

Math 258 Lecture Notes, Fall 2020

Harmonic Analysis

Professor: Michael Christ

Vishal Raman

Contents

1 August 27th, 2020	3
1.1 Introduction	3
1.2 Fourier Analysis	3
1.3 On Tori of Arbitrary Dimension	3
1.4 Euclidean Spaces	4
2 September 1st, 2020	7
2.1 Proof of Plancherel's Theorem	7
2.2 Introduction to Convolution	8
2.3 General Convolution	9
3 September 3rd, 2020	11
3.1 Convolution and Continuity	11
3.2 Convolution and Differentiation	11
3.3 Approximation	12
4 September 8th, 2020	15
4.1 Fourier Transform Identities	15
4.2 The Gaussian	16
4.3 Schwartz Spaces	17
5 September 10th, 2020	19
5.1 Schwartz Space, continued	19
5.2 Tempered Distributions	19
6 September 15th, 2020	23
6.1 Poisson Summation Formula	23
6.2 Size of Fourier Coefficients	23
7 September 17th, 2020	27
7.1 Size of Fourier Coefficients, continued	27
7.2 Comparing Size of Functions to Size of Fourier Coefficients	28

§1 August 27th, 2020

§1.1 Introduction

We begin by considering the problem of conduction of heat in a circle. We use the map $x \mapsto e^{ix}$, $x \in [0, 2\pi)$. Where u is the temperature, t is the time, we believed that $u_t = \gamma u_{xx}$, where subscripts denote partial derivatives. We also have an initial condition, $f(x) = u(x, 0)$.

There are some simple solutions $e^{inx}e^{-\gamma n^2 t}|_{t=0} = e^{inx}$. The product of solutions, the sum of solutions, and scalar multiple of solutions are all solutions, so he wrote the solution as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, u(x, t) = \sum_n a_n e^{-\gamma n^2 t} e^{inx}.$$

§1.2 Fourier Analysis

We take a circle $\{z \in \mathbb{C} : |z| = 1\}$, which can also be thought of as $\mathbb{R}/(2\pi\mathbb{Z})$, with the map $x \mapsto e^{ix}$. Suppose we have G a finite abelian group, and $\widehat{G} = \{\text{hom } \varphi : G \rightarrow \mathbb{R}/\mathbb{Z}\}$, the dual group. \widehat{G} is also a group, formally known as the set of characters.

Example 1.1

If we take $G = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, with the map $x \mapsto e^{2\pi i x n/N}$, for $n \in \mathbb{Z}_N$.

Similarly, taking $G \cong \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \dots$, we take $x \mapsto \prod e^{2\pi i x n/N_i}$.

Take $e_\xi(x) = e^{2\pi i \xi(x)}$, where $\xi : G \rightarrow \mathbb{R}/\mathbb{Z}$. Working in $L^2(G)$, we note the following:

Fact 1.2. If $\xi \neq \varphi$, then $\langle e_\xi, e_\varphi \rangle = 0$.

Proof.

$$\sum_{x \in G} \xi(x) \overline{\varphi(x)} = \sum_u \xi(u+y) \overline{\varphi(u+y)} - \left(\sum_u \xi(u) \overline{\varphi(u)} \right) \xi(y) \overline{\varphi(y)}.$$

Hence, either $\langle \xi, \varphi \rangle = 0$ or $\xi(y) \overline{\varphi(y)} = 1$ for all $y \in G$, which implies $\xi = \varphi$. \square

It follows that $\{e_f : f \in \widehat{G}\}$ is an orthonormal set in $L^2(G)$. Then, the dimension is $|\widehat{G}| = |G| = \dim(L^2(G))$. Hence, the set forms an orthonormal basis for $L^2(G)$.

Then, for all $f \in L^2(G)$, we have

$$\|f\|_{L^2(G)}^2 = \sum_{\varphi \in \widehat{G}} |\langle f, e_\varphi \rangle|^2,$$

$$f = \sum_{e_\xi \in \widehat{G}} \langle f, e_\xi \rangle e_\xi.$$

§1.3 On Tori of Arbitrary Dimension

We define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, from $[0, 2\pi]$. We then work on \mathbb{T}^d , $d \geq 1$.

For $f \in L^2(\mathbb{T}^d)$, we define

$$\widehat{f}(n) = (2\pi)^{-d} \int f(x) e^{-inx} dx.$$

We have an inner product $\langle f, g \rangle = \int_{\mathbb{T}^d} f(x) \overline{g(x)} d\mu(x)$ defined over a Lebesgue measure or Euclidean measure on \mathbb{T}^d .

Theorem 1 (Parseval's Theorem)

For all $f \in L^2(\Pi^d)$,

$$\|f\|_{L^2}^2 = (2\pi)^d \sum_{n \in \mathbb{Z}^d} |\widehat{f}(n)|^2,$$

and we have

$$f = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx},$$

in the sense that

$$\|f - \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx}\|_L^2 \rightarrow 0.$$

Note: you can usually figure out the constant with the simplest example, $f = 1$.

Proof. Take $\mathbb{T}^d, e_n(x) = e^{in \cdot x}$. The $\{(2\pi)^{-d/2} e_n : n \in \mathbb{Z}^d\}$ is orthonormal (left as an exercise). Then, for all f , $\sum_n \langle f, (2\pi)^{-d/2} e_n \rangle \leq \|f\|_{L^2}^2$, with equality if the set is a basis (Bessel's inequality).

It suffices to show that $\text{span}\{e_n\}$ is dense in L^2 . Take $P = \text{span}\{e_n\}$, and note that P is an algebra of continuous functions on Π^d , closed under conjugation, contains 1, and separates points. Hence, the Stone-Weierstrass theorem implies that P is dense in $C^o(\Pi^d)$ with respect to $\|\cdot\|_{C^o}$. Then $C^o \subset L^2$ is dense (general theory about Compact Hausdorff spaces, Radon Measures).

The statement $\|f - \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx}\|_L^2 \rightarrow 0$ follows from the general theory of orthonormal systems. \square

§1.4 Euclidean Spaces

We work in \mathbb{R}^d , ($d \geq 1$). Take $\xi \in \mathbb{R}^d$, and $x \mapsto x\xi \in \mathbb{R}$ is a homomorphism from $\mathbb{R}^d \rightarrow \mathbb{R}$, but if we take $x \mapsto e^{ix\xi}$, we have a homomorphism from $\mathbb{R}^d \mapsto \Gamma$. We try to define the following:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx = \langle f, e_\xi \rangle_{L^2(\mathbb{R}^d)},$$

where $e_{xi}(x) = e^{ix\xi}$.

Some problems:

1. $e_\xi \notin L^2(\mathbb{R}^d)$
2. $f(x) e^{-ix\xi}$ need not be in L^1 if $f \in L^2$.

We fix this by imposing extra conditions.

Definition 1.3. For $f \in L^1(\mathbb{R}^d)$, we define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx.$$

Note that $f \in L^1$ implies that \widehat{f} is bounded, continuous. We see this as follows: $\widehat{f}(\xi + u) - \widehat{f}(\xi) = \int f(x) e^{-ix\xi} (e^{-ixu} - 1) dx$. If we let $u \rightarrow 0$, the right goes to 0 pointwise, and $(2|f|) \in L^1$ dominates the integral, it goes to 0.

Proposition 1.4

If $f \in L^1 \cap L^2(\mathbb{R}^d)$, $\widehat{f} \in L^2(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2.$$

Theorem 2 (Plancherel's Theorem)

$\pi : L^1 \cap L^2 \rightarrow L^2$ extends uniquely to $\widehat{\pi} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, linear, bounded, $\|\widehat{\pi}f\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2$, and for all $f \in L^2$, we have an inverse Fourier Transform, $\check{f}(y) = \int f(\xi) e^{iy\xi} d\xi$ for $f \in L^1 \cap L^2$, and $\check{\cdot}$ also extends.

Finally,

$$\|f - (2\pi)^{-d} \int_{|\xi| \leq R} \widehat{f}(\xi) e^{ix\xi} d\xi\|_{L^2} \rightarrow 0.$$

Note that $\check{f}(y) = \widehat{f}(-y)$.

Proof. We first prove that $\|f\|_{L^2}^2 = (2\pi)^{-d} \|\widehat{f}\|_{L^2}^2$ for all $f \in L^1 \cap L^2$. We prove this for a dense subspace \mathcal{P} of L^2 . We will show later that there exists a subspace $V \subset L^2(\mathbb{R}^d)$ so that V is dense in L^2 , $V \subset L^1$, $\forall f \in V$, there exists $C_f < \infty$, so for all $\xi \in \mathbb{R}^d$, $|\widehat{f}(\xi)| \leq C_f (f(\xi))^{-d}$ and f is continuous with compact support.

We are given $f : \mathbb{R}^d \rightarrow \mathbb{C}$ supported where $|x| \leq R = R_f < \infty$. For large $t \geq 0$, define $f_t(x) = f(tx)$ (this shrinks the support of f), supported where $|x| \leq R/t < \pi$. We can then think of $f_t : \mathbb{T}^d \rightarrow \mathbb{C}$.

Now, we calculate

$$\begin{aligned} \widehat{f}_t(n) &= (2\pi)^d \int_{\mathbb{T}^d} f_t(x) e^{-inx} dx \\ &= t^{-d} (2\pi)^d \int_{\mathbb{R}^d} f(x) e^{-in/ty} dy \\ &= t^{-d} (2\pi)^{-d} \widehat{f}(t^{-1}n), \end{aligned}$$

where the first hat is on \mathbb{T}^d and the second is on \mathbb{R}^d , so the Fourier coefficients in the euclidean case are scalar multiples of the Fourier coefficients in the Tori case.

Thus,

$$\|f_t\|_{L^2(\mathbb{T}^d)}^2 = t^{-d} \|f\|_{L^2(\mathbb{R}^d)}^2 = c_d \sum_{n \in \mathbb{Z}^d} |\widehat{f}_t(n)|^2 = c'_d t^{-2d} \sum_n |\widehat{f}(t^{-1}n)|^2$$

Hence,

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c'_d t^{-d} \sum_n |\widehat{f}(t^{-1}n)|^2.$$

This has a nice tiling Riemann sum interpretation: if we take \mathbb{R}^d and tile it with cubes of sidelength $1/t$ where one corner is at $t^{-1}n$ for $n \in \mathbb{Z}^d$, then

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c'_d t^{-d} \sum_n \left| \widehat{f}(t^{-1}n) \right|^2 = \int_{\mathbb{R}^d} |g_t|^2 dx,$$

where $g(x) = \widehat{f}(t^{-1}n)$.

We claim

$$\int_{\mathbb{R}^d} |g_t|^2 \rightarrow \int_{\mathbb{R}^d} |\widehat{f}|^2,$$

which follows from the dominated convergence theorem: where we take a sequence over t going to infinity, with dominator $C_f^2(1 + |\xi|)^{-2d}$ in L^1 and $|\widehat{f}(\xi)| \leq C_f^2(1 + |\xi|)^{-2d}$. Furthermore, we have $g_t(\xi) \rightarrow \widehat{f}(\xi)$ as $t \rightarrow 0$, and \widehat{f} is continuous so g_t is pointwise convergent, and we have

$$|g_t(\xi)| = |\widehat{f}(t^{-1}n)| \leq C_f(1 + |t^{-1}n|)^{-d} \leq C'(1 + |\xi|)^{-d}.$$

□

§2 September 1st, 2020

§2.1 Proof of Plancherel's Theorem

Last time

- \mathbb{R}^d ,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

- $V = \{f \in L_1 \cap L_2(\mathbb{R}^d) : |\widehat{f}(\xi)| \langle \xi \rangle^d \text{ is a bounded linear function, } \langle x \rangle = (1+|x|^2)^{1/2} \geq 1, = |x| \text{ for } x \text{ large.}\}$
- Claim: V is dense in $L^2(\mathbb{R}^d)$. Then $\|\widehat{f}\|_{L^2} = (2\pi)^{d/2} \|f\|_{L^2}$ for all $f \in V$ so there exists a unique bounded linear operator \mathcal{F} on $L^2(\mathbb{R}^d)$, where \mathcal{F} takes a function to its fourier transform.
- We discussed some properties of \mathcal{F} .
 - $\|\mathcal{F}f\|_2 = (2\pi)^{d/2} \|f\|_2$
 - \mathcal{F} is onto.
 - For all $f \in L^2$,

$$\left\| f - (2\pi)^{-d} \int_{|\xi| \leq R} e^{ix \cdot \xi} \mathcal{F}(f)(\xi) d\xi \right\|_{L^2} \rightarrow 0,$$

in the limit where $R \rightarrow \infty$.

First note that \mathcal{F} has closed range (this was an exercise). It suffices to show: If $g \in L^2, g \perp \mathcal{F}(f)$ for all $f \in V$, then $g = 0$.

Proof. First, note that

$$0 = \langle g, \mathcal{F}(f) \rangle = \langle \mathcal{F}^*(g), f \rangle,$$

and for all $g \in V$,

$$\mathcal{F}^*g(x) = \int g(\xi) e^{ix \cdot \xi} d\xi$$

Therefore, $\mathcal{F}^*(g)(x) = (\mathcal{F}g)(-x)$ for all $g \in V$, which is dense in L^2 . Hence, $\mathcal{F}g = 0$, and the Fourier transform preserves norms, so $g = 0$. \square

We also claimed the following: Let $f \in L^2$:

$$\|f(x) - (2\pi)^{-d} \int_{|\xi| \leq R} (\mathcal{F}f)(\xi) e^{ix \cdot \xi} d\xi\|_2^2 \rightarrow 0.$$

Proof. Let $g_r = (2\pi)^{-d} \int_{|\xi| \leq R} (\mathcal{F}f)(\xi) e^{ix \cdot \xi} d\xi$. We have to show $\langle f, g_r \rangle \rightarrow \|f\|_2^2$. Then

$$\|f - g_r\|_2^2 = \|f\|_2^2 + \|g_r\|_2^2 - 2\operatorname{Re}\langle f, g_r \rangle \rightarrow \|f\|_2^2 + \|f\|_2^2 - 2\|f\|_2^2.$$

$$\begin{aligned} \langle f, g_r \rangle &= (2\pi)^{-d} \int f(x) \overline{\int_{|\xi| \leq R} (\mathcal{F}f)(\xi) e^{ix \cdot \xi} d\xi} dx \\ &= (2\pi)^{-d} \int_{|\xi| \leq R} \left(\int f(x) e^{-ix \cdot \xi} dx \right) \overline{(\mathcal{F}f)(\xi) d\xi} \\ &= (2\pi)^{-d} \int_{|\xi| \leq R} |\mathcal{F}f(\xi)|^2 d\xi \rightarrow (2\pi)^{-d} \|\mathcal{F}f\|_2^2 = \|f\|_2^2. \end{aligned}$$

However, it's not clear that we can use Fubini's theorem. We would need $f \in L^1 \cap L^2$. But this is not an issue as $L^1 \cap L^2 \subset L^2$ is dense, so if we let $\epsilon > 0$, $f = G + h$, $\|h\|_2 \leq \epsilon$ and $G \in L^1 \cap L^2$. Showing the convergence from here is an exercise. \square

We still need $V = (f \in L^1 \cap L^2 : \langle \xi \rangle^d (\widehat{f}(\xi)) \text{ is bounded})$ is dense in L^2 . We'll discuss this in the future.

§2.2 Introduction to Convolution

Our meta definition is $f * g(x) = \int f(x-y)g(y)dy$, but it will depend on the conditions of the function for the integral to be defined.

Convolution is generally associated to a group, where

$$\int_G f(xy^{-1}g(y)d\mu(y)),$$

with the Haar measure(done in 202b).

If we substitute $y = x - u$, then

$$f * g(x) = \int f(u)g(x-u)du = g * f(x).$$

It is also associative: $(f * g) * h = f * (g * h)$ for all f, g, h (involves Fubini's theorem).

We can formally write

$$f * g(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u)g(v)d\lambda_x(u, v),$$

where λ_x is supported on $\Lambda = \{(u, v) : u + v = x\}$ (an affine subspace). If we have a subset $E \subset \Lambda$, $\lambda_x(E) = |\pi_1(E)| = |\pi_2(E)|$, where π_i are Lebesgue measure s of projections on the i -th factor. Note the following: suppose that f, g are continuous with compact support. Then $\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)$, where $A + B = \{a + b : (a, b) \in A \times B\}$.

Let $T : C_0^0(\mathbb{R}^d) \rightarrow C_b^0(\mathbb{R}^d)$ be bounded, linear and $T \circ \tau_y = \tau_y \circ T$ for all $x \in \mathbb{R}^d$ ($\tau_y f(x) = f(x + y)$, a translation). Then, there exists a Complex Radon measure μ on \mathbb{R}^d so that for all $f \in C_0^0$, $T(f) = f * \mu$, where

$$f * \mu(x) = \int f(x-y)d\mu(y).$$

In the case of \mathbb{T}^1 , $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}$ for all $f \in L^2$. Suppose we wanted to consider the partial sums,

$$\sum_{n=-N}^N \widehat{f}(n)e^{inx} = S_N(f)(x).$$

In what sense does $S_N f \rightarrow f$, and for which functions f do we have convergence?

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^N e^{inx}(2\pi)^{-1} \int_{-\pi}^{\pi} f(y)e^{-iny}dy = (2\pi)^{-1} \int f(y) \sum_{n=-N}^N e^{in(x-y)}dy \\ &= (2\pi)^{-1} \int_{-\pi}^{\pi} f(y)D_N(x-y)dy. \end{aligned}$$

The Dirichlet Kernels, $D_N(x) = \sum_{n=-N}^N e^{inx} = \frac{\sin((N+1/2)x)}{\sin(x/2)}$ if $x \neq 0$ or $D_N(x) = 2N+1$ if $x = 0$.

§2.3 General Convolution

Theorem 3

Let $f, g \in L^1(\mathbb{R}^d)$. Then, the following are true:

- $y \mapsto f(x - y)g(y) \in L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x - y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- If $f, g \geq 0$, then $\|f * g\|_1 = \int f * g = \int f \int g$.
- The operation commutative and associative, so L^1 is an algebra, but it no multiplicative identity, so no inverses.
- For $f, g \in L^1$, $\widehat{(f \star g)} = \widehat{f} \cdot \widehat{g}$.

In other words, convolution is a nice bilinear operation.

Proof. Let $F(x, y) = f(x - y)g(y)$, $F : \mathbb{R}^{d+d} \rightarrow \mathbb{C}$ is Lebesgue measurable. We claim that $F \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. It follows from

$$\int |F(x, y)| dx dy = \int |f(x - y)| |g(y)| dx dy = \int |g(y)| dy \int |f(x)| dx = \|g\|_1 \|f\|_1 < \infty.$$

Now, $F \in L^1$, so by Fubini's theorem, for almost every $x, y \mapsto f(x - y)g(y) \in L^1$ and $x \mapsto \int f(x - y)g(y)dy$ is Lebesgue measurable.

$$\|f * g\|_1 = \int |f * g(x)| dx = \int \left| \int f(x - y)g(y) dy \right| dx \leq \int \int |f(x - y)| |g(y)| dy dx = \|f\|_1 \|g\|_1.$$

Note that $\int (f * g)(x) dx = \|f\|_1 \|g\|_1$, for non-negative functions.

Finally,

$$\begin{aligned} (f * g)^\wedge(\xi) &= \int e^{-ix \cdot \xi} \left(\int f(x - y)g(y) dy \right) dx \\ &= \int \left(\int e^{-ix \cdot \xi} f(x - y) dx \right) dy, x = u + y \\ &= \int \left(e^{-i(u+y) \cdot \xi} f(u) du \right) g(y) dy \\ &= \int e^{-iy \cdot \xi} \widehat{f}(u) g(y) dy \\ &= \widehat{f}(\xi) \cdot \widehat{g}(\xi). \end{aligned}$$

□

Example 2.1 (A Warning)

In \mathbb{R}^1 , $f(x) = |x|^{-2/3} 1_{|x| \leq 1}$, which has an asymptote at 0. $f \in L^1$, and

$$(f * f)(0) = \int_{-1}^1 |u|^{-4/3} dy = +\infty.$$

Proposition 2.2

Let $p \in [1, \infty]$. Let $f \in L^1, g \in L^p$. Then,

- $y \mapsto f(x - y)g(y) \in L^1$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x - y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^p(\mathbb{R}^d)$, $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. For $p = \infty$, $\int f(x - y)g(y)dy \in C_0(\mathbb{R}^d)$.

If $1 < p < \infty$, $L^p \subset L^1 + L^\infty$, as follows:

$$f(x) = f(x)1_{|f(x)| \leq 1} + f(x)1_{|f(x)| > 1}.$$

We can prove the rest with Minkowski's inequality, or a simpler way. Let $q = p' = \frac{p}{p-1}$ (hence $\frac{1}{q} + \frac{1}{p} = 1$). We use the norm definition,

$$\|f * g\|_p = \sup_{\|h\|_q \leq 1} \int |g * f| \cdot |h|.$$

$$\begin{aligned} \int |g * f| \cdot |h| &\leq \int (|g| * |f|) \cdot |h| = \int \int |g(x - y)| |f(y)| dy h(x) dx \\ &= \int |f(y)| \int |g(x - y)| h(x) dx dy \leq \int |f(y)| \|g\|_p * 1 dy = \|f\|_1 \|g\|_p. \end{aligned}$$

□

§3 September 3rd, 2020

§3.1 Convolution and Continuity

Recall convolution:

$$f * g(x) = \int f(x-y)g(y)dy, f * \mu(x) = \int_{\mathbb{R}^d} f(x-y)d\mu(y),$$

where f is continuous, bounded, μ is a complex Radon measure ($|\mu|$ is finite)

Proposition 3.1

Let $T : C_0^0 \rightarrow C_b^0$. Suppose T is translation invariant: $T \circ \tau_y = \tau_y \circ T$ for all $y \in \mathbb{R}^d$. [There exists $A < \infty : \|Tf\|_{C_0} \leq A\|f\|_{C_0}$ for all f . Recall $\|f\|_{C_0} = \sup_x |f(x)|$, and C_0^0, C_b^0 are Banach spaces.] There exists a complex radon measure μ such that $Tf = f * \mu$ for all f .

Proof. Given $T : C_0^0 \rightarrow C_b^0$, consider the map $\ell : C_0^0 \rightarrow \mathbb{C}$ given by $f \mapsto (Tf)(0)$. It is clear that ℓ is linear. Furthermore, ℓ is bounded, since

$$|Tf(0)| \leq \|Tf\|_{C_0} \leq A\|f\|_{C_0},$$

so $\ell \in (C_0^0)^*$. Recall the Riesz Representation Theorem, there exists ν , a complex Radon measure, such that for all $f \in C_0^0$

$$\ell(f) = \int f d\nu.$$

Let $y \in \mathbb{R}^d$. We have

$$Tf(-y) = Tf(0 - y) = (\tau_y Tf)(0) = T(\tau_y f)(0) = \int \tau_y f(x) d\nu(x) = \int f(x - y) d\nu(x).$$

Similarly, for all x , $(Tf)(-x) = \int f(y - x) d\nu(y)$. [See lecture notes for correct algebra, sad]. \square

§3.2 Convolution and Differentiation

Informally,

$$\frac{\partial}{\partial x_j} \int f(x-y)g(y)dy = \int \frac{\partial f}{\partial x_j}(x-y)g(y)dy.$$

Proposition 3.2

Assume $f \in C^1(\mathbb{R}^d)$, $g \in L^1$ and $f, \nabla f$ is bounded. Then

$$f * g \in C^1, \frac{\partial}{\partial x_j}(f * g) = \left(\frac{\partial f}{\partial x_j} \right) * g.$$

Proof. We assume $d = 1$ for clarity.

$$\frac{(f * g)(x+t) - (f * g)(x)}{t} = \int \frac{f(x+t-y) - f(x-y)}{t} g(y) dy.$$

Let $t \rightarrow 0$. Use DCT, with dominator

$$|g(y)| \cdot \sup_{t,u} \frac{|f(u+t) - f(u)|}{|t|}.$$

The supremum is finite by the mean value theorem. \square

Example 3.3

Take $g \in L^\infty$, $f \in C_1$, and there exists $a < \infty$ such that for all x ,

$$|f(x)| + |\nabla f(x)| \leq A\langle x \rangle^{-\gamma}.$$

Hence, $f, \nabla f \in L^1$. Then $f * g \in C^1$, $\nabla(f * g) = (\nabla f) * g$.

We can iterate this: Under appropriate conditions

$$\begin{aligned} \frac{\partial^\alpha(f * g)}{\partial x^\alpha} &= \frac{\partial^\alpha f}{\partial x^\alpha} * g, \\ \frac{\partial^{\alpha+\beta}(f * g)}{\partial x^{\alpha\beta}} &= \frac{\partial^\alpha f}{\partial x^\alpha} * \frac{\partial^\beta g}{\partial x^\beta}. \end{aligned}$$

Proposition 3.4

If $f \in L^1$ and $g \in L^\infty$, then $f * g \in C_b^0$.

Proof. Recall: If $f \in L^1(\mathbb{R}^d)$, then $y \mapsto \tau_y f \in L^1$ is continuous: As $y \rightarrow 0$,

$$\|\tau_y f - f\|_1 \rightarrow 0.$$

Then,

$$(f * g)(x) - (f * g)(x') = \int (f(x-y) - f(x'-y))g(y)dy = \int [f(x-y) - (\tau_u f)(x-y)]g(y)dy,$$

where $u = x' - x$. As $u \rightarrow 0$, $\|f - \tau_u f\|_1 \rightarrow 0$, and $g \in L^\infty$, so the integral approaches 0, as desired. \square

§3.3 Approximation

Definition 3.5 (Approximate Identity Sequence). An approximate identity sequence for \mathbb{R}^d is $(\varphi_n)_{n \in \mathbb{N}}$, $\varphi_n \in L^1(\mathbb{R}^d)$ with the following conditions:

- $\int_{\mathbb{R}^d} \varphi_n = 1$.
- For all $\delta > 0$, $\int_{|x| \geq \delta} |\varphi_n| dx \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 4

Let (φ_n) be an approximate identity sequence in \mathbb{R}^d .

1. Let $f \in C_b^0$ be uniformly continuous. Then $f * \varphi_n \rightarrow f$ uniformly.
2. Let $f \in C_b^0$. Then $f * \varphi_n \rightarrow f$ uniformly on every compact set.
3. If $1 \leq p \leq \infty$, then for all $f \in L^p$, $\|f * \varphi_n - f\|_p \rightarrow 0$.

[All the above limits are taken for $n \rightarrow \infty$.]

Proof.

$$\begin{aligned} f * \varphi_n(x) - f(x) &= \int f(x-y)\varphi_n(y)dy - f(x) \\ &= \int (f(x-y) - f(x))\varphi_n(y)dy \end{aligned}$$

Then,

$$|f * \varphi_n(x) - f(x)| \leq \int |f(x-y) - f(x)|\varphi_n(y)dy.$$

Let $\delta > 0$. Then,

$$\int |f(x-y) - f(x)|\varphi_n(y)dy = \int_{|y| \leq \delta} |f(x-y) - f(x)|\varphi_n(y)dy + \int_{|y| \geq \delta} |f(x-y) - f(x)|\varphi_n(y)dy.$$

$$\begin{aligned} \int_{|y| \leq \delta} |f(x-y) - f(x)|\varphi_n(y)dy &\leq \|\varphi_n\|_1 \cdot \sup_{x, |y| \leq \delta} |f(x-y) - f(x)| \\ &= \|\varphi_n\|_1 \cdot \omega_f(\delta) \\ &\leq A \cdot \omega_f(\delta). \end{aligned}$$

Then

$$\begin{aligned} \int_{|y| \geq \delta} |f(x-y) - f(x)|\varphi_n(y)dy &\leq \int_{|y| \geq \delta} 2\|f\|_{C^0} \cdot |\varphi_n(y)|dy \\ &\leq 2\|f\|_{C^0} \int_{|y| \geq \delta} |\varphi_n|dy. \end{aligned}$$

Hence

$$|f * \varphi_n(x) - f(x)| \leq A\omega_f(\delta) + 2\|f\|_{C^0} \int_{|y| \geq \delta} |\varphi_n|dy.$$

Taking the lim sup, the second term goes to 0, so for all $\delta > 0$,

$$\lim_{n \rightarrow \infty} \sup \|f * \varphi_n - f\|_{C^0} \leq A\omega_f(\delta).$$

Since f is uniformly continuous, $\lim_{\delta \rightarrow 0} \omega_f(\delta) = 0$, which proves the claim. \square

Corollary 3.6

$C^\infty \cap L^p$ is dense in L^p for all $1 \leq p \leq \infty$.

Proof. We want to construct (φ_n) with $\varphi_n \in C_0^\infty$.

We claim there exists a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\int \varphi = 1$ and $\varphi \geq 0$. In $d = 1$, take $h(x) = 1x > 0e^{-\|x\|}$. Then, define $\varphi(x) = h(x)h(1-x) \in C_0^\infty$. Then, we normalize φ .

Now, take $\varphi_n(x) = n^d \varphi(nx)$. □

Example 3.7

Let $\varphi \geq 0$, $\int \varphi = 1$. Define $\varphi_n(x) = n^d \varphi(nx)$. Then $\int \varphi_n = 1$.

Furthermore,

$$\int_{|x| \geq \delta} n^d \varphi(nx) dx = \int_{|y| \geq n\delta} \varphi(y) dy \rightarrow 0.$$

Example 3.8

Let $\varphi(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^d$. Let $t > 0$ and $\varphi_t(x) = (2\pi)^{-d/2} t^{-d/2} e^{-|x|^2/(2t)}$. Now $t \rightarrow 0^+$ and

$$\int_{|x| \geq \delta} \varphi_t(x) dx \rightarrow 0.$$

This is an approximate identity family.

Example 3.9 (Interpretation of $f * g$)

$$f * g = \int \tau_y f(x) \cdot g(y) dy.$$

If $g \geq 0$ and $\int g = 1$, then we have an **average** of translates of f .

As $n \rightarrow \infty$, $g = \varphi_n$ so the weight concentrates asymptotically at the origin.

§4 September 8th, 2020

§4.1 Fourier Transform Identities

We have many functorial identities.

1. For $f \in L^1$,

$$(\tau_y f)^\wedge(\xi) = e^{-iy \cdot \xi} \widehat{f}(\xi).$$

2. For $f, g \in L^1(\mathbb{R})$,

$$(f * g)^\wedge = \widehat{f} \cdot \widehat{g}.$$

3. For $f \in L^1$,

$$(e^{ix \cdot \eta} f)^\wedge(\xi) = \widehat{f}(\xi - \eta).$$

4. We use the notation

$$\xi^\alpha = \prod_{j=1}^d \xi_j^{\alpha_j}.$$

For $f \in C^0, C^{|\alpha|}, C_0^0$,

$$(\partial^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \widehat{f}(\xi).$$

This comes from the fact that

$$\int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_k} f(x) \right) e^{-ix \cdot \xi} dx,$$

so we integrate by parts, use Fubini in \mathbb{R}^d and induct on $|\alpha|$.

5. For $f \in C_0^\infty$,

$$(X^\beta f(x))^\wedge(\xi) = (i\partial_\xi)^\beta \widehat{f}(\xi),$$

where

$$x^\beta = \prod_{j=1}^d x_j^{\beta_j}, (i\partial_\xi)^\beta = i^{|\beta|} \partial^\beta.$$

6. For $f \in C_0^\infty$,

$$(\partial_x^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \widehat{f}(\xi).$$

7. If $L \in GL(d)$, $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$, linear invertible, then for all $f \in L^1$,

$$(f \circ L)^\wedge(\xi) = |\det(L)|^{-1} \widehat{f} \circ ((L^*)^{-1})(\xi).$$

The proof follows from the substitution $x = L^{-1}(y)$ and $(L^{-1})^* = (L^*)^{-1}$.

Corollary 4.1

$$V = \{f \in (L^1 \cap L^2)(\mathbb{R}^d) : \exists A = A_f < \infty, |\widehat{f}(\xi)| \leq A_f \langle \xi \rangle^{-d}\}$$

is dense in $L^2(\mathbb{R}^d)$.

Proof. We showed last time that C_0^∞ is dense in $L^2(\mathbb{R}^d)$. We need to show that $f \in C_0^\infty$ implies that $\widehat{f}(\xi) = O(\langle \xi \rangle^{-N})$ for all $N \leq \infty$.

WLOG, assume $\xi \neq 0$, $\xi_d \neq 0$, $|\xi_d| \geq \frac{|\xi|}{d}$. Then,

$$\begin{aligned} \int f(x) e^{-ix \cdot \xi} dx &= (-i\xi_d)^{-1} \int f(x) \frac{\partial}{\partial x_d} (e^{-ix \cdot \xi}) dx \\ &= (-i\xi_d)^{-1} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_d}(x) e^{-ix \cdot \xi} dx \leq \infty. \end{aligned}$$

We can pick up as many factors of ξ_d as we'd like to get arbitrary bounds. \square

§4.2 The Gaussian

Fact 4.2. ($d \geq 1$) Take $e^{-z|x|^2/2} = f(x) = f_z(x)$. Assume $\operatorname{Re}(z) \geq 0 \rightarrow f_z \in L^1$.

$$(e^{-z|x|^2/2})^\wedge(\xi) = (2\pi)^{d/2} z^{-d/2} e^{-|\xi|^2/(2z)}.$$

We consider $z^{-d/2}$ in the principal branch. When $z = 1$, $(e^{-|x|^2/2})^\wedge(\xi) = (2\pi)^{d/2} e^{-|\xi|^2/2}$. Note the fact

$$I = \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}.$$

In order to calculate

$$\int_{\mathbb{R}} e^{-x^2/2} e^{-ix\xi} dx,$$

we have

$$x^2/2 + ix\xi = \frac{1}{2}(x^2 + 2ix\xi) = 1/2(x + i\xi)^2 + \xi^2/2,$$

so

$$e^{-\xi^2/2} \int_{\mathbb{R}} e^{-(x+i\xi)^2/2} = e^{-\xi^2/2} \sqrt{2\pi}.$$

If $F(x) = \prod_{j=1}^d f_j(x_j)$, then $\widehat{F}(\xi) = \prod_{j=1}^d \widehat{f}_j(\xi_j)$.

For $z \in \mathbb{R}^+$, $e^{-z|x|^2/2} = e^{-|L(x)|^2/2}$, where

$$L(x) = z^{1/2}x.$$

Then, we use $(f \circ L)^\wedge(\xi) = |\det(L)|^{-1} \widehat{f}((L^*)^{-1}(\xi))$. For $\operatorname{Re}(z) \geq 0$,

$$\int f(x) e^{-ix \cdot \xi} dx = \int e^{-z|x|^2/2} e^{-ix \cdot \xi} dx.$$

We claim that this is a homomorphic function of z in $\operatorname{Re}(z) > 0$.

Fact 4.3. If $f \in L^1(\mathbb{R}^d)$ and $\widehat{f} \in L^1$, then

$$f = (2\pi)^{-d} (\widehat{f})^\vee, \check{g}(x) = \int g(\xi) e^{ix \cdot \xi} d\xi.$$

Corollary 4.4

If $f \in L^1$, $\widehat{f} = 0$, then $f = 0$ almost everywhere.

Proof. Given $f, \widehat{f} \in L^1$. Let $\varphi \in C_0^\infty$ with $\int \varphi = 1$. Let $\varphi_n(x) = n^d \varphi(nx)$. Define $f_n = f * \varphi_n$. We know that $f_n \rightarrow f$ in L^1 as $n \rightarrow \infty$.

Moreover, $f_n \in L^2$, since $f_n \in L^1 * L^2$. For each n , we have

$$\|(2\pi)^{-d} \int_{|\xi| \leq R} \widehat{f}_n(\xi) e^{ix \cdot \xi} d\xi - f_n(x)\|_{L^2} \rightarrow 0,$$

as $R \rightarrow \infty$.

Note that

$$\widehat{f}_n(\xi) = \widehat{f}(\xi) \widehat{\varphi}_n(\xi) = \widehat{f}(\xi) \widehat{\varphi}(n^{-1}\xi).$$

As $n \rightarrow \infty$, $\widehat{\varphi}(n^{-1}\xi) \rightarrow \widehat{\varphi}(0) = \int \varphi = 1$. Hence,

$$\widehat{f}_n(\xi) \rightarrow \widehat{f}(\xi).$$

Furthermore

$$\int_{|\xi| \leq R} \widehat{f}_n(\xi) e^{ix \cdot \xi} d\xi \rightarrow \int_{\mathbb{R}^d} \widehat{f}_n(\xi) e^{ix \cdot \xi} d\xi,$$

since $\widehat{f}_n \in L^1$ as $R \rightarrow \infty$.

Hence, we have that

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi) \widehat{\varphi}(n^{-1}\xi) e^{ix \cdot \xi} d\xi = f_n(x),$$

in the L^2 norm. Now, letting $n \rightarrow \infty$, $f_n = f * \varphi_n \rightarrow f$ in the L^1 norm.

$$\int_{\mathbb{R}^d} \widehat{f}(\xi) \widehat{\varphi}(n^{-1}\xi) e^{ix \cdot \xi} d\xi \rightarrow \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix \cdot \xi} d\xi = (\widehat{f})^\vee(x),$$

by the dominated convergence theorem. Thus,

$$f(x) = (2\pi)^{-d} (\widehat{f})^\vee(x).$$

But we actually proved a stronger result: $g \in L^1 \implies \check{g} \in C^0$, so if $g = \widehat{f}$, $(\widehat{f})^\vee \in C^0$ if $f \in L^1$, so if f, \widehat{f} are in L^1 , then f agrees almost everywhere with $(2\pi)^{-d} (\widehat{f})^\vee \in C^0$. \square

Example 4.5

Take $f(x) = 1_{[0,1]}(x)$. Hence $\widehat{f} \notin L^1$. Essentially, we have that $|\widehat{f}(\xi)| \approx \frac{1}{|\xi|}$ as $|\xi| \rightarrow \infty$.

§4.3 Schwartz Spaces

Definition 4.6 (Schwartz Space).

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^d) = \{f : \mathbb{R}^d \rightarrow \mathbb{C}, f \in C^\infty, \forall N, \alpha, x \mapsto \langle x \rangle^N \frac{\partial^\alpha f}{\partial x^\alpha} \text{ is bounded.}\}.$$

It is clear that \mathcal{S} is a vector space over \mathbb{C} . Furthermore, \mathcal{S} is a topological vector space.

The topology on \mathcal{S} is defined by a countable family of seminorms.

$$\|f\|_{M,N} = \sup_{x \in \mathbb{R}^d} \langle x \rangle^N \sum_{0 \leq |\beta| \leq M} \left| \frac{\partial^\beta f}{\partial x^\beta}(x) \right|.$$

We have that $f \in \mathcal{S}$ if and only if $f \in C^\infty$ and for all $M, N \in \mathbb{N}$, $\|f\|_{M,N} < \infty$.

A neighborhood base for the topology at g would be

$$V(g, M, N, \epsilon) = \{f \in \mathcal{S} : \|f - g\|_{M,N} < \epsilon\}.$$

Note that if ρ_n is a metric,

$$\sum_{n=1}^{\infty} 2^{-n} \left(\frac{\rho_n}{1 + \rho_n} \right)$$

is also a metric, but it wouldn't preserve the vector space condition. Next time, we will prove the following theorem:

Theorem 5

$\wedge : \mathcal{S} \rightarrow \mathcal{S}$ is a linear, bijective homeomorphism.

§5 September 10th, 2020

§5.1 Schwartz Space, continued

Last time, we introduced the Schwartz space,

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty : \forall M, N \|f\|_{M,N} < \infty\},$$

$$\|f\|_{M,N} = \sup_x \{ \langle x \rangle^M \sum_{|\alpha|=0}^N \left| \frac{\partial^\alpha f}{\partial x^\alpha} \right| \}.$$

An equivalent formulation is $x^\beta \partial^\alpha f$ is bounded for all α, β .

Theorem 6

The fourier transform, $\wedge : \mathcal{S} \rightarrow \mathcal{S}$ is a linear, bijective homeomorphism.

Proof. Note that if $f \in \mathcal{S}$, then $\widehat{f} \in C^\infty$. This is clear since

$$\partial_\xi^\alpha \int f(x) e^{-ix \cdot \xi} dx = \int f(x) \partial_{xi}^\alpha (e^{-ix \cdot \xi}) dx.$$

Hence $f \cdot \langle x \rangle^N$ is in L^1 for all N .

Note the following identities:

$$(\partial_x^\alpha f)^\wedge = (i\xi)^\alpha \widehat{f}(\xi), (x^\beta f)^\wedge = (i\partial_{xi}^\beta) \widehat{f}(\xi),$$

which can be verified from repeated integration by parts.

We claim that $\xi^\beta \partial_\xi^\alpha \widehat{f}$ is bounded for all α, β . Moreover, there exists M, N such that

$$\sup_{xi} |\xi^\beta \partial_\xi^\alpha \widehat{f}(\xi)| \leq C_{\alpha,\beta} \|f\|_{M,N}.$$

Note that

$$|\xi^\beta \partial_\xi^\alpha \widehat{f}(\xi)| = |(\partial_x^\beta x^\alpha f)^\wedge(\xi)|,$$

so

$$\sup_{xi} |\xi^\beta \partial_\xi^\alpha \widehat{f}(\xi)| \leq \|(\partial_x^\beta x^\alpha f)^\wedge(\xi)\|_{L^1} \leq C_d \sup_x |\langle x \rangle^{d+1} \partial_x^\beta (x^\alpha f)|.$$

By the Leibniz rule, we can commute ∂_x^β , which gives the result.

Hence, we have proved that $\widehat{\mathcal{S}} \subset \mathcal{S}$, and $\wedge : \mathcal{S} \rightarrow \mathcal{S}$ is continuous. and the same holds for $f \mapsto \check{f}$, so $f \in \mathcal{S} \Rightarrow f \in L^1$ and $\widehat{f} \in L^1$, so \wedge is 1-1 on \mathcal{S} and \vee is onto, so we get that \wedge is onto. \square

§5.2 Tempered Distributions

We will consider the dual of the Schwartz space,

$$\mathcal{S}' = \{\varphi : \mathcal{S} \rightarrow \mathbb{C}, \text{ linear and continuous}\}.$$

Recall, continuity by definition is given by the existence of $M, N, C < \infty$ so that for all $f \in \mathcal{S}$, $|\varphi(f)| \leq C \|f\|_{M,N}$.

Example 5.1 (Dirac Mass)

We can take $\varphi(f) = f(0)$, the dirac mass. We can also take $\varphi(f) = \partial^\alpha f(y_0)$.

Let μ be a complex Radon measure, $h \in L^1_{loc}$, $\int_{|x| \leq R} |h| dx \leq C_h \langle R \rangle^{A_h}$. We can define

$$\varphi(f) = \int \partial^\alpha f(x) \cdot h(x) d\mu(x) \in \mathbb{C}.$$

Theorem 7

Every $\varphi \in \mathcal{S}'$ is a finite linear combination of $f \mapsto \int \partial^\alpha f \cdot h d\mu$, with h, μ, α as before.

The proof is left as an exercise. The key ingredient is the Riesz Representation theorem and the Hahn-Banach theorem.

\mathcal{S}' is given a weak topology: a neighborhood base of $\varphi \in \mathcal{S}'$ is given by choosing J , a finite index set, $\epsilon > 0$ and $f_j \in \mathcal{S} (j \in J)$. Then

$$V = \{\psi \in \mathcal{S}' : |\psi(f_j) - \varphi(f_j)| < \epsilon \forall j \in J\}.$$

Definition 5.2. For $\varphi \in \mathcal{S}'$, $\widehat{\varphi}$ is a map $f \in \mathcal{S} \mapsto \varphi(\widehat{f})$. Then $\widehat{\varphi} : \mathcal{S} \mapsto \mathbb{C}$ is linear. Similarly, we can define $\check{\varphi} : \mathcal{S} \rightarrow \mathbb{C}$, linear.

We can verify that $\widehat{\varphi} \in \mathcal{S}'$. Note that

$$|\widehat{\varphi}(f)| = |\varphi(\widehat{f})| \leq C_\varphi \|\widehat{f}\|_{M,N} \leq C' \|f\|_{M',N'}.$$

Theorem 8

$\wedge : \mathcal{S}' \rightarrow \mathcal{S}'$ is a bijective homeomorphism.

Proof. We first show that $\varphi \mapsto \widehat{\varphi}$ is continuous at ψ . Given V , a neighborhood of ψ : J finite, $\epsilon > 0$, $f_j : j \in J$, we need to control $|\widehat{\varphi}(f_j) - \psi(f_j)| < \epsilon$ for every $j \in J$. The neighborhood $W = \{\varphi : |\varphi(\widehat{f}_j) - \psi(\widehat{f}_j)| < \epsilon \forall j \in J\}$ gives the desired bound.

Now we claim for all $\varphi \in \mathcal{S}'$, $(\widehat{\varphi})^\vee = (2\pi)^d \varphi$. This comes from

$$(\widehat{\varphi})^\vee(f) = \widehat{\varphi}(\check{f}) = \varphi((\check{f})^\wedge) = \varphi((2\pi)^d f).$$

Hence \wedge is 1-1 and onto, so we conclude that it is a bijective homeomorphism. \square

We can define a partial derivative of a distribution, $\partial^\alpha \varphi$, with $\partial^\alpha : \mathcal{S}' \rightarrow \mathcal{S}'$ continuous, linear. This is a bit shocking: Take $\varphi = h \in L^1_{loc}$ with $\int_{|x| \leq R} |h| dx \leq C_h R^{A_h}$. This defines a distribution $f \mapsto \int f h = \varphi(f)$. That means, we have a way of essentially differentiating anything.

Note that we have a natural map $i : \mathcal{S} \rightarrow \mathcal{S}'$ injective, where $i(g)(f) = \int_{\mathbb{R}^d} f g$. Then, we take $g \mapsto i(g)$. Note that i is a continuous map.

Given some linear operator $T : \mathcal{S} \rightarrow \mathcal{S}$, we want to associate an extension $\tilde{T} : \mathcal{S}' \rightarrow \mathcal{S}'$ for all $g \in \mathcal{S}$.

Define $T' : \mathcal{S}' \rightarrow \mathcal{S}'$, where $T'(\varphi)(f) = \varphi(T(f))$. It's easy to check that $T' \in \text{End}(\mathcal{S}')$, but there are some bad examples.

Example 5.3

If we take $T(f) = \frac{df}{dx}$, $\int f \cdot g' = -\int f' \cdot g$, then

$$T(i(g)) = -i(T(g)).$$

Suppose we have some $T \in \text{End}(\mathcal{S})$ and a transpose $A \in \text{End}(\mathcal{S})$ in the sense that

$$\int T(f)g = \int fA(g) \forall f, g \in \mathcal{S}.$$

For example, $T = \frac{d}{dx}$, $A = -\frac{d}{dx}$. With $T, A \in \text{End}(\mathcal{S})$, we can define $\tilde{T}(\varphi)(f) = \varphi(A'(f))$, which defines our extension.

Proposition 5.4

$i(\mathcal{S})$ is dense in \mathcal{S}' .

Definition 5.5 (Convolution for Distributions). If $f \in \mathcal{S}$ and $\varphi \in \mathcal{S}'$, then

$$\varphi * f(x) = \varphi(f_x), f_x(y) = f(x - y).$$

One can show that $\varphi * f \in C^\infty$ if $f \in \mathcal{S}$.

Proposition 5.6

Let $(\varphi_n) \in \mathcal{S}'$. If $\varphi_n \rightarrow \varphi$ in \mathcal{S}' , then $\varphi_n f \rightarrow \varphi(f) \forall f \in \mathcal{S}$.

Proposition 5.7

Let $(\varphi_n) \in \mathcal{S}'$. If $\varphi_n \rightarrow 0$ in \mathcal{S}' . Then there exists $M, N < \infty$ such that for all n and for all $f \in \mathcal{S}$,

$$|\varphi_n(f)| \leq C_n \|f\|_{M,N},$$

and $C_n \rightarrow 0$ as $n \rightarrow \infty$.

The proof uses the Baire Category Theorem. Recall \mathcal{S} is a complete metrizable space, where we define a norm from

$$\sum_{M,N} 2^{-M-N} \frac{\|f\|_{M,N}}{1 + \|f\|_{M,N}}.$$

For $d \geq 1$, define $g(x) = e^{-i\lambda|x|^2/2}$, $\lambda \in \mathbb{R}$. Note that $g \in L^\infty$, $|g| \equiv 1$.

We define $\hat{g}(\xi) = (2\pi)^{d/2} (i\lambda)^{-d/2} e^{-i|\xi|^2/(2\lambda)}$, for $\lambda \neq 0$. If we take $g \mapsto i(g) \in \mathcal{S}'$, note that $(i(g))^\wedge = i$, so we are in fact doing a normal fourier transform.

Define $g_z(x) = e^{-z\lambda|x|^2/2}$, for $z \in \mathbb{C}$, $\text{Re}(z) \geq 0$. We claim that as $z \rightarrow i\lambda$, $g_z \rightarrow g$ in the topology of \mathcal{S}' . Furthermore,

$$\int f g_z \rightarrow \int f g$$

for all $f \in \mathcal{S}$ by the dominated convergence theorem, with dominator $|f|$, since $|g_z| \leq 1$, $|g| \equiv 1$.

We know that $\widehat{g}_z \rightarrow \widehat{g}$ in \mathcal{S}' as $z \rightarrow i\lambda$. Note that

$$\widehat{g}_z(\xi) = (2\pi)^{d/2} z^{-d/2} e^{-|\xi|^2/(2z)}.$$

If $\operatorname{Re}(z) > 0$, then $g_z \in \mathcal{S}$.

Then as $z \rightarrow i\lambda$,

$$(2\pi)^{d/2} z^{-d/2} e^{-|\xi|^2/(2z)} \rightarrow (2\pi)^{d/2} (i\lambda)^{-d/2} e^{-|\xi|^2/(2i\lambda)}.$$

So $\widehat{g}_z \rightarrow \widehat{g}$ in \mathcal{S}' , so we have the result.

§6 September 15th, 2020

§6.1 Poisson Summation Formula

Define $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$. We have that $\|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}$ for all $f \in L^2 \cap L^1$.

Theorem 9

For all $f \in \mathcal{S}$,

$$\sum_{n \in \mathbb{Z}^d} \mathcal{F}(f)(n) = \sum_{k \in \mathbb{Z}^d} f(k).$$

This has a nice interpretation: suppose we define $\delta_n(g) = g(n)$. We have $\delta_n \in \mathcal{S}'$, and

$$\mathcal{F}\left(\sum_{n \in \mathbb{Z}^d} \delta_n\right) = \sum_{k \in \mathbb{Z}^d} \delta_k.$$

Proof. Given $f \in \mathcal{S}$, set $g : \mathbb{R}^d / \mathbb{Z}^d \rightarrow \mathbb{C}$, defined by $g(x) = \sum_{n \in \mathbb{Z}^d} f(x + n)$. Note that g is periodic: $g(x + e_j) = g(x)$ for all $1 \leq j \leq d$.

$$g(x) = \sum_{k \in \mathbb{Z}^d} \left(\int g(y) e^{-2\pi i k \cdot y} dy \right) e^{ik \cdot x}.$$

Note that

$$\begin{aligned} \sum_n f(n) &= g(0) = \sum_k \int e^{-2\pi i k \cdot y} \sum_n f(y + n) dy \\ &= \sum_k \int_{[0,1]^d} \sum_n e^{-2\pi i k \cdot (y+n)} f(y + n) = \sum_k \int_{\mathbb{R}^d} f(u) e^{-2\pi i k \cdot u} du = \sum_k \hat{f}(k). \end{aligned}$$

Because f is a Schwartz function, all these series converge and we can easily swap sums and integrals. \square

Example 6.1

There are lots of functions that are their own Fourier transforms. Take $x^n e^{-x^2/2}$, for $n \in \mathbb{Z}_{\geq 0}$. Apply Gram-Schmidt in the order of $\mathbb{Z}_{\geq 0}$. We get an orthonormal basis $P_n(x) e^{-x^2/2}$, where $P_n = c_n x^n + O(|x|^{n-2})$.

If $n \equiv 0 \pmod{4}$,

$$(P_n e^{-x^2/2})^\wedge = (2\pi)^{1/2} P_n e^{-x^2/2}.$$

§6.2 Size of Fourier Coefficients

Remark: If $f \in C_c^k(\mathbb{R}^d)$ or $C^k(\mathbb{T}^d)$, then

$$\hat{f}(\xi) = O(\langle \xi \rangle^{-k}).$$

This comes from $\left(\frac{\partial f}{\partial x_j}\right)^\wedge \xi_j = i \xi_j \hat{f}(\xi)$.

We can have a stronger bound,

$$\langle \xi \rangle^k \widehat{f} \in L^2, \ell^2.$$

The proof is the same since $\xi^\alpha \widehat{f} \in L^2/\ell^2$ whenever $0 \leq |\alpha| \leq k$.

Recall the class

$$\text{Lip} = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C} : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.$$

Proposition 6.2

Assume $f \in \text{Lip}$ and has compact support. Then,

$$\begin{aligned} \widehat{f}(\xi) &= O(\langle \xi \rangle^{-1}), \\ \langle \xi \rangle \widehat{f} &\in L^2. \end{aligned}$$

Proof. We have $f \in C_0^0(\mathbb{R}^d) \cap \text{Lip}$. Assuming $\xi \neq 0$,

$$\widehat{f}(\xi) = \int f(x) e^{-ix \cdot \xi} dx = \frac{1}{2} \int f(x) e^{-ix \cdot \xi} dx + \frac{1}{2} \int f(x + \frac{\pi}{\xi}) e^{-i(x + \frac{\pi}{\xi}) \cdot \xi} dx.$$

Since $e^{-i(\pi/\xi)\xi} = -1$, we have

$$\frac{1}{2} \int [f(x) - f(x + \pi/\xi)] e^{-ix \cdot \xi} dx.$$

Because f is Lipschitz, $f(x) - f(x + \pi/\xi) \in O(|\xi|^{-1})$, so it's clear the whole integral is bounded.

Definition 6.3 (Holder Class). Define Λ_α ($0 < \alpha < 1$), as $f : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$.

Note that $\alpha > \beta \Rightarrow \Lambda_\alpha \subset \Lambda_\beta$. Furthermore $\text{Lip} \subset \Lambda_\alpha$.

We can state a similar proposition as above for Holder classes.

Example 6.4

Let $0 < \alpha < 1$,

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}.$$

The function $f \in \Lambda_\alpha$, but not Λ_β for any $\beta > \alpha$, since $\widehat{f}(2^n) = (2^n)^{-\alpha}$.

Let $f \in \text{Lip} \cap C_0^0$. Claim $f' \in L^\infty$ in the \mathcal{S}' sense. In other words, there exists $g \in L^\infty$ such that $\int f \varphi' = - \int g \varphi$ for all $\varphi \in \mathcal{S}$.

The claim immediately implies that $\xi \widehat{f}(\xi) \in L^2$, since $\widehat{g} \in L^2 = i \xi \widehat{f}$ and has compact support.

$$\lim_{t \rightarrow 0} \int f(x) \frac{\varphi(x+t) - \varphi(x)}{t} dx = \lim_{t \rightarrow 0} \int \frac{f(x) - f(x-t)}{t} \varphi(x) dx$$

Let $f_t = \frac{f(x) - f(x-t)t}{t}$. Note that $f_t \in L^\infty(\mathbb{R})$ and $L^\infty = (L^1)^*$, so by Alaoglu's theorem, there exists a sequence $t_{\nu} \rightarrow 0$ and $g \in (L^1)^*$ with $f_t \rightarrow -g$ in the weak star topology.

Therefore, $\int f_{t_{\nu}} \varphi \rightarrow -\int g \varphi$ as $\nu \rightarrow \infty$. Thus, $\int f \varphi' = -\int g \varphi$. \square

Example 6.5

Take

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}$$

with $\alpha = 1$. f is not Lipschitz, since

$$\sum_{\xi=2^n} |\xi| |\widehat{f}(\xi)| = \sum_n 1 = \infty.$$

Remark: For $\alpha < 1$, f is nowhere differentiable.

Example 6.6

Take $f \in BV(\mathbb{R}^1)$ with compact support, the class with bounded variation. Then $|\widehat{f}(\xi)| \leq \pi V(f) |\xi|^{-1}$.

Lemma 6.7 (Riemann-Lebesgue Lemma)

If $f \in L^1(\mathbb{R}^d)$ or (\mathbb{T}^d) (then $\widehat{f} \in C^0$ bounded), then $|\widehat{f}(\xi)| \rightarrow 0$ as $|\xi| \rightarrow \infty$.

Proof. Note that

$$\widehat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}^d} (f(x) - f(x + \frac{\pi \xi}{|\xi|^2})) e^{-ix \cdot \xi} dx$$

Then

$$|\widehat{f}(\xi)| \leq \frac{1}{2} \|f(x) - f(x + \frac{\pi \xi}{|\xi|^2})\|_{L^1} \rightarrow 0.$$

\square

How fast do they go to zero? Is there a quantitative bound? (Nope) How do we characterize $\widehat{L^1}$? Is $C_{\rightarrow 0}^0 = (L^1)^\wedge$? (Nope).

Proposition 6.8

The map $\wedge : L^1(\mathbb{R}^d) \rightarrow C_{\rightarrow 0}^0(\mathbb{R}^d)$ is not onto. Equivalently, $\vee : C_{\rightarrow 0}^0(\mathbb{R}^d) \not\rightarrow L^1$.

Proof. $\wedge : L^1 \rightarrow C_{\rightarrow 0}^0$ is linear, bounded, and an injective mapping between Banach spaces. We can apply the Open Mapping Theorem: if the map was onto, there would exist $A < \infty$ such that $\|f\|_{L^1} \leq A \|\widehat{f}\|_{C^0}$.

We claim that $\frac{\|\widehat{f}\|_{C^0}}{\|f\|_{L^1}}$ can be arbitrarily small. Define $f_t(x) = e^{-(1+it)|x|^2/2}$ for $t \in \mathbb{R}$ going to ∞ .

We know that

$$\widehat{f}_t(\xi) = (2\pi)^{d/2}(1+it)^{-d/2}e^{-(1-it)|\xi|^2/(2(1+t^2))}.$$

Hence,

$$|\widehat{f}_t| = (2\pi)^{d/2}(1+t^2)^{-d/4}e^{-|\xi|^2/(2(1+t^2))} \leq (2\pi)^{d/2}(1+t^2)^{-d/4} \rightarrow 0.$$

On the other hand $\|f_t\|_{L^1}$ is independent of t . □

Theorem 10

Let $w : \mathbb{R}^d \rightarrow (0, \infty)$ and $w(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$. There exists $f \in L^1$ with

$$|\widehat{f}(\xi)| \geq w(\xi) \forall \xi.$$

Proof. We have a key lemma: Let $w : \mathbb{R}^1 \rightarrow (0, \infty)$ continuous, even, piecewise, $C^2(\mathbb{R} \setminus \{0\})$, convex on $(0, \infty)$ with compact support. Then, $\widehat{w} \in L^1$ and $\widehat{w} \geq 0$, hence, $\|\widehat{w}\|_{L^1} = \int \widehat{w} = (2\pi)^d w(0)$. □

§7 September 17th, 2020

§7.1 Size of Fourier Coefficients, continued

Theorem 11

Let $w : \mathbb{R}^d \rightarrow (0, \infty)$ and $w(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$. There exists $f \in L^1$ with

$$|\widehat{f}(\xi)| \geq w(\xi) \forall \xi.$$

Proof. We have a key lemma:

Lemma 7.1

Let $w : \mathbb{R}^1 \rightarrow (0, \infty)$ continuous, even, piecewise $C^2(\mathbb{R} \setminus \{0\})$, convex on $(0, \infty)$ with compact support and nondecreasing. Then, $\widehat{w} \in L^1$ and $\widehat{w} \geq 0$, hence,

$$\|\widehat{w}\|_{L^1} = \int \widehat{w} = (2\pi)^d w(0).$$

Proof. Note that

$$\widehat{w}(\xi) = \int_{\mathbb{R}} w(x) e^{-ix \cdot \xi} dx = \int_{\mathbb{R}} w(x) \cos(x\xi) dx.$$

Furthermore, note that $|x| \cdot |w'(x)|$ is a bounded function (as $x \rightarrow 0$). It follows from Jensen's inequality.

$$\begin{aligned} \widehat{w}(\xi) &= 2 \int_0^\infty w(x) \cos(x\xi) dx \\ &= 2\xi^{-2} \int_0^\infty w''(x)(1 - \cos(x\xi)) dx \geq 0. \end{aligned}$$

It suffices to show the equality $\int_0^\infty w(x) \cos(x\xi) dx = \xi^{-2} \int_0^\infty w''(x)(1 - \cos(x\xi)) dx$. We integrate by parts twice:

$$\begin{aligned} \widehat{w}(\xi) &= 2 \int_0^\infty w'(x) \xi^{-1} \sin(x\xi) dx \\ &= 2 \int_0^\infty w''(x) \xi^{-2} (1 - \cos(x\xi)) dx. \end{aligned}$$

We might have issues at 0, but we can take a limit for integrating from ϵ to ∞ with boundary terms $w''(\epsilon)(1 - \cos(\epsilon\xi)) \in O(\epsilon^2)$. Hence, $\widehat{w} \geq 0$.

Note that $\widehat{w} \in L^1$ and for $|\xi| \geq 1$,

$$|\widehat{w}(\xi)| \leq 2\xi^{-1} \int_0^\infty |w''(x)| dx \cdot 2$$

. Assume $|w'(0)| < \infty$, where the derivative is the right-hand derivative at 0.

Then

$$\int_0^\infty w''(x) dx = -w'(0)$$

so it follows that $\widehat{w} \in L^1$.

Finally,

$$w(0) = (2\pi)^{-1}(\widehat{w})^\vee(0) = (2\pi)^{-1} \int \widehat{w}(\xi) d\xi = (2\pi)^{-1} \|\widehat{w}\|_{L^1},$$

which gives the desired bound. \square

Let $g : \mathbb{R} \rightarrow [0, \infty]$ continuous, with $g(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$.

Lemma 7.2

There exists $w : \mathbb{R} \rightarrow (0, \infty)$ so that $w \geq g$ and w is even, convex on $(0, \infty)$, $w(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$, and w is piecewise C^2 , where we may have infinity many breaks.

To prove the theorem, it suffices to find a function $f \in L^1$ such that $\widehat{f}(\xi) \geq w(\xi)$ for all ξ .

WLOG, g is even (replace $g(\xi) + g(-\xi)$), nonincreasing (we can replace $\tilde{g}(x) = \sup_{y \geq x} g(y)$ for $x \geq 0$). Note that $\tilde{w}(\xi) = \widehat{w}(-\xi)$ so define $f = \widehat{w}$. $\widehat{f} = (2\pi)w \geq 2\pi g$.

To treat w , we approximate it with functions of compact support. Let $t > 0$ and define $w_t = \max(w - t, 0)$. We conclude that $\widehat{w}_t \in L^1$ and $\|\widehat{w}_t\|_{L^1} = (2\pi)w_t(0)$. As $t \rightarrow 0^+$, $w_t \rightarrow w$ in \mathcal{S}' so $\widehat{w}_t \rightarrow \widehat{w}$ in \mathcal{S}' . We have that \widehat{w} is a complex radon measure.

Fact 7.3. If μ is a complex Radon measure and if $\mu|_{\mathbb{R} \setminus 0}$ is absolutely continuous, then $\mu = c\delta_0 + h$ for $h \in L^1$.

We know that $w(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ and $\widehat{\mu}(\xi) = c + \widehat{h}(\xi)$ so $c = 0$ and $\widehat{w} \in L^1$ as desired. \square

§7.2 Comparing Size of Functions to Size of Fourier Coefficients

We have that $\|\widehat{f}\|_{L^2} = (2\pi)^{-d/2} \|f\|_{L^2}$ and $\|\widehat{f}\|_{C^0} \leq \|f\|_{L^1}$.

Theorem 12 (Hausdorff-Young)

Let $p \in [1, 2]$. The $f \in L^p(\mathbb{R}^d)$ implies that $\widehat{f} \in L^q$ for $q = p' = \frac{p}{p-1}$, and

$$\|\widehat{f}\|_q \leq C(p, d) \|f\|_p.$$

For \mathbb{T}^d ,

$$\|\widehat{f}\|_{\ell^q} \leq C(p)^d \|f\|_{L^p(\mathbb{T}^d)}.$$

Note that for \mathbb{R}^d , $\wedge : L^p \rightarrow L^r$ is bounded.

Proof. We must have that $r = p'$. Fix a function $0 \neq f \in \mathcal{S}$. Define $f_t(x) = f(tx)$ for $t \in \mathbb{R}^+$.

$$\widehat{f}_t(\xi) = t^{-d} \widehat{f}(t^{-1}\xi).$$

Note that

$$\|f_t\|_p^p = \int |f(tx)|^p dx = t^{-d} \int |f(y)|^p dy = t^{-d} \|f\|_p^p.$$

Then $\|\widehat{f}_t\|_r = t^{-d} t^{d/r} \|\widehat{f}\|_r$, so

$$\frac{\|\widehat{f}_t\|_r}{\|f_t\|_p} = t^\gamma \frac{\|\widehat{f}\|_r}{\|f\|_p}$$

where $\gamma = -d + d/r + d/p$. We must have that $\gamma = 0$ for the ratio to be bounded, which gives $1 = \frac{1}{p} + \frac{1}{r}$.

For \mathbb{T}^d , we can only take $t \rightarrow +\infty$ so $\gamma \leq 0$, and we can only conclude that $r \geq p'$. But $r \geq p'$ implies that $\ell^{p'} \subset \ell^r$, so $\wedge : L^p \rightarrow \ell^{p'} \subset \ell^r$. \square

Theorem 13 (Riesz-Thoren)

Let $(X, \mu), (Y, \nu)$ be σ -finite measure spaces. Suppose we have exponents $p_0, p_1, q_0, q_1 \in [1, \infty]$. Let $S(X)$ be the set of simple functions from $X \rightarrow \mathbb{C}$. Assume $T : S(X) \rightarrow (L^1 + L^\infty)(Y)$ is linear and there exists $A_0, A_1 < \infty$ so that for all $f \in S(X)$,

$$\|Tf\|_{L^{q_j}} \leq A_j \|f\|_{L^{p_j}}.$$