

Olympiad Notebook

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Abstract

An overview of topics from math olympiads with selected problems and solutions. The sources for handouts and expositions are provided when available. Any typos or mistakes are my own - kindly direct them to my inbox.

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1 Combinatorics

1.1 Invariants and Monovariants

1.2 Bijections

1.3 Pigeonhole Principle

Theorem 1.1 (Pigeonhole Principle). *Let m, n be positive integers with $m \geq n$. If $m + 1$ pigeons fly to n pigeonholes, then at least one pigeonhole contains at least $\lfloor \frac{m}{n} \rfloor + 1$ pigeons.*

1.4 Extremal Principle

1.5 Combinatorial Games

The main strategies for analyzing combinatorial games are:

- Play the game: try to find some forced moves.
- Reduce the game to a simpler game.
- Start at the end of the game: find endgame positions which are winning and losing and work backwards.
- Find an invariant or monovariant that a player can control.

Problem 1.2. Four heaps contain 38, 45, 61, and 70 matches respectively. Two players take turns choosing any two of the heaps and removing a non-zero number of matches from each heap. The player who cannot make a move loses. Which one of the players has a winning strategy?

Proof. Denote the heaps with a 4-tuple (w, x, y, z) with $w \leq x \leq y \leq z$. We claim the winning positions are of the form (w, x, y, z) with $w < y$. It is clear that $(0, 0, y, z)$ leads to a win by removing y and z and $(0, x, y, z)$ leads to a win by reducing to $(0, 1, 1, z)$ which is forced to leave either 1 or 2 heaps.

Since we remove tiles on each move, the game must terminate. If we have (w, x, y, z) with $w < y$, we can reduce to (w, w, w, x) by sending y and z to w .

We show that (w, w, w, z) is a losing position. We have three cases:

1. If we remove from two of the w -heaps, we are left with (w', w'', w, z) .
2. If we remove from a w -heap and the z -heap, we are left with either (w', z', w, w) or (w', w, z', w) or (w', w, w, z') .
3. If we remove any number of heaps entirely, the resulting position is clearly winning.

It follows that (w, x, y, z) with $w < y$ is a winning position as desired. □

Problem 1.3. The number 10^{2015} is written on a blackboard. Alice and Bob play a game where each player can do one of the following on each turn:

- replace an integer x on the board with integers $a, b > 1$ so that $x = ab$
- erase one or both of two equal integers on the blackboard.

The player who is not able to make a move loses the game. Who has a winning strategy?

Proof. We claim Alice has a winning strategy. First, it is clear that the game must eventually terminate. On the first turn, Alice can replace 10^{2015} with 2^{2015} and 5^{2015} . We claim that after any of Bob's turns, Alice can move the board into the state

$$2^{\alpha_1} 2^{\alpha_2} \dots 2^{\alpha_k} 5^{\alpha_1} 5^{\alpha_2} \dots 5^{\alpha_k}.$$

If Bob sends 2^{α_j} to $2^{\beta_1}, 2^{\beta_2}$, then Alice can send 5^{α_j} to $5^{\beta_1}, 5^{\beta_2}$ and vice versa. Otherwise, if Bob removes one or two integers $2^{\alpha_j}, 2^{\alpha_k}$, then we have $\alpha_j = \alpha_k$ so Alice can remove one or two of $5^{\alpha_j}, 5^{\alpha_k}$ or vice versa. Since Alice can always follow the copycat strategy and the game eventually terminates, we must have that Bob is unable to make a move at some point, which implies that Alice wins the game as desired. \square

1.6 Algorithms

1.7 Generating Functions

Problem 1.4 (Putnam 2020 A2). Let k be a non-negative integer. Evaluate

$$\sum_{j=0}^k 2^{k-j} \binom{k+j}{j}.$$

Proof. We claim the sum evaluates to 4^k . Note that $\binom{k+j}{j} = \binom{k+j}{k}$. It follows that the sum is the coefficient of x^k in the power series $\sum_{j=0}^n 2^{k-j} (1+x)^{k+j}$. Evaluating this, we find

$$\begin{aligned} \sum_{j=0}^n 2^{k-j} (1+x)^{k+j} &= 2^k (1+x)^k \sum_{j=0}^k 2^{-j} (1+x)^j \\ &= 2^k (1+x)^k \frac{1 - (1+x)^{k+1}/2^{k+1}}{1 - (1+x)/2} \\ &= \frac{2^{k+1} (1+x)^k - (1+x)^{2k+1}}{1-x} \\ &= 2^{k+1} (1+x)^k - (1+x)^{2k+1} \sum_{n \geq 0} x^n. \end{aligned}$$

It follows that the coefficient of x^k is given by

$$2^{k+1} \sum_{j=0}^k \binom{k}{j} - \sum_{j=0}^k \binom{2k+1}{j} = 2^{2k+1} - 2^{2k} = 4^k.$$

\square

Problem 1.5. (CJMO 2020/1) Let N be a positive integer, and let S be the set of all tuples with positive integer elements and a sum of N . For all tuples t , let $p(t)$ denote the product of all the elements of t . Evaluate

$$\sum_{t \in S} p(t).$$

Proof. We claim the sum evaluates to F_{2N} , where F_k denotes the k -th Fibonacci number. Note that the sum can be represented as the coefficient of x^N in $\sum_{k=1}^N \left(\sum_{n \geq 0} nx^n \right)^k$. Evaluating this, we find

$$\begin{aligned} \sum_{k=1}^N \left(\sum_{n \geq 0} nx^n \right)^k &= \sum_{k=1}^N \left(\frac{x}{(1-x)^2} \right)^k \\ &= \sum_{k=1}^N \frac{x^k}{(1-x)^{2k}} \\ &= \sum_{k=1}^N \sum_{j \geq 0} \binom{2k-1+j}{2k-1} x^{j+k}. \end{aligned}$$

The coefficient of x^N is given by

$$\sum_{k=1}^N \binom{N+k-1}{2k-1} = \sum_{k=1}^N \binom{N+k-1}{N-k} = \sum_{j \geq 0} \binom{2N-1-j}{j} = F_{2N}.$$

□

Problem 1.6 (IMO 1995/6). Let p be an odd prime number. How many p -element subsets A of $\{1, 2, \dots, 2p\}$ are there, the sum of whose elements is divisible by p ?

Proof. Define $f(x, y) = \prod_{k=1}^{2p} (1 + x^k y)$. We wish to find the sum of the coefficients of terms of the form $x^{p\ell} y^p$. We do this by first considering f as a generating function in x using the root of unity filter associated to $\omega = e^{\frac{2\pi i}{p}}$. Then, we read off the coefficient of y^p to find the desired expression.

Note that for $1 \leq k \leq p-1$,

$$f(\omega^k, y) = \prod_{k=1}^{2p} (1 + \omega^k y) = \prod_{k=1}^p (1 + \omega^k y)^2 = (1 + y^p)^2.$$

It follows that

$$\begin{aligned} \frac{1}{p} \sum_{i=0}^{p-1} f(\omega^i, y) &= \frac{1}{p} \left((1 + y)^{2p} + \sum_{i=1}^{p-1} f(\omega^i, y) \right) \\ &= \frac{(1 + y)^{2p} + (p-1)(1 + y^p)^2}{p}. \end{aligned}$$

Finally, the coefficient of y^p is given by

$$\frac{\binom{2p}{p} + 2(p-1)}{2}.$$

□

1.8 Enumerative Combinatorics

1.9 Probabilistic Method

Some tips for using the probabilistic method:

- A statement E can be true by showing that its probability is greater than 0. item Show that E is true is the same as showing $P(\neg E) < 1$.
- Show that X can be at least or at most a by showing $E[X] \geq a$ or $E[X] \leq a$ respectively.
- Show that it is possible for $|X|$ to be at least or at most $a > 0$ by showing $E[X] = 0$ and $\text{Var}(X) \geq a^2$ or $\text{Var}(X) \leq a^2$ respectively.

1.10 Algebraic Combinatorics

1.11 Combinatorial Geometry

1.11.1 Convex Hull

Problem 1.7 (Happy-Ending Problem). Suppose we have five points in the plane with no three collinear. Show that we can find four points whose convex hull is a quadrilateral.

Proof. Take the convex hull of the five points. If it is a quadrilateral or pentagon, we are done (choose any 4 points in the latter case). Suppose the convex hull is a triangle. Label the points with A through E and without loss of generality, let the points A, B, C form the triangle and D, E , be the points inside the hull.

Extend the line DE . Note that two points must lie on one side of the line - if not then we have three collinear points. It is easy to show that these four points form a convex quadrilateral. \square

Problem 1.8. There are $n > 3$ coplanar points, no three collinear and every four of them are the vertices of a convex quadrilateral. Prove that the n points are the vertices of a convex n -sided polygon.

Proof. Suppose that some point P is inside the convex hull of the n points. Let Q be some vertex of the convex hull. The diagonals from Q to the other vertices divide the convex hull into triangles and since no three points are collinear, P must lie inside some triangle $\triangle QRS$. But this is a contradiction since P, Q, R, S do not form a convex quadrilateral. \square

Problem 1.9 (1985 IMO Longlist). Let A, B be finite disjoint sets of points in the plane such that any three distinct points in $A \cup B$ are not collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices are contained in A or in B that does not contain in its interior any point from the other set.

Proof. Suppose A has at least five points. Take $A_1 A_2$ on the boundary of the convex hull of A . For any other $A_i \in A$, define $\theta_i = \angle A_1 A_2 A_i$. Without loss of generality, $\theta_3 < \theta_4 < \dots < 180^\circ$. It follows that $\text{conv}(\{A_1, A_2, A_3, A_4, A_5\})$ contains no other points of A . \square

Problem 1.10 (Putnam 2001 B6). Assume that $(a_n)_{n \geq 1}$ is an increasing sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-i} + a_{n+i} < 2a_n$$

for $i = 1, \dots, n-1$?

Proof. We claim such a subsequence exists. Let $A = \text{conv}\{(n, a_n) : n \in \mathbb{N}\}$ and let ∂A denote the set of points on the boundary of the convex hull.

We claim that ∂A contains infinitely many elements. Suppose not. Then, ∂A has a last point (N, a_N) . If we let $m = \sup_{n > N} \frac{a_n - a_N}{n - N}$, the slope of the line between (N, a_N) and (n, a_n) , then the line through (N, a_N) with slope m lies above (or contains) each point (n, a_n) for $n > N$. However, since $a_n/n \rightarrow 0$ and a_N, N are fixed, we have that

$$\frac{a_n - a_N}{n - N} \rightarrow 0.$$

This implies that the set of slopes attains a maximum, i. e. there is some point (M, a_M) with $M > N$ so that $m = \frac{a_M - a_N}{M - N}$. But then, we must also have that $(M, a_M) \in \partial A$, contradicting the fact that (N, a_N) is the last point in ∂A .

For each point on the boundary $(n, a_n) \in \partial A$, we must have that midpoint of the line through $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ for $i \in [n-1]$ must lie below (n, a_n) . From this, it follows that $a_n > \frac{a_{n-i} + a_{n+i}}{2}$, which implies the result. \square

2 Algebra

2.1 Polynomials

Problem 2.1 (Putnam 2005/A3). Let $p(z)$ be a polynomial of degree n , all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = \frac{p(z)}{z^{n/2}}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Proof. Note that we can write $p(z) = a \prod_{j=1}^n (z - \omega_j)$ where $|\omega_j| = 1$ for all j . It follows that

$$\log g(z) = \log a + \sum_{j=1}^n \log(z - \omega_j) - \frac{n}{2} \log z = \log a + \sum_{j=1}^n \left(\log(z - \omega_j) - \frac{\log z}{2} \right).$$

Taking the derivative of both sides, we obtain

$$\begin{aligned} \frac{g'(z)}{g(z)} &= \sum_{j=1}^n \left(\frac{1}{z - \omega_j} - \frac{1}{2z} \right) \\ &= \frac{1}{2z} \sum_{j=1}^n \frac{z + \omega_j}{z - \omega_j} \\ &= \frac{1}{2z} \sum_{j=1}^n \frac{|z|^2 - 1 + \omega_j \bar{z} - z \bar{\omega}_j}{|z - \omega_j|^2} \\ &= \frac{1}{2z} \sum_{j=1}^n \left(\frac{|z|^2 - 1}{|z - \omega_j|^2} + i \frac{\operatorname{Im}(\omega_j \bar{z})}{|z - \omega_j|^2} \right). \end{aligned}$$

It follows that

$$\operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) = \frac{|z|^2 - 1}{2} \sum_{j=1}^n \frac{1}{|z - \omega_j|^2}.$$

Since $\sum_{j=1}^n \frac{1}{|z - \omega_j|^2} > 0$, it follows that the real part of $\frac{zg'(z)}{g(z)}$ is zero if and only if $|z|^2 - 1 = 0$, which implies that $|z|^2 = 1$. It follows that all the zeros of $g'(z)$ must either satisfy $|z|^2 = 1$ or $g(z) = 0$ which gives the desired result since the zeros of $g(z)$ lie on the unit circle on the complex plane. \square

2.2 Inequalities

Theorem 2.2 (QM-AM-GM-HM). *Let $x_1, \dots, x_n \in \mathbb{R}^+$. Then,*

$$\sqrt[n]{\frac{x_1^2 + \dots + x_n^2}{n}} \geq \frac{x_1 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \dots x_n} \geq \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}$$

with equality if and only if $x_1 = \dots = x_n$.

Definition 2.3 (Power Mean). Given $p \in \mathbb{R}$, $x_1, \dots, x_n \in \mathbb{R}^+$, define

$$M_p(x_1, \dots, x_n) = \begin{cases} (\sum_{i=1}^n w_i x_i^p)^{1/p} & \text{if } p \neq 0 \\ \prod_{i=1}^n x_i^{w_i} & \text{else} \end{cases}.$$

Definition 2.4 (Weighted Power Mean). Given $(w_i)_{i=1}^n$ with $\sum_i w_i = 1$, define

$$M_p^w(x_1, \dots, x_n) = \begin{cases} (\sum_{i=1}^n w_i x_i^p)^{1/p} & \text{if } p \neq 0 \\ \prod_{i=1}^n x_i^{w_i} & \text{else} \end{cases}.$$

Theorem 2.5. Given $x_1, \dots, x_n \in \mathbb{R}^+$, the following properties hold:

- $\min(x_1, \dots, x_n) \leq M_p(x_1, \dots, x_n) \leq M_p(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n)$
- $M_p(x_1, \dots, x_n) = M_p(\sigma(x_1, \dots, x_n))$ for $\sigma \in S_n$
- $M_p(bx_1, \dots, bx_n) = bM_p(x_1, \dots, x_n)$
- $M_p(x_1, \dots, x_{nk}) = M_p(M_p(x_1, \dots, x_n), M_p(x_{k+1}, \dots, x_{2k}), \dots, M_p(x_{(n-1)k+1}, \dots, x_{nk}))$

Theorem 2.6 (Power Mean Inequality). If $p < q$,

$$M_p(x_1, \dots, x_n) \leq M_q(x_1, \dots, x_n),$$

with equality if and only if $x_1 = \dots = x_n$.

2.3 Functional Equations

2.4 Linear Algebra

Problem 2.7. Let $A \in M_n(\mathbb{R})$ be skew-symmetric. Show that $\det(A) \geq 0$.

Proof. If n is odd, note that

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A).$$

It follows that $\det(A) = 0$.

Otherwise, suppose n is even and let $p(\lambda) = \det(A - I_n \lambda)$. If $\lambda \neq 0$ is an eigenvalue, note that $p(\lambda) = 0$ by the Cayley-Hamilton Theorem. Moreover,

$$p(-\lambda) = \det(A + I_n \lambda) = \det(A^T + I_n^T \lambda) = \det(-A + I_n \lambda) = 0.$$

Moreover, let v be an eigenvector with corresponding eigenvalue λ . Note that

$$\langle Av, v \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2,$$

$$\langle Av, v \rangle = \langle v, A^T v \rangle = \langle v, -Av \rangle = -\bar{\lambda} \langle v, v \rangle = -\bar{\lambda} \|v\|^2.$$

It follows that $\lambda = -\bar{\lambda}$, which implies that $\lambda = ri$ for $r \in \mathbb{R}$. Hence,

$$\det(A) = \prod_{j=1}^{n/2} (i\lambda_j)(-i\lambda_j) = \prod_{j=1}^n \lambda_j^2 \geq 0.$$

□

Problem 2.8. Let $A \in M_n(\mathbb{R})$ with $A^3 = A + I_n$. Show that $\det(A) > 0$.

Proof. Let $p(x) = x^3 - x - 1$. Note that $p(0) = -1$, $p(2) = 5$, so the polynomial has a root in the interval $(0, 2)$ by the intermediate value theorem. Furthermore, $p'(x) = 3x^2 - 1$ so the polynomial has critical points at $\pm \frac{1}{\sqrt{3}}$. It is easy to see that at both of these values, $p(x) < 0$ so it follows that the other roots of $p(x)$ are conjugate complex numbers. Let the roots be $\lambda_1, \lambda_2, \lambda_3$ with λ_1 being the positive real root and λ_2, λ_3 the conjugate complex ones. If A satisfies $A^3 = A + I_n$, then we must have the eigenvalues of A are λ_1, λ_2 and λ_3 , with multiplicity $\alpha_1, \alpha_2, \alpha_3$ respectively. Since λ_2, λ_3 are complex conjugates, we must have $\alpha_2 = \alpha_3$, so it follows that

$$\det(A) = \lambda_1^{\alpha_1} (\lambda_2 \lambda_3)^{\alpha_2} = \lambda_1^{\alpha_1} |\lambda_2|^{\alpha_2} > 0.$$

□

Problem 2.9. If $A, B \in M_n(\mathbb{R})$ such that $AB = BA$, then $\det(A^2 + B^2) \geq 0$.

Proof.

$$\det(A^2 + B^2) = \det(A + iB) \det(A - iB) = \det(A + iB) \overline{\det(A + iB)} = |\det(A + iB)|^2 \geq 0.$$

□

Problem 2.10. Let $A, B \in M_2(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + B^2) = 0$. Show that $\det(A) = \det(B)$.

Proof. Let $p_{A,B}(\lambda) = \det(A + \lambda B) = \det(B)\lambda^2 + (\text{tr } A + \text{tr } B - \text{tr}(AB))\lambda + \det(A)$. By Problem 1.3, we have $\det(A + iB)$ and $\det(A - iB) = 0$, which implies that $p_{A,B}(\lambda) = c(\lambda - i)(\lambda + i) = c(\lambda^2 + 1)$. It follows that $c = \det B = \det A$. □

Problem 2.11. Let $A \in M_2(\mathbb{R})$ with $\det A = -1$. Show that $\det(A^2 + I_2) \geq 4$. When does equality hold?

Proof. First, note the identity

$$\det(X + Y) + \det(X - Y) = 2(\det X + \det Y).$$

This follows from writing $p(z) = \det(X + zY) = \det(Y)z^2 + (\text{tr } X + \text{tr } Y - \text{tr}(XY))z + \det(X)$ and taking

$$p(1) + p(-1) = \det(X + Y) + \det(X - Y) = 2 \det Y + 2 \det X.$$

Then, taking $X = A^2 + I$ and $Y = 2A$, we have

$$0 \leq \det(A + I)^2 + \det(A - I)^2 = 2(\det(A^2 + I) + \det(2A)) = 2(\det(A^2 + I) - 4).$$

It follows that $\det(A^2 + I) \geq 4$ as desired. We have equality when the eigenvalues of A are 1 and -1. □

Problem 2.12. Let $A, B \in M_3(\mathbb{C})$ with $\det(A) = \det(B) = 1$. Show that $\det(A + \sqrt{2}B) \neq 0$.

2.5 Group Theory

Theorem 2.13 (Lagrange's Theorem). *Let G be a finite field. If H is a subgroup of G , then $|G| = [G : H]|H|$.*

2.6 Field Theory

3 Number Theory

3.1 Orders

3.2 P-adic Valuation

Definition 3.1. Let p be a prime and let n be a non-zero integer. We define $\nu_p(n)$ to be the exponent of p in the prime factorization of n .

Theorem 3.2 (Legendre's Theorem).

$$\nu_p(n!) = \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Problem 3.3 (Putnam 2003/B3). Show that for each positive integer n ,

$$n! = \prod_{i=1}^n \text{lcm} \left\{ 1, 2, \dots, \left\lfloor \frac{n}{i} \right\rfloor \right\}$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Proof. Note that

$$\begin{aligned} \nu_p \left(\prod_{k=1}^n \text{lcm} \{1, 2, \dots, \lfloor n/k \rfloor\} \right) &= \sum_{k=1}^n \nu_p (\text{lcm} \{1, 2, \dots, \lfloor n/k \rfloor\}) \\ &= \sum_{k=1}^n \lfloor \log_p \lfloor n/k \rfloor \rfloor \\ &= \sum_{k=1}^n \sum_{\ell: \lfloor n/k \rfloor \geq p^\ell} 1 \\ &= \sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{p^\ell} \right\rfloor. \end{aligned}$$

This is exactly $\nu_p(n!)$ by Legendre's Theorem. □

Theorem 3.4 (Lifting-the-Exponent(LTE) Lemma). *Let p be prime, $x, y \in \mathbb{Z}$, $n \in \mathbb{N}$ and $p \mid (x - y)$, $p \nmid x$, $p \nmid y$.*

- if p is odd, $\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n)$,
- for $p = 2$ and even n , $\nu_2(x^n - y^n) = \nu_2(x - y) + \nu_2(n) + \nu_2(x + y) - 1$.

3.3 Cyclotomic Polynomials

3.4 Finite Field Arithmetic

Refer to [Evan Chen, Summations](#).

Theorem 3.5 (Fermat's Little Theorem). *Let p be a prime. Then $a^{p-1} \equiv 1 \pmod{p}$ whenever $\gcd(p, a) = 1$.*

Theorem 3.6 (Lagrange's Theorem). *If p is prime and $f(x) \in \mathbb{Z}[x]$, then either*

- *every coefficient of $f(x)$ is divisible by p , or*
- *$f(x) \equiv 0 \pmod{p}$ has at most $\deg(f)$ incongruent solutions.*

Theorem 3.7 (Wilson's Theorem). *For any prime p ,*

$$(p-1)! \equiv -1.$$

Proof. Let $g(x) = (x-1)(x-2)\dots(x-(p-1))$ and $h(x) = x^{p-1} - 1$. Both polynomials have degree $p-1$ and leading term x^{p-1} . The constant term for $g(x)$ is $(p-1)!$. By Fermat's little theorem, $h(x)$ has roots $1, 2, \dots, p-1$ in \mathbb{F}_p .

Now, consider $f(x) = g(x) - h(x)$. Note that $\deg(f) \leq p-2$ since the leading terms cancel. In \mathbb{F}_p , it also has the same roots $1, 2, \dots, p-1$. By Lagrange's Theorem(3.2), we must have that $f(x) \equiv 0 \pmod{p}$. It follows that $f(0) = (p-1)! + 1 \equiv 0 \pmod{p}$ which proves the result. \square

Theorem 3.8 (Sums of Powers). *Let p be a prime and n an integer. Then,*

$$\sum_{k=1}^{p-1} k^m \equiv \begin{cases} 0 \pmod{p} & \text{if } p-1 \nmid m \\ -1 \pmod{p} & \text{if } p-1 \mid m \end{cases}$$

Proof. If $p-1 \mid m$, then $(p-1)\ell = m$ for some ℓ , so it follows that

$$\sum_{k=1}^{p-1} k = 1^{p-1} k^m \equiv \sum_{k=1}^{p-1} (k^{p-1})^\ell \equiv \sum_{k=1}^{p-1} 1 \equiv p-1 \equiv -1 \pmod{p}.$$

Otherwise, if we let g be a generator for $(\mathbb{Z}/p\mathbb{Z})^\times$, we have

$$\sum_{k=1}^{p-1} k^m \equiv \sum_{k=0}^{p-2} g^{km} \equiv \frac{g^{(p-1)m} - 1}{g^m - 1} \equiv 0 \pmod{p}$$

since $g^m - 1 \not\equiv 0 \pmod{p}$. \square

Theorem 3.9 (Wolstenholme's Theorem). *Let $p > 3$ be prime. Then*

$$(p-1)! \left(\frac{1}{1} + \dots + \frac{1}{p-1} \right) \equiv 0 \pmod{p^2}.$$

Theorem 3.10 (Harmonic modulo p). *For any integer $k = 1, 2, \dots, p-1$, we have*

$$\frac{1}{k} \equiv (-1)^{k-1} \frac{1}{p} \binom{p}{k} \pmod{p}.$$

Problem 3.11 (ELMO 2009). Let p be an odd prime and x be an integer such that $p \mid x^3 - 1$ but $p \nmid x - 1$. Prove that p divides

$$(p-1)! \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \frac{x^{p-1}}{p-1} \right).$$

Proof. Note that $p \mid x^3 - 1$ and $x \nmid x - 1$ implies that $p \mid x^2 + x + 1$, so we have $1 + x \equiv -x^2 \pmod{p}$. Using Theorem 3.6, we can rewrite the expression as

$$\begin{aligned} x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \frac{x^{p-1}}{p-1} &\equiv \frac{x}{p} \binom{p}{1} + \frac{x^2}{p} \binom{p}{2} + \dots + \frac{x^{p-1}}{p} \binom{p}{p-1} \pmod{p} \\ &= \frac{1}{p} ((1+x)^p - 1 - x^p) \pmod{p} \\ &= -\frac{1}{p} (1 + x^p + x^{2p}). \end{aligned}$$

Note that $x^{2p} + x^p + 1 \equiv (x^2 + x)^p + 1 \pmod{p}$. By the Lifting-The-Exponent(LTE) lemma,

$$\nu_p((x^2 + x)^p + 1^p) = \nu_p(x^2 + x + 1) + \nu_p(p) \geq 2.$$

It follows that $1 + x^p + x^{2p} \equiv 0 \pmod{p^2}$, which proves the result. \square

3.5 Arithmetic Functions

Definition 3.12. A function $f : \mathbb{N} \rightarrow \mathbb{C}$ is **multiplicative** if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. It is **completely multiplicative** if $f(mn) = f(m)f(n)$ for any $m, n \in \mathbb{N}$.

Definition 3.13 (Möbius Function). The Möbius Function, μ , is defined by

$$\mu(n) = \begin{cases} (-1)^m & \text{if } n \text{ has } m \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Definition 3.14 (Dirichlet Convolution). Given two arithmetic functions, $f, g : \mathbb{N} \rightarrow \mathbb{C}$, we define

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) = \sum_{de=n} f(d)g(e).$$

Theorem 3.15 (Möbius Inversion). *Given two arithmetic functions $f, g : \mathbb{N} \rightarrow \mathbb{C}$,*

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d)g(n/d).$$

*In other words, $g = f * 1$ if and only if $f = g * \mu$.*

Problem 3.16 (Bulgaria 1989). Let $\Omega(n)$ denote the number of prime factors of n , counted with multiplicity. Evaluate

$$\sum_{n=1}^{1989} (-1)^{\Omega(n)} \left\lfloor \frac{1989}{n} \right\rfloor.$$

Proof. Note that $g(n) = -1^{\Omega(n)}$ is (completely) multiplicative. Then,

$$\begin{aligned} \sum_{n=1}^{1989} (-1)^{\Omega(n)} \left\lfloor \frac{1989}{n} \right\rfloor &= \sum_{n=1}^{1989} \sum_{k \leq 1989, n|k} (-1)^{\Omega(n)} \\ &= \sum_{k=1}^{1989} \sum_{n|k} (-1)^{\Omega(n)}. \end{aligned}$$

Note that $g * 1$ is multiplicative so it suffices to evaluate $(g * 1)(k) = \sum_{n|k} (-1)^{\Omega(n)}$ for prime powers. Note that

$$(g * 1)(p^k) = \sum_{r=0}^k (-1)^r = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{else} \end{cases}.$$

It follows that $(g * 1)(n) = 1$ when n is a perfect square and is 0 otherwise. Hence, the sum evaluates to $\lfloor \sqrt{1989} \rfloor = 44$. \square

4 Analysis

4.1 Sequences and Series

Problem 4.1 (Putnam 2020/A3). Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Proof. We claim the series diverges. It suffices to show that $a_n \geq \frac{1}{\sqrt{n}}$. We proceed by induction. It is clear that $a_1 = 1 \geq \frac{1}{\sqrt{1}} = 1$. Suppose that $a_k \geq \frac{1}{\sqrt{k}}$. Since $\sin x \geq x - x^3/6$ and $\sin x$ is monotonically increasing in $[0, \pi/2]$, we have

$$a_{k+1} \geq \sin\left(\frac{1}{\sqrt{k}}\right) > \frac{1}{\sqrt{k}} - \frac{1}{6k\sqrt{k}} = \frac{6k-1}{6k\sqrt{k}}.$$

It suffices to show that

$$\frac{6k-1}{6k} \geq \frac{\sqrt{k}}{\sqrt{k+1}} \Leftrightarrow 24k^2 - 11k + 1 \geq 0,$$

which is true for $k \geq 1$. □

4.2 Measure Theory and Integration

Problem 4.2 (Putnam 2002/A6). Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base- b digits of n . For which values of b does the sum $\sum_{n \geq 1} 1/f(n)$ converge?

Proof. The sum converges for $b = 2$ and diverges for $b \geq 3$.

We first consider $b \geq 3$. Suppose the sum converges. Note that we can write

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} = \sum_{d=1}^{\infty} \frac{1}{f(d)} \sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n}.$$

Note that $\sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n}$ is a left-endpoint Riemann approximation for the integral $\int_{b^{d-1}}^{b^d} \frac{1}{x}$ and the function $\frac{1}{x}$ is monotonically decreasing on this interval so it follows that

$$\sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n} > \int_{b^{d-1}}^{b^d} \frac{1}{x} = \log b.$$

However, this implies that

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} > \log b \sum_{d=1}^{\infty} \frac{1}{f(d)},$$

which is a contradiction since $\log b > 1$.

Now, we show that the sum converges in the case of $b = 2$. Let $C = \log 2 + \frac{1}{8} < 1$. We prove by induction that for each $m \in \mathbb{N}$,

$$\sum_{n=1}^{2^m-1} \frac{1}{f(n)} < 1 + \frac{1}{2} + \frac{1}{6(1-C)} = L.$$

For $m = 1, 2$, the result is clear. Suppose it is true for all $m \in \{1, 2, \dots, N-1\}$. Note that

$$\sum_{n=1}^{2^N-1} \frac{1}{f(n)} = 1 + \frac{1}{2} + \frac{1}{6} + \sum_{d=3}^N \frac{1}{f(d)} \sum_{n=2^{d-1}}^{2^d-1} \frac{1}{n}.$$

Then, using a right-endpoint Riemann approximation, we have

$$\begin{aligned} \sum_{n=2^{d-1}}^{2^d-1} \frac{1}{n} &= \frac{1}{2^{d-1}} - \frac{1}{2^d} + \sum_{n=2^{d-1}+1}^{2^d} \frac{1}{n} \\ &< 2^{-d} + \int_{2^{d-1}}^{2^d} \frac{dx}{x} \\ &< \frac{1}{8} + \log 2 = C. \end{aligned}$$

It follows that

$$1 + \frac{1}{2} + \frac{1}{6} + \sum_{d=3}^N \frac{1}{f(d)} < 1 + \frac{1}{2} + \frac{1}{6} + C \sum_{d=3}^N \frac{1}{f(d)} \quad (1)$$

$$< 1 + \frac{1}{2} + \frac{1}{6} + \frac{C}{6(1-C)} \quad (2)$$

$$= 1 + \frac{1}{2} + \frac{1}{6(1-C)} = L, \quad (3)$$

where we used the strong induction hypothesis to obtain (2). \square

Problem 4.3 (Putnam 2003/B6). Let $f(x)$ be a continuous real-valued function defined on $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx.$$

Proof. Let $f^+ = \max(f(x), 0)$ and $f^- = f^+ - f$. Let $A = \text{supp } f^+$, $B = \text{supp } f^-$. We will denote $\|g\| = \int_0^1 |g(x)| dx$.

Note that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy = \left(\iint_{A \times A} + \iint_{B \times B} + 2 \iint_{A \times B} \right) |f(x) + f(y)| dx dy.$$

Note that

$$\begin{aligned} \iint_{A \times A} |f(x) + f(y)| dx dy &= \iint_{A \times A} (f(x) + f(y)) dx dy \\ &= \iint_{A \times A} f(x) dx dy + \iint_{A \times A} f(y) dx dy \\ &= 2|A|\|f^+\|. \end{aligned}$$

Similarly, $\iint_{B \times B} |f(x) + f(y)| \, dx dy = 2|B|\|f^-\|$.

Finally, note that

$$\begin{aligned} \iint_{A \times B} |f(x) + f(y)| \, dx dy &= \iint_{A \times B} |f^+(x) - f^-(y)| \, dx dy \\ &\geq \left| \iint_{A \times B} (f^+(x) - f^-(y)) \, dx dy \right| \\ &= ||B|\|f^+\| - |A|\|f^-\||. \end{aligned}$$

Combining the results, we have that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx dy \geq 2|A|\|f^+\| + 2|B|\|f^-\| + 2||B|\|f^+\| - |A|\|f^-\||.$$

Squaring both sides of the expression, we have that

$$\begin{aligned} \left(\int_0^1 \int_0^1 |f(x) + f(y)| \, dx dy \right)^2 &\geq (2|A|\|f^+\| + 2|B|\|f^-\| + 2||B|\|f^+\| - |A|\|f^-\||)^2 \\ &= 4(|A|\|f^+\| + |B|\|f^-\| + ||B|\|f^+\| - |A|\|f^-\||)^2 \\ &= 4(|A|\|f^+\| + |B|\|f^-\|)^2 + 4(|B|\|f^+\| - |A|\|f^-\||)^2 + 8(|A|\|f^+\| + |B|\|f^-\|)(|B|\|f^+\| - |A|\|f^-\||) \\ &\geq 4(|A|^2\|f^+\|^2 + |B|^2\|f^-\|^2 + |A|^2\|f^-\|^2 + |B|^2\|f^+\|^2) \\ &\geq 4(|A|^2 + |B|^2)(\|f^+\|^2 + \|f^-\|^2) \\ &\geq (|A| + |B|)^2(\|f^+\| + \|f^-\|)^2 \\ &= (1)^2(\|f\|)^2 \\ &= \left(\int_0^1 |f(x)| \, dx \right)^2. \end{aligned}$$

□

4.3 Vector Calculus

4.4 Complex Analysis

5 Geometry

5.1 Classical Results

Theorem 5.1 (Incenter-Excenter Lemma).

Theorem 5.2 (Euler's Theorem). *Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote the circumcenter and incenter. then $OI^2 = R(R - 2r)$. In particular, $R \geq 2r$.*

5.2 Complex Numbers

Theorem 5.3 (Complex Special Points). *Let (ABC) be the unit circle. We have*

- the circumcenter, $o = 0$.
- the orthocenter, $h = a + b + c$.
- the centroid, $g = \frac{a+b+c}{3}$.
- the nine-point center, $n_9 = \frac{a+b+c}{2}$.

Theorem 5.4 (Complex Incenter). *Given ABC on the unit circle, it is possible to pick u, v, w such that*

- $a = u^2, b = v^2, c = w^2$,
- the midpoint of \widehat{BC} is $-vw$, the midpoint of \widehat{CA} is $-wu$ and the midpoint of \widehat{ab} is $-uv$,
- the incenter $I = -(uv - vw - wu)$.

Theorem 5.5 (Complex Foot). *If $a \neq b$ are on the unit circle and $z \in \mathbb{C}$, then the foot from Z to AB is given by*

$$\frac{a + b + z - ab\bar{z}}{2}$$

Theorem 5.6 (Complex Shoelace). *If $a, b, c \in \mathbb{C}$, the signed area of $\triangle ABC$ is given by*

$$\frac{i}{4} \begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}.$$

Theorem 5.7 (Concyclic Complex Numbers). *Let a, b, c, d be distinct complex numbers, not all collinear. Then A, B, C, D are concyclic if and only if*

$$\frac{b-a}{c-a} \div \frac{b-d}{c-d} \in \mathbb{R}.$$

Problem 5.8 (Putnam 2003/B5). *Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O , and let P be any point in the circle's interior. Let a, b, c be the distances from P to A, B, C respectively. Show that there is a triangle with side lengths a, b, c , and that the area of this triangle depends only on the distance from P to O .*

Proof. Let $\omega = e^{2\pi i/3}$, $A = 1$, $B = \omega$, $C = \omega^2$, $P = z \in \mathbb{C}$ with $|z| < 1$. We have

$$a = |z - 1|, b = |z - \omega|, c = |z - \omega^2|.$$

Note that

$$(z - 1) + \omega(z - \omega) + \omega^2(z - \omega^2) = z(1 + \omega + \omega^2) - (1 + \omega^2 + \omega^4) = 0.$$

The corresponding triangle, where we visualize the complex numbers as vectors that are sides of the triangle, has side lengths of a, b, c as desired.

The area of the triangle is given by

$$\begin{aligned} |(z - 1)\omega(\bar{z} - \bar{\omega}) - z\bar{1}\omega(\bar{z} - \bar{\omega})|/4 &= |(z - 1)(\omega^2\bar{z} - \bar{\omega}) - (\bar{z} - 1)(\omega z - \omega^2)|/4 \\ &= |z\bar{z}\omega^2 - \omega^2\bar{z} - z\omega + \omega - z\bar{z}\omega + \omega z + \bar{z}\omega^2 - \omega^2|/4 \\ &= |(z\bar{z} - 1)(\omega^2 - \omega)|/4 \\ &= \frac{(1 - |z|^2)\sqrt{3}}{4}, \end{aligned}$$

which is a function of z , as desired. \square

Problem 5.9. Let H be the orthocenter of $\triangle ABC$. Let X be the reflection of H over \overline{BC} and Y the reflection over the midpoint of \overline{BC} . Prove that X and Y lie on (ABC) , and \overline{AY} is a diameter.

Proof. Let $A = a, B = b, C = c$ be the complex number representation and without loss of generality, suppose that (ABC) is the unit circle. Note that the orthocenter is given by $h = a + b + c$. Then,

$$\begin{aligned} x &= b + (c - b)\overline{\left(\frac{h - b}{c - b}\right)} \\ &= b + (c - b)\left(\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{b}}{\frac{1}{c} - \frac{1}{b}}\right) \\ &= b - bc\left(\frac{a + c}{ac}\right) \\ &= b\left(1 - 1 - \frac{c}{a}\right) \\ &= -\frac{bc}{a} \in (ABC). \end{aligned}$$

Next, if we let $m = \frac{b+c}{2}$, note that we have

$$y - m = -(h - m) \Rightarrow y = 2m - h = -a \in (ABC).$$

Furthermore, since $y = -a$, we have that \overline{AY} is a diameter of (ABC) . \square

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