}'$.
  Then,
$$\mathcal{F}(fe^{ix\xi_0})(\xi) = \frac{1}{2(\pi)^{n/2}}\int e^{-ix(\xi-\xi_0)}f(x)dx = \widehat{f}(\xi - \xi_0).$$
  This is called a shift operator for f.

Example 16.3 (Fourier Transform of $PV(1/x)$)

Let $f = PV(1/x)$. We can try to argue that
$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} PV(\frac{1}{x})e^{-ix\xi}.$$

The principle value handles convergence at 0, but what happens at infinity? It is clear that the integral is not absolutely convergent; however, integrating by parts reveals that the integral is convergent, provided that $\xi \neq 0$.

If f is homogeneous of degree -1, then $\widehat{f}$ is homogeneous of degree 0. Suppose f as being an odd, real-valued function. Then,

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}f(x)dx = \frac{1}{\sqrt{2\pi}}\int [cos(x\xi) - i\sin(x\xi)]f(x)dx = \frac{i}{\sqrt{2\pi}}\int \sin(x\xi)f(x)dx.$$

Hence, the Fourier transform takes odd, real functions to odd, purely imaginary functions. It also would take even, real functions to even, real functions.

Hence $\mathcal{F}(PV(1/x)) = ic\,\text{sgn}(\xi)$. Taking $PV(1/x) \cdot x = 1$ gives that the constant is $\frac{1}{2\sqrt{2\pi}}$.

Example 16.4 (Fourier Transform of $\frac{1}{1+x^2}$)

Note that $f = \frac{1}{1+x^2} \in L^1$, so $\widehat{f} \in C_0$. We also know that f is real, even, so $\widehat{f}$ is real, even. Then,

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int \frac{e^{-ix^2}}{1+x^2} dx.$$

We can compute this integral via complex-analytic methods.

§17 October 22nd, 2020

§17.1 PDEs via Fourier Transforms

Let $P(D)u = f$ be a constant coefficient linear PDE. Let $D_j = \frac{1}{i}\partial_j$. Then,

$$P(\xi)\widehat{u}(\xi) = \widehat{f}(\xi).$$

Formally,

$$\widehat{u}(\xi) = \widehat{f}(\xi)\frac{1}{P(\xi)},$$

so applying the inverse FT gives

$$u = \frac{1}{(2\pi)^{n/2}} f * \left(\frac{1}{P(\xi)}\right)^\vee.$$

This is a formal computation because $P(\xi)$ might have zeros, and it only applies if $u, f \in \mathcal{S}'$. In particular, it excludes all real exponentials.

Recall the *Fundamental Solution*, a convolution operator with the kernel K so that

$$P(D)K = \delta_0.$$

Taking FT gives

$$\widehat{K}(\xi) = \frac{1}{2\pi}^{n/2}\frac{1}{P(\xi)}.$$

We conclude that we can find a fundamental solution by computing the inverse FT of a rational function. This is easy when $P(\xi) \neq 0$ for all ξ, and we can actually conclude the uniqueness of the solution in $\mathcal{S}'$.

Definition 17.1. P is called elliptic if $P(\xi) \neq 0$ for $\xi \neq 0$(this is not quite what is used in the literature though).

§17.2 Elliptic PDEs

Definition 17.2. P is called "strongly elliptic" if P has order m and

$$|P(\xi)| \geq C(1 + |\xi|)^m.$$

Some examples:

- For $m = 1$, $P = \partial + 1 = iD + 1$, so $P(\xi) = i\xi + 1$ and

$$|P(\xi)| = \sqrt{1 + \xi^2}.$$

  The solution to the homogeneous equation is $u(x) = e^{-x} \notin \mathcal{S}$.

- For $m = 2$, take $P = -\partial^2 + 1 = D^2 + 1$. Then

$$|P(\xi)| = |1 + \xi^2|.$$

  The fundamental solution is $K(x) = -\frac{1}{2}e^{-|x|}$. The solution to the homogeneous equation is $e^{\pm x}$. We also saw that

$$K = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1 + \xi^2}\right)^\vee.$$

Definition 17.3. An operator P is locally elliptic if

$$|P(\xi)| \geq c|\xi|^m, |\xi| \geq R.$$

The behavior at small ξ corresponds to the behavior at infinity of the fundamental solution. The behavior of P for ξ in a compact set corresponds to the local behavior for K.

§17.3 Fundamental Solutions for PDEs

We start with first order equations:

- Transport Equation: We have $P(D) = D_1$. Then, $P(\xi) = \xi_1$ and $\frac{1}{p(\xi)} = \frac{1}{\xi_1}$ has zeros, so we need to interpret it as a distribution. We can take $PV(1/\xi_1) + c\delta_{\xi_1=0}$. The inverse FT(ignoring 2π factors) gives

$$K(x) = (\text{sgn}(x_1) + c)\delta_{x'=0}, x^1 = (x_2, \ldots, x_n).$$

- Consider the Cauchy-Riemann Equations,

$$\bar{\partial} = \partial_1 + i\partial_2,$$

$$\partial = \partial_1 - i\partial_2.$$

A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $\bar{\partial}f = 0$.

Then

$$\bar{\partial} = i(D_1 + iD_2),$$

so

$$P(\xi) = i(\xi_1 + i\xi_2).$$

This vanishes only at the origin. Hence, it follows under the class of elliptic operators, but not strongly elliptic operators. Finally,

$$\frac{1}{P(\xi)} = \frac{-i}{\xi_1 + i\xi_2} \in L^1_{loc}.$$

One strategy to compute the inverse FT is by computing it in each variable. Instead of that, note that $1/P(\xi)$ is -1 homogeneous, so $\mathcal{F}^{-1}(1/P(\xi))$ has -1 homogeneity. Hence $K(x) = \frac{1}{r}K(\theta)$. We rotate coordinates by angle θ so that $z \to ze^{i\theta}$. Then $\bar{\partial}$ stays the same. If we let $A(\theta)$ denote the matrix of rotations, $(A^T)^{-1} = A$ in the Fourier space, so $AA^T = A^TA = I$.

Hence $K(x) = \frac{c}{x_1 + ix_2}$, so we find that

$$\left(\frac{1}{\xi_1 + u\xi_2}\right)^{\vee} = \frac{c}{x_1 + ix_2}.$$

We can find that $|c|^2 = 1$, so it follows that the fundamental solution of $\bar{\partial}$ is $1/z \in L^1_{loc}$.

§18 October 27th, 2020

§18.1 Fundamental Solutions for Partial Differential Operators

If we had $P(D)u = f$, where $u, f \in \mathcal{S}$, then the fundamental solution K is given by

$$\widehat{K} = \frac{1}{P(\xi)},$$

where $P(\xi)$ is called the symbol of $P(D)$ and $D = \frac{1}{i}\partial$. The issue: how do we interpret $\widehat{K}$ in the sense of distributions?

§18.2 The Laplacian, n ≥ 3

Suppose $P = -\Delta = \sum_{j=1}^{n} \partial_j^2$. Then $P(D) = D^2$. The symbol is given by $P(\xi) = \xi^2$ and we consider $\frac{1}{\xi^2}$. This is locally integrable if and only if $n \geq 3$. For $n = 1$, we computed $K = \frac{1}{2}|x|$. For $n = 2$, we have logarithmic divergence. For $n \geq 3$, we can integrate this.

First, note that $1/|\xi|^2$ is homogeneous of degree -2. We also have spherical symmetry. What can we say about $K = \mathcal{F}^{-1}(1/|\xi|^2)$? This is homogeneous of degree $2 - n > -n$. It also must be spherically symmetric. Hence $K(x) = |x|^{2-n} \cdot k(\theta)$. Then $K(x) = c|x|^{2-n}$ by spherical symmetry. To find c, we compute $-\Delta|x|^{2-n} = 0$ away from $x = 0$. If we compute

$$-\Delta|x|^{2-n}(\varphi) = |x|^{2-n}(-\Delta\varphi) = -\int_{\mathbb{R}^n} |x|^{2-n}\Delta\varphi \, dx = \lim_{\epsilon \to 0} -\int_{\mathbb{R}^n \setminus B_\epsilon} |x|^{2-n}\Delta\varphi \, dx.$$

If we integrate by parts, we find

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon} \nabla|x|^{2-n} \cdot \nabla\varphi \, dx - \int_{\partial B_\epsilon} |x|^{2-n} \cdot \nabla\varphi \cdot \nu \, d\sigma$$

$$= \lim_{\epsilon \to 0} -\int_{\mathbb{R}^n \setminus B_\epsilon} \Delta|x|^{2-n} \cdot \varphi \, dx - \int_{\partial B_\epsilon} |x|^{2-n}\nabla\varphi \cdot \nu \, d\sigma + \int_{\partial B_\epsilon} \nabla|x|^{2-n} \cdot \nu \cdot \varphi \, d\sigma.$$

where $\nu = -\frac{x}{|x|}$ is the outer normal vector.

The first term is 0 since $\Delta|x|^{2-n} = 0$. The second term can is bounded by

$$O(\epsilon^{2-n}|\partial B_\epsilon|) = O(\epsilon^{2-n}\epsilon^{n-1}) = O(\epsilon) \to 0.$$

Note that $\nabla|x| = \frac{x}{|x|} = \nu$. It follows that

$$\nabla|x|^{2-n} = (2-n)\frac{x}{|x|}|x|^{1-n} = (2-n)x|x|^{-n}.$$

Hence

$$\nabla|x|^{2-n} \cdot \nu = -(2-n)v|x|^{1-n}.$$

Then the last contribution is The last contribution is

$$\lim_{\epsilon \to 0} \int_{\partial B_1} (n-2)|x|^{1-n} \cdot \varphi \, dx.$$

If we think of $|x| = \epsilon y$ for $|y| = 1$, then we have

$$\lim_{\epsilon \to 0} \int_{\partial B_1} (n-2)\epsilon^{1-n}\varphi(\epsilon y) \, dy \cdot \epsilon^{n-1} = \int_{\partial B_1} (n-2)\varphi(0)dy = (n-2)\varphi(0)|\partial B_1|.$$

If we let $c_n = |\partial B_1|$, then

$$-\Delta|x|^{2-n}(\varphi) = (n-2)c_n\varphi(0) \implies -\Delta|x|^{2-n} = (n-2)c_n\delta_0.$$

Therefore,

$$K = -\frac{1}{(2-n)c_n}|x|^{2-n}.$$

§18.3 The Laplacian, n = 2

In the case where $n = 2$, $\frac{1}{|\xi|^2}$ is not integrable. For a simpler example, think of $\frac{1}{|x|}$ in 1D. We considered $PV(1/x)$ for this computation. The problem is that it kills the homogeneity.

$$\frac{1}{|x|}(\varphi) = \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus [-\epsilon, \epsilon]} \varphi(x)/|x| dx - 2|\log \epsilon| \cdot \varphi(0).$$

Then

$$f(\lambda x) = \frac{1}{\lambda} f(x) - 2 \log \lambda \cdot \delta_0.$$

Applying a fourier transform, we find that

$$\widehat{f} = c \log |\xi|.$$

We have that

$$\frac{1}{|\xi|^2}(\varphi) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon} \varphi(\xi)/|\xi|^2 \, d\xi - 2\pi \varphi(0)|\log \epsilon|.$$

If follows that

$$\mathcal{F}^{-1} PV(1/|\xi|^2) = -2\pi \log |x|.$$

We conclude that $K(X) = c \log |x|$. To exactly determine c, note that

$$\begin{aligned}
-\Delta K(\varphi) &= \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \nabla(\log |x|) \cdot \nu \cdot \varphi \, d\sigma \\
&= \int_{\epsilon \to 0} \int_{C(0,\epsilon)} -\frac{1}{|x|} \cdot \varphi ds \\
&= -\varphi(0) \frac{2\pi \epsilon}{\epsilon} \\
&= -2\pi \varphi(0).
\end{aligned}$$

It follows that

$$K(x) = -\frac{1}{2\pi} \log |x|.$$

Example 18.1 (More General Operators)

Suppose we have an operator $P(D) = -a^{ij} D_i D_j$. For example, for $-\Delta$, $A = (a^{ij}) = I_n$. Then $P(\xi) = a^{ij} \xi_i \xi_j$. Note that the operator is elliptic if A is positive definite. After a suitable change of coordinates, this is equivalent to $-\Delta$.

For the fundamental solution, we look at the inverse matrix $A^{-1} = (a_{ij})$, so that

$$K = c_m \cdot (a_{ij} x^i x^j)^{1-n/2} (\det A)^{-1/2}.$$

More details can be found in Hormander.

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§18.4 The Heat Operator

We work in $\mathbb{R} \times \mathbb{R}^n$, where $\mathbb{R}$ corresponds to time and $\mathbb{R}^n$ corresponds to space. Let $P = \partial_t - \Delta_x$. If we denote (τ, ξ) as the Fourier variables, we have

$$P(D) = iD_t + D_x^2 \implies P(\xi) = i\tau + |\xi|^2.$$

Is $\frac{1}{P(\xi)} = \frac{1}{i\tau+\xi^2}$ integrable? Consider the region where $|i\tau + \xi^2| = R^2$ for some $R > 0$. Hence $|\xi| \leq R$ and $|\tau| \leq R^2$, which corresponds to a parabolic ball, which we will denote B_R. Note that $|B_R| = R^{n+2}$, so the integral over B_R is

$$\int_{B_R} \left| \frac{1}{P(\xi)} \right| \approx \frac{1}{R^2} R^{n+2} \approx R^n$$

Then, if we let $\mathbb{R} \times \mathbb{R}^n = \bigcap_{R=2^j} B_R$, then

$$\int_{|\tau,\xi| \leq 1} 1/|P(\xi)| \approx \sum_{R \leq 1, R = 2^j} R^n < \infty.$$

Is $\frac{1}{P(\xi)} = \frac{1}{i\tau+\xi^2}$ integrable? Consider the region where $|i\tau + \xi^2| = R^2$ for some $R > 0$.

§19 October 29th, 2020

§19.1 Heat Operator, Continued

We observed that $\frac{1}{p(\tau,\xi)} \in L^1_{loc}$, so can we find $\mathcal{F}^{-1}\frac{1}{i\tau+\xi^2}$?

First, take the Fourier transform in τ. For $\xi \neq 0$, $1/(i\tau + \xi^2)$ is smooth on $\mathbb{R}$, but has a pole at $i\tau = -\xi^2$.

Then, the inverse FT is given by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{i\tau + \xi^2}\, d\tau.$$

If $t < 0$, $e^{it\tau}$ decays exponentially as $\text{Im}(\tau) \to -\infty$. It follows that the integral is 0 in this case, pushing the contour down towards $-\infty$.

If $t > 0$, $e^{it\tau}$ decays exponentially as $\text{Im}(\tau) \to \infty$. The integral is the residue at the pole $\tau = i\xi^2$.

We find that

$$\mathcal{F}^{-1}\left(\frac{1}{i\tau + \xi^2}\right) = e^{-t\xi^2}.$$

Finally,

$$\mathcal{F}_x^{-1} e^{-it\xi^2} = 2^{-n/2} e^{-x^2/4t},$$

from our previous work on the Fourier transforms of Gaussians. Hence,

$$K(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-x^2/4t} \cdot 1_{t \geq 0}.$$

Remarks

- supp $K = \{t \geq 0\}$, since $Pu = f$, which gives $u = K * f$ which adds the supports. Then, the support of f is moved only in the positive time direction. We say that K is a forward fundamental solution.

- Suppose $\widehat{K_0} \cdot p = 1$. Then $(\widehat{K} - \widehat{K_0}) \cdot p = 0$. Since p vanishes only at 0, we have supp $(\widehat{K} - \widehat{K_0}) = \{(0,0)\}$. Hence, $K - K_0$ is a polynomial. This further implies that there is a unique forward fundamental solution in $\mathcal{S}'$.

§19.2 Heat Equation, Cauchy Problem

We have the following Cauchy problem:

$$\begin{cases} (\partial_t - \Delta)u = f, t \geq 0 \\ u(t = 0) = u_0 \end{cases}.$$

Considering the homogeneous solution, if we take the FT in x,

$$\begin{cases} (\partial_t + \xi^2)\widehat{u}(t,\xi) = 0 \\ \widehat{u}(0,\xi) = \widehat{u_0}(\xi). \end{cases}$$

The solution is

$$\widehat{u}(t,\xi) = \widehat{u_0}(\xi)e^{-t\xi^2}.$$

$$u(t,x) = u_0 * K(t,x).$$

To connect this with the solution to the inhomogeneous problem:

- We extend to $t < 0$,

$$\tilde{u}(t, x) = \begin{cases} u(t, x), t = 0 \\ 0, t < 0. \end{cases}$$

- We compute

$$(\partial_t - \Delta)\tilde{u} = \delta_{t=0} \cdot u_0(x).$$

We solve this by using the fundamental solution,

$$\tilde{u} = u_0(x)\delta_{t=0} *_{t,x} K(t, x)$$
$$= u_0(x) *_x K(t, x).$$

We could also solve,

$$(\partial_t - \Delta)K = \delta_{(0,0)},$$

which means that $K(t, x)$ is the solution to the Cauchy problem with initial data $u_0 = \delta_{x=0}$.

Some observations:

- To solve $Pu = f$, we can set $u = f * K$. Because K decays exponentially as $x \to \infty$, the convolution is well defined even if f has exponential growth. For example, if $|f| \leq e^{x^2/4T}$, the convolution makes sense for $|t| < T$.

- We cannot solve the heat equation backward in time.

$$\widehat{u}(t, \xi) = \widehat{u}_0(\xi)e^{-t\xi^2}.$$

This is not in $\mathcal{S}'$ since $e^{-t\xi^2}$ grows exponentially in ξ for $t < 0$.

- The heat equation is a diffusion equation.

§19.3 Schrodinger Equation in $\mathbb{R} \times \mathbb{R}^n, n \geq 1$

We have the following Cauchy Problem:

$$\begin{cases} (i\partial_t + \Delta)u = 0 \\ u(0) = u_0, \end{cases},$$

where u is complex valued.

We first find the fundamental solution. Take a spatial Fourier Transform,

$$\begin{cases} (i\partial_t - \xi^2)\widehat{u}(t, \xi) = 0 \\ \widehat{u}(0, \xi) = \widehat{u}_0(\xi) \end{cases}.$$

We find that

$$\widehat{u}(t, \xi) = e^{-it\xi^2} \cdot \widehat{u}_0(\xi).$$

Taking an inverse FT, we (formally) find that

$$u(t, x) = \mathcal{F}^{-1}(e^{-it\xi^2}) * u_0 = \frac{1}{(4\pi it)^{n/2}} e^{-ix^2/4t}.$$

However, we are working with complex arguments which requires further inquiry.

Consider

$$F(\epsilon, \xi) = e^{-\epsilon\xi^2/2}.$$

For $\mathrm{Re}(\epsilon) \geq 0$, $F \in \mathcal{S}'$ and for $\mathrm{Re}(\epsilon) > 0$, $F \in \mathcal{S}$. As a function of ϵ, this is an analytic function.

Then $\check{F}(\epsilon, x)$ is also analytic in ϵ with values in $\mathcal{S}$ for $\mathrm{Re}(\epsilon) > 0$. The same holds for $\mathrm{Re}(\epsilon) > 0$. The trick is to extend to $\epsilon \in i\mathbb{R}$ by using continuity.

Some observations:

- The fundamental solution for the Schrodinger equation corresponds to $u_0 = \delta_0$, so the forward solution is given by

$$K_+(t, x) = (4\pi i t)^{-n/2} e^{ix^2/4t} \cdot 1_{t \geq 0}$$

- The backward fundamental solution is given by

$$K_-(t, x) = -(4\pi i t)^{-n/2} e^{ix^2/4t} \cdot 1_{t \leq 0}.$$

- K is smooth away from $t = 0$ but does not decay.

§20 November 3rd, 2020

§20.1 Schrodinger Equation, Physical Interpretation

Recall the equation

$$\begin{cases} (i\partial_t + \Delta)u = 0 \\ u(0) = u_0, \end{cases},$$

with forward fundamental solution

$$K_+(t, x) = (4\pi i t)^{-n/2} e^{ix^2/4t} \cdot 1_{t \geq 0}.$$

The equation comes from quantum mechanics. Suppose we want to describe the state of an electron at a given time. Classically, we would need to specify the position and the velocity. However, in quantum mechanics, we can only describe these up to probability measures.

Suppose the states of the electron correspond to

$$S = \{u \in L^2(\mathbb{R}^n), \|u\|_{L^2} = 1\}.$$

The probability distribution for position is given by $|u|^2 dx$, in the sense that

$$P(e \in D) = \int_D |u|^2 dx,$$

$$P(v \in E) = \int_E |\widehat{u}(\xi)|^2 d\xi.$$

The key dynamical property is that the state function u evolves along the Schrodinger flow:

$$iu_t + \hbar \Delta u = 0,$$

where $\hbar$ is Planck's constant.

How well can we predict both the position and velocity of the particle? In other words, what is the measure of the localization of a particle near a point. The expected position is given by

$$x_0 = \int x |u(x)|^2 dx.$$

The mean square deviation from the expected position is

$$(\delta x)^2 \int (x - x_0)^2 |\widehat{u}(x)|^2 dx.$$

The expected velocity is

$$\xi_0 = \int \xi |u(x)|^2 dx,$$

and the mean square deviation is

$$(\delta \xi)^2 = \int (\xi - \xi_0) |\widehat{u}(x)|^2 dx.$$

Can we make both $\delta x, \delta \xi$ small?

> **Theorem 9** (Uncertainty Principle)
>
> $$\delta x \cdot \delta \xi \geq c_n[\hbar].$$

Proof. Observe that we can reduce the problem to the case $x_0 = 0$ and $\xi_0 = 0$. We want to compare

$$\int x^2 |u|^2 dx, \int \xi^2 |\widehat{u}|^2 d\xi.$$

By Plancherel's theorem, the second integral is $\int |\partial_x u|^2 dx$.

Then,

$$\begin{aligned}
(\delta x) \cdot (\delta \xi) &= \left(\int x^2 |u|^2 \, dx \right)^{1/2} \cdot \left(\int |\partial_x u|^2 \, dx \right)^{1/2} \\
&\geq -\mathrm{Re} \int x u \cdot \partial_x \overline{u} \, dx \\
&= -\frac{1}{2} \int x \cdot \partial_x |u|^2 \, dx \\
&= \frac{n}{2} \int |u|^2 \, dx = n.
\end{aligned}$$

Hence,

$$\delta x \cdot \delta \xi \geq \frac{n}{2}.$$

$\square$

For which questions is the inequality sharp? This happens when Cauchy-Schwartz is sharp, namely $x \cdot u = a \partial_x u$. Solving this, we find that $u = e^{\frac{x^2}{2a}}$, for $a < 0$. With respect to all variables, we find

$$u = c_n a^{-n/2} e^{-x^2/2a},$$
$$\widehat{u} = c_n a^{n/2} e^{-a\xi^2/2}.$$

More generally,

$$u_{x_0, \xi_0, a} = c_n a^{-n/2} e^{-(x-x_0)^2/2a} e^{i(x-x_0)\xi_0}.$$

In physics, these states are called **Coherent States**.

§20.2 Paley-Wiener Theorem

Suppose we have a function f supported in a ball $B(0, R)$. Note that

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \, dx.$$

Extend ξ to the complex plane, $\xi \in \mathbb{C}^n$. Because f is compactly supported, $\widehat{f}$ is defined for every $\xi \in \mathbb{C}^n$. Note that $\widehat{f}$ is an entire function in $\xi \in \mathbb{C}^n$. We showed that for $\xi \in \mathbb{R}^n$, then

$$|\widehat{f}(\xi)| \leq c(1 + |\xi|)^N.$$

For $\xi \in \mathbb{C}^n$, we cannot bound the exponential bt 1, and instead,

$$|e^{-ix\xi}| = e^{x \cdot \mathrm{Im}\, \xi} \leq e^{|x||\mathrm{Im}\, \xi|} \leq e^{R|\mathrm{Im}\, \xi|}.$$

> **Theorem 10** (Paley-Wiener)
>
> The supp $f \subset B(0, R)$ if and only if
>
> $$|\widehat{f}(\xi)| \leq c(1 + |\xi|)^N e^{R|\text{Im } \xi|}$$
>
> for some large enough N.

What if supp f is not in a ball? Suppose supp $f \in D$.

$$|e^{-ix\xi}| \leq \sup_{x \in D} e^{x \cdot Im(\xi)} = e^{\sup_{x \in D} x \cdot Im(\xi)} = e^{\omega_D Im(\xi)},$$

with ω_D homogeneous of order 1. What happens when we maximize a linear function over a set? We maximize over the convex hull of D so that ω_D depends on the extreme points of the convex hull.

Can we use Paley-Wiener for computing fundamental solutions? Note that the forward fundamental solution is supported in $t \geq 0$. What can we say about the Fourier transform of distributions with $t \geq 0$ support?

$$\widehat{u}(\xi) = \int e^{-x\xi} u(x) \, dx.$$

We first take $\xi \in \mathbb{R} \to \xi \in \mathbb{C}$. If $x > 0$, then $Re(-ix\xi) = xIm(\xi) \leq 0$ if $x > 0, Im(\xi) < 0$.

> **Theorem 11** (Paley-Wiener Extension)
>
> u is supported in $[0, \infty)$ if and only if $\widehat{u}$ admits a holomorphic extension to the lower half-plane, s. t.
> $$|\widehat{u}(\xi)| \leq (1 + |\xi|)^N \cdot |Im(\xi)|^{-N}.$$

> **Example 20.1**
>
> Let $P = D_x = \frac{1}{i}\partial_x$. The fundamental solution is $K = \mathcal{F}^{-1}(1/\xi)$, up to constants.
>
> By the above theorem, we should make the choice with a holomorphic extension to the upper half-plane. Take the limit coming from the upper half-plane:
>
> $$K_f = \mathcal{F}^{-1}(1/(\xi - io)), K_b = \mathcal{F}^{-1}(1/(\xi + io)).$$

> **Example 20.2**
>
> For the heat equation,
> $$K = \mathcal{F}^{-1}(1/(i\tau + \xi^2)).$$
>
> This has a unique holomorphic extension to $Im\tau > 0$.

§21 November 5th, 2020

§21.1 The Wave Equation

Recall the equation

$$(\partial_t^2 - \Delta_x)u = 0, \mathbb{R} \times \mathbb{R}^n.$$

We discussed this in the case $n = 1$, namely $K(t, x) = \frac{1}{2}1_{\{t>|x|\}}$.

In the case for $n \geq 2$, we make general considerations. Note that $P = \partial_t^2 - \Delta_x = -D_\tau^2 + D_x^2$. The Symbol is $P(\tau, \xi) = -\tau^2 + \xi^2$. For a fundamental solution $K = \mathcal{F}^{-1}((\xi^2 - \tau^2)^{-1})$. We have a zero set for P, namely $\{\tau^2 = \xi^2\}$, which is a cone.

For a forward fundamental solution, we need to look at the interpretation of with holomorphic extension in the lower half-plane:

$$K_f = \mathcal{F}^{-1}\left(\frac{1}{-(\tau - i\epsilon)^2 + \xi^2}\right).$$

In the limit as $\epsilon \to 0$, we approach the real line.

§21.2 Symmetries of the d'Alembertian

We need to make some symmetry considerations.

- From the Homogeneity, we have

$$\frac{1}{-(\tau + io)^2 + \xi^2}$$

  which corresponds to -2 homogeneity. Then, K has $-n + 1$ homogeneity.

- Note that the equation is spherically symmetric in ξ, so K is spherically symmetry in x.

- Are there other linear symmetries from there? If $y = Ax$, then $\Delta_x = b^{ij}\partial_{y_i}\partial_{y_j}$ and $B = A^T A$. This is a symmetry from Δ if $A^T A = I$.

Denote the wave equation operator as $\Box$. We write $(t, x) = (x_0, x_1, \ldots, x_n)$. Then $\Box = -m^{\alpha\beta}\partial_\alpha\partial_\beta$, where m is the Minkowski Metric, namely:


Under a linear change of coordinates, $y^\alpha = A_{\alpha\beta}x^\beta$, so $\Box_x = g^{\alpha\beta}\partial_\alpha\partial_\beta$. Then $G = A^T mA$. $\Box$ is left unchanged if $A^T mA = m$. What matrices have this property? We can take O_x orthogonal matrices, so that

$$A = \begin{bmatrix} 1, 0 \\ 0, O_x \end{bmatrix}.$$

There are many more matrices satisfying the property. Namely, we have the **Lorentz group**, which contains rotations in x and hyperbolic rotations in (t, x_j). Any element

in the Lorentz can be generated by the spacial and hyperbolic rotations. The semi-norm associated with the group is $\|x\|_m^2 = m_{\alpha\beta}x^\alpha x^\beta$, which is the Minkowski space $(M = \mathbb{R}^n \times \mathbb{R}, m)$. This space is a pseudo-Riemannian manifold. For the fundamental solution, we can observe that the Lorentz group leaves the quadratic form $x^2 - t^2$ unchanged.
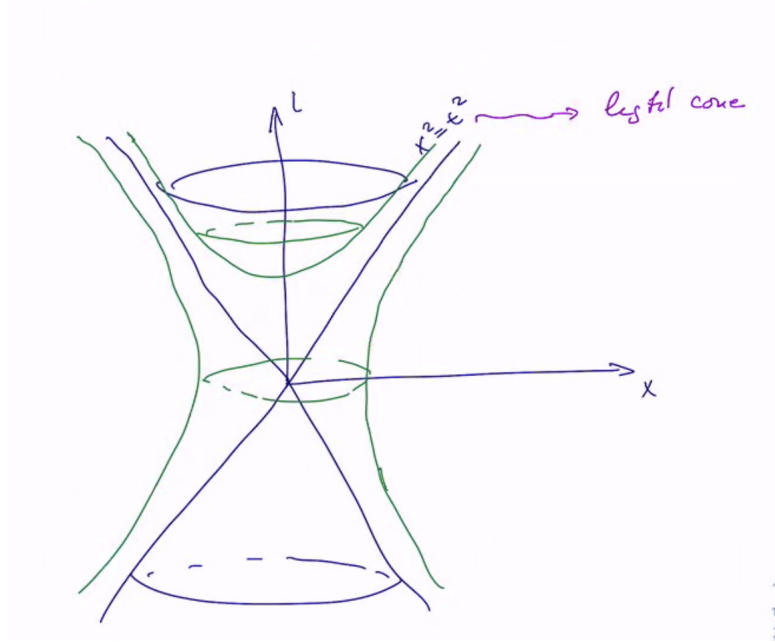
We conclude that $K = K(t^2 - x^2)$. Also note that K is forward, so it is supported in $t \geq 0$. But t is symmetric about the time axis, so it follows that $K = 0$ outside the light cones. Hence, K has to be supported inside the forward light cone.

§21.3 Fundamental Solutions

From the earlier symmetry considerations, we find that $K = c_n(t^2 - x^2)^{(1-n)/2}1_{C^+}$. This is locally integrable if $n = 2$. Namely,

$$K = \frac{c_2}{\sqrt{t^2 - x^2}}1_{c^+}.$$

In 3D, $K = K(t^2 - x^2)$, which is homogeneous of degree -1, so it is supported in $t^2 - x^2 \geq 0$. The $K = \delta_{t^2-x^2=0}$. If $\varphi_\epsilon \to \delta$ in $\mathcal{D}'$, then $K = \lim_{\epsilon \to 0}, \varphi_\epsilon(t^2 - x^2)$.

> **Corollary 21.1** (Huygen's Principle)
> If $\Box u = f$, then f is supported exactly at the origin of the light cone.

For odd dimensions $n \geq 3$,
$$K_n = c_n\delta_{t^2-x^2=0}^{(n-3/2)}.$$

For even dimensions,
$$K_n = c_n(t^2 - x^2)_+^{-(n-1/2)}\delta_{t^2-x^2=0}^{(n-1/2)}.$$

§22 November 10th, 2020

§22.1 Sobolev Spaces

We have discussed C, C^k, and Holder Spaces, C^γ for $\gamma \in [0, 1]$. Lipschitz functions are some times denoted as $C^{0,1}$. We can extend this by $C^{k,\gamma}$, for $\gamma \in [0, 1]$.

The spaces we consider here are $L^p(\mathbb{R}^n)$ spaces for $1 \leq p < \infty$, which gave a norm

$$\|u\|_{L_p}^p = \int_{\mathbb{R}^n} |u|^p.$$

Definition 22.1. The Sobolev space $W^{k,p}$ is the space of functions $u \in L^p$, such that $\partial^\alpha u \in L^p$ for all $|\alpha| \leq k$.

We let $k \in \mathbb{N}$. For $k = 0$, we have the L^p spaces, but for $k > 0$, we have the Sobolev spaces. We let $1 \leq p \leq \infty$. For $p = \infty$,

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{\mathbb{R}^n} |u|,$$

the **essential supremum**. We will ignore the difference between the essential supremum and the regular supremum for the notes.

Proposition 22.2

We have a norm in $W^{k,p}$ for $1 \leq p \leq \infty$:

$$\|u\|_{W^{k,p}}^p = \sum_{|\alpha| < k} \|\partial^\alpha u\|_{L^p}^p.$$

For $p = \infty$,

$$\|u\|_{W^{k,\infty}} = \sup_{|\alpha| < k} \sup_x |\partial^\alpha u(x)|.$$

Remark 22.3. The triangle inequality follows from the triangle inequality in L^p.

Proposition 22.4

$W^{k,p}$ is a Banach space.

Proof. Suppose we have a Cauchy sequence u^n in $W^{k,p}$. Then $\partial^\alpha u^n$ form a Cauchy sequence in L^p. In particular, because L^p is complete, there exists functions u^α such that $\partial^\alpha u^n \to u^\alpha$ in L^p. But then $u_n \to u$ in L^p, where $u^\alpha = \partial^\alpha u$ considering distributional limits. $\square$

§22.2 Alternate construction of the Sobolev Spaces

We begin as follows. Start with $\mathcal{D}$, the space of test functions, and endow it with the $W^{k,p}$ norm. Then, take the completion with respect to this norm. It is clear that $\mathcal{D} \subset W^{k,p}$, but is it dense?

> **Proposition 22.5**
> $\overline{\mathcal{D}} = W^{k,p}$.

Proof. We first show that $W^{k,p}$ functions can be approximated by compactly supported $W^{k,p}$ functions. Define χ_R as

$$\chi_R(x) = \begin{cases} 1, |x| < R \\ (0,1), R \leq |x| \leq 2R \\ 0, |x| > 2R \end{cases} \quad .$$

We can define $\chi_R(x) = \chi_1(x/R)$. Then, $|\partial^\alpha \chi_R| \leq c_\alpha \cdot R^{-|\alpha|}$. If $u \in W^{k,p}$, then we can localize it $u_R = \chi_R \cdot u$, where u_R have compact support.

Then, if $u \in L^p$, then $\chi_R u \to u$ in L^p. Then,

$$\|u - \chi_R u\|_{L^p} \leq \int_{\mathbb{R}^n \setminus B(0,R)} |u|^p dx \to 0.$$

We know that

$$\partial^\alpha u_R = \partial^\alpha(\chi_R u) = \sum_{\beta < \alpha} \binom{\alpha}{\beta} \partial^\beta \chi_R \cdot \partial^{\alpha \cdot \beta} u.$$

For $\beta = 0$, $\chi_R \partial^\alpha u \to \partial^\alpha u$ in L^p. For $\beta > 0$, $\partial^\beta \chi_R \to 0$ uniformly.

We now show the regularity part. If $u \in W^{k,p}$ with compact support, then define $\mathcal{D} \ni u_\epsilon = u * \varphi_\epsilon$, where φ_ϵ are the standard mollifiers. For $u \in L^p$, we have $u_\epsilon \to u$ in L^p for $1 \leq p < \infty$. Then

$$\partial^\alpha u_\epsilon = \partial^\alpha(u * \varphi_\epsilon) = \partial^\alpha u * \varphi_\epsilon \to \partial^\alpha u$$

in L^p, for $|\alpha| \geq K$. If $p = \infty$, then this fails since $u_n \to u$ in L^∞, so $u \in C$, but there are bounded functions that are not continuous. $\qquad \square$

§22.3 Large Families of Spaces

Take $\Omega \subset \mathbb{R}^n$ open, and consider $\mathcal{D}(\Omega)$. Then,

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega); \partial^\alpha u \in L^p, |\alpha| \leq k\}.$$

But what is the closure of $\mathcal{D}$ in $W^{k,p}$. This is a smaller subset of $W^{k,p}$ denoted by $W_0^{k,p}$.

Alternative question: Can we approximate $W^{k,p}$ functions by smooth functions? We could take $\mathcal{E}(\Omega) = C^\infty(\Omega)$, which are smooth functions in Ω with no information on the boundary. We could also consider $C^\infty(\overline{\Omega})$, which is the restriction to Ω of $C^\infty(\mathbb{R}^n)$ functions. We also have $\bigcap C^k(\overline{\Omega})$, which are k-times differentiable functions with bounded derivatives.

Question: Can $u \in \bigcap C^k(\overline{\Omega})$ be extended to $u \in C^\infty(\mathbb{R}^n)$. The answer depends on the regularity properties of $\partial\Omega$. We can have cusps and other singularities depending on the boundary conditions.

> **Proposition 22.6**
> $W^{k,p}(\Omega)$ functions can be approximated by smooth functions in Ω(with no restriction on what happens on the boundary).

> **Proposition 22.7**
>
> $W^{k,p}(\Omega)$ functions extend to $W^{k,p}(\mathbb{R}^n)$ functions if $\partial\Omega$ is of class C^k.

The C^k boundary allows us to flatten it under a change of coordinates.

§22.4 Extensions with a Flat Boundary

Take a function $u : \mathbb{R}^+ \to \mathbb{R}$ and extend it to $\mathbb{R}$. Take $u \in L^p(\mathbb{R}^+)$. A corresponding extension could be $u_* = u$ in $\mathbb{R}^+$ and 0 in $\mathbb{R}^-$. However, we would lose continuity at 0. We could also reflect across the y-axis or take a constant extension.

We can apply the same extension for $W^{1,p}$ functions. We take

$$u_* = \begin{cases} u(x), x \geq 0 \\ a_1 u(-x) + a_2 u(-2x), x \leq 0 \end{cases} .$$

We need to ensure that $a_1 + a_2 = 1$ and $-a_1 - 2a_2 = 1$ to ensure matching functions and matching derivatives respectively. The same works in $W^{2,p}$, where we add a third coefficient for $u(-3x)$.

§22.5 Sobolev Embeddings

The easy part is for 1D. We look at $W^{1,p}(\mathbb{R})$. For $p = 1$, we have $W^{1,1}(\mathbb{R})$, the absolutely continuous functions. We have

$$u(x) - u(y) = \int_x^y \partial_x u(z) dz.$$

For test functions, this is clear, and we argue by density.

$$|u(x) - u(y)| \leq \int_x^y |\partial_x u(z)| dz \leq \|\partial_x u\|_{L^1}.$$

Hence $\|u\|_{L^\infty} \leq \|u\|_{W^{1,1}}$. By density, $\|u\|_C \leq \|u\|_{W^{1,1}}$, so it follows that $W^{1,1} \subset C$.

§23 November 12th, 2020

§23.1 Morrey Inequalities, 1D

We continue in 1D. Suppose that $u \in \dot{W}^{1,p}$, the homogeneous Sobolev space. That means that $\partial u \in L^p$. We have

$$u(x) - u(y) = \int_x^y \partial_x u(z) dz.$$

Let q be the dual exponent for p. It follows that

$$|u(x) - u(y)| \leq \int_x^y |\partial_x u(z)| dz = \int_{\mathbb{R}} 1_{[x,y]} |\partial u(z)| dz \leq |x - y|^{1/q} \|\partial u\|_{L^p}.$$

Denote $1/q = \sigma = 1 - 1/p$. We find that

$$\|u\|_{\dot{C}^\sigma} := \sup_{x,y} \frac{|u(x) - u(y)|}{|x - y|^\sigma} \leq \|\partial u\|_{L^p} := \|u\|_{\dot{W}^{1,p}}$$

This is one of the **Morrey Inequalities**.

Proposition 23.1

Consider the special case where $p = \infty$. We have

$$W^{1,\infty} = \text{Lip}.$$

If we work in a homogeneous space, we yield scale invariant inequality:

$$u(x) \to u(\lambda x) = u_\lambda,$$

so the inequality for u corresponds to the inequality for u_λ. Conversely, if he inequality holds uniformly for all u_λ, then we must have

$$\sigma = 1 - \frac{1}{p}.$$

§23.2 More Sobolev Inequalities, $n \geq 2$

Take $\dot{W}^{1,p} \subset L^q$ in $\mathbb{R}^n$. Then

$$\|u\|_{L^q} \leq \|\nabla u\|_{L^p}.$$

We prove this first for test functions $u \in \mathcal{D}$.

Consider the transformation $u \to u_\lambda$.

$$\|u_\lambda\|_{L^q} = \left(\int |u(\lambda x)|^q dx \right)^{1/q}$$

$$= \left(\int |u(y)|^q dy/\lambda^n \right)^{1/q}$$

$$= \|u\|_{L^q} \cdot \lambda^{-n/q}.$$

Note that $\nabla u_\lambda = \lambda \nabla u(\lambda x)$. Then

$$\|\nabla u_\lambda\|_{L^p} = \|\nabla u\|_{L^p} \cdot \lambda^{1-n/p}.$$

Hence
$$\|u\|_{L^q} \cdot \lambda^{-n/q} \le \|\nabla u\|_{L^p} \cdot \lambda^{1-n/p}$$

for all $\lambda > 0$. It follows that
$$\frac{n}{q} = \frac{n}{p} - 1.$$

This gives us the scaling relation between exponents, noting that $1 \le p < n$.

> **Theorem 12** (Gagliardo-Nirenberg-Sobolev Inequality)
> For $\dot{W}^{1,p} \subset L^q$,
> $$\|u\|_{L^q} \le \|u\|_{\dot{W}^{1,p}}$$
> if
> $$\frac{n}{q} = \frac{n}{p} - 1.$$

Proof. We use various inequalities:

- Holder's Inequality

- Interpolation Inequalities in L^p spaces: for $p_0 < p < p_1$,
  $$L^{p_0} \cap L^{p_1} \subset L^p,$$
  and
  $$\|u\|_{L^p} \le \|u\|_{L^{p_0}}^{1-\theta} \cdot \|u\|_{L^{p_1}}^{\theta},$$
  for some $\theta \in [0, 1]$. We do a scaling computation:
  $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \theta \in [0, 1].$$

We begin with dimension $n = 2$ and proceed by induction on n. We start with $p = 1$. We already have that
$$\|u\|_{L^{n/n-1}} \le \|\nabla u\|_1.$$

For $n = 1$, this is just $\|u\|_\infty \le \|\partial u\|_1$.
   For $n = 2$, $\nabla u \in L^1_{x_1,x_2} = L^1_{x_1} L^1_{x_2}$. Hence,
$$\|\nabla u\|_{L^1} = \iint |\nabla u| \, dx_2 dx_1 = \int \|\nabla u(x_1, \cdot)\|_{L^1} \, dx_1.$$

> **Remark 23.2.** The order of the exponents matter if they are not equal.

Then, for $\nabla u \in L^1$, we have $\partial_1 u \in L^1$ and $\partial_2 u \in L^1$. From the first, it follows that $\partial_1 u \in L^1_{x_1} L^1_{x_2} \implies u \in L^\infty_{x_1} L^1_{x_2}$. and from the second, it follows that $u \in L^1_{x_1} \dot{W}^{1,1} \implies u \in L^1_{x_1} L^\infty_{x_2}$. Interpolating the two inequalities by picking $\theta = 1/2$, we find that
$$\|u\|_{L^2} \le \|u\|_{L^\infty_{x_1} L^1_{x_2}}^{1/2} \cdot \|u\|_{L^1_{x_1} L^\infty_{x_2}}^{1/2}.$$

We could have also found this directly:

$$\|u\|_2^2 = \int u^2 dx_1 dx_2$$

$$= \iint u \cdot u dx_2 dx_1$$

$$\leq \int \|u(x_1)\|_{L_{x_2}^\infty} \|u(x_1)\|_{L_{x_2}^1} dx_1$$

$$\leq \|u\|_{L_{x_1}^1 L_{x_2}^\infty} \|u\|_{L_{x_1}^\infty L_{x_2}^1}.$$

From n to $n+1$, we have $x = (x_1, x^1)$. We start by noting that $\nabla u \in L^1$ implies that $\partial_1 u \in L_{x_1}^1 L_{x^1}^1$ and $u \in L_{x_1}^1 \dot{W}_{x^1}^{1,1}$. From the first, we have that $u \in L_{x_1}^\infty L_{x^1}^1$ and $u \in L_{x_1}^1 L_{x^1}^{n/n-1}$. We interpolate from here to find that $u \in L^p L^p$.

We choose

$$x_1 : \frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{1},$$

$$x^1 : \frac{1}{p} = \frac{1-\theta}{1} + \theta \frac{n-1}{n}.$$

It follows that $\frac{1}{p} = \theta$, $\frac{1}{p} = 1 - \frac{\theta}{n}$. Solving this system, we find that

$$\frac{1}{p}\left(1 + \frac{1}{n}\right) = 1 \implies p = \frac{n+1}{n}.$$

Hence. $\|u\|_{L^q} \leq \|\nabla u\|_{L^p}, u \in \mathcal{D}$ for $p = 1$. If $p \geq 1$, we do the following: denote $\nu = u^\gamma$, $\nu > 1$. Apply the $p = 1$ case of GNS for to ν.

$$\|u^\gamma\|_{L^{n/n-1}} \leq \|\nabla u \cdot u^{\gamma-1}\|_{L^1}.$$

The LHS is $\|u\|_{L^{\gamma n/n-1}}^\gamma$. On the RHS, we have $\|\nabla u\|_{L^p} \cdot \|u^{\gamma-1}\|_{L^{p'}}$, using Holder's inequality. It suffices to choose γ appropriately, namely

$$\frac{\gamma n}{n-1} = q \Leftrightarrow (\gamma - 1)p' = q.$$

We have the more general case of GNS with $\dot{W}^{k,p} \subset L^q$ provided that

$$\frac{n}{q} = \frac{n}{p} - k,$$

with $1 \leq p < n/k$, $k < n$. For $k = n$, we have $\dot{W}^{n,1} \subset L^\infty$.   □

Look at $\mathcal{D}$ with norm

$$\|u\|_{\dot{W}^{1,p}} = \|\nabla u\|_{L^p},$$

and take the completion - we call this $\dot{W}^{1,p}$.

§24 November 17th, 2020

§24.1 GNS Inequalities

We discussed the embedding $\dot{W}^{1,p} \subset L^q$, which is the sharp Sobolev embedding. We have a scaling relation

$$\frac{n}{p} - 1 = \frac{n}{q},$$

for $1 \le p < n$. This gives the embeddings

$$W^{1,p} \subset L^q, W^{1,p} \subset L^q \to W^{1,p} \subset L^r, r \in [p,q].$$

The embedding in the case of $p = n$ would be $\dot{W}^{1,n} \subset L^\infty$, but this is **false**: take $u(x) = |\log^\gamma |x||$ for $\gamma << 1$.

What happens when $p > n$? Remember the case $\dot{W}^{1,p} \subset \dot{C}^\gamma$ for $\gamma = 1 - \frac{1}{p}$.

§24.2 Morrey's Inequalities

> **Theorem 13** (Morrey)
> We have that $\dot{W}^{1,p}(\mathbb{R}^n) \subset \dot{C}^\gamma(\mathbb{R}^n)$, for $\gamma = 1 - \frac{n}{p}$, $n < p < \infty$.

Proof. In 1D, we want

$$|u(x) - u(y)| \le |x - y|^\gamma \|\partial_x u\|_{L^p}.$$

In higher dimensions, we want

$$|u(x) - u(y)| \le |x - y|^\gamma \|\nabla u\|_{L^p}.$$

As an intermediate step, we compare $u(x)$ with $u|_{B(x,r)}$. If we take $z \in B(x,r)$, we would like to show that

$$\fint_{B(x,r)} |u(x) - u(z)| \, dz \le \fint_{B(x,r)} \frac{1}{|x - z|^{n-1}} |\nabla u(z)| \, dz.$$

We show that this implies the Morrey Inequality. Suppose we have two points x, y with $D = B(x,r) \cap B(y,r)$, where $r = |x - y|$. By the inequality,

$$|u(x) - u(y)| \le |u(x) - u(z)| + |u(y) - u(z)|$$
$$\le \fint_D |u(x) - u(z)| \, dz + \fint_D |u(y) - u(z)| \, dz$$
$$\le C \fint_{B(x,r)} |u(x) - u(z)| \, dz + C \fint_{B(y,r)} |u(y) - u(z)| \, dz$$
$$\le \fint_{B(x,r)} \frac{|Du(z)|}{|x - z|^{n-1}} \, dz + \fint_{B(y,r)} \frac{|Du(z)|}{|x - z|^{n-1}} \, dz$$
$$\le \frac{1}{r^n} \|\nabla u\|_{L^p} \cdot \||x - z|^{1-n}\|_{L^q} + \frac{1}{r^n} \|\nabla u\|_{L^p} \cdot \||y - z|^{1-n}\|_{L^q} \le r^\gamma \|\nabla u\|_{L^p},$$

where both q norms are finite since $n < p < \infty$ so $1 < q < \frac{n}{n-1}$.

Hence,

$$|u(x) - u(y)| \le |x - y|^\gamma \|\nabla u\|_{L^p}.$$

Instead of computing γ, we conclude that it must be the correct one by scaling.

It suffices to show that

$$\fint_{B(x,r)} |u(x) - u(z)| \, dz \leq \fint_{B(x,r)} \frac{1}{|x - z|^{n-1}} |\nabla u(z)| \, dz.$$

We parameterize $[x, z]$ as $w = (1 - h)x + hz$ for $h \in [0, 1]$. Then,

$$u(z) - u(x) = u(w(1)) - u(w(0)) = \int_0^1 \frac{d}{dh} u(w(h)) \, dh = \int_0^1 \nabla u(w(h)) \cdot (z - x) \, dz$$

so

$$|u(x) - u(z)| \leq \int_0^q |\nabla u(w(h))| \cdot |z - x| \, dh$$

It follows that

$$\fint |u(x) - u(z)| \leq \int_0^1 \int_B |\nabla u(w(h))| \cdot |z - x| \, dh \, dz.$$

Applying a change of coordinates, we find that

$$\fint |u(x) - u(z)| \leq \frac{1}{|B|} \int_B \int_{|w-x|/r}^\infty |\nabla u(w)| \frac{1}{h} |w - x| h^n \, dh \, dw$$

Then $r \geq |z - x| = \frac{1}{h} |w - x|$, so $h \geq |w - x|/r$, Integrating in h, we find that

$$|u(x) - u(z)| \leq \frac{1}{|B|} \int_B |\nabla u(w)| \cdot |w - x| \cdot \left(\frac{r}{|w - x|} \right)^n \leq \int_B |\nabla u(w)| \cdot \frac{1}{|w - x|^{n-1}} \, dw.$$

$\square$

§24.3 GNS with Bounded Domains

We have in general that $\|u\|_q \leq \|\nabla u\|_p$ in $\mathbb{R}^n$. What happens for $\Omega \subset \mathbb{R}^n$? Is it true that

$$\|u\|_{L^q(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}?$$

This true if u is compactly supported in Ω since we can extend u be 0 outside Ω. This inequality will be very false otherwise, simply take a constant function.

Define $u_\Omega = \overline{u} = \frac{1}{|\Omega|} \int_\Omega u \, dx$.

> **Theorem 14** (Poincare Inequalities)
>
> $$\|u - \overline{u}\|_{L^r} \leq \|\nabla u\|_{L^p},$$
>
> for $1 \leq r \leq q$.
> This is true for $r = p$, $p > n, 1 \leq r \leq \infty$ and $p = n, 1 \leq r < \infty$.

Proof. If $p > n$,

$$\|u - \overline{u}\|_{L^\infty} \leq \|\nabla u\|_{L^p}.$$

This is true since by Holder, $\|u - \overline{u}\|_r \leq \|u - \overline{u}\|_\infty$. Applying the previous proof,

$$\fint_{B(x,r) \cap \Omega} |u(x) - u(z)| \leq \int_{B(x,r) \cap \Omega} \frac{|\nabla u(z)|}{|x - z|^{n-1}} \, dz$$

It follows that we have

$$|u(x) - u(y)| \leq |x - y|^\gamma \|\nabla u\|_{L^p},$$

so

$$
\begin{aligned}
|u - \overline{u}| &= \left| u(x) - \fint_\Omega u \, dy \right| \\
&= \left| \fint_\Omega u(x) - u(y) \, dy \right| \\
&\leq \fint_\Omega |u(x) - u(y)| \, dy \\
&\leq \|\nabla u\|_{L^p}.
\end{aligned}
$$

$\square$

Consider the special case of Poincare's inequality when $p = n$.

$$\left(\fint_B |u - \overline{u}|^p \right)^{1/p} \leq \|\nabla u\|_{L^n},$$

for $1 \leq p < \infty$.

The true embedding is that $\dot{W}^{1,n} \subset \mathrm{BMO}$ where

$$\|u\|_{\mathrm{BMO}} = \sup_B \fint |u - \overline{u}| \, dx.$$

§24.4 Compact Sobolev Embeddings

An embedding $X \subset Y$ is compact if any bounded sequence in X has a convergent subsequence in Y.

When are Sobolev embeddings compact? Start with $\dot{W}^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$. Take $u_n \in \dot{W}^{1-p}$ and $\|u_n\|_{\dot{W}^{1-p}=1}$. Is there a convergent subsequence?

We have two obvious enemies. Take $\varphi \in \mathcal{D}$, and change it using the symmeteries of $\dot{W}^{1,p}, L^q$. We have translation invariance: $u_n(x) = \varphi(x - x_n)$.

Then if $\|u_n\|_p = C$, $\|u_n\|_{\dot{W}^{1,p}} = \tilde{C}$. If $u_n \to u$ in L^p on a subsequence, then $u_n \to u$ in $\mathcal{D}'$. But $u_n \to 0$ in $\mathcal{D}'$, so it follows that $u = 0$. But this is a contradiction since $\|u\|_p = \lim \|u_n\|_p = c \neq 0$.