Notes on Free Probability Advisor: Dan-Virgil Voiculescu, Fall 2021

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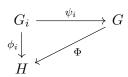
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§1 Free Products

We first define the notion of a free product for objects. They can all be boiled down to universal products in the categorical sense.

Definition 1.1 (Free Product of Groups). let $(G_i)_{i\in I}$ denote a family of groups. The group free product, denoted $*_{i\in I}G_i$ is the unique group G up to isomorphism with homomorphisms $\psi_i: G_i \to G$ so that given any group H and $\phi_i: G_i \to H$, there exists a unique homomorphism $\Phi = *_{i\in I}\phi_i: G \to H$ making the diagram commute:

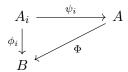


The free product can also be constructed as

$$H = \{g_1 g_2 \dots g_n : g_j \in G_{i_j} \setminus \{e\}, i_1 \neq i_2 \neq \dots \neq i_n\} \cup \{\emptyset\},\$$

with multiplication defined by concatenation followed by reduction.

Definition 1.2 (Free Product of Groups). let $(A_i)_{i\in I}$ denote a family of unital algebras. The *unital algebra free product*, denoted $*_{i\in I}A_i$ is the unique unital algebra A up to isomorphism with homomorphisms $\psi_i: A_i \to A$ so that given any group B and $\phi_i: A_i \to B$, there exists a unique homomorphism $\Phi: *_{i\in I}A \to B$ making the diagram commute:



As a vector space, we construct the unital algebra free product by first taking a basis formed by

$$B = \{a_1 a_2 \dots a_n : a_j \in A_{i_j} : i_1 \neq i_2 \neq \dots i_n\},\$$

then taking the quotient with the subspace generated by relations

$$a_1 \dots a_{j_1} (\lambda a_j^{(0)} + \mu a_j^{(1)}) a_{j+1} \dots a_n, a_j = 1.$$

We also have the following proposition:

Proposition 1.3. If $A_i = \mathbb{C}1 \oplus V_i$, then

$$*_{i \in I} A_i \cong \mathbb{C}1 \oplus \bigoplus_{n \geq 1} \left(\bigoplus_{i_1 \neq i_2 \neq \cdots \neq i_n} V_{i_1} \otimes \cdots \otimes V_{i_n} \right).$$

References: [1] Free Random Variables, D. V. Voiculescu, J. H. Dykema, A. Nica