Math 219, Lecture Notes

Dynamical Systems

Professor: Maciej Zworski, Spring 2022
Scribe: Vishal Raman

Contents

§1 Lecture 1: 1/18/2022

§1.1 Introduction

We are concerned with a transformation $T : X \to X$ and we are concerned with the behavior of $T^n = T \circ \cdots \circ T$, $n \in \mathbb{N}_0$. This is the discrete setting. We could also consider $n \in \mathbb{Z}$ if T^{-1} exists, and we take $T^0 = \text{id}$ by convention.

In the continuous setting, we have a family $\varphi_t : (\mathbb{R}^+)_t \times X \to X$ and we want the property that $\varphi_{t+k} = \varphi_t \circ \varphi_s$ for $t, s \geq 0$. This gives us a semigroup structure. If φ_t is invertible, we take $\varphi_{-t} = \varphi_t^{-1}$ giving us a group. It is already clear that $\varphi_0 = \text{id}$.

Some questions we are concerned with:

- How many (if any) periodic orbits? These are points with $T^n(x_0) = x_0$. If we define $T^j(x_0) := x_j$, then $x_0 \to x_1 \to \ldots x_n$ forms a periodic orbit of order n.
- Do we have invariant sets - subsets $Y \subset X$ such that $T(Y) \subset Y$.
- Do we have invariant measures - a measure on X so that the measure is preserved under the transformation T. We will talk about this more precisely in the next section.

§1.2 Invariant Measures

We have a measure space $(X, \mathcal{M}, \mu)$, where $\mu : \mathcal{M} \to \overline{\mathbb{R}}_+$ a measure and a measurable function $T : X \to X$.

Definition 1.1 (Invariant Measure)**.** μ is an invariant measure for T if $T^{-1} : \mathcal{M} \to \mathcal{M}$ and $\mu(T^{-1}(A)) = \mu(A)$. We also say that T is measure preserving.

Why do we say $\mu(T^{-1}(A)) = \mu(A)$ and not $\mu(T(A)) = \mu(A)$?

- $T_*\mu(A) := \mu(T^{-1}(A))$ is always a measure because $T^{-1}(A \cap B) = T^{-1}(A) \cap T^{-1}(B)$. This is not true for $\mu \circ T$.
- If we have a function $f : X \to \mathbb{R}$, we can consider the pullback $T^*f(x) = f(T(x))$. We have that
$$\int f \, d(T_*\mu) = \int T^*f \, d\mu.$$
This follows by checking it on indicator functions and extending by linearity. Note that $T^*(\mathbf{1}_A)(x) = \mathbf{1}_A(T(x)) = \mathbf{1}_{T^{-1}(A)}(x)$, which gives the result.

§1.3 Ergodicity and Mixing

As before, we have $(X, \mathcal{M}, \mu), T : X \to X$ measure preserving.

Definition 1.2 (Ergodic)**.** We say that T is **ergodic** if $T^{-1}(A) = A$ implies that $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Definition 1.3 (Mixing)**.** We say that T is **mixing** if for all $A, B \subset \mathcal{M}$, $\mu(T^{-n}(A) \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$.

- When X is a measure space and T is measure preserving, we have Measurable Dynamics, or Ergodic Theory.
- When X is a metric space and T is continuous, we have Topological Dynamics.

- When X is a manifold and $T \in C^1$, $dT(x) : T_x X \to T_{T(x)} X$ continuous, we have Smooth Dynamics. We have additional structure because we can also consider differentiability properties and use the properties of the tangent space.

- When (X, ω) is a symplectic manifold, ω a nondegenerate closed 2-form. For a function $f \in C^\infty(X) \to H_f$ a vector field on x, $\omega(\cdot, H_f) = df$. We also obtain a flow $\varphi_t = \exp tH_f$, where $\varphi_t(x) = x(t)$, $\dot{x}(t) = H_f(x(t))$, $x(0) = 0$. This is Hamiltonian Dynamics.

Example 1.4 (Newton's Equations)
Taking $X = \mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, $f : \mathbb{R}^{2n} \to \mathbb{R}$. We take

$$H_f = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j} \partial_{x_j} - \frac{\partial f}{\partial x_j} \partial_{\xi_j},$$

$$\omega = \sum_{j=1}^n d\xi_j \wedge dx_j.$$

Taking $f(x, \xi) = \frac{1}{2}\xi^2 + V(x)$, $H_f = \xi \cdot \partial_x - V'(x)\partial_\xi$, $\varphi_t(x, \xi) = (x(t), \xi)(t))$. We obtain $\dot{x} = \xi, \dot{\xi} = -V'(x)$, Newton's Equations.

Example 1.5 (Invariant Measure on $\mathbb{S}^1$)
Take $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Define $T : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $T(x) = x + 2\pi\alpha \pmod{2\pi}$. The invariant measure is given by the Lebesgue Measure, which is translation invariant. We will see that this is the only invariant Borel measure(a measure generated by open sets).

T is continuous and smooth. We will find that it is ergodic when $\alpha \notin \mathbb{Q}$. In other words, there are no invariant sets that are not measure 0 or measure 2π.

§1.4 Equivalence

Definition 1.6 (Semiconjugate)**.** $S : Y \to Y$ is **semiconjugate** to $T : X \to X$ if there exists $\pi : Y \to X$ surjective such that $T \circ \pi = \pi \circ S$. We say that T is a **factor** of S. If π is bijective, then we say S is conjugate to T.

Note that if we have $S^n(x) = x$ semiconjugate to T, then $T^n(\pi(x)) = \pi(x)$.

Example 1.7
$T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto x + 2\pi\alpha \pmod{2\pi}$. Take $S : \mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$, $(x, y) \mapsto R_{2\pi\alpha}(x, y)$, the rotation by $2\pi\alpha$. The map $\pi : (x, y) \mapsto \arg(x + iy)$.

§1.5 Poincaré Recurrence Theorem

Theorem 1.8 (Poincaré Recurrence Theorem)

We have $(X, \mathcal{M}, \mu)$ a measure space, $\mu(X) < \infty$, $T : X \to X$, $T_*\mu = \mu$, $A \in \mathcal{M}$ with $\mu(A) > 0$. For almost every $x \in A$, there exists a subsequence n_j such that $T^{n_j}(x) \in A$.

The above theorem is a consequence of the following lemma:

Lemma 1.9

Suppose $\mu(A) > 0$. Then, for almost every x, there exists $n > 0$ such that $T_*^x \in A$.

Proof. Take $A, T^{-1}A, T^{-2}A, \ldots$. The claim is that there exists $j > i$ such that $\mu(T^{-i}A \cap T^{-j}A) > 0$. Suppose not. Then, the sequence $A, T^{-1}A, \ldots$ is disjoint modulo sets of measure 0. It would follow that $\mu\left(\bigcup_{j=0}^{\infty} T^{-j}A\right) = \sum \mu(T^{-j}(A)) = \sum \mu(A) = \infty$, but $\mu(X) < \infty$ contradicting monotonicity. It follows that $\mu(T^{-i}A \cap T^{-j}A) - \mu(T^{-i}(A \cap T^{-j+i}A)) = \mu(A \cap T^{i-j}A) > 0$. $\square$

Example 1.10

Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. $A : \mathbb{R}^2 \to \mathbb{R}^2$. It also takes $\mathbb{Z}^2 \to \mathbb{Z}^2$ and has an inverse taking $\mathbb{Z}^2 \to \mathbb{Z}^2$. It follows that $A : \mathbb{R}^2 \setminus \mathbb{Z}^2 \to \mathbb{R}^2 \setminus \mathbb{Z}^2$, which is a torus action. Taking μ to be the Lebesgue measure on the torus. Since $\det A = 1$, it follows that $A_*\mu = \mu$.

Consider $T : (\mathbb{T}^2)^{N^2} \to (\mathbb{T}^2)^{N^2}$, $T((x_j)) = T((Ax_j))$. We have a set of measure 1 so we can apply Poincare's Theorem.

§2 Lecture 2: 1/20/2022

§2.1 Measurable Dynamics on $\mathbb{S}^1$

Recall last time we were discussing measurable dynamics: $(X, \mathcal{M}, \mu)$, where $\mathcal{M}$ is a σ-algebra of measurable sets, $\mu : \mathcal{M} \to \overline{\mathbb{R}_+}$ is a measure, and $T : X \to X$ is measure preserversing.

Definition 2.1 (Invariant Set). *A such that $T^{-1}(A) = A$.*

Last time, we considered the example $T : \mathbb{S}^1 \to \mathbb{S}^1$ with $Tx = x + 2\pi\alpha \pmod{2\ pi\mathbb{Z}}$. We characterize the invariant sets of T. If $\alpha = \frac{p}{q}, (p, q) = 1$, then $T^q x = x + 2\pi q \frac{p}{q} = x + 2\pi p = x \pmod{2\pi\mathbb{Z}}$. We can use this to easily construct sets of positive measure by extending the points to small arcs.

Suppose we have T_α, $\alpha, \beta \in \mathbb{Q}$ with $\alpha \neq \beta$. When are T_α, T_β conjugate?

If $T_\alpha^q = \text{id}$, then $T_\beta^q = \text{id}$ and similarly if $T_\alpha^{q-1} \neq 0$, then $T_\beta^{q-1} \neq 0$ when they are conjugate. This gives the necessary condition that $\alpha = \frac{p_1}{q}, \beta = \frac{p_2}{q}$ where $(p_1; q) = (p_2; q) = 1$. However, we must have exactly, $\alpha = \beta$. We will prove this later.

Now, we consider $\alpha \notin \mathbb{Q}$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, T_α which maps $x \mapsto x + 2\pi\alpha \pmod{2\pi\mathbb{Z}}$ for $x \in \mathbb{S}^1$ is ergodic, and for every $x \in \mathbb{S}^1$, $T^n x$ is dense in $\mathbb{S}^1$.

Theorem 2.2 (Weyl, Khinchin)

Suppose $f \in C(\mathbb{S}^1)$. Then for all $x \in \mathbb{S}^1$,

$$\frac{1}{N} \sum_{j=0}^{N-1} f(T^j(x)) \xrightarrow{N\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f(y)\, dy =: \overline{f}.$$

Corollary 2.3

Every orbit $\{T^n x\}$ is dense on $\mathbb{S}^1$.

Proof. If not, then there exists K and an open set U such that $T^n x \notin U$ for $n > K$. We can take $U = (a, b)$, f to be 0 outside (a, b) and a cone from a to b.

$$\frac{1}{N} \sum_{j=0}^{K} f(T^j(x)) + \frac{1}{N} \sum_{j=K+1}^{N-1} f(T^j(x)) = \frac{1}{N} \sum_{j=0}^{K} f(T^j(x)) \to 0,$$

a contradiction. □

Now, we prove the main theorem.

Proof. We first prove the result for $f \in \mathcal{P} \subset C(S^1)$, $\mathcal{P} = \{\sum_{|j| \leq J} a_j e^{ijx} : a_j \in \mathbb{C}\}$, the trignometric polynomials. It is enough to prove it for $f(x) = e^{ijx}$, since elements of $\mathcal{P}$ are finite linear combinations of e^{ijx}. Note that the theorem statement becomes

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1} e^{ixk}e^{i2\pi\alpha jk} &= e^{ixj}\frac{1}{N}\sum_{j=0}^{N-1} e^{(2\pi\alpha ij)k} \\ &= \begin{cases} 1, & j=0 \\ \frac{1}{N}\frac{1-e^{2\pi i\alpha jN}}{1-e^{2\pi i\alpha j}}, & j\neq 0\end{cases} \\ &\to \delta_{\{j=0\}} = \overline{f}.\end{aligned}$$

Finally, $\mathcal{P}\subset C(\mathbb{S}^1)$ is dense: if $f\in C(S^1)$, there exists $p\in\mathcal{P}$ such that $\|f-p\|_{\mathbb{S}^1}<\epsilon$. This implies that

$$\begin{aligned}\left|\frac{1}{N}\sum_{j=0}^{N-1} f(T^j(x)) - \frac{1}{2\pi}\int_0^{2\pi} f(y)\,dy\right| &\le \left|\frac{1}{N}\sum_{j=0}^{N-1}(f-p)(T^j(x)) - \overline{f-p}\right| + \left|\frac{1}{N}\sum_{k=0}^{N-1} p(T^k x) - \overline{p}\right| \\ &\le 2\epsilon + \left|\frac{1}{N}\sum_{k=0}^{N-1} p(T^k x)-\overline{p}\right| \to 2\epsilon.\end{aligned}$$

□

We denote $S_N f(x) = \sum_{k=0}^N f(T^k(x))$, and we call $\frac{1}{N}S_N f(x)$ the **ergodic average**.

Theorem 2.4 ($L^2(\mathbb{S}^1)$ Ergodic Theorem)
Suppose $f\in L^2(\mathbb{S}^1)$. Then

$$\frac{1}{N}S_N f \xrightarrow{L^2} \overline{f}$$

Recall that $L^2(\mathbb{S}^1) = \{f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} : 2\pi\sum_{n\in\mathbb{Z}}|a_n|^2<\infty\}$, where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

Note that $\{\frac{1}{\sqrt{n}}e^{inx} : n\in\mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{S}^1)$.

Proof.

$$\begin{aligned}\|\frac{1}{N}S_n\sum_{n\neq 0} f_n\|_{L^2} &= \|\sum_{n\neq 0}\frac{1}{N}S_N f_n\| L^2 \\ &\le \sum_{0<|n|<K}\|1/N S_N f_n\|_{L^2} + \|\sum_{|n|\ge K} f_n\|_{L^2}\end{aligned}$$

Take $\epsilon>0$ and choose K so that the tail is bounded by ϵ. Using the result with trignometric polynomials,

$$1/NS_N f_n \to 0, |n|<K, n\neq 0, N\to\infty.$$

In other words,

$$\limsup \|\frac{1}{N}S_N\sum_{n\neq 0} f_n\| \le \sum_{0<|n|<K}\lim\|\frac{1}{N}S_N f_n\| + \epsilon \le \epsilon.$$

Finally, $\sum_{n\neq 0} f_n = f-\overline{f}$, so $\|\frac{1}{N}S_N(f-\overline{f})\|_{L^2}\to 0$ and $\frac{1}{N}S_N f\to\overline{f}$. □

Theorem 2.5 (No Invariant Sets on $\mathbb{S}^1$)
. If $A = T^{-1}(A)$, then $m(A) = 0$ or $m(A) = 2\pi$.

Proof. Suppose not: $T^{-1}(A) = A$, $0 < m(S^1 \setminus A) < 2\pi$. Take $f = \mathbf{1}_A(x) \in L^2(\mathbb{S}^1)$. □

§3 Lecture 3: 1/25/2022

§3.1 Comments about Invariant Sets

- $T : X \to X$, for $A \subset X$, $T^{-1}(A) = \{x : Tx \in A\}$.
- A is invariant under T if and only if $T^{-1}(A) = A$.
- "Everything that lands in A comes from A."
- Take $f : X \to \mathbb{C}$. $T^*f = f \circ T : X \to \mathbb{C}$ is well-defined. We also have $f_A = f|_A : A \to C$. If A is invariant, then $(T^*f)_A = T^*(f_A)$.
- If T is invertible, then this is the same as $T(A) = A$.
- Take $X = [0, \infty)$, $Tx = mx$, $0 < m < 1$. $T([0, 1)) = [0, m) \subset [0, 1]$. $T^{-1}([0, 1)) = [0, 1/m)$. An example of an invariant set is $A = \{m^k : k \in \mathbb{Z}\}$. This is not invariant when you take a one-sided set.

§3.2 Irrational Translations on $\mathbb{S}^1$, continued

We have $T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto x + 2\pi\alpha \pmod{2\pi\mathbb{Z}}$ for $\alpha \notin \mathbb{Q}$.

- For $f \in C(\mathbb{S}^1)$, $\frac{1}{N}S_N f(x) \to \overline{f} := \frac{1}{2\pi}\int_0^{2\pi} f$. This is uniform in x but not in f, which can even be seen by taking trignometric polynomials.
- Every orbit $\{T^j x\}_{j\in\mathbb{N}}$ is dense in $\mathbb{S}^1$.
- (Unique Ergodicity)The Lebesgue measure is the only Radon measure invariant under T.

  Proof. Suppose $d\mu$ is invariant. Then, $\int f(Tx)d\mu(x) = \int f d\mu$, which implies that $\frac{1}{N}\int S_N f(x)\, d\mu(x) = \int f d\mu$. But the left-hand side converges uniformly to $\frac{1}{2\pi}\int f\, dx$ which is the Lebesgue measure, so it must follows that $d\mu$ is the Lebesgue measure. □

- For every $f \in L^2$, $\frac{1}{N}S_N f \xrightarrow{L^2} \overline{f}$.
- Corollary: $T^{-1}(A) = A \implies m(A)m(S^1 \setminus A) = 0$.

  Proof. Take $f = \mathbf{1}_A(x) \in L^2(\mathbb{S}^1)$. Take $g = \mathbf{1}_{S^1\setminus A} \in L^2(\mathbb{S}^1)$. Note that $T^*\mathbf{1}_A = \mathbf{1}_{T^{-1}A} = \mathbf{1}_A$.

  $$\begin{aligned} \langle g, \frac{1}{N}S_N f\rangle_{L^2} &= \langle \mathbf{1}_{\mathbb{S}^1\setminus A}, \mathbf{1}_A\rangle = 0, \\ \langle g, \frac{1}{N}S_N f\rangle_{L^2} &\to \langle g, \overline{f}\rangle = m(A)m(\mathbb{S}^1 \setminus A)/2\pi. \end{aligned}$$

  □

§3.3 General Theory

We have $(X, \mathcal{M}, \mu)$, μ a measure, $\mathcal{M}$ a σ-algebra, $T : X \to X$, with $T^{-1} : \mathcal{M} \to \mathcal{M}$, $T_*\mu = \mu(T_*\mu(A) = \mu(T^{-1}(A)))$. We also have $\int T^*f d\mu = \int f d\mu$, $T^*f = f \circ T$.

Recall $L^2(X, d\mu) = \{f : X \to \mathbb{C} | \int |f|^2 d\mu < \infty\}$, with the inner product $\langle f, g\rangle = \int f\overline{g}d\mu(x)$. This defines a complete metric topology. From T, we obtain an operator $Uf = T^*f$.

Note that

$$\langle Uf, Uf\rangle = \int T^* f\overline{T^* f} d\mu = \int T^* |f|^2\, d\mu = \langle f, f\rangle.$$

As before, we can take $S_N f = \sum_{j=0}^{N-1} U^j f$.

Theorem 3.1 (Mean Ergodic Theorem)
$\mathcal{H}$, a Hilbert space, $U : \mathcal{H} \to \mathcal{H}$ linear, $\|Uf\| \le \|f\|$ for all $f \in \mathcal{H}$. Define $\mathrm{Inv} = \{f : f = Uf\} = \ker(I - U) \subset \mathcal{H}$, a closed subspace. Let $\mathcal{P} : H \to \mathrm{Inv}$ be the orthogonal projection($\mathcal{P}(\mathcal{H}) = \mathrm{Inv}, P^2 = P, P = P^*$). Then,

$$\|\frac{1}{N} S_N f - \mathcal{P} f\|_{L^2} \xrightarrow{N\to\infty} 0.$$

Proof. We first prove a lemma:

Lemma 3.2
For all $g \in \mathcal{H}$, $Ug = g$ if and only if $U^* g = g$.

Remark 3.3. Note that if $Uf = T^* f$, then $U^* f = T_* f$. This follows from

$$\langle f, Ug\rangle = \int_X f(x)\overline{g(y)}\, d\mu(x) = \int_X f(T^{-1}(x))\overline{g(y)} |D(T^{-1})(y)| dy = \langle U^* f, g\rangle.$$

Proof. If $Ug = g$,

$$\|U^* g - g\|^2 = \langle U^* g - g, U^* g - g\rangle = \|U^* g\|^2 + \|g\|^2 - 2\,\mathrm{Re}\langle Ug, g\rangle$$
$$\le 2\|g\|^2 - 2\,\mathrm{Re}\langle Ug, g\rangle = 2\,\mathrm{Re}\langle g - Ug, g\rangle = 0.$$

The opposite implication is given by reversing U and U^*. □

If $f \in \mathrm{Inv}$, the result is obvious, so we need to show $\frac{1}{N} S_N f \xrightarrow{L^2} 0$ for all $f \in \mathrm{Inv}^\perp$.

$$\|\frac{1}{N} S_N f\|^2 = \langle f, \frac{1}{N} S_N^* \frac{1}{N} S_N f\rangle =: \langle f, g_N\rangle.$$

It is enough to show that $g_N \to 0$ weakly in L^2. Note that $\|g_N\| \le \|f\|$ since $\|Uf\| \le \|f\|$, which implies that $\{g_N\}$ is weakly compact(has a weakly converging subsequence). If we know in some topology that $\{g_N\}$ is weakly compact, then $g_N \to 0$ weakly if every weak limit point of g_n is 0.

It is enough to show that weak limits are invariant under U(or U^*). Put $h = \frac{1}{N} S_N f$.

$$(I - U^*)\frac{1}{N} S_N^* h = \frac{1}{N}(I - U^*) \sum_{j=0}^{N-1} (U^*)^j h = \frac{1}{N}(I - U^{*N}) h,$$

and

$$\|\frac{1}{N}(I - U^{*N}) h\| \le \frac{1}{N}\|I - U^{*N}\| \|\frac{1}{N} S_N f\| \le \frac{2}{N}\|f\| \to 0.$$

It follows that $g = U^* g$, so $g = Ug$ and $g \in \mathrm{Inv}$ which implies that $g = 0$. □

An immediate consequence is the following:

Theorem 3.4 (Von Neumann, Ergodic Theorem)

$(X, \mathcal{M}, \mu)$, $T : X \to X$ measure-preserving. Then,

$$\frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j \xrightarrow{L^2(X)} \mathcal{P} f,$$

where $\mathcal{P} : L^2 \xrightarrow{\perp} \{f \in L^2 : f \circ T = f\}$.

§4 Lecture 4: 1/27/2022

§4.1 Invariant Everywhere from Invariant Almost Everywhere

Proposition 4.1

Suppose $f(x) = f(T(x))$ almost everywhere where f is measurable. Then, there exists g measurable such that $f = g$ almost everywhere and $g(x) = g(T(x))$ everywhere.

Proof. We write $g(x) = \limsup_{n\to\infty} f(T^n(x))$. This can potentially be infinite at certain points, which is not a problem. Note that $g(x) = g(T(x))$, since

$$g(T(x)) = \limsup_{n\to\infty} f(T^{n+1}(x)) = g(x).$$

Furthermore, note that $g(x) = f(x)$ if $f(T^n(x)) = f(x)$ for all $n \geq 0$, or equivalently, $f(T^{n+1}x) = f(T^n x)$ for all $n \geq 0$. Equivalently, $T^n(x) \in \{y : Tf(y) = f(y)\}$ or equivalently,

$$x \in \bigcap T^{-n}(\{y : Tf(y) = f(y)\}) =: Y.$$

Taking complements, $X \setminus Y = \bigcup T^{-n}(\{y : Tf(y) \neq f(y)\})$, but this set has measure zero. This implies that $\mu(X \setminus Y) = 0$, so the set where $g(x) = f(x)$ has full measure. □

In particular, if we take $f = \mathbf{1}_A$, $f = f \circ T$ almost everywhere is equivalent to $\mathbf{1}_A = \mathbf{1}_{T^{-1}(A)}$ almost everywhere. This is equivalent to saying that the symmetric difference has measure zero:

$$\mu((A \setminus T^{-1}(A)) \cup (T^{-1}(A) \setminus A)) = 0.$$

If we take g as constructed in the proposition, $g = \limsup f(T^n(x)) = \mathbf{1}_B$ for some B. Since $\mathbf{1}_B = \mathbf{1}_A$ almost everywhere, $\mu((B \setminus A) \cup (A \setminus B)) = 0$ and $T^{-1}(B) = B$. Hence

Note that $\sup_{k\leq n} \mathbf{1}_{B_k} = \mathbf{1}_{\bigcup_{k\leq n} B_k}$, and $\int_{k\leq n} \mathbf{1}_{B_k} = \mathbf{1}_{\bigcap_{k\leq n} B_k}$. It follows that

$$g(x) = \bigcap \bigcup T^{-1}.$$

§4.2 σ-algebra of Invariant Sets

Take $\mathscr{J} \subset \mathscr{M}$, $\mathscr{J} = \{A \in \mathscr{M} : T^{-1}(A) = A\}$.

Proposition 4.2

A measurable function $f : X \to \mathbb{R}$ is invariant almost everywhere if and only if f is measurable with respect to $\mathscr{J}$ - for every $(a, b) \subset \mathbb{R}$, $f^{-1}((a, b)) \in \mathscr{J}$.

Proof. The forward direction is obvious. For the backward direction, fix some $y \in \mathbb{R}$. Then $f^{-1}(\{y\}) \in \mathscr{J}$. This implies that $T^{-1}(f^{-1}(y)) = f^{-1}(y)$, which is the same as $(f \circ T)^{-1}(y) = f^{-1}(y)$. This implies that $f \circ T(x) = f(x)$. □

§4.3 Properties of $\mathcal{P} : L^2(X, d\mu) \to \mathrm{Inv}(T)$

- For every $f \in L^2$, $g \in \mathrm{Inv}$, $\int Pf \cdot \overline{g} = \int f \cdot \overline{g}$.

  Proof. $Pg = g$, $P = P^*$. ☐

- $f \in L^2$, $T^{-1}(A) = A$, $\mu(A) < \infty$. Then $\int_A Pf d\mu = \int_A f d\mu$.

  Proof. Take $g = \mathbf{1}_A$ and apply the previous result. ☐

- $\mu(X) < \infty$, $f \in L^2$, then $\int Pf d\mu = \int f d\mu$

  Proof. Take $X = A$ in the previous result. ☐

- $\mu(X) < \infty$, for every $f \in L^2$, $f \geq 0$, then $f(x) > 0$ implies $Pf(x) > 0$.

  Proof. If we have $a < 0$,

$$\begin{aligned}
\mu(\{x : g(x) < a\}) &\leq \mu(\{x : |g(x)|^2 \geq a^2\}) \\
&= \int_{|g(x)^2 \geq a^2|} d\mu \\
&\leq \int_{|g(x)^2 \geq a^2|} \frac{|g(x)|^2}{a^2} d\mu \\
&\leq \frac{1}{a^2} \int_X |g(x)|^2 d\, mu \\
&= \|g\|_{L^2}/a^2.
\end{aligned}$$

  This is known as Chebyshev's Inequality.

  Hence,

$$\mu(\{x : Pf(x) < -1/N\}) \leq N^2 \int |Pf|^2 < \infty.$$

  Furthermore, note that $a\{x : g(x) < a\} \geq \int_{g(x)<a} g(x) d\mu$. Applying this with $g = Pf$, $a = -1/N$, we have

$$\frac{-1}{N}\mu\{x : Pf(x) < -1/N\} \geq \int_{Pf<-1/N} Pf\, d\mu = \int_{Pf<-1/N} f\, d\mu \geq 0.$$

  It follows that $\mu\{x : Pf(x) < -1/N\} = 0$. Taking a union over N, we have that $\mu\{x : Pf(x) < 0\} = 0$.

  This implies that

$$\int_{Pf=0} f = \int_{Pf=0} Pf = 0.$$

  Hence, $Pf = 0$ implies that $f = 0$. But this is the same as saying $f(x) > 0 \Rightarrow Pf(x) > 0$. ☐

> **Remark 4.3.** If we don't assume $\mu(X) < \infty$, then $f \geq 0$ implies $Pf \geq 0$. But if $\mu(X) < \infty$, we have the stronger statement that $f(x) > 0$ implies that $Pf(x) > 0$.
>
> This also follows from the mean ergodic theorem, $S_N f \geq 0$, $g = g^+ - g_-$, $g_+, g_- \geq 0$, $|g| = g^+ + g^-$.

Recall the statement:

Theorem 4.4 (Poincare Recurrence)

$B \in M$, $\mu(B) > 0$, for almost every $x \in B$, there exists a subsequence n_k such that $T^{n_k}(x) \in B$.

Proof. If $\frac{1}{n} S_n f \xrightarrow{L^2} Pf$, then there exists a subsequence so that $1/n_k S_{n_k} f \to Pf(x)$ almost everywhere. Taking $f = \mathbf{1}_B$, the last property of $\mathcal{P}$ shows that $\mathcal{P}\mathbf{1}_B(x) > 0$ almost everywhere in B.

Hence for almost every x,

$$\frac{1}{n_k} \sum_{k=0}^{n_k} \mathbf{1}_B \circ T^{n_k}(x) \to \mathcal{P}(x).$$

But if no such subsequence existed, the LHS would converge to 0. $\square$

§4.4 Examples of Ergodic Functions

- Recall the example $T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto x + \theta \pmod{\mathbb{Z}}$. If θ is irrational, then T is ergodic.
- Take $Tx = mx$, $m \in \mathbb{N}$, $m \geq 2$, $T : \mathbb{S}^1 \to \mathbb{S}^1$. This is an m-to-1 map of $\mathbb{S}^1 \to \mathbb{S}^1$. Is T measure-preserving? Take $[x, y) \subset \mathbb{S}^1$, $d(x, y) < 1/m$. Then, $T^{-1}([x, y)) = \bigcup_{j=0}^{m-1} [x/m + j/m, y/m + j/m]$, each of which has length $(y - x)/m$, so $\mu(T^{-1}([x, y])) = y - x = \mu([x, y))$.

List of Definitions and Theorems