Math 205: Complex Variables

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§1.1 Intro to Riemann Mapping Theorem

Our first goal is to proof a fundamental theorem of Riemann on conformal mappings. We start with several preparations, including some detours. The theorem essentially says that lots of open sets in $\mathbb{C}$ are holomorphically isomorhpic, given that they satisfy some simple topological conditions.

§1.2 Cauchy's Integral Formula

Recall Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, dz$$

where Γ is a simple closed curve, piecewise differentiable, $z_0 \in \mathrm{Int}(\Gamma)$, and $f : \Omega \to \mathbb{C}$ is a holomorphic function, with Ω is open, $\Omega \supset \Gamma \cup \mathrm{Int}(\Gamma)$.

If Γ is the circle $|z - z_0| = R$, we parameterize with $z = Re^{i\theta} + z_0$ with $\theta \in [0, 2\pi)$. This gives

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) \, d\theta,$$

which represents the average of f on the circle.

It follows that

$$|f(z_0)| \leq \max_{\partial B_R(z_0)} |f(z)|,$$

with equality if and only if f is constant.

If $f : \Omega \to \mathbb{C}$ is holomorphic for Ω connected, open and $z_0 \in \Omega$, then

$$|f(z_0)| \leq \sup_{z \in \Omega} |f(z)|$$

with equality if and only if f is constant.

§1.3 Schwarz Lemma

Theorem 1 (Schwarz Lemma)
For $f : B_1(0) \to \mathbb{C}$ holomorphic with $|f(z)| \leq 1$ for all z and $f(0) = 0$. Then

$$|f(z)| \leq |z|, |f'(0)| \leq 1.$$

If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = cz$ for some $|c| = 1$.

Proof. Define a function

$$g(z) = \begin{cases} f(z)/z, \text{if } 0 \leq |z| \leq 1 \\ f'(0), \text{if } z = 0 \end{cases}.$$

Note that $g(z)$ is continuous since at zero,

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Hence, $|g(z)| \leq C < \infty$ using the Weierstrass Extreme Value theorem. If $0 < \epsilon < |w| < r < 1$, note that taking a Keyhole Contour, we have

$$g(w) = \frac{1}{2\pi i}\left(\int_{|z|=r} - \int_{|z|=\epsilon}\right)\frac{g(z)}{z-w}\,dz.$$

Note that

$$\left|\int_{|z|=\epsilon}\frac{g(z)}{z-w}\,dz\right| \leq (2\pi\epsilon)\cdot C\frac{1}{|w|-\epsilon} \xrightarrow{\epsilon\to 0} 0.$$

It follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r}\frac{g(z)}{z-w}\,dz$$

for $0 < |w| < r$. The right side is holomorphic in w if $|w| < r$, so it follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r}\frac{g(z)}{z-w}\,dz$$

is holomorphic in $|z| < 1$.

This can also be proved by taking a Taylor series about the origin. Since there is no constant term, we can divide by z to still have a convergent Taylor series.

If $r < 1$,

$$\sup_{|z|\leq r}|g(z)| = \sup_{|z|=r}|g(z)| \leq \sup_{|z|=r}\frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

If we let $r \uparrow 1$, then we get $\sup_{|z|<1}|g(z)| \leq 1$. It follows that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

If $|f(z_0)| = z_0$ for some $0 < |z_0| < 1$ then $|g(z_0)| = 1$ and g is constant by the maximum principle so $g(z) = c$, $f(z) = cz$. If $|f'(0)| = 1$, then $|g(0)| = 1$ so g is constant and $f = cz$. □

§1.4 Maximum Principles

In the above proof, we used the maximum principle. Some other versions we will use are the following:

If $K \subset \mathbb{C}$ compact and $f : K \to \mathbb{C}$ continuous, and the restriction of f to the interior of K is holomorphic, then

$$\sup_{z\in K}|f(z)| = \sup_{z\in\partial K}|f(z)|.$$

If Ω is open and connected, $f : \Omega \to \mathbb{C}$, $z_0 \in \Omega$, and $|f(z_0)| = \sup_{z\in\Omega}|f(z)|$, then f is constant. Applying this to e^f and using that $|e^f| = e^{\mathrm{Re}\, f}$, we find that

$$\mathrm{Re}\ f(z_0) = \sup_{z\in\Omega}\mathrm{Re}\ f(z),$$

implies that f is constant. We have the same result for $\mathrm{Im}\ f$ by replacing f with $-if$.

§1.5 Homework I

Show the Automorphisms of the unit disk are fractional linear transformations.

Proof. Following the hint, define $h = g \circ f$. Note that $h : \{|z| < 1\} \to \{|z| < 1\}$ is a holomorphic bijection as a composition of holomorphic bijections. Furthermore, note that

$$h(0) = \frac{f(0) - z_0}{1 - \overline{z_0}f(0)} = \frac{z_0 - z_0}{1 - \overline{z_0}f(0)} = 0.$$

Note that we can apply the Schwarz lemma to h and h^{-1} since the ranges of both functions are $\{|z| < 1\}$. It follows that $|h'(0)| \leq 1$ and

$$1 \geq |(h^{-1})'(0)| = \left|\frac{1}{h'(h^{-1}(0))}\right| = \left|\frac{1}{h'(0)}\right|.$$

It follows that $|h'(0)| = 1$, so by the equality case of the Schwarz lemma, $h(z) = cz$ for some $c \in \mathbb{C}$.

It follows that

$$h(z) = \frac{f(z) - z_0}{1 - \overline{z_0}f(z)} = cz \Rightarrow f(z) = \frac{z_0 + cz}{1 + c\overline{z_0}z} = \frac{a + bz}{c + dz}$$

for $a, b, c, d \in \mathbb{C}$. □

§2 January 25th, 2021

§2.1 Uniform Convergence

> **Remark 2.1.** They sometimes call open connected sets "regions".

Definition 2.2 (Uniform Convergence)**.** Let $\Omega \subset \mathbb{C}$ be open. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic and $f : \Omega \to \mathbb{C}$ a function so that $\lim_{n\to\infty} \sup_{z\in K} |f(z) - f_n(z)| = 0$ for all $K \subset \Omega$ compact(also denoted $K \subset\subset \Omega$).

> **Remark 2.3.** Recall from real analysis that f is a continuous function.

Some further remarks:

- It suffices to check the result for a sequence of compact subsets K_m so that $\bigcup_m K_m^\circ = \Omega$, the it suffices to check those. If $K \subset\subset \Omega$, then K is compact and covered by the union of the subsets so there exists a finite subcovering, and uniform convergence on the subcovering implies uniform convergence on K.

- It is often convenient to introduce $\|g\|_K = \sup_{z\in K} |g(z)|$. Uniform convergence can be restated as $\|f_n - f\|_K \to 0$ for all $K \subset\subset \Omega$.

- If $\|f_n - f\|_K \to 0$ for all $K \subset\subset \Omega$, then f is also holomorphic. It follows by passing to the limit in the Cauchy Integral formula. Namely, take $\{z : |z - z_0| \le R\} \subset \Omega$ and consider the points in $|z_0 - \zeta| < R$.

$$\begin{aligned}\left| f_n(\zeta) - \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{z-\zeta}\,dz \right| &= \left| \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f_n(z)}{z-\zeta}\,dz - \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{z-\zeta}\,dz \right| \\ &\le \frac{1}{2\pi}\frac{1}{R - |z_0 - \zeta|} \cdot (2\pi R)\|f_n - f\|_{|z-z_0|=R} \to 0.\end{aligned}$$

  So it follows that

$$f(\zeta) = \lim_{n\to\infty} f_n(\zeta) = \frac{1}{2\pi i}\int_{|z-z_0|} \frac{f(z)}{z-\zeta}\,dz.$$

  It follows that f continuous on $|z - z_0| = R$ is holomorphic in $\zeta \in \{|z - z_0| < R\}$, so it follows that f is holomorphic.

- We can similarly show that

$$f_n^(j)(\zeta) = \frac{n!}{2\pi i}\int_{|z-z_0|=R} \frac{f_n(z)}{(z-\zeta)^{n+1}}\,dz$$

  and $\|f_n^{(j)} - f^(j)\|_K \to 0$.

From the last item, we have the following theorem.

> **Theorem 2**
>
> If $f_n \to f$ on compact subsets of Ω, the if f_n is holomorphic we find that f is holomorphic and $f_n^(j) \to f^{(j)}$ uniformly on compact subsets of Ω.

Theorem 3 (Hurwitz)

Let Ω be a region, $f : \Omega \to \mathbb{C}$ and $f_n : \Omega \to \mathbb{C}$ holomorphic with $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$ and $\|f_n - f\|_K \to 0$ for all compact subsets. Then either $f \equiv 0$ or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

Proof. If f is not identically zero on ω, then since f is holomorphic, its zeros are isolated. If $z_0 \in \Omega$, $f(z_0) = 0$, then there is $\epsilon > 0$ so that when $0 < |z - z_0| < \epsilon$, $f(z) \neq 0$.

Since $f(z) \neq 0$ for $|z - z_0| = \epsilon/2$, by the Weierstrass theorem applied to $|f|$ on $|z - z_0| = \epsilon$, we have $|f(z)| \geq m > 0$ on $\{|z - z_0| = \epsilon/2\} = \Gamma$. If $\|f_n - f\|_\Gamma \leq m/2$ for $n \geq N$, then

$$|f_n(z)| \geq |f(z)| - m/2 \geq m - m/2 = m/2$$

for $z \in \Gamma$. Hence, it follows that $\|1/f_n - 1/f\|_\Gamma \to 0$(we leave this as an exercise).

Since $\|f_n' - f'\|_\Gamma \to 0$, we find that $\|f_n'/f_n - f'/f\| \to 0$(another exercise) and hence

$$\frac{1}{2\pi i} \int_\Gamma \frac{f_n'}{f_n}\, dz \to \frac{1}{2\pi i} \int_\Gamma \frac{f'}{f}\, dz.$$

The integrand of the left hand side is $(\log f_n)'$, whose integral is 0, and the right side is the order of the zero of f at z_0 by the argument principle. It follows that the order of z_0 as a possible zero is 0, so $f(z_0) \neq 0$. $\square$

Theorem 4

For $\Omega \subset \mathbb{C}$ open, $\mathcal{F}$ a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \ldots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

Proof. We first show 2 implies 1. If $\sup_{f \in \mathcal{F}} \|f\|_K = \infty$, then we can find for each $n \in \mathbb{N}$ $f_n \in \mathcal{F}$ so that $\|f_n\|_K \geq n$. If we abstract a convergence subsequence, then $\|f_{n_j} - f\|_K \leq C < \infty$ and $\|f_{n_j}\|_K \leq \|f\|_K + C$, while $\|f_{n_j}\|_K \to \infty$, a contradiction. $\square$