Math 222a Lecture Notes, Fall 2020

Partial Differential Equations

Professor: Daniel Tataru
Vishal Raman

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§1.1 Introduction

Partial differential equations apply to functions $u : \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$, where u refers to the space dimension. Usually, $n \geq 2 (n = 1$ corresponds to ODEs).

We present the following notation:

- $\frac{\partial}{\partial x_i} u = \partial_i u$
- There is also multi-index notation, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u$. The size of α is given by $|\alpha| = \sum_{i=1}^n \alpha_i$.
- $C(\mathbb{R}^n)$, continuous functions in $\mathbb{R}^n$.
- $C(\Omega)$, $\Omega \subset \mathbb{R}^n$, continuous functions in Ω.
- $C^1(\mathbb{R}^n), C^1(\Omega)$, continuously differentiable functions.
- $C^k(\mathbb{R}^n), C^k(\Omega)$, k-times differentiable.
- $C^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty C^k(\mathbb{R}^n)$.

We consider an example PDE,

$$F(u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0.$$

In the above, $k \geq 1$ and k is the **order** of the equation. We also have the shorthand $F(\partial^{\leq k} u) = 0$.

§1.2 Classification of PDE's

Definition 1.1 (Linear PDE)**.** The PDE is a linear function of its arguments. We can apply multi-index notation, as follows:

$$\sum_{|\alpha| < k} c_\alpha \partial^\alpha u = f(x).$$

If $f(x) = 0$, the PDE is **homogeneous**, otherwise it is **inhomogeneous**.

This can be separated into linear PDEs with constant coefficients, $c_\alpha \in \mathbb{R}, \mathbb{C}$ and variables coefficients, $c_\alpha = c_\alpha(x)$. [In this class, we focus on constant coefficient PDEs, but many of the techniques can be extended to variable coefficient PDEs.]

Definition 1.2 (Nonlinear PDE)**.** We look at a function $F = F(u, \partial u, \ldots, \partial^k u)$. The highest order terms are take the *leading role.*

- Semilinear PDE's: F is linear, with constant or variable coefficients in $\partial^k u$:

  $$\sum_{|\alpha| = k} c_\alpha(x) \partial^\alpha u = N(\partial^{\leq k-1} u).$$

  The LHS is called the principal part, and the RHS is the perturbative role.

- Quasilinear PDE's:

  $$\sum_{|\alpha| = k} c_\alpha(\partial^{\leq k-1} u) \partial^\alpha u = N(\partial^{\leq k-1} u).$$

- Fully Nonlinear PDE's: $F(\partial^{\leq k}u) = 0$, with a nonlinear dependence on $\partial^k u$.

Some examples:

- Linear, homogeneous, variable coefficients, order 1:

$$\sum_{k=1}^{u} c_k(x)\partial_k(u) = 0.$$

- Define $\Delta = \partial_1^2 + \cdots + \partial_n^2$, the Laplacian operator. We have a linear, constant coefficients, inhomogeneous, order 2:

$$\Delta u = f.$$

- Semilinear, order 2:

$$\Delta u = u^3.$$

  [Note that translation invariance makes homogeneous vs inhomogeneous not useful for classification in the case of nonlinear PDE's.]

- Harmonic Map Equation:

$$\Delta u = u|\nabla u|^2.$$

  It is still semilinear, but with a stronger nonlienarity.

- Monge Ampere Equation:

$$\mathbb{R}^2, \partial_1^2 u \partial_2^2 u - (\partial_1\partial_2 u)^2 = 0.$$

  It is a fully nonlinear equation.

§1.3 Initial Value Problems

We have various types of problems:

- (Stationary Problems) With $u : \mathbb{R}^n \to \mathbb{R}$,

$$F(\partial^{\leq k}u) = 0,$$

  might describe an equilibrium configuration of a physical system.

- (Evolution Equations) With $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $u(t, x)$ describes the state at time t. We can think about the order in x or in t.

Definition 1.3 (Initial Value Problem/Cauchy Problem)**.** A PDE with initial conditions.

Example 1.4

Consider the heat equation:

$$\partial_t u = \Delta_x u,$$

$$u(t = 0, x) = u_o(x).$$

The equation is first order in t, but second order in x.

Example 1.5

In $[\mathbb{R} \times \mathbb{R}]$, the vibrating string:

$$\partial_t^2 u = \partial_x^2 u,$$

$$u(t=0, x) = u_0(x),$$
$$\partial_t u(t=0, x) = u_1(x).$$

Note that this equation is second order in time, and requires 2 pieces of initial data.

An easier problem: Compute the Taylor series of u at some point $(0, x_0)$. It requires $\partial_t^\alpha \partial_x^\beta u(0, x_0)$.

- This is obvious if we have no time derivative or exactly 1.
- Second order time derivatives come from the equation.
- Third order or higher time derivatives come from differentiating the equation:

$$\partial_t^3 u = \partial_x^2 \partial_t u.$$

§1.4 Boundary Value Problems

We begin with an example.

Example 1.6

Take $\Delta u = f$ in $\Omega \subset \mathbb{R}^3$, which represents equilibrium for temperature in a solid. To solve, we need information about the boundary of Ω. For example,

$$\Delta u = f \in \Omega,$$

$$u = g \in \partial\Omega.$$

§1.5 Fluid Classification

We take $u : \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$, and

$$F(\partial^{\leq k} u) = 0.$$

This is considered to be a **scalar equation**.

We could also take a **system** of equations, where $u : \mathbb{R}^n \to \mathbb{R}^m(\mathbb{C}^m)$, where $u = [u_i]$ a column of equations. These are often more difficult than scalar equation. We should have

$$F(\partial^{\leq k} u) = 0,$$

but $F : \mathbb{R}^{(\cdot)} \to \mathbb{R}^m(\mathbb{C}^m)$.

Example 1.7

A 2-system:

$$\Delta u = v,$$

$$\Delta v = -u.$$

We can often reduce the order of a scalar equation by turning it into a system:

Example 1.8

Consider the vibrating string,
$$\partial_t^2 u = \partial_x^2 u.$$
If we take $v = \partial_t u$, the it suffices to solve the system,
$$\partial_t u = v,$$
$$\partial_t v = \partial_x^2 u.$$
We van reduce it further by saying $u_1 = \partial_x u, u_2 = \partial_t u$ for the system,
$$\partial_t u_1 = \partial_x u_2,$$
$$\partial_t u_2 = \partial_x u_1.$$