Analytic Methods in Geometry

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We present several computation results and theorems that prove to be useful in geometry problems at the Olympiad level. This includes Cartesian coordinates, complex numbers, barycentric coordinates, etc.

Contents

§1 Computational Geometry

§1.1 Cartesian Coordinates

- (Shoelace Formula) Given three points $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$, the signed area of triangle ABC is given by
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$
The area of a triangle ABC is positive if A, B, C appear in counterclockwise order.

- Three points are collinear if and only if the area of the triangle they determine is zero. This gives a symmetric formula for determining collinearity.

- (Point-to-Line Distance) If ℓ is the line determined by $Ax + By + C = 0$, the shortest distance from $P = (x_1, y_1)$ to ℓ is given by
$$d(P, \ell) = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

- (Point-to-Plane Distance) If H is the plane determined by $Ax + By + Cz + D = 0$ the shortest distance between the point $P = (x_0, y_0, z_0)$ and the plane is given by
$$d(P, H) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Problems that are most effectively solved using Cartesian coordinates usually have defining characteristics, such as

- a prominent right angle at the origin.
- many intersections or perpendicular lines.

§1.2 Areas

The area of a triangle ABC is given by

- $\frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$
- $\frac{a^2 \sin B \sin C}{2 \sin A}$
- $\frac{abc}{4R}$, R is the circumradius.
- sr, s is the semi-perimeter, r is the inradius.
- $\sqrt{s(s-a)(s-b)(s-c)}$.

A useful result from trignometry:

Lemma 1.1

If x, y, z satisfy $x + y + z = 180°$ and $0° < x, y, z < 90°$, then $\tan x + \tan y + \tan z = \tan x \tan y \tan z$.

Example 1.2

In ABC, $AB = 13$, $BC = 14$, $CA = 15$. Find the length of the altitude from A onto $\overline{BC}$.

Proof. Note that $s = (13 + 14 + 15)/2 = 21$. It follows that

$$[ABC] = \sqrt{(21)(21-13)(21-14)(21-15)} = \sqrt{(21)(8)(7)(6)} = 84.$$

Then,

$$[ABC] = \frac{1}{2} h_A \cdot BC = 84$$

so it follows that $h_A = \frac{2 \cdot 84}{BC} = 12$. □

§1.3 Trignometry

- (Law of Sines)
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
- (Law of Cosines) Given a triangle ABC, we have
$$a^2 = b^2 + c^2 - 2bc \cos A.$$
- (Product-Sum)
$$\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\ 2\sin\alpha\sin\beta &= \cos(\alpha-\beta) - \cos(\alpha+\beta) \\ 2\sin\alpha\cos\beta &= \sin(\alpha-\beta) + \sin(\alpha+\beta) \end{aligned}$$

§1.4 Ptolemy's Theorem

Theorem 1 (Ptolemy's Theorem)
Let $ABCD$ be a cyclic quadrilateral. Then

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Proof. We prove the result using trignometry. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the angles $\angle ADB$, $\angle BAC$, $\angle CBD$, $\angle DCA$ respectively. WLOG let $(ABCD)$ have unit diameter. It follows from the law f sines that

$$AB = \sin\alpha_1, BC = \sin\alpha_2, CD = \sin\alpha_3, DA = \sin\alpha_4.$$

Furthermore,

$$AC = \sin\angle ABC = \sin(\alpha_3 + \alpha_4),$$

$$CD = \sin\angle DAB = \sin(\alpha_2 + \alpha_3).$$

It suffices to show that for $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 180^\circ$,

$$\sin\alpha_1 \sin\alpha_3 + \sin\alpha_2 \sin\alpha_4 = \sin(\alpha_3 + \alpha_4)\sin(\alpha_2 + \alpha_3).$$

This result can be easily verified through the product to sum identities. □

Theorem 2 (Strong Form of Ptolemy's Theorem)
In a cyclic quadrilateral $ABCD$ with $AB = a$, $BC = b$, $CD = c$, $DA = d$, we have

$$AC^2 = \frac{(ac + bd)}{ab + cd}$$

and

$$BD^2 = \frac{(ac + bd)(ab + cd)}{ad + bc}.$$

Proof. The proof follows from setting

$$AC^2 = a^2 + b^2 - 2ab\cos\angle ABC = c^2 + d^2 - 2cd\cos\angle ADC$$

and noting that $\angle ADC + \angle ABC = 180°$. $\square$

§1.5 Problems