Math 258 Lecture Notes, Fall 2020

Harmonic Analysis

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§1.1 Introduction

We begin by considering the problem of conduction of heat in a circle. We use the map $x \mapsto e^{ix}, x \in [0, 2\pi)$. Where u is the temperature, t is the time, we believed that $u_t = \gamma u_{xx}$, where subscripts denote partial derivatives. We also have an initial condition, $f(x) = u(x, 0)$.

There are some simple solutions $e^{inx}e^{-\gamma n^2 t}|_{t=0} = e^{inx}$. The product of solutions, the sum of solutions, and scalar multiple of solutions are all solutions, so he wrote the solution as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, u(x,t) = \sum_n a_n e^{-\gamma n^2 t} e^{inx}.$$

§1.2 Fourier Analysis

We take a circle $\{z \in \mathbb{C} : |z = 1|\}$, which can also be thought of as $\mathbb{R}/(2\pi\mathbb{Z})$, with the map $x \mapsto e^{ix}$. Suppose we have G a finite abelian group, and $\widehat{G} = \{\text{hom } \varphi : G \to \mathbb{R}/\mathbb{Z}\}$, the dual group. $\widehat{G}$ is also a group, formally known as the set of characters.

Example 1.1

If we take $G = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, with the map $x \mapsto e^{2\pi i x n/N}$, for $n \in Z_n$.

Similarly, taking $G \cong Z_{N_1} \times \mathbb{Z}_{N_2} \times \dots$, we take $x \mapsto \prod e^{2\pi i x n/N_i}$.

Take $e_\xi(x) = e^{2\pi i \xi(x)}$, where $\xi : G \mapsto \mathbb{R}/\mathbb{Z}$. Working in $L^2(G)$, we note the following:

Fact 1.2. If $\xi \neq \varphi$, then $\langle e_\xi, e_\varphi \rangle = 0$.

Proof.

$$\sum_{x \in G} \xi(x)\overline{\varphi(x)} = \sum_u \xi(u+y)\overline{\varphi(u+y)} - \left(\sum_u \xi(u)\overline{\varphi(u)}\right) \xi(y)\overline{\varphi(u)}.$$

Hence, either $\langle \xi, \varphi \rangle = 0$ or $\xi(y)\overline{\varphi}(y) = 1$ for all $y \in G$, which implies $\xi = \varphi$. □

If follows that $\{e_f : f \in \widehat{G}\}$ is an orthonormal set in $L^2(G)$ Then, the dimension is $|\widehat{G}| = |G| = \dim(L^2(G))$. Hence, the set forms an orthonormal basis for $L^2(G)$.

Then, for all $f \in L^2(G)$, we have

$$\|f\|^2_{L^2(G)} = \sum_{\varphi \in \widehat{G}} |\langle f, e_\xi \rangle|^2,$$

$$f = \sum_{e_\xi \in \widehat{G}} \langle f, e_\xi \rangle e_\varphi.$$

§1.3 On Tori of Arbitrary Dimension

We define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, from $[0, 2\pi]$. We then work on $\mathbb{T}^d$, $d \geq 1$.

For $f \in L^2(\mathbb{T}^d)$, we define

$$\widehat{f}(n) = (2\pi)^{-d} \int f(x)e^{-inx}dx.$$

We have an inner product $\langle f, g \rangle = \int_{\mathbb{T}^d} f(x)\overline{g(x)}d\mu(x)$ defined over a Lebesgue measure or Euclidean measure on $\mathbb{T}^d$.

Theorem 1 (Parseval's Theorem)
For all $f \in L^2(\Pi^d)$,

$$\|f\|_{L^2}^2 = (2\pi)^d \sum_{n \in \mathbb{Z}^d} |\widehat{f}(n)|^2,$$

and we have

$$f = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx},$$

in the sense that

$$\|f - \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx}\|_L^2 \to 0.$$

Note: you can usually figure out the constant with the simplest example, $f = 1$.

Proof. Take $\mathbb{T}^d, e_n(x) = e^{in \cdot x}$. The $\{(2\pi)^{-d/2} e^n : n \in \mathbb{Z}^d\}$ is orthonormal(left as an exercise). Then, for all f, $\sum_n \langle f, (2\pi)^{-d/2} e_n \rangle \leq \|f\|_{L^2}^2$, with equality if the set is a basis(Bessel's inequality).

It suffices to show that $\operatorname{span}\{e_n\}$ is dense in L^2. Take $P = \operatorname{span}\{e_n\}$, and note that P is an algebra of continuous functions on Π^d, closed under conjugation, contains 1, and separates points. Hence, the Stone-Weierstrass theorem implies that P is dense in $C^o(\Pi^d)$ with respect to $\|\cdot\|_{C^o}$. Then $C^o \subset L^2$ is dense(general theory about Compact Hausdorff spaces, Radon Measures).

The statement $\|f - \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx}\|_L^2 \to 0$ follows from the general theory of orthonormal systems. $\square$

§1.4 Euclidean Spaces

We work in $\mathbb{R}^d, (d \geq 1)$. Take $\xi \in R^d$, and $x \mapsto x\xi \in \mathbb{R}$ is a homomorphism from $\mathbb{R}^d \to \mathbb{R}$, but if we take $x \mapsto e^{ix\xi}$, we have a homomorphism from $\mathbb{R}^d \mapsto \Gamma$. We try to define the following:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx = \langle f, e_\xi \rangle_{L^2(\mathbb{R}^d)},$$

where $e_{xi}(x) = e^{ix\xi}$.

Some problems:

1. $e_\xi \notin L^2(\mathbb{R}^d)$
2. $f(x)e^{-ix\xi}$ need not be in L^1 if $f \in L^2$.

We fix this by imposing extra conditions.

Definition 1.3. For $f \in L^1(\mathbb{R}^d)$, we define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx.$$

Note that $f \in L^1$ implies that $\widehat{f}$ is bounded, continuous. We see this as follows: $\widehat{f}(\xi + u) - \widehat{f}(\xi) = \int f(x) e^{-ix\xi}(e^{-ixu} - 1) dx$. If we let $u \to 0$, the right goes to 0 pointwise, and $(2|f|) \in L^1$ dominates the integral, it goes to 0.

Proposition 1.4
If $f \in L^1 \cap L^2(\mathbb{R}^d)$, $\widehat{f} \in L^2(\mathbb{R}^d)$,
$$\|\widehat{f}\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2.$$

Theorem 2 (Plancherel's Theorem)
$\pi : L^1 \cap L^2 \to L^2$ extends uniquely to $\widehat{\pi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, linear, bounded, $\|\widehat{\pi} f\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2$, and for all $f \in L^2$, we have an inverse Fourier Transform, $\check{f}(y) = \int f(\xi) e^{iy\xi} d\xi$ for $f \in L^1 \cap L^2$, and $\check{\cdot}$ also extends.

Finally,
$$\|f - (2\pi)^{-d} \int_{|\xi| \le R} \widehat{f}(\xi) e^{ix\xi} d\xi \|_{L^2} \to 0.$$

Note that $\check{f}(y) = \widehat{f}(-y)$.

Proof. We first prove that $\|f\|_{L^2}^2 = (2\pi)^{-d} \|\widehat{f}\|_{L^2}^2$ for all $f \in L^1 \cap L^2$. We prove this for a dense subspace $\mathscr{P}$ of L^2. We will show later that there exists a subspace $V \subset L^2(\mathbb{R}^d)$ so that V is dense in L^2, $V \subset L^1$, $\forall f \in V$, there exists $C_f < \infty$, so for all $\xi \in \mathbb{R}^d$, $|\widehat{f}(\xi)| \le C_f (f(\xi))^{-d}$ and f is continuous with compact support.

We are given $f : \mathbb{R}^d \to \mathbb{C}$ supported where $|x| \le R = R_f < \infty$. For large $t \ge 0$, define $f_t(x) = f(tx)$(this shrinks the support of f), supported where $|x| \le R/t < \pi$. We can then think of $f_t : \mathbb{T}^d \to \mathbb{C}$.

Now, we calculate
$$\begin{aligned} \widehat{f_t}(n) &= (2\pi)^d \int_{\mathbb{T}^d} f_t(x) e^{-inx} dx \\ &= t^{-d} (2\pi)^d \int_{R^d} f(x) e^{-in/ty} dy \\ &= t^{-d} (2\pi)^{-d} \widehat{f}(t^{-1} n), \end{aligned}$$

where the first hat is on $\mathbb{T}^d$ and the second is on $\mathbb{R}^d$, so the Fourier coeficients in the euclidean case are scalar multiples of the Fourier coefficients in the Tori case.

Thus,
$$\|f_t\|_{L^2(\mathbb{T}^d)}^2 = t^{-d} \|f\|_{L^2(\mathbb{R}^d)}^2 = c_d \sum_{n \in \mathbb{Z}^d} |\widehat{f_t}(n)|^2 = c_d' t^{-2d} \sum_n |\widehat{f}(t^{-1} n)|^2$$

Hence,
$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n |\widehat{f}(t^{-1} n)|^2.$$

This has a nice tiling Riemann sum interpretation: if we take $\mathbb{R}^d$ and tile it with cubes of sidelength $1/t$ where one corner is at $t^{-1} n$ for $n \in \mathbb{Z}^d$, then
$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n \left| \widehat{f}(t^{-1} n) \right|^2 = \int_{\mathbb{R}^d} |g_t|^2 dx,$$

where $g(x) = \widehat{f}(t^{-1} n)$.

We claim
$$\int_{\mathbb{R}^d} |g_t|^2 \to \int_{\mathbb{R}^d} |\widehat{f}|^2,$$
which follows from the dominated convergence theorem: where we take a sequence over t going to infinity, with dominator $C_f^2(1+|\xi|)^{-2d}$ in L^1 and $|\widehat{f}(\xi)| \leq C_f^2(1+|\xi|)^{-2d}$. Furthermore, we have $g_t(\xi) \to \widehat{f}(\xi)$ as $t \to 0$, and $\widehat{f}$ is continuous so g_t is pointwise convergent, and we have
$$|g_t(\xi)| = |\widehat{f}(t^{-1}n)| \leq C_f(1+|t^{-1}n|)^{-d} \leq C'(1+|\xi|)^{-d}.$$

$\square$

§2 September 1st, 2020

§2.1 Proof of Plancherel's Theorem

Last time

- $\mathbb{R}^d$,
$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx.$$
- $V = (f \in L_1 \cap L_2(\mathbb{R}^d)) : |\widehat{f}(\xi)| \langle\xi\rangle^d$ is a bounded linear function, $\langle x\rangle = (1+|x|^2)^{1/2} \geq 1, = |x|$ for x large.
- Claim: V is dense in $L^2(\mathbb{R}^d)$. Then $\|\widehat{f}\|_{L^2} = (2\pi)^{d/2}\|f\|_{L^2}$ for all $f \in V$ so there exists a unique bounded linear operator $\mathscr{F}$ on $L^2(\mathbb{R}^d)$, where $\mathscr{F}$ takes a function to it's fourier transform.
- We discussed some properties of $\mathscr{F}$.
  - $\|\mathscr{F} f\|_2 = (2\pi)^{d/2}\|f\|_2$
  - $\mathscr{F}$ is onto.
  - For all $f \in L^2$,
  $$\left\| f - (2\pi)^{-d} \int_{|\xi|\leq R} e^{ix\cdot\xi} \mathscr{F}(f)(\xi) d\xi \right\|_{L^2} \to 0,$$
  in the limit where $R \to \infty$.

First note that $\mathscr{F}$ has closed range(this was an exercise). It suffices to show: If $g \in L^2, g \perp \mathscr{F}(f)$ for all $f \in V$, then $g = 0$.

Proof. First, note that
$$0 = \langle g, \mathscr{F}(f)\rangle = \langle \mathscr{F}^*(g), f\rangle,$$
and for all $g \in V$,
$$\mathscr{F}^* g(x) = \int g(\xi) e^{ix\cdot\xi} d\xi$$
Therefore, $\mathscr{F}^*(g)(x) = (\mathscr{F} g)(-x)$ for all $g \in V$, which is dense in L^2. Hence, $\mathscr{F} g = 0$, and the Fourier transform preserves norms, so $g = 0$. □

We also claimed the following: Let $f \in L^2$:
$$\|f(x) - (2\pi)^{-d} \int_{|\xi|\leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi\|_2^2 \to 0.$$

Proof. Let $g_r = (2\pi)^{-d} \int_{|\xi|\leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi$.We have to show $\langle f, g_r\rangle \to \|f\|_2^2$. Then
$$\|f - g_r\|_2^2 = \|f\|_2^2 + \|g_r\|_2^2 - 2\mathrm{Re}\langle f, g_r\rangle \to \|f\|_2^2 + \|f\|_2^2 - 2\|f\|_2^2.$$

$$\begin{aligned}
\langle f, g_R\rangle &= (2\pi)^{-d} \int f(x) \overline{\int_{|\xi|\leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi dx} \\
&= (2\pi)^{-d} \int_{|\xi|\leq R} \left(\int f(x) e^{-ix\cdot\xi} dx \right) \overline{(\mathscr{F} f)(\xi) d\xi} \\
&= (2\pi)^{-d} \int_{|\xi|\leq R} |\mathscr{F} f(\xi)|^2 d\xi \to (2\pi)^{-d} \|\mathscr{F} f\|_2^2 = \|f\|_2^2.
\end{aligned}$$

However, it's not clear that we can use Fubini's theorem. We would need $f \in L^1 \cap L^2$. But this is not an issue as $L^1 \cap L^2 \subset L^2$ is dense, so if we let $\epsilon > 0, f = G + h, \|h\|_2 \leq \epsilon$ and $G \in L^1 \cap L^2$. Showing the convergence from here is an exercise. □

We still need $V = (f \in L^1 \cap L^2 : \langle \xi \rangle^d (\widehat{f}(\xi))$ is bounded$)$ is dense in L^2. We'll discuss this in the future.

§2.2 Introduction to Convolution

Our meta definition is $f * g(x) = \int f(x-y)g(y)dy$, but it will depend on the conditions of the function for the integral to be defined.

Convolution is generally associated to a group, where

$$\int_G f(xy^{-1}g(y)d\mu(y)),$$

with the Haar measure(done in 202b).

If we substitute $y = x - u$, then

$$f * g(x) = \int f(u)g(x-u)du = g * f(x).$$

It is also associative: $(f * g) * g = f * (g * h)$ for all f, g, h(involves Fubini's theorem).

We can formally write

$$f * g(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u)g(v)d\lambda_x(u,v),$$

where λ_x is supported on $\Lambda = \{(u,v) : u + v = \lambda\}$(an affline subspace). If we have a subset $E \subset \Lambda$, $\lambda_x(E) = |\pi_1(E)| = |\pi_2(E)|$, where π_i are Lebesgue measure s of projections on the i-th factor. Note the following: suppose that f, g are continuous with compact support. Then $\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)$, where $A + B = \{a + b : (a,b) \in A \times B\}$.

Let $T : C_0^0(\mathbb{R}^d) \to C_b^0(\mathbb{R}^d)$ be bounded, linear and $T \circ \tau_y = \tau_y \circ T$ for all $x \in \mathbb{R}^d$ ($\tau_y f(x) = f(x+y)$, a translation). Then, there exists a Complex Radon measure μ on $\mathbb{R}^d$ so that for all $f \in C_0^0$, $T(f) = f * \mu$, where

$$f * \mu(x) = \int f(x-y)d\mu(y).$$

In the case of $\mathbb{T}^1$, $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}$ for all $f \in L^2$. Suppose we wanted to consider the partial sums,

$$\sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = S_N(f)(x).$$

In what sense does $S_N f \to f$, and for which functions f do we have convergence?

$$S_N(f)(x) = \sum_{n=-N}^{N} e^{inx}(2\pi)^{-1}\int_{-\pi}^{\pi} f(y)e^{-iny}dy = (2\pi)^{-1}\int f(y)\sum_{n=-N}^{N} e^{in(x-y)}dy$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi} f(y)D_n(x-y)dy.$$

The Dirichlet Kernels, $D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin{(N+1/2)x}}{\sin{(x/2)}}$ if $x \neq 0$ or $D_N(x) = 2N+1$ if $x = 0$.

§2.3 General Convolution

Theorem 3

Let $f, g \in L^1(\mathbb{R}^d)$. Then, the following are true:

- $y \mapsto f(x-y)g(y) \in L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- If $f, g \geq 0$, then $\|f * g\|_1 = \int f * g = \int f \int g$.
- The operation commutative and associative, so L^1 is an algebra, but it no multiplicative identity, so no inverses.
- For $f, g \in L^1$, $\widehat{(f \star g)} = \widehat{f} \cdot \widehat{g}$.

In other words, convolution is a nice bilinear operation.

Proof. Let $F(x, y) = f(x-y)g(y)$, $F : \mathbb{R}^{d+d} \to \mathbb{C}$ is Lebesgue measurable. We claim that $F \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. It follows from

$$\int |F(x,y)|dxdy = \int |f(x-y)||g(y)|dxdy = \int |g(y)|dy \int |f(x)|dx = \|g\|_1 \|f\|_1 < \infty.$$

Now, $F \in L^1$, so by Fubini's theorem, for almost every $x, y \to f(x-y)g(y) \in L^1$ and $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.

$$\|f*g\|_1 = \int |f*g(x)|dx = \int \left| \int f(x-y)g(y)dy \right| dx \leq \int \int |f(x-y)||g(y)|dydx = \|f\|_1 \|g\|_1.$$

Note that $\int (f * g)(x)dx = \|f\|_1 \|g\|_1$, for non-negative functions.

Finally,

$$\begin{aligned}
(f * g)^\wedge(\xi) &= \int e^{-ix\cdot\xi} \left(\int f(x-y)g(y)dy \right) dx \\
&= \int \left(\int e^{-ix\cdot\xi} f(x-y)dx \right) dy, x = u + y \\
&= \int \left(e^{-i(u+y)\cdot\xi} f(u)du \right) g(y)dy \\
&= \int e^{-iy\cdot\xi} \widehat{f}(u) g(y)dy \\
&= \widehat{f}(\xi) \cdot \widehat{g}(\xi).
\end{aligned}$$

□

Example 2.1 (A Warning)

In $\mathbb{R}^1$, $f(x) = |x|^{-2/3} 1_{|x| \leq 1}$, which has an asymptote at 0. $f \in L^1$, and

$$(f * f)(0) = \int_{-1}^{1} |u|^{-4/3} dy = +\infty.$$

Proposition 2.2

Let $p \in [1, \infty]$. Let $f \in L^1, g \in L^p$. Then,

- $y \mapsto f(x-y)g(y) \in L^1$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^p(\mathbb{R}^d)$, $\|f * g\|_p \leq \|f\|_1\|g\|_p$.

Proof. For $p = \infty$, $\int f(x-y)g(y)dy \in C_0(\mathbb{R}^d)$.

If $1 < p < \infty$, $L^P \subset L^1 + L^\infty$, as follows:

$$f(x) = f(x)1_{|f(x)|\leq 1} + f(x)1_{f(x)>1}.$$

We can prove the rest with Minkowski's inequality, or a simpler way. Let $q = p' = \frac{p}{p-1}$ (hence $\frac{1}{q} + \frac{1}{p} = 1$). We use the norm definition,

$$\|f * g\|_p = \sup_{\|h\|_q \leq 1} \int |g * f| \cdot |h|.$$

$$\int |g * f| \cdot h \leq \int (|g| * |f|) \cdot h = \int \int |g(x-y)||f(y)|dyh(x)dx$$

$$= \int |f(y)| \int |g(x-y)|h(x)dxdy \leq \int |f(y)|\|g\|_p * 1dy = \|f\|_1\|g\|_p.$$

$\square$

§3 September 3rd, 2020

§3.1 Convolution and Continuity

Recall convolution:

$$f * g(x) = \int f(x-y)g(y)dy, f * \mu(x) = \int_{\mathbb{R}^d} f(x-y)d\mu(y),$$

where f is continuous, bounded, μ is a complex Radon measure($|\mu|$ is finite)

Proposition 3.1

Let $T : C_0^0 \to C_b^0$. Suppose T is translation invariant: $T \circ \tau_y = \tau_y \circ T$ for all $y \in \mathbb{R}^d$. [There exists $A < \infty$: $\|Tf\|_{C_0} \leq A\|f\|_{C_0}$ for all f. Recall $\|f\|_{C_0} = \sup_x |f(x)|$, and C_0^0, C_b^0 are Banach spaces.] There exists a complex radon measure μ such that $Tf = f * \mu$ for all f.

Proof. Given $T : C_0^0 \to C_b^0$, consider the map $\ell : \mathbb{C}_0^0 \to \mathbb{C}$ given by $f \mapsto (Tf)(0)$. It is clear that ℓ is linear. Furthermore, ℓ is bounded, since

$$|Tf(0)| \leq \|Tf\|_{C_0} \leq A\|f\|_{C_0},$$

so $\ell \in (C_0^0)^*$. Recall the Riesz Representation Theorem, there exists ν, a complex Radon measure, such that for all $f \in C_0^0$

$$\ell(f) = \int f d\nu.$$

Let $y \in \mathbb{R}^d$. We have

$$Tf(-y) = Tf(0-y) = (\tau_y Tf)(0) = T(\tau_y f)(0) = \int \tau_y f(x)d\nu(x) = \int f(x-y)d\nu(x).$$

Similarly, for all x, $(Tf)(-x) = \int f(y-x)d\nu(y)$. [See lecture notes for correct algebra, sad]. □

§3.2 Convolution and Differentiation

Informally,

$$\frac{\partial}{\partial x_j} \int f(x-y)g(y)dy = \int \frac{\partial f}{\partial x_j} f(x-y)g(y)dy.$$

Proposition 3.2

Assume $f \in C^1(\mathbb{R}^d), g \in L^1$ and $f, \nabla f$ is bounded. Then

$$f * g \in C^1, \frac{\partial}{\partial x_j}(f \star g) = \left(\frac{\partial f}{\partial x_j}\right) * g.$$

Proof. We assume $d = 1$ for clarity.

$$\frac{(f*g)(x+t) - (f*g)(x)}{t} = \int \frac{f(x+t-y) - f(x-y)}{t} g(y)dy.$$

Let $t \to 0$. Use DCT, with dominator

$$|g(y)| \cdot \sup_{t,u} \frac{|f(u+t) - f(u)|}{|t|}.$$

The supremum is finite by the mean value theorem. $\square$

Example 3.3

Take $g \in L^\infty$, $f \in C_1$, and there exists $a < \infty$ such that for all x,

$$|f(x)| + |\nabla f(x)| \leq A\langle x\rangle^{-\gamma}.$$

Hence, $f, \nabla f \in L^1$. Then $f * g \in C^1, \nabla(f * g) = (\nabla f) * g$.

We can iterate this: Under appropriate conditions

$$\frac{\partial^\alpha (f * g)}{\partial x^\alpha} = \frac{\partial^\alpha f}{\partial x^\alpha} * g,$$

$$\frac{\partial^{\alpha+\beta}(f * g)}{\partial x^{\alpha_\beta}} = \frac{\partial^\alpha f}{\partial x^\alpha} * \frac{\partial^\beta g}{\partial x^\beta}.$$

Proposition 3.4

If $f \in L^1$ and $g \in L^\infty$, then $f * g \in C_b^0$.

Proof. Recall: If $f \in L^1(\mathbb{R}^d)$, then $y \mapsto \tau_y f \in L^1$ is continuous: As $y \to 0$,

$$\|\tau_y f - f\|_1 \to 0.$$

Then,

$$(f*g)(x)-(f*g)(x') = \int (f(x-y)-f(x'-y))g(y)dy = \int [f(x-y)-(\tau_u f)(x-y)]g(y)dy,$$

where $u = x' - x$. As $u \to 0$, $\|f - \tau_u f\|_1 \to 0$, and $g \in \text{Ł}^\infty$, so the integral approaches 0, as desired. $\square$

§3.3 Approximation

Definition 3.5 (Approximate Identity Sequence)**.** An approximate identity sequence for $\mathbb{R}^d$ is $(\varphi_n)_{n\in\mathbb{N}}, \varphi_n \in L^1(\mathbb{R}^d)$ with the following conditions:

- $\int_{\mathbb{R}^d} \varphi_n = 1$.
- For all $\delta > 0$, $\int_{|x|\geq\delta} |\varphi_n| dx \to 0$ as $n \to \infty$.

Theorem 4

Let (φ_n) be an approximate identity sequence in $\mathbb{R}^d$.

1. Let $f \in C_b^0$ be uniformly continuous. Then $f * \varphi_n \to f$ uniformly.
2. Let $f \in C_b^0$. Then $f * \varphi_n \to f$ uniformly on every compact set.
3. If $1 \leq p \leq \infty$, then for all $f \in L^p$, $\|f * \varphi_n - f\|_p \to 0$.

[All the above limits are taken for $n \to \infty$.]

Proof.

$$\begin{aligned} f * \varphi_n(x) - f(x) &= \int f(x-y)\varphi_n(y)dy - f(x) \\ &= \int (f(x-y) - f(x))\varphi_n(y)dy \end{aligned}$$

Then,

$$|f * \varphi_n(x) - f(x)| \leq \int |f(x-y) - f(x)||\varphi_n(y)|dy.$$

Let $\delta > 0$. Then,

$$\int |f(x-y)-f(x)||\varphi_n(y)|dy = \int_{|y \leq \delta|} |f(x-y)-f(x)||\varphi_n(y)|dy + \int_{|y \geq \delta|} |f(x-y)-f(x)||\varphi_n(y)|dy.$$

$$\begin{aligned} \int_{|y \leq \delta|} |f(x-y) - f(x)||\varphi_n(y)|dy &\leq \|\varphi_n\|_1 \cdot \sup_{x, |y| \leq \delta} |f(x-y) - f(x)| \\ &= \|\varphi_n\|_1 \cdot \omega_f(\delta) \\ &\leq A \cdot \omega_f(\delta). \end{aligned}$$

Then

$$\begin{aligned} \int_{|y \geq \delta|} |f(x-y) - f(x)||\varphi_n(y)|dy &\leq \int_{|y| \geq \delta} 2\|f\|_{C^0} \cdot |\varphi_n(y)|dy \\ &\leq 2\|f\|_{C^0} \int_{|y| \geq \delta} |\varphi_n|dy. \end{aligned}$$

Hence

$$|f * \varphi_n(x) - f(y)| \leq A\omega_f(\delta) + 2\|f\|_{C^0} \int_{|y| \geq \delta} |\varphi_n|dy.$$

Taking the $\limsup$, the second term goes to 0, so for all $\delta > 0$,

$$\lim_{n \to \infty} \sup \|f * \varphi_n - f\|_{C^0} \leq A\omega_f(\delta).$$

Since f is uniformly continuous, $\lim_{\delta \to 0} \omega_f(\delta) = 0$, which proves the claim. $\square$

Corollary 3.6

$C^\infty \cap L^p$ is dense in L^p for all $1 \le p \le \infty$.

Proof. We want to construct (φ_n) with $\varphi_n \in C_0^\infty$.

We claim there exists a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\int \varphi = 1$ and $\varphi \ge 0$. In $d = 1$, take $h(x) = 1x > 0e^{-\|x\|}$. Then, define $\varphi(x) = h(x)h(1-x) \in C_0^\infty$. Then, we normalize φ.

Now, take $\varphi_n(x) = n^d\varphi(nx)$. □

Example 3.7

Let $\varphi \ge 0$, $\int \varphi = 1$. Define $\varphi_n(x) = n^d\varphi(nx)$. Then $\int \varphi_n = 1$.

Furthermore,

$$\int_{|x|\ge\delta} n^d\varphi(nx)dx = \int_{|y|\ge n\delta} \varphi(y)dy \to 0.$$

Example 3.8

Let $\varphi(x) = (2\pi)^{-d/2}e^{-|x^2|/2}, x \in \mathbb{R}^d$. Let $t > 0$ and $\varphi_t(x) = (2\pi)^{-d/2}t^{-d/2}e^{-|x|^2/(2t)}$. Now $t \to 0^+$ and

$$\int_{|x|\ge\delta} \varphi_t(x)dx \to 0.$$

This is an approximate identity family.

Example 3.9 (Interpretation of $f * g$)

$$f * g = \int \tau_y f(x) \cdot g(y)dy.$$

If $g \ge 0$ and $\int g = 1$, then we have an **average** of translates of f.

As $n \to \infty$, $g = \varphi_n$ so the weight concentrates asymptotically at the origin.

§4 September 8th, 2020

§4.1 Fourier Transform Identities

We have many functorial identities.

1. For $f \in L^1$,
$$(\tau_y f)^\wedge(\xi) = e^{-iy\cdot\xi}\widehat{f}(\xi).$$

2. For $f, g \in L^1(\mathbb{R})$,
$$(f * g)^\wedge = \widehat{f} \cdot \widehat{g}.$$

3. For $f \in L^1$,
$$(e^{ix\cdot\eta} f)^\wedge(\xi) = \widehat{f}(\xi - \eta).$$

4. We use the notation
$$\xi^\alpha = \prod_{j=1}^d \xi_j^{\alpha_j}.$$
For $f \in C^0, C^{|\alpha|}, C_0^0$,
$$(\partial^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \widehat{f}(\xi).$$
This comes from the fact that
$$\int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_k} f(x)\right) e^{-ix\cdot\xi} dx,$$
so we integrate by parts, use Fubini in $\mathbb{R}^d$ and induct on $|\alpha|$.

5. For $f \in C_0^\infty$,
$$(X^\beta f(x))^\wedge(\xi) = (i\partial_\xi)^\beta \widehat{f}(\xi),$$
where
$$x^\beta = \prod_{j=1}^d x_j^{\beta_j}, (i\partial_\xi)^\beta = i^{|\beta|}\partial^\beta.$$

6. For $f \in C_0^\infty$,
$$(\partial_x^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \widehat{f}(\xi).$$

7. If $L \in GL(d)$, $L : \mathbb{R}^d \to \mathbb{R}^d$, linear invertible, then for all $f \in L61$,
$$(f \circ L)^\wedge(\xi) = |\det(L)|^{-1}\widehat{f} \circ ((L^*)^{-1})(\xi).$$
The proof follows from the substitution $x = L^{-1}(y)$ and $(L^{-1})^* = (L^*)^{-1}$

Corollary 4.1

$$V = \{f \in (L^1 \cap L^2)(\mathbb{R}^d) : \exists A = A_f < \infty, |\widehat{f}(\xi)| \leq A_f \langle \xi \rangle^{-d}\}$$
is dense in $L^2(\mathbb{R}^d)$.

Proof. We showed last time that C_0^∞ is dense in $L^2(\mathbb{R}^d)$. We need to show that $f \in C_0^\infty$ implies that $\widehat{f}(\xi) = O(\langle \xi \rangle^{-N})$ for all $N \leq \infty$.

WLOG, assume $\xi \neq 0$, $\xi_d \neq 0, |\xi_d| \geq \frac{|\xi|}{d}$. Then,

$$\begin{aligned}\int f(x)e^{-ix\cdot\xi}dx &= (-i\xi_d)^{-1}\int f(x)\frac{\partial}{\partial x_d}(e^{-ix\cdot\xi})dx \\ &= (-i\xi_d)^{-1}\int_{\mathbb{R}^d}\frac{\partial f}{\partial x_d}(x)e^{-ix\cdot\xi}dx \leq \infty.\end{aligned}$$

We can pick up as many factors of ξ_d as we'd like to get arbitrary bounds. □

§4.2 The Gaussian

Fact 4.2. $(d \geq 1)$ Take $e^{-z|x|^2/2} = f(x) = f_z(x)$. Assume $Re(z) \geq 0 \to f_z \in L^1$.

$$(e^{-z|x|^2/2})^\wedge(\xi) = (2\pi)^{d/2}z^{-d/2}e^{-|\xi|^2/(2z)}.$$

We consider $z^{-d/2}$ in the principal branch. When $z = 1$, $(e^{-|x|^2/2})^\wedge(\xi) = (2\pi)^{d/2}e^{-|\xi|^2/2}$.

Note the fact

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}.$$

In order to calculate

$$\int_R e^{-x^2/2}e^{-ix\xi}dx,$$

we have

$$x^2/2 + ix\xi = \frac{1}{2}(x^2 + 2ix\xi) = 1/2(x+i\xi)^2 + \xi^2/2,$$

so

$$e^{-\xi^2/2}\int_{\mathbb{R}} e^{-(x+i\xi)^2/2} = e^{-\xi^2/2}\sqrt{2\pi}.$$

If $F(x) = \prod_{j=1}^d f_j(x_j)$, then $\widehat{F}(\xi) = \prod_{j=1}^d \widehat{f}_j(\xi_j)$.

For $z \in \mathbb{R}^+$, $e^{-z|x|^2/2} = e^{-|L(x)|^2/2}$, where

$$L(x) = z^{1/2}x.$$

Then, we use $(f\circ L)^\wedge(\xi) = |\det(L)|^{-1}\widehat{f}((L^*)^{-1}(\xi))$. For $Re(z) \geq 0$,

$$\int f(x)e^{-ix\cdot\xi}dx = \int e^{-z|x|^2/2}e^{-ix\cdot\xi}dx.$$

We claim that this is a homomorphic function of z in $Re(z) > 0$.

Fact 4.3. If $f \in L^1(\mathbb{R}^d)$ and $\widehat{f} \in L^1$, then

$$f = (2\pi)^{-d}(\widehat{f})^\vee, \check{g}(x) = \int g(\xi)e^{ix\cdot\xi}d\xi.$$

Corollary 4.4

If $f \in L^1$, $\widehat{f} = 0$, then $f = 0$ almost everywhere.

Proof. Given $f, \widehat{f} \in L^1$. Let $\varphi \in C_0^\infty$ with $\int \varphi = 1$. Let $\varphi_n(x) = n^d \varphi(nx)$. Define $f_n = f * \varphi_n$. We know that $f_n \to f$ in L^1 as $n \to \infty$.

Moreover, $f_n \in L^2$, since $f_n \in L^1 * L^2$. For each n, we have

$$\|(2\pi)^{-d} \int_{|\xi| \le R} \widehat{f_n}(\xi) e^{ix\cdot\xi} d\xi - f_n(x)\|_{L^2} \to 0,$$

as $R \to \infty$.

Note that
$$\widehat{f_n}(\xi) = \widehat{f}(\xi)\widehat{\varphi_n}(\xi) = \widehat{f}(\xi)\widehat{\varphi}(n^{-1}\xi).$$
As $n \to \infty$, $\widehat{\varphi}(n^{-1}\xi) \to \widehat{\varphi}(0) = \int \varphi = 1$. Hence,

$$\widehat{f_n}(\xi) \to \widehat{f}(\xi).$$

Furthermore
$$\int_{|\xi| \le R} \widehat{f_n}(\xi) e^{ix\cdot\xi} d\xi \to \int_{\mathbb{R}^d} \widehat{f_n}(\xi) e^{ix\cdot\xi} d\xi,$$

since $\widehat{f_n} \in L^1$ as $R \to \infty$.

Hence, we have that

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{\varphi}(n^{-1}\xi) e^{ix\cdot\xi} d\xi = f_n(x),$$

in the L^2 norm. Now, letting $n \to \infty$, $f_n = f * \varphi_n \to f$ in the L^1 norm.

$$\int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{\varphi}(n^{-1}\xi) e^{ix\cdot\xi} d\xi \to \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi} d\xi = (\widehat{f})^\vee(x),$$

by the dominated convergence theorem. Thus,

$$f(x) = (2\pi)^{-d}(\widehat{f})^\vee(x).$$

But we actually proved a stronger result: $g \in L^1 \implies \check{g} \in C^0$, so if $g = \widehat{f}$, $(\widehat{f})^\vee \in C^0$ if $f \in L^1$, so if $f, \widehat{f}$ are in L^1, then f agrees almost everywhere with $(2\pi)^{-d}(\widehat{f})^\vee \in C^0$. $\square$

Example 4.5

Take $f(x) = 1_{[0,1]}(x)$. Hence $\widehat{f} \notin L^1$. Essentially, we have that $|\widehat{f}(\xi)| \approx \frac{1}{|\xi|}$ as $|\xi| \to \infty$.

§4.3 Schwartz Spaces

Definition 4.6 (Schwartz Space)**.**

$$\mathscr{S} = \mathscr{S}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C}, f \in C^\infty, \forall N, \alpha, x \mapsto \langle x\rangle^N \frac{\partial^\alpha f}{\partial x^\alpha} \text{ is bounded.}\}.$$

It is clear that $\mathscr{S}$ is a vector space over $\mathbb{C}$. Furthermore, $\mathscr{S}$ is a topological vector space.

The topology on $\mathscr{S}$ is defined by a countable family of seminorms.

$$\|f\|_{M,N} = \sup_{x \in \mathbb{R}^d} \langle x\rangle^N \sum_{0 \le |\beta| \le M} \left|\frac{\partial^\beta f}{\partial x^\beta}(x)\right|.$$

We have that $f \in \mathscr{S}$ if and only if $f \in C^\infty$ and for all $M, N \in \mathbb{N}$, $\|f\|_{M,N} < \infty$.

A neighborhood base for the topology at g would be

$$V(g, M, N, \epsilon) = \{f \in \mathscr{S} : \|f - g\|_{M,N} < \epsilon\}.$$

Note that if ρ_n is a metric,

$$\sum_{n=1}^{\infty} 2^{-n} \left(\frac{\rho_n}{1 + \rho_n}\right)$$

is also a metric, but it wouldn't preserve the vector space condition. Next time, we will prove the following theorem:

Theorem 5

$\wedge : \mathscr{S} \to \mathscr{S}$ is a linear, bijective homeomorphism.

§5 September 10th, 2020

§5.1 Schwartz Space, continued

Last time, we introduced the Schwartz space,

$$\mathscr{S} = \mathscr{S}(\mathbb{R}^d) = \{f \in C^\infty : \forall M, N \|f\|_{M,N} < \infty\},$$

$$\|f\|_{M,N} = \sup_x \{\langle x \rangle^M \sum_{|\alpha|=0}^{N} \left|\frac{\partial^\alpha f}{\partial x^\alpha}\right|\}.$$

An equivalent formulation is $x^\beta \partial^\alpha f$ is bounded for all α, β.

Theorem 6

The fourier transform, $\wedge : \mathscr{S} \to \mathscr{S}$ is a linear, bijective homeomorphism.

Proof. Note that if $f \in \mathscr{S}$, then $\widehat{f} \in C^\infty$. This is clear since

$$\partial_\xi^\alpha \int f(x) e^{-ix\cdot\xi} dx = \int f(x) \partial_{xi}^\alpha (e^{-ix\cdot\xi}) dx.$$

Hence $f \cdot \langle x \rangle^N$ is in L^1 for all N.

Note the following identities:

$$(\partial_x^\alpha f)^\wedge = (i\xi)^\alpha \widehat{f}(\xi), (x^\beta f)^\wedge = (i\partial_x i)^\beta \widehat{f}(\xi),$$

which can be verified from repeated integration by parts.

We claim that $\xi^\beta \partial_\xi^\alpha \widehat{f}$ is bounded for all α, β. Moreover, there exists M, N such that

$$\sup_{xi} |\xi^\beta \partial_\xi^{\alpha \widehat{f}(\xi)}| \le C_{\alpha,\beta} \|f\|_{M,N}.$$

Note that

$$|\xi^\beta \partial_\xi^{\alpha \widehat{f}(\xi)}| = |(\partial_x^\beta x^\alpha f)^\wedge(\xi)|,$$

so

$$\sup_{xi} |\xi^\beta \partial_\xi^{\alpha \widehat{f}(\xi)}| \le \|(\partial_x^\beta x^\alpha f)^\wedge(\xi)\|_{L^1} \le C_d \sup_x |\langle x \rangle^{d+1} \partial_x^\beta (x^\alpha f)|.$$

By the Leibniz rule, we can commute ∂_x^β, which gives the result.

Hence, we have proved that $\widehat{\mathscr{S}} \subset \mathscr{S}$, and $\wedge : \mathscr{S} \to \mathscr{S}$ is continuous. and the same holds for $f \mapsto \check{f}$, so $f \in \mathscr{S} \Rightarrow f \in L^1$ and $\widehat{f} \in L^1$, so $\wedge$ is 1-1 on § and $\vee$ is onto, so we get that $\wedge$ is onto. □

§5.2 Tempered Distributions

We will consider the dual of the Schwartz space,

$$\mathscr{S}' = \{\varphi : \mathscr{S} \to \mathbb{C}, \text{ linear and continuous}\}.$$

Recall, continuity by definition is given by the existence of $M, N, C < \infty$ so that for all $f \in \mathscr{S}$, $|\varphi(f)| \le C\|f\|_{M,N}$.

Example 5.1 (Dirac Mass)

We can take $\varphi(f) = f(0)$, the dirac mass. We can also take $\varphi(f) = \partial^\alpha f(y_0)$.

Let μ be a complex Radon measure, $h \in L^1_{loc}$, $\int_{|x|\le R} |h| dx \le C_h \langle R\rangle^{A_h}$. We can define

$$\varphi(f) = \int \partial^\alpha f(x) \cdot h(x) d\mu(x) \in \mathbb{C}.$$

Theorem 7

Every $\varphi \in \mathscr{S}'$ is a finite linear combination of $f \mapsto \int \partial^\alpha f \cdot h d\mu$, with h, μ, α as before.

The proof is left as an exercise. The key ingredient is the Riesz Representation theorem and the Hahn-Banach theorem.

$\mathscr{S}'$ is given a weak topology: a neighborhood hood base of $\varphi \in \mathscr{S}'$ is given by choosing J, a finite index set, $\epsilon > 0$ and $f_j \in \mathscr{S}(j \in J)$. Then

$$V = \{\psi \in \mathscr{S}' : |\psi(f_j) - \varphi(f_j)| < \epsilon \; \forall j \in J\}.$$

Definition 5.2. For $\varphi \in \mathscr{S}'$, $\widehat{\varphi}$ is a map $f \in \mathscr{S} \mapsto \varphi(\widehat{f})$. Then $\widehat{\varphi} : \mathscr{S} \mapsto \mathbb{C}$ is linear. Similarly, we can define $\check{\varphi} : \mathscr{S} \to \mathbb{C}$, linear.

We can verify that $\widehat{\varphi} \in \mathscr{S}'$. Note that

$$|\widehat{\varphi}(f)| = |\varphi(\widehat{f})| \le C_\varphi \|\widehat{f}\|_{M,N} \le C' \|f\|_{M',N'}.$$

Theorem 8

$\wedge : \mathscr{S}' \to \mathscr{S}'$ is a bijective homeomorphism.

Proof. We first show that $\varphi \mapsto \widehat{\varphi}$ is continuous at ψ. Given V, a neighborhood of ψ: J finite, $\epsilon > 0$, $f_j : j \in J$, we need to control $|\widehat{\varphi}(f_j) - \widehat{\psi}(f_j)| < \epsilon$ for every $j \in J$. The neighborhood $W = \{\varphi : |\varphi(\widehat{f_j}) - \psi(\widehat{f_j})| < \epsilon \forall j \in J\}$ gives the desired bound.

Now we claim for all $\varphi \in \mathscr{S}'$, $(\widehat{\varphi})^\vee = (2\pi)^d \varphi$. This comes from

$$(\widehat{\varphi})^\vee(f) = \widehat{\varphi}(\check{f}) = \varphi((\check{f})^\wedge) = \varphi((2\pi)^d f).$$

Hence $\wedge$ is 1-1 and onto, so we conclude that it is a bijective homeomorphism. □

We can define a partial derivative of a distribution, $\partial^\alpha \varphi$, with $\partial^\alpha : \mathscr{S}' \to \mathscr{S}'$ continuous, linear. This is a bit shocking: Take $\varphi = h \in L^1_{loc}$ with $\int_{|x|\le R} |h| dx \le C_h R^{A_h}$. This defines a distribution $f \mapsto \int fh = \varphi(f)$. That means, we have a way of essentially differentiating anything.

Note that we have a natural map $i : \mathscr{S} \to \mathscr{S}'$ injective, where $i(g)(f) = \int_{\mathbb{R}^d} fg$. Then, we take $g \mapsto i(g)$. Note that i is a continuous map.

Given some linear operator $T : \mathscr{S} \to \mathscr{S}$, we want to associate an extension $\tilde{T}$: $\tilde{T}(i(g)) = i(T(g))$ for all $g \in \mathscr{S}$.

Define $T' : \mathscr{S}' \to \mathscr{S}'$, where $T'(\varphi)(f) = \varphi(T(f))$. It's easy to check that $T' \in \mathrm{End}(\mathscr{S}')$, but there are some bad examples.

Example 5.3

If we take $T(f) = \frac{df}{dx}$, $\int f \cdot g' = -\int f' \cdot g$, then

$$T(i(g)) = -i(T(g)).$$

Suppose we have some $T \in \text{End}(\mathscr{S})$ and a transpose $A \in \text{End}(\mathscr{S})$ in the sense that

$$\int T(f)g = \int fA(g) \forall f, g \in \mathscr{S}.$$

For example, $T = \frac{d}{dx}, A = -\frac{d}{dx}$. With $T, A \in \text{End}(\mathscr{S})$, we can define $\tilde{T}(\varphi)(f) = \varphi(A'(f))$, which defines our extension.

Proposition 5.4

$i(\mathscr{S})$ is dense in $\mathscr{S}'$.

Definition 5.5 (Convolution for Distributions)**.** If $f \in \mathscr{S}$ and $\varphi \in \mathscr{S}'$, then

$$\varphi * f(x) = \varphi(f_x), f_x(y) = f(x-y).$$

One can show that $\varphi * f \in C^\infty$ if $f \in \mathscr{S}$.

Proposition 5.6

Let $(\varphi_n) \in \mathscr{S}'$. If $\varphi_n \to \varphi$ in $\mathscr{S}'$, then $\varphi_n f \to \varphi(f) \forall f \in \mathscr{S}$.

Proposition 5.7

Let $(\varphi_n) \in \mathscr{S}'$. If $\varphi_n \to 0$ in $\mathscr{S}'$. Then there exists $M, N < \infty$ such that for all n and for all $f \in \mathscr{S}$,

$$|\varphi_n(f)| \le C_n \|f\|_{M,N},$$

and $C_n \to 0$ as $n \to \infty$.

The proof uses the Baire Category Theorem. Recall $\mathscr{S}$ is a complete metrizable space, where we define a norm from

$$\sum_{M,N} 2^{-M-N} \frac{\|f\|_{M,N}}{1 + \|f\|_{M,N}}.$$

For $d \ge 1$, define $g(x) = e^{-i\lambda|x|^2/2}, \lambda \in \mathbb{R}$. Note that $g \in L^\infty$, $|g| \equiv 1$.

We define $\widehat{g}(\xi) = (2\pi)^{d/2}(i\lambda)^{-d/2}e^{-i|\xi|^2/(2\lambda)}$, for $\lambda \ne 0$. If we take $g \mapsto i(g) \in \mathscr{S}'$, note that $(i(g))^\wedge = i$, so we are in fact doing a normal fourier transform.

Define $g_z(x) = e^{-z\lambda|x|^2/2}$, for $z \in \mathbb{C}$, $Re(z) \ge 0$. We claim that as $z \to i\lambda$, $g_z \to g$ in the topology of $\mathscr{S}'$. Furthermore,

$$\int fg_z \to \int fg$$

for all $f \in \mathscr{S}$ by the dominated convergence theorem, with dominator $|f|$, since $|g_z| \le 1$, $|g| \equiv 1$.

We know that $\widehat{g}_z \to \widehat{g}$ in $\mathscr{S}'$ as $z \to i\lambda$. Note that

$$\widehat{g}_z(\xi) = (2\pi)^{d/2} z^{-d/2} e^{-|\xi|^2/(2z)}.$$

If $Re(z) > 0$, then $g_z \in \mathscr{S}$.
Then as $z \to i\lambda$,

$$(2\pi)^{d/2} z^{-d/2} e^{-|\xi|^2/(2z)} \to (2\pi)^{d/2} (i\lambda)^{-d/2} e^{-|\xi|^2/(2i\lambda)}.$$

So $\widehat{g}_z \to \widehat{g}$ in $\mathscr{S}'$, so we have the result.

§6 September 15th, 2020

§6.1 Poisson Summation Formula

Define $\mathscr{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx$. We have that $\|\mathscr{F}(f)\|_{L^2} = \|f\|_{L^2}$ for all $f \in L^2 \cap L^1$.

Theorem 9

For all $f \in \mathscr{S}$,

$$\sum_{n\in\mathbb{Z}^d} \mathscr{F}(f)(n) = \sum_{k\in\mathbb{Z}^d} f(k).$$

This has a nice interpretation: suppose we define $\delta_n(g) = g(n)$. We have $\delta_n \in \mathscr{S}'$, and

$$\mathscr{F}\left(\sum_{n\in\mathbb{Z}^d} \delta_n\right) = \sum_{k\in\mathbb{Z}^d} \delta_k.$$

Proof. Given $f \in \mathscr{S}$, set $g : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C}$, defined by $g(x) = \sum_{n\in\mathbb{Z}^d} f(x+n)$. Note that g is periodic: $g(x+e_j) = g(x)$ for all $1 \le j \le d$.

$$g(x) = \sum_{k\in\mathbb{Z}^d} \left(\int g(y)e^{-2\pi i k\cdot y}dy\right) e^{ik\cdot x}.$$

Note that

$$\begin{aligned}\sum_n f(n) = g(0) &= \sum_k \int e^{-2\pi i k\cdot y} \sum_n f(y+n)dy \\ &= \sum_k \int_{[0,1]^d} \sum_n e^{-2\pi i k\cdot(y+n)} f(y+n) \quad = \sum_k \int_{R^d} f(u)e^{-2\pi i k\cdot u}du = \sum_k \widehat{f}(k).\end{aligned}$$

Because f is a Schwartz function, all these series converge and we can easily swap sums and integrals. $\square$

Example 6.1

There are lots of functions that are their own Fourier transforms. Take $x^n e^{-x^2/2}$, for $n \in \mathbb{Z}_{\ge 0}$. Apply Gram-Schmidt in the order of $\mathbb{Z}_{\ge 0}$. We get an orthonormal basis $P_n(x)e^{-x^2/2}$, where $P_n = c_n x^n + O(|x|^{n-2})$.

If $n \equiv 0 \pmod 4$,

$$(P_n e^{-x^2/2})^\wedge = (2\pi)^{1/2} P_n e^{-x^2/2}.$$

§6.2 Size of Fourier Coefficients

Remark: If $f \in C_c^k(\mathbb{R}^d)$ or $C^k(\mathbb{T}^d)$, then

$$\widehat{f}(\xi) = O(\langle\xi\rangle^{-k}).$$

This comes from $\widehat{\left(\frac{\partial f}{\partial x_j}\right)}\xi = i\xi_j \widehat{f}(\xi)$.

We can have a stronger bound,

$$\langle \xi \rangle^k \widehat{f} \in L^2, \ell^2.$$

The proof is the same since $\xi^\alpha \widehat{f} \in L^2/\ell^2$ whenever $0 \le |\alpha| \le k$.

Recall the class

$$\mathrm{Lip} = \left\{ f : \mathbb{R}^d \to \mathbb{C} : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.$$

Proposition 6.2

Assume $f \in \mathrm{Lip}$ and has compact support. Then,

$$\widehat{f}(\xi) = O(\langle \xi \rangle^{-1}),$$

$$\langle \xi \rangle \widehat{f} \in L^2.$$

Proof. We have $f \in C_0^0(\mathbb{R}^d) \cap \mathrm{Lip}$. Assuming $\xi \neq 0$,

$$\widehat{f}(\xi) = \int f(x) e^{-ix\cdot\xi} dx = \frac{1}{2} \int f(x) e^{-ix\cdot\xi} dx + \frac{1}{2} \int f(x + \frac{\pi}{\xi}) e^{-i(x + \frac{\pi}{\xi})\xi} dx.$$

Since $e^{-i(\pi/\xi)\xi} = -1$, we have

$$\frac{1}{2} \int [f(x) - f(x + \pi/\xi)] e^{-ix\cdot\xi} dx.$$

Because f is Lipschitz, $f(x) - f(x + \pi/\xi) \in O(|\xi|^{-1})$, so it's clear the whole integral is bounded.

Definition 6.3 (Holder Class)**.** Define $\Lambda_\alpha (0 < \alpha < 1)$, as $f : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$.

Note that $\alpha > \beta \Rightarrow \Lambda_\alpha \subset \Lambda_\beta$. Furthermore $\mathrm{Lip} \subset \Lambda_\alpha$.

We can state a similar proposition as above for Holder classes.

Example 6.4

Let $0 < \alpha < 1$,

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}.$$

The function $f \in \Lambda_\alpha$, but not Λ_β for any $\beta > \alpha$, since $\widehat{f}(2^n) = (2^n)^{-\alpha}$.

Let $f \in \mathrm{Lip} \cap C_0^0$. Claim $f' \in L^\infty$ in the $\mathscr{S}'$ sense. In other words, there exists $g \in L^\infty$ such that $\int f\varphi' = -\int g\varphi$ for all $\varphi \in \mathscr{S}$.

The claim immediately implies that $\xi \widehat{f}(\xi) \in L^2$, since $\widehat{g} \in L^2 = i\xi \widehat{f}$ and has compact support.

$$\lim_{t \to 0} \int f(x) \frac{\varphi(x+t) - \varphi(x)}{t} dx = \lim_{t \to 0} \int \frac{f(x) - f(x-t)}{t} \varphi(x) dx$$

Let $f_t = fracf(x) - f(x-t)t$. Note that $f_t \in L^\infty(\mathbb{R})$ and $L^\infty = (L^1)^*$, so by Alouglu's theorem, there exists a sequence $t_\nu \to 0$ and $g \in (L^1)^8$ with $f_t \to -g$ in the weak star topology.

Therefore, $\int f_{t_\nu}\varphi \to -\int g\varphi$ as $\nu \to \infty$. Thus, $\int f\varphi' = -\int g\varphi$. □

Example 6.5

Take

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}$$

with $\alpha = 1$. f is not Lipschitz, since

$$\sum_{\xi=2^n} |\xi||\widehat{f}(\xi)| = \sum_n 1 = \infty.$$

Remark: For $\alpha < 1$, f is nowhere differentiable.

Example 6.6

Take $f \in BV(\mathbb{R}^1)$ with compact support, the class with bounded variation. Then $|\widehat{f}(\xi)| \le \pi V(f)|\xi|^{-1}$.

Lemma 6.7 (Riemann-Lebesgue Lemma)

If $f \in L^1(\mathbb{R}^d)$ or $(\mathbb{T}^d)$(then $\widehat{f} \in C^0$ bounded), then $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

Proof. Note that

$$\widehat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} (f(x) - f(x + \frac{\pi\xi}{|\xi|^2}))e^{-ix\cdot\xi} dx$$

Then

$$|\widehat{f}(\xi)| \le \frac{1}{2}\|f(x) - f(x + \frac{\pi\xi}{|\xi|^2})\|_{L^1} \to 0.$$

□

How fast do they go to zero? Is there a quantitative bound? (Nope) How do we characterize $\widehat{L^1}$? Is $C^0_{\to 0} = (L^1)^\wedge$? (Nope).

Proposition 6.8

The map $\wedge : L^1(\mathbb{R}^d) \to C^0_{\to 0}(\mathbb{R}^d)$ is not onto. Equivalently, $\vee : C^0_{\to 0}(\mathbb{R}^d) \not\to L^1$.

Proof. $\wedge : L^1 \to C^0_{\to 0}$ is linear, bounded, and an injective mapping between Banach spaces. We can apply the Open Mapping Theorem: if the map was onto, there would exist $A < \infty$ such that $\|f\|_{L^1} \le A\|\widehat{f}\|_{C^0}$.

We claim that $\frac{\|\widehat{f}\|_{C_0}}{\|f\|_{L^1}}$ can be arbitrarily small. Define $f_t(x) = e^{-(1+it)|x|^2/2}$ for $t \in \mathbb{R}$ going to ∞.

We know that

$$\widehat{f_t}(\xi) = (2\pi)^{d/2}(1+it)^{-d/2}e^{-(1-it)|\xi|^2/(2(1+t^2))}.$$

Hence,

$$|\widehat{f_t}| = (2\pi)^{d/2}(1+t^2)^{-d/4}e^{-|\xi|^2/(2(1+t^2))} \le (2\pi)^{d/2}(1+t^2)(-d/4) \to 0.$$

On the other hand $\|f_t\|_{L^1}$ is independent of t. $\square$

Theorem 10

Let $w : \mathbb{R}^d \to (0, \infty)$ and $w(\xi) \to 0$ as $|\xi| \to \infty$. There exists $f \in L^1$ with

$$|\widehat{f}(\xi)| \ge w(\xi) \forall \xi.$$

Proof. We have a key lemma: Let $w : \mathbb{R}^1 \to (0, \infty)$ continuous, even, piecewise, $C^2(\mathbb{R} \setminus \{0\})$, convex on $(0, \infty)$ with compact support. Then, $\widehat{w} \in L^1$ and $\widehat{w} \ge 0$, hence, $\|\widehat{w}\|_{L^1} = \int \widehat{w} = (2\pi)^d w(0)$. $\square$

§7 September 17th, 2020

§7.1 Size of Fourier Coefficients, continued

Theorem 11

Let $w : \mathbb{R}^d \to (0, \infty)$ and $w(\xi) \to 0$ as $|\xi| \to \infty$. There exists $f \in L^1$ with

$$|\widehat{f}(\xi)| \geq w(\xi) \forall \xi.$$

Proof. We have a key lemma:

Lemma 7.1

Let $w : \mathbb{R}^1 \to (0, \infty)$ continuous, even, piecewise $C^2(\mathbb{R} \setminus \{0\})$, convex on $(0, \infty)$ with compact support and nondecreasing. Then, $\widehat{w} \in L^1$ and $\widehat{w} \geq 0$, hence,

$$\|\widehat{w}\|_{L^1} = \int \widehat{w} = (2\pi)^d w(0).$$

Proof. Note that

$$\widehat{w}(\xi) = \int_{\mathbb{R}} w(x) e^{-ix \cdot \xi} dx = \int_{\mathbb{R}} w(x) \cos(x\xi) dx.$$

Furthermore, note that $|x| \cdot |w'(x)|$ is a bounded function(as $x \to 0$). It follows from Jensen's inequality.

$$\begin{aligned}
\widehat{w}(\xi) &= 2 \int_0^\infty w(x) \cos(x\xi) \\
&= 2\xi^{-2} \int_0^\infty w''(x)(1 - \cos(x\xi)) dx \geq 0.
\end{aligned}$$

It suffices to show the equality $\int_0^\infty w(x) \cos(x\xi) = \xi^{-2} \int_0^\infty w''(x)(1 - \cos(x\xi))$. We integrate by parts twice:

$$\begin{aligned}
\widehat{w}(\xi) &= 2 \int_0^\infty w'(x) \xi^{-1} \sin(x\xi) dx \\
&= 2 \int_0^\infty w''(x) \xi^{-2} (1 - \cos(x\xi)) dx.
\end{aligned}$$

We might have issues at 0, but we can take a limit for integrating from ϵ to ∞ with boundary terms $w''(\epsilon)(1 - \cos(\epsilon\xi)) \in O(\epsilon^2)$. Hence, $\widehat{w} \geq 0$.

Note that $\widehat{w} \in L^1$ and for $|\xi| \geq 1$,

$$|\widehat{w}(\xi)| \leq 2\xi^{-1} \int_0^\infty |w''(x)| dx \cdot 2$$

. Assume $|w'(0)| < \infty$, where the derivative is the right-hand derivative at 0.

Then

$$\int_0^\infty w''(x) dx = -w'(0)$$

so it follows that $\widehat{w} \in L^1$.

Finally,

$$w(0) = (2\pi)^{-1}(\widehat{w})^\vee(0) = (2\pi)^{-1}\int \widehat{w}(\xi)d\xi = (2\pi)^{-1}\|\widehat{w}\|_{L^1},$$

which gives the desired bound. □

Let $g : \mathbb{R} \to [0,\infty]$ continuous, with $g(\xi) \to 0$ as $\xi \to \infty$.

Lemma 7.2

There exists $w : \mathbb{R} \to (0,\infty)$ so that $w \geq g$ and w is even, convex on $(0,\infty)$, $w(\xi) \to 0$ as $|\xi| \to \infty$, and w is piecewise C^2, where we may have infinity many breaks.

To prove the theorem, it suffices to find a function $f \in L^1$ such that $\widehat{f}(\xi) \geq w(\xi)$ for all ξ.

WLOG, g is even(replace $g(\xi)+g(-\xi)$), nonincreasing(we can replace $\tilde{g}(x) = \sup_{y\geq x} g(y)$ for $x \geq 0$). Note that $\check{w}(\xi) = \widehat{w}(-\xi)$ so define $f = \widehat{w}$. $\widehat{f} = (2\pi)w \geq 2\pi g$.

To treat w, we approximate it with functions of compact support. Let $t > 0$ and define $w_t = \max(w - t, 0)$. We conclude that $\widehat{w_t} \in L^1$ and $\|\widehat{w}_t\|_{L^1} = (2\pi)w_t(0)$. As $t \to 0^+$, $w_t \to w$ in $\mathscr{S}'$ so $\widehat{w}_t \to \widehat{w}$ in $\mathscr{S}'$. We have that $\widehat{w}$ is a complex radon measure.

Fact 7.3. If μ is a complex Radon measure and if $\mu|_{\mathbb{R}\setminus 0}$ is absolutely continuous, then $\mu = c\delta_0 + h$ for $h \in L^1$.

We know that $w(\xi) \to 0$ as $|\xi| \to \infty$ and $\widehat{\mu}(\xi) = c + \widehat{h}(\xi)$ so $c = 0$ and $\widehat{w} \in L^1$ as desired. □

§7.2 Comparing Size of Functions to Size of Fourier Coefficients

We have that $\|\widehat{f}\|_{L^2} = (2\pi)^{-d/2}\|f\|_{L^2}$ and $\|\widehat{f}\|_{C^0} \leq \|f\|_{L^1}$.

Theorem 12 (Hausdorff-Young)

Let $p \in [1,2]$. The n $f \in L^P(\mathbb{R}^d)$ implies that $\widehat{f} \in L^1$ for $q = p' = \frac{p}{p-1}$, and

$$\|\widehat{f}\|_q \leq C(p,d)\|f\|_p.$$

For $\mathbb{T}^d$,

$$\|\widehat{f}\|_{\ell^q} \leq C(p)^d\|f\|_{L^p(\mathbb{T}^d)}.$$

Note that for $\mathbb{R}^d$, $\wedge : L^p \to L^r$ is bounded.

Proof. We must have that $r = p'$. Fix a function $0 \neq f \in \mathscr{S}$. Define $f_t(x) = f(tx)$ for $t \in \mathbb{R}^+$.

$$\widehat{f}_t(\xi) = t^{-d}\widehat{f}(t^{-1}\xi).$$

Note that

$$\|f_t\|_p^p = \int |f(tx)|^p dx = t^{-d}\int |f(y)|^p dy = t^{-d}\|f\|_p^p.$$

Then $\|\widehat{f}_t\|_r = t^{-d}t^{d/r}\|\widehat{f}\|_r$, so

$$\frac{\|\widehat{f}_t\|_r}{\|f_t\|_p} = t^\gamma \frac{\|\widehat{f}\|_r}{\|f\|_p}$$

where $\gamma = -d + d/r + d/p$. We must have that $\gamma = 0$ for the ratio to be bounded, which gives $1 = \frac{1}{p} + \frac{1}{r}$.

For $\mathbb{T}^d$, we can only take $t \to +\infty$ so $\gamma \leq 0$, and we can only conclude that $r \geq p'$. But $r \geq p'$ implies that $\ell^{p'} \subset \ell^r$, so $\wedge : L^p \to \ell^{p'} \subset \ell^r$. $\square$

Theorem 13 (Riesz-Thoren)

Let $(X, \mu), (Y, \nu)$ be σ-finite measure spaces. Suppose we have exponents $p_0, p_1, q_0, q_1 \in [1, \infty]$. Let $S(X)$ be the set of simple functions from $X \to \mathbb{C}$. Assume $T : S(X) \to (L^1 + L^\infty)(Y)$ is linear and there exists $A_0, A_1 < \infty$ so that for all $f \in S(X)$,

$$\|Tf\|_{L^{q_j}} \leq A_j \|f\|_{L^p_j}.$$

§8 September 22nd, 2020

§8.1 Comparing Size of Functions to Size of Fourier Coefficients, continued

Recall

Theorem 14 (Riesz-Thoren)

Let $(X, \mu), (Y, \nu)$ be σ-finite measure spaces. Suppose we have exponents $p_0, p_1, q_0, q_1 \in [1, \infty]$. Let $S(X)$ be the set of simple functions from $X \to \mathbb{C}$. Assume $T : S(X) \to (L^1 + L^\infty)(Y)$ is linear and there exists $A_0, A_1 < \infty$ so that for all $f \in S(X)$,

$$\|Tf\|_{L^{q_j}} \le A_j \|f\|_{L^p_j}.$$

We will prove this later, with an elegant application of complex analysis.

Remark: $(\mathbb{R}^d)$ Is it true that $\widehat{L^p} \subset L^q$ $(2 < p, q = p')$? No. We sketch the proof. Suppose it was true. For $f \in L^p$ with $\|f\|_p \le 1$, define $\ell^f \in (L^{q'})^*$ by

$$\ell_f(g) = \int g\widehat{f}.$$

This defines a bounded linear functional as desired. We claim that $\{\ell_f\}$ is pointwise bounded. Then, by the Uniform Boundedness Principle, it follows that ℓ_f are uniformly bounded. We know that

$$\|\ell_f\|_{(L^{q'})^*} = \|\widehat{f}\|_{L^{(q')'}} = \|\widehat{f}\|_{L^p}$$

by the Reverse Holder's Inequality. This would give the desired inequality.

Finally,

$$\ell_f(g) = \int g\widehat{f} = \int \widehat{g} f,$$

and $\widehat{g} \in L^q$. Then

$$|\ell_f(g)| = |\int \widehat{g} f| \le \|\widehat{g}\|_q \|f\|_p \le \|\widehat{g}\|_{L^q}.$$

§8.2 Rademacher Functions

Theorem 15 (Kahane)

If $a \in \ell^2$, there exists $f \in L^\infty$ such that for all n, $|\widehat{f}(n)| \ge |a_n|$.

We prove a weaker result.

Theorem 16

For $\mathbb{T}^d$, $d \ge 1$. For any $a \in \ell^2$, there exists $f \in \cap_{p<\infty} L^p$ such that for all $n \in \mathbb{N}$,

$$|\widehat{f}(n)| = |a_n|$$

We will use **Rademacher Functions**: $r_n : [0, 1] \to \{-1, 1\}$, with $n \ge 0$. We let $r_0(x) = 1$, for r_n, we split $[0, 1]$ into 2^n intervals and alternate between 1 and -1.Note that $\|r_n\|_{L^2([0,1])} = 1$. If $n > m$, then

$$\int r_n r_m dx = 0.$$

Lemma 8.1

For $a_j \in \{1, 2, 3, \dots\}$,
$$\int_0^1 \prod_{j=1}^N r_{n_j}^{a_j} dx = 0,$$
unless every a_j is even.

We can now form a Rademacher Series:
$$f(x) = \sum_{n=0}^{\infty} c_n r_n(x).$$
If $c \in \ell^2$, then $f \in L^2$ and $\|c\|_{\ell^2} = \|f\|_{L^2}$.

Theorem 17 (Khinchine's Inequality)

If $c \in \ell^2$ then $f \in \bigcap_{p<\infty} L^p$. For all $p, q \in (0, \infty)$, there exists $A_{p,q} < \infty$ such that for all c, $\|f\|_{L^q} \le A\|f\|_{L^p}$.

Proof. WLOG, $p = 2q$.
$$\int |f|^{2q} = \int f^q \overline{f}^q = \int \sum_{n_1, \dots, n_q} \prod_{j=1}^q c_{n_j} r_{n_j} \sum_{m_1, \dots, m_q} \prod_{i=1}^q \overline{c_{m_i}} r_{m_i}.$$
which is
$$\sum \sum \int_0^1 (\prod_{j=1}^q r_{n_j})(\prod_{i=1}^q r_{m_i}) dx.$$
The integral is 0 unless all the $\{m_i\} = \{n_j\}$. We can bound this above by $q! \|c\|_{\ell^2}^{2q}$. $\square$