Math 222b Lecture Notes

Partial Differential Equations II

Professor: Maciej Zworski, Spring 2021
Scribe: Vishal Raman

Contents

§1 January 19th, 2021

§1.1 Review of Sobolev Spaces

Definition 1.1. Given $u \in \mathcal{D}'(U)$ for $U \subseteq \mathbb{R}^n$ open: that means that $u : C_c^\infty(U) \to C$ and for every compact set $K \subset\subset U$, $\exists C, N$ for all $\varphi \in C_0^\infty(K)$ such that

$$|u(\varphi)| \leq C \sup_{|\alpha| \leq N} |\partial^\alpha \varphi|.$$

Examples:

- Take $U = (0, 1)$ and take $u = \sum_{\mathbb{N}} \delta_{1/n}$, where $\delta_{1/n}(\varphi) = \varphi(1/n)$.
- Take $u \in L^1_{\text{loc}}(U)$, where $u(\varphi) = \int u\varphi$. Differentiation is defined formally though integration by parts as $\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$.

Definition 1.2. The Sobolev spaces $W^{k,p}(U) = \{u \in L^1_{loc}(U) : \partial^\alpha u \in L^p(U), \forall |\alpha| \leq k\}$, for $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$. Note that differentiation is in the sense of distributions. We write $H^k(U) = W^{k,2}(U)$, which are Hilbert spaces with the inner product

$$\langle u, v \rangle = \sum_{|\alpha| \leq k} \int_U \partial^\alpha u \overline{\partial^\alpha v}.$$

Definition 1.3. $W_0^{k,p}(U) = \overline{C_c^\infty(U)}$, where the closure is with respect to the $W^{k,p}$ norm.

Theorem 1 (Approximation)

For $U \subset\subset \mathbb{R}^n$,

$$\overline{C^\infty(U) \cap W^{k,p}(U)} = W^{k,p}(U)$$

where the closure is with respect to the $W^{k,p}$.

If $\partial U \in C^1$, then we can improve up to

$$\overline{C^\infty(\overline{U}) \cap W^{k,p}(U)} = W^{k,p}(U)$$

Theorem 2 (Extension)

If $U \subset\subset \mathbb{R}^n$ and $\partial U \in C^1$, for $U \subset\subset V \subset\subset \mathbb{R}^n$, there exists $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu|_U = u$ and the supp $u \subset\subset V$.

We can extend this to $W^{k,p}$ if the boundary is C^k.

Theorem 3 (Traces)

For $U \subset\subset \mathbb{R}^n$ with $\partial U \in C^1$, there exists $T : W^{1,p}(U) \to L^p(\partial U)$ which is linear and boundary such that for $u \in C(\overline{U}) \cap W^{1,p}$ $Tu = u|_{\partial U}$.

Example 1.4

For $U \subset\subset \mathbb{R}^n$, ∂U bounded,

$$H_0^1(U) = \{u \in H^1 : Tu = 0 \in L^2(\partial U)\}.$$

The converse of showing $Tu = 0$ implies H_0^1 is the more difficult one.

§1.2 Fourier Transform

We first review the Fourier Transform. We define the Schwartz space:

$$\mathcal{S} = \{\varphi \in C^\infty(\mathbb{R}^n) : x^\alpha \partial^\beta \varphi \in L^\infty \forall \alpha, \beta \in \mathbb{N}^n\}.$$

For $\varphi \in \mathcal{S}$, we define

$$\widehat{\varphi}(\xi) = \int \varphi(x) e^{-ix\cdot\xi}\, dx.$$

Note that $\mathcal{F}$, the Fourier transform is invertible on §. The key properties of the fourier transform are

$$\mathcal{F}(1/i\partial x\varphi) = \xi\mathcal{F}\varphi, F(x\varphi) = -1/i\partial_\xi \mathcal{F}\varphi.$$

We also have

$$\mathcal{F}^{-1} = \frac{R\mathcal{F}}{(2\pi)^n}, R\varphi(x) = \varphi(-x).$$

We define $\mathcal{S}'$ onto $\mathbb{C}$ so that for $u \in \mathcal{S}'$, there exists C, N such that

$$|u(\varphi)| \le C \sup_{|\alpha|,|\beta|\le N} |x^\alpha \partial^\beta \varphi|.$$

Note that $\mathcal{S}' \subset \mathcal{D}'$.

Definition 1.5. $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ by $\widehat{u}(\varphi) = u(\widehat{\varphi})$.

Examples:

- $\widehat{\delta_0}(\varphi) = \delta_0(\widehat{\varphi}) = \widehat{\varphi}(0) = \int \varphi = 1(\varphi)$.
- Take $\mathbb{R}^2$ and consider $u(x) = \frac{1}{|x|}$. This function is in L^1_{loc}. If we multiply by $(1 + |x|)^{-2} u \in L^1(\mathbb{R}^n)$, it follows that $u \in \mathcal{S}'$, since

$$|u(\varphi)| = \left| \int (1 + |x|)^{-2} u(1 + |u|)^2 \varphi \right| \le C \sup(1 + |x|)^2 \varphi.$$

  Now, we compute $\widehat{u} \in \mathcal{S}'$. Since $\mathcal{F}$ is continuous on $\mathcal{S}'$, we approximate u and hope the result converges to the desired result. Define $u_\epsilon \to u$ in $\mathcal{S}'$ for $u_\epsilon \in L^1$.

  Try $u_\epsilon(x) = \frac{e^{-\epsilon|x|^2/2}}{|x|} \in L^1$ for $\epsilon > 0$. We want to calculate $\widehat{u}_\epsilon$ and take the limit as $\epsilon \to 0^+$. We can evaluate the integral by converting to polar coordinates and completing the square. Unfortunately, it reduces to an integral that is too hard, but we will learn asymptotics of the integral as $\epsilon \to 0$. We find that $\widehat{u}(\xi) = 2\pi/|\xi|$.

  We can approach this differently. Note that $u = 1/|x|$ is homogeneous: $u(tx) = t^a u(x)$ for $t > 0$, for functions. For distributions, we have that for $\varphi \in S$, $u(\varphi(\cdot/t)t^{-n}) = t^a u(\varphi)$ for $t > 0$. For the Fourier Transform, if $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree a, then $\widehat{u}$ is homogeneous of degree $-n - a$. It follows that our Fourier transform is of degree -1.

  Furthermore, note that $1/|x|$ is spherically symmetric, and the Fourier transform preserves spherical symmetry(note that the Jacobian factor for rotations is 1). It follows that the fourier transform is also spherically symmetric. It follows that

$$\mathcal{F}(1/|x|) = C/|\xi| + \sum_{|\alpha \le N|} c_\alpha \delta_0^{(\alpha)},$$

  but delta terms have too much homogeneity.

§2 December 21st, 2021

§2.1 Plancherel's Theorem

Recall that the Fourier transform is an isomorphism on $\mathcal{S}$ - it is a bounded linear operator whose inverse is also bounded.

Note that
$$\int \widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi = \iiint u(x)\overline{\varphi(y)}e^{-i(x-y)\xi}\,dxdyd\xi$$
In the sense of distributions, $\int e^{-i(x-y)\xi}\,d\xi = (2\pi)^n\delta(x-y)$. Hence,
$$\iiint u(x)\overline{\varphi(y)}e^{-i(x-y)\xi}\,dxdyd\xi = (2\pi)^n\int u(x)\overline{\varphi(x)}\,dx.$$

For $u, \varphi \in \mathcal{S}$, we have the following:
$$\langle \widehat{u}, \widehat{\varphi}\rangle = (2\pi)^n\langle u, \varphi\rangle.$$

This implies that
$$\|\widehat{u}\|_2 = (2\pi)^{n/2}\|u\|_2, u \in \S.$$
If $u_n \to u$ in L^2 then $u_n \to u$ in $\mathcal{S}'$ by the Cauchy-Schwartz inequality. It follows that $\widehat{u_n} \to \widehat{u}$ in $\mathcal{S}'$ but our formula shows that $\widehat{u}$ is in L^2. Hence, $\mathcal{F} : L^2 \to L^2$ and for $u, v \in L^2$, $\langle \widehat{u}, \widehat{v}\rangle = (2\pi)^n\langle u, v\rangle$.

Recall last time, we were finding the Fourier transform of $u(x) = 1/|x|$ in $\mathbb{R}^2$. For $u \in S'(\mathbb{R}^n)$ homogeneous of degree a, $\widehat{u} \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree $-n-a$. In our example, It follows that $\widehat{u}(\xi)$ is homogeneous of degree -1. We also observed that u is invariant under rotations so it follows that $\widehat{u}$ is invariant under rotations.

A function is homogeneous of degree -1 if $v(k\theta) = \frac{a(\theta)}{r}$. Since our function is invariant under rotations, $\widehat{u}(\xi) = \frac{c}{|\xi|}$ away from zero. It follows from our previous argument that $\widehat{u}(\xi) = \frac{c}{|\xi|}$ since δ terms have homogeneity of at least -2.

Note that $\langle u, \varphi\rangle = (2\pi)^2\langle \widehat{u}, \widehat{\varphi}\rangle$ and we find $\widehat{u}$ by choosing an appropriate φ.

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{\varphi(x)}{|x|}\,dx &= \int_0^{2\pi}\int_0^\infty \varphi(r)\,drd\theta\\ &= 2\pi\int_0^\infty \varphi(r)\,dr.\end{aligned}$$

Choosing $\varphi(r) = e^{-r^2/2}$, we find that the integral is $(2\pi)^{3/2}$.

Evaluating the other side,
$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^2} e^{-|x|^2/2-ix\cdot\xi}\,dx = \int e^{-\frac{1}{2}(x+i\xi)^2-\frac{1}{2}|\xi|^2} = 2\pi\int e^{-|\xi|^2/2} = (2\pi)^{5/2}.$$

It follows that $c = 2\pi$.

§2.2 Fourier Characterization of H^k spaces

Theorem 4

$H^k(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : (1+|\xi|^2)^{k/2}\widehat{u} \in L^2\}$.

Proof. Suppose that $\partial^\alpha u \in L^2$ for $|\alpha| \le k$. We know that $\|u\|_2 = (2\pi)^{-n/2}\|\widehat{u}\|$. It follows that $\widehat{\partial^\alpha u} \in L^2$. Note that $\widehat{\partial^\alpha u} = i^{|\alpha|}$
$xi^\alpha \widehat{u} \in L^2$ for all $|\alpha| \le k$.

Hence,

$$(1+|\xi|^2)^{k/2} \le C_{n,k} \sup_{|\alpha|\le k} |\xi^\alpha|.$$

So it follows that $(1+|\xi|^2)^{k/2}\widehat{u} \in L^2$.

Now, suppose $(1+|\xi|^2)^{k/2}\widehat{u} \in L^2$. It follows that $|\xi^\alpha| \le C_{k,\alpha}(1+|\xi|^2)^{k/2}$ for $|\alpha| \le k$. Hence $\xi^\alpha \widehat{u} \in L^2$ so it follows that $\partial^\alpha u \in L^2$ by Plancherel's Theorem. □

Remark 2.1. We use the notation $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$.

Note that the definition does not require $k \in \mathbb{N}$.

Definition 2.2. $H^s(\mathbb{R}^n) = \{u \in \mathcal{S}' : \langle\xi\rangle^s \widehat{u} \in L^2\}, s \in \mathbb{R}$.

Theorem 5

Suppose $u \in H^s(\mathbb{R}^n)$ and $s > \frac{1}{2}$. Then $v(y) = u(0,y), y \in \mathbb{R}^{n-1}$ satisfies $v \in H^{s-1/2}(\mathbb{R}^{n-1})$.

Remark 2.3. We should define $Tu(y) = u(0,y)$ for $u \in \mathcal{S}$. Then $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ if $s > 1/2$.

Proof. Take $u \in \mathcal{S}$. We wish to show that $\|v\|_{H^{s-1/2}(\mathbb{R}^{n-1})} \le C\|u\|_{H^s(\mathbb{R}^n)}$.

Note that

$$\widehat{v}(\eta) = \int_{\mathbb{R}^{n-1}} u(0,y)e^{-y\cdot\eta}\, dy$$

and by the Fourier Inversion Formula

$$u(0,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{u}(\xi_1,\xi')e^{iy\cdot\xi'}\, d\xi_1 d\xi',$$

so it follows that

$$\begin{aligned}
\widehat{v}(\eta) &= (2\pi)^{-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n} \widehat{u}(\xi_1,\xi')e^{-iy\cdot(\eta-\xi')}\, d\xi dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}} \widehat{u}(\xi_1,\xi')e^{iy\cdot(\xi'-\eta)}\, dy d\xi \\
&= (2\pi)^{-1}\int_{\mathbb{R}^n} \widehat{u}(\xi_1,\xi')\delta_{\xi'=\eta} d\xi \\
&= (2\pi)^{-1}\int_{\mathbb{R}} \widehat{u}(\xi_1,\eta) d\xi_1.
\end{aligned}$$

Note that up to constants

$$\|v\|_{H^{s-1/2}}^2 = \int_{\mathbb{R}^{n-1}} \langle\eta\rangle^{2s-1}|\widehat{v}(\xi)|^2\, d\eta = \int_{\mathbb{R}^{n-1}} \langle\eta\rangle^{2s-1}\left|\int \widehat{u}(\xi_1,\eta)\, d\xi_1\right|^2 d\eta.$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \langle \eta \rangle^{2s-1} \left| \int \widehat{u}(\xi_1, \eta)\, d\xi_1 \right|^2 d\eta \\
&\quad = \int \langle \eta \rangle^{2s-1} \left| \int \widehat{u}(\xi, \eta)(1 + |\xi_1|^2 + |\eta|^2)^{s/2}(1 + |\xi_1|^2 + |\eta|^2)^{-s/2} d\xi_1 \right|^2 d\eta \\
&\quad \leq \int \langle \eta \rangle^{2s-1} \int |\widehat{u}(\xi_1, \eta)|^2 (1 + |\xi_1|^2 + |\eta|^2)^s d\xi_1 \int (1 + |\xi_1|^2 + |\eta|^2)^{-s} d\xi_1\, d\eta \\
&\quad \leq \int \langle \eta \rangle^{2s-1} \langle \eta \rangle^{-2s+1} \int |\widehat{u}(\xi_1, \eta)|^2 (1 + |\xi_1|^2 + |\eta|^2)^s d\xi_1 \int (1 + u^2)^{-s} du d\eta \\
&\quad = \int |\widehat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi = \|u\|_{H^s}^2,
\end{aligned}$$

since

$$\int |\widehat{u}(\xi_1, \eta)|^2 (1 + |\xi_1|^2 + |\eta|^2)^s d\xi_1 d\eta = \int |\widehat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi.$$

$\square$

§3 January 26th, 2021

§3.1 Sobolev Spaces, continued

Recall, we have $U \subset \mathbb{R}^n$ open. We typically assume U is bounded and $\partial U \in C^1$. For these spaces, we define

$$W^{k,p}(U) = \{u \in \mathcal{D}' : \partial^\alpha u \in L^p(U), |\alpha| \le k\}.$$

Recall the extension property: there exists a map $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu|_U = u$ and $u = 0$ for $|x| > R$ for some R with $U \subset\subset B(0,R)$.

We also consider the $H^s(\mathbb{R}^n)$, the fractional Sobolev spaces: $\{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s \widehat{u} \in L^2\}$. This is a Hilbert space with the norm

$$\|u\|_{H^s}^2 = \int \langle\xi\rangle^{2s} |\widehat{u}(\xi)|^2 \, d\xi.$$

Last time, we showed that If we have $u \in H^s(\mathbb{R}^n)$ and $s > 1/2$, then $v(y) : u(0,y)$, $y \in \mathbb{R}^{n-1}$ satisfies $v \in H^{s-1/2}(\mathbb{R}^{n-1})$. Today, we will show that $H^s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ if $s > n/2$, where C_0 denotes continuous functions vanishing at infinity. This means that there exists $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ such that for $u \in \mathcal{S}$, $Tu(y) = u(0,y)$.

Theorem 6

$H^s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ if $s > n/2$.

Proof. We first prove that if $\langle\xi\rangle^s \widehat{u} \in L^2, s > n/2$ then $\widehat{u} \in L^1(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |\widehat{u}| d\xi = \int_{\mathbb{R}^n} \langle\xi\rangle^{-s} \langle\xi\rangle^2 |\widehat{u}| d\xi \le \|\langle\xi\rangle^{-s}\|_2 \|u\|_{H^s}.$$

The first term is finish precisely when $s > n/2$ [exercise: convert to polar coordinates]. This implies that $u \in L^\infty(\mathbb{R}^n)$, following from the Fourier Inversion formula.

We know that $x \mapsto \widehat{u}(\xi)e^{ix\xi}$ is continuous so it follows that $x \mapsto u(x)$ is continuous by the dominated convergence theorem. Finally $u(x) \to 0$ as $|x| \to \infty$ by the Riemann-Lebesgue lemma: if $\widehat{u} \in L^1(\mathbb{R}^n)$, then $u(x) \to 0$ as $|x| \to \infty$.

Proof. Recall $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense. Taking $v \in L^1$, taking $v_R = v(x)1_{B(0,R)}(x)$. Then $v_R \to v$ y the dominated convergence theorem. Now take $\varphi \in C_c^\infty$ with $\varphi \ge 0$, $\int \varphi = 1$ wth $\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(x/\epsilon)$. Taking $v_{R,\epsilon} = v_R * \varphi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ and $v_R * \varphi_\epsilon \to v_R$ in L^1 as $\epsilon \to 0$.

Hence, we can take $v \in \mathcal{S}$ so that $\|\widehat{v} - \widehat{u}\|_{L^1} < \epsilon/2$. Now, $|v(x)| < \epsilon/2$ if $|x| > R$, hence

$$|u(x)| \le |u(x) - v(x)| + |v(x)| < C\epsilon + \epsilon/2$$

which goes to 0 as we send $\epsilon \to 0$. □

□

§3.2 Gagliardo-Nirenberg-Sobolev(GNS) Inequalities

Theorem 3.1

If $1 \leq p < n$ and we define $p^* = \frac{np}{n-p}$, then there exists $C = C(p,n)$ so that for all $u \in C_c^\infty(\mathbb{R}^n)$,
$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_p.$$

> **Remark 3.2.** We can find the value of p^* without doing the computation through scaling.Take $u_\lambda(x) = u(\lambda x)$. We have that $\|u_\lambda\|_{p^*} \leq C\|\nabla(u_\lambda)\|_p$. Then, evaluate both sides and compare the exponent on λ.
>
> Note that the result is not true for $p = n > 1$. It is true for $p = n = 1$.

Theorem 3.3 (Morrey's Inequality)

For $n < p \leq \infty$, there exists $C = C(p,n)$ such that for $u \in C^1(\mathbb{R}^n)$, we have
$$\|u\|_{C^\gamma(\mathbb{R}^n)} \leq C(\|u\|_p + \|\nabla u\|_p),$$
where $\gamma = 1 - \frac{n}{p}$, where
$$\|u\|_{C^\gamma(\mathbb{R}^n)} = \sup |u| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\gamma}.$$

Theorem 7 (General Formulation)

Take $U \subset\subset \mathbb{R}^n$ with $\partial U \in C^1$. Take $n \in W^{k,p}(U)$.

- if $k < n/p$, then $u \in L^q(U)$ where $1/q \geq 1/p - k/n$ and $\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}}$.
- $k > n/p$, then $u \in C^{k-[n/p]-1,\gamma}(\overline{U})$ where $\gamma = [n/p] + 1 - n/p$ if $n/p \notin \mathbb{N}$ and $1 - \delta$ for all $\delta > 0$ if $n/p \in \mathbb{N}$.

§3.3 Compactness

Definition 3.4. Let B be a Banach space. A subset $K \subset B$ is compact if for every sequence $\{u_n\} \subset K$ such that $\|u_n\|_B \leq C$, there exists a convergence subsequence $u_{n_k} \to u \in B$.

> **Remark 3.5.** If $\{u : \|u\|_B \leq 1\} \subset B$ is compact, then B is finite dimensional. We can have a space $B' \subset B$ and $\{u \in B' : \|u\|_{B'} \leq 1\}$ compact in B. If we have a sequence $\{u_n\} \subset B'$ and $\|u_n\|_{B'} \leq C$ then there exists n_k, $u \in B$ such that $\|u_{n_k} - u\|_B \to 0$.

We will take $B = L^q(U)$ where $1 \leq q < p^*$ and $B' = W^{1,p}(U)$.

Theorem 8 (Rellich-Kondrachov)

The unit ball in $W^{1,p}(U)$ is compact in $L^q(U)$ for **bounded** U.

§4 January 28th, 2021

Recall the GNS inequality: if $1 \le p < n$, $p^* = \frac{np}{n-p}$, there exists $C = C(n,p)$ for all $u \in C_c^\infty(\mathbb{R}^n)$ so that $\|u|_{L^{p^*}} \le C\|\nabla u\|_{L^p}$.

If $U \Subset \mathbb{R}^n$, $\partial U \in C^1$, then there exists $C = C(n,p,U)$ such that $L^q(U) \supset W^{1,p}(U)$ for $1 \le q \le p^*$.

§4.1 Compactness

Suppose B is a Banach space and $B' \subset B$ another Banach space. We say that the inclusion $B' \subset B$ is compact if bounded sets in B' are precompact in B. In other words, for a sequence $\{u_n\} \subset B'$ with $\|u_n\|_{B'} \le M$, there exists a subsequence u_{n_k} and $u \in B$ such that $u_{n_k} \to u$ in B.

Example 4.1

Take $B = C([-1,1])$, $B' = C^1([-1,1])$ with the supremum norm on B and $\|u\|_{B'} = \sup_{|x|\le 1}(|u(x)| + |u'(x)|)$.

The inclusion is compact: if we have $\|u_n\|_{B'} \le C$, by the mean value theorem, $|u_n(x)| \le C$ and $|u_n(x) - u_n(y)| \le C|x-y|$. By Arzela-Ascoli, there exists a subsequence u_{n_k} and $u \in C$ so that $\|u_{n_k} - u\|_{C([-1,1])} \to 0$.

Example 4.2

In the previous example, take $u_n(x) = |x|1_{|x|>1/n} + (\frac{nx^2}{2} + \frac{1}{n})1_{|x|\le 1/n}$.

Then $u_n \in C^1[-1,1]$ and $\|u_n\|_{C^1[-1,1]} \le 2$. We can take a subsequence $n_k = k$ and $u(x) = |x| \in C[-1,1] \setminus C^1[-1,1]$ where $u_{n_k} \to u \in C$.

Given a Banach space B, we have the dual space $B^* = \{\text{linear } u : B \to \mathbb{C} | \forall x \in B, |u(x)| \le C\|x\|_B\}$. The is also a Banach space.

Theorem 9 (Banach-Alaoglu)

Suppose $\|u_n\|_{B^*} \le M$. Then, there exists a subsequence u_{n_k} and $u \in B^*$ such that for all $x \in B$, $u_{n_k}(x) \to u(x)$.

§4.2 Rellich-Kondrachov

Theorem 10 (Rellich-Kondrachov)

If $U \Subset \mathbb{R}^n$, $\partial U \in C^1$, then for $1 \le q < p^*$, $L^q(U) \supset W^{1,p}(U)$ is a compact inclusion.

Proof. Take $p = 2$. Then $p^* = \frac{2n}{n-2} > 2$. First, suppose $\overline{U} \Subset B(0,R)$. We can assume $R = 1$. Suppose we have a sequence $\|v_n\|_{H^1(U)} \le 1$. There exists a sequence $u_n \in H^1(\mathbb{R}^n)$ such that $u_n|_U = v_n$, $\|u_n\|_{H^1(\mathbb{R}^n)} \le 1$ and supp $u_n \subset B(0,1)$(this is the extension operator).

We have $u_n \in H^1(\mathbb{R}^n)$, $\|u_n\|_{H^1} \le 1$, supp $u_n \subset B(0,R)$. We want n_k, $u \in L^2(\mathbb{R}^n)$ such that $u_{n_k} \to u$ in L^2. We claim that $u(x) = (2\pi)^{-n}\sum_{m\in\mathbb{Z}^n} \widehat{u}(m)e^{im\cdot x}$ for $x \in B(0,1) \subset [-\pi,\pi]^n$ with convergence in $\mathcal{D}'$.

Alternatively,

$$\int u(x)\overline{\varphi(x)}\,dx = (2\pi)^{-n}\sum_{m\in\mathbb{Z}^n}\widehat{u}(m)\int\overline{\varphi(x)}e^{im\cdot x}\,dx$$
$$= (2\pi)^{-n}\sum_{m\in\mathbb{Z}^n}\widehat{u}(m)\overline{\widehat{\varphi}(m)}.$$

For $u, v \in L^2$, $\int u(x)\overline{v}(x)\,dx = (2\pi)^{-n}\sum_{m\in\mathbb{Z}^n}\widehat{u}(m)\overline{\widehat{v}(m)}$.

Recall the Poisson summation formula: for $n = 1, a \neq 0$,

$$\sum_{k\in\mathbb{Z}} e^{ikax} = \frac{2\pi}{a}\sum_{k\in\mathbb{Z}}\delta(x - 2\pi k/a)$$

in the distributional sense.

Note that $(1 - e^{iax})\sum_k e^{ikax} = \sum_k e^{ikax} - \sum_k e^{i(k+1)ax} = 0$. We can rewrite this as $-2ie^{-iax/2}\sin(ax/2)\sum_k e^{ikax} = 0$. Let $w(x) = \sum_k e^{ikax}$, so it follows that supp $w \subset \{\frac{2\pi}{a}k\}_{k\in Z}$. It follows that $w(x)$ is the sum of delta functions supported at $2\pi k/a$ for $k \in \mathbb{Z}$ up to constants.

Furthermore, note that $w(x + 2\pi a) = w(x)$. So it follows that the constants are independent of the index. To find the constant, for some function, replace $\varphi(\cdot)$ with $\varphi(\cdot + x)$. Then the right side is $c\sum_{k\in\mathbb{Z}}\varphi(2\pi k/a + x)$. Note that $\widehat{\varphi}(\cdot + x)(\xi) = e^{ix\xi}\widehat{\varphi}(\xi)$. It follows that the left hand side is $\sum_{k\in\mathbb{Z}}\widehat{\varphi}(ka)e^{ikax}$. Now suppose supp $\varphi \in C_c^\infty((0, 2\pi/a))$. Integrating both sides, the left side is $2\pi/a\widehat{\varphi}(0)$. The right side is $c\int\varphi(x)\,dx = c(a)\widehat{\varphi}(0)$. Thus, $c = \frac{2\pi}{a}$.

The Poisson summation formula is more generally $\sum_{k\in\mathbb{Z}^k} e^{iak\cdot x} = (2\pi/a)^n\sum_{k\in\mathbb{Z}^d}\delta(x - 2\pi a/k)$.

Applying this to a φ gives our desired claim from earlier. it follows that $\frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n}\widehat{u}(k)\overline{\widehat{v}(k)} = \int u(x)\overline{v(x)}\,dx$ with $u, v \in L^2$, supp $u, v \in [-\pi, \pi]^n$.

Note that for $u \in L^2$, we have the Plancherel formula,

$$\|u\|_2^2 = \int |u(x)|^2\,dx = \frac{1}{2\pi}^n\sum_{n\in\mathbb{Z}^n}|\widehat{u}(m)|^2.$$

For $u \in H^1(\mathbb{R}^n)$ and supp $u \subset B(0, 1)$, then $u \in L^2$ and $\partial^\alpha u \in L^2$, for all $|\alpha| = 1$,

$$\|\partial^\alpha u\|_{L^2}^2 = (2\pi)^n\sum_{m\in\mathbb{Z}^n}|\widehat{\partial^\alpha u}(m)|^2 = (2\pi)^{-n}\sum_{m\in\mathbb{Z}^n}|m^\alpha\widehat{u}(m)|^2.$$

Claim: Under these assumptions, $\|u\|_{H^1}^2 = \Theta(\sum_{m\in\mathbb{Z}^n}\langle m\rangle^2|\widehat{u}(m)|^2)$ $\square$

§5 February 2nd, 2021

Recall the following:

- GNS inequality: For $U \Subset \mathbb{R}^n$, $\partial U \in C^1$, $1 \le p < n$, $p^* = \frac{np}{n-p} > p$,

$$\|u\_p^* \le C(\|u\|_p + \|\nabla u\|_p).$$

- R-K Theorem: For $1 \le p < n$, $1 \le q < p^*$,

$$W^{1,p}(U) \subset L^q(U)$$

  is compact: If we have $\{u_n\} \subset W^{1,p}(U)$ and $\|u_n\|_{W^{1,p}} \le C$, there exists a subsequence u_{n_k}, $u \in L^q$ such that $\|u_{n_k} - u\|_q \to 0$.

§5.1 Rellich-Kondrachov, continued

Last time, we considered the special case of $\{u_n\} \subset H^1(\mathbb{R}^n)$ such that $\operatorname{supp} u_n \subset B(0,R)$ and $\|u_n\|_{H^1} \le C$, which implies that there exists a subsequence u_{n_k} and $u \in L^2(\mathbb{R}^n)$ such that $\|u_n - u\|_{L^2} \to 0$. We continue the proof of the special case.

Proof. Recall that we showed that if $u \in C_0^\infty((-\pi,\pi)^n)$, we can write $u(x) = (2\pi)^{-n}\sum \widehat{u}(n)e^{in\cdot x}$. Then

$$\int u(x)\overline{v(x)}\,dx = \frac{1}{(2\pi)^n}\sum \widehat{u}(n)\overline{\widehat{v}(n)}$$

and

$$\int |\nabla u(x)|^2\,dx = \frac{1}{(2\pi)^n}\sum |n|^2|\widehat{u}(n)|^2.$$

For $u \in H^1$ with $\operatorname{supp} u \in B(0,1)$,

$$\|u\|_{H^1}^2 = \frac{1}{(2\pi)^n}\sum \langle n\rangle^2|\widehat{u}(n)|^2.$$

$$\|u_n\|_{H^1}^2 = \sum_{\ell\in\mathbb{Z}^n}\langle \ell\rangle^2|\widehat{u_n}(\ell)|^2 \le C.$$

$$\|v\|_{L^2}^2 = \sum_{\ell\in\mathbb{Z}^n}|\widehat{v}(\ell)|^2.$$

We want to show that there exists n_k such that $\|u_{n_k} - u_{n_p}\|_{L^2} \to 0$ as $k,p\to\infty$.

We introduce an operator $\Pi_p u(x) = (2\pi)^{-n}\sum_{|\ell|\le p}\widehat{u}(\ell)e^{i\ell\cdot x}$. We can think of $\Pi_p : L^2([-\pi,\pi]^n) \to \mathbb{C}^{N_p}$. N^p can be found through combinatorial methods(left as an exercise)[should be $\binom{n+p}{p}$ or something like that].

We have the following estimate:

$$\|(I-\Pi_p)u\|_2 \le \langle p\rangle^{-2}\|u\|_{H^1}^2.$$

This is because

$$(2\pi)^{-n}\sum_{|\ell|>p}|\widehat{u}(\ell)|^2 = (2\pi)^{-n}\sum_{|\ell|>p}\langle\ell\rangle^{-2}\langle\ell\rangle^2|\widehat{u}(\ell)|^2 \le \langle p\rangle^{-2}\|u\|_{H^1}^2.$$

Now, we find the Cauchy subsequence.

1. For all p, we have $\|\Pi_p u_n\|_{\mathbb{C}^{N_p}} \leq \|u_n\|_2 \leq \|u_n\|_{H^1} \leq C$. Then $\{|z| \leq C\} \subset \mathbb{C}^{N_p}$ is compact. It follows that we can choose subsequences $\{n_k^{p+1}\} \subset \{u_k^p\}$ such that $\Pi_p u_k^p$ converges and $\limsup_{k,\ell} \|u_k^p - u_\ell^p\| \leq C\langle p\rangle^{-2}$, which follows from the triangle inequality. [let $u_a^b = u_{n_a^b}$]

2. We choose $n_k = n_k^k$. It follows that $\limsup_{k,\ell\to\infty} \|u_{n_k} - u_{n_\ell}\|_2 = 0$, since

□

§5.2 Morrey's Inequality

Theorem 11 (Morrey's Inequality)

Suppose $u \in L^p(\mathbb{R}^n), \nabla u \in L^p(\mathbb{R}^n)$ and $n < p \leq \infty$. Then there exists $u^* \in C^{0,\gamma}(\mathbb{R}^n)$, with $\gamma = 1 - \frac{n}{p}$ such that $u = u^*$ almost everywhere and $\|u^*\|_{C^{0,\gamma}} \leq \|u\|_p + \|\nabla u\|_p$.

Remark 5.1. Recall

$$\|u\|_{C^{0,\gamma}} = \sup |u(x)| + \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\gamma}.$$

Proof. We use the Littlewood-Paley Decomposition.

Lemma 5.2 (Dyadic Partitions of Identity)

There exists a function $\psi_0 \in C_c^\infty(\mathbb{R}), \psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ such that

$$\psi_0(\xi) + \sum_{j=0}^\infty \psi(2^{-j}|\xi|) = 1.$$

Proof. Choose $\varphi_0 \in C_c^\infty((-1,1))$ with $0 \leq \psi_0 \leq 1$ and $\varphi_0(p) = 1$, $|p| \leq 1/2$.

Choose a new function

$$\varphi_1(p) = \sum_{j\in\mathbb{Z}} \varphi_0(p-j) \geq 1.$$

Note that $\varphi_1(p-k) = \varphi_1(p)$ for $k \in \mathbb{Z}$. Choose $\varphi(p) = \frac{\varphi_0(p)}{\varphi_1(p)}$.

Then

$$\sum_{j\in\mathbb{R}} \varphi(p-j) = \sum \frac{\varphi_0(p-j)}{\varphi_1(p-j)} = \frac{1}{\varphi_1(p)} \sum \varphi_0(p-j) = 1.$$

Define $\psi(r) = \varphi(\frac{\log r}{\log 2})$ for positive r. Notice that $\psi \in C_c^\infty((0,\infty))$. This gives that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}r) = 1.$$

Define $\psi_0(r) = 1 - \sum_{j=0}^\infty \psi(2^{-j}r)$. Note that $\psi_0(r) = 1$ for $r < 1/2$ and $\psi_0(r) = 0$ for $r > 1$. It follows that ψ_0 and ψ satisfy the conditions. □

We can extend the Dyadic Partitions of Identity in $\mathbb{R}^n$ in the natural way. We then define the Littlewood-Paley Decomposition as

$$u = \psi_0(D)u + \sum_{j=1}^\infty \psi(2^{-j}D)u$$

where for $a \in L^\infty(\mathbb{R}^n)$, $a(D)u = \mathcal{F}^{-1}(a(\xi)\widehat{u}(\xi))$ where $D_x = 1/i\partial_x$ and $\widehat{Du} = \xi\widehat{u}$. $\square$

§6 February 4th, 2021

§6.1 Morrey's Inequality, continued

Recall the statement of the theorem.

Theorem 12 (Morrey's Inequality)

Suppose $u \in L^p(\mathbb{R}^n), \nabla u \in L^p(\mathbb{R}^n)$ and $n < p \leq \infty$. Then there exists $u^* \in C^{0,\gamma}(\mathbb{R}^n)$, with $\gamma = 1 - \frac{n}{p}$ such that $u = u^*$ almost everywhere and $\|u^*\|_{C^{0,\gamma}} \leq \|u\|_p + \|\nabla u\|_p$.

Proof. Recall for $u \in \mathcal{S}(\mathbb{R})$, $\widehat{D_{x_j} u}(\xi) = \xi_j \widehat{u}(\xi)$ where $D_{x_j} = \frac{1}{i}\partial_{x_j}$.

We define a **Fourier multipler** $a \in L^\infty(\mathbb{R}^n)$ so that $a(D)u = \mathcal{F}^{-1}(a(\xi)\widehat{u}(\xi))$ for $u \in \mathcal{S}$. Note that for $a \in L^\infty$, $\|a(D)u\|_{L^2} \leq \sup |a| \|u\|_{L^2}$. For $\psi \in \mathcal{S}(\mathbb{R}^n)$, if we take $u \in \mathcal{S}'$, then $\psi(D)u \in \mathcal{S}'$, and $\psi(\xi)\widehat{u}(\xi) \in \mathcal{S}'$.

Recall the Littlewood - Paley Decomposition. We had a lemma: there exists $\psi_0 \in C_c^\infty(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ such that for all $\xi \in \mathbb{R}^n$,

$$\psi_0(|\xi|) + \sum_{j=0}^{\infty} \psi(2^{-j}|\xi|) = 1.$$

Slightly abusing notation, we will write $\psi_0(\xi) = \psi_0(|\xi|)$ and $\psi(\xi) = \psi(|\xi|)$.

The full L-P Decomposition is given as follows: given $u \in \mathcal{S}'$, $a = \psi_0(D)u + \sum_{j=1}^\infty \psi(2^{-j}D)u$. We will write $h = 2^{-j}$ as a shorthand.

Lemma 6.1

Suppose $\chi \in C_c^\infty(\mathbb{R}^n)$. Then for $u \in \mathcal{S}(\mathbb{R}^n)$, $\|\chi(hD)u\|_{L^\infty} \leq Ch^{-n/p}\|u\|_{L^p}$ and $\|\chi(hD)u\|_{L^p} \leq (2\pi)^{-n}\|\widehat{\chi}\|_1\|u\|_p$.

Proof. Recall the following inequalities

- Holder's Inequality: $\|fg\|_1 \leq \|f\|_p\|g\|_q$ for $1/p + 1/q = 1$ for $1 \leq p \leq \infty$.
- Minkowski's Inequality: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ and
$$\left\| \int F(x,t)\, dt \right\|_p \leq \int \|F(\cdot, t)\|_p\, dt.$$
- Young's inequality: $\|f * g\|_p \leq \|f\|_1\|g\|_p$.

We have

$$\begin{aligned}
\chi(hD)u(x) = \mathcal{F}^{-1}(\chi(h\xi)\widehat{u}(\xi)) &= (2\pi)^{-n} \iint e^{i(x-y)\xi}\chi(h\xi)u(y)dyd\xi \\
&= (2\pi h)^{-n} \int \widehat{\chi}\left(\frac{x-y}{h}\right) u(y)\, dy \\
&\leq (2\pi h)^{-n} \frac{C}{h^n} \|\widehat{\chi}(\cdot/h)\|_q \|u\|_p
\end{aligned}$$

Then,

$$\|\widehat{\chi}(\cdot/h)\|_q = \left(\int |\widehat{\chi}(y/h)|^q \, dy\right)^{1/q} = h^{n/q}\|\widehat{\chi}\|_q.$$

It follows that

$$|\chi(hD)u(x)| \leq Ch^{-n+n/q}\|u\|_p = Ch^{-n/p}\|u\|_p.$$

For the second inequality, note that $\chi(hD)u(x) = (2\pi h)^n \widehat{\chi}(\cdot/h) * u$. Applying Young's Inequality,

$$\|\chi(hD)u\|_p \leq \frac{1}{(2\pi h)^n}\|\widehat{\chi}(\cdot/h)\|_1\|u\|_p \leq \frac{1}{(2\pi)^n}\|\widehat{\chi}\|_1\|u\|_p.$$

□

Theorem 6.2

For $u \in L^p$, $1 \leq p \leq \infty$, $u \in C^{0,\gamma}(\mathbb{R}^n)$ if and only if for every $\chi \in C_c^\infty(\mathbb{R}^n \setminus 0)$, $\|\chi(hD)u\|_\infty \leq Ch^\gamma$.

Proof. We start with the forward direction. Note that

$$\begin{aligned}
\chi(hD)u(x) &= \frac{1}{(2\pi h)^n}\int \widehat{\chi}((x-y)/h)u(y)\, dy \\
&= (2\pi)^{-n}\int \widehat{\chi}(y)u(x-yh)\, dy \\
&= (2\pi)^{-n}\int \widehat{\chi}(y)(u(x-yh)-u(x))\, dy
\end{aligned}$$

So it follows that

$$|\chi(hD)u(x)| \leq C\|u\|_{C^{0,\gamma}}\int |\widehat{\chi}(y)|(hy)^\gamma dy \leq C\|u\|_{C^{0,\gamma}}h^\gamma\int |\widehat{\chi}(y)||y|^\gamma \, dy$$

□

and the last integral is bounded since $\widehat{\chi}$ is a Schwartz function, so it follows that $|\chi(hD)u(x)| \leq C\|u\|_{C^{0,\gamma}}h^\gamma$. □

§7 February 9th, 2021

§7.1 Fourier Transform proof of Morrey's Inequality

Recall the Littlewood-Paley decomposition: There exists $\psi_0 \in C_c^\infty(\mathbb{R}^n)$, $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$1 = \psi_0(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi).$$

From this, we have for $u \in \mathcal{S}'$,

$$u = \psi_0(D)u + \sum_{j=0}^{\infty} \psi(2^{-j}D)u.$$

More generally, for $a \in \mathcal{S}(\mathbb{R}^n)$, $a(D)u = \mathcal{F}^{-1}(a(\xi)\widehat{u}(\xi))$. We were proving the following theorem:

Theorem 7.1

For $u \in L^p$, $1 \leq p \leq \infty$, $u \in C^{0,\gamma}(\mathbb{R}^n)$ if and only if for every $\chi \in C_c^\infty(\mathbb{R}^n \setminus 0)$, $\|\chi(hD)u\|_\infty \leq Ch^\gamma$.

Proof. We proved the forward direction last time. We now show the converse.

Denote

$$\Lambda_\gamma(u) = \sup_{0<h<1} h^{-\gamma}(\|\psi(hD)u\|_\infty + \max \|\psi_k(hD)u\|_\infty)$$

where $\psi_k(\xi) = \xi_k \psi(\xi)$.

We have the hypothesis: $\|u\|_p + \Lambda_\gamma(u) < \infty$. We want to show that $\|u\|_{C^{\gamma,0}} \leq C(\|u\|_p + \Lambda_\gamma(u))$. We first bound $\|u\|_\infty$. Note that

$$\begin{aligned} \|u\|_\infty &\leq \|\psi_0(D)u\|_\infty + \sum_j \|\psi(2^{-j}D)u\|_\infty \\ &\leq \|\psi_0(D)u\|_\infty + \sum_j 2^{-j\gamma}\Lambda_\gamma(u) \\ &\leq C\|u\|_p + (2^\gamma - 1)^{-1}\Lambda_\gamma(u). \end{aligned}$$

Now, we bound the quotient term, $|u(x) - u(y)|/|x-y|^\gamma$. In order words, we want

$$|u(x) - u(y)| \leq C(\|u\|_p + \Lambda_\gamma(u))r^\gamma,$$

if $|x - y| \leq r$.

Note that

$$u(x) - u(y) = \psi_0(D)u(x) - \psi_0(D)u(y) + \sum_j \left(\psi(2^{-j}D)u(x) - \psi(2^{-j}D)u(y)\right)$$

It is enough to prove that $|\psi_0(D)u(x) - \psi_0(D)u(y)| \leq Cr^\gamma\|u\|_p$ and

$$\sum_{j=0}^{\infty} |\psi(2^{-j}D)u(x) - \psi(2^{-j}D)u(y)| \leq Cr^\gamma\Lambda_\gamma(u).$$

Note that

$$\begin{aligned}
|\psi_0(D)u(x) - \psi_0(D)u(y)| &\le \sup(\nabla(\psi_0(D)u))|x-y| \\
&\le |x-y|\frac{1}{(2\pi)^n}\sup\int \nabla|\widehat{\psi}_0(x-y)||u(y)|\,dy \\
&\le |x-y|\frac{1}{(2\pi)^n}\|\nabla\widehat{\psi}_0\|_q\|u\|_p.
\end{aligned}$$

For the second inequality, we prove for both high frequency and low frequency estimates. For the high ones,

$$|\psi(hD)u(x) - \psi(hD)u(y)| \le 2\|\psi(hD)u\|_\infty \le 2h^\gamma\Lambda_\gamma(u).$$

For low frequencies,

$$\begin{aligned}
|\psi(hD)u(x) - \psi(hD)u(y)| &\le Cr\max_k \|D_{x_k}\psi(hD)u\|_\infty \\
&= Crh^{-1}\max_k \|hD_{x_k}\psi(hD)u\|_\infty \\
&= Crh^{-1}\max_k \|\psi_k(hD)u\|_\infty \\
&\le Crh^{-1}\max_k \|\psi_k(hD)u\|_\infty \\
&\le Crh^{\gamma-1}\Lambda_\gamma(u).
\end{aligned}$$

Then, note that

$$\sum_{2^j\le s} Cr2^{-j(\gamma-1)} \le C'rs^{1-\gamma}$$

and

$$\sum_{2^j>s} C2{-}j\gamma \le C''s^{-\gamma}.$$

It follows that

$$\sum_{j=0}^\infty |\psi(2^{-j}D)u(x) - \psi(2^{-j}D)u(y)| \le C\Lambda_\gamma(u)(rs^{1-\gamma}+s^{-\gamma}) \le Cr^\gamma\Lambda_\gamma(u)$$

□

§8 February 11th, 2021

§8.1 Finishing Morrey's Inequality

The original statement of the theorem.

Theorem 13 (Morrey's Inequality)
Suppose $u \in L^p(\mathbb{R}^n), \nabla u \in L^p(\mathbb{R}^n)$ and $n < p \leq \infty$. Then there exists $u^* \in C^{0,\gamma}(\mathbb{R}^n)$, with $\gamma = 1 - \frac{n}{p}$ such that $u = u^*$ almost everywhere and $\|u^*\|_{C^{0,\gamma}} \leq C(\|u\|_p + \|\nabla u\|_p)$.

Last time, we showed the following theorem:

Theorem 14
For $u \in L^p$, $1 \leq p \leq \infty$, $u \in C^{0,\gamma}(\mathbb{R}^n)$ if and only if for every $\chi \in C_c^\infty(\mathbb{R}^n \setminus 0)$, $\|\chi(hD)u\|_\infty \leq Ch^\gamma$.

Recall that

$$\Lambda_\gamma(u) = \sup_{0<h<1} h^{-\gamma}(\|\psi(hD)u\|_\infty + \max \|\psi_k(hD)u\|_\infty)$$

where $\psi_k(\xi) = \xi_k\psi(\xi)$. We proved this by showing that $\|u\|_{C^{0,\gamma}} \leq C(\|u\|_p + \Lambda_\gamma(u))$. We now show the complete proof of Morrey's Inequality.

Proof. It suffices to show that $\Lambda_\gamma(u) \leq C\|\nabla u\|_p$. Recall that for all $\chi \in C_c^\infty(\mathbb{R}^n)$, we showed that $\|\chi(hD)u\| \leq Ch^{-n/p}\|u\|_p$. Note that

$$\begin{aligned} &\|\varphi(hD)hD_{x_j}u\|_\infty \leq ch^{1-n/p}\|\nabla u\|_p \\ &\Rightarrow \|\varphi_j(hD)u\|_\infty \leq Ch^\gamma\|\nabla u\|_p. \end{aligned}$$

We would like to write $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, $\psi(\xi) = \sum \xi_j\chi_j(\xi)$ with $\chi_j \in C_c^\infty$. We can do this with $\sum \xi_j \frac{\xi_j}{|\xi|^2}\psi(\xi)$.

It follows that

$$\|\psi(hD)u\|_\infty \leq \sum_{j=1}^\infty \|\xi_j\chi(hD)u\|_\infty \leq Ch^\gamma\|\nabla u\|_p.$$

$\square$

We can use this result to show regularity properties for solutions to PDEs. For example, one statement is as follows: suppose $u \in L^1$, $\Delta u = f \in C^{k,\gamma}$ for $0 < \gamma < 1$. We could show that $u \in C^{k+2,\gamma}$.

§8.2 Final Comments about Sobolev Spaces

Definition 8.1. Suppose $U \subset\subset \mathbb{R}^n$ with $\partial U \in C^1$. Then $W_0^{1,p}(U) = \overline{C_c^\infty(U)}$ where the closure is respect to the $W^{1,p}$ norm.

Fact 8.2. $W_0^{1,p} = \{u \in W^{1,p}(U) : Tu = 0\}$, where $T : W^{1,p}(U) \to L^p(\partial U)$ linear and bounded and for $u \in W^{1,p}(U) \cap C(\overline{U})$, $Tu = u|_{\partial U}$.

Fact 8.3 (Poincare Inequality)**.** Suppose $1 \leq p < n$ and $1 \leq q \leq p^* = \frac{np}{n-p}$. Then $\|u\|_q \leq C\|\nabla u\|_p$.

Theorem 15 (Poincare Inequality(v2))
For all $1 \leq p \leq \infty$, $\|u\|_p \leq C\|u\|_p$.

Proof. Suppose $p < n$. This follows from the version 1. Suppose $\infty > p \geq n$. In this case, take $q = n - \epsilon$. Then $q^* = \frac{(n-\epsilon)n}{\epsilon}$. Choose small enough ϵ so that $q^* \geq p$. Then, we apply Poincare 1: $\|u\|_p \leq \|u\|_{q^*} \leq C\|\nabla u\|_q \leq C\|\nabla u\|_p$. For $p = \infty$, the result follows from Morrey's inequality. □

§8.3 Duality

Recall the Riesz Representation Theorem for Hilbert Spaces:

Theorem 16 (Riesz Representation)
For $\Phi : H \to \mathbb{C}$ with a Hilbert space H, if $|\Phi(u)| \leq C\|u\|$, there exists $v \in H$ such that $\Phi(u) = \langle u, v\rangle$.

Fact 8.4. $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^*$: if $u \in H^{-s}(\mathbb{R}^n)$ and $v \in H^s(\mathbb{R}^n)$, $u \in H^s(\mathbb{R}^n)$, then $\langle u, v\rangle_{L^2} = \int u\overline{v}$ is well defined, and for any $\Phi : H^s(\mathbb{R}^n) \to \mathbb{C}$ such that $|\Phi(u)| \leq C\|u\|_{H^s}$, there exists $v \in H^{-s}$ such that $\Phi(u) = \langle u, v\rangle_{L^2}$.

Proof. First assume $u, v \in \mathcal{S}$. Then

$$\int u\overline{v} = (2\pi)^n \int \widehat{u}(\xi)\overline{\widehat{v}(\xi)} = (2\pi)^n \int \langle\xi\rangle^s\langle\xi\rangle^{-s}\overline{\widehat{v}(\xi)}$$

so it follows that

$$|\langle u, v\rangle|_2 \leq (2\pi)^n\|u\|_{H^s}\|v\|_{H^{-s}}.$$

Conversely, suppose we have Φ as above. Riesz implies that

$$\Phi(u) = \langle u, w\rangle_{H^s} = (2\pi)^{-n}\int \langle\xi\rangle^{2s}\widehat{u}\overline{\widehat{w}} = (2\pi)^{-n}\int \widehat{u}\overline{\langle\xi\rangle^{2s}\widehat{w}}$$

and we finish by setting $\widehat{v} = \langle\cdot\rangle^{2s}\widehat{w}$. □

§8.4 Duality on Bounded Domains

We define $H^{-1}(U) = \{u \in \mathcal{D}'(U) : \forall\varphi \in C_c^\infty(U), |u(\varphi)| \leq C\|\varphi_{H^1}\}$. The norm on H^{-1} is given by

$$\|u\|_{H^{-1}(U)} = \sup\{|u(\varphi)| : \varphi \in H_0^1, \|\varphi\|_{H^1} \leq 1\},$$

which is the usual operator norm, treating u as a linear functional on $H^1(U)$.

Example 8.5

We claim

$$H_0^1((0,\pi)) = \{u(x) = \sum_{n=1}^{\infty} a_n \sin nx : \sum |a_n|^2 n^2 < \infty \sim \|u\|_{H^1}\}.$$

Then,

$$H^{-1}((0,\pi)) = \{v(x) = \sum_{n=1}^{\infty} a_n \sin nx : \sum_{n=1}^{\infty} |a_n|^2 n^{-2} < \infty\},$$

where we take convergence in the sense of distributions.

Then,

$$\langle u, v\rangle = \sum a_n \overline{b_n} = \sum_{n=1}^{\infty} n a_n n^{-1} \overline{b}_n \le \|u\|_{H^1} \|v\|_{H^{-1}}.$$

§9 February 16th, 2021

§9.1 Calculus of Variations: Minimizing Distance in the Plane

We start with a motivating example. Take points a, b in the x-axis, c, d in the y-axis. We wish to find a function $y = f(x)$ such that $f(a) = c$, $f(b) = d$ and the graph of f has the shortest length. Recall that

$$L(f) = \int_a^b (1 + f'(x)^2)^{1/2} \, dx.$$

We wish to minimize L over all paths from a to b. If f is a minimizer, then for all $\varphi \in C_c^\infty((a, b))$, $L(f + t\varphi)$ has a minimum at $t = 0$. This implies that $\frac{d}{dt} L(f + t\varphi)|_{t=0} = 0$ for all φ as above. Then,

$$\begin{aligned}\frac{d}{dt} L(f + t\varphi) &= \frac{d}{dt} \int_a^b (1 + (f + t\varphi)'^2)^{1/2} \, dx \\ &\int_a^b \frac{\partial}{\partial t} [(1 + (f + t\varphi)'^2)^{1/2}] \, dx \\ &= \int_a^b \frac{\varphi'(x)(f'(x) + t\varphi'(x))}{(1 + (f'(x) + t\varphi'(x)^2)^{1/2}}\end{aligned}$$

Applying $t = 0$, we have

$$0 = \int_a^b \varphi'(x) \frac{f'(x)}{(1 + f'(x)^2)^{1/2}} \, dx$$

for all $\varphi \in C_c^\infty((a, b))$.

Integrating by parts, we get that

$$\int_a^b \varphi(x) \left(\frac{f'(x)}{(1 + (f'(x))^2)^{1/2}} \right)' \, dx = 0.$$

The implies that

$$\frac{d}{dx} \left(\frac{f'(x)}{(1 + f'(x)^2)^{1/2}} \right) = 0,$$

with $f(a) = c$, $f(b) = d$.

We find that $f'(x) = \alpha$ so $f(x) = \alpha x + \beta$.

§9.2 Calculus of Variations: Minimizing Area in $\mathbb{R}^3$

Take $U \subset\subset \mathbb{R}^2$, $\partial U \in C^1$. We wish to minimize the area of the graph with the condition that $f = g$ on ∂U.

We have

$$A(f) = \iint_U (1 + |\nabla f(x)|^2)^{1/2} \, dx.$$

We wish to minimize $A(f)$ over f satisfying $f = g$ on ∂U.

If f is a minimizer, $t \mapsto A(f + t\varphi)$, $\varphi \in C_c^\infty(U)$ has a minimum at $t = 0$. So,

$$\frac{d}{dt} A(f + t\varphi)|_{t=0} = 0.$$

Doing the same calculation as before, we have

$$\begin{aligned}\frac{d}{dt}A(f+t\varphi)|_{t=0} &= \int_U \frac{\partial}{\partial t}(1+|\nabla f + t\nabla\varphi|^2)^{1/2}\,dx \\ &= \int_U \frac{\nabla\varphi\cdot\nabla f}{(1+|\nabla f|^2)^{1/2}}\,dx \\ &= \int_U \varphi\left[\left(\frac{f_{x_1}}{(1+|\nabla f|^2)^{1/2}}\right)_{x_1} + \left(\frac{f_{x_2}}{(1+|\nabla f|^2)^{1/2}}\right)_{x_2}\right]dx.\end{aligned}$$

As before, this implies that

$$\left(\frac{f_{x_1}}{(1+|\nabla f|^2)^{1/2}}\right)_{x_1} + \left(\frac{f_{x_2}}{(1+|\nabla f|^2)^{1/2}}\right)_{x_2} = 0.$$

This is called the **Minimal Surface Equation**.

We will not solve this, but how could we do it? Consider $f \in H^1(U)$, and note that $Tf = g \in L^2(\partial U)$ is well-defined. If we take $m = \inf\{A(f) = f \in H^1(U), f|_{\partial U} = g\}$. Then, there exists $f_j \in H^1(U), f_j|_{\partial U} = g$ with $A(f_j) \to m$. Could we find $f_{j_k} \to f$?

§9.3 Calculus of Variations: General Setup

Take $U \subset\subset \mathbb{R}^n$. Take $L : \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$ in C^∞, written as $L(p, z, x)$. We introduce the functional $I[w] = \int_U L(D_w(x), w(x), x)\,dx$, with $w|_{\partial U} = g$.

Example 9.1

In the minimal surface problem, $L(p, z, x) = (1 + |p|^2)^{1/2}$.

We first derive an equation satisfied by the minimizer. As before, we have $I[w] = \int_U L(Dw, w, x)\,dx$, a minimizer. This implies that $\frac{d}{dt}I[w + t\varphi]|_{t=0} = 0$ for all $\varphi \in C_c^\infty(U)$.
Then,

$$\begin{aligned}\int_U \frac{d}{dt}\left[L(Dw + tD\varphi, w + t\varphi, x)\right]\,dx|_{t=0} &= \int_U \left(D\varphi\cdot D_pL(Dw, w, x) + \varphi D_zL\right)\,dx \\ &= \int_U \left(-\sum (L_{p_j}(Dw, w, x))_{x_j} + D_zL(Dw, w, x)\right)\varphi\,dx \\ &\implies \boxed{-\sum_{j=1}^n (L_{p_j}(Dw, w, x))_{x_j} + D_zL(Dw, w, x) = 0},\end{aligned}$$

the **Euler-Lagrange Equation**.

Example 9.2

Take $L(p, z, x) = |p|^2/2 - f(x)z$.

$$I[w] = \int_U \left(|\nabla w(x)|^2/2 - f(x)w(x)\right)\,dx.$$

Since $L_{p_j} = p_j$, the Euler-Lagrange equation is given by

$$-\sum (w_{x_j})_{x_j} - f(x) = 0 \implies -\Delta w = f, w|_{\partial U} = g.$$

We can generalize this as follows: If we take $L = |p|^2/2 + F(z)$ and $f(z) = F'(z)$. The Euler-Lagrange equation is then $-\nabla w = f(w)$. For example, if we take $f(z) = z^p$, $F(z) = \frac{z^{p+1}}{p+1}$.

We could also take non-constant coefficients: $L(p, z, x) = \frac{1}{2}\sum a_{ij}(x)p_i p_j - f(x)z$, where $a_{ij} = a_{ji}$.

Then, $L_{p_j} = \frac{1}{2}\sum_{i=1}^n a_{ij}(x)p_i$. Then, the Euler-Lagrange equation is given by

$$-\sum_{i,j=1}^n \partial_{x_j}(a_{ij}\partial_{x_i}w(x)) = f(x).$$

When $\sum a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$ for all $\xi \in \mathbb{R}^n$, $x \in \overline{U}$, this is solvable.

§9.4 Existence of Minimizers

- Coercivity: There exists $\alpha.0, \beta \geq 0$ with $L(p, z, u) \geq \alpha|p|^q - \beta$, for $1 < q < \infty$, for all $z \in \mathbb{R}$, $x \in \overline{U}$.

  The condition gives the following bound: $I[w] \geq \alpha\|Dw\|_q^q - \beta\mu(U)$. We can always set $\beta = 0$ by translating L by a constant. Taking $\mathcal{A} = \{w \in W^{1,q}(U) : u|_{\partial U} = g\}$, we minimize $I[w]$ over $\mathcal{A}$.

- Lower semicontinuity: Suppose we have $u_k \rightharpoonup u$ weakly in $W^{1,q}$. Then,

$$I[u] \leq \liminf I[u_k].$$

§10 February 18th, 2021

Recall, we have $L : \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$ for $U \Subset \mathbb{R}^n$, $\partial U \in C^1$. We denote $L = L(p, z, x)$, $D_p L = (\partial_{p_1} L, \ldots, \partial_{p_n} L)$, etc. We also defined

$$I[w] = \int_U L(Dw(x), w(x), x)\, dx, \quad w|_{\partial U} = g.$$

As an example, $L(p, x) = \frac{1}{2} \sum a_{ij}(x) p_i p_j - f(x) z$. Last time, we used the principle that if w is a minimizer, for every $\varphi \in C_c^\infty(U)$, $t \mapsto I[w + t\varphi]$ has a local minimum at $t = 0$. This implies that if w is a minimizer, L satisfies the Euler-Lagrange equation:

$$-\sum_{j=1}^n (L_{p_j}(Dw, w, x))_{x_j} + D_z L(Dw, w, x) = 0.$$

§10.1 Second-Derivative Test

If $i'(0) = 0, i''(0) > 0$, then we have a local minimum at 0.

By definition $i(t) = I[w + t\varphi]$, where $\varphi \in C_c^\infty$. Recall that

$$i'(t) = \int \left(\sum \varphi_{x_j} \partial_{p_j} L(Dw + t\varphi, w + t\varphi, x) + \varphi \partial_z L(Dw + t\varphi, w + t\varphi, x) \right) dx.$$

Then,

$$i''(0) = \int \left(\sum_{i,j} \varphi_{x_j} \varphi_{x_i} \partial_{p_j p_i} L + \sum_j \varphi \varphi_{x_j} \partial_z \partial_{p_j} L + \varphi^2 L_{zz} \right) dx$$

If this is at least 0 for all φ, what do we get about L? This makes sense for φ that is Lipschitz and 0 at the boundary. If we choose $\varphi(x) = \epsilon \rho(\frac{x\xi}{\epsilon}) \zeta(x)$, where $\zeta \in C_c^\infty(U)$ and ρ consists of triangles with slope ± 1 starting at 0. Then $|\rho'(x)| = 1$ almost everywhere.

Using this φ, we get $\varphi_{x_j} = \epsilon \rho(x\xi/\epsilon) \zeta'(x) + \xi_j \rho'(x\xi/\epsilon) \zeta(x) = \xi_j \rho'(x\xi/\epsilon) \zeta(x) + O(\epsilon)$ and

$$\begin{aligned} 0 &\le \int_U \sum_{i,j} (\xi_i \xi_j \partial^2_{p_i p_j} L)((\rho')^2 \zeta^2) + O(\epsilon) \\ &\xrightarrow{\epsilon \to 0} \int_U \sum_{i,j} (\xi_i \xi_j \partial^2_{p_i p_j} L)(\zeta^2) \end{aligned}$$

for any $\zeta \in C_c^\infty(U)$, so it follows that for all $\xi \in \mathbb{R}^n$, $\sum \xi_i \xi_j L_{p_i p_j}(Dw(x), w(x), x) \ge 0$. Hence, it is useful to assume convexity:

$$\sum_{i,j}^n \xi_i, \xi_j L_{p_i p_j} L(p, z, x) \ge 0$$

for all $\xi \in \mathbb{R}^n$, $(p, z, x) \in \mathbb{R}^n \times \mathbb{R} \times U$. (

$$L(p + t\xi) = L(p) + t \sum \xi_j L_{p_j} L(p) + t^2 \int_0^1 (1 - s) \sum \xi_i \xi_j L_{p_i p_j}(p + st\xi)\, ds$$

§10.2 Convexity

For smooth L, convexity is the statement

$$\sum L_{p_ip_j}(p,z,x)\xi_i\xi_j \geq c|\xi|^2 \geq 0.$$

for all $\xi \in \mathbb{R}^n$.

Example 10.1

for $L = 1/2\sum a_{ij}(x)p_ip_j$, $a_{ij} = a_{ji}$, convexity is that

$$\sum a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$$

We call this an Elliptic operator.

Example 10.2

For the minimal surface equation, $L = (1+|p|^2)^{1/2}$. Note that $L_{p_i} = \frac{p_i}{(1+|p|^2)^{1/2}}$.

$$L_{p_ip_j} = \frac{\delta_{ij}}{(1+|p|^2)^{1/2}} - \frac{p_ip_j}{(1+|p|^2)^{3/2}} = \frac{\delta_{ij}(1+|p|^2) - p_ip_j}{(1+|p|^2)^{3/2}}.$$

Then,

$$\begin{aligned}\sum L_{p_ip_j}\xi_i\xi_j &= \frac{1}{(1+|p|^2)^{3/2}}\left(\sum |\xi|^2(1+|p|^2) - \sum \xi_ip_i\xi_jp_j\right)\\ &= (1+|p|^2)^{3/2}(|\xi|^2 + |\xi|^2|p|^2 - \langle \xi, p\rangle^2) \geq 0\end{aligned}$$

This is not strictly convex, since as $p \to \infty$ our term goes to 0.

§10.3 Existence of Minimizers

Recall our conditions:

- Coercivity: There exists $\alpha > 0, \beta \geq 0$ with $L(p,z,u) \geq \alpha|p|^q - \beta$, for some $1 < q < \infty$, for all $z \in \mathbb{R}$, $x \in \overline{U}$.

  The condition gives the following bound: $I[w] \geq \alpha\|Dw\|_q^q - \beta\mu(U)$. We can always set $\beta = 0$ by translating L by a constant. Taking $\mathcal{A} = \{w \in W^{1,q}(U) : u|_{\partial U} = g\}$, we minimize $I[w]$ over $\mathcal{A}$.

- Lower semicontinuity: Suppose we have $u_k \rightharpoonup u$ weakly in $W^{1,q}$. Then,

$$I[u] \leq \liminf I[u_k].$$

As we will see, the coercivity leads to nice compactness results via Rellich-Kondrachov. How can we use lower semicontinuity? Assume $\mathcal{A}$ is nonempty. Take $m = \inf_{w \in mcA} I[w]$. Then, we have $I[w_j] \to m$. Assuming coercivity, we have $\|Dw_j\|_q$ is bounded. If $w_0 \in \mathcal{A}$, then $\|w - w_0\|_q \leq \|Dw - Dw_0\|_q$ by the Poincare inequality. So it follows that $\|w_j\|_q \leq C$. From the Banach-Alaoglu Theorem, we have w_j is weakly compact in $W^{1,q}$. Passing to a subsequence, $w_j \rightharpoonup w$ in $W^{1,q}$. From lower semicontinuity, we have $I[w] \leq \liminf I[w_j] = m$. This implies that $I[w] = m$.

§11 February 23rd, 2021

§11.1 Weak Convergence

We have a Banach space B with dual B^* with $u : B \to \mathbb{C}$ linear in the dual if for all $x \in B$, $|u(x)| \leq C\|x\|_B$.

Theorem 17 (Banach - Alaoglou)
The unit ball $\{u \in B^* : \|u\|_{B^*} \leq 1\}$ is weak-* compact: if we have $\|u_j\|_{B^*} \leq 1$, then there exists a subsequence and $u \in B^*$ such that for every $x \in B$, $u_{j_k}(x) \to u(x)$.

Corollary 11.1
If B is reflexive, $(B^*)^* = B$, then $\{x : \|x\|_B \leq 1\}$ is weakly compact. Given $\|x_j\| \leq 1$, there exists $x \in B$ and a subsequence so that $u(x_{j_k}) \to u(x)$ for $u \in B^*$.

Example 11.2
Take $B = L^q(U)$ for $1 < q < \infty$. This is reflexive since $B^* = L^{q'}(U)$ with $q^{-1} + (q')^{-1} = 1$ for $1 < q' < \infty$.

Remark 11.3. For $u_j \in B$, we say $x_j \rightharpoonup x \in B$ iff for all $u \in B^*$, $u(x_j) \to u(x)$.

- If B reflexive and $x_j \rightharpoonup x$, then $\|x\| \leq \liminf \|x_j\|$. This is because $|x(u)| = \lim |x_j(u)| \leq \liminf \|u\|_{B^*}\|x_j\|_B$ and $\|x\|_B = \sup_{\|u\|_{B^*}=1} |x(u)|$.
- If B reflexive and $x_j \rightharpoonup x$, there exists C such that $\|x_j\|_B \leq C$. For every $u \in B^*$, $|x_j(u)| \leq C(u)$, which implies by the Uniform Boundedness Principle that $\|x_j\|_B \leq C$.
- (We don't assume B is reflexive) Suppose $V \subset B$ is a closed subspace. Then V is weakly closed. This is a special case of Mazur's Theorem.

  Proof. We need to show that if $x_j \in V$, $x_j \rightharpoonup x \in B$, then $x \in V$. For $u \in B^*$, $u(x_j) \to u(x)$. So if $x \notin V$, we want to construct $u \in B^*$ so that $u(x_j) = 0$ and $u(x) = 1$.

  Recall Hahn-Banach: If we have a subspace $\tilde{V} \subset B$ and $\tilde{\varphi} : \tilde{V} \to \mathbb{C}$ with $|\tilde{\varphi}(x)| \leq C\|x\|_B$, with $x \in \tilde{V}$, then there exists $\varphi \in B^*$ so that $\varphi|_{\tilde{V}} = \tilde{\varphi}$.

  Take $\tilde{V} = V + \mathbb{C}x$. Define $\tilde{\varphi} : \tilde{V} \to \mathbb{C}$ and define $\tilde{\varphi}(y + \alpha x) = \alpha$, $y \in V$, $\alpha \in \mathbb{C}$. It suffices to check that it is bounded. We need $\tilde{\varphi}(y + \alpha x) \leq C\|y + \alpha x\|$. Suppose not - for every n, there exists y_n, α_n such that $|\alpha_n| = |\tilde{\varphi}(y_n + \alpha_n x)| > n\|y_n + \alpha_n x\|$.

  Dividing by α_n, we get
  $$1/n > \|y_n/\alpha_n + x\|_B.$$
  But this would imply that $-y_n/\alpha_n \to x \notin V$, but V is closed.

  □

§11.2 Calculus of Variations

We now move back to calculus of variations. We have $I[w] = \int_U L(Dw(x), w(x), x)\, dx$ with $U \Subset \mathbb{R}^n$, $\partial U C^1$. We have $L = L(p, z, x) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times \overline{U})$. We wish to minimize L under the constraint that $w|_{\partial U} = g$.

Recall that we introduced $i(t) = I[w + t\varphi]$, $\varphi \in \mathbb{C}_c^\infty(U)$. If w is a minimizer of $I[w]$, then

- $i'(0) = 0$ for all φ implies that $-\sum_{j=1}^n \partial_{x_j}(\partial_{p_j} L(Dw, w, x)) + \partial_z L(Dw, w, x) = 0$(Euler-Lagrange equation).
- If $i''(0) \geq 0$ for all φ, then we have the convexity condition: for all $\xi \in \mathbb{R}^n$, $\sum_{i,j} \partial_{p_i}\partial_{p_j} L(Dw, w, x)\xi_i\xi_j \geq 0$.

We introduced the conditions:

- Coercivity: there exists $1 < q < \infty$, $\alpha > 0$, $\beta \geq 0$ such that $L(p, z, x) \geq \alpha |p|^q - \beta$. This implies that $I[w] \geq \alpha \|Dw\|_q^q - \beta$.
- (Weak) Lower semicontinuity: If $u_j \rightharpoonup u$ and $Du_j \rightharpoonup u$ weakly in $L^q(U)$, then $I[u] \leq \liminf I[u_j]$.

Remark 11.4. Take $\mathcal{A} = \{u \in W^{1,q} : u|_{\partial U} = g\} \neq \emptyset$. If we put $m = \inf_{\mathcal{A}} I[w]$, there is a sequence $w_k \in \mathcal{A}$ such that $I[w_k] \to m$. Using weak compactness, we have a subsequence $w_{k_j} \rightharpoonup w$ in $W^{1,q}$ with $w \in \mathcal{A}$. Then $m \leq I[w] \leq \liminf I[w_{j_k}] = m$, so $I[w] = m$.

§11.3 Getting around Lower Semicontinuity

Theorem 18

Suppose $L \geq -C$ and $p \mapsto L(p, z, x)$ is convex for all $(z, x) \in \mathbb{R} \times \overline{U}$. Then, for any $1 < q < \infty$, $w \mapsto I[w]$ is weakly lower semicontinuous in $W^{1,q}$: if $w_j \rightharpoonup w$ in L^q, $Dw_j \rightharpoonup Dw$ in L^q, then $I[w] \leq \liminf I[w_j]$.

Remark 11.5. Convexity implies that for all p_1, p_2, $L(p_1) \geq L(p_2) + D_pL(p_2) \cdot (p_1 - p_2)$.

Proof. We assume that $u_k \rightharpoonup u$ in L^q and $Du_k \rightharpoonup Du$ in L^q. We have $\ell = \liminf I[u_k]$ and we want $I[u] \leq \ell$.

By taking a subsequence, we can say that $\ell = \lim I[u_k]$. By taking another subsequence, we can say $u_k \to u \in L^q$, since weak convergence implies that $\|u_k\|_q \leq C$ and $\|Du_k\|_q \leq C$ and using compactness of $W^{1,q}$ in L^q. By taking a subsequence we can use $u_k \to u$ almost everywhere[this is the Riesz-Fisher theorem]. By Egorov's Theorem, for every ϵ, there exists a set E_ϵ such that $m(U \setminus E_\epsilon) \leq \epsilon$ so that $u_k \to u$ uniformly on E_ϵ. Note that $m(U) < \infty$.

We define a set $F_\epsilon = \{x \in U : |u(x)| + |Du(x)| \leq \frac{1}{\epsilon}\}$. Then $m(U \setminus F_\epsilon) \to 0$ as $\epsilon \to 0$. We define $G_\epsilon = E_\epsilon \cap F_\epsilon$, with $m(U \setminus G_\epsilon) \leq m(U \setminus E_\epsilon) + m(U \setminus F_\epsilon) \to 0$ as $\epsilon \to 0$.

Without loss of generality, we can assume $L \geq 0$. Note that

$$\begin{aligned}
I[u_k] &= \int_U L(Du_k, u_k, x) \\
&\geq \int_{G_\epsilon} L(Du_k, u_k, x) \\
&\geq \int_{G_\epsilon} L(Du, u_k, x) + D_pL(Du, u_k, x)(Du_k - Du)\, dx
\end{aligned}$$

Then, $\lim \int_{G_\epsilon} L(Du, u_k, x) = \int_{G_\epsilon} L(Du, u, x)$ since $u_k \to u$ uniformly on G_ϵ and Du is uniformly bounded on G_ϵ. For the second term, $D_pL(Du, u_k, x) \to D_pL(Du, u, x)$ uniformly on G_ϵ. Then $Du_k \rightharpoonup Du$ in L^q. Then writing $\int (D_pL(Du, u_k, x) - D_pL(Du, u, x))(Du_k - Du) + D_pL(Du, u, x) \cdot (Du_k - Du)$, $Du_k - Du$ is bounded in L^q and $(D_pL(Du, u_k, x) - D_pL(Du, u, x))$ converges uniformly to 0, so the first term goes to 0. For the second term, $D_pL(Du, u, x)$ is bounded and $Du_k - Du$ converges weakly to 0.

It follows that

$$\ell = \liminf I[u_k] \geq \int_{G_\epsilon} (Du, u, x)\, dx \xrightarrow{\epsilon \to 0} \int_U L(Du, u, x)\, dx = I[u].$$

Hence, $I[u] \leq \liminf I[u_k]$, as desired.

$\square$

§12 February 25th, 2021

§12.1 Existence of Minimizers

We proved last time that a convexity condition was sufficient for showing the weak lower semicontinuity condition.

Example 12.1

A simple example is $L(p, z, x) = \sum a_{ij}(x)p_i p_j$ where $a_i j = a_{ji}$ and $\sum a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$. for all $\xi \in \mathbb{R}^n$, $x \in \overline{U}$. In this case, $\mathcal{A} = \{u \in H^1(U) : u|_{\partial U} = g\}$, if $g \in H^{1/2}(\partial U)$. Then, we minimize $\int_U \sum a_{ij}(x)\partial_{x_j}u\partial_{x_i}u\,dx = 0$ with $u|_{\partial U} = g$.

Theorem 19 (Existence of Minimizers)

Suppose $p \mapsto L(p, z, x)$ is convex and $L(p, z, x) \geq \alpha|p|^q - \beta$, $\alpha > 0$, $\beta \geq 0$, $1 < q < \infty$. Suppose that $\mathcal{A} = \{w \in W^{1,q}(U) : w|_{\partial U} = g\} \neq \emptyset$ with $g \in L^q(\partial U)$, then there exists $u \in \mathcal{A}$ such that $I[u] = \min_{w\in\mathcal{A}} I[w]$.

Proof. We can assume without loss of generality that $\beta = 0$. Put $m = \inf_A I[w] \neq \infty$. Choose a sequence $u_k \in \mathcal{A}$ such that $I[u_k] \to m$. Then $I[u_k] \geq \alpha \int |Du_k|^q$. This implies that $\|Du_k\|_{L^q} \leq C$.

Fix $w \in \mathcal{A}$ and note that $u_k - w \in W_0^{1,q}$. Recall the Poincare Inequality: if $v \in W_0^{1,q}$, then $\|v\|_q \leq C\|Dv\|_q$. Hence

$$\|u_k\|_q \leq \|u_k - w\|_q + \|w\|_q \leq \|Du_k - Dw\|_q + \|w\|_q \leq \|Du_k\|_q + \|w\|_{W^{1,q}} \leq C'.$$

This implies that $\|u_k\|_{W^{1,q}} \leq C$. Hence, there exists a subsequence $u_k \rightharpoonup u$ in $W^{1,q}$. This means that $u_k - w \rightharpoonup u - w$ in $W^{1,q}$ but $u_k - w \in W_0^{1,q}$, a closed subspace in $W^{1,q}$ which implies that $u - w \in W_0^{1,q}$. Hence, $u \in \mathcal{A}$. So, it suffices to show that u is a minimizer.

From the convexity of $p \mapsto L(p, z, x)$, we have I is weakly lower-semicontinuous. In other words, $I[u] \leq \liminf_{k\to 0} I[u_k] = \inf_{w\in\mathcal{A}} I[w]$. Hence, $I[u] = \inf_{w\in\mathcal{A}} I[w]$. □

§12.2 Uniqueness of Minimizers

Theorem 20 (Uniqueness of Minimizers)

Suppose $L = L(p, x)$ and there exists $\theta > 0$ such that for all $\xi \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $x \in \overline{U}$, we have $\sum L_{p_i p_j}(p, x)\xi_i\xi_j \geq \theta|\xi|^2$(uniform convexity). Then, any minimizer of $I[u]$ is unique.

Proof. We have that $L(p, x) \geq L(q, x) \geq D_p L(q, x)(p - q) + \frac{\theta}{2}|p - q|^2$ from strict convexity(this follows from the Taylor remainder). Let $u, \tilde{u}$ be minimizers of $I[w]$. Take $v = \frac{u+\tilde{u}}{2}$. From strict convexity, we have

$$I[u] \geq I[v] + \int D_p L(Du/2 + D\tilde{u}/2, x)(Du/2 - D\tilde{u}/2) + \theta/8|Du - D\tilde{u}|^2\,dx.$$

Similarly,

$$I[\tilde{u}] \geq I[v] + \int D_p L(Du/2 + D\tilde{u}/2, x)(D\tilde{u}/2 - Du/2) + \theta/8|Du - D\tilde{u}|^2\,dx.$$

Then,

$$m = I[u]/2 + I[\tilde{u}]/2 \geq I[v] + \theta/8 \int |Du - D\tilde{u}|^2\,dx \geq m + \theta/8 \int |Du - D\tilde{u}|^2\,dx.$$

This would imply that $\theta/8 \int |Du - D\tilde{u}|^2\,dx \leq 0$, which implies that $Du = D\tilde{u}$ almost everywhere but $u|_{\partial U} = \tilde{u}|_{\partial U}$ so $u = \tilde{u}$ almost everywhere. □

§12.3 The Euler-Lagrange Equation

Recall the example $L(p, x) = \sum a_{ij}(x)p_ip_j \geq \theta|p|^2$ with $a_{ij} = a_{ji}$. This has the Euler-Lagrange Equation:

$$\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j}u) = 0, \quad u|_{\partial U} = g.$$

For all $g \in H^{1/2}(\partial U)$ we can find $u \in H^1(U)$ such that the Euler-Lagrange equation weakly(in the sense of distributions). Furthermore, u is unique. We will show this next time.

Today, we show that we can solve the equation:

$$\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j}u) = f, \quad u|_{\partial U} = g, f \in H^{-1}(U)$$

with $u \in H^1(U)$. We can write

$$I[w] = \int \sum \left(a_{ij}(x)\partial_{x_i}w\partial_{x_j}w - f(x)w\right)\,dx$$

In this case $L(p, z, x) = \sum a_{ij}(x)p_ip_j - f(x)z$ for $z \in \mathbb{R}$, so this is not bounded below. However, $p \mapsto L(p, z, x)$ is convex. So it suffices to deal with the coercivity issue.

What we really need is that $I[w] \geq \int_U |Dw|^2 - \beta$.

Recall the Peter Paul inequality:

$$2ab \leq a^2/\epsilon + \epsilon b^2, \quad \forall \epsilon > 0.$$

Hence,

$$I[w] \geq \theta \int |Dw|^2 - \int |f||w| \geq \theta \int |Dw|^2 - \frac{1}{2\epsilon}\int |f|^2 - \frac{\epsilon}{2}\int |w|^2.$$

If we fix $w_0 \in H^1$ with $w_0|_{\partial U} = g \in H^{1/2}$, we have

$$I[w] \geq \theta \int |Dw|^2 - \frac{1}{2\epsilon}\int |f|^2 - \frac{\epsilon}{2}\left(\int |w - w_0|^2 + \int |w_0|^2\right) \geq \theta \int |Dw|^2 - \frac{C}{2\epsilon} - \frac{\epsilon}{2}|Dw|^2.$$

It follows that

$$I[w] \geq \theta \int |Dw|^2 - C_\epsilon - \epsilon \int |Dw|^2 = (\theta - \epsilon)\int |Dw|^2 - C_\epsilon.$$

Choosing $\epsilon < \theta/2$, we have

$$I[w] \geq \frac{\theta}{2} \int |Dw|^2 - C_\epsilon \geq \alpha \int |Dw|^2 - C.$$

One problem: we assumed $f \in L^2$. How do we fix this? We solve 2 problems:

$$\sum (a_{ij} u_{x_j})_{x_i} = f \in H^{-1}, u|_{\partial U} = 0,$$

and $u \in H_0^1(U)$. Then, $\int fu \leq \|f\|_{H^{-1}} \|u\|_{H_0^1} \leq \frac{1}{\epsilon}\|f\|_{H^{-1}}^2 + \epsilon \|u\|_{H_0^1}$. For $u \in H_0^1$, $\|u\|_{H_0^1} \leq C\|Du\|_2$. Then, we apply the same argument.

If not, we take $v = u - \tilde{u}$ and we have

$$-\sum (a_{ij} v_{x_j})_{x_i} = 0, v \in H_0^1(U).$$

Otherwise, we multiply by v and we use the definition of weak solution. This implies that $\int \sum a_{ij} v_{x_i} v_{x_j}$, but it is also at least $\theta \int |Dv|^2$, which shows that it is exactly zero.

§13 March 2nd, 2021

Today, we find conditions on the Lagrangian so that the the E-L condition holds:

$$-\sum(\partial_{p_j} L(Du(x), u(x), x))_{x_j} + \partial_z L(Du, u, x) = 0.$$

§13.1 Euler-Lagrange Equation, continued

We will make the following assumptions: for all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x \in \overline{U}$,

- $|L(p, z, x)| \leq C(|p|^q + |z|^q + 1)$,
- $|D_p L|, |D_z L| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$.

It is natural to consider the E-L equation in a weak sense, that is for all $v \in C_c^\infty(U)$,

$$\int \sum \partial_{p_j} L v_{x_j} + \partial_z L v = 0.$$

The conditions imply that $|\partial_{p_j} L| \leq C(|Du|^{q-1} + |u|^{q-1}) \in L^{q'}(U)$, and the same with $\partial_z L$. This implies that our integral condition would make sense for $v \in W_0^{1,q}(U)$.

Definition 13.1. Suppose our assumptions from above hold(bounds on $|L|$ and $|DL|$) and $u \in \mathcal{A} = \{w \in W^{1,q}(U) : w|_{\partial U} = g\}$. We then say that the E-L equation holds weakly if for all $v \in W_0^{1,q}(U)$, we have

$$\int \sum \partial_{p_j} L v_{x_j} + \partial_z L v = 0.$$

Theorem 21

Suppose $u \in \mathcal{A}$ is a minimizer for L satisfying the bounds. Then u is a weak solution to the Euler Lagrange equation.

Proof. Define $i(t) = I[u + tv]$ where $v \in W_0^{1,q}$. Let

$$\frac{i(t) - i(0)}{t} = \int_U \frac{L(Du + tDv, u + tv, x) - L(Du, u, x)}{t} \, dx$$

and call the integrand $L^t(x)$.

We have that $L^t(x) \xrightarrow{t \to 0} \sum L_{p_i}(Du, u, x) v_{x_i} + L_z v$ almost everywhere in x. We want to bound $|L^t(x)|$ by a function in L^1 so that we can apply the dominated convergence theorem.

Note that $f(\xi + t\eta) - f(\xi) = \eta \int_0^t f'(\xi + t\eta) \, dt$.This means that

$$L(Du + tDv, u + tv, x) - L(Du, u, x) = \int_0^t \sum L_{p_j}(Du + sDv, u + sv, x) v_{x_j} + L_z v \, ds.$$

Now, we bound this using our assumptions. Namely, recall that $|DL| \le C(|p|^{q-1} + |z|^{q-1} + 1)$. It follows that

$$\begin{aligned}
|L(Du + tDv, u + tv, x) - L(Du, u, x)| &\le \int_0^t \sum |L_{p_j}(Du + sDv, u + sv, x)v_{x_j}| + |L_z v| \, ds \\
&\le C \int_0^t (|Du + sDv|^{q-1} + |u + sv|^{q-1} + 1)(|Dv| + |v|) \, ds \\
&\le C \int_0^t (1 + |Du|^{q-1} + |Dv|^{q-1} + |u|^{q-1} + |v|^{q-1})(|Dv + |v||) \, ds \\
&\le Ct \left(|Du|^{q-1}(|Dv| + |v|) + |u|^{q-1}(|Dv| + |v|) + |Dv|^q + |v|^q + 1\right)
\end{aligned}$$

We would like to say

$$\left(|Du|^{q-1}(|Dv| + |v|) + |u|^{q-1}(|Dv| + |v|) + |Dv|^q + |v|^q + 1\right) \le C \left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1\right).$$

We do this via Young's inequality: $ab \le \frac{a^{q'}}{q'} + \frac{b^q}{q}$. This implies that

$$|Du|^{q-1}|Dv| \le C(|Du|^{(q-1)q'} + |Dv|^q),$$

and doing this for the other product terms gives the desired inequality.

It follows that $|L^t(x)|$ is bounded by an L^1 function, so we can apply the dominated convergence theorem, which gives the result. □

Remark 13.2. The converse is not necessarily true. However, we have the following theorem:

Theorem 22

Suppose $u \in \mathcal{A}$ is a weak solution to the Euler-Lagrange equation. If $(p, z) \mapsto L(p, z, x)$ is convex for all $x \in U$, then u is a minimizer.

Proof. From convexity, we have that $L(p, z, x) + D_p L(p, z, x) \cdot (q - p) + D_z L(p, z, x)(w - z) \le L(q, w, z)$. Upon integrating, if we set $p = Du$, $q = Dw$, $z = u$, $w = w(x)$ for $w \in \mathcal{A}$:

$$I[u] + \int_U D_p(Du, u, x) \cdot (Dw - Du) + D_z L(Du, u, x) \cdot (w - u) \, dx \le I[w].$$

But $w - u \in W_0^{1,q}$ and u weakly satisfies the equation so it follows that the integral is 0 and for every $w \in \mathcal{A}$, $I[u] \le I[w]$. □

§14 March 4th, 2021

§14.1 Regularity

We will make the assumption that $L(p, z, x) = L(p) - zf(x)$. We also assume that $|L(p)| \leq C(|p|^2 + 1)$ and $|D_p L(p)| \leq C(|p| + 1)$, $|D_p^2 L(p)| \leq C$. Finally, we assume strong convexity: $\sum L_{p_i p_j}(p)\xi_i \xi_j \geq \theta |\xi|^2$ for all $p, \xi \in \mathbb{R}^n$.

Example 14.1

$L(p) = \frac{1}{2}|p|^2$ is an example of a function which satisfies the above conditions.

Last time, we showed (with weaker assumptions) that if $I[u]$ is a minimum, then u satisfies the E-L equation weakly. With a convexity condition, we have the converse as well. We assume that $u|_{\partial U} = 0$ for simplicity.

Proposition 14.2

There exists a constant $C = C(L, n, U)$ so that $\|u\|_{H^1} \leq C\|f\|_2$.

Proof. We use the weak E-L with $v = u$. Namely,

$$\int_U \sum L_{p_j}(Du)u_{x_j} = \int_U fu.$$

Then, strict convexity implies that $(DL(p) - DL(0)) \cdot p \geq \theta |p|^2$, with $p = Du$. Hence,

$$\theta \int_U |Du|^2 \leq \int_U D_p L(Du) \cdot Du - \int_U DL(0) \cdot Du = \int_U fu,$$

where the second term is 0 by the divergence theorem. Then

$$\theta \int_U |Du|^2 \leq \int_U fu \leq \int_U \frac{f^2}{\epsilon} + \epsilon u^2 = \|f\|^2/\epsilon + \epsilon \|u\|^2.$$

Hence, $\|Du\|_2^2 \leq \frac{1}{\epsilon\theta}\|f\|^2 + \frac{\epsilon}{\theta}\|u\|^2$. By the Poincare inequality, $\|u\|_2 \leq C\|Du\|_2^2$, so it follows that by taking epsilon small enough. $\square$

§14.2 Interior Regularity

Theorem 23

Suppose that $-\sum (L_{p_j}(Du))_{x_j} = f$ weakly, $f \in L^2$, $u \in H_0^1$ with the same bounds as before. Then $u \in H^2_{loc}(U)$.

Proof. Take $V \Subset W \Subset U$ open sets. Choose a function $\zeta \in C_c^\infty(W)$ and $\zeta = 1$ near V.

Choose $D_k^h u(x) = \frac{u(x+he_k) - u(x)}{h}$. If h is small enough then it is well defined on U. Note that for $v \in C_c^\infty(W)$, $\int uD_k^{-h} v = -\int vD_k^h u$. Then, we define $v = -D_k^{-h}(\zeta^2 D_k^h u) \in H_0^1(W) \subset H_0^1(U)$.

Then noting that $(D_k^{-h} v)_{x_i} = D_k^{-h}(v_{x_i})$,

$$\int_U \sum D_k^h (L_{p_i}(Du))(\zeta^2 D_k^h u)_{x_i} = -\int_U fD_k^{-h}(\zeta^2 D_k^h u).$$

$$\begin{aligned} D_k^h(L_{p_i}(Du)) &= \frac{1}{h}\int_0^1 \sum_j L_{p_ip_j}(sDu(x+he_k)+(1-s)Du(x))\,ds\cdot(D_{x_j}u(x+he_k)-D_{x_j}u(x)) \\ &= \sum_{j=1}^n a_{ij}^h(x)D_k^h u_{x_j}(x) \end{aligned}$$

where

$$a_{ij}^h(x) = \int_0^1 L_{p_ip_j}(sDu(x+he_k)+(1-s)Du(x)).$$

So, we have

$$\int_U \sum_{j=1}^n a_{ij}^h(x)D_k^h u_{x_j}(\zeta^2 D_k^h u)_{x_i} = -\int_U f D_k^{-h}(\zeta^2 D_k^h u).$$

The left hand side is

$$\int \sum a_{ij}^h D_k^h u_{x_j} D_k^h u_{x_i}\zeta^2 + \sum a_{ij}^h D_k^h u_{x_j} 2\zeta\zeta_{x_i} D_k^h u.$$

The first term is bounded from below by $\theta|D_k^h Du|^2\zeta^2$ by strict convexity. The second term is bounded by $-\int \zeta|D_k^h Du||D_k^h u| \geq -\epsilon\int \zeta^2|D_k^h Du|^2 - \frac{1}{\epsilon}|D_k^h u|^2$.

We end up with

$$\int \zeta^2|D_k^h Du|^2\,dx \leq C\int_W (f^2+|D_k^h u|^2).$$

□

§15 March 9th, 2021

§15.1 Inner Regularity

Last time, we discussed regularity in the special case where $L = L(p) - f(x)z$. We also assumed that $|D_p^k L(p)| \leq C(1 + |p|^{2-k})$ and strict convexity: for all $p, \xi \in \mathbb{R}^n$, $\sum L_{p_i p_j}(p)\xi_i \xi_j \geq \theta |\xi|^2$.

We know that for $U \Subset \mathbb{R}^n$, ∂U C^1 and $f \in L^2$, for all $g \in H^{1/2}(\partial U)$, there exists u such that for all $v \in H_0^1(U)$,

$$\int_U (\sum L_{p_j}(Du) v_{x_j} - f(x)v)\, dx = 0, \quad u|_{\partial U} = g.$$

Theorem 15.1

Suppose that $-\sum (L_{p_j}(Du))_{x_j} = f$ weakly, $f \in L^2$, $u \in H_0^1$ with the same bounds as before. Then $u \in H^2_{loc}(U)$.

Remark 15.2. The main idea in the proof is that you can estimate derivatives with cutoff functions if you only need local results. This requires carefully choosing the width of the quotients to stay away from the boundary.

Proof. We had a function $D_k^h u(x) = \frac{u(x+he_k) - u(x)}{h}$, and we chose a ζ so that $\zeta \equiv 1$ in V and $\zeta \in C_c^\infty(W)$ where $V \subset W \subset U$.

Then, we defined $v = -D_k^{-h}(\zeta^2 D_k^h u)$. Last time, we showed that

$$\int_U \sum a_{ij}^h(x) D_k^h u_{x_j} (\zeta^2 D_x^h u)_{x_i} = \int_U f D_k^{-h}(\zeta^2 D_k^h u),$$

where we have the bounds

$$\theta |\xi|^2 \leq \sum a_{ij}^h(x)\xi_i \xi_j \leq C|\xi|^2.$$

Differentiating the expression, the LHS gives

$$\int \sum a_{ij}^h(x)(D_k^h u_{x_j} D_k^h u_{x_i})\zeta^2 + \int \sum a_{ij}^h(x) D_k^h u_{x_j} 2\zeta \zeta_{x_i} D_k^h u.$$

The first term is bounded below by $\theta \int_U |D_k^h Du|^2 \zeta^2$. The second term is bounded below by $-C \int \zeta |D_k^h Du||D_k^h u|$.

Now, the RHS is bounded above by

$$\int_W |f||D_k^{-h} Du|\zeta^2 + \int_W |f|\zeta|D_k^h u|.$$

Note that

$$D_k^h u = \int_0^1 u_{x_k}(x + the_k)\, dt.$$

So it follows that $\int_W |D_k^h D_k^{-h} u|\zeta^2 \leq \int |D_k^h Du|\zeta^2$. It follows that

$$\begin{aligned} \int_W |f||D_k^h Du|\zeta^2 + \int_W |f|\zeta|D_k^h u| &\leq \frac{2}{\epsilon}\int |f|^2 + \epsilon \int |D_k^h Du|^2 \zeta^2 + \epsilon \int \zeta^2 |D_k^h u|^2 \\ &\leq \frac{2}{\epsilon}\int |f|^2 + \epsilon \int \zeta^2 |D_k^h u|^2 + \epsilon \int |D_k^h Du|^2 \zeta^2. \end{aligned}$$

By using Peter Paul on the LHS bound and making $\epsilon << 1$, we obtain the inequality

$$\theta/2\int |D_k^h Du|^2\zeta^2 \leq 2/\epsilon\int |f|^2 + \epsilon\int \zeta^2|D_k^h u|^2 \leq 2/\epsilon\int |f|^2 + \epsilon\int |Du|^2.$$

Finally, we claim that $\int |D_k^h w|^2 \leq C \Rightarrow w_{x_k} \in L^2_{loc}$. We know that $D_k^h w$ is bounded in L^2 for all h, so it is weakly compact. Hence, $D_k^{h_j} w \rightharpoonup v \in L^2$. It follows that

$$\int (D_k^h w)\varphi = -\int w D_k^{-h}\varphi \to -\int w\varphi_{x_k} = \int w_{x_k}\varphi,$$

which is the weak derivative. □

§15.2 Higher Regularity

Take $f \equiv 0$ and $\int L_{p_i}(Du)v_{x_i} = 0$ for all $v \in H_0^1$. Take $w \in C_0^\infty(U)$ and set $v = -w_{x_k}$. We have

$$-\int \sum L_{p_i}(Du)\partial_{x_k}(w_{x_i})\,dx.$$

But $u \in H^2_{loc}$, so it follows that

$$\int \sum_{i,j} L_{p_i p_j}(Du)u_{x_j x_k}w_{x_i}\,dx.$$

Setting $\tilde{u} = u_{x_k} \in H^1$. We get that $\sum \partial_{x_j}(a_{ij}(x)\partial_{x_i}\tilde{u}) = 0$ where $a_{ij} = L_{p_i p_j}(Du)$. But the bounded coefficients don't give a strong enough condition to prove the result.

Theorem 15.3

Suppose that $w \in H^1_{loc}(U)$, $\sum \partial_{x_j}(a_{ij}(x)\partial_{x_i}w) = 0$ weakly, and $\theta|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq C|\xi|^2$. Then, there exists $\gamma > 0$ such that $w \in C^{0,\gamma}_{loc}$(DeGiorgi-Nash, Moser). Applying with $w = \tilde{u} = u_{x_k}$ gives that $u \in C^{1,\gamma}_{loc}$, which implies that $a_{ij} = L_{p_i p_j}(Du) \in C^{0,\gamma}$. Finally, using Schauder estimates, we have that $u \in C^{2,\gamma}$.

§16 March 11th, 2021

We follow the book *Grigis-Sjostraund: Microlocal Analysis for Differential Operators.*

§16.1 Oscillatory Integrals

We denote $X \subset \mathbb{R}^n$ as an open set and $\mathcal{D}'(X) = \{u : C_c^\infty(X) \to \mathbb{C} : \forall K \subset\subset X, \exists C, N, \forall \varphi \in C_c^\infty(K), |u(\varphi)| \le C \sup_{|\alpha|\le N} |\partial^\alpha \varphi|\}$.

We wish to generalize expressions like $\delta_0(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi}\, d\xi$. This is an "oscillatory integral" in the sense that we are integrating something that oscillates rapidly. This means that for all $\psi \in C_c^\infty$, $\delta_0(\psi) = \psi(0) = (2\pi)^{-n} \int\int e^{ix\cdot\xi}\psi(x)\, dx d\xi$.

We change the phase of $x \cdot \xi$ to a function $\varphi(x, \theta)$ where $x \in X$, $\theta \in \mathbb{R}^N$ so that $\varphi(x, \lambda\theta) = \lambda\varphi(x, \theta)$ for $\lambda > 0$.

The amplitude $(2\pi)^{-n}$ is generalized to $a(x, \theta)$, and we try to consider which properties we need so that we can define $I(a, \varphi) = \int_{\mathbb{R}^N} a(x, \theta)e^{o\varphi(x,\theta)}\, d\theta$ to be a distribution.

Example 16.1

Take $X = \mathbb{R}^n$, $N = n$. Define $P(\xi)$ to be a homogeneous polynomial satisfying $P(\xi) \neq 0$ whenever $\xi \neq 0$. For example, we could take $P(\xi) = |\xi|^2$. Define $\chi \in C_c^\infty(\mathbb{R}^n)$ so that $\chi \equiv 1$ near zero. Take $E(x) = (2\pi)^{-n}\chi(x) \int_{\mathbb{R}^n} \frac{1-\chi(\xi)}{P(\xi)} e^{ix\cdot\xi}\, d\xi$. This doesn't converge, but the integrand is a smooth homogeneous function of degree $-m$ away from zero. If we define $D_x = \partial_x/i$, then

$$P(D)E(x) = \frac{1}{2\pi}\chi(x) \int_{\mathbb{R}^n} \frac{1-\chi(\xi)}{P(\xi)} P(\xi)e^{ix\cdot\xi} + [P(D), \chi]u \int_{\mathbb{R}^n} \frac{1-\chi}{R} e^{ix\xi}\, d\xi,$$

where $[P(D), \chi]$ is the commutator.

The first term is

$$\frac{1}{2\pi}\chi(x) \int e^{ix\xi} d\xi + (2\pi)^{-n}\chi(x) \int (-\chi(\xi))e^{ix\xi} d\xi.$$

The second term is

$$\sum_{|\alpha|>1} C\partial^\alpha X(2\pi)^{-n} \int (1-\xi)/P\xi^\beta e^{ix\xi} d\xi.$$

The first term is $\delta_0(x)$, and the second is compactly supported. For the last term, for $x \neq 0$, $1/|x|^2\langle x, \partial_\xi\rangle e^{ix\ xi} = e^{ix\cdot\xi}$. The idea is that we can integrate by parts and the ∂_ξ derivatives will decay rapidly. Then, we can replace this term with something that decays rapidly so that we have $\delta_0(x) + K(x)$ for a compactly supported $K \in C_c^\infty(\mathbb{R}^n)$. This is called a "Parametrix" for $P(D)$.

In our above example, $\varphi(x, \theta) = x \cdot \theta$ and $\theta = \xi$, with $a(x, \xi) = \frac{1-\chi}{P(\xi)}$. Our function satisfies the estimate

$$|\partial_\xi^\alpha((1-\chi)/P)| \le C_\alpha \langle\xi\rangle^{-m-|\alpha|}.$$

§16.2 General Theory: Amplitudes

Definition 16.2. $S^m_{\rho,\delta}(X \times \mathbb{R}^N) = \{a \in C^\infty(X \times \mathbb{R}^N) : \forall K \Subset X, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^N, \exists C = C(K, \alpha, \beta) : |\partial_X^\alpha \partial_\xi^\beta a| \le C\langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}\}$.

In our previous example, when P is a homogeneous polynomial of degree q, $\frac{1-\chi(\xi)}{P(\xi)} \in S^{-q}_{1,0}$.

> **Remark 16.3.** This only makes sense for $0 \le \rho \le 1$ and $0 \le \delta \le 1$. For $\rho > 1$. Then $|\partial^\alpha a| \le C_{N,\alpha}\langle \xi \rangle^{-N}$. Suppose ξ has dimension $[in]$ and a has dimension $[in]^k$. $\partial_\xi^\alpha a$ has dimensions $[in]^{k-|\alpha|}$. On the other hand, $|\partial_\xi^\alpha a| \le C\langle \xi \rangle^{m-\rho|\alpha|}$, which has dimension $[in]^{m-\rho|\alpha|}$. Then $k - |\alpha| = m - \rho|\alpha|$ so $k = m - (\rho - 1)|\alpha|$, but since α is arbitrary, the units of a are any negative number.
>
> The same analysis would work for $\delta < 0$.

$S^m_{\rho,\delta}$ is a Frechet space, one that is generated by seminorms. Namely, note that $\|a\|_{K,\alpha,\beta} = \sup_{(x,\theta)} \langle \theta \rangle^{-m+\rho|\beta|+|\alpha|\delta} |\partial_x^\alpha \partial_\xi^\beta a|$. Then $a \in S^m_{\rho,\delta}$ if and only if for all $K \Subset X$, $\|a\|_{K,\alpha,\beta} < \infty$. The space is a Frechet space if it is complete with respect to this norm. This is also a meterizable space, with a metric defined in the obvious way.

Some properties:

- If $m \le m'$, $\delta \le \delta'$, $\rho \ge \rho'$, then $S^m_{\rho,\delta} \subset S^{m'}_{\rho',\delta'}$.
- We define $S^{-\infty}(X \times \mathbb{R}^N) = \{a \in C^\infty(X \times \mathbb{R}^N) : K \Subset X, \forall N, \exists C |\partial_x^\alpha \partial_\xi^\beta a| \le C\langle \xi \rangle^{-N}\}$.
- $S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_m S^m_{\rho,\delta}(X \times \mathbb{R}^N)$. We call this the residual space.

Example 16.4

Take $a \in C^\infty(X \times \mathbb{R}^N)$ and for $|\theta| \ge 1$, $\lambda > 0$, $a(x, \lambda\theta) = \lambda^m a(x, \theta)$. We claim that $a \in S^m_{1,0}$. If we differentiate, we have $\partial_\theta^\alpha a(x, \lambda\theta) = \lambda^{m-|\alpha|}\partial_\theta^\alpha a(x, \theta)$.

§17 March 16th, 2021

§17.1 Amplitudes of Oscillatory Integrals

We are making sense of $I(a,\varphi) = \int_{\mathbb{R}^N} a(x,\theta)e^{i\varphi(x,\theta)}\,d\theta$ on a distribution. We defined a class of functions for $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$. $S^m_{\rho,\delta}(X \times \mathbb{R}^N) = \{a \in C^\infty(X \times \mathbb{R}^N) : \forall K \Subset X, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^N, \exists C = C(K,\alpha,\beta) : |\partial_X^\alpha \partial_\xi^\beta a| \le C\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}\}$. These are called the **symbols** of order m and type (ρ,δ). We write $S^m = S^m_{1,0}$, which is the case where $|\langle\theta\rangle^{-m+|\beta|}\partial_x^\alpha\partial_\xi^\beta a| \le C_{\alpha\beta}$.

Why are these called symbols? Suppose we have $P(x,0)u = \sum_{|\alpha|\le m} a_\alpha(x)D_k^\alpha u$. We can also write this as $(2\pi)^{-n}\int\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha e^{i(x-y)\xi}u(y)\,dyd\xi$, $u \in \S$. The current order of integration makes sense but we can also just consider the integral in $d\xi$, which is an oscillatory integral, whose integrand is $p(x,\xi)$ is the symbol of $P(x,D)$. Then $p \in S^m(X \times \mathbb{R}^n)$, which is of type $(1,0)$.

§17.2 A Cool Example

Suppose $f \in C^\infty(X \times \mathbb{R}^n; [0,\infty))$ homogeneous - $f(x,\lambda\theta) = \lambda f(x,\theta)$, $\lambda > 0$, $|\theta| \ge 1$. Define $a(x,\theta) = e^{-f(x,\theta)}$. Note that $0 \le a \le 1$. This is also a smooth function.

We claim that

$$\partial_x^\alpha\partial_\theta^\beta(e^{-f}) = \sum_{|\tilde\alpha|\le|\alpha|, |\tilde\beta|\le|\beta|} a_{\alpha\beta}(x,\theta)(\partial_x f^)\tilde\alpha(\partial_\theta f)^{\tilde\beta}e^{-f}.$$

We can estimate the bad term $(\partial_x f^)\tilde\alpha(\partial_\theta f)^{\tilde\beta}e^{-f}$. For this, we use Landau's inequality: If $g \in C^2(U)$, $g \ge 0$, for all $K \Subset U$, there exists C such that $|\nabla g(x)| \le C\sqrt{g(x)}$, $x \in K$.

Proof. Note that $0 \le g(x+y) = g(x) + \nabla g(x) + y + O(|y|^2)$. This implies that $-\nabla g(x)\cdot y \le g(x) + O(|y|^2)$. Taking $y = -\epsilon\nabla g(x)$, it follows that

$$\epsilon|\nabla g(x)| \le g(x) + O(\epsilon^2|\nabla g(x)|^2),$$

which implies the result. □

We have that $f \ge 0$ so we have that $|\partial_x f| + |\partial_\theta f| \le Cf^{1/2}$, with $1 \le \theta \le 2$, $x \in K$. For $\lambda > 0$, note that

$$\lambda^{-1}\partial_x f(x,\lambda\theta) + \partial_\theta f(x,\theta) \le C\lambda^{-1/2}f(x,\lambda\theta)^{1/2}.$$

It follows by taking $\tilde\theta = \lambda\theta$,

$$|\theta|^{-1/2}\partial_x f(x,\theta) + |\theta|^{1/2}\partial_\theta f(x,\theta) \le C(f,\theta)^{1/2}, \qquad |\theta| \le 1, x \in K.$$

Now, we estimate

$$|(\partial_x f)^{\tilde\alpha}(\partial_\theta f)^{\tilde\beta}e^{-f}| \le |\theta|^{|\tilde\alpha|/2}|\theta|^{-|\tilde\beta|/2}f^{1/2(|\tilde\alpha+\tilde\beta|)}e^{-f}.$$

Finally $f^k e^{-f} \le k!$, which is some constant, so it follows that our term is bounded by $C_{\tilde\alpha\tilde\beta}|\theta|^{|\tilde\alpha|/2}|\theta|^{-\tilde\beta/2} \in S^0_{1/2,1/2}(X \times \mathbb{R}^N)$.

§17.3 Some Lemmas

Proposition 17.1

Suppose $\{a_j\}$ is bounded in $S^m_{\rho,\delta}$ and $a_j(x,\theta) \to a(x,\theta)$ for all $x, \theta \in X \times \mathbb{R}^N$. Then, $a \in S^m_{\rho,\delta}$ and $a_j \to a$ in $S^{m'}_{\rho,\delta}$ for all $m' > m$.

Proof. We first prove a lemma.

Lemma 17.2

Suppose $f \in C^2([-\epsilon, \epsilon])$. Then $|f'(0)| \le 2\|f\|^{1/2}_{L^\infty}\|f''\|^{1/2}_{L^\infty} + (2/\epsilon + 1/2)\|f\|_{L^\infty}$, where $\|g\|_{L^\infty} = \sup_{|x|\le\epsilon}|g|$.

Proof. $f(x) = f(0) + xf'(0) + x^2 + \int_0^1 (1-t)f''(tx)\,dt$. We have the estimate

$$|xf'(0)| \le 2\|f\|_\infty + \frac{x^2}{2}\|f''\|_\infty.$$

Dividing by x, we we have $|f'(0)| \le 2/x\|f\|_\infty + x/2\|f''\|_\infty$. Then, we take $x = \min(2\|f\|^{1/2}_\infty/\|f''\|^{1/2}_\infty, \epsilon) \le 2\|f\|^{1/2}_\infty\|f''\|^{1/2}_\infty + (\frac{2}{\epsilon} + \frac{1}{2})\|f\|_\infty$, ☐

From the lemma,

$$\|a_j' - a_k'\| \le C\|a_j - a_k\|^{1/2}_\infty\|a_j'' - a_k''\|^{1/2}_\infty + C\|a_j - a_k\|_\infty.$$

Hence, a Cauchy sequence in L^∞ implies Cauchy for higher symbols.

Example 17.3

Take $a = 1$, $a_j(\theta) = \chi(\theta/j)$. $a_j(\theta) \to a(\theta)$ for all θ. Do we have convergence of $a_j \to a$ in S^0? No! because $\|a_j - a\|_\infty = 1$. This is similar to the statement C_0^∞ is dense in L^p for $1 \le p < \infty$.

Now, $\|\langle\theta\rangle^{-\delta}(a_j - a)\|_\infty \to 0$ as $j \to \infty$.

Define

$$b_j = \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{\langle\theta\rangle^{m' - \rho|\beta| + \delta|\alpha|}} = \frac{1}{\langle\theta\rangle^{m'-m}} \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{\langle\theta\rangle^{m - \rho|\beta| + \delta|\alpha|}}.$$

We know that $\partial_x^\alpha \partial_\theta^\beta (a_j - a)$ goes to 0 on compact sets, and $\langle\theta\rangle^{m-m'} \to 0$ as $|\theta| \to \infty$. Then, for all $\epsilon > 0$, there exists R_ϵ so that $|k_j| < \epsilon/2$ if $|\theta| > R_\epsilon$.

☐