Math 258 Lecture Notes, Fall 2020

Harmonic Analysis

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Contents

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§1.1 Introduction

We begin by considering the problem of conduction of heat in a circle. We use the map $x \mapsto e^{ix}, x \in [0, 2\pi)$. Where u is the temperature, t is the time, we believed that $u_t = \gamma u_{xx}$, where subscripts denote partial derivatives. We also have an initial condition, $f(x) = u(x, 0)$.

There are some simple solutions $e^{inx}e^{-\gamma n^2 t}|_{t=0} = e^{inx}$. The product of solutions, the sum of solutions, and scalar multiple of solutions are all solutions, so he wrote the solution as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, u(x,t) = \sum_n a_n e^{-\gamma n^2 t} e^{inx}.$$

§1.2 Fourier Analysis

We take a circle $\{z \in \mathbb{C} : |z = 1|\}$, which can also be thought of as $\mathbb{R}/(2\pi\mathbb{Z})$, with the map $x \mapsto e^{ix}$. Suppose we have G a finite abelian group, and $\hat{G} = \{\hom \varphi : G \to \mathbb{R}/\mathbb{Z}\}$, the dual group. $\hat{G}$ is also a group, formally known as the set of characters.

Example 1.1

If we take $G = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, with the map $x \mapsto e^{2\pi i x n/N}$, for $n \in Z_n$.

Similarly, taking $G \cong Z_{N_1} \times \mathbb{Z}_{N_2} \times \dots$, we take $x \mapsto \prod e^{2\pi i x n/N_i}$.

Take $e_\xi(x) = e^{2\pi i \xi(x)}$, where $\xi : G \mapsto \mathbb{R}/\mathbb{Z}$. Working in $L^2(G)$, we note the following:

Fact 1.2. If $\xi \neq \varphi$, then $\langle e_\xi, e_\varphi \rangle = 0$.

Proof.

$$\sum_{x \in G} \xi(x)\overline{\varphi(x)} = \sum_u \xi(u+y)\overline{\varphi(u+y)} - \left(\sum_u \xi(u)\overline{\varphi(u)}\right) \xi(y)\overline{\varphi(u)}.$$

Hence, either $\langle \xi, \varphi \rangle = 0$ or $\xi(y)\overline{\varphi}(y) = 1$ for all $y \in G$, which implies $\xi = \varphi$. □

If follows that $\{e_f : f \in \hat{G}\}$ is an orthonormal set in $L^2(G)$ Then, the dimension is $|\hat{G}| = |G| = \dim(L^2(G))$. Hence, the set forms an orthonormal basis for $L^2(G)$.

Then, for all $f \in L^2(G)$, we have

$$\|f\|^2_{L^2(G)} = \sum_{\varphi \in \hat{G}} |\langle f, e_\xi \rangle|^2,$$

$$f = \sum_{e_\xi \in \hat{G}} \langle f, e_\xi \rangle e_\varphi.$$

§1.3 On Tori of Arbitrary Dimension

We define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, from $[0, 2\pi]$. We then work on $\mathbb{T}^d$, $d \geq 1$.

For $f \in L^2(\mathbb{T}^d)$, we define

$$\hat{f}(n) = (2\pi)^{-d} \int f(x) e^{-inx} dx.$$

We have an inner product $\langle f, g \rangle = \int_{\mathbb{T}^d} f(x)\overline{g(x)} d\mu(x)$ defined over a Lebesgue measure or Euclidean measure on $\mathbb{T}^d$.

Theorem 1 (Parseval's Theorem)
For all $f \in L^2(\Pi^d)$,

$$\|f\|_{L^2}^2 = (2\pi)^d \sum_{n \in \mathbb{Z}^d} |\hat{f}(n)|^2,$$

and we have

$$f = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{inx},$$

in the sense that

$$\|f - \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{inx}\|_L^2 \to 0.$$

Note: you can usually figure out the constant with the simplest example, $f = 1$.

Proof. Take $\mathbb{T}^d, e_n(x) = e^{in \cdot x}$. The $\{(2\pi)^{-d/2} e^n : n \in \mathbb{Z}^d\}$ is orthonormal(left as an exercise). Then, for all f, $\sum_n \langle f, (2\pi)^{-d/2} e_n \rangle \leq \|f\|_{L^2}^2$, with equality if the set is a basis(Bessel's inequality).

It suffices to show that $\operatorname{span}\{e_n\}$ is dense in L^2. Take $P = \operatorname{span}\{e_n\}$, and note that P is an algebra of continuous functions on Π^d, closed under conjugation, contains 1, and separates points. Hence, the Stone-Weierstrass theorem implies that P is dense in $C^o(\Pi^d)$ with respect to $\| \cdot \|_{C^o}$. Then $C^o \subset L^2$ is dense(general theory about Compact Hausdorff spaces, Radon Measures).

The statement $\|f - \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{inx}\|_L^2 \to 0$ follows from the general theory of orthonormal systems. $\square$

§1.4 Euclidean Spaces

We work in $\mathbb{R}^d, (d \geq 1)$. Take $\xi \in R^d$, and $x \mapsto x\xi \in \mathbb{R}$ is a homomorphism from $\mathbb{R}^d \to \mathbb{R}$, but if we take $x \mapsto e^{ix\xi}$, we have a homomorphism from $\mathbb{R}^d \mapsto \Gamma$. We try to define the following:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx = \langle f, e_\xi \rangle_{L^2(\mathbb{R}^d)},$$

where $e_{xi}(x) = e^{ix\xi}$.

Some problems:

1. $e_\xi \notin L^2(\mathbb{R}^d)$
2. $f(x)e^{-ix\xi}$ need not be in L^1 if $f \in L^2$.

We fix this by imposing extra conditions.

Definition 1.3. For $f \in L^1(\mathbb{R}^d)$, we define

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx.$$

Note that $f \in L^1$ implies that $\hat{f}$ is bounded, continuous. We see this as follows: $\hat{f}(\xi + u) - \hat{f}(\xi) = \int f(x) e^{-ix\xi}(e^{-ixu} - 1) dx$. If we let $u \to 0$, the right goes to 0 pointwise, and $(2|f|) \in L^1$ dominates the integral, it goes to 0.

Proposition 1.4

If $f \in L^1 \cap L^2(\mathbb{R}^d)$, $\hat{f} \in L^2(\mathbb{R}^d)$,

$$\|\hat{f}\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2.$$

Theorem 2 (Plancherel's Theorem)

$\pi : L^1 \cap L^2 \to L^2$ extends uniquely to $\hat{\pi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, linear, bounded, $\|\hat{\pi} f\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2$, and for all $f \in L^2$, we have an inverse Fourier Transform, $\check{f}(y) = \int f(\xi) e^{iy\xi} d\xi$ for $f \in L^1 \cap L^2$, and $\check{\cdot}$ also extends.

Finally,

$$\|f - (2\pi)^{-d} \int_{|\xi| \le R} \hat{f}(\xi) e^{ix\xi} d\xi \|_{L^2} \to 0.$$

Note that $\check{f}(y) = \hat{f}(-y)$.

Proof. We first prove that $\|f\|_{L^2}^2 = (2\pi)^{-d} \|\hat{f}\|_{L^2}^2$ for all $f \in L^1 \cap L^2$. We prove this for a dense subspace $\mathscr{P}$ of L^2. We will show later that there exists a subspace $V \subset L^2(\mathbb{R}^d)$ so that V is dense in L^2, $V \subset L^1$, $\forall f \in V$, there exists $C_f < \infty$, so for all $\xi \in \mathbb{R}^d$, $|\hat{f}(\xi)| \le C_f (f(\xi))^{-d}$ and f is continuous with compact support.

We are given $f : \mathbb{R}^d \to \mathbb{C}$ supported where $|x| \le R = R_f < \infty$. For large $t \ge 0$, define $f_t(x) = f(tx)$(this shrinks the support of f), supported where $|x| \le R/t < \pi$. We can then think of $f_t : \mathbb{T}^d \to \mathbb{C}$.

Now, we calculate

$$\begin{aligned}
\hat{f}_t(n) &= (2\pi)^d \int_{\mathbb{T}^d} f_t(x) e^{-inx} dx \\
&= t^{-d} (2\pi)^d \int_{R^d} f(x) e^{-in/ty} dy \\
&= t^{-d} (2\pi)^{-d} \hat{f}(t^{-1} n),
\end{aligned}$$

where the first hat is on $\mathbb{T}^d$ and the second is on $\mathbb{R}^d$, so the Fourier coeficients in the euclidean case are scalar multiples of the Fourier coefficients in the Tori case.

Thus,

$$\|f_t\|_{L^2(\mathbb{T}^d)}^2 = t^{-d} \|f\|_{L^2(\mathbb{R}^d)}^2 = c_d \sum_{n \in \mathbb{Z}^d} |\hat{f}_t(n)|^2 = c_d' t^{-2d} \sum_n |\hat{f}(t^{-1} n)|^2$$

Hence,

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n |\hat{f}(t^{-1} n)|^2.$$

This has a nice tiling Riemann sum interpretation: if we take $\mathbb{R}^d$ and tile it with cubes of sidelength $1/t$ where one corner is at $t^{-1} n$ for $n \in \mathbb{Z}^d$, then

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n \left| \hat{f}(t^{-1} n) \right|^2 = \int_{\mathbb{R}^d} |g_t|^2 dx,$$

where $g(x) = \hat{f}(t^{-1} n)$.

We claim
$$\int_{\mathbb{R}^d} |g_t|^2 \to \int_{\mathbb{R}^d} |\hat{f}|^2,$$
which follows from the dominated convergence theorem: where we take a sequence over t going to infinity, with dominator $C_f^2(1+|\xi|)^{-2d}$ in L^1 and $|\hat{f}(\xi)| \le C_f^2(1+|\xi|)^{-2d}$. Furthermore, we have $g_t(\xi) \to \hat{f}(\xi)$ as $t \to 0$, and $\hat{f}$ is continuous so g_t is pointwise convergent, and we have
$$|g_t(\xi)| = |\hat{f}(t^{-1}n)| \le C_f(1+|t^{-1}n|)^{-d} \le C'(1+|\xi|)^{-d}.$$

$\square$

§2 September 1st, 2020

§2.1 Proof of Plancherel's Theorem

Last time

- $\mathbb{R}^d$,
$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$
- $V = (f \in L_1 \cap L_2(\mathbb{R}^d)) : |\hat{f}(\xi)| \langle \xi \rangle^d$ is a bounded linear function, $\langle x \rangle = (1+|x|^2)^{1/2} \geq 1, = |x|$ for x large.
- Claim: V is dense in $L^2(\mathbb{R}^d)$. Then $\|\hat{f}\|_{L^2} = (2\pi)^{d/2}\|f\|_{L^2}$ for all $f \in V$ so there exists a unique bounded linear operator $\mathscr{F}$ on $L^2(\mathbb{R}^d)$, where $\mathscr{F}$ takes a function to it's fourier transform.
- We discussed some properties of $\mathscr{F}$.
  - $\|\mathscr{F} f\|_2 = (2\pi)^{d/2}\|f\|_2$
  - $\mathscr{F}$ is onto.
  - For all $f \in L^2$,
  $$\left\| f - (2\pi)^{-d} \int_{|\xi| \leq R} e^{ix\cdot\xi} \mathscr{F}(f)(\xi) d\xi \right\|_{L^2} \to 0,$$
  in the limit where $R \to \infty$.

First note that $\mathscr{F}$ has closed range(this was an exercise). It suffices to show: If $g \in L^2, g \perp \mathscr{F}(f)$ for all $f \in V$, then $g = 0$.

Proof. First, note that
$$0 = \langle g, \mathscr{F}(f) \rangle = \langle \mathscr{F}^*(g), f \rangle,$$
and for all $g \in V$,
$$\mathscr{F}^* g(x) = \int g(\xi) e^{ix\cdot\xi} d\xi$$
Therefore, $\mathscr{F}^*(g)(x) = (\mathscr{F} g)(-x)$ for all $g \in V$, which is dense in L^2. Hence, $\mathscr{F} g = 0$, and the Fourier transform preserves norms, so $g = 0$. □

We also claimed the following: Let $f \in L^2$:
$$\|f(x) - (2\pi)^{-d} \int_{|\xi| \leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi\|_2^2 \to 0.$$

Proof. Let $g_r = (2\pi)^{-d} \int_{|\xi| \leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi$.We have to show $\langle f, g_r \rangle \to \|f\|_2^2$. Then
$$\|f - g_r\|_2^2 = \|f\|_2^2 + \|g_r\|_2^2 - 2\mathrm{Re}\langle f, g_r \rangle \to \|f\|_2^2 + \|f\|_2^2 - 2\|f\|_2^2.$$

$$\begin{aligned}
\langle f, g_R \rangle &= (2\pi)^{-d} \int f(x) \overline{\int_{|\xi| \leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi dx} \\
&= (2\pi)^{-d} \int_{|\xi| \leq R} \left(\int f(x) e^{-ix\cdot\xi} dx \right) \overline{(\mathscr{F} f)(\xi) d\xi} \\
&= (2\pi)^{-d} \int_{|\xi| \leq R} |\mathscr{F} f(\xi)|^2 d\xi \to (2\pi)^{-d} \|\mathscr{F} f\|_2^2 = \|f\|_2^2.
\end{aligned}$$

However, it's not clear that we can use Fubini's theorem. We would need $f \in L^1 \cap L^2$. But this is not an issue as $L^1 \cap L^2 \subset L^2$ is dense, so if we let $\epsilon > 0, f = G + h, \|h\|_2 \leq \epsilon$ and $G \in L^1 \cap L^2$. Showing the convergence from here is an exercise. $\square$

We still need $V = (f \in L^1 \cap L^2 : \langle \xi \rangle^d (\hat{f}(\xi))$ is bounded) is dense in L^2. We'll discuss this in the future.

§2.2 Introduction to Convolution

Our meta definition is $f * g(x) = \int f(x-y)g(y)dy$, but it will depend on the conditions of the function for the integral to be defined.

Convolution is generally associated to a group, where

$$\int_G f(xy^{-1}g(y)d\mu(y)),$$

with the Haar measure(done in 202b).

If we substitute $y = x - u$, then

$$f * g(x) = \int f(u)g(x-u)du = g * f(x).$$

It is also associative: $(f * g) * g = f * (g * h)$ for all f, g, h(involves Fubini's theorem).

We can formally write

$$f * g(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u)g(v)d\lambda_x(u,v),$$

where λ_x is supported on $\Lambda = \{(u,v) : u + v = \lambda\}$(an affline subspace). If we have a subset $E \subset \Lambda$, $\lambda_x(E) = |\pi_1(E)| = |\pi_2(E)|$, where π_i are Lebesgue measure s of projections on the i-th factor. Note the following: suppose that f, g are continuous with compact support. Then $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$, where $A + B = \{a + b : (a,b) \in A \times B\}$.

Let $T : C_0^0(\mathbb{R}^d) \to C_b^0(\mathbb{R}^d)$ be bounded, linear and $T \circ \tau_y = \tau_y \circ T$ for all $x \in \mathbb{R}^d$ ($\tau_y f(x) = f(x+y)$, a translation). Then, there exists a Complex Radon measure μ on $\mathbb{R}^d$ so that for all $f \in C_0^0$, $T(f) = f * \mu$, where

$$f * \mu(x) = \int f(x-y)d\mu(y).$$

In the case of $\mathbb{T}^1$, $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ for all $f \in L^2$. Suppose we wanted to consider the partial sums,

$$\sum_{n=-N}^{N} \hat{f}(n)e^{inx} = S_N(f)(x).$$

In what sense does $S_N f \to f$, and for which functions f do we have convergence?

$$S_N(f)(x) = \sum_{n=-N}^{N} e^{inx}(2\pi)^{-1}\int_{-\pi}^{\pi} f(y)e^{-iny}dy = (2\pi)^{-1}\int f(y)\sum_{n=-N}^{N} e^{in(x-y)}dy$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi} f(y)D_n(x-y)dy.$$

The Dirichlet Kernels, $D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin{(N+1/2)x}}{\sin{(x/2)}}$ if $x \neq 0$ or $D_N(x) = 2N+1$ if $x = 0$.

§2.3 General Convolution

Theorem 3

Let $f, g \in L^1(\mathbb{R}^d)$. Then, the following are true:

- $y \mapsto f(x-y)g(y) \in L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- If $f, g \geq 0$, then $\|f * g\|_1 = \int f * g = \int f \int g$.
- The operation commutative and associative, so L^1 is an algebra, but it no multiplicative identity, so no inverses.
- For $f, g \in L^1$, $(f \hat{\star} g) = \hat{f} \cdot \hat{g}$.

In other words, convolution is a nice bilinear operation.

Proof. Let $F(x,y) = f(x-y)g(y)$, $F : \mathbb{R}^{d+d} \to \mathbb{C}$ is Lebesgue measurable. We claim that $F \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. It follows from

$$\int |F(x,y)| dx dy = \int |f(x-y)||g(y)| dx dy = \int |g(y)| dy \int |f(x)| dx = \|g\|_1 \|f\|_1 < \infty.$$

Now, $F \in L^1$, so by Fubini's theorem, for almost every $x, y \to f(x-y)g(y) \in L^1$ and $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.

$$\|f*g\|_1 = \int |f*g(x)| dx = \int \left| \int f(x-y)g(y)dy \right| dx \leq \int \int |f(x-y)||g(y)| dy dx = \|f\|_1 \|g\|_1.$$

Note that $\int (f*g)(x)dx = \|f\|_1 \|g\|_1$, for non-negative functions.

Finally,

$$\begin{aligned}
(f*g)^\wedge(\xi) &= \int e^{-ix\cdot\xi} \left(\int f(x-y)g(y)dy \right) dx \\
&= \int \left(\int e^{-ix\cdot\xi} f(x-y)dx \right) dy, x = u + y \\
&= \int \left(e^{-i(u+y)\cdot\xi} f(u)du \right) g(y)dy \\
&= \int e^{-iy\cdot\xi} \hat{f}(u) g(y) dy \\
&= \hat{f}(\xi) \cdot \hat{g}(\xi).
\end{aligned}$$

$\square$

Example 2.1 (A Warning)

In $\mathbb{R}^1$, $f(x) = |x|^{-2/3} 1_{|x| \leq 1}$, which has an asymptote at 0. $f \in L^1$, and

$$(f*f)(0) = \int_{-1}^{1} |u|^{-4/3} dy = +\infty.$$

Proposition 2.2

Let $p \in [1, \infty]$. Let $f \in L^1, g \in L^p$. Then,

- $y \mapsto f(x-y)g(y) \in L^1$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^p(\mathbb{R}^d)$, $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. For $p = \infty$, $\int f(x-y)g(y)dy \in C_0(\mathbb{R}^d)$.

If $1 < p < \infty$, $L^P \subset L^1 + L^\infty$, as follows:

$$f(x) = f(x)1_{|f(x)| \leq 1} + f(x)1_{f(x) > 1}.$$

We can prove the rest with Minkowski's inequality, or a simpler way. Let $q = p' = \frac{p}{p-1}$ (hence $\frac{1}{q} + \frac{1}{p} = 1$). We use the norm definition,

$$\|f * g\|_p = \sup_{\|h\|_q \leq 1} \int |g * f| \cdot |h|.$$

$$\int |g * f| \cdot h \leq \int (|g| * |f|) \cdot h = \int \int |g(x-y)||f(y)|dyh(x)dx$$

$$= \int |f(y)| \int |g(x-y)|h(x)dxdy \leq \int |f(y)| \|g\|_p * 1dy = \|f\|_1 \|g\|_p.$$

$\square$