The Hardy-Littlewood Maximal Function

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The Hardy-Littlewood maximal operator is a non-linear operator that takes a locally integrable function and returns another function corresponding to the maximum average value the original function can have on balls cenetered at a given point. It has several applications in Real Analysis and Harmonic Analysis. We present the lectures notes and solutions to exercises from Math 258(Christ).

§1 Weak L^p and Distribution Functions

We work in a measure space (X, μ) that is σ-finite. Let $S(X) = S(X, \mu)$ denote the space of simple functions $f : X \to \mathbb{C}$ and $\mathcal{M}(X)$ denote the space of measure functions.

Definition 1.1. The distribution function λ_f of $f \in \mathcal{M}(X)$ is

$$\lambda_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}.$$

This gives us a way to think about norms in the measure space. For example, consider the following lemma:

Lemma 1.2

For $p \in (0, \infty)$ and $f \in \mathcal{M}(X)$,

$$\|f\|_p^p = \int |f|^p d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha.$$

Proof. Denote $E = \{(x, \alpha) : |f(x)| > \alpha\}$.

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha &= p \int_0^\infty \alpha^{p-1} \int_X 1_E(x, \alpha) d\mu(x) d\alpha \\ &= \int_X \int_0^\infty p\alpha^{p-1} 1(\alpha < |f(x)|) d\alpha d\mu(x) \\ &= \int_X |f(x)|^p d\mu(x) = \|f\|_p^p. \end{aligned}$$

□

Exercise 1.3. Present an alternate proof for simple functions and use the monotone convergence theorem to pass to general functions.

Proof. Let $f = \sum_{i=1}^{n} c_j 1_{E_j}$ be a simple function. Then, $\|f\|_p^p = \sum_{i=1}^{n} |c_j|^p \mu(E_j)$. Note that

$$\begin{aligned} p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha &= \int_0^\infty p\alpha^{p-1}\sum_{i=1}^{n}\mu(E_j)1(|c_j| > \alpha) \\ &= \sum_{i=1}^{n}\mu(E_j)\int_0^\infty p\alpha^{p-1}1(|c_j| > \alpha) \\ &= \sum_{i=1}^{n}\mu(E_j)\int_0^{|} c_j|p\alpha^{p-1} \\ &= \sum_{i=1}^{n}\mu(E_j)|c_j|^p \\ &= \|f\|_p^p. \end{aligned}$$

For a general nonnegative function f, we can write $f_n \uparrow f$, where $f_n = \sum_{i=1}^{n} c_{in} 1_{E_i n}$. By the monotone convergence theorem, it follows that

$$\int |f|^p = \lim_{n\to\infty}\int |f_n|^p = \lim_{n\to\infty}\int_0^\infty p\alpha^{p-1}\lambda_{f_n}(\alpha)d\alpha = \int_0^\infty p\alpha^{p-1}\lambda_f(\alpha)d\alpha,$$

by noting that $\lambda_{f_n} \uparrow \lambda_f$ and using the monotone convergence theorem. □

Lemma 1.4 (Chebyshev's Inequality)
If $p \in (0, \infty)$ and $f \in L^p$, then for $\alpha > 0$,

$$\lambda_f(\alpha) \le \alpha^{-p}\|f\|_p^p.$$

For $p = 1$, then gives *Markov's Inequality*:

$$\lambda_f(\ell) \le \ell^{-1}\|f\|_1.$$

Proof.

$$\lambda_f(\alpha) = \int_X 1(|f(x)| > \alpha)d\mu(x) \le \int_X \alpha^{-p}|f(x)|^p d\mu(x) = \alpha^{-p}\|f\|_p^p.$$

□

Chebyshev's inequality loses information, in the sense that

$$p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha \le p\|f\|_p^p\int_0^\infty \alpha^{p-1}\alpha^{-p}d\alpha,$$

and the latter integral diverges. However, it does allow us to extract useful information from the finiteness of the L^p norms.

Definition 1.5. For $p \in [1, \infty)$, define $L^{p,\infty}(X, \mu)$ as the set of functions $f \in \mathcal{M}(X)$ for which there exists $C < \infty$ with $\lambda_f(\alpha) \le C^p\alpha^{-p}$.

Note that $L^{p,\infty}$ is a quasi-normed vector space. We prove the triangle inequality:

Proof.

$$\begin{aligned}\|f+g\|_{p,\infty} &= \inf\{C : \lambda_{f+g}(\alpha) \leq C^p\alpha^{-p}\} \\ &\leq \inf\{C : \lambda_f(\alpha/2) + \lambda_g(\alpha/2) \leq C^p\alpha^{-p}\} \\ &\leq \inf\{C : \lambda_f(\alpha/2) \leq C^p\alpha^{-p}/2\} + \inf\{C : \lambda_g(\alpha/2) \leq C^p\alpha^{-p}/2\} \qquad \leq\end{aligned}$$

□

§1.1 The Hardy-Littlewood Maximal Operator

Definition 1.6 (Hardy-Littlewood Maximal Operator)**.** Let $f \in L^1_{loc}(\mathbb{R}^d)$, and define $Mf : \mathbb{R}^d \to [0, \infty]$ by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)| dy.$$