EE222, Lecture Notes

Nonlinear Systems

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Contents

§1 Lecture 1: 1/18/2022

To dive into analysis, we first discuss how nonlinear systems differ from linear systems.

§1.1 Multiple Equilibria

Write $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $x = \theta_n = 0 \in \mathbb{R}^n$ is the only equilibrium if A is nonsingular. However, if A is singular, (A) are all equilibrium points.

Example 1.1 (Euler's Buckling Beam)

This happens when we take a beam and apply a stress to each side. This is governed by the equation

$$m\ddot{x} + d\dot{x} - \mu x + \lambda x + x^3 = 0.$$

We can equivalently solve the system

$$\dot{x} = y,$$

$$\dot{y} = -\frac{d}{m}y + \frac{\mu - \lambda}{m}x - \frac{x^3}{m}.$$

If $\mu > \lambda$, we have $y = 0$, $x = \sqrt{\mu - \lambda}, x = -\sqrt{\mu - \lambda}, x = 0$ as equilibrium points. If $\mu \leq \lambda$, we only have $(0, 0)$.

Example 1.2 (Nonlinear Damping)

When we bow a violin string, we can think of it as a mass on a spring sitting on a conveyor belt moving at a constant velocity. The nonlinearity comes from the friction which is a function of $F(\dot{x} - b)$. We have sticky friction:

$$m\ddot{x} + kx + F(\dot{x} - b) = 0.$$

§1.2 Qualitative Analysis: Equilibrium Points

Take $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, x_0 and equilibrium point of $f(x_0) = 0$, $x(0) = x_0, x(t) = x_0$ for all $t \geq 0$.

Recall the Jacobian

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

We take $A = Df(x_0)$. Consider the equation $\dot{z} = Az$, with $z \in \mathbb{R}^2$, $A = Df(x_0)$.

Theorem 1.3 (Hartmann Grobman Theorem)

Under certain conditions, if no eigenvalues of A are imaginary or zero, then the flow of $\dot{x} = f(x)$ can be mapped to the flow of $\dot{z} = Az$.