§1 Problems

Problem 1

Determine all possible values of the expression

$$A^3+B^3+C^3-3ABC,$$

where A, B, C are nonnegative integers.

Proof. Let $S = A^3+B^3+C^3-3ABC$. We claim that S attains all values such that $S \not\equiv 3, 6 \pmod 9$.

Note that the expression can be factored as

$$A^3+B^3+C^3-3ABC = \left(\frac{A+B+C}{2}\right)\left((A-B)^2+(B-C)^2+(C-A)^2\right).$$

If $(A, B, C) = (A, A+1, A+2)$, then

$$S = \frac{3A+3}{2}(1^2+1^2+2^2) = (3A+3)(3) = 9A+9,$$

so we can achieve all $S \equiv 0 \pmod 9$.

If $(A, B, C) = (A, A, A+1)$, then

$$S = \frac{3A+1}{2}(0^2+1^2+1^2) = 3A+1,$$

and if $(A, B, C) = (C+1, C+1, C)$, then

$$S = \frac{3C+2}{2}(0^2+1^2+1^2) = 3C+2,$$

so we can achieve all $S \equiv 1, 2 \pmod 3$.

It suffices to show that if $S \equiv 0 \pmod 3$, then $S \equiv 0 \pmod 9$. This implies that we cannot have $S \not\equiv 3, 6 \pmod 9$ as desired. If $S \equiv 0 \pmod 3$, then we must have $A+B+C \equiv 0 \pmod 3$ or $(A-B)^2+(B-C)^2+(C-A)^2 \equiv 0 \pmod 3$. In the first case, then without loss of generality, we must have either $(A, B, C) \in \{(0, 0, 0), (1, 1, 1), (2, 2, 2), (0, 1, 2)\}$. In each of these cases, we can show that $(A-B)^2+(B-C)^2+(C-A)^2 \equiv 0 \pmod 3$. Similarly, in the second case, we must have that $(A-B)^2 = (B-C)^2 = (C-A)^2 = 0, 1$. In the first case $A = B = C$, which gives that $A+B+C \equiv 0 \pmod 3$. In the second case, the remainders of A, B, C must be distinct mod 3, which, without loss of generality, gives $(A, B, C) = (0, 1, 2)$ which implies that $A+B+C \equiv 0 \pmod 3$, as desired. In all cases, we show that both terms in the product are $0 \pmod 3$, which implies that the product is $0 \pmod 9$. $\square$

Problem 2

In the triangle ABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\arctan(1/3)$. Find α.

Proof. We use complex numbers. Let $B = 0$. Then $\arg(I) = \beta/2 = \arctan(1/3)$, so $I = k(3+i)$ for some $k \in \mathbb{R}^+$. Without loss of generality, let $k = 1$. Let $A = a$. Then, IG is parallel to AB which implies that $\text{Im}(B-A) = \text{Im}(I-G)$. Then $\text{Im}(B-A) = 0$, so $\text{Im}(I) = \text{Im}(G) = 1$.

Then, note that $\arg(I^2) = \arg(C)$, so $C = \ell(3+i)^2 = \ell(8+6i)$ for some $\ell \in \mathbb{R}^+$. Then $G = \frac{A+B+C}{3} = \frac{A+C}{3}$, so

$$1 = \text{Im}(G) = \text{Im}((A+C)/3) = \text{Im}(C/3),$$

which implies that $\ell = \frac{1}{2}$. Thus, $C = 4+3i$.

Finally,

$$I = \frac{|CB|A + |AC|B + |AB|C}{|AB| + |BC| + |CA|} = \frac{5a + a(4+3i)}{5 + a + \sqrt{(4-a)^2+9}} = 3+i.$$

Hence,

$$5 + a + \sqrt{(4-a)^2+9} = 3a,$$

which has solutions $a = 0, a = 4$. Taking the positive solution, we have $A = 4$. Then, note that ABC is a right triangle with right angle at A, so $\alpha = \frac{\pi}{2}$. $\square$

Problem 3

Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Let $\mu = \frac{1}{2019}\sum_{k=1}^{2019} |z_k|$. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ satisfying

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

Proof. By the AM-GM inequality,

$$\mu = \frac{\sum_{k=1}^{2019} |z_k|}{2019} = \left(\prod_{k=1}^{2019} |z_k|\right)^{1/2019} = \left|\frac{b_0}{b_{2019}}\right|^{1/2019} \leq (2019)^{-1/2019}.$$

We show that $M = (2019)^{-1/2019}$. Let $\zeta = e^{\frac{2\pi i}{2020}}$ and let $z_i = M\zeta^i$. Notice that $|z_i| = M$ for each i and the roots $z_1, z_2, \ldots, z_{2019}$ satisfy the polynomial

$$0 = \frac{(z_i/M)^{2020} - 1}{(z_i/M) - 1} = M^{-2019}\left(\frac{z_i^{2020} - M^{2020}}{z_i - M}\right) = \sum_{k=0}^{2019} z_i^k M^{-k}.$$

Hence, the polynomial

$$P(z) = \sum_{k=1}^{2019} z_i^k 2019^{k/2019}$$

satisfies the equality case $\mu = M$. Furthermore, note that $b_0 = 1$, $b_{2019} = 2019$ and and $2019^{i/2019} < 2019^{j/2019}$ for all $i < j$. Hence, P satisfies the conditions. □

Problem 4

Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x, y, z)$ over the surface of S equals 0. Must $f(x, y, z)$ be identically 0?

Proof. No. Take $f(x, y, z) = \sin(\pi z)$. □