Ch 3. Meromorphic Functions and the Logarithm

3.1 Zeros and Poles

Def. A point singularity of f is a complex number z_0 s.t. f is defined in a neighborhood of z_0 but not at z_0.

Def. A complex number z_0 is a zero of holomorphic f is $f(z_0)=0$. The zeros of a non-trivial holomorphic function are isolated.

Theorem 1.1. Suppose f is holomorphic in connected, open Ω, has a zero at $z_0 \in \Omega$, and does not vanish identically in Ω. Then $\exists U \subset \Omega$ w/ $z_0 \in U$, a non-vanishing holomorphic function g on U and $n \in \mathbb{N}$ s.t.

$$f(z) = (z-z_0)^n g(z) \qquad \text{for all } z \in U.$$

Proof. Since Ω is connected, $f \not\equiv 0$, f has an expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$

$\exists$ smallest $a_n \neq 0$. Then

$$f(z) = (z-z_0)^n \left(a_n + a_{n+1}(z-z_0) + \cdots \right) = (z-z_0)^n g(z).$$

Def. Deleted Neighborhood of z_0 : $D_r(z_0) \setminus \{z_0\}$. f defined in a deleted neighborhood of z_0 has a pole at z_0, if $1/f$, defined to be zero at z_0, is holomorphic in a full neighborhood of z_0.

Theorem 1.2 If f has a pole at $z_0 \in \Omega$, then in a neighborhood of z_0, $\exists h$ nonvanishing, holomorphic and $\exists!\, n$ s.t.

$$f(z) = (z-z_0)^{-n} h(z).$$

Theorem 1.3 If f has a pole of order n at z_0,

$$f(z) = \overbrace{\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)}}^{\text{principle}} + G(z),$$

where G is holomorphic on a neighborhood of z_0.

$a_{-1} = \operatorname{res}_{z_0} f$, the residue.

Theorem 1.4 If f has a pole of order n at z_0,

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

3.2 Residue Formula

Theorem 2.1 Suppose f is holomorphic in an open set containing circle C and its interior, except for a pole at $z_0 \in C^\circ$. Then

$$\int_C f(z)\,dz = 2\pi i \ \mathrm{res}_{z_0}(f).$$

Proof. Choose a keyhole contour about pole and let width $\to 0$. Then

$$\int_C f(z)\,dz = \int_{C_\varepsilon} f(z)\,dz.$$

Then $$\frac{1}{2\pi i}\int \frac{a_{-1}}{z-z_0} = a_{-1},$$

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{a_{-k}}{(z-z_0)^k} = 0,$$

$$f(z) = G(z) + \sum_{k=1}^{n} \frac{a_{-k}}{(z-z_0)^k}, \quad \int_{C_\varepsilon} G(z) = 0$$

$$\Rightarrow \int_{C_\varepsilon} f(z)\,dz = a_{-1} = \mathrm{res}_{z_0}(f). \ \blacksquare$$

Corollary 2.2 If f is holomorphic in an open set conteing circle C, C°, except for poles at $z_1, \dots, z_N$, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k}(f).$$

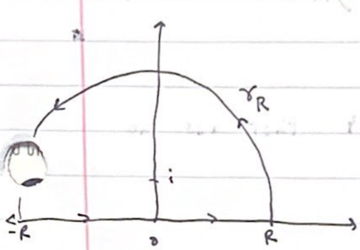
The same result holds for general toy contours.

Examples

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

~~If we take $x \mapsto x+iy$,~~ Consider $f(z) = \frac{1}{1+z^2}$ holomorphic except for at $-i, i$.

Take



$$\mathrm{res}_{(i)}(f) = \lim_{z\to i} \left(\frac{\cancel{z-i}}{z+i}\right) = \frac{1}{2i}$$

Thus $$\int_{\gamma_R} (f(z)\,dz = \frac{2\pi i}{2i} = \pi$$

$$\left|\int_{C_R^+} f(z)\,dz\right| \le \pi R \frac{B}{R^2} \le \frac{M}{R} \longrightarrow 0$$

So $$\int_{-\infty}^{\infty} f(z)\,dz = \pi.$$

- $$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin(\pi a)} \qquad 0 < a < 1$$
- $$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh(\pi x)}\,dx = \frac{1}{\cosh(\pi \xi)}.$$

3.3 Singularities and Meromorphic Functions

Thm (Riemann's Theorem on Removable Singularities) Suppose that f is holomorphic in an open set Ω except at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.

Sketch. Take a keyhole about $z \neq z_0$, z_0 and let width $\longrightarrow 0$.

Corollary. Suppose f has an isolated singularity at z_0. Then z_0 is a pole iff $|f(z)| \longrightarrow \infty$ as $z \longrightarrow z_0$.

A singularity that is not isolated or a pole is an <u>essential singularity</u>.

Thm (Casorati-Weierstrauss) Suppose f is holomorphic in $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) \setminus \{z_0\}$ under f is dense in $\mathbb{C}$.

Proof. Assume not. $\exists w \in \mathbb{C}$ w/ $\delta > 0$ s.t.

$$|f(z) - w| > \delta \quad \text{for} \quad z \in D_r(z_0) \setminus \{z_0\}.$$

Define

$$g(z) = \frac{1}{f(z) - w} \quad \text{on} \quad D_r(z_0) \setminus \{z_0\}$$

which is holomorphic and bounded by $1/\delta$. g has a removable sing. at z_0. If $g(z_0) \neq 0$, $f(z) - w$ is holomorphic at z_0, contradicting the fact that z_0 is essential. Thus, $g(z_0) = 0$ so $f(z) - w$ has a pole at z_0, contradicting essential. ∎

Def. A function f on open Ω is <u>meromorphic</u> if $\exists \{z_i\}$ that has no limit points in Ω, such that
 (i) f is holomorphic in $\Omega \setminus \{z_0, \dots\}$
 (ii) f has poles at $\{z_0, z_1, z_2, \dots\}$

Thm. The meromorphic fuctions in the extended complex place are rational fuctions.

Thm (Argument Principle) Suppose f is meromorphic in open set containing a circle C and its interior. If f has no poles and never vanishes on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \left(\#\text{ of zeros of } f \text{ inside } C\right) - \left(\#\text{ of poles of } f \text{ inside } C\right).$$

Thm. (Rouché's) If f, g are holomorphic in open set containing circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f, $f+g$ have the same number of zeroes inside C.

Thm (Open Mapping Theorem) If f is holomorphic and nonconstant in Ω, then f is open.

Thm (Maximum Modulus) If f is non-constant holomorphic in Ω, then f cannot attain a maximum in Ω.

Corollary. If Ω is a region w/ compact closure $\bar{\Omega}$, if f is holomorphic on Ω and continuous on $\bar{\Omega}$, then

$$\|f\|_{\Omega} \leq \|f\|_{\bar{\Omega}\setminus\Omega}.$$