

Math 205: Complex Variables

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§1 January 20th, 2021

§1.1 Intro to Riemann Mapping Theorem

Our first goal is to prove a fundamental theorem of Riemann on conformal mappings. We start with several preparations, including some detours. The theorem essentially says that lots of open sets in \mathbb{C} are holomorphically isomorphic, given that they satisfy some simple topological conditions.

§1.2 Cauchy's Integral Formula

Recall Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz$$

where Γ is a simple closed curve, piecewise differentiable, $z_0 \in \text{Int}(\Gamma)$, and $f : \Omega \rightarrow \mathbb{C}$ is a holomorphic function, with Ω open, $\Omega \supset \Gamma \cup \text{Int}(\Gamma)$.

If Γ is the circle $|z - z_0| = R$, we parameterize with $z = Re^{i\theta} + z_0$ with $\theta \in [0, 2\pi)$. This gives

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta,$$

which represents the average of f on the circle.

It follows that

$$|f(z_0)| \leq \max_{\partial B_R(z_0)} |f(z)|,$$

with equality if and only if f is constant.

If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic for Ω connected, open and $z_0 \in \Omega$, then

$$|f(z_0)| \leq \sup_{z \in \Omega} |f(z)|$$

with equality if and only if f is constant.

§1.3 Schwarz Lemma

Theorem 1 (Schwarz Lemma)

For $f : B_1(0) \rightarrow \mathbb{C}$ holomorphic with $|f(z)| \leq 1$ for all z and $f(0) = 0$. Then

$$|f(z)| \leq |z|, |f'(0)| \leq 1.$$

If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = cz$ for some $|c| = 1$.

Proof. Define a function

$$g(z) = \begin{cases} f(z)/z, & \text{if } 0 < |z| \leq 1 \\ f'(0), & \text{if } z = 0 \end{cases}.$$

Note that $g(z)$ is continuous since at zero,

$$\lim_{z \rightarrow 0} \frac{f(z)}{z} = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Hence, $|g(z)| \leq C < \infty$ using the Weierstrass Extreme Value theorem. If $0 < \epsilon < |w| < r < 1$, note that taking a Keyhole Contour, we have

$$g(w) = \frac{1}{2\pi i} \left(\int_{|z|=r} - \int_{|z|=\epsilon} \right) \frac{g(z)}{z-w} dz.$$

Note that

$$\left| \int_{|z|=\epsilon} \frac{g(z)}{z-w} dz \right| \leq (2\pi\epsilon) \cdot C \frac{1}{|w|-\epsilon} \xrightarrow{\epsilon \rightarrow 0} 0.$$

It follows that

$$g(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(z)}{z-w} dz$$

for $0 < |w| < r$. The right side is holomorphic in w if $|w| < r$, so it follows that

$$g(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(z)}{z-w} dz$$

is holomorphic in $|z| < 1$.

This can also be proved by taking a Taylor series about the origin. Since there is no constant term, we can divide by z to still have a convergent Taylor series.

If $r < 1$,

$$\sup_{|z| \leq r} |g(z)| = \sup_{|z|=r} |g(z)| \leq \sup_{|z|=r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

If we let $r \uparrow 1$, then we get $\sup_{|z| < 1} |g(z)| \leq 1$. It follows that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

If $|f(z_0)| = z_0$ for some $0 < |z_0| < 1$ then $|g(z_0)| = 1$ and g is constant by the maximum principle so $g(z) = c$, $f(z) = cz$. If $|f'(0)| = 1$, then $|g(0)| = 1$ so g is constant and $f = cz$. \square

§1.4 Maximum Principles

In the above proof, we used the maximum principle. Some other versions we will use are the following:

If $K \subset \mathbb{C}$ compact and $f : K \rightarrow \mathbb{C}$ continuous, and the restriction of f to the interior of K is holomorphic, then

$$\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|.$$

If Ω is open and connected, $f : \Omega \rightarrow \mathbb{C}$, $z_0 \in \Omega$, and $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$, then f is constant. Applying this to e^f and using that $|e^f| = e^{\operatorname{Re} f}$, we find that

$$\operatorname{Re} f(z_0) = \sup_{z \in \Omega} \operatorname{Re} f(z),$$

implies that f is constant. We have the same result for $\operatorname{Im} f$ by replacing f with $-if$.

§1.5 Homework I

Show the Automorphisms of the unit disk are fractional linear transformations.

Proof. Following the hint, define $h = g \circ f$. Note that $h : \{|z| < 1\} \rightarrow \{|z| < 1\}$ is a holomorphic bijection as a composition of holomorphic bijections. Furthermore, note that

$$h(0) = \frac{f(0) - z_0}{1 - \overline{z_0}f(0)} = \frac{z_0 - z_0}{1 - \overline{z_0}f(0)} = 0.$$

Note that we can apply the Schwarz lemma to h and h^{-1} since the ranges of both functions are $\{|z| < 1\}$. It follows that $|h'(0)| \leq 1$ and

$$1 \geq |(h^{-1})'(0)| = \left| \frac{1}{h'(h^{-1}(0))} \right| = \left| \frac{1}{h'(0)} \right|.$$

It follows that $|h'(0)| = 1$, so by the equality case of the Schwarz lemma, $h(z) = cz$ for some $c \in \mathbb{C}$.

It follows that

$$h(z) = \frac{f(z) - z_0}{1 - \overline{z_0}f(z)} = cz \Rightarrow f(z) = \frac{z_0 + cz}{1 + c\overline{z_0}z} = \frac{a + bz}{c + dz}$$

for $a, b, c, d \in \mathbb{C}$. □

§2 January 25th, 2021

§2.1 Uniform Convergence

Remark 2.1. They sometimes call open connected sets "regions".

Definition 2.2 (Uniform Convergence). Let $\Omega \subset \mathbb{C}$ be open. Let $f_n : \Omega \rightarrow \mathbb{C}$ be holomorphic and $f : \Omega \rightarrow \mathbb{C}$ a function so that $\lim_{n \rightarrow \infty} \sup_{z \in K} |f(z) - f_n(z)| = 0$ for all $K \subset \Omega$ compact (also denoted $K \subset\subset \Omega$).

Remark 2.3. Recall from real analysis that f is a continuous function.

Some further remarks:

- It suffices to check the result for a sequence of compact subsets K_m so that $\bigcup_m K_m^\circ = \Omega$, then it suffices to check those. If $K \subset\subset \Omega$, then K is compact and covered by the union of the subsets so there exists a finite subcovering, and uniform convergence on the subcovering implies uniform convergence on K .
- It is often convenient to introduce $\|g\|_K = \sup_{z \in K} |g(z)|$. Uniform convergence can be restated as $\|f_n - f\|_K \rightarrow 0$ for all $K \subset\subset \Omega$.
- If $\|f_n - f\|_K \rightarrow 0$ for all $K \subset\subset \Omega$, then f is also holomorphic. It follows by passing to the limit in the Cauchy Integral formula. Namely, take $\{z : |z - z_0| \leq R\} \subset \Omega$ and consider the points in $|z_0 - \zeta| < R$.

$$\begin{aligned} \left| f_n(\zeta) - \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z-\zeta} dz \right| &= \left| \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f_n(z)}{z-\zeta} dz - \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z-\zeta} dz \right| \\ &\leq \frac{1}{2\pi} \frac{1}{R - |z_0 - \zeta|} \cdot (2\pi R) \|f_n - f\|_{|z-z_0|=R} \rightarrow 0. \end{aligned}$$

So it follows that

$$f(\zeta) = \lim_{n \rightarrow \infty} f_n(\zeta) = \frac{1}{2\pi i} \int_{|z-z_0|} \frac{f(z)}{z-\zeta} dz.$$

It follows that f continuous on $|z - z_0| = R$ is holomorphic in $\zeta \in \{|z - z_0| < R\}$, so it follows that f is holomorphic.

- We can similarly show that

$$f_n^{(j)}(\zeta) = \frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f_n(z)}{(z-\zeta)^{n+1}} dz$$

$$\text{and } \|f_n^{(j)} - f^{(j)}\|_K \rightarrow 0.$$

From the last item, we have the following theorem.

Theorem 2

If $f_n \rightarrow f$ on compact subsets of Ω , then if f_n is holomorphic we find that f is holomorphic and $f_n^{(j)} \rightarrow f^{(j)}$ uniformly on compact subsets of Ω .

Theorem 3 (Hurwitz)

Let Ω be a region, $f : \Omega \rightarrow \mathbb{C}$ and $f_n : \Omega \rightarrow \mathbb{C}$ holomorphic with $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$ and $\|f_n - f\|_K \rightarrow 0$ for all compact subsets. Then either $f \equiv 0$ or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

Proof. If f is not identically zero on ω , then since f is holomorphic, its zeros are isolated. If $z_0 \in \Omega$, $f(z_0) = 0$, then there is $\epsilon > 0$ so that when $0 < |z - z_0| < \epsilon$, $f(z) \neq 0$.

Since $f(z) \neq 0$ for $|z - z_0| = \epsilon/2$, by the Weierstrass theorem applied to $|f|$ on $|z - z_0| = \epsilon$, we have $|f(z)| \geq m > 0$ on $\{|z - z_0| = \epsilon/2\} = \Gamma$. If $\|f_n - f\|_\Gamma \leq m/2$ for $n \geq N$, then

$$|f_n(z)| \geq |f(z)| - m/2 \geq m - m/2 = m/2$$

for $z \in \Gamma$. Hence, it follows that $\|1/f_n - 1/f\|_\Gamma \rightarrow 0$ (we leave this as an exercise).

Since $\|f'_n - f'\|_\Gamma \rightarrow 0$, we find that $\|f'_n/f_n - f'/f\| \rightarrow 0$ (another exercise) and hence

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'_n}{f_n} dz \rightarrow \frac{1}{2\pi i} \int_\Gamma \frac{f'}{f} dz.$$

The integrand of the left hand side is $(\log f_n)'$, whose integral is 0, and the right side is the order of the zero of f at z_0 by the argument principle. It follows that the order of z_0 as a possible zero is 0, so $f(z_0) \neq 0$. \square

Theorem 4

For $\Omega \subset \mathbb{C}$ open, \mathcal{F} a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω .

Proof. We first show 2 implies 1. If $\sup_{f \in \mathcal{F}} \|f\|_K = \infty$, then we can find for each $n \in \mathbb{N}$ $f_n \in \mathcal{F}$ so that $\|f_n\|_K \geq n$. If we abstract a convergence subsequence, then $\|f_{n_j} - f\|_K \leq C < \infty$ and $\|f_{n_j}\|_K \leq \|f\|_K + C$, while $\|f_{n_j}\|_K \rightarrow \infty$, a contradiction. \square

§3 January 27th, 2021

Theorem 5

For $\Omega \subset \mathbb{C}$ open, \mathcal{F} a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω .

I missed the beginning of the class, but I will add the proof of the theorem once notes are posted.

One can put a metric on holomorphic functions so that convergence in the metric is uniform convergence on compact sets. For $f : \Omega \rightarrow \mathbb{C}$, but $K_n \Subset \Omega$ so that $\bigcup_n K_n^\circ = \Omega$ and take

$$d(f, g) = \sum_{n=1}^{\infty} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}} 2^{-n}.$$

§3.1 Riemann Sphere

On the set $\mathbb{C} \cup \{\infty\}$, we consider the topology which makes it the Alexandroff (one-point) compactification of \mathbb{C} . If $z \in \mathbb{C}$, a neighborhood is one that contains a neighborhood in \mathbb{C} and a neighborhood of ∞ is of the form $\{\infty\} \cup (\mathbb{C} \setminus K)$ for $K \Subset \mathbb{C}$.

Let $U_+ = \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ and $U_- = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$. Note that the union of the two sets covers the Riemann Sphere. Define $\psi_+ : U_+ \rightarrow \mathbb{C}$ by $\psi_+(z) = z$ and $\psi_- : U_- \rightarrow \mathbb{C}$ is given by $\psi_-(w) = 1/w$ if $w \in \mathbb{C} \setminus \{\infty\}$ and 0 if $w = \infty$. Notice that these two functions are bijections.

If $V \subset \mathbb{C} \cup \{\infty\}$ is open, a function $f : V \rightarrow \mathbb{C}$ is holomorphic if

$$f|_{V \cup U_{\pm}} \circ (\psi_{\pm}|_{V \cup U_{\pm}})^{-1} : \psi_{\pm}(V \cup U_{\pm}) \rightarrow \mathbb{C}$$

is holomorphic. In this way, we know what holomorphic functions are on open sets of $\mathbb{C} \cup \{\infty\}$.

More generally, we can describe a Riemann surface in the following way - Let X be a topological space. Take $\{(U_{\alpha}, z_{\alpha})\}_{\alpha \in I}$ where $U_{\alpha} \subset X$ is open, and $\bigcup_{\alpha \in I} U_{\alpha} = X$ and $z_{\alpha} : U_{\alpha} \rightarrow \mathbb{C}$ is continuous, $z_{\alpha}(U_{\alpha})$ is open and z_{α} is a homeomorphism. The key requirement is that the maps $z_{\alpha} \circ z_{\beta}^{-1} : z_{\beta}(U_{\alpha} \cap U_{\beta}) \rightarrow z_{\alpha}(U_{\alpha} \cap U_{\beta})$ are holomorphic.

Then, if $U \subset X$ is open, $f : U \rightarrow \mathbb{C}$ is holomorphic if for all $\alpha \in I$,

$$f|_{U \cap U_{\alpha}} \circ (z_{\alpha}|_{U \cap U_{\alpha}})^{-1}$$

is holomorphic. Two such atlases give the same Riemann surface if put together, we get an atlas.