Math 205: Complex Variables

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§1 January 20th, 2021

§1.1 Intro to Riemann Mapping Theorem

Our first goal is to proof a fundamental theorem of Riemann on conformal mappings. We start with several preparations, including some detours. The theorem essentially says that lots of open sets in $\mathbb{C}$ are holomorphically isomorhpic, given that they satisfy some simple topological conditions.

§1.2 Cauchy's Integral Formula

Recall Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz$$

where Γ is a simple closed curve, piecewise differentiable, $z_0 \in \text{Int}(\Gamma)$, and $f : \Omega \to \mathbb{C}$ is a holomorphic function, with Ω is open, $\Omega \supset \Gamma \cup \text{Int}(\Gamma)$.

If Γ is the circle $|z - z_0| = R$, we parameterize with $z = Re^{i\theta} + z_0$ with $\theta \in [0, 2\pi)$. This gives

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta,$$

which represents the average of f on the circle.

It follows that

$$|f(z_0)| \leq \max_{\partial B_R(z_0)} |f(z)|,$$

with equality if and only if f is constant.

If $f : \Omega \to \mathbb{C}$ is holomorphic for Ω connected, open and $z_0 \in \Omega$, then

$$|f(z_0)| \leq \sup_{z \in \Omega} |f(z)|$$

with equality if and only if f is constant.

§1.3 Schwarz Lemma

Theorem 1 (Schwarz Lemma)
For $f : B_1(0) \to \mathbb{C}$ holomorphic with $|f(z)| \leq 1$ for all z and $f(0) = 0$. Then

$$|f(z)| \leq |z|, |f'(0)| \leq 1.$$

If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = cz$ for some $|c| = 1$.

Proof. Define a function

$$g(z) = \begin{cases} f(z)/z, \text{if } 0 \leq |z| \leq 1 \\ f'(0), \text{if } z = 0 \end{cases}.$$

Note that $g(z)$ is continuous since at zero,

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Hence, $|g(z)| \leq C < \infty$ using the Weierstrass Extreme Value theorem. If $0 < \epsilon < |w| < r < 1$, note that taking a Keyhole Contour, we have

$$g(w) = \frac{1}{2\pi i}\left(\int_{|z|=r} - \int_{|z|=\epsilon}\right)\frac{g(z)}{z-w}\,dz.$$

Note that

$$\left|\int_{|z|=\epsilon}\frac{g(z)}{z-w}\,dz\right| \leq (2\pi\epsilon)\cdot C\frac{1}{|w|-\epsilon} \xrightarrow{\epsilon\to 0} 0.$$

It follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r}\frac{g(z)}{z-w}\,dz$$

for $0 < |w| < r$. The right side is holomorphic in w if $|w| < r$, so it follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r}\frac{g(z)}{z-w}\,dz$$

is holomorphic in $|z| < 1$.

This can also be proved by taking a Taylor series about the origin. Since there is no constant term, we can divide by z to still have a convergent Taylor series.

If $r < 1$,

$$\sup_{|z|\leq r}|g(z)| = \sup_{|z|=r}|g(z)| \leq \sup_{|z|=r}\frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

If we let $r \uparrow 1$, then we get $\sup_{|z|<1}|g(z)| \leq 1$. It follows that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

If $|f(z_0)| = z_0$ for some $0 < |z_0| < 1$ then $|g(z_0)| = 1$ and g is constant by the maximum principle so $g(z) = c$, $f(z) = cz$. If $|f'(0)| = 1$, then $|g(0)| = 1$ so g is constant and $f = cz$. □

§1.4 Maximum Principles

In the above proof, we used the maximum principle. Some other versions we will use are the following:

If $K \subset \mathbb{C}$ compact and $f : K \to \mathbb{C}$ continuous, and the restriction of f to the interior of K is holomorphic, then

$$\sup_{z\in K}|f(z)| = \sup_{z\in\partial K}|f(z)|.$$

If Ω is open and connected, $f : \Omega \to \mathbb{C}$, $z_0 \in \Omega$, and $|f(z_0)| = \sup_{z\in\Omega}|f(z)|$, then f is constant. Applying this to e^f and using that $|e^f| = e^{\mathrm{Re}\, f}$, we find that

$$\mathrm{Re}\ f(z_0) = \sup_{z\in\Omega}\mathrm{Re}\ f(z),$$

implies that f is constant. We have the same result for Im f by replacing f with $-if$.

§2 January 25th, 2021

§2.1 Uniform Convergence

Remark 2.1. They sometimes call open connected sets "regions".

Definition 2.2 (Uniform Convergence)**.** Let $\Omega \subset \mathbb{C}$ be open. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic and $f : \Omega \to \mathbb{C}$ a function so that $\lim_{n\to\infty} \sup_{z\in K} |f(z) - f_n(z)| = 0$ for all $K \subset \Omega$ compact(also denoted $K \subset\subset \Omega$).

Remark 2.3. Recall from real analysis that f is a continuous function.

Some further remarks:

- It suffices to check the result for a sequence of compact subsets K_m so that $\bigcup_m K_m^\circ = \Omega$, the it suffices to check those. If $K \subset\subset \Omega$, then K is compact and covered by the union of the subsets so there exists a finite subcovering, and uniform convergence on the subcovering implies uniform convergence on K.

- It is often convenient to introduce $\|g\|_K = \sup_{z\in K} |g(z)|$. Uniform convergence can be restated as $\|f_n - f\|_K \to 0$ for all $K \subset\subset \Omega$.

- If $\|f_n - f\|_K \to 0$ for all $K \subset\subset \Omega$, then f is also holomorphic. It follows by passing to the limit in the Cauchy Integral formula. Namely, take $\{z : |z - z_0| \le R\} \subset \Omega$ and consider the points in $|z_0 - \zeta| < R$.

$$\left| f_n(\zeta) - \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z-\zeta}\, dz \right| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f_n(z)}{z-\zeta}\, dz - \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z-\zeta}\, dz \right|$$
$$\le \frac{1}{2\pi} \frac{1}{R - |z_0 - \zeta|} \cdot (2\pi R) \|f_n - f\|_{|z-z_0|=R} \to 0.$$

  So it follows that

$$f(\zeta) = \lim_{n\to\infty} f_n(\zeta) = \frac{1}{2\pi i} \int_{|z-z_0|} \frac{f(z)}{z-\zeta}\, dz.$$

  It follows that f continuous on $|z - z_0| = R$ is holomorphic in $\zeta \in \{|z - z_0| < R\}$, so it follows that f is holomorphic.

- We can similarly show that

$$f_n^(j)(\zeta) = \frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f_n(z)}{(z-\zeta)^{n+1}}\, dz$$

  and $\|f_n^{(j)} - f^(j)\|_K \to 0$.

From the last item, we have the following theorem.

Theorem 2

If $f_n \to f$ on compact subsets of Ω, the if f_n is holomorphic we find that f is holomorphic and $f_n^(j) \to f^{(j)}$ uniformly on compact subsets of Ω.

Theorem 3 (Hurwitz)

Let Ω be a region, $f : \Omega \to \mathbb{C}$ and $f_n : \Omega \to \mathbb{C}$ holomorphic with $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$ and $\|f_n - f\|_K \to 0$ for all compact subsets. Then either $f \equiv 0$ or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

Proof. If f is not identically zero on ω, then since f is holomorphic, its zeros are isolated. If $z_0 \in \Omega$, $f(z_0) = 0$, then there is $\epsilon > 0$ so that when $0 < |z - z_0| < \epsilon$, $f(z) \neq 0$.

Since $f(z) \neq 0$ for $|z - z_0| = \epsilon/2$, by the Weierstrass theorem applied to $|f|$ on $|z - z_0| = \epsilon$, we have $|f(z)| \geq m > 0$ on $\{|z - z_0| = \epsilon/2\} = \Gamma$. If $\|f_n - f\|_\Gamma \leq m/2$ for $n \geq N$, then

$$|f_n(z)| \geq |f(z)| - m/2 \geq m - m/2 = m/2$$

for $z \in \Gamma$. Hence, it follows that $\|1/f_n - 1/f\|_\Gamma \to 0$(we leave this as an exercise).

Since $\|f_n' - f'\|_\Gamma \to 0$, we find that $\|f_n'/f_n - f'/f\| \to 0$(another exercise) and hence

$$\frac{1}{2\pi i}\int_\Gamma \frac{f_n'}{f_n}\,dz \to \frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz.$$

The integrand of the left hand side is $(\log f_n)'$, whose integral is 0, and the right side is the order of the zero of f at z_0 by the argument principle. It follows that the order of z_0 as a possible zero is 0, so $f(z_0) \neq 0$. $\square$

Theorem 4

For $\Omega \subset \mathbb{C}$ open, $\mathcal{F}$ a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

Proof. We first show 2 implies 1. If $\sup_{f \in \mathcal{F}} \|f\|_K = \infty$, then we can find for each $n \in \mathbb{N}$ $f_n \in \mathcal{F}$ so that $\|f_n\|_K \geq n$. If we abstract a convergence subsequence, then $\|f_{n_j} - f\|_K \leq C < \infty$ and $\|f_{n_j}\|_K \leq \|f\|_K + C$, while $\|f_{n_j}\|_K \to \infty$, a contradiction. $\square$

§3 January 27th, 2021

§3.1 Uniform Convergence, continued

Theorem 5

For $\Omega \subset \mathbb{C}$ open, $\mathcal{F}$ a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

I missed the beginning of the class, but I will add the proof of the theorem once notes are posted.

§3.2 Metric Convergence

One can put a metric on holomorphic functions so that convergence in the metric is uniform convergence on compact sets. For $f : \Omega \to \mathbb{C}$, but $K_n \Subset \Omega$ so that $\bigcup_n K_n^\circ = \Omega$ and take

$$d(f, g) = \sum_{n=1}^{\infty} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}} 2^{-n}.$$

§3.3 Riemann Sphere

On the set $\mathbb{C} \cup \{\infty\}$, we consider the topology which makes it the Alexandroff(one-point) compactification of $\mathbb{C}$. If $z \in \mathbb{C}$, a neighborhood is one that contains a neighborhood in $\mathbb{C}$ and a neighborhood of ∞ is of the form $\{\infty\} \cup (\mathbb{C} \setminus K)$ for $K \Subset \mathbb{C}$.

Let $U_+ = \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ and $U_- = (C \setminus \{0\}) \cup \{\infty\}$. Note that the union of the two sets covers the Riemann Sphere. Define $\psi_+ : U_+ \to \mathbb{C}$ by $\psi_+(z) = z$ and $\psi_i : U_- \to \mathbb{C}$ is given by $\psi_-(w) = 1/w$ if $w \in \mathbb{C} \setminus \{\infty\}$ and 0 if $w = \infty$. Notice that these two functions are bijections.

If $V \subset \mathbb{C} \cup \{\infty\}$ is open, a function $f : V \to \mathbb{C}$ is holomorphic if

$$f|_{V \cup U_\pm} \circ (\psi_\pm|_{V \cup U_\pm})^{-1} : \psi_\pm(V \cup U_\pm) \to \mathbb{C}$$

is holomorphic. In this way, we know what holomorphic functions are on open sets of $\mathbb{C} \cup \{\infty\}$.

More generally, we can describe a Riemann surface in the following way - Let X be a topological space. Take $\{(U_\alpha, z_\alpha)\}_{\alpha \in I}$ where $U_\alpha \subset X$ is open, and $\bigcup_{\alpha \in I} U_\alpha = X$ and $z_\alpha : U_\alpha \to \mathbb{C}$ is continuous, $z_\alpha(U_\alpha)$ is open and z_α is a homeomorphism. The key requirement is that the maps $z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cup U_\beta) \to z_\alpha(U_\alpha \cup U_\beta)$ are holomorphic.

Then, if $U \subset X$ is open, $f : U \to \mathbb{C}$ is holomorphic if for all $\alpha \in I$,

$$f|_{U \cup U_\alpha} \circ (z_\alpha|_{u \cup U_\alpha})^{-1}$$

is holomorphic. Two such atlases give the same Riemann surface if put together, we get an atlas.

§4 February 1st, 2021

§4.1 Connectivity

Definition 4.1. $\Omega \subset \mathbb{C}$ open is connected if $\Omega = \Omega_1 \cup \Omega_2$ open with $\Omega_1 \cap \Omega_2 = \emptyset$ implies that one of the two is empty. For open sets, this is equivalent to arcwise connected.

Definition 4.2. An set is arcwise connected if for every $z_1, z_2 \in \Omega$, there is a path $\varphi : [0,1] \to \Omega$ which is continuous and $\varphi(0) = z_1$, $\varphi(1) = z_2$.

Definition 4.3. Ω is simply connected if for $z_0 \in \Omega$, $\Gamma : [0,1] \to \Omega$ continuous and $\Gamma(0) = \Gamma(1) = z_0$, then there is $G : [0,1] \times [0,1] \to \Omega$ continuous with $G(t,0) = \Gamma(t)$ for $t \in [0,1]$ and $G(t,1) = z_0$, for $t \in [0,1]$.

Simply connected corresponds to the idea of being able to continuously deform the set to a point for each point.

In $\mathbb{R}^2 \cong \mathbb{C}$, Ω-open simply connected is equivalent to $(C \cup \{\infty\}) \setminus \Omega$ is connected in $\mathbb{C} \cup \{\infty\}$. That is, if $F = \mathbb{C} \cup \{\infty\} \setminus \Omega$, which is closed in $\mathbb{C} \cup \{\infty\}$, with $F \cap V_1 \cap V_2 = \emptyset$, then at least one of the $F \cap V_k = \emptyset$. If $0 \in \Omega$, then Ω is simply connected if and only if $\{0\} \cup \{1/z : z \in \mathbb{C} \setminus \Omega\}$ is connected(this is a local representation).

- Take $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^m \{tz_j : t \in [1, \infty)\}$ for $z_1, \ldots, z_n \in \mathbb{C} \setminus \{0\}$.
- $\mathbb{C} \setminus$ spirals.

Theorem 6 (Riemann Mapping Theorem)

If $\Omega \subset \mathbb{C}$ open, connected, simply connected, $\emptyset \neq \Omega \neq \mathbb{C}$, then Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic isomorphisms.

§4.2 Fractional Linear Transformations

Recall that if $f \in \operatorname{Aut}(\mathbb{D})$ then $f(z) = \frac{az+b}{xz+d}$, which was proved using the Schwarz lemma. We view the fractional linear maps from a different context.

We define a map $p : \mathbb{C}^2 \setminus \{\binom{0}{0}\} \to \mathbb{C} \cup \{\infty\}$ given by

$$p\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{cases} z_1/z_2 \text{ if } z_2 \neq 0 \\ \infty \text{ if } z_2 = 0 \end{cases} .$$

Then $p(\xi) = p(\eta)$ if and only if $\xi = \lambda \eta$ for $\lambda \in C^\times = C \setminus \{0\}$.

There is a larger group acting on $C^2 \setminus \{\binom{0}{0}\}$ given by $GL(2, \mathbb{C})$ the invertible 2×2 matrices in the natural way so that

$$A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \frac{A_{11} p(\xi) + A_{12}}{A_{21} p(\xi) + A_{22}}.$$

Define $T_g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by

$$T_g z = \frac{az+b}{cz+d},$$

with $T_g(\infty) = \frac{a}{c}$. We have the action $T_g p(\xi) = p(g\xi)$ for $g \in GL(2, \mathbb{C})$.

This gives

$$T_{g_1} \circ T_{g_2} = T_{g_1 g_2},$$
$$(T_g)^{-1} = T_{g^{-1}}.$$

We can also ask about the fixed point:

$$T_g p(\xi) = p(\xi) \leftrightarrow p(\xi) = p(g\xi) \Leftrightarrow g\xi = \lambda \xi \ , \lambda \in C^\times$$

It follows that the fixed points of T_g correspond to the eigenvectors of $GL(2, \mathbb{C})$.

§4.3 Fractional Linear Transformations, Unit Disk

If we have $\xi = \binom{z_1}{z_2}$, then $p(\xi) \in \mathbb{D}$ if and only if $|z_1| < |z_2|$ if and only if $z_1\overline{z_1} - z_2\overline{z_2} < 0$. If we let

$$J = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix},$$

we consider the sesquilinear form $\langle J\binom{\xi_1}{\xi_2}, \binom{\eta_1}{\eta_2}\rangle$, where it is linear in the first coordinate and conjugate linear in the second coordinate. Note that

$$\langle J\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\rangle = \xi_1\overline{\eta_1} - \xi_2\overline{\eta_2}.$$

When does $g \in GL(2, \mathbb{C})$ preserve $\langle J\xi, \xi\rangle$?

This means that

$$\langle Jg\xi, g\xi\rangle = \langle J\xi, \xi\rangle$$

for all $\xi \in C^2 \setminus \{0\}$. Then,

$$\langle g^* Jg\xi, \xi\rangle = \langle J\xi, \xi\rangle$$

so it follows that $g^*Jg = J$. (We prove this by transforming ξ in polar coordinates, $\xi = x + i^k y$, and considering $k = 0, 1, 2, 3$. These four equations allow us to determine the equality). Note that $U(1, 1) = \{g : g^*Jg = J\}$ forms a group structure where J has eigenvalues ± 1 for this reason, we denote $U(1, 1) \subset GL_2(\mathbb{C})$.

We claim the following: $T_g \in \mathrm{Aut}(\mathbb{D}) \Leftrightarrow g \in C^\times \cdot U(1, 1)$.

§5 February 3rd, 2021

§5.1 Remark on the Zeta Function

Theorem 5.1 (S.M. Voronin 1975)
For $D = \{\frac{1}{2} < \mathrm{Re}(z) < 1\}$, $f : D \to \mathbb{C} \setminus \{0\}$. If $K \subset\subset D$ and $\epsilon > 0$, then there exists $t \in \mathbb{R}$ such that
$$\|f(\cdot) - \zeta(\cdot + it)\|_K < \epsilon.$$

This theorem essentially says that if I slide around the zeta function in the strip D, I can uniformly approximate pretty much any function I want.

§5.2 Fractional Linear Transformations, continued

Note that $\mathrm{Ker}(g \mapsto T_g) = C^\times I_2$. We define $SL(2; C) = \{g \in GL(2; \mathbb{C}) : \det g = 1\}$, the special linear group.

Theorem 5.2
For $g \in SL(2; C)$, $T_g \in \mathrm{Aut}(\mathbb{D})$ if and only if $g \in U(1, 1)$.

Proof. We start with the forward direction. From the first homework, we showed that $f \in \mathrm{Aut}(\mathbb{D})$ implies that $f(z) = T_g z$ where g is the composition of a rotation g_1 and $g_2 = \begin{pmatrix} 1 & z_0 \\ \overline{z_0} & 1 \end{pmatrix}$ for $z_0 \in \mathbb{D}$. It suffices to check that $g_1, g_2 \in U(1, 1) \times \mathbb{C}^\times I_2$. This is easy to check.

Now, we show the converse. If $g \in U(1, 1)$, then $g^{-1} \in U(1, 1)$. If $z \in \mathbb{D}$, then $z = p(\xi), \langle J\xi, \xi\rangle < 0$. We have $T_g z = p(g\ xi)$ and $\langle J\xi, \xi\rangle < 0$ implies that $\langle g^* jg\xi, \xi\rangle < 0$, which implies that $\langle Jg\xi, g\xi\rangle < 0$, which shows that $T_g z = p(g\xi) \in \mathbb{D}$. Hence $T_g\mathbb{D} \subset \mathbb{D}$. The same argument holds for $T_g^{-1}\mathbb{D} \subset \mathbb{D}$ so we have $T_g\mathbb{D} = \mathbb{D}$ exactly, so $T_g = \mathrm{Aut}(\mathbb{D})$. □

§5.3 Automorphisms of the Half Plane

There is a conformal map from $\mathbb{H}_+ \to \mathbb{D}$ given by $f : z \mapsto \frac{z-i}{z+i}$. This corresponds to
$$f = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Note that
$$f^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Now, $\mathrm{Aut}(\mathbb{H}_+) = \{(T_f)^{-1} T_g T_f | T_g \in \mathrm{Aut}(\mathbb{D})\} = \{T_{f^{-1}gf} | g \in SU(1, 1)\}$. it follows that $\mathrm{Aut}(\mathbb{H}_+) = \{T_h | fhf^{-1} \in SU(1, 1)\}$(assuming $h \in SL(2, \mathbb{C})$, $fhf^{-1} \in SL(2, \mathbb{C})$). It follows that $(fhf^{-1})^* J(fhf^{-1}) = J$, so $h^*(f^*Jf)h = f^*Jf$. We can compute
$$f^*Jf = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}.$$

It follows that
$$h^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h = I_2.$$

If we check the computation, we find that $a, b, c, d \in \mathbb{R}$, so it follows that $h \in SL(2, \mathbb{R})$.

§5.4 The Cross Ratio

Note that T_g is completely determined by $T_g 0, T_g 1, T_g \infty$. Suppose $T_g 0 = T_h 0$, $T_g 1 = T_h 1$, $T_g \infty = T_h \infty$. If we let $r = g^{-1}h$, we have $T_r 0 = 0, T_r 1 = 1, T_r \infty = \infty$, so it follows that $r \in C^\times I_2$ (carry out the matrix multiplication for an arbitrary matrix).

if we look at g^{-1} instead of g, we find that T_g is completely determined by $a, b, c \in C \cup \infty$ so that $Ta = 1, Tb = 0, Tc = \infty$. Given, a, b, c, such a T_g is the map

$$z \mapsto \frac{z-b}{z-c} : \frac{a-b}{a-c}.$$

We denote the RHS by (z, a, b, c), which is a fractional linear map taking a, b, c to $1, 0, \infty$. This is called the cross ratio of z, a, b, c.

Theorem 5.3

If T_g is a fractional linear transformation and z_1, z_2, z_3, z_4 are distinct points in $\mathbb{C} \cup \infty$, then

$$(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4).$$

Remark 5.4. The above theorem shows that cross ratios are invariant under fractional linear transformations.