# USAJMO 2011 - Problems and Solutions

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## §1 Problems

**Problem 1.1** (USAJMO 2011/1)**.** Find, with proof, all positive integers $n$ for which $2^n + 12^n + 2011^n$ is a perfect square.

**Problem 1.2** (USAJMO 2011/2)**.** Let $a, b, c$ be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

**Problem 1.3** (USAJMO 2011/3)**.** For a point $P = (a, a^2)$ in the coordinate plane, let $l(P)$ denote the line passing through $P$ with slope $2a$.. Consider the set of triangles with vertices of the form $P_1 = (a_1, a_1^2), P_2 = (a_2, a_2^2), P_3 = (a_3, a_3^2)$, such that the intersection of the lines $l(P_1), l(P_2), l(P_3)$ form an equilateral triangle $\triangle$. Find the locus of the center of $\triangle$ as $P_1P_2P_3$ ranges over all such triangles.

**Problem 1.4** (USAJMO 2011/4)**.** A word is defined as any finite string of letters. A word is a palindrome if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, ...$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, $W_n$ is the word formed by writing $W_{n-2}$ followed by $W_{n-1}$. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, W_3, ..., W_n$ in succession is a palindrome.

**Problem 1.5** (USAJMO 2011/5)**.** Points $A, B, C, D, E$ lie on a circle $\omega$ and point $P$ lies outside the circle. The given points are such that (i) lines $PB$ and $PD$ are tangent to $\omega$, (ii) $P, A, C$ are collinear, and (iii) $DE \parallel AC$. Prove that $BE$ bisects $AC$.

**Problem 1.6** (USAJMO 2011/6)**.** Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing $2^{2^n}$ by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

# §2 Solutions

**Problem 1** (USAJMO 2011/1)

Find, with proof, all positive integers $n$ for which $2^n + 12^n + 2011^n$ is a perfect square.

*Proof.* The only possible value of $n$ is $n = 1$. Note that $2^1 + 12^1 + 2011^1 = 2025 = 45^2$ is a perfect square.

Suppose there exists $m > 1$ so that $2^m + 12^m + 2011^m$ is a perfect square. Then

$$2^m + 12^m + 2011^m \equiv 2^m + (-1^m) \equiv (-1)^m \pmod 4.$$

Since the quadratic residues of 4 are 0 and 1, $m$ must be even so that $2^m + 12^m + 2011^m \equiv 1 \pmod 4$. Furthermore,

$$2^m + 12^m + 2011^m \equiv (-1)^m + (1)^m \equiv 1 + (-1)^m \pmod 3.$$

Since the quadratic resides of 3 are 0 and 1, $m$ must be odd so that $2^m + 12^m + 2011^m \equiv 1 + (-1) \equiv 0 \pmod 3$. However, $m$ cannot be simultaneously odd and even. Therefore, there $n = 1$ is the only value of $n$ so that $2^n + 12^n + 2011^n$ is a perfect square. $\square$

**Problem 2** (USAJMO 2011/2)

Let $a, b, c$ be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

*Proof.* Note that

$$\begin{aligned} a^2 + b^2 + c^2 + (a + b + c)^2 &= 2a^2 + 2b^2 + 2c^2 + 2ab + 2bc + 2ca \\ &= (a^2 + 2ab + b^2) + (b^2 + 2bc + c^2) + (c^2 + 2ca + c^2) \\ &= (a + b)^2 + (b + c)^2 + (c + a)^2. \end{aligned}$$

Let $x = a+b, y = b+c, z = c+a$. Then the given condition is equivalent to $x^2+y^2+z^2 \leq 4$ which implies $4 - y^2 - z^2 \geq x^2$.

Also note that $a = \frac{x-y+z}{2}, b = \frac{x+y-z}{2}, c = \frac{-x+y+z}{2}$.

Thus, we have

$$\begin{aligned} \sum_{\text{cyc}} \frac{ab + 1}{(a + b)^2} &= \sum_{\text{cyc}} \frac{4 + (x - y + z)(x + y - z)}{4x^2} \\ &= \sum_{\text{cyc}} \frac{4 + (x - (y - z))(x + (y - z))}{4x^2} \\ &= \sum_{\text{cyc}} \frac{4 + x^2 - (y - z)^2}{4x^2} \\ &= \sum_{\text{cyc}} \frac{4 - y^2 - z^2 + x^2 + 2yz}{4x^2} \\ &\geq \sum_{\text{cyc}} \frac{2x^2 + 2yz}{4x^2} \\ &= \sum_{\text{cyc}} \frac{1}{2} + \frac{1}{2}\frac{yz}{x^2} \\ &= \frac{3}{2} + \frac{1}{2}\sum_{\text{cyc}} \frac{yz}{x^2}, \end{aligned}$$

and

$$\sum_{\text{cyc}} \frac{yz}{x^2} = \sum_{\text{cyc}} \frac{(b + c)(c + a)}{(a + b)^2} \geq 3\sqrt[3]{\prod_{\text{cyc}} \frac{(b + c)(c + a)}{(a + b)^2}} = 3,$$

from the AM-GM Inequality.

Therefore,

$$\sum_{\text{cyc}} \frac{ab + 1}{(a + b)^2} \geq \frac{3}{2} + \frac{1}{2}(3) = 3,$$

as desired. □

**Problem 3** (USAJMO 2011/3)

For a point $P = (a, a^2)$ in the coordinate plane, let $l(P)$ denote the line passing through $P$ with slope $2a$.. Consider the set of triangles with vertices of the form $P_1 = (a_1, a_1^2), P_2 = (a_2, a_2^2), P_3 = (a_3, a_3^2)$, such that the intersection of the lines $l(P_1), l(P_2), l(P_3)$ form an equilateral triangle $\triangle$. Find the locus of the center of $\triangle$ as $P_1P_2P_3$ ranges over all such triangles.

*Proof.* Note that $l(P) = a^2 + 2a(x - a) = 2ax - a^2$.

Therefore, we have

$$l(P_1) = 2a_1x - a_1^2,$$
$$l(P_2) = 2a_2x - a_2^2,$$
$$l(P_3) = 2a_3x - a_3^2.$$

Let $Q_1 = l(P_1) \cap l(P_2), Q_2 = l(P_2) \cap l(P_3), Q_3 = l(P_3) \cap l(P_1)$. Solving for the intersection points, we have

$$Q_1 = \left(\frac{a+b}{2}, ab\right), Q_2 = \left(\frac{b+c}{2}, bc\right), Q_3 = \left(\frac{c+a}{2}, ca\right).$$

Denote $H$ as the orthocenter and $G$ as the centroid of $\triangle Q_1Q_2Q_3$. If $\triangle Q_1Q_2Q_3$ is equilateral, then $H = G$.

Firstly,

$$G = \left(\frac{a+b+c}{3}, \frac{ab+bc+ca}{3}\right),$$

the average of the coordinates.

Let $l(H_3)$ and $l(H_1)$ denote the altitudes from $Q_3$ and $Q_1$ respectively. The slope of $\overline{Q_1Q_2} = \frac{a_1a_2 - a_2a_3}{\frac{a_1+a_2}{2} - \frac{a_2+a_3}{2}} = 2a_2$, which means the slope of $l(H_3)$ is $-\frac{1}{2a_2}$, which gives

$$l(H_3) = a_1a_3 - \frac{1}{2a_2}\left(x - \frac{a_1+a_3}{2}\right),$$

and similarly,

$$l(H_1) = a_1a_2 - \frac{1}{2a_3}\left(x - \frac{a_1+a_2}{2}\right).$$

Then,

$$H = l(H_3) \cap l(H_1) = \left(\frac{a_1+a_2+a_3+4a_1a_2a_3}{2}, -\frac{1}{4}\right).$$

Call the $x$ and $y$ coordinates of $G$, $x$ and $y$ respectively. Since $G = H$, $x = \frac{a_1+a_2+a_3}{3} = \frac{a_1+a_2+a_3+4a_1a_2a_3}{2}$, which gives $x = -4a_1a_2a_3$.

Also, $y = \frac{a_1a_2+a_2a_3+a_3a_1}{3} = -\frac{1}{4}$, so $-12a_1y = -4a_1^2a_2 - 4a_1a_2a_3 - 4a_1^2a_3 = 3a_1$, and since $-4a_1a_2a_3 = x$, we find $x(a_1) = \frac{3a_1 - 4a_1^3}{1-12a_1^2}$, where $x(a_1)$ denotes $x$ as a function of $a_1 \in \mathbb{R} - \{-\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}\}$.

Note that $\lim_{a_1 \to -\frac{1}{\sqrt{12}}^-} x = -\infty$, and $\lim_{a_1 \to \frac{1}{12}} x = \infty$. Therefore $x$ takes on all reals in the interval $\left(-\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}^+\right)$. Therefore, the locus of centers of the equilateral triangles is the line $y = -\frac{1}{4}$.

$\square$

**Problem 4** (USAJMO 2011/4)

A word is defined as any finite string of letters. A word is a palindrome if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, \ldots$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, $W_n$ is the word formed by writing $W_{n-2}$ followed by $W_{n-1}$. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, W_3, \ldots, W_n$ in succession is a palindrome.

*Proof.* We will use strong induction on $n$. We will let the $AB$ represent the word formed by writing the words $A$ and $B$ in succession. From the problem statement, $W_{k+1} = W_k W_{k-1}$. Let $W(n) = W_1 W_2 \ldots W_n$, and let $r(l_1 l_2 l_3 \ldots l_n) = l_n \ldots l_3 l_2 l_1$, where $l_1, l_2, \ldots, l_n$ are letters. Note that $r(XY) = r(Y)r(X)$ and $W_(n) = W_1 W_2 \ldots W_{n-2} W_{n-1} W_n = W(n-2) W_{n-1} W_n = W_{n-2} W_{n+1}$. It suffices to show $r(W(n)) = W(n)$ for all $n \in \mathbb{N}$.

First, note that $W(1) = W_1 = b$ is a palindrome. Also, $W(2) = W_1 W_2 = bab$ is a palindrome. Now, suppose $W(m) = r(W(m))$ for all $3 \leq m \leq k$. Then

$$\begin{aligned} r(W(k+1)) &= r(W(k)W_{k+1}) \\ &= r(W_{k+1})r(W(k)) \\ &= r(W_{k+1})W(k) \\ &= r(W_{k+1})W(k-2)W_{k+1}. \end{aligned}$$

So

$$\begin{aligned} r(r(W(k+1))) &= r(r(W_{k+1})W(k-2)W_{k+1}) \\ &= r(W_{k+1})r(W(k-2))r(r(W_{k+1})) \\ &= r(W_{k+1})W(k-2)W_{k+1}. \end{aligned}$$

But $r(r(W(k+1))) = W(k+1)$. Therefore, $W(k+1) = r(W(k+1))$, which means $W(n)$ is a palindrome for all $n \in \mathbb{N}$. $\square$

**Problem 5** (USAJMO 2011/5)

Points $A, B, C, D, E$ lie on a circle $\omega$ and point $P$ lies outside the circle. The given points are such that (i) lines $PB$ and $PD$ are tangent to $\omega$, (ii) $P, A, C$ are collinear, and (iii) $DE \parallel AC$. Prove that $BE$ bisects $AC$.

*Proof.* Let $\measuredangle$ denote directed angles (mod 180°). Let $O$ be the center of the circle, and let $F = BE \cap AC$.

Since $PB$ and $PD$ are tangents, $OB \perp BP$ and $OD \perp DP$, so $OBPD$ is cyclic.

Now,

$$\measuredangle BOP = \measuredangle BDP = \measuredangle BAD = \measuredangle BED = \measuredangle BFP,$$

where the last equality follows from $DE||AC$, and the others follow from the Inscribed Angles Theorem. Since $\measuredangle BOP = \measuredangle BFP$, $BOFP$ is cyclic, which means $OF \perp AC$. Since $OA = OC$, we must have $AF = FC$. Therefore, $BE$ bisects $AC$.

□

**Problem 6** (USAJMO 2011/6)

Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing $2^{2^n}$ by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

*Proof.* Let $r(a, b)$ denote the remainder upon dividing $a$ by $b$.

From long division,

$$2^{2^n} = (2^n - 1)(2^{2^n - n} + 2^{2^n - 2n} + ... + 2^{2^n - (k-1)n}) + 2^{2^n - kn},$$

where $k$ is the largest integer so that $2^n - kn \geq 0$. In other words, $2^n - kn = r(2^n, n)$. Thus, $r(2^{2^n}, n) = 2^{r(2^n, n)}$.

If $n$ is even, $r(2^n, n)$ is clearly even, so $r(2^{2^n}, n)$ will be a power of 4.

Thus, it suffices to check odd composite numbers. One quickly finds a counterexample $n = 25$, since $r(2^25, 25) = 7$, so $r(2^{2^n}) = 2^7$, which is not a power of 4.

$\square$