Putnam Solutions

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My sketches to problems from the Putnam exams. I tend to leave out a lot of the details for routine checks or brute force calculations. Any typos or mistakes found are mine - kindly direct them to my inbox.

Contents

§1 Putnam - 2001

§1.1 A1 - Algebra

Consider a set S and a binary operation $*$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Proof. Note that

$$b = ((b * a) * b) * (b * a) = a * (b * a).$$

□

§1.2 A2 - Combinatorics

You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that when tossed, is has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd?

Proof. We claim the probability is $P(n) = \boxed{\dfrac{n}{2n+1}}$. We prove it by induction. We are given that $P(1) = \frac{1}{3}$, which satisfies the claim. Suppose $P(k) = \frac{k}{2k+1}$ for $k \geq 1$. In order to find $P(k+1)$, we condition on the result of the first k coin tosses. Namely, suppose the number of heads is even after k tosses. Then, the total number of heads is odd if we flip a head on the $k+1$-th toss. Similarly, if the number of heads is odd after k tosses, then the total number of heads is odd if we flip a tail on the $k+1$-th toss.

Putting this together gives

$$\begin{aligned}
P(k+1) &= (1-P(k))p_{k+1} + P(k)(1-p_{k+1}) \\
&= P(k)\left(1-2p_{k+1}\right) + p_{k+1} \\
&= P(k)\left(1-\frac{2}{2k+3}\right) + \frac{1}{2k+3} \\
&= P(k)\frac{2k+1}{2k+3} + \frac{1}{2k+3} \\
&= \frac{k}{2k+1}\frac{2k+1}{2k+3} + \frac{1}{2k+3} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

which proves the result. □

§1.2.1 Official Solution

There is another remarkable proof using generating functions.

Proof. Consider the polynomial $f(x) = \prod_{k=1}^{n}\left(\frac{x}{2k+1} + \frac{2k}{2k+1}\right)$. The coefficient of x^m gives the probability of exactly m heads. The sum of the odd coefficients is given by $\frac{f(1)-f(-1)}{2}$. It is clear that $f(1) = 1$ and note that

$$f(-1) = \prod_{k=1}^{n} \frac{2k-1}{2k+1} = \frac{1}{2n+1}.$$

The overall probability is $\frac{n}{2n+1}$ as desired. □

§1.3 A3 - Algebra

For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Proof. We claim that m is the square of an integer or twice the square of an integer. Set $y = x^2$. We look for square-integer solutions for y. From the quadratic formula,

$$\begin{aligned} y &= \frac{2m+4 \pm \sqrt{(2m+4) - 4(m-2)^2}}{2} \\ &= m+2 \pm \sqrt{(m+2)^2 - (m-2)^2} \\ &= m+2 \pm \sqrt{4(2m)} \\ &= m+2 \pm 2\sqrt{2m} \\ &= (\sqrt{m} \pm \sqrt{2})^2. \end{aligned}$$

Hence, $x = \pm\sqrt{m} \pm \sqrt{2}$. Note that if m is neither the square of an integer nor twice the square of an integer then the field $\mathbb{Q}(\sqrt{m}, \sqrt{2})$ is of degree 4 and the Galois group acts transitively on the roots $\{\pm\sqrt{m} \pm \sqrt{2}\}$. It follows that the polynomial is irreducible.

It is easy to verify that if m is a square or twice a square, then $P_m(x)$ reduces into the product of non-constant integer polynomials.

□

§1.4 A4 - Geometry

Triangle ABC has area 1. Points E, F, G lie on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

Proof. We claim that $[RST] = \frac{7-\sqrt{5}}{4}$. Let $EC/BC = r$, $FA/CA = s$, $GB/AB = t$.

Note that $[ABE] = [AFE]$ since they share a base AE and $BR = FR$ implies that the share the same altitude length as well(drop altitudes from F and B and use the congruent triangles).

Then, $[ABE] = [ABE]/[ABC] = BE/BC = 1 - EC/BC = 1 - r$. We also have $[ACE] = r$. It follows that $[FCE] = [ACE](FC/AC) = r(1 - s)$.

Now,

$$1 = [ABC] = [ABE] + [AFE] + [EFC] = (1 - r) + (1 - r) + r(1 - s) \implies r(1 + s) = 1.$$

Arguing similarly for the other sides, we have $s(1 + t) = 1$, and $t(1 + r) = 1$.

It follows that

$$r = \frac{1}{1+s} = \frac{1}{1+\frac{1}{1+t}} = \frac{1}{1+\frac{1}{1+\frac{1}{r}}}.$$

Simplifying this, we find that $r = \frac{2+r}{3+2r}$, which gives $3r + 2r^2 = 2 + r$, or equivalently, $r^2 + r - 1 = 0$. Plugging into the quadratic formula and taking the positive root gives

$$r = \frac{-1+\sqrt{5}}{2},$$

and by repeating the argument, we have $r = s = t = \frac{-1+\sqrt{5}}{2}$.

Now, note that $[ATC] = [AEC]/2 = r/2$, $[ATG] = [ACG] - [ATC] = 1 - t - r/2$. Similarly, $[BSC] = t/2$ and $[BRE] = 1 - r - s/2$, so it follows that $[BRTG] = [ABE] - [ATG] - [BRE] = r/2 + s/2 + t - 1$.

$$\begin{aligned}
[RST] &= [ABC] - [ACG] - [BSC] - [BRTG] \\
&= 1 - (1 - t) - (t/2) - (r/2 + s/2 + t - 1) \\
&= 1 - \frac{r+s+t}{2} \\
&= 1 - \frac{3\frac{\sqrt{5}-1}{2}}{2} \\
&= \frac{7-\sqrt{5}}{4}.
\end{aligned}$$

□

Proof. A brute-force calculation through vectors. Define A to be the origin and take B, C to be basis vectors from A. We can set $G = \beta B$, $F = (1-\gamma)C$, $E = \alpha C + (1-\alpha)C$. Furthermore, we set $R = (1-\rho)E$, $S = \sigma B + (1-\sigma)F$, $T = \tau C + (1-\tau)G$. To satisfy the conditions of the problem, we must have that

$$2R = B + F, 2S = C + G, 2T = E.$$

After messy algebra, we obtain that

$$\alpha = \frac{1-\beta}{2-\beta}, \beta = \frac{1-\gamma}{2-\gamma}, \gamma = \frac{1-\gamma}{2-\gamma},$$

which has an obvious solution $\alpha = \beta = \gamma = \frac{3-\sqrt{5}}{2}$.

It follows that

$$(R-T, S-T) = \frac{1}{2}\begin{pmatrix} \alpha & 2\alpha - 1 \\ 1-2\alpha & 1-\alpha \end{pmatrix},$$

so we can evaluate

$$[RST] = \frac{[ABC]}{4}\begin{vmatrix} \alpha & 2\alpha - 1 \\ 1-2\alpha & 1-\alpha \end{vmatrix} = \frac{\alpha^2}{2} = \frac{7-3\sqrt{5}}{4}.$$

$\square$

§1.5 A5 - Number Theory

Show that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

Proof. We claim the unique pair of positive integers satisfying the claim is $(a, n) = (13, 2)$. It is easy to verify that this is indeed a solution.

Considering the equation in $\mathbb{Z}_3$, we see that $a \equiv 1 \pmod 3$ - in the other cases, one of the terms vanishes and the other term is non-vanishing, so the difference cannot vanish.

Considering the equation in $\mathbb{Z}_4$, we cannot have $a \equiv 0 \pmod 4$, for the same reason as above. If $a \equiv 1 \pmod 4$, we must have that $n > 1$ in order for the equivalence to be satisfied. If $a \equiv 2, 3 \pmod 4$, we must have that n is even.

In the case with $a \equiv 1 \pmod 3$ and $a \equiv 1 \pmod 4$, we obtain $a \equiv 1 \pmod{12}$ and $n > 1$. We see easily that $a = 1$ does not satisfy the equation for any $n > 1$. For $a = 13$, we have a solution as above for $n = 2$. It is easy to see that no higher value of n also satisfies the equation since the function $f(x) = 13^{x+1} - 14^x$ is monotonically increasing. We can repeatedly apply similar arguments for the other cases to show that this is the unique solution. □

§1.6 A6 - Calculus

Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Proof. We claim that it is possible. Take a unit circle given by $x^2 + (y-1)^2 = 1$ and a parabola $y = kx^2$. The length of the curve inside the arc is given by

$$L(k) = 2\int_0^{\sqrt{2k-1}/k} \sqrt{1 + 4k^2x^2}\, dx = \frac{1}{4k}\int_0^{2\sqrt{2k-1}} \sqrt{1 + x^2}\, dx.$$

It is easy to show that $\sqrt{1+x^2} \geq \frac{1}{x+1}$ so it follows that $\lim_{k\to\infty} L(k) = +\infty$. It follows that there exists some k so that $L(k) > 4$ as desired. $\square$

§1.7 B1 - Combinatorics

Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid from left to right. Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black. Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Proof. I have two proofs. An outline of the first follows the approach of invariants. Namely, we can start from a checkerboard pattern and repeatedly swap squares so that the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares. It suffices to show that the group of colorings with the given conditions is transitive under the transposition. This is easy to show with an algorithm approach: for each square on a given board, we assign 1 if it differs from the checkerboard, otherwise we assign 0. Then, we take the sum of the values. We can choose transpositions so that the sum decreases on each turn, which must eventually terminate.

The other approach is as follows. For convenience, we subtract 1 from each square so that it starts at 0. We can take the expansion in base n, so the value of each square s is given by $nf(s) + g(s)$ where $0 \leq f(s), g(s) < n$. If we let R denote the set of red numbers and B the set of black numbers, note that $\sum_{s \in R} f(s) = \sum_{s \in B} f(s)$ since the number of red and black squares in each row is the same. Similarly, $\sum_{s \in R} g(s) = \sum_{s \in B} g(s)$ since the number of red and black squares in each column is the same. It follows that

$$\sum_{s \in R} nf(s) + g(s) = \sum_{s \in B} nf(s) + g(s).$$

$\square$

§1.8 B2 - Algebra

Find all pairs $(x, y) \in \mathbb{R}^2$ satisfying the system

$$\begin{aligned}\frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2)\\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4).\end{aligned}$$

Proof. An easy algebra exercise. Expanding the right-hand sides, then adding/subtracting the equations, we obtain $(x + y)^5 = 3$ and $(x - y)^5 = 1$ respectively. This has a unique solution in $\mathbb{R}^2$, $\left(\frac{3^{1/5}+1}{2}, \frac{3^{1/5}-1}{2}\right)$. □

§1.9 B3 - Analysis

For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate $\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle}+2^{-\langle n \rangle}}{2^n}$.

Proof. We reindex the sum by summing over the fixed values of $\langle n \rangle$, which is non-decreasing. Namely, we have

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle}+2^{-\langle n \rangle}}{2^n} = \sum_{m=1}^{\infty} \sum_{\langle n \rangle = m} \frac{2^{\langle n \rangle}+2^{-\langle n \rangle}}{2^n} = \sum_{m=1}^{\infty} \left((2^m + 2^{-m}) \sum_{\langle n \rangle = m} 2^{-n} \right).$$

Note that $\langle n \rangle = m$ whenever $n \in ((m-1/2)^2, (m+1/2)^2) = (m^2-m+1/4, m^2+m+1/4)$. This happens for $n \in [m^2-m+1, m^2+m]$. Then, note that

$$\sum_{n=m^2-m+1}^{m^2+m} 2^{-n} = (1-2^{-m^2-m}) - (1-2^{-m^2+m}) = 2^{-m^2+m} - 2^{-m^2-m}.$$

Combining the results, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} (2^m + 2^{-m})(2^{-m^2+m} - 2^{-m^2-m}) &= \sum_{m=1}^{\infty} 2^{-m^2+2m} - 2^{-m^2-2m} \\
&= \sum_{m=1}^{\infty} 2^{-m(m-2)} - 2^{-m(m+2)} \\
&= \sum_{m=1}^{\infty} 2^{-m(m-2)} - \sum_{m=3}^{\infty} 2^{-(m-2)m} \\
&= 2^1 + 2^0 = 3.
\end{aligned}$$

□

§1.10 B4 - Number Theory

Let $S = \mathbb{Q} \setminus \{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset.$$

Proof. The claim is true. Suppose we had $x = \frac{p}{q} \in \bigcap_{n=1}^{\infty} f^{(n)}(S)$, where $(p, q) = 1$. Then, there is some $m \in \mathbb{N}$ so that $f^{(m)}(x) = x$. However, note that

$$f\left(\frac{p}{q}\right) = \frac{p^2 - q^2}{pq},$$

and $|pq| > |q|$ since $p \notin \{-1, 0, 1\}$. Furthermore, note that $(p^2 - q^2, pq) = 1$. If there is some prime r diving both $p^2 - q^2$ and pq, then we have that r divides one of $p - q, p + q$ and one of p, q. However, from these we could conclude that r divides both p, q which contradicts the fact that $(p, q) = 1$. It follows that since the denominators strictly increase, we cannot have $f^{(m)}(x) = x$. □

§1.11 B5 - Algebra

let $a, b \in (0, 1/2)$ and let g be a ontinuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Proof. First, note that g is injective. This is because $g(x) = g(y)$ implies that $g(g(x)) = g(g(y))$, which implies that $ag(x) + bx = ag(y) + by$, which implies that $x = y$. Since g is continuous and injective, it follows that g is monotone.

We claim that g is unbounded, which implies that it is surjective. Suppose $|g(x)| \leq M$ for all $x \in \mathbb{R}$. Then,

$$(a+1)M \geq |g(g(x)) - ag(x)| = |bx|,$$

which is a contradiction since bx is unbounded.

Now, let $x_0 \in \mathbb{R}$ be arbitrary and define $x_{n+1} = g(x_n)$, $x_{n-1} = g^{-1}(x_n)$. The original functional equation gives a linear recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n.$$

The corresponding characteristic polynomial is $\lambda^2 - a\lambda - b$, which has two distinct roots $\lambda_\pm = \frac{a \pm \sqrt{a^2+4b}}{2}$, since $a, b \in (0, 1/2)$. It follows that

$$x_n = c_+\lambda_+^n + c_-\lambda_-^n$$

for constants c_+, c_-.

Note that $\lambda_+ > 0, \lambda_- < 0$ and $1 > |\lambda_+| > |\lambda_-|$.

Suppose f is monotone increasing(the case where f is monotone decreasing is similar). If $c_- \neq 0$ then as n gets sufficiently small, λ_-^n dominates λ_+^n, so there is some large enough n so that $0 < x_n < x_{n+2}$ and $x_{n+3} < x_{n+1} < 0$, which would give that $g(x_n) > g(x_{n+2})$, a contradiction. It follows that $c_- = 0$, so we have $x_0 = c_+$ and $g(x_0) = x_1 = c_+\lambda_+ = \lambda_+ x_0$, which gives the result.

□

§1.12 B6 - Algebra/Combinatorics

Assume $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

Proof. Let $A = \mathrm{Conv}\{(n, a_n) : n \in \mathbb{N}\}$ and let ∂A denote the set of points on the boundary of the convex hull.

We claim that ∂A contains infinitely many elements. Suppose not. Then, ∂A has a last point (N, a_N). If we let $m = \sup_{n>N} \frac{a_n - a_N}{n-N}$, the slope of the line between (N, a_N) and (n, a_n), then the line through (N, a_N) with slope m lies above(or contains) each point (n, a_n) for $n > N$. However, since $a_n/n \to 0$ and a_N, N are fixed, we have that

$$\frac{a_n - a_N}{n - N} \to 0.$$

This implies that the set of slopes attains a maximum, i. e. there is some point (M, a_M) with $M > N$ so that $m = \frac{a_M - a_N}{M-N}$. But then, we must also have that $(M, a_M) \in \partial A$, contradicting the fact that (N, a_N) is the last point in ∂A.

For each point on the boundary $(n, a_n) \in \partial A$, we must have that midpoint of the line through $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ for $i \in [n-1]$ must lie below (n, a_n). From this, it follows that $a_n > \frac{a_{n-i}+a_{n+i}}{2}$, which implies the result. □

§2 Putnam - 2002

§2.1 A1 - Algebra

Let $k \in \mathbb{N}$. The n-th derivative of $1/(x^k - 1)$ has the form $P_n(x)/(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Proof. We can write

$$\begin{aligned}\frac{P_n(x)}{(x^k-1)^{n+1}} &= \frac{d}{dx}\left(\frac{P_{n-1}(x)}{(x^k-1)^n}\right)\\ &= \frac{(x^k-1)^n P'_{n-1}(x) - nkx^{k-1}(x^k-1)^{n-1}P_{n-1}(x)}{(x^k-1)^{2n}}\\ \implies P_n(x) &= (x^k-1)P'_{n-1}(x) - nkx^{k-1}P_{n-1}(x).\end{aligned}$$

Plugging in $x = 1$ gives a recurrence relation $P_n(1) = -nkP_{n-1}(x)$. It follows that

$$P_n(1) = n!(-k)^n P_0(x) = n!(-k)^n.$$

□

§2.1.1 Official Solution

An alternate solution comes from expanding $\frac{1}{x^k-1}$ in a Laurent series around 1.

Proof. It suffices to keep track of the $O((x-1)^{-1})$ terms since the others vanish upon plugging in 1.

Note that

$$\frac{1}{x^k-1} = \frac{1}{k(x-1)+\dots} = \frac{1}{k}(x-1)^{-1} + \dots.$$

Taking the n-th, derivative, we have obtain

$$\frac{d^n}{dx^n}\frac{1}{x^k-1} = \frac{(-1)^n n!}{k(x-1)^{-n-1}} + \dots.$$

It follows that

$$\begin{aligned}P_n(x) &= (x^k-1)^{n+1}\frac{d^n}{dx^n}\frac{1}{x^k-1}\\ &= (k(x-1)+\dots)^{n+1}\left(\frac{(-1)^n n!}{k(x-1)^{-n-1}} + \dots\right)\\ &= k^n(-1)^n n! + \dots.\end{aligned}$$

□

§2.2 A2 - Combinatorics

Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Proof. Draw a great circle through any two points and consider the remaining three. By the pigeonhole principle, there is closed hemisphere with at least two points, and choosing this hemisphere gives the result. □

§2.3 A3 - Combinatorics

Let $n \geq 2$ be an integer and T_n the number of nonempty subsets S of $\{1, 2, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Proof. Note that each one element subset $\{1\}, \{2\}, \ldots, \{n\}$ has the property that the average of the element is an integer. It suffices to consider the subsets at least 2 elements. For set of size at least 2, we can pair them into $(S, S \cup \{a\})$, where $a \notin S$ and the average of the elements in S is a. Each subset is contained in exactly one pair, so each of them don't contribute to the parity of $T_n - n$. It follows that $T_n - n$ is even as desired. $\square$

§2.4 A4 - Combinatorics

in Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Who wins and how?

Proof. Player 0 wins. After exchanging rows and columns(which doesn't change the norm of the determinant), we can assume without loss of generality that player 1 enters a 1 in the a_{11} square.

In the optimal strategy, player 0 enters a 0 in the a_{22} square. There are 3 possible cases to check for player 1's next move:

1. a_{12} or a_{21},
2. a_{13} or a_{31},
3. a_{23} or a_{32},
4. a_{33}.

For each of these cases, it suffices to check the first since we can exchange a_{ij} with a_{ji} in order to obtain the strategy in the other corresponding case. Note that if player 0 creates a row/column of 0's or a 2×2 block of 0's, the determinant of the matrix will be 0.

(1) When player 1 enters a 1 in a_{12}, player 0 enters a 0 in square a_{32}. If player 1 enters in a_{21} or a_{23}, player 0 enters in a_{33} or a_{31} respectively. In this position, player 1 cannot stop player 0 from creating a 2×2 block or a row of 0's. Alternatively, if player 1 enters in a_{13}, player 0 enters in a_{21}. In this position, player 1 cannot stop player 0 from creating a 2×2 block or a row of 0's. In the other cases, we take a knight's move across whatever player 1 plays and in this position, player 1 cannot stop player 0 from creating a 2×2 block or a row of 0's.

The other cases follow a similar analysis, creating a triangle block of 0's in other to create two threats. □

§2.5 A5 - Number Theory

Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set $\{a_n/a_{n+1} : n \geq 0\}$.

Proof. We proceed by induction on $k = \max\{p, q : \gcd(p, q) = 1\}$. For $k = 1$, we know that $a_0 = a_1 = 1$ so we have $1/1$, which contains all the rational numbers p/q with $\max\{p, q\} \leq 1$. Suppose the set contains all the rationals p/q with $\max\{p, q : \gcd(p, q) = 1\} \leq n$. Then it contains $\frac{n+1-k}{k}$ for $1 \leq k \leq n$ whenever $\gcd(n+1-k, k) = 1$. Note that $\gcd(n+1-k, k) = 1 \Leftrightarrow \gcd(n+1, k) = 1$.

It follows that we have $a_m = \ell(n+1-k), a_{m+1} = \ell(k)$ for some $\ell \in \mathbb{N}$. Then, $a_{2m+1} = \ell(n+1-k)$, $a_{2n+2} = \ell(n+1)$ and $a_{2n+3} = \ell(k)$. It follows that the set contains

$$\frac{n+1-k}{n+1}, \frac{n+1}{k}$$

for each $1 \leq k \leq n$ so that $\gcd(n+1, k) = 1$, which proves the inductive step. □

§2.6 A6 - Analysis

Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does the sum $\sum_{n\geq 1} 1/f(n)$ converge?

Proof. The sum converges for $b = 2$ and diverges for $b \geq 3$.

We first consider $b \geq 3$. Suppose the sum converges. Note that we can write

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} = \sum_{d=1}^{\infty} \frac{1}{f(d)} \sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n}.$$

Note that $\sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n}$ is a left-endpoint Riemann approximation for the integral $\int_{b^{d-1}}^{b^d} \frac{1}{x}$ and the function $\frac{1}{x}$ is monotonically decreasing on this interval so it follows that

$$\sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n} > \int_{b^{d-1}}^{b^d} \frac{1}{x} = \log b.$$

However, this implies that

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} > \log b \sum_{d=1}^{\infty} \frac{1}{f(d)},$$

which is a contradiction since $\log b > 1$.

Now, we show that the sum converges in the case of $b = 2$. Let $C = \log 2 + \frac{1}{8} < 1$. We prove by induction that for each $m \in \mathbb{N}$,

$$\sum_{n=1}^{2^m-1} \frac{1}{f(m)} < 1 + \frac{1}{2} + \frac{1}{6(1-C)} = L.$$

For $m = 1, 2$, the result is clear. Suppose it is true for all $m \in \{1, 2, \ldots, N-1\}$. Note that

$$\sum_{n=1}^{2^N-1} \frac{1}{f(n)} = 1 + \frac{1}{2} + \frac{1}{6} + \sum_{d=3}^{N} \frac{1}{f(d)} \sum_{n=2^{d-1}}^{2^d-1} \frac{1}{n}.$$

Then, using a right-endpoint Riemann approximation, we have

$$\begin{aligned} \sum_{n=2^{d-1}}^{2^d-1} \frac{1}{n} &= \frac{1}{2^{d-1}} - \frac{1}{2^d} + \sum_{n=2^{d-1}+1}^{2^d} \frac{1}{n} \\ &< 2^{-d} + \int_{2^{d-1}}^{2^d} \frac{dx}{x} \\ &< \frac{1}{8} + \log 2 = C. \end{aligned}$$

It follows that

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{6} + \sum_{d=3}^{N} \frac{1}{f(d)} &< 1 + \frac{1}{2} + \frac{1}{6} + C\sum_{d=3}^{N} \frac{1}{f(d)} && (1) \\ &< 1 + \frac{1}{2} + \frac{1}{6} + \frac{C}{6(1-C)} && (2) \\ &= 1 + \frac{1}{2} + \frac{1}{6(1-C)} = L, && (3) \end{aligned}$$

where we used the strong induction hypothesis to obtain (2). □