Tensors

Vishal Raman

April 22, 2021

These notes correspond to Chapter 12 of Lee, *Smooth Manifolds* on Tensors. We define multilinear maps in order to construct tensors and tensors fields on manifolds. We also introduce symmetric and alternating tensors, as well as tensor fields and bundles on smooth manifolds.

Contents

§1 Multilinear Algebra and the Tensor Product

Definition 1.1. Suppose $V_1, \ldots, V_k$ and W are vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is **multilinear** if it is linear as a function of each variable separately when the others are held fixed.

Some common examples include:

- The dot product
- The cross product
- The determinant
- The bracket in a Lie algebra

Example 1.2 (Tensor Products of Covectors)
Suppose V is a vector space, and $\omega, \eta \in V^*$. Define $\omega \otimes \eta : V \times V \to R$ by

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2).$$

The linearity of ω and η implies that $\omega \otimes \eta$ is a bilinear function of v_1 and v_2.

Definition 1.3. Given $V_1, \ldots, V_k, W_1, \ldots, W_\ell$ real vector spaces and functions $F \in L(V_1, \ldots, V_k; \mathbb{R})$, $G \in L(W_1, \ldots, W_\ell; \mathbb{R})$, define the **tensor product** $F \otimes G$ by

$$F \otimes G(v_1, \ldots, v_k, w_1, \ldots, w_\ell) = F(v_1, \ldots, v_k)G(w_1, \ldots, w_\ell).$$

Proposition 1.4
Given $V_1, \ldots, V_k$ real vector spaces of dimensions $n_1, \ldots, n_k$, if $(E_1^{(j)}, \ldots, E_{n_j}^{(j)})$ is a basis for V_j with corresponding dual basis $(\epsilon_{(j)}^1, \ldots, \epsilon_{(j)}^{n_j})$, the set

$$\mathcal{B} = \{e_{(1)}^{i_1} \otimes \cdots \otimes \epsilon_{(k)}^{i_k} : 1 \leq i_1 \leq n_1, \ldots, 1 \leq i_k \leq n_k\}$$

is a basis for $L(V_1, \ldots, V_k; \mathbb{R})$, which has dimension equal to $n_1 \ldots n_k$.

§2 Tensors on Vector Spaces

Definition 2.1. Given a vector space V, $\dim V < \infty$, define $T^k(V^*) = V^* \otimes \cdots \otimes V^*$ to be the space of multilinear maps on $V \times \cdots \times V \to \mathbb{R}$.

Definition 2.2. We can define a mixed tensor $T^{(k,\ell)}(V) = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$.

§3 Symmetric and Alternating Tensors

Definition 3.1. Let V be a finite dimensional space. $\alpha \in T^k(V^*)$ is said to be symmetric is if it invariant under interchanging pairs of elements. The set of symmetric covariant k-tensors is a linear subspace denoted by $\Sigma^k(V^*)$.

We can construct $\Sigma^k(V^*)$ as the projection from $T^k(V^*)$ under the symmetric group given by

$$\operatorname{Sym}\alpha = \frac{1}{k!}\sum_{\sigma \in S_k} {}^{\sigma}\alpha.$$

More explicitly,

$$(\operatorname{Sym}\alpha)(v_1, \ldots, v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Note that $\operatorname{Sym}\alpha$ is symmetric and it is the identity if and only if α is symmetric.

Definition 3.2. If α and β are symmetric tensors on V, we define $\alpha\beta = \operatorname{Sym}(\alpha \otimes \beta)$, the symmetric product.

Definition 3.3. $\alpha \in T^k(V^*)$ is said to be alternating if it is skew-symmetric. These are also called exterior forms.

§4 Tensors and Tensor Fields on Manifolds