Math 210a Lecture Notes, Fall 2020

Theoretical Statistics

Professor: William Fithian

Scribe: Vishal Raman

Contents

§1 August 27th, 2020

§1.1 Measure Theory Basics

Definition 1.1 (Informal Measure)**.** Given a set X, a measure μ is a map from the subsets $A \subseteq X$ to $[0, \infty]$.

Some examples include:

- The Counting Measure: X is countable, $\nu(A) = |A|$, the cardinality of A.
- The Lebesgue Measure: $X = \mathbb{R}^n$, $\lambda(A) = \int_A dA = \text{Volume}(A)$.
- Standard Gaussian Distribution: Take $\phi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum x_i^2}$. Then

$$P(A) = \int_A \phi(x) dA.$$

For $Z \sim N_n(0, I_n)$, it is denoted $\mathbb{P}$.

In general, the domain of a measure μ is not be all subsets of X. We require $\mathcal{F} \subseteq 2^X$ to be a σ-field/algebra. Examples of σ-algebras:

- If X is countable, $\mathcal{F} = 2^X$ is a sigma-field.
- if $X = \mathbb{R}^n$, $\mathcal{F}$ is the smallest sigma-field containing all the open rectangles of $\mathbb{R}^n$.

Definition 1.2 (Formal Measure)**.** Given a **measurable space** $(X, \mathcal{F})$, a measure is a map $\mu : \mathcal{F} \to [0, \infty]$ satisfying the property that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for disjoint A_i.

Definition 1.3 (Probability Measure)**.** μ is a probability measure if it is a measure with $\mu(X) = 1$.

Measures give us a "weight" of sets in a space. We first define

$$\int 1\{x \in A\} d\mu(x) = \mu(A).$$

Then,

$$\int \sum_{i=1}^{\infty} c_i 1\{x \in A_i\} d\mu(x) = \sum_{i=1}^{\infty} c_i \mu(A_i).$$

For other nice functions, we can approximate them by taking limits of functions of the above form. Examples of integrals:

- Counting: $\int f d\nu = \sum_{x \in X} f(x)$.
- Lebesgue: $\int f d\mu = \int_X f(x) dx_1 \ldots dx_n$.
- Gaussian: $\int f dP = \int_X f(x)\phi(x) dx_1 \ldots dx_n = \mathbb{E}[f(z)]$ for random normal variables.

§1.2 Densities

We can see that the Lebesgue and Gaussian measures are very similar.

Definition 1.4 (Absolutely Continuous)**.** Given $(X, \mathcal{F})$, two measures P, μ, P is absolutely continuous with respect to μ(or $P << \mu$ or μ dominates P) if $P(A) = 0$ whenever $\mu(A) = 0$.

Definition 1.5 (Radon-Nikodym Derivative)**.** If $P << \mu$, we can also define a density function $p : X \to [0, \infty]$ with $P(A) = \int_A p(x) d\mu(x)$. We write $p(x) = \frac{dP}{d\mu}(x)$, and it is called the **Radon-Nikodym derivative**.

If P is a probability measure, μ Lebesgue, then $p(x)$ is called a **probability density function**(pdf). If μ is a counting measure, then $p(x)$ is called a **probability mass function**.