

Math 218a Lecture Notes, Fall 2020

Probability Theory

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§1 August 27th, 2020

§1.1 Introduction

Consider a **random experiment** - this involves a state space Ω and some "probability" on it. The outcome of an experiment would be $\omega \in \Omega$.

Example 1.1 (Fair Coin Toss)

$\Omega = \{0, 1\}$, $P(0) = 1/2$, $P(1) = 1/2$ models a fair coin toss. The outcomes are $\omega \in \Omega$, $\omega = 0$ or $\omega = 1$.

Example 1.2 (Continuous State Space)

$\Omega = [0, 1]$, X is the outcome of a random experiment. Suppose X is uniformly distributed random variable. $P(X \in [0, \frac{1}{2}]) = 1/2$. Take $A = \mathbb{Q} \cap [0, 1]$. $P(x \in A) = 0$, since A has no "volume". Similarly, taking $A_1 = \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$, then $P(x \in A_1) = 1 - P(x \in A) = 1$. Finally, take $E \subset [0, 1]$. $P(x \in E)$ = "volume" of E .

The issue: we need to define some notion of volume. Some properties we would like are the following:

- Translation Invariance
- Countable Additivity: A_1, A_2, \dots disjoint with $A = \bigcup A_i$, then $P(A) = \sum_{i=1}^{\infty} P(A_i)$.

§1.2 Nonmeasurable Sets

Take $I = [-1, 2]$, and define $x \sim y$ iff $x - y \in \mathbb{Q}$. [Exercise: check that \sim is an equivalence relation.] This decomposes I into equivalence classes I/\sim . Note that the equivalence classes are countable, since any class is $x + A$, $A \subset \mathbb{Q}$.

For each equivalence class B , pick $x_B \in B \cap [0, 1]$. Define $E = \{x_B\}$ over all the equivalence classes. Note that x_B is a representative of B in E , so $B = \{x_b + q : x_b + q \in I, q \in \mathbb{Q}\}$.

Now, consider the set $[0, 1] \subset \bigcup_{q \in [-1, 1]} E + q \subset [-1, 2]$. Equality doesn't hold, because there can be B s. t. x_b is close to 0. Then $E + (\mathbb{Q} \cap [-1, 1])$ will only recover elements of B near 1 and will not go up to 2.

Proposition 1.3

We claim that $E + q$ are disjoint for different values of q .

Proof. Suppose $E + q_1 \cap E + q_2 \neq \emptyset$ for some q_1, q_2 . Then, there exists $x, y \in E$ such that $x + q_1 = y + q_2$. This implies that $x - y = q_2 - q_1 \in \mathbb{Q}$, so $x \sim y$, but by definition, there is exactly one member of each equivalence class in E . \square

The big question: What is $P(E)$? Suppose $P(E) > 0$. Then $\bigcup_{q \in [-1, 1]} E + q \subset [-1, 2]$ and $P(E + q_1) = P(E + q_2) = P(E)$ for all q_1, q_2 . Furthermore, by countable additivity,

$$1 \geq P\left(\bigcup_{q \in [-1, 1]} E + q\right) = \sum_{q \in [-1, 1]} P(E + q) = \infty \cdot P(E).$$

This would imply that $P(E) = 0$. However,

$$[0, 1] \subseteq \bigcup_{q \in [-1, 1]} E + q \Rightarrow P([0, 1]) = 1/3 \leq \sum_{q \in [-1, 1]} P(E + q) = 0.$$

Hence, $P(E)$ cannot be defined.

The issue is the step where we pick x_B , since we need to pick x_B from uncountably many points, which assumes the axiom of choice. It was proved by Robert M. Solovay that all models of set theory excluding the axiom of choice have the property that all sets are Lebesgue measurable.

Our goal is thus to come up with a general framework where things can be consistently defined for a large class of sets.

§1.3 Measure Theory Beginnings

For the definitions, we take Ω to be the state space.

Definition 1.4 (Sigma-Algebra). Suppose Σ follows the following properties:

1. $\emptyset \in \Sigma$
2. $A \in \Sigma \Rightarrow A^c \in \Sigma$
3. $A_1, A_2, \dots \in \Sigma$, then $\bigcup A_i \in \Sigma$

Note that 2 and 3 imply 1 since $(A \cup A^c)^c = \emptyset$. Then Σ is a sigma-algebra.

Note that we also have countable intersections (this is an easy exercise).

§2 September 1st, 2020

Last time:

- We discussed the notation of a Σ -algebra, a reasonable class of sets on which we will define measures.
- Properties: $\emptyset \in \mathcal{A}$, $A \in \mathcal{A}$, $A^c \in \mathcal{A}$, $\bigcup A_i \in \mathcal{A}$.

§2.1 Measures

We are working in a space (Ω, Σ) .

Definition 2.1 (Measure). A measure is a function $\mu : \Sigma \rightarrow [0, \infty]$ with the following properties:

- $\mu(\emptyset) = 0$
- "Countable Additivity": $\mu(\bigcup A_i) = \sum \mu(A_i)$ for disjoint $A_i \in \Sigma$.

Example 2.2

If Ω is finite, $1, 2, \dots, n$, $\Sigma = 2^\Omega$, then all possible measures on (Ω, Σ) are given by fixing $a_1, a_2, \dots, a_n \in [0, \infty]$ and $\mu(A) = \sum_{i \in A} a_i$.

Properties of measures:

- Monotonicity: $A \subset B$, then $\mu(A) \leq \mu(B)$.

Proof. $B = A \cup (B \setminus A)$ and $B \setminus A \in \Sigma$, so

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

□

- Countable Subadditivity: $A \subseteq \bigcup_{i=1}^{\infty} B_i$, then $\mu(A) \leq \sum \mu(B_i)$.

Proof. We disjointify the B_i : Define $C_1 = B_1$, $C_i = B_i \setminus B_{i-1}$. Then

$$\mu(A) \leq \mu\left(\bigcup C_i\right) = \sum \mu(C_i) \leq \sum \mu(B_i).$$

□

- Continuity from below: If $A_i \uparrow A$, then $\mu(A_i) \uparrow \mu(A)$.

Proof. $A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \dots$, so by countable additivity

$$\mu(A) = \sum_{i=1}^{\infty} \mu(C_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(C_i) = \lim_{n \rightarrow \infty} \mu(A_n).$$

□

- Continuity from above, if $A_i \downarrow A$, and $\mu(A_1) < \infty$, then $\mu(A_i) \rightarrow \mu(A)$

Proof. We need the condition $\mu(A_i) < \infty$. Take $A_i = [i, \infty)$ as a counterexample if we don't have that condition.

Define $A_1 \setminus A_i = B_i$, so $B_i \uparrow A_1 \setminus A$. Then, use the continuity from below. □

§2.2 Sigma algebras

Fact 2.3. For any $A \subset 2^\Omega$, define

$$\Sigma(A) = \bigcap_{A \in \Sigma} \Sigma.$$

Then, $\Sigma(A)$ is a sigma-algebra.

Note that $\Sigma(A)$ is the smallest sigma-algebra containing A . For this reason, we call it the sigma-algebra **generated** by A .

Example 2.4

Take $X, Y \subset 2^\Omega$. We want to prove $\Sigma(X) = \Sigma(Y)$. It suffices to show $X \subseteq \Sigma(Y)$ and $Y \subseteq \Sigma(X)$.

Definition 2.5 (Borel Sigma-Algebra). (Ω, \mathcal{U}) , a topological space with a family of open sets. The **Borel Sigma-Algebra** is $\mathcal{B} = \Sigma(\mathcal{U})$.

Example 2.6

For $\Omega = \mathbb{R}$, \mathcal{B} is the sigma algebra generated by open sets in \mathbb{R} . We also have \mathcal{B} is the sigma-algebra generated by open intervals in \mathbb{R} , which follows from the fact that any open set can be written as a countable union of open intervals. Furthermore,

$$\Sigma((a, b) : a, b \in \mathbb{Q}, \mathbb{R}) = \Sigma([a, b] : a, b \in \mathbb{Q}, \mathbb{R}),$$

since $[a, b] = \bigcap (a - 1/n, b + 1/n)$ and $(a, b) = \bigcup [a + 1/n, b - 1/n]$.

§2.3 Uniform Measure on the Borel Sets

We will attempt to define the uniform measure on Borel sets of \mathbb{R} . Broadly, we do it as follows:

1. Define it on a semi-algebra containing the intervals.
2. Extend the definition to an algebra.
3. Extend it to a sigma-algebra.

Definition 2.7 (Semi-algebra). $\Sigma \subset 2^\Omega$ is a semi-algebra if

- $A_1, A_2 \in \Sigma$ implies $A_1 \cap A_2 \in \Sigma$
- $A_1 \in \Sigma$ implies that $A_1^c = \bigcup_{i=1}^n B_i$ for $B_i \in \Sigma$.

Note: The set of intervals $\{(a, b) : a, b \in \mathbb{R}\}$ is not a semi-algebra. If $(a, b)^c = [b, \infty)$ which is not finitely coverable by disjoint open sets. Similarly, $\{[a, b] : a, b \in \mathbb{R}\}$ is not a semi-algebra.

Claim: $\Sigma = \{(a, b] : a, b \in \mathbb{R}\}$ is a semi-algebra. [This is left as an exercise].

Now, $\mu((a, b]) = b - a$. The proof that μ is countable additive on Σ . If $A = \bigcup_{i=1}^\infty B_i$, B_i disjoint, $A, B_i \in \Sigma$, then $\mu(A) = \sum_{i=1}^\infty \mu(B_i)$.

Proof. We first show that $\mu(A) \geq \sum_{i=1}^{\infty} \mu(B_i)$. This is an easy exercise, show $\mu(A) \geq \sum_{i=1}^n \mu(B_i)$, and we pass to the limit.

It suffices to show $\mu(A) \leq \sum_{i=1}^{\infty} \mu(B_i)$. We do this by exploiting compactness.

Let $A = (a, b] \supset [a + 1/n, b] = A'$, take $B_i = (c_i, d] \subset c_i, d + \frac{\epsilon}{2^i} = B'_i$. Note that

$$A' \subset \bigcup_{i=1}^{\infty} B'_i,$$

so there exists a finite subcover $A' \subset \bigcup_{j=1}^k B'_{i_j}$. It is easy to show that $b - (a + 1/n) \leq \sum_{j=1}^k (d'_{i_j} - c'_{i_j})$. But note that

$$\sum_{j=1}^k (d'_{i_j} - c'_{i_j}) \leq \sum_{j=1}^k d_{i_j} - c_{i_j} + \epsilon,$$

which implies that

$$\mu(A) - 1/n \leq \sum_{i=1}^{\infty} \mu(B_i) + \epsilon \Rightarrow \mu(A) \leq \sum_{i=1}^{\infty} \mu(B_i).$$

□

Definition 2.8. \mathcal{A} is an algebra if

- $\emptyset \in \mathcal{A}$
- $A_1 \in \mathcal{A}$ implies $A^c \in \mathcal{A}$
- $A_1, \dots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^n A_i \in \mathcal{A}$.

The algebra generated by a semi-algebra is given by taking all possible disjoint finite unions.

Claim: $\Sigma_a = \{\bigcup_{i=1}^n A_i\}$ for disjoint A_i semialgebras is an algebra.

Proof. We show $A, B \in \Sigma_a \Rightarrow A \cup B \in \Sigma_a$ and $A^c \in \Sigma_a$. Note that $A = \bigcup_{i=1}^n C_i, B = \bigcup_{j=1}^k D_j$, so

$$A \cap B = \bigcup_{i=1}^n \bigcup_{j=1}^k C_i \cap D_j,$$

and $C_i \cap D_j$ are disjoint. Then $C_i, D_j \in \Sigma$ implies $C_i \cap D_j \in \Sigma$.

Then, if $A = \bigcup_{i=1}^k C_i$, then $A^c = \bigcap_{i=1}^k C_i^c$, and $C_i^c = \bigcup_{j=1}^{\ell} E_j \in \Sigma_a$. □

We extend μ to an algebra by $\mu(A) = \sum_{i=1}^k \mu(C_i)$, where $A = \bigcup C_i$ in the semi-algebra.

§3 September 3rd, 2020

Recall that we are aiming to define the uniform measure on $(\mathbb{R}, \mathcal{B})$.

Last time:

1. We defined a **premeasure** on a semi-algebra, which $\Sigma_{semi} = \{(a, b] : -\infty \leq a \leq b \leq \infty\}$, $\mu((a, b]) = b - a$ was countably additive.
2. Extend μ to an algebra $\Sigma_a =$ disjoint union of elements of Σ_{semi} .
3. For $A = \bigcup_{i=1}^k C_i \in \Sigma_a$,

$$\mu(A) = \sum_{i=1}^k \mu(C_i).$$

§3.1 Uniform Measure on the Borel Sets

We first need show show μ is well defined. Suppose $A = \bigcup_{i=1}^l C_i, \bigcup_{j=1}^\ell B_j$ for $C_i, B_i \in \Sigma_{semi}$. We want

$$\mu(A) = \sum_{i=1}^k \mu(C_i) = \sum_{j=1}^\ell \mu(B_j).$$

Note that $C_i = \bigcup_{j=1}^\ell (C_i \cap B_j)$, which are all disjoint. Similarly, $B_j = \bigcup_{i=1}^k (B_j \cap C_i)$, disjoint. Thus, from the finite additivity of Σ_{semi} , we have

$$\sum_{i=1}^k \mu(C_i) = \sum_{i=1}^k \sum_{j=1}^\ell \mu(C_i \cap B_j) = \sum_{j=1}^\ell \mu(B_j),$$

as desired.

We will next show that μ is finitely additive. First, if we have $A_1, A_2, \dots, A_n \in \Sigma_a$ disjoint, we show $\mu(\bigcup A_i) = \sum \mu(A_i)$.

Note that each $A_i = \bigcup_{j=1}^{m_i} C_j^i$, which are disjoint, so

$$\mu\left(\bigcup A_i\right) = \mu\left(\bigcup_i \bigcup_j C_j^i\right) = \sum_{i=1}^n \sum_{j=1}^{m_i} \mu(C_j^i) = \sum_{i=1}^n \mu(A_i).$$

Next, we show μ is monotonic. For $A, B \in \Sigma_a, A \subseteq B, B = A \cup (B \setminus A)$, so

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

We show countably additivity: Let $A = \bigcup_{i=1}^\infty A_i$. We need to show $\mu(A) = \sum_{i=1}^\infty \mu(A_i)$.

We first show $\mu(A) \geq \sum_{i=1}^\infty \mu(A_i)$. It suffices to show $\mu(A) \geq \sum_{i=1}^n \mu(A_i)$. Since $\bigcup_{i=1}^n A_i \subseteq A$, monotonicity gives $\mu(\bigcup_{i=1}^n A_i) \leq \mu(A)$.

Next, we show $\mu(A) \leq \sum_{i=1}^\infty \mu(A_i)$. First $A = \bigcup_{j=1}^k C_j$ for $C_j \in \Sigma_{semi}$, $A_i = \bigcup_{\ell}^{m_i} C_\ell^i$ for $C_\ell^j \in \Sigma_{semi}$.

Thus,

$$\mu(A) = \sum_{j=1}^k \mu(C_j).$$

Hence, it suffices to show $\mu(C_j) \leq \sum_{i=1}^{\infty} \mu(C_j \cap A_i)$, since

$$\mu(A) = \sum \mu(C_j) \leq \sum_{j=1}^k \sum_{i=1}^{\infty} \mu(C_j \cap A_i) = \sum_{i=1}^{\infty} \sum_{j=1}^k \mu(C_j \cap A_i).$$

Note that $C_j = \bigcup_{i=1}^{\infty} \bigcup_{\ell}^{m_i} C_j \cap C_{\ell}^i$, and we finish by using countable additivity for Σ_{semi} .

It suffices extend to $\Sigma(\Sigma_a)$ which is the sigma-algebra generated by Σ_a .

Theorem 1 (Caratheodory's Extension Theorem)

We have the following:

- Given a countably additive measure μ on an algebra Σ_a , it can be extended to a measure on $\Sigma(\Sigma_a)$.
- If μ is σ -finite on Σ_a , the extension is unique.

A measure μ is σ -finite on Σ_a if there exists $A_1 \subseteq A_2 \subseteq \dots \in \Sigma_a$ so that $\bigcup A_i = \Omega$, $\mu(A_i) \leq \infty$ for all i .

Proof. For example, consider $\Sigma_{semi} = \{(a, b] \cap \mathbb{Q}\}$. Then $\Sigma = 2^{\mathbb{Q}}$. The cardinality of every element in Σ_{semi} is either ∞ or 0. Define $\mu(A) = \infty$ if $|A| = \infty$, else 0. One can check μ is a measure on Σ_{semi} . We can also take the counting measure ν . This agrees on Σ_{semi} , but not on Σ . We can check that ν is not sigma-finite.

We now show uniqueness, given σ -finiteness. For simplicity, assume $\mu(\Omega) < \infty$. If we have two measures μ_1, μ_2 on Σ with $\mu_1(A) = \mu_2(A)$ for all $A \in \Sigma_a$, then we show $\mu_1(B) = \mu_2(B)$ for all $B \in \Sigma$.

A general strategy to show some property is true for a sigma-algebra is to show that the sets satisfying those properties must be closed under some natural operations and that any such family of sets must contain a sigma-algebra.

Theorem 2 (π - λ)

A class of sets P is said to be a π -system if $A, B \in P$ implies $A \cap B \in P$. A class of sets G is said to be a λ -system if $\Omega \in G$, $A \subset B$, $A, B \in G$, then $B \setminus A \in G$, and $A_i \in G$, $A_i \uparrow A \rightarrow A \in G$. If P is a π system contained in G , a λ -system, then $\Sigma(P) \subset G$.

Note that a semi-algebra is a π system. It suffices to consider the set $\mathcal{C} = \{A : \mu_1(A) = \mu_2(A)\}$. We know that $\Sigma_{semi} \subset \mathcal{C}$. To show $\Sigma \subset \mathcal{C}$, it suffices to show that \mathcal{C} is a λ -system.

Note that a sigma-algebra is a λ -system, so given any π -system P , $\Sigma(P)$ is the smallest λ -system containing P .

We have already verified $\Sigma_a \subset \mathcal{C}$. Furthermore, $\Omega \in \mathcal{C}$ because Ω is an algebra. Finally, suppose we have $A \subset B$, $A, B \in \mathcal{C}$. We need $B \setminus A \in \mathcal{C}$. $\mu_1(A) = \mu_2(A)$, $\mu_1(B) = \mu_2(B)$ and $\mu(\Omega) < \infty$. Since $\mu_1(\Omega) = \mu(\Omega) = \mu_2(\Omega) < \infty$,

$$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A).$$

For $A_i \uparrow A$, by continuity from below, $\mu_1(A_i) \rightarrow \mu_1(A)$, $\mu_2(A_i) \rightarrow \mu_2(A)$, so $A \in \mathcal{C}$. \square

An easy exercise is to modify the above proof to include the sigma-finite case. The proof of the $\pi - \lambda$ theorem involves some set theory.

We'll sketch the existence proof. Suppose we have μ on Σ_a . For example, take $B \subset \mathbb{R}$. How do we define $\mu(B)$? We could try to approximate B by the union of intervals.

Define the outer measure, $\mu_*(B) = \inf \sum_{i=1}^{\infty} \mu(A_i)$ defined over covers of B . Some properties are $\mu_*(B_1) \leq \mu_*(B_2)$ if $B_1 \subseteq B_2$, $\mu_*(\emptyset) = 0$, and $\mu_*(\bigcup C_i) \leq \sum_{i=1}^{\infty} \mu_*(C_i)$.

Define $\mathcal{A} = \{A : \mu_*(E) = \mu_*(E \cap A) + \mu_*(E \cap A^C) \forall E\}$. \mathcal{A} is a sigma algebra containing Σ_a and μ_* is a measure when restricted to \mathcal{A} .