Putnam Solutions

Vishal Raman

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I present some solutions from various Putnam Exams. Problems are not necessarily posted in chronological order. Any typos or mistakes found are mine - kindly direct them to my inbox.

Contents

§1 Putnam 1985

§1.1 A1 - Combinatorics

Problem 1

Determine with proof, the number of ordered triples (A, B, C) with $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A \cap B \cap C = \emptyset$.

Proof. The Venn Diagram for A, B, C has 7 regions and we can place integers 1 through 10 in any of the regions except for the one corresponding to $A \cap B \cap C$. This leads to $6^{10} = \boxed{2^{10}3^{10}5^07^0}$ possible ordered triples, since each configuration corresponds to a unique triple. $\square$

§1.2 A2 - Geometry

Problem 2

Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

Proof. Drop a perpendicular from a vertex. It suffices to maximize the ratio of areas for the left half since the ratio stays the same reflecting the triangle about the axis. Let the height of the resulting triangle be H, the length L. Split the height into 3 smaller heights h_1, h_2, h_3 so that $h_1 + h_2 + h_3 = H$. Drawing a line parallel to the length through the heights gives the rectangles. The lengths of the rectangles are

$$\ell_1 = \frac{L}{H}h_1, \ell_2 = \frac{L}{H}(h_1 + h_2).$$

The ratio is then given by

$$\gamma = \frac{A(R) + A(S)}{A(T)} = \frac{h_2\ell_1 + h_3\ell_2}{LH/2} = \frac{h_1h_2L/H + h_3(h_1 + h_2)L/H}{LH/2} = \frac{2h_1h_2 + 2h_2h_3 + 2h_3h_1}{H^2}.$$

Note that

$$H^2 = (h_1 + h_2 + h_3)^2 = h_1^2 + h_2^2 + h_3^2 + 2h_1h_2 + 2h_2h_3 + 2h_3h_1,$$

so it follows that

$$\gamma = 1 - \frac{h_1^2 + h_2^2 + h_3^2}{(h_1 + h_2 + h_3)^2}.$$

By the Cauchy-Schwarz Inequality,

$$(h_1^2 + h_2^2 + h_3^2)(1 + 1 + 1) \geq (h_1 + h_2 + h_3)^2$$

with equality when $h_1 = h_2 = h_3$. Hence, $\gamma \leq \boxed{2/3}$. It is easy to show the equality case for an equilateral triangle of height 1 with $h_1 = h_2 = h_3 = 1/3$. $\square$

§1.3 A3 - Analysis

Proof. We claim that
$$\lim_{n\to\infty} a_n(n) = \begin{cases} 0, d = 0 \\ e^d - 1, d \neq 0 \end{cases}.$$
The $d = 0$ case is clear, so we show the case where $d \neq 0$. First, we prove that $a_m(n) + 1 = (a_n(0) + 1)^{2^m}$. Note that $a_1(1) + 1 = a_1(0)^2 + 2a_1(0) + 1 = (a_1(0) + 1)^2$. Suppose $a_n(k) + 1 = (a_n(0) + 1)^{2^k}$ for some $k \in \mathbb{N}$. Then
$$a_n(k+1) + 1 = a_n(k)^2 + 2a_n(k) + 1 = (a_n(k) + 1)^2 = ((a_n(0) + 1)^{2^k})^2 = (a_n(0) + 1)^{2^{k+1}},$$
as desired.

Plugging in the value of $a_n(0)$, we find that
$$a_n(n) = \left(\frac{d}{2^n} + 1\right)^{2^n} - 1 = \left(\frac{d}{2^n} + 1\right)^{\frac{2^n}{d}\cdot d} - 1.$$
Taking the limit as $n \to \infty$, we find that
$$\lim_{n\to\infty} a_n(n) = \lim_{n\to\infty} \left(\frac{d}{2^n} + 1\right)^{\frac{2^n}{d}\cdot d} - 1 \xrightarrow{m=2^n/d} \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{md} - 1 = e^d - 1.$$
□

§2 Putnam - 2001

§2.1 A1 - Algebra

Problem 3 (2001-A1)

Consider a set S and a binary operation $*$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Proof. Note that

$$b = ((b * a) * b) * (b * a) = a * (b * a).$$

□

§2.2 A2 - Combinatorics

Problem 4 (2001-A2)

You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that when tossed, is has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd?

Proof. We claim the probability is $P(n) = \boxed{\frac{n}{2n+1}}$. We prove it by induction. We are given that $P(1) = \frac{1}{3}$, which satisfies the claim. Suppose $P(k) = \frac{k}{2k+1}$ for $k \geq 1$. In order to find $P(k+1)$, we condition on the result of the first k coin tosses. Namely, suppose the number of heads is even after k tosses. Then, the total number of heads is odd if we flip a head on the $k+1$-th toss. Similarly, if the number of heads is odd after k tosses, then the total number of heads is odd if we flip a tail on the $k+1$-th toss.

Putting this together gives

$$\begin{aligned}
P(k+1) &= (1 - P(k))p_{k+1} + P(k)(1 - p_{k+1}) \\
&= P(k)\left(1 - 2p_{k+1}\right) + p_{k+1} \\
&= P(k)\left(1 - \frac{2}{2k+3}\right) + \frac{1}{2k+3} \\
&= P(k)\frac{2k+1}{2k+3} + \frac{1}{2k+3} \\
&= \frac{k}{2k+1}\frac{2k+1}{2k+3} + \frac{1}{2k+3} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

which proves the result. □

§2.3 A3 - Algebra

Problem 5 (2001 - A3)

For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Proof. We claim that m is the square of an integer or twice the square of an integer. Set $y = x^2$. We look for square-integer solutions for y. From the quadratic formula,

$$\begin{aligned} y &= \frac{2m+4 \pm \sqrt{(2m+4) - 4(m-2)^2}}{2} \\ &= m+2 \pm \sqrt{(m+2)^2 - (m-2)^2} \\ &= m+2 \pm \sqrt{4(2m)} \\ &= m+2 \pm 2\sqrt{2m} \\ &= (\sqrt{m} \pm \sqrt{2})^2. \end{aligned}$$

Hence, $x = \pm\sqrt{m} \pm \sqrt{2}$. Note that if m is neither the square of an integer nor twice the square of an integer then the field $\mathbb{Q}(\sqrt{m}, \sqrt{2})$ is of degree 4 and the Galois group acts transitively on the roots $\{\pm\sqrt{m} \pm \sqrt{2}\}$. It follows that the polynomial is irreducible.

It is easy to verify that if m is a square or twice a square, then $P_m(x)$ reduces into the product of non-constant integer polynomials.

□

§2.4 A4 - Geometry

Problem 6

Triangle ABC has area 1. Points E, F, G lie on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

Proof. We claim that $[RST] = \frac{7-\sqrt{5}}{4}$. Let $EC/BC = r$, $FA/CA = s$, $GB/AB = t$.

Note that $[ABE] = [AFE]$ since they share a base AE and $BR = FR$ implies that the share the same altitude length as well(drop altitudes from F and B and use the congruent triangles).

Then, $[ABE] = [ABE]/[ABC] = BE/BC = 1 - EC/BC = 1 - r$. We also have $[ACE] = r$. It follows that $[FCE] = [ACE](FC/AC) = r(1-s)$.

Now,

$$1 = [ABC] = [ABE] + [AFE] + [EFC] = (1-r) + (1-r) + r(1-s) \Longrightarrow r(1+s) = 1.$$

Arguing similarly for the other sides, we have $s(1+t) = 1$, and $t(1+r) = 1$.

It follows that

$$r = \frac{1}{1+s} = \frac{1}{1+\frac{1}{1+t}} = \frac{1}{1+\frac{1}{1+\frac{1}{r}}}.$$

Simplifying this, we find that $r = \frac{2+r}{3+2r}$, which gives $3r + 2r^2 = 2 + r$, or equivalently, $r^2 + r - 1 = 0$. Plugging into the quadratic formula and taking the positive root gives

$$r = \frac{1 + \sqrt{5}}{2},$$

and by repeating the argument, we have $r = s = t = \frac{-1+\sqrt{5}}{2}$.

Now, note that $[ATC] = [AEC]/2 = r/2$, $[ATG] = [ACG] - [ATC] = 1 - t - r/2$. Similarly, $[BSC] = t/2$ and $[BRE] = 1 - r - s/2$, so it follows that $[BRTG] = [ABE] - [ATG] - [BRE] = r/2 + s/2 + t - 1$.

$$\begin{aligned}
[RST] &= [ABC] - [ACG] - [BSC] - [BRTG] \\
&= 1 - (1 - t) - (t/2) - (r/2 + s/2 + t - 1) \\
&= 1 - \frac{r + s + t}{2} \\
&= 1 - \frac{3\frac{\sqrt{5}-1}{2}}{2} \\
&= \frac{7 - \sqrt{5}}{4}.
\end{aligned}$$

$\square$