Math 222b Lecture Notes
Partial Differential Equations II

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§1 January 19th, 2021

§1.1 Review of Sobolev Spaces

Definition 1.1. Given $u \in \mathcal{D}'(U)$ for $U \subseteq \mathbb{R}^n$ open: that means that $u : C_c^\infty(U) \to C$ and for every compact set $K \subset\subset U$, $\exists C, N$ for all $\varphi \in C_0^\infty(K)$ such that

$$|u(\varphi)| \le C \sup_{|\alpha| \le N} |\partial^\alpha \varphi|.$$

Examples:

- Take $U = (0, 1)$ and take $u = \sum_{\mathbb{N}} \delta_{1/n}$, where $\delta_{1/n}(\varphi) = \varphi(1/n)$.
- Take $u \in L^1_{\text{loc}}(U)$, where $u(\varphi) = \int u\varphi$. Differentiation is defined formally though integration by parts as $\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$.

Definition 1.2. The Sobolev spaces $W^{k,p}(U) = \{u \in L^1_{loc}(U) : \partial^\alpha u \in L^p(U), \forall |\alpha| \le k\}$, for $k \in \mathbb{N}_0$, $1 \le p \le \infty$. Note that differentiation is in the sense of distributions. We write $H^k(U) = W^{k,2}(U)$, which are Hilbert spaces with the inner product

$$\langle u, v \rangle = \sum_{|\alpha| \le k} \int_U \partial^\alpha u \overline{\partial^\alpha v}.$$

Definition 1.3. $W_0^{k,p}(U) = \overline{C_c^\infty(U)}$, where the closure is with respect to the $W^{k,p}$ norm.

Theorem 1 (Approximation)

For $U \subset\subset \mathbb{R}^n$,

$$\overline{C^\infty(U) \cap W^{k,p}(U)} = W^{k,p}(U)$$

where the closure is with respect to the $W^{k,p}$.

If $\partial U \in C^1$, then we can improve up to

$$\overline{C^\infty(\overline{U}) \cap W^{k,p}(U)} = W^{k,p}(U)$$

Theorem 2 (Extension)

If $U \subset\subset \mathbb{R}^n$ and $\partial U \in C^1$, for $U \subset\subset V \subset\subset \mathbb{R}^n$, there exists $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu|_U = u$ and the supp $u \subset\subset V$.

We can extend this to $W^{k,p}$ if the boundary is C^k.

Theorem 3 (Traces)

For $U \subset\subset \mathbb{R}^n$ with $\partial U \in C^1$, there exists $T : W^{1,p}(U) \to L^p(\partial U)$ which is linear and boundary such that for $u \in C(\overline{U}) \cap W^{1,p}$ $Tu = u|_{\partial U}$.

Example 1.4

For $U \subset\subset \mathbb{R}^n$, ∂U bounded,

$$H_0^1(U) = \{u \in H^1 : Tu = 0 \in L^2(\partial U)\}.$$

The converse of showing $Tu = 0$ implies H_0^1 is the more difficult one.

§1.2 Fourier Transform

We first review the Fourier Transform. We define the Schwartz space:

$$\mathcal{S} = \{\varphi \in C^\infty(\mathbb{R}^n) : x^\alpha \partial^\beta \varphi \in L^\infty \forall \alpha, \beta \in \mathbb{N}^n\}.$$

For $\varphi \in \mathcal{S}$, we define

$$\widehat{\varphi}(\xi) = \int \varphi(x) e^{-ix\cdot\xi}\, dx.$$

Note that $\mathcal{F}$, the Fourier transform is invertible on §. The key properties of the fourier transform are

$$\mathcal{F}(1/i\partial x\varphi) = \xi\mathcal{F}\varphi, F(x\varphi) = -1/i\partial_\xi \mathcal{F}\varphi.$$

We also have

$$\mathcal{F}^{-1} = \frac{R\mathcal{F}}{(2\pi)^n}, R\varphi(x) = \varphi(-x).$$

We define $\mathcal{S}'$ onto $\mathbb{C}$ so that for $u \in \mathcal{S}'$, there exists C, N such that

$$|u(\varphi)| \le C \sup_{|\alpha|,|\beta|\le N} |x^\alpha \partial^\beta \varphi|.$$

Note that $\mathcal{S}' \subset \mathcal{D}'$.

Definition 1.5. $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ by $\widehat{u}(\varphi) = u(\widehat{\varphi})$.

Examples:

- $\widehat{\delta_0}(\varphi) = \delta_0(\widehat{\varphi}) = \widehat{\varphi}(0) = \int \varphi = 1(\varphi)$.
- Take $\mathbb{R}^2$ and consider $u(x) = \frac{1}{|x|}$. This function is in L^1_{loc}. If we multiply by $(1+|x|)^{-2}u \in L^1(\mathbb{R}^n)$, it follows that $u \in \mathcal{S}'$, since

  $$|u(\varphi)| = \left|\int (1+|x|)^{-2}u(1+|u|)^2\varphi\right| \le C\sup(1+|x|)^2\varphi.$$

  Now, we compute $\widehat{u} \in \mathcal{S}'$. Since $\mathcal{F}$ is continuous on $\mathcal{S}'$, we approximate u and hope the result converges to the desired result. Define $u_\epsilon \to u$ in $\mathcal{S}'$ for $u_\epsilon \in L^1$.

  Try $u_\epsilon(x) = \frac{e^{-\epsilon|x|^2/2}}{|x|} \in L^1$ for $\epsilon > 0$. We want to calculate $\widehat{u}_\epsilon$ and take the limit as $\epsilon \to 0^+$. We can evaluate the integral by converting to polar coordinates and completing the square. Unfortunately, it reduces to an integral that is too hard, but we will learn asymptotics of the integral as $\epsilon \to 0$. We find that $\widehat{u}(\xi) = 2\pi/|\xi|$.

  We can approach this differently. Note that $u = 1/|x|$ is homogeneous: $u(tx) = t^a u(x)$ for $t > 0$, for functions. For distributions, we have that for $\varphi \in S$, $u(\varphi(\cdot/t)t^{-n}) = t^a u(\varphi)$ for $t > 0$. For the Fourier Transform, if $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree a, then $\widehat{u}$ is homogeneous of degree $-n-a$. It follows that our Fourier transform is of degree -1.

  Furthermore, note that $1/|x|$ is spherically symmetric, and the Fourier transform preserves spherical symmetry(note that the Jacobian factor for rotations is 1). It follows that the fourier transform is also spherically symmetric. It follows that

  $$\mathcal{F}(1/|x|) = C/|\xi| + \sum_{|\alpha \le N|} c_\alpha \delta_0^{(\alpha)},$$

  but delta terms have too much homogeneity.

§2 December 21st, 2021

§2.1 Plancherel's Theorem

Recall that the Fourier transform is an isomorphism on $\mathcal{S}$ - it is a bounded linear operator whose inverse is also bounded.

Note that
$$\int \widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi = \iiint u(x)\overline{\varphi(y)}e^{-i(x-y)\xi}\,dxdyd\xi$$

In the sense of distributions, $\int e^{-i(x-y)\xi}\,d\xi = (2\pi)^n\delta(x-y)$. Hence,

$$\iiint u(x)\overline{\varphi(y)}e^{-i(x-y)\xi}\,dxdyd\xi = (2\pi)^n\int u(x)\overline{\varphi(x)}\,dx.$$

For $u,\varphi\in\mathcal{S}$, we have the following:

$$\langle\widehat{u},\widehat{\varphi}\rangle = (2\pi)^n\langle u,\varphi\rangle.$$

This implies that
$$\|\widehat{u}\|_2 = (2\pi)^{n/2}\|u\|_2, u\in\S.$$

If $u_n\to u$ in L^2 then $u_n\to u$ in $\mathcal{S}'$ by the Cauchy-Schwartz inequality. It follows that $\widehat{u_n}\to\widehat{u}$ in $\mathcal{S}'$ but our formula shows that $\widehat{u}$ is in L^2. Hence, $\mathcal{F}:L^2\to L^2$ and for $u,v\in L^2$, $\langle\widehat{u},\widehat{v}\rangle = (2\pi)^n\langle u,v\rangle$.

Recall last time, we were finding the Fourier transform of $u(x) = 1/|x|$ in $\mathbb{R}^2$. For $u\in S'(\mathbb{R}^n)$ homogeneous of degree a, $\widehat{u}\in\mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree $-n-a$. In our example, It follows that $\widehat{u}(\xi)$ is homogeneous of degree -1. We also observed that u is invariant under rotations so it follows that $\widehat{u}$ is invariant under rotations.

A function is homogeneous of degree -1 if $v(k\theta) = \frac{a(\theta)}{r}$. Since our function is invariant under rotations, $\widehat{u}(\xi) = \frac{c}{|\xi|}$ away from zero. It follows from our previous argument that $\widehat{u}(\xi) = \frac{c}{|\xi|}$ since δ terms have homogeneity of at least -2.

Note that $\langle u,\varphi\rangle = (2\pi)^2\langle\widehat{u},\widehat{\varphi}\rangle$ and we find $\widehat{u}$ by choosing an appropriate φ.

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{\varphi(x)}{|x|}\,dx &= \int_0^{2\pi}\int_0^\infty\varphi(r)\,drd\theta\\ &= 2\pi\int_0^\infty\varphi(r)\,dr.\end{aligned}$$

Choosing $\varphi(r) = e^{-r^2/2}$, we find that the integral is $(2\pi)^{3/2}$.

Evaluating the other side,

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^2}e^{-|x|^2/2-ix\cdot\xi}\,dx = \int e^{-\frac{1}{2}(x+i\xi)^2-\frac{1}{2}|\xi|^2} = 2\pi\int e^{-|\xi|^2/2} = (2\pi)^{5/2}.$$

It follows that $c = 2\pi$.

§2.2 Fourier Characterization of H^k spaces

Theorem 4

$H^k(\mathbb{R}^n) = \{u\in S'(\mathbb{R}^n) : (1+|\xi|^2)^{k/2}\widehat{u}\in L^2\}$.

Proof. Suppose that $\partial^\alpha u \in L^2$ for $|\alpha| \le k$. We know that $\|u\|_2 = (2\pi)^{-n/2}\|\widehat{u}\|$. It follows that $\widehat{\partial^\alpha u} \in L^2$. Note that $\widehat{\partial^\alpha u} = i^{|\alpha|}$
$xi^\alpha \widehat{u} \in L^2$ for all $|\alpha| \le k$.

Hence,

$$(1+|\xi|^2)^{k/2} \le C_{n,k} \sup_{|\alpha|\le k} |\xi^\alpha|.$$

So it follows that $(1+|\xi|^2)^{k/2}\widehat{u} \in L^2$.

Now, suppose $(1+|\xi|^2)^{k/2}\widehat{u} \in L^2$. It follows that $|\xi^\alpha| \le C_{k,\alpha}(1+|\xi|^2)^{k/2}$ for $|\alpha| \le k$. Hence $\xi^\alpha \widehat{u} \in L^2$ so it follows that $\partial^\alpha u \in L^2$ by Plancherel's Theorem. □

Remark 2.1. We use the notation $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$.

Note that the definition does not require $k \in \mathbb{N}$.

Definition 2.2. $H^s(\mathbb{R}^n) = \{u \in \mathcal{S}' : \langle \xi \rangle^s \widehat{u} \in L^2\}, s \in \mathbb{R}$.

Theorem 5

Suppose $u \in H^s(\mathbb{R}^n)$ and $s > \frac{1}{2}$. Then $v(y) = u(0,y), y \in \mathbb{R}^{n-1}$ satisfies $v \in H^{s-1/2}(\mathbb{R}^{n-1})$.

Remark 2.3. We should define $Tu(y) = u(0,y)$ for $u \in \mathcal{S}$. Then $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ if $s > 1/2$.

Proof. Take $u \in \mathcal{S}$. We wish to show that $\|v\|_{H^{s-1/2}(\mathbb{R}^{n-1})} \le C\|u\|_{H^s(\mathbb{R}^n)}$.

Note that

$$\widehat{v}(\eta) = \int_{\mathbb{R}^{n-1}} u(0,y)e^{-y\cdot\eta}\, dy$$

and by the Fourier Inversion Formula

$$u(0,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{u}(\xi_1,\xi')e^{iy\cdot\xi'}\, d\xi_1 d\xi',$$

so it follows that

$$\begin{aligned}
\widehat{v}(\eta) &= (2\pi)^{-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n} \widehat{u}(\xi_1,\xi')e^{-iy\cdot(\eta-\xi')}\, d\xi dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}} \widehat{u}(\xi_1,\xi')e^{iy\cdot(\xi'-\eta)}\, dy d\xi \\
&= (2\pi)^{-1}\int_{\mathbb{R}^n} \widehat{u}(\xi_1,\xi')\delta_{\xi'=\eta} d\xi \\
&= (2\pi)^{-1}\int_{\mathbb{R}} \widehat{u}(\xi_1,\eta) d\xi_1.
\end{aligned}$$

Note that up to constants

$$\|v\|^2_{H^{s-1/2}} = \int_{\mathbb{R}^{n-1}} \langle \eta \rangle^{2s-1}|\widehat{v}(\xi)|^2\, d\eta = \int_{\mathbb{R}^{n-1}} \langle \eta \rangle^{2s-1}\left|\int \widehat{u}(\xi_1,\eta)\, d\xi_1\right|^2 d\eta.$$

Then,

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \langle \eta \rangle^{2s-1} \left| \int \widehat{u}(\xi_1, \eta)\, d\xi_1 \right|^2 d\eta \\
&\quad = \int \langle \eta \rangle^{2s-1} \left| \int \widehat{u}(\xi, \eta)(1 + |\xi_1|^2 + |\eta|^2)^{s/2}(1 + |\xi_1|^2 + |\eta|^2)^{-s/2} d\xi_1 \right|^2 d\eta \\
&\quad \leq \int \langle \eta \rangle^{2s-1} \int |\widehat{u}(\xi_1, \eta)|^2 (1 + |\xi_1|^2 + |\eta|^2)^s d\xi_1 \int (1 + |\xi_1|^2 + |\eta|^2)^{-s} d\xi_1\, d\eta \\
&\quad \leq \int \langle \eta \rangle^{2s-1} \langle \eta \rangle^{-2s+1} \int |\widehat{u}(\xi_1, \eta)|^2 (1 + |\xi_1|^2 + |\eta|^2)^s d\xi_1 \int (1 + u^2)^{-s} du d\eta \\
&\quad = \int |\widehat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi = \|u\|_{H^s}^2,
\end{aligned}
$$

since

$$
\int |\widehat{u}(\xi_1, \eta)|^2 (1 + |\xi_1|^2 + |\eta|^2)^s d\xi_1 d\eta = \int |\widehat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi.
$$

□

§3 January 26th, 2021

§3.1 Sobolev Spaces, continued

Recall, we have $U \subset \mathbb{R}^n$ open. We typically assume U is bounded and $\partial U \in C^1$. For these spaces, we define

$$W^{k,p}(U) = \{u \in \mathcal{D}' : \partial^\alpha u \in L^p(U), |\alpha| \le k\}.$$

Recall the extension property: there exists a map $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu|_U = u$ and $u = 0$ for $|x| > R$ for some R with $U \subset\subset B(0, R)$.

We also consider the $H^s(\mathbb{R}^n)$, the fractional Sobolev spaces: $\{u \in \mathcal{S}'(\mathbb{R}^n) : \langle \xi \rangle^s \widehat{u} \in L^2\}$. This is a Hilbert space with the norm

$$\|u\|_{H^s}^2 = \int \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \, d\xi.$$

Last time, we showed that If we have $u \in H^s(\mathbb{R}^n)$ and $s > 1/2$, then $v(y) : u(0, y)$, $y \in \mathbb{R}^{n-1}$ satisfies $v \in H^{s-1/2}(\mathbb{R}^{n-1})$. Today, we will show that $H^s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ if $s > n/2$, where C_0 denotes continuous functions vanishing at infinity. This means that there exists $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ such that for $u \in \mathcal{S}$, $Tu(y) = u(0, y)$.

Theorem 6

$H^s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ if $s > n/2$.

Proof. We first prove that if $\langle \xi \rangle^s \widehat{u} \in L^2, s > n/2$ then $\widehat{u} \in L^1(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |\widehat{u}| d\xi = \int_{\mathbb{R}^n} \langle \xi \rangle^{-s} \langle \xi \rangle^2 |\widehat{u}| d\xi \le \|\langle \xi \rangle^{-s}\|_2 \|u\|_{H^s}.$$

The first term is finish precisely when $s > n/2$ [exercise: convert to polar coordinates]. This implies that $u \in L^\infty(\mathbb{R}^n)$, following from the Fourier Inversion formula.

We know that $x \mapsto \widehat{u}(\xi)e^{ix\xi}$ is continuous so it follows that $x \mapsto u(x)$ is continuous by the dominated convergence theorem. Finally $u(x) \to 0$ as $|x| \to \infty$ by the Riemann-Lebesgue lemma: if $\widehat{u} \in L^1(\mathbb{R}^n)$, then $u(x) \to 0$ as $|x| \to \infty$.

Proof. Recall $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense. Taking $v \in L^1$, taking $v_R = v(x)1_{B(0,R)}(x)$. Then $v_R \to v$ y the dominated convergence theorem. Now take $\varphi \in C_c^\infty$ with $\varphi \ge 0$, $\int \varphi = 1$ wth $\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(x/\epsilon)$. Taking $v_{R,\epsilon} = v_R * \varphi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ and $v_R * \varphi_\epsilon \to v_R$ in L^1 as $\epsilon \to 0$.

Hence, we can take $v \in \mathcal{S}$ so that $\|\widehat{v} - \widehat{u}\|_{L^1} < \epsilon/2$. Now, $|v(x)| < \epsilon/2$ if $|x| > R$, hence

$$|u(x)| \le |u(x) - v(x)| + |v(x)| < C\epsilon + \epsilon/2$$

which goes to 0 as we send $\epsilon \to 0$. □

□

§3.2 Gagliardo-Nirenberg-Sobolev(GNS) Inequalities

Theorem 3.1

If $1 \leq p < n$ and we define $p^* = \frac{np}{n-p}$, then there exists $C = C(p, n)$ so that for all $u \in C_c^\infty(\mathbb{R}^n)$,

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_p.$$

Remark 3.2. We can find the value of p^* without doing the computation through scaling.Take $u_\lambda(x) = u(\lambda x)$. We have that $\|u_\lambda\|_{p^*} \leq C\|\nabla(u_\lambda)\|_p$. Then, evaluate both sides and compare the exponent on λ.

Note that the result is not true for $p = n > 1$. It is true for $p = n = 1$.

Theorem 3.3 (Morrey's Inequality)

For $n < p \leq \infty$, there exists $C = C(p, n)$ such that for $u \in C^1(\mathbb{R}^n)$, we have

$$\|u\|_{C^\gamma(\mathbb{R}^n)} \leq C(\|u\|_p + \|\nabla u\|_p),$$

where $\gamma = 1 - \frac{n}{p}$, where

$$\|u\|_{C^\gamma(\mathbb{R}^n)} = \sup |u| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

Theorem 7 (General Formulation)

Take $U \subset\subset \mathbb{R}^n$ with $\partial U \in C^1$. Take $n \in W^{k,p}(U)$.

- if $k < n/p$, then $u \in L^q(U)$ where $1/q \geq 1/p - k/n$ and $\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}}$.
- $k > n/p$, then $u \in C^{k-[n/p]-1,\gamma}(\overline{U})$ where $\gamma = [n/p] + 1 - n/p$ if $n/p \notin \mathbb{N}$ and $1 - \delta$ for all $\delta > 0$ if $n/p \in \mathbb{N}$.

§3.3 Compactness

Definition 3.4. Let B be a Banach space. A subset $K \subset B$ is compact if for every sequence $\{u_n\} \subset K$ such that $\|u_n\|_B \leq C$, there exists a convergence subsequence $u_{n_k} \to u \in B$.

Remark 3.5. If $\{u : \|u\|_B \leq 1\} \subset B$ is compact, then B is finite dimensional. We can have a space $B' \subset B$ and $\{u \in B' : \|u\|_{B'} \leq 1\}$ compact in B. If we have a sequence $\{u_n\} \subset B'$ and $\|u_n\|_{B'} \leq C$ then there exists n_k, $u \in B$ such that $\|u_{n_k} - u\|_B \to 0$.

We will take $B = L^q(U)$ where $1 \leq q < p^*$ and $B' = W^{1,p}(U)$.

Theorem 8 (Rellich-Kondrachov)

The unit ball in $W^{1,p}(U)$ is compact in $L^q(U)$.