EECS 208, Lecture Notes

Computational Principles of High-Dimensional Data Analysis

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§1 Lecture 2, 8/31/21

We would like to recover x from linear measurements given by $y = Ax$, where A are the measurements and y are the observations. We would like the vector x to be sparse: most components are 0. Of course, this is extremely ideal, so we also allow noise in our measurements.

§1.1 Finding Sparse Solutions

We are finding x where $y = Ax$, $A \in \mathbb{R}^{m\times n}$ with $m \ll n$ and x k-sparse.

Definition 1.1. A norm on a vector space V over $\mathbb{R}$ is a function $\|\cdot\| : V \to \mathbb{R}$ such that

- Positive Definite: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
- non-negatively homogeneous: $\|\alpha x\| = |\alpha|\|x\|$
- Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Recall the classical ℓ^p norms. We will consider these even for $0 \leq p < 1$.
Given $y = Ax_0$, we wish to find

$$\widehat{x} = \arg\min \|x\|_0 \quad s.t. \quad Ax = y.$$

Definition 1.2 (Support)**.**

$$\text{supp }(x) = \{i : x(u) \neq 0\} \subset [n].$$

Definition 1.3 (Kruskal Rank)**.** The Kruskal rank of a matrix A is the largest number r such that every subset of r columns of A are linearly independent, denoted $\text{krank}(A)$.

Theorem 1.4 (ℓ^0 Recovery)

Suppose $y = Ax_0$ with $\|x_0\|_0 \leq \text{krank}(A)/2$. then x_0 is the unique optimal solution to the ℓ^0 minimization problem

$$\min \|x\|_0, \quad Ax = y.$$

Theorem 1.5

The ℓ^0-minimization problem is (strongly) NP-hard.

§1.2 Two Fundamental Questions

- Sample Complexity: how many measurements are needed for the problem to become computationally tractable?
- Computational Complexity: Once tractable, what is the precise computational complexity in finding the correct solution?

§2 Lecture 3, September 2nd

§2.1 Convex Functions

We assume basic familiarity with convex functions.

Definition 2.1 (Lower Convex Envelope)**.** A function $f_c(x)$ is said to be a (lower) convex envelope of $f(x)$ if for all convex functions $g \leq f$, $g \leq f_c$.

Example 2.2

For all $x \in \mathbb{R}^n$, we have $\|x\|_0 = \sum_{i=1}^n 1_{x(i) \neq 0}$, $\|x\|_1 = \sum_{i=1}^n |x(i)|$. The ℓ^1 norm is the envelope of the ℓ^0 norm.