Putnam Solutions

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I present some solutions from various Putnam Exams. Problems are not necessarily posted in chronological order. Any typos or mistakes found are mine - kindly direct them to my inbox.

Contents

§1 Putnam - 2001

§1.1 A1 - Algebra

Problem 1 (2001-A1)

Consider a set S and a binary operation $*$. Assume $(a*b)*a = b$ for all $a, b \in S$. Prove that $a*(b*a) = b$ for all $a, b \in S$.

Proof. Note that

$$b = ((b*a)*b)*(b*a) = a*(b*a).$$

□

§1.2 A2 - Combinatorics

Problem 2 (2001-A2)

You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that when tossed, is has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd?

Proof. We claim the probability is $P(n) = \boxed{\dfrac{n}{2n+1}}$. We prove it by induction. We are given that $P(1) = \frac{1}{3}$, which satisfies the claim. Suppose $P(k) = \frac{k}{2k+1}$ for $k \geq 1$. In order to find $P(k+1)$, we condition on the result of the first k coin tosses. Namely, suppose the number of heads is even after k tosses. Then, the total number of heads is odd if we flip a head on the $k+1$-th toss. Similarly, if the number of heads is odd after k tosses, then the total number of heads is odd if we flip a tail on the $k+1$-th toss.

Putting this together gives

$$\begin{aligned}
P(k+1) &= (1-P(k))p_{k+1} + P(k)(1-p_{k+1}) \\
&= P(k)\left(1-2p_{k+1}\right) + p_{k+1} \\
&= P(k)\left(1 - \frac{2}{2k+3}\right) + \frac{1}{2k+3} \\
&= P(k)\frac{2k+1}{2k+3} + \frac{1}{2k+3} \\
&= \frac{k}{2k+1}\frac{2k+1}{2k+3} + \frac{1}{2k+3} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

which proves the result. □

§1.3 A3 - Algebra

Problem 3 (2001 - A3)

For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Proof. We claim that m is the square of an integer or twice the square of an integer. Set $y = x^2$. We look for square-integer solutions for y. From the quadratic formula,

$$\begin{aligned} y &= \frac{2m+4 \pm \sqrt{(2m+4) - 4(m-2)^2}}{2} \\ &= m+2 \pm \sqrt{(m+2)^2 - (m-2)^2} \\ &= m+2 \pm \sqrt{4(2m)} \\ &= m+2 \pm 2\sqrt{2m} \\ &= (\sqrt{m} \pm \sqrt{2})^2. \end{aligned}$$

Hence, $x = \pm\sqrt{m} \pm \sqrt{2}$. Note that if m is neither the square of an integer nor twice the square of an integer then the field $\mathbb{Q}(\sqrt{m}, \sqrt{2})$ is of degree 4 and the Galois group acts transitively on the roots $\{\pm\sqrt{m} \pm \sqrt{2}\}$. It follows that the polynomial is irreducible.

It is easy to verify that if m is a square or twice a square, then $P_m(x)$ reduces into the product of non-constant integer polynomials.

□

§1.4 A4 - Geometry

Problem 4 (2001 - A4)

Triangle ABC has area 1. Points E, F, G lie on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

Proof. We claim that $[RST] = \frac{7-\sqrt{5}}{4}$. Let $EC/BC = r$, $FA/CA = s$, $GB/AB = t$.

Note that $[ABE] = [AFE]$ since they share a base AE and $BR = FR$ implies that the share the same altitude length as well(drop altitudes from F and B and use the congruent triangles).

Then, $[ABE] = [ABE]/[ABC] = BE/BC = 1 - EC/BC = 1 - r$. We also have $[ACE] = r$. It follows that $[FCE] = [ACE](FC/AC) = r(1-s)$.

Now,

$$1 = [ABC] = [ABE] + [AFE] + [EFC] = (1-r) + (1-r) + r(1-s) \implies r(1+s) = 1.$$

Arguing similarly for the other sides, we have $s(1+t) = 1$, and $t(1+r) = 1$.

It follows that

$$r = \frac{1}{1+s} = \frac{1}{1+\frac{1}{1+t}} = \frac{1}{1+\frac{1}{1+\frac{1}{r}}}.$$

Simplifying this, we find that $r = \frac{2+r}{3+2r}$, which gives $3r + 2r^2 = 2 + r$, or equivalently, $r^2 + r - 1 = 0$. Plugging into the quadratic formula and taking the positive root gives

$$r = \frac{1+\sqrt{5}}{2},$$

and by repeating the argument, we have $r = s = t = \frac{-1+\sqrt{5}}{2}$.

Now, note that $[ATC] = [AEC]/2 = r/2$, $[ATG] = [ACG] - [ATC] = 1 - t - r/2$. Similarly, $[BSC] = t/2$ and $[BRE] = 1 - r - s/2$, so it follows that $[BRTG] = [ABE] - [ATG] - [BRE] = r/2 + s/2 + t - 1$.

$$\begin{aligned} [RST] &= [ABC] - [ACG] - [BSC] - [BRTG] \\ &= 1 - (1-t) - (t/2) - (r/2 + s/2 + t - 1) \\ &= 1 - \frac{r+s+t}{2} \\ &= 1 - \frac{3\frac{\sqrt{5}-1}{2}}{2} \\ &= \frac{7-\sqrt{5}}{4}. \end{aligned}$$

$\square$

§1.5 A5 - Number Theory

Problem 5 (2001 - A5)

§1.6 A6 - Calculus

Problem 6 (2001 - A6)

§2 Putnam - 2019

§2.1 A1 - Number Theory

Problem 7 (2019 - A1)

Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC,$$

where A, B, C are nonnegative integers.

Proof. Let $S = A^3 + B^3 + C^3 - 3ABC$. We claim that S attains all values such that $S \not\equiv 3, 6 \pmod 9$.

Note that the expression can be factored as

$$A^3 + B^3 + C^3 - 3ABC = \left(\frac{A+B+C}{2}\right)\left((A-B)^2 + (B-C)^2 + (C-A)^2\right).$$

If $(A, B, C) = (A, A+1, A+2)$, then

$$S = \frac{3A+3}{2}(1^2 + 1^2 + 2^2) = (3A+3)(3) = 9A + 9,$$

so we can achieve all $S \equiv 0 \pmod 9$.

If $(A, B, C) = (A, A, A+1)$, then

$$S = \frac{3A+1}{2}(0^2 + 1^2 + 1^2) = 3A + 1,$$

and if $(A, B, C) = (C+1, C+1, C)$, then

$$S = \frac{3C+2}{2}(0^2 + 1^2 + 1^2) = 3C + 2,$$

so we can achieve all $S \equiv 1, 2 \pmod 3$.

It suffices to show that if $S \equiv 0 \pmod 3$, then $S \equiv 0 \pmod 9$. This implies that we cannot have $S \not\equiv 3, 6 \pmod 9$ as desired. If $S \equiv 0 \pmod 3$, then we must have $A+B+C \equiv 0 \pmod 3$ or $(A-B)^2 + (B-C)^2 + (C-A)^2 \equiv 0 \pmod 3$. In the first case, then without loss of generality, we must have either $(A, B, C) \in \{(0,0,0), (1,1,1), (2,2,2), (0,1,2)\}$. In each of these cases, we can show that $(A-B)^2 + (B-C)^2 + (C-A)^2 \equiv 0 \pmod 3$. Similarly, in the second case, we must have that $(A-B)^2 = (B-C)^2 = (C-A)^2 = 0, 1$. In the first case $A = B = C$, which gives that $A + B + C \equiv 0 \pmod 3$. In the second case, the remainders of A, B, C must be distinct mod 3, which, without loss of generality, gives $(A, B, C) = (0, 1, 2)$ which implies that $A + B + C \equiv 0 \pmod 3$, as desired. In all cases, we show that both terms in the product are $0 \pmod 3$, which implies that the product is $0 \pmod 9$. □

§2.2 A2 - Geometry

Problem 8 (2019 - A2)

In the triangle ABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\arctan(1/3)$. Find α.

Proof. We use complex numbers. Let $B = 0$. Then $\arg(I) = \beta/2 = \arctan(1/3)$, so $I = k(3+i)$ for some $k \in \mathbb{R}^+$. Without loss of generality, let $k = 1$. Let $A = a$. Then, IG is parallel to AB which implies that $\text{Im}(B - A) = \text{Im}(I - G)$. Then $\text{Im}(B - A) = 0$, so $\text{Im}(I) = \text{Im}(G) = 1$.

Then, note that $\arg(I^2) = \arg(C)$, so $C = \ell(3+i)^2 = \ell(8+6i)$ for some $\ell \in \mathbb{R}^+$. Then $G = \frac{A+B+C}{3} = \frac{A+C}{3}$, so

$$1 = \text{Im}(G) = \text{Im}((A+C)/3) = \text{Im}(C/3),$$

which implies that $\ell = \frac{1}{2}$. Thus, $C = 4 + 3i$.

Finally,

$$I = \frac{|CB|A + |AC|B + |AB|C}{|AB| + |BC| + |CA|} = \frac{5a + a(4+3i)}{5 + a + \sqrt{(4-a)^2 + 9}} = 3 + i.$$

Hence,

$$5 + a + \sqrt{(4-a)^2 + 9} = 3a,$$

which has solutions $a = 0, a = 4$. Taking the positive solution, we have $A = 4$. Then, note that ABC is a right triangle with right angle at A, so $\alpha = \frac{\pi}{2}$. $\square$

Problem 9 (2019 - A3)

Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Let $\mu = \frac{1}{2019} \sum_{k=1}^{2019} |z_k|$. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ satisfying

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

Proof. By the AM-GM inequality,

$$\mu = \frac{\sum_{k=1}^{2019} |z_k|}{2019} = \left(\prod_{k=1}^{2019} |z_k|\right)^{1/2019} = \left|\frac{b_0}{b_{2019}}\right|^{1/2019} \leq (2019)^{-1/2019}.$$

We show that $M = (2019)^{-1/2019}$. Let $\zeta = e^{\frac{2\pi i}{2020}}$ and let $z_i = M\zeta^i$. Notice that $|z_i| = M$ for each i and the roots $z_1, z_2, \ldots, z_{2019}$ satisfy the polynomial

$$0 = \frac{(z_i/M)^{2020} - 1}{(z_i/M) - 1} = M^{-2019}\left(\frac{z_i^{2020} - M^{2020}}{z_i - M}\right) = \sum_{k=0}^{2019} z_i^k M^{-k}.$$

Hence, the polynomial

$$P(z) = \sum_{k=1}^{2019} z_i^k 2019^{k/2019}$$

satisfies the equality case $\mu = M$. Furthermore, note that $b_0 = 1$, $b_{2019} = 2019$ and and $2019^{i/2019} < 2019^{j/2019}$ for all $i < j$. Hence, P satisfies the conditions. $\square$