Olympiad Geometry

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I first present my notes from the Art of Problem Solving(AoPS) Olympiad Geometry courses. This includes problems, solutions and expository material.

Contents

§1 Week 1: Fundamentals of Geometry

§1.1 Similar Triangles

The first fundamental tool at our disposal is similar triangles, which give us relationships between the lengths and angles of segments.

Definition 1.1. Two triangles $\triangle ABC, \triangle DEF$ and similar (denoted $\triangle ABC \sim \triangle DEF$) if $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. If the above relations hold, then we also have

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

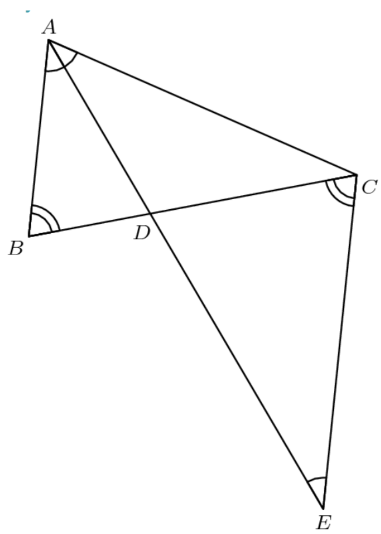
Similar triangles can be useful if a problem involves ratios or products of lengths. Another use(though rare) is that we show triangles are similar by showing $AB/DE = AC/DF = BC/EF$ and deduce the angles are equal. We could also show that pair of sides have equal ratio and the included angle is equal: $AB/DE = AC/DF$ and $\angle BAC = \angle EDF$, then $\triangle ABC \sim \triangle DEF$.

We begin by present some applications.

Theorem 1 (Angle Bisector)

Take $\triangle ABC$. If $D \in BC$ so that AD bisects $\angle BAC$, then $AB/BD = AC/CD$.

Proof. Draw a line through C parallel to AB and mark E as the insertion of the parallel line through C and the extension of AD.



Then $\angle ABC = \angle ECD$ and $\angle DEC = \angle DAC$ so it follows that $\triangle ABC \sim \triangle ECD$. Thus, $AB/BD = EC/CD$. Finally, $\angle CED = \angle DAC$ so it follows that $\triangle ACE$ is isosceles and $AC = EC$ so we find that

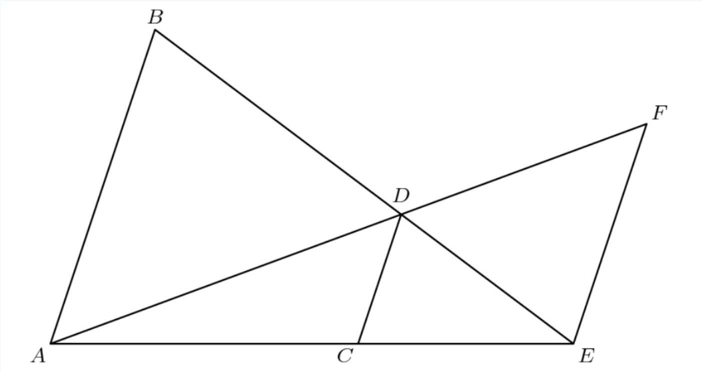
$$\frac{AB}{BD} = \frac{AC}{CD},$$

as desired. $\square$

Remark 1.2. We also could prove this using the Law of Sines or the ratio of areas of the two triangles.

Problem 1

Given that $AB||CD||EF$, prove that $\frac{1}{AB} + \frac{1}{EF} = \frac{1}{CD}$ in the following diagram:



Proof. Multiplying through by CD, we get that

$$CD/AB + CD/EF = 1.$$

Note that $\triangle ACD \sim \triangle AEF$ and $\triangle ECD \sim \triangle EAB$ so it follows that $CD/AB = CE/AE$ and $CD/EF = CA/AE$.

Finally,

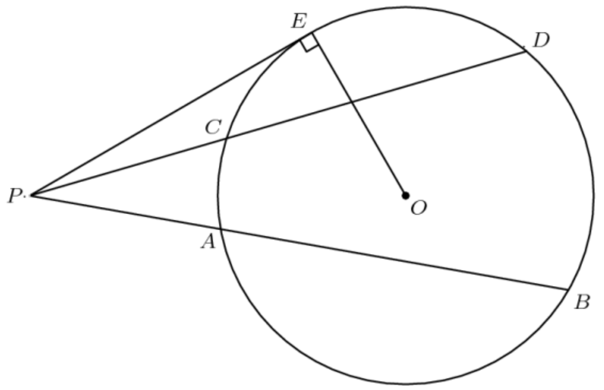
$$\frac{CD}{AB} + \frac{CD}{EF} = \frac{CE}{AE} + \frac{CA}{AE} = \frac{AE}{AE} = 1.$$

$\square$

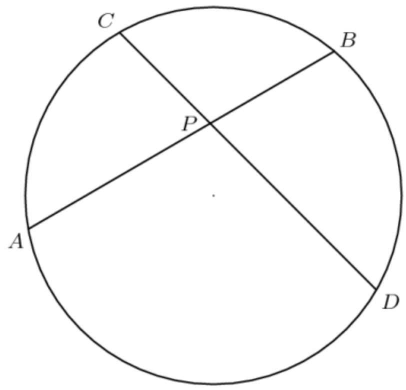
§1.2 Power of a Point

Theorem 2 (Power of a Point)

Take a point P and circle O. For any line that passes through P and intersects O at two points X and Y, the product $(PX)(PY)$ is constant. We call this product the **power of point** P with respect to circle O.

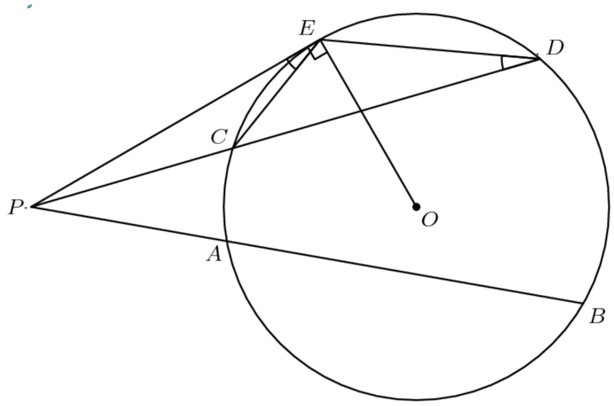


$PC \cdot PD = PA \cdot PB = PE^2$

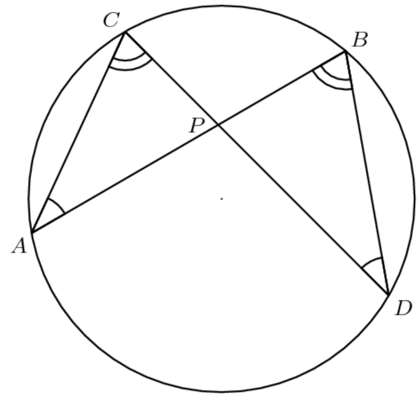


$PC \cdot PD = PA \cdot PB$

The Power of a Point Theorem follows from similar triangles:



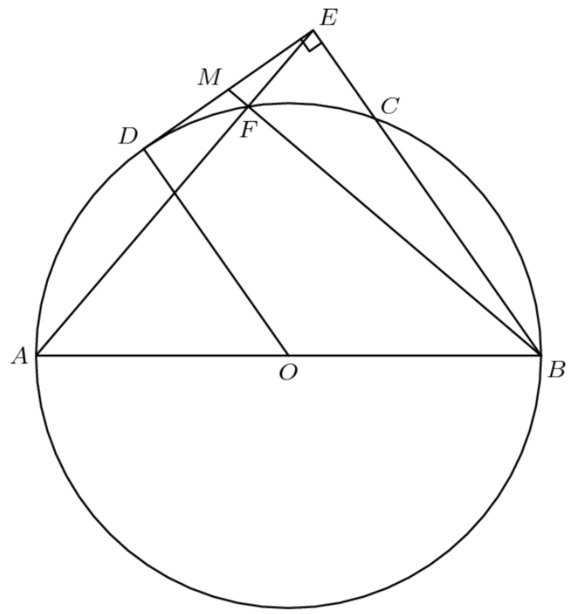
$\triangle PCE \sim \triangle PED$



$\triangle PCA \sim \triangle PBD$

Problem 2

AB is a diameter of circle O. Points C and D are on the circle such that D bisects arc AC. Point E is on the extension of BC that that BE is perpendicular to DE. F is the intersection of AE and circle O. Prove that the extension of BF bisects segment DE at M.



Proof. We first claim that $OD||EB$. This is because

$$\angle AOD = \text{arc}(AD) = \text{arc}(AC)/2 = \angle ABE.$$

It follows that ED is tangent to the circle, since $\angle ODE$ is a right angle. Furthermore, $\angle AFB$ is a right angle since AB is the diameter of the circle. Now, note that $\text{Pow}_O(M) = MD^2 = MF \cdot FB$. It suffices to show that $EF^2 = MF \cdot FB$. This follows from the fact that $MFE \sim EFB$, so it follows that

$$\frac{EF}{FB} = \frac{ME}{FE} \implies EF^2 = ME \cdot FB.$$

$\square$

§1.3 Cyclic Quadrilaterals

Definition 1.3. A quadrilateral is called **cyclic** if a circle can be drawn that passes through all four vertices.

There are 4 equivalent methods to showing a quadrilateral $ABCD$ is cyclic, namely:

- Showing $\angle ABD = \angle ACD$(or any of the other pairs of similarly defined angles).
- Showing a pair of opposite angles sum to 180 degrees.
- The converse of the Power of a Point: if P is the intersection of lines AB and CD and
$$PA \cdot PB = PC \cdot PD$$
or
$$QC \cdot QD = QB \cdot QA,$$
then A, B, C, D are all on a circle.
- The equality condition of **Ptolemy's Inequality**: In a quadrilateral $ABCD$,
$$AB \cdot CD + BC \cdot DA \geq AC \cdot BC,$$
with equality if and only if $ABCD$ is cyclic.

I omit the basic examples but present some of the interesting ones:

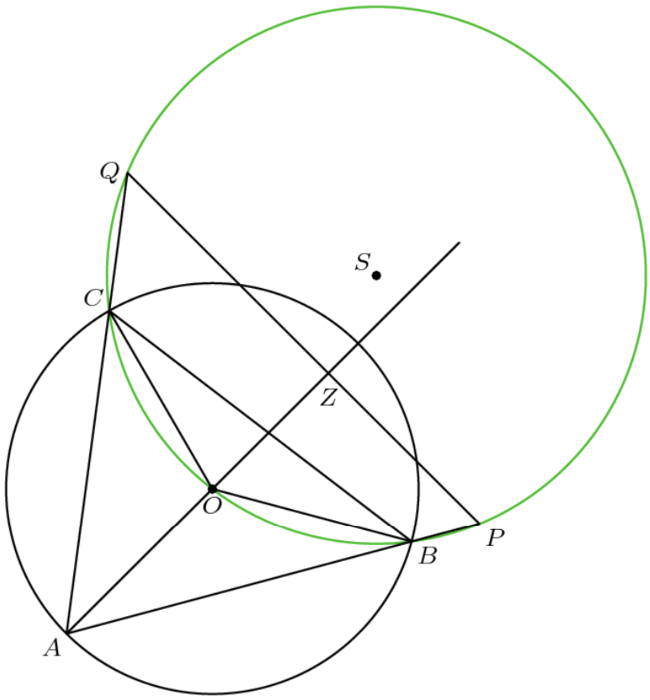
Proposition 1.4

A chord ST of constant length slides around a semicircle with diameter AB. M is the midpoint of ST and P is the foot of the perpendicular from S to AB. Prove that the angle SPM is constant for all positions of ST.

Proof. If $SM = MT$, then it follows M is the perpendicular bisector of $\triangle OST$. Thus, $OPSM$ is cyclic and $\angle SPM = \angle SOM$. Finally, the length of SM is constant, so it follows that the arc between intersection of the extension of OM and the circle and S is constant. Thus, $\angle SPM$ is constant, as desired. □

Proposition 1.5

ABC is an acute triangle with O as its circumcenter. Let S be the circle through C, O, B. The lines AB and AC meet circle S again at P and Q, respectively. Show that AO and PQ are perpendicular.



Proof. It suffices to show that $\angle AZP$ is right, where $Z = AO \cap PQ$. This reduces to showing that $\angle ZPA + \angle ZAP = 90$. Since $PBCQ$ is cyclic, note that

$$\angle ZPA = 180 - \angle BCQ = \angle ACB,$$

so it suffices to show that $ACB + OAB = 90$. Mark D as the intersection of AO with the original circle. Then,

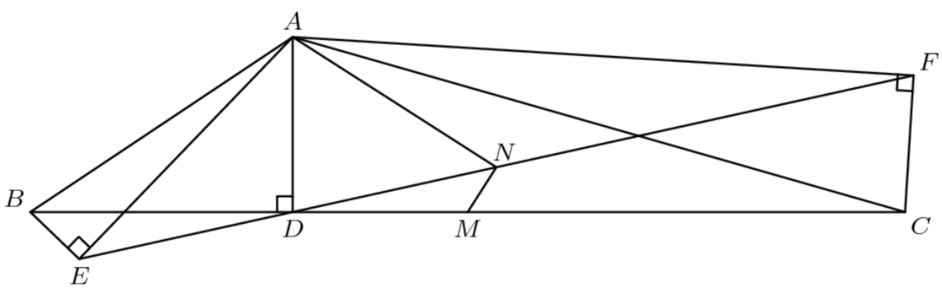
$$\angle ACB + \angle OAB = \frac{\text{arc}(AB) + \text{arc}(BD)}{2} = \frac{\text{arc}(AD)}{2} = 90.$$

□

§1.4 Problems

Problem 3

Let ABC be a triangle and D be the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BC, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.



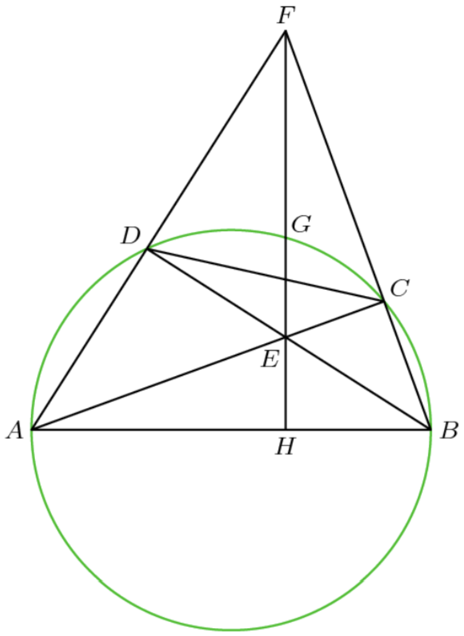
Proof. Note that $ABED$ and $AFCD$ are cyclic quadrilaterals. It follows that $ABC \sim AEF$ since $\angle ABD = \angle AED$ and $\angle AFD = \angle ACD$. Similarly, we can show that $ABM \sim AEN$ since

$$\frac{AB}{AE} = \frac{BC}{EF} = \frac{2BN}{2EM} = \frac{BN}{EM}.$$

Therefore, $\angle AND = \angle AMD$ and it follows that $ANMD$ is cyclic. Therefore $\angle ANM = 180 - \angle AD = 90$, as desired. □

Problem 4

Let $ABCD$ be a convex quadrilateral inscribed in a semicircle with diameter AB. The lines AC and BD intersect at E and the lines AD and BC meet at F. The line EF meets the semicircle at G and AB at H. Prove that E is the midpoint of GH if and only if G is the midpoint of the line segment FH.



Proof. Note that $\angle ADB = \angle ACB = 90$. It follows that E is the orthocenter of FAB and $\angle FAH = 90$. We obtain many similar triangles, with one notable one begin $\triangle AEH = FBH$ which gives the relation

$$HE \cdot HF = HA \cdot HB.$$

However, note that

$$\text{Pow}(H) = HG^2 = HA \cdot HB,$$

so it follows that

$$\frac{HG}{HF} = \frac{HE}{HG},$$

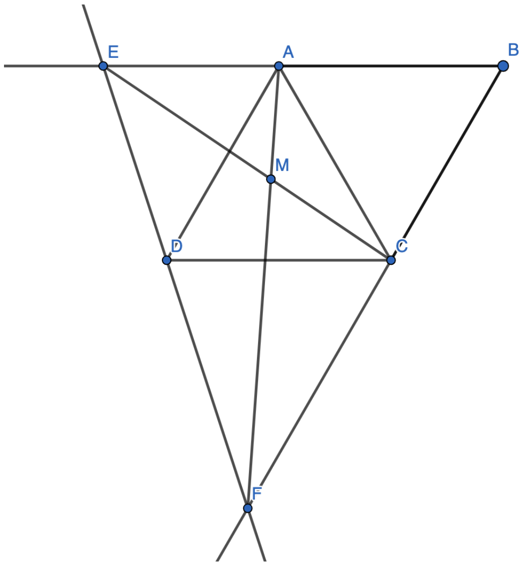
which proves the result. □

§2 Homework 1: Fundamentals of Geometry

§2.1 Problem 1

Problem 5

Let $ABCD$ be a quadrilateral such that all sides have equal length and $\angle ABC = 60$. Let k be a line through D and not intersecting the quadrilateral. Let E and F be the intersection of k with lines AB and BC respectively. Let M be the point of intersection of CE and AF. Prove that $CA^2 = CM \cdot CE$.



Proof. It suffices to show that $\triangle MCA \sim \triangle ACE$. We already have that $\angle MCA = \angle ACE$ so we finish by showing that $\angle CAM = \angle CEA$.

We first claim that $AD||CB$ and $AB||DC$. Note that $AB = BC$ and $\angle ABC = 60$ so it follows that $\triangle ABC$ is equilateral. Hence $AB = CB = CA$. But note that $AD = DC = AB = CA$, so it follows that $\triangle ADC$ is also equilateral. Hence $\angle DAB = 120$ and $\angle ADC = \angle ABC = 60$ showing that $AD||CB$ and $AB||DC$.

Note that $\angle EAC = \angle ACE = 120$, so it suffices to show that $\frac{EA}{AC} = \frac{AC}{CF}$, since it follows that $\triangle EAC = \triangle ACF$ and $\angle CAM = \angle CEA$. Furthermore, we have that $\triangle DCF \sim \triangle EAD$ since $\angle EAD = \angle DCA$ and $\angle AED = \angle CDF$. It follows that

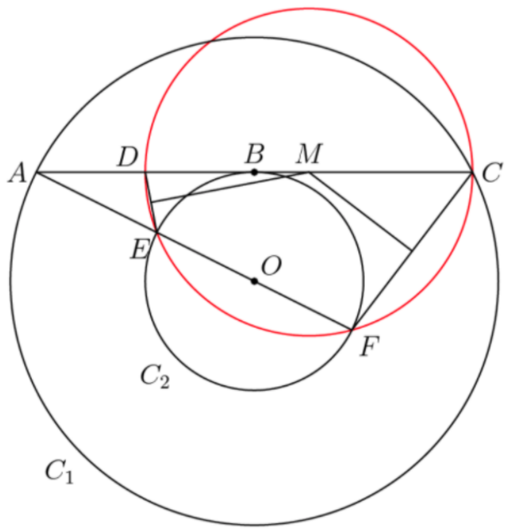
$$\frac{EA}{AC} = \frac{DA}{FC} = \frac{AC}{FC},$$

since $DA = AC$, which completes the proof. □

§2.2 Problem 2

Problem 6

Let C_1 and C_2 be concentric circles with C_2 inside C_1. Let A and C be on C_1 such that AC is tangent to C_2 at B. Let D be the midpoint of AB. A line passing through A meets C_2 at E and F such that the perpendicular bisectors of DE and CF meet at a point M on a segment DC. Find the ratio AM/MC.



Proof. Note that $\text{Pow}_{C_2}(A) = AB^2 = AE \cdot AF$. Furthermore, since $AD = \frac{1}{2}AB$ and $AC = 2AB$, it follows that

$$AD \cdot AC = \frac{1}{2}AB \cdot 2AB = AB^2 = AE \cdot AF.$$

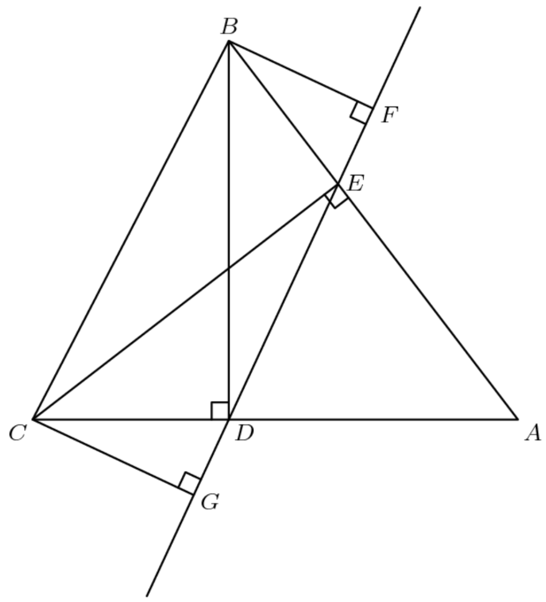
Hence, $DCFE$ is a cyclic quadrilateral. Furthermore, Since M is the perpendicular bisector of the chords DE and CF, it follows that M is the center of the corresponding circle. Hence M is the midpoint of DC. It follows that $AM = \frac{5}{8}AC$ and $MC = \frac{3}{8}AC$ so $AM/MC = 5/3$. □

§3 Week 2: Fundamentals, Continued

§3.1 Warm-up Problem

Problem 7

$\triangle ABC$ is acute; BD and CE are altitudes. Points F and G are the feet of perpendiculars BF and CG to line DE. Prove that $EF = DG$.



We present two proofs for the problem, though there are many. The first uses basic facts about cyclic quadrilaterals and similar triangles.

Proof. Note that $BEDC$ is a cyclic quadrilateral. Note that $\angle BCD = \angle BEF = 180 - \angle BED$. Hence, $\triangle BEF \sim \triangle BCD$. Similarly, $\triangle CGD \sim \triangle CEB$. Therefore,

$$\frac{EF}{CD} = \frac{BE}{BC} = \frac{DG}{CD},$$

so it follows that $EF = DG$. □

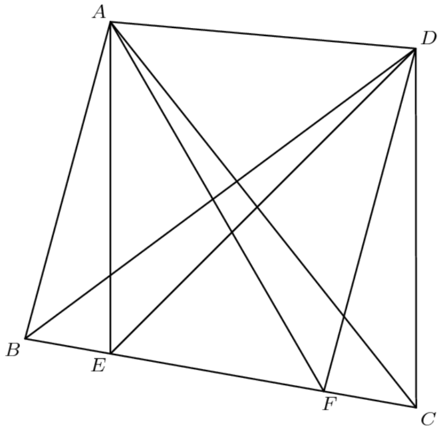
The second proof uses properties of projections.

Proof. The midpoint of BC is the circumcenter of circle $BCDE$, so it projects to the midpoint of DE. On the other hand, the midpoint of BC projects to the midpoint of FG, since $BFGC$ is a trapezoid. It follows that DE and GF have the same midpoint, so $DG = EF$. □

§3.2 Russia

Problem 8 (Russia)

Points E and F are on side BC of a convex quadrilateral $ABCD$ with $BE < BF$. Given that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$, prove that $\angle FAC = \angle EDB$.



Proof. Note that $\angle EAF = \angle FDE$ implies that $AEFD$ is cyclic. It suffices to show that $ABCD$ is cyclic. Note that $\angle ADC = \angle ADF + \angle FDC$, so we have

$$\angle ABC + \angle ADC = \angle ABC + \angle ADF + \angle FDC.$$

Then, $\angle ABC = \angle AEF - \angle BAE$, so it follows that

$$\begin{aligned}\angle ABC + \angle ADC &= \angle ABC + \angle ADF + \angle FDC \\ &= \angle AEF - \angle BAE + \angle ADF + \angle FDC \\ &= \angle AEF + \angle ADF \\ &= 180,\end{aligned}$$

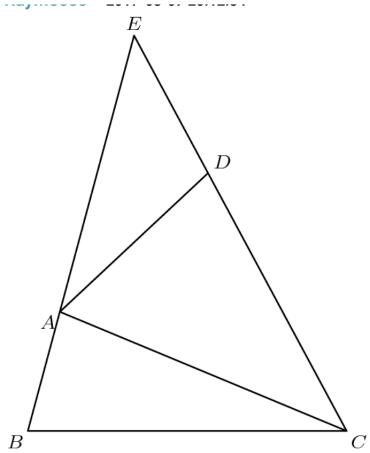
which shows that $ABCD$ is cyclic, as desired. □

§3.3 Bulgaria

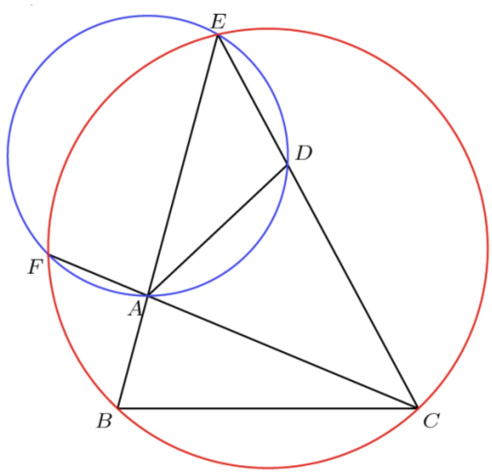
Problem 9 (Bulgaria)

A convex quadrilateral $ABCD$ is given for which $\angle ABC + \angle BCD < 180$. AB and CD extended meet at E. Prove that $\angle ABC = \angle ADC$ if and only if $AC^2 = CD \cdot CE - AB \cdot AE$.

Remark 3.1. After drawing the diagram for the problem, one should check that it corresponds to the solution in the problem. One can enter a trap proceeding without checking for this problem specifically.



Proof. Let ω_1 be the circumcircle of ADE and ω_2 be the circumcircle of EBC. Note that $\text{Pow}_{\omega_1}(C) = CD \cdot CE$ and $\text{Pow}_{\omega_2}(A) = AB \cdot AE$. Extend CA to ω_2 and label the intersection F.



Assuming that $AC^2 = CD \cdot CE - AB \cdot AE = CA \cdot CF - AB \cdot AE$, it follows that

$$AC(CF - AC) = AC \cdot AF = AB \cdot AE,$$

so from the converse of the Power of a Point, it follows that $F \in \omega_2$.

Finally,

$$\angle ABC = \angle AFE = 180 - \angle ADE = \angle ADC.$$

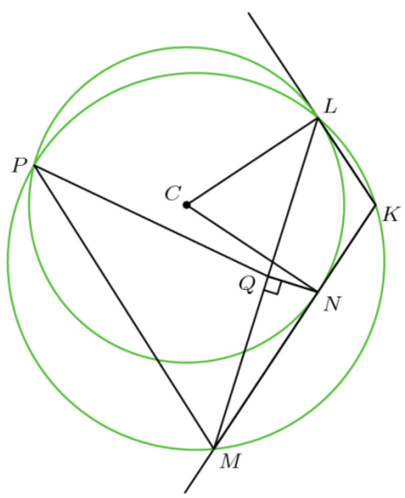
We can go back and show that each of the steps are reversible, but this is left as an exercise. □

§3.4 Iran

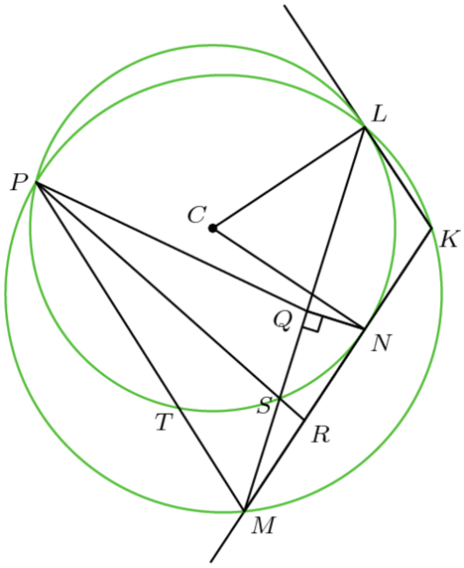
Warning: This is a very difficult problem.

Problem 10 (Iran)

Point K is outside circle C and points L and N are on C such that KL and KN are tangent to C. Let M be on ray KN beyond N, and let P be the second intersection of the circumcircle of KLM and C. Let Q be the foot of the perpendicular from N to ML. Prove that $\angle MPQ = 2\angle KML$.

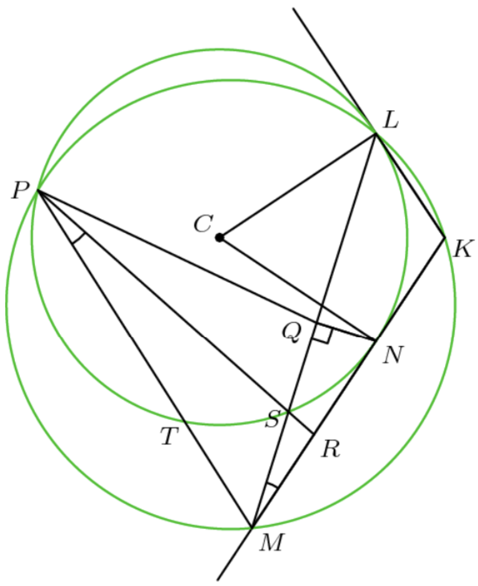


Proof. Let S be the intersection of QM and circle C. We show that PS bisects P. Let T be the intersection of PM and (PNL).



We would like to show that $\angle QPS = \angle MPS = \angle KML$. First, note that $\angle KML = \angle KPL$ since they are inscribed in the same arc LK of $(KLPM)$. If we can show $\angle MPK = \angle SPL$, this shows that $\angle KPL = \angle MPS$ since they share a common angle $\angle SPK$, and hence $\angle KML = \angle MPS$.

Firstly, $\angle MLK = \angle MPL$ from cyclic quadrilateral $MKLP$. Then, $\angle MLK = \angle SLK = \angle SPL$ since they are inscribed in arc LS of circle C. Thus, $\angle KML = \angle MPS$.



It suffices to show that either $\angle KML = \angle QPS$ or $\angle MPS = \angle QPS$. To show the first, we can show that $PQRM$ is cyclic. A good candidate to show this is to show that $\angle RQM = \angle RPM$, since we already know that $\angle RPM = \angle RMS$. To show $\angle RQM = \angle RMS$, it suffices to show that RQM is isosceles, or $RQ = RM$.

Note that $\triangle PRM \sim \triangle MRS$ since they share $\angle SRM$ and $\angle SMR = \angle MPR$. From this, we find that

$$\frac{PR}{MR} = \frac{RM}{RS} = \frac{MP}{SM} \implies MR^2 = RP \cdot RS.$$

Then,

$$\mathrm{Pow}_C(R) = RN^2 = RS \cdot RP = RM^2,$$

so it follows that $RM = RN$ so R is the center of (MQN) and it follows that $RQ = RM$, as desired. Therefore,

$$\angle QPM = \angle QPR + \angle RPM = \angle KML + \angle KML = 2\angle KML,$$

as desired. $\square$