Math 222a Lecture Notes, Fall 2020

Partial Differential Equations

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§1 September 1st, 2020

§1.1 Introduction

Partial differential equations apply to functions $u : \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$, where u refers to the space dimension. Usually, $n \geq 2 (n = 1$ corresponds to ODEs).

We present the following notation:

- $\frac{\partial}{\partial x_i} u = \partial_i u$
- There is also multi-index notation, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u$. The size of α is given by $|\alpha| = \sum_{i=1}^n \alpha_i$.
- $C(\mathbb{R}^n)$, continuous functions in $\mathbb{R}^n$.
- $C(\Omega)$, $\Omega \subset \mathbb{R}^n$, continuous functions in Ω.
- $C^1(\mathbb{R}^n), C^1(\Omega)$, continuously differentiable functions.
- $C^k(\mathbb{R}^n), C^k(\Omega)$, k-times differentiable.
- $C^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty C^k(\mathbb{R}^n)$.

We consider an example PDE,

$$F(u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0.$$

In the above, $k \geq 1$ and k is the **order** of the equation. We also have the shorthand $F(\partial^{\leq k} u) = 0$.

§1.2 Classification of PDE's

Definition 1.1 (Linear PDE)**.** The PDE is a linear function of its arguments. We can apply multi-index notation, as follows:

$$\sum_{|\alpha| < k} c_\alpha \partial^\alpha u = f(x).$$

If $f(x) = 0$, the PDE is **homogeneous**, otherwise it is **inhomogeneous**.

This can be separated into linear PDEs with constant coefficients, $c_\alpha \in \mathbb{R}, \mathbb{C}$ and variables coefficients, $c_\alpha = c_\alpha(x)$. [In this class, we focus on constant coefficient PDEs, but many of the techniques can be extended to variable coefficient PDEs.]

Definition 1.2 (Nonlinear PDE)**.** We look at a function $F = F(u, \partial u, \ldots, \partial^k u)$. The highest order terms are take the *leading role.*

- Semilinear PDE's: F is linear, with constant or variable coefficients in $\partial^k u$:

  $$\sum_{|\alpha| = k} c_\alpha(x) \partial^\alpha u = N(\partial^{\leq k-1} u).$$

  The LHS is called the principal part, and the RHS is the perturbative role.

- Quasilinear PDE's:

  $$\sum_{|\alpha| = k} c_\alpha(\partial^{\leq k-1} u) \partial^\alpha u = N(\partial^{\leq k-1} u).$$

- Fully Nonlinear PDE's: $F(\partial^{\leq k}u) = 0$, with a nonlinear dependence on $\partial^k u$.

Some examples:

- Linear, homogeneous, variable coefficients, order 1:
$$\sum_{k=1}^{u} c_k(x)\partial_k(u) = 0.$$
- Define $\Delta = \partial_1^2 + \cdots + \partial_n^2$, the Laplacian operator. We have a linear, constant coefficients, inhomogeneous, order 2:
$$\Delta u = f.$$
- Semilinear, order 2:
$$\Delta u = u^3.$$
[Note that translation invariance makes homogeneous vs inhomogeneous not useful for classification in the case of nonlinear PDE's.]
- Harmonic Map Equation:
$$\Delta u = u|\nabla u|^2.$$
It is still semilinear, but with a stronger nonlienarity.
- Monge Ampere Equation:
$$\mathbb{R}^2, \partial_1^2 u \partial_2^2 u - (\partial_1\partial_2 u)^2 = 0.$$
It is a fully nonlinear equation.

§1.3 Initial Value Problems

We have various types of problems:

- (Stationary Problems) With $u : \mathbb{R}^n \to \mathbb{R}$,
$$F(\partial^{\leq k}u) = 0,$$
might describe an equilibrium configuration of a physical system.
- (Evolution Equations) With $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $u(t, x)$ describes the state at time t. We can think about the order in x or in t.

Definition 1.3 (Initial Value Problem/Cauchy Problem)**.** A PDE with initial conditions.

Example 1.4

Consider the heat equation:
$$\partial_t u = \Delta_x u,$$
$$u(t = 0, x) = u_o(x).$$
The equation is first order in t, but second order in x.

Example 1.5

In $[\mathbb{R} \times \mathbb{R}]$, the vibrating string:

$$\partial_t^2 u = \partial_x^2 u,$$

$$u(t = 0, x) = u_0(x),$$
$$\partial_t u(t = 0, x) = u_1(x).$$

Note that this equation is second order in time, and requires 2 pieces of initial data.

An easier problem: Compute the Taylor series of u at some point $(0, x_0)$. It requires $\partial_t^\alpha \partial_x^\beta u(0, x_0)$.

- This is obvious if we have no time derivative or exactly 1.
- Second order time derivatives come from the equation.
- Third order or higher time derivatives come from differentiating the equation:

$$\partial_t^3 u = \partial_x^2 \partial_t u.$$

§1.4 Boundary Value Problems

We begin with an example.

Example 1.6

Take $\Delta u = f$ in $\Omega \subset \mathbb{R}^3$, which represents equilibrium for temperature in a solid. To solve, we need information about the boundary of Ω. For example,

$$\Delta u = f \in \Omega,$$

$$u = g \in \partial\Omega.$$

§1.5 Fluid Classification

We take $u : \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$, and

$$F(\partial^{\leq k} u) = 0.$$

This is considered to be a **scalar equation**.

We could also take a **system** of equations, where $u : \mathbb{R}^n \to \mathbb{R}^m(\mathbb{C}^m)$, where $u = [u_i]$ a column of equations. These are often more difficult than scalar equation. We should have

$$F(\partial^{\leq k} u) = 0,$$

but $F : \mathbb{R}^{(\cdot)} \to \mathbb{R}^m(\mathbb{C}^m)$.

Example 1.7

A 2-system:

$$\Delta u = v,$$

$$\Delta v = -u.$$

We can often reduce the order of a scalar equation by turning it into a system:

Example 1.8

Consider the vibrating string,

$$\partial_t^2 u = \partial_x^2 u.$$

If we take $v = \partial_t u$, the it suffices to solve the system,

$$\partial_t u = v,$$

$$\partial_t v = \partial_x^2 u.$$

We van reduce it further by saying $u_1 = \partial_x u, u_2 = \partial_t u$ for the system,

$$\partial_t u_1 = \partial_x u_2,$$

$$\partial_t u_2 = \partial_x u_1.$$

§2 September 3rd, 2020

§2.1 Picard-Lindeloff Theorem

Consider the example, $x' = f(x), x(0) = x_0$, $x : \mathbb{R} \to \mathbb{R}^n$. We ask for existence, uniqueness, continuous dependence on initial data.

Definition 2.1 (Locally Lipschitz)**.** A **Lipschitz** continuous function f is one that satisfies,

$$|f(x) - f(y)| \leq c|x - y|.$$

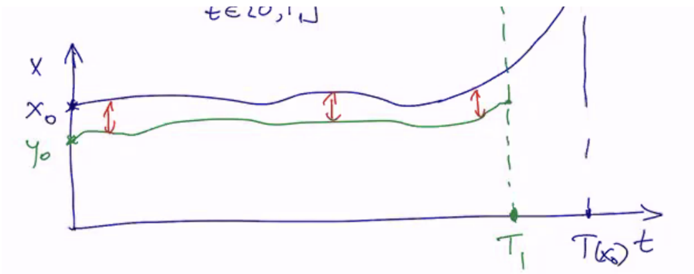
A function is **Locally Lipschitz** if for each R, there exists $c(R)$ such that

$$|f(x) - f(y)| \leq c(r)|x - y|, x, y \in \text{Ball}(0, R).$$

As examples, $f(x) = x$ is Lipschitz, $f(x) = x^2$ is not Lipschitz, but is locally Lipschitz.

Definition 2.2 (Locally well-posed)**.** For each $x_0 \in \mathbb{R}^n$, there exists $T > 0$(lifespan) and a unique solution $u \in C^1[0, T; \mathbb{R}^n]$ with the property that $u_0 = x_0$ and the solution has a Lipschitz dependence on the data: x_0, y_0 initial data, $T = T(x_0)$. For $T_1 < T$, there exists $\epsilon > 0$ such that if $|y_0 - x_0| \leq \epsilon$ then $T(y) > T_1$ and

$$\sup_{t \in [0, T_1]} |x(t) - y(t)| \leq \tilde{C}|x_0 - y_0|.$$



> **Theorem 1** (Picard-Lindelof)
> Assume that f is locally Lipschitz continuous. Then the ODE is locally well-posed.

§2.2 Contraction Principle

We will use the "Contraction principle" - recall the following definitions:

Definition 2.3 (Fixed-point Problem)**.** Let X be a Banach space,let $D \subset X$ be a closed subset of X, and let $F : D \to D$. Question: Can we solve the equation $F(u) = u$ where $u \in D$.

Definition 2.4 (Contraction)**.**

$$\|F(u) - F(v)\|_X \leq L\|u - v\|,$$

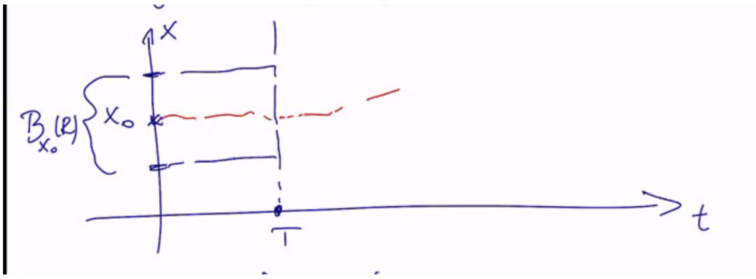
where $L < 1$.

If F is a contraction, then it has a unique fixed point. The existence proof follows an iterative construction: start with an arbitrary element $u_0 \in D$ and define $u_{n+1} = F(u_n)$. We would show $\{u_n\}$ is a Cauchy sequence, so it converges.

We now prove the theorem. We have $x' = f(x), x(0) = x_0$, so

$$x(t) = x_0 + \int_0^t f(x(s))ds, t \in [0, T].$$

We choose $X = C[0, T; \mathbb{R}^n]$, $F(x)(t) = x_0 + \int_0^t f(x(s))ds$. Then x solves the ODE in $(0, T)$ if $F(x) = x$.



We have to choose R, T. Then

$$D = \{x \in X : \|x - x_0\|_X \leq R\}.$$

Let $R = |x_0|$. Next, we choose T so that $F : D \to D$ is Lipschitz. For $F : D \to D$, we estimate the size of $F(x) - x_0$.

$$\begin{aligned} F(x)(t) - x_0| &= \left| \int_0^t f(x(s))ds \right| \\ &\leq \left| \int_0^t f(x_0(s))ds \right| + \left| \int_0^t f(x) - f(x_0)ds \right| \\ &\leq T|f(x_0)| + CT\|x - x_0\|_X \end{aligned}$$

Hence,

$$\|F(x) - x_0\| \leq T(|f(x_0)| + CR).$$

Thus, we choose T such that $T(|f(x_0)| + CR) \leq R$.

Now look at differences: For $x, y \in D$,

$$\begin{aligned} |F(x)(t) - F(y)(t)| &\leq \int_0^t |f(x(s)) - f(y(s))|ds \\ &\leq TC \sup_{s \in [0,T]} |x(s) - y(s)| \end{aligned}$$

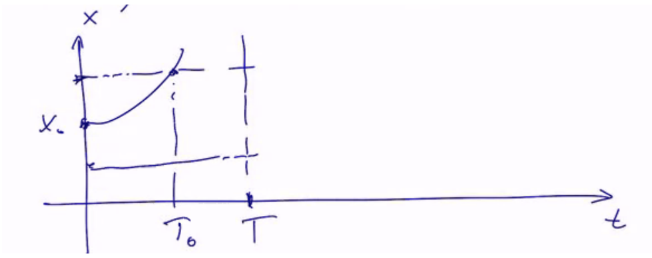
thus,

$$\|F(x) - F(y)\|_X \leq CT\|x - y\|_X,$$

so we can choose T so that $CT\|x - y\|_X < 1$.

By the contraction principle, there exists a unique solution $x \in D$.

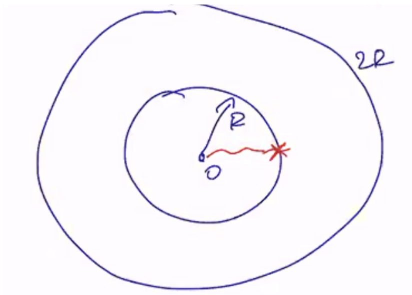
To prove uniqueness of a solution, we have to show that any solution has to stay in D, up to time T.



Suppose a solution $\tilde{x}$ leave the ball before time T. We repeat the above computation up to the exit time T_0. Then, $T_0(|f(x_0) + CR|) < T$, since $T_0 < T$. This is a contradiction since T_0 is the exit time.

§2.3 Bootstrap Argument

Consider a bootstrap argument: try to solve an equation and show that the solution x satisfies some bound $\|x\|_T \leq R$. The difficulty is that a priori, we do not know any bound on $\|x\|_T$. The solution: make a bootstrap assumption, $\|x\|_T \leq 2R$ and show that $\|x\|_T \leq R$ under this assumption.



So far, we know uniqueness in $[0, T]$, where $T = T(x_0)$ given by the contraction argument. We now show global uniqueness: Suppose we have a solution x_0 with maximal lifespan $T_{max}(x_0)$. Suppose y is another solution. We look at the maximal T so that $x = y$ in $[0, T)$. We now think of T as the initial time. We $x(T) = y(T)$ from continuity. Then, the solution is unique up to some time $T + T_0$, so $x = y$ in $[T, T + T_0]$, contradicting the maximality of T. This is called a "continuity argument".

Next, we compare two solutions: We have $x(0) = x_0, x : [0, T) \to R^n$. We choose $T_1 < T$. Then $x : [0, T_1] \to \mathbb{R}^n$. We compare x with a "nearby" solution $y(0) = y_0$ close to x_0. We have$\|x\|_{X_{T_1}} \leq R$ since we have continuity on a compact set. We claim the following: if $|y_0 - x_0| < \epsilon$, then x, y stay close. We make a bootstrap assumption $\|y\|_{X_{T_1}} \leq 2R$.

$$\frac{d}{dt}|x-y|^2 = 2(x-y)(f(x) - f(y)) \leq 2C|x-y|^2.$$

This is the *Gronwall Inequality.* It follows that

$$|x-y|^2(t) \leq e^{2ct}|x-y|^2(0) = e^{2ct}|x_0 - y_0|^2.$$

To close the bootstrap:

$$\|y\|_{X_{T_1}} \leq \|x\|_{X_{T_1}} + \|x-y\|_{X_{T_1}} \leq R + e^{cT_1}\|x_0 - y_0\| \leq \frac{3R}{2},$$

which is better than the bootstrap assumption.

§3 September 8th, 2020

Last lecture, we discussed the ordinary differential equation $x' = f(x)$ in R^n with $x(0) = x_0$. We proved the Pircard-Lindelof theorem: if f is locally Lip. then this problem is locally well-posed and the solution has a local Lip. dependence on the initial data. We proved this by the contraction principle, using Picard iterations.

§3.1 Observations regarding Picard-Lindelof

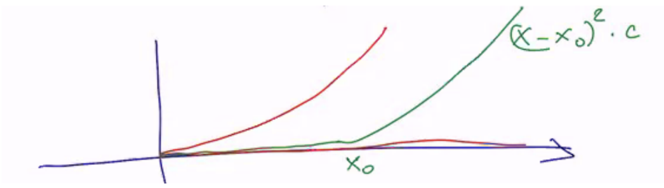
We note the following observations:

1. The result is local, so it can blow up in finite time.

   For example, take $x' = x^2, x(0) = x_0 > 0$. The positive solutions to the ODE are $x(t) = \frac{1}{T-t}, T \geq 0$, where T is the blow up time. In this case, it is $T = \frac{1}{x_0}$.

2. If f is not Lipschitz, then uniqueness might fail.

   Take $x' = \sqrt{x}, x(0) = 0$. An obvious solution is $x = 0$. Other solutions are like $x(t) = ct^2$. We can generate infinitely many solutions from here.

   But solutions might still exist:

   Theorem 2 (Peano)
   If f is continuous, then a local solution exists.

   The proof uses Schauder's fixed point theorem.

3. What if $f \in C^1_{loc}$, the space of differentiable functions on a compact set?

   Theorem 3
   If $f \in C^1_{loc}$, then the flow map $x_0 \mapsto x(t, x_0) = \Phi(t, x_0)$ is of class C^1.

   Proof. We give a sketch. Take x_0, x_0^h and assume $\frac{d}{dh}x_0^h(0)$ exists and show that $\frac{d}{dh}x^h$ exists. The linearized equation about $h = 0$ is $\dot{y} = Df(x_0)y, y_0 = \frac{d}{dh}x_0^h$. We expect that
   $$x^h(t) = x(t) + hy(t) + o(h).$$
   Let $\tilde{x}^h(t) = x(t) + hy(t)$. We claim that this is an "approximate solution", in the sense that
   $$\dot{\tilde{x}}^h(t) = f(\tilde{x}^h(t)) + o(h).$$
   Furthermore, we have close initial data in the sense that
   $$|x_0^h - \tilde{x}_0^h| \leq o(h).$$
   We repeat the difference bound for one exact and one approximate solution and show that
   $$|x^h(t) - \tilde{x}^h(t)| \leq o(h)$$
   □

This implies that the Flow map is a group of local diffeomorphisms:

$$\Phi(t) \circ \Phi(s) = \Phi(t+s).$$

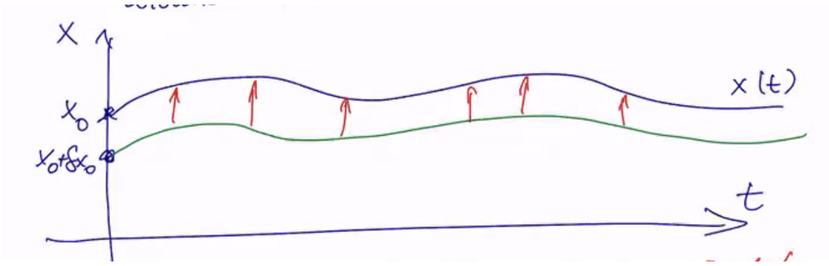
§3.2 Linearization of an ODE

The leads us to the notion of the linearization of the ODE: If we consider $x_0 \to x_0^h$, a one parameter family of data, assume this is C^1 in h. The corresponding solution $x_0^h \to x^h(t)$ also in C^1 in h.

What can we say about

$$y^h(t) = \frac{d}{dh} x^h(t)?$$

We have $\dot{x}^h = f(x^h), x^h(0) = x_0$. If we differentiate with respect to h, we have $\dot{y}^h = Df(x^h)y^h, y^h(0) = \frac{d}{dh}x_0^h.$, where $Df(x^h)$ is the differential of f, $\left(\frac{\partial f_i}{\partial x_j}\right)_{n \times m}$. This is a linear ODE with variable coefficients.



Proposition 3.1

If the linearized equation is well-posed, then we have Lip. dependence of solutions on the initial data.

§3.3 Our First Partial Differential Equation

Our first example is scalar first order equations in $\mathbb{R}^n$,

$$F(x, u, Du) = 0 \in \mathbb{R}^n, y : \mathbb{R}^n \to \mathbb{R}.$$

Today, we look at the case of linear, constant coefficients:

$$\sum a^i \partial_i u = f(x).$$

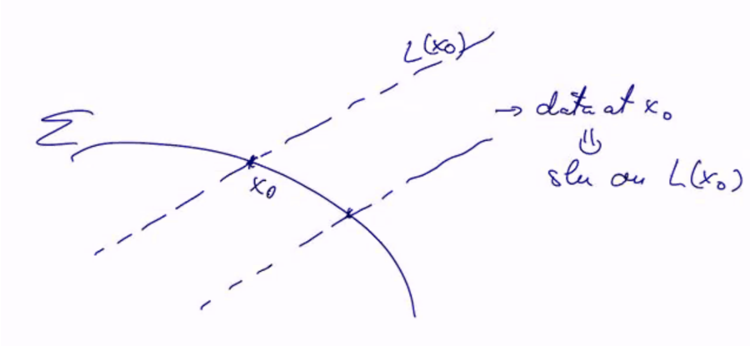
We will write this as $a^i\partial_i u$ following the Einstein summation convention. Take $A = (a_1, \ldots, a_n)$, so we have $A \cdot Du = f(x)$, with $A \neq 0$. This can be interpreted as a directional derivative of u in the direction A.

$$\frac{d}{dt}u(x(t)) = A \cdot Du(x(t)) = f(x(t)).$$

Note the fundamental theorem of calculus,

$$u(x(t)) = u(x_0) + \int_0^t f(x(t))dx.$$

Suppose we have a C^1 surface Σ and we are asked to solve a PDE with inidtial data $u = u_0$ on Σ.

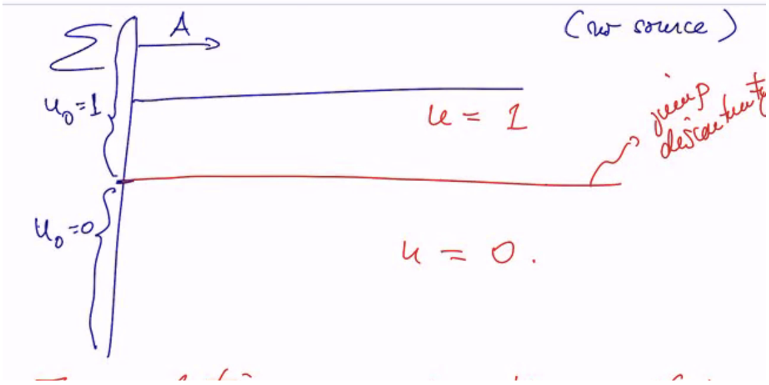


But things can go wrong. If Σ is a circle, we'd could have two intersection points. Furthermore, we could miss the circle entirely and have no solutions. Our solution in this case would be to assume that each line intersects Σ exactly once. However, if solutions are tangent, perturbations of the surface cause problems.

To solve all these issues, we assume that A is always transversal to Σ. This can be written in terms of N, the normal vector to Σ, namely,

$$A \cdot N \neq 0.$$

Definition 3.2 (Noncharacteristic Surface)**.** If $A \cdot N \neq 0$, then we say the surface Σ is noncharacteristic.



We can have solutions that solve the equation at every point but not differentiable everywhere. We learn 2 lessons from this example:

1. We need to enlarge the notion of what is a solution, this leads to the theory of distributions.

2. There are solutions to our PDE with a jump discontinuity along characteristic surfaces. (Γ in the picture)

After applying a change of coordinates, we have a Cauchy problem:

$$u_t + AD_x u = f, u(t=0) = u_0,$$

where u_t is nonzero, corresponding to the condition that the surface is noncharacteristic.