The Cotangent Bundle

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These notes correspond to Chapter 11 of Lee, *Smooth Manifolds*. Roughly, we define tangent covectors and the cotangent space. We use this to define line integrals of covector fields, which leads to the study of differential forms. Note the usage of the EInstein Summation convention.

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§1 Covectors

Let V be a finite-dimensional vector space over $\mathbb{R}$.

Definition 1.1. A **covector** is a real-valued linear functional on V; a linear map $\omega : V \to \mathbb{R}$.

The space of all covectors on V is a real vector space under scalar multiplication and pointwise addition, denoted V^*, the **dual space**.

Proposition 1.2

Let V be a finite-dimensional vector space. given any bases $(E_1, \ldots, E_n)$ for V, let $\varepsilon^1, \ldots \varepsilon^n \in V^*$ be the covectors defined by $\varepsilon^i(E_j) = \delta^i_j$, where δ^i_j is the usual Kronecker delta symbol. Then $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis for V^*, called the dual bases to (E_j). Hence $\dim V^* = \dim V$.

Definition 1.3. If V, W are vector spaces and $A : V \to W$ is a linear map, we defined $A^* : W^* \to V^*$ by $(A^*\omega)(v) = \omega(Av)$, for $\omega \in W^*, v \in V$.

Proposition 1.4

The dual map satisfies $(A \circ B)^* = B^* \circ A^*$ and $(\mathrm{id}_V)^* = id_{V^*}$.

Corollary 1.5

The assignment that sends a vector space to its dual space and a linear map to its dual map is a contravariant functor from the category of real vector spaces to itself.

Proposition 1.6

For any finite dimensional vector space V, the map $\xi : V \to V^{**}$ defined by

$$\xi(v)(\omega) = \omega(v),$$

for $\omega \in V^*$ is an isomorphism.

§2 Tangent Covectors on Manifolds

Definition 2.1. Let M be a smooth manifold with or without boundary. For each $p \in M$, we define the cotangent space at p, denoted by $T^*_pM = (T_pM)^*$.

Given smooth local coordinates (x^i) on an open subset $U \subset M$, for each $p \in U$, the coordinate basis $(\frac{\partial}{\partial x^i}|_p)$ gives rise to a dual basis for T^*_pM, denoted by $(\lambda^i|p)$. It follows that any $\omega \in T^*_pM$ can be written as $\omega = \omega_i\lambda^i|_p$ where $\omega_i = \omega(\frac{\partial}{\partial x^i}|_p)$.

§3 Covector Fields

Definition 3.1. Let M be a smooth manifold with or without boundary. The disjoint union

$$T^*M = \bigsqcup_{p \in N} T_p^*M$$

is called the **cotangent bundle of M**. It is equipped with a natural projection map $\pi : T^*M \to M$ sending $\omega \in T_p^*M$ to $p \in M$.

Definition 3.2. Given smooth local coordinates (x^i) to $U \subset M$, for each $p \in U$, we denote the basis for T_p^*M dual to $(\frac{\partial}{\partial x^i}|_P)$ by $(\lambda^i|_p)$. This defines n maps $\lambda^1, \dots, \lambda^n : U \to T^*M$, the **coordinate covector fields.**

Proposition 3.3

Let M be a smooth n-manifold with or without boundary. With its standard projection map and the natural vector space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth rank-n vector bundle over M for which all coordiante covector fields are smooth local sections.

Definition 3.4. Given the covector bundle, we can define (smooth) covector fields, which are sections of T^*M. This is also called a differential 1-form.

We could also define coframes in a similar way, and they satisfy the same usual properties of frames for bundles.

§4 The Differential

Definition 4.1. Let f be a smooth real-valued function on a smooth manifold M with or without boundary. Define a covector field df. called the **differential** of f by

$$df_p(v) = vf \quad v \in T_pM.$$

Given smooth coordinates (x^i) on an open subset $U \subset M$ and a coordinate coframe (λ^i), we can write

$$df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i|_p.$$

If we apply the formula to one of the coordinate functions $x^j : U \to \mathbb{R}$, we obtain $dx^j|_p = \lambda^j|_p$, so it follows that λ^j is the differential dx^j. Hence, we have

$$df_p = \frac{\partial f}{\partial x^i}(p)dx^i|_p, \quad df = \frac{\partial f}{\partial x^i}dx^i.$$

Proposition 4.2

Let M be a smooth manifold with or without boundary, and let $f, g \in C^\infty(M)$.

- If a and b are constants, then $d(af + bg) = a\,df + b\,dg$.
- $d(fg) = f\,dg + g\,df$.
- $d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.
- If $J \subset \mathbb{R}$ is an interval containing the image of f, and $h : J \to \mathbb{R}$ is smooth, then $d(h \circ f) = (h' \circ f)\,df$.
- If f is constant, $df = 0$.

Proposition 4.3

If f is a smooth real-valued function on a smooth manifold M, then $df = 0$ if and only if f is constant on each component of M.

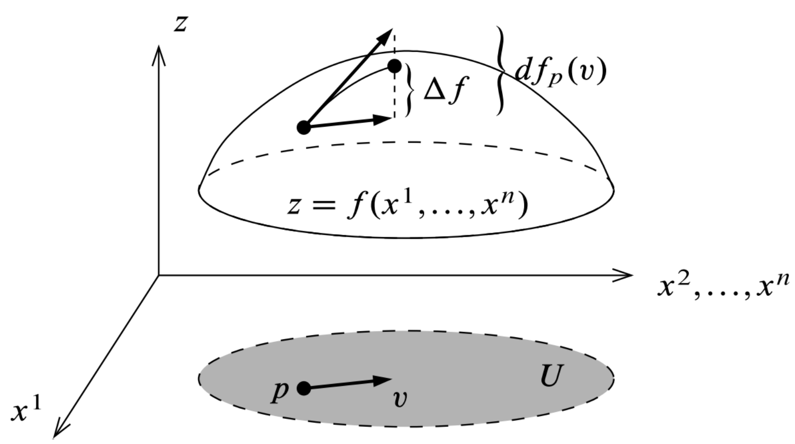

Fig. 11.2 The differential as an approximation to Δf

If we write $\Delta f = f(p + v) - f(p)$ for $v \in \mathbb{R}^n$ Taylor's theorem shows that

$$\Delta f \approx \frac{\partial f}{\partial x_i}(p)v^i = df_p(v).$$

Proposition 4.4

Suppose M is a smooth manifold, $\gamma : J \to M$ is a smooth curve, and $f : M \to \mathbb{R}$ is a smooth function. Then the derivative of the real-valued function $f \circ \gamma : J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

§5 Pullbacks of Covector Fields

Definition 5.1. Let $F : M \to N$ be a smooth map between smooth manifolds and let $p \in M$. The differential $dF_p : T_pM \to T_{F(p)}N$ yields a dual map

$$dF_p^* : T_{F(p)}^*N \to T_p^*M$$

called the **pullback at F by p** or the **cotangent map of F**. Note that

$$dF_p^*(\omega)(v) = \omega(dF_p(v)).$$

Definition 5.2. Given a smooth map $F : M \to N$ and a covector field ω on N, define a rough covector field $F^*\omega$ on M, called the **pullback of ω by F** by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}).$$

Proposition 5.3

Let $F : M \to N$ be a smooth map between smooth manifolds. Suppose u is a continuous real valued function on N and ω is a covector field on N. Then

$$F^*(u\omega) = (u \circ F)F^*\omega.$$

If u is smooth, then

$$F^*du = d(u \circ F).$$

Proposition 5.4

Suppose $F : M \to N$ is a smooth map between smooth manifolds, and let ω be a covector field on N. Then $F^*\omega$ is a covector field on M. If ω is smooth, then so if $F^*\omega$.

§6 Line Integrals

Definition 6.1. Suppose $[a, b] \subset \mathbb{R}$ is a compact interval, and ω is a smooth covector field on $[a, b]$. Note that ω can be written $\omega_t = f(t)\, dt$ for a smooth function $f : [a, b] \to \mathbb{R}$. We define **the integral of ω over** $[a, b]$ to be

$$\int_{[a,b]} \omega = \int_a^b f(t)\, dt.$$

Proposition 6.2

Let ω be a smooth covector field on the compact interval $[a, b] \subset \mathbb{R}$. If $\varphi : [c, d] \to [a, b]$ is an increasing diffeomorphism, then

$$\int_{[c,d]} \varphi^*\omega = \int_{[a,b]} \omega.$$

Definition 6.3. If $\gamma : [a, b] \to M$ is a smooth curve segment and ω is a smooth covector field on M, we define **the line integral of ω over γ** to be the real number

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega.$$

If γ is piecewise smooth, we define

$$\int_\gamma \omega = \sum_{i=1}^{k} \int_{[a_{i-1},a_i]} \gamma^*\omega,$$

where $[a_{i-1}, a_i]$ are the subintervals on which γ is smooth.

Proposition 6.4

Let M be a smooth manifold. Suppose $\gamma : [a, b] \to M$ is a piecewise smooth curve segment, and $\omega, \omega_1, \omega_2 \in \mathfrak{X}^*(M)$.

- For any $c_1, c_2 \in \mathbb{R}$,

$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

- If γ is a constant map, then $\int_\gamma \omega = 0$.
- If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ with $a < c < b$, then

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

- If $F : M \to N$ is a smooth map and $\eta \in \mathfrak{X}^*(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Proposition 6.5

If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment, the line integral of ω over γ can also be expresesd as the ordinary integral

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt$$

Proposition 6.6

Let M be a smooth manifold. Suppose f is a smooth real-valued function on M and $\gamma : [a, b] \to M$ is a piecewise smooth curve segment in M. Then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

§7 Conservative Covector Fields

Definition 7.1. A smooth covector field ω on a smooth manifold M with or without boundary is said to be exact on M if there is a function $f \in C^\infty(M)$ such that $\omega = df$. The function f is called a potential function for ω.

Definition 7.2. A smooth covector field ω is conservative if the line integral of ω over every piecewise smooth closed curve segment is zero.

Proposition 7.3

A smooth covector field is conservative if and only if its line integrals are path-independent.

Proposition 7.4

A smooth covector field on M is conservative if and only if it is exact.

Proposition 7.5

Every exact covector field is closed, that is it's components in every smooth chart satisfy

$$\frac{\partial \omega_j}{dx^i} = \frac{\partial \omega_i}{\partial x^j}.$$

Definition 7.6. If V is a finite-dimensional vector space, a subset $U \subset V$ is said to be star-shaped if there exists a point $c \in U$ such that for every $x \in U$, the line segment from $c \to x$ is contained in U.

Theorem 7.7 (Poincare Lemma)

If U is a star-shaped open subset of $\mathbb{R}^n$, then every closed covector field on U is exact.