Math 258 Lecture Notes, Fall 2020

Harmonic Analysis

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§1.1 Introduction

We begin by considering the problem of conduction of heat in a circle. We use the map $x \mapsto e^{ix}, x \in [0, 2\pi)$. Where u is the temperature, t is the time, we believed that $u_t = \gamma u_{xx}$, where subscripts denote partial derivatives. We also have an initial condition, $f(x) = u(x, 0)$.

There are some simple solutions $e^{inx}e^{-\gamma n^2 t}|_{t=0} = e^{inx}$. The product of solutions, the sum of solutions, and scalar multiple of solutions are all solutions, so he wrote the solution as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, u(x,t) = \sum_n a_n e^{-\gamma n^2 t} e^{inx}.$$

§1.2 Fourier Analysis

We take a circle $\{z \in \mathbb{C} : |z = 1|\}$, which can also be thought of as $\mathbb{R}/(2\pi\mathbb{Z})$, with the map $x \mapsto e^{ix}$. Suppose we have G a finite abelian group, and $\hat{G} = \{\text{hom } \varphi : G \to \mathbb{R}/\mathbb{Z}\}$, the dual group. $\hat{G}$ is also a group, formally known as the set of characters.

Example 1.1

If we take $G = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, with the map $x \mapsto e^{2\pi i x n/N}$, for $n \in Z_n$.

Similarly, taking $G \cong Z_{N_1} \times \mathbb{Z}_{N_2} \times \dots$, we take $x \mapsto \prod e^{2\pi i x n/N_i}$.

Take $e_\xi(x) = e^{2\pi i \xi(x)}$, where $\xi : G \mapsto \mathbb{R}/\mathbb{Z}$. Working in $L^2(G)$, we note the following:

Fact 1.2. If $\xi \neq \varphi$, then $\langle e_\xi, e_\varphi \rangle = 0$.

Proof.

$$\sum_{x \in G} \xi(x)\overline{\varphi(x)} = \sum_u \xi(u+y)\overline{\varphi(u+y)} - \left(\sum_u \xi(u)\overline{\varphi(u)}\right) \xi(y)\overline{\varphi(u)}.$$

Hence, either $\langle \xi, \varphi \rangle = 0$ or $\xi(y)\overline{\varphi}(y) = 1$ for all $y \in G$, which implies $\xi = \varphi$. $\square$

If follows that $\{e_f : f \in \hat{G}\}$ is an orthonormal set in $L^2(G)$ Then, the dimension is $|\hat{G}| = |G| = \dim(L^2(G))$. Hence, the set forms an orthonormal basis for $L^2(G)$.

Then, for all $f \in L^2(G)$, we have

$$\|f\|^2_{L^2(G)} = \sum_{\varphi \in \hat{G}} |\langle f, e_\xi \rangle|^2,$$

$$f = \sum_{e_\xi \in \hat{G}} \langle f, e_\xi \rangle e_\varphi.$$

§1.3 On Tori of Arbitrary Dimension

We define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, from $[0, 2\pi]$. We then work on $\mathbb{T}^d$, $d \geq 1$.

For $f \in L^2(\mathbb{T}^d)$, we define

$$\hat{f}(n) = (2\pi)^{-d} \int f(x) e^{-inx} dx.$$

We have an inner product $\langle f, g \rangle = \int_{\mathbb{T}^d} f(x)\overline{g(x)} d\mu(x)$ defined over a Lebesgue measure or Euclidean measure on $\mathbb{T}^d$.

Theorem 1 (Parseval's Theorem)
For all $f \in L^2(\Pi^d)$,
$$\|f\|_{L^2}^2 = (2\pi)^d \sum_{n\in\mathbb{Z}^d} |\hat{f}(n)|^2,$$
and we have
$$f = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{inx},$$
in the sense that
$$\|f - \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{inx}\|_L^2 \to 0.$$

Note: you can usually figure out the constant with the simplest example, $f = 1$.

Proof. Take $\mathbb{T}^d, e_n(x) = e^{in\cdot x}$. The $\{(2\pi)^{-d/2}e^n : n \in \mathbb{Z}^d\}$ is orthonormal(left as an exercise). Then, for all f, $\sum_n \langle f, (2\pi)^{-d/2}e_n\rangle \le \|f\|_{L^2}^2$, with equality if the set is a basis(Bessel's inequality).

It suffices to show that $\operatorname{span}\{e_n\}$ is dense in L^2. Take $P = \operatorname{span}\{e_n\}$, and note that P is an algebra of continuous functions on Π^d, closed under conjugation, contains 1, and separates points. Hence, the Stone-Weierstrass theorem implies that P is dense in $C^o(\Pi^d)$ with respect to $\|\cdot\|_{C^o}$. Then $C^o \subset L^2$ is dense(general theory about Compact Hausdorff spaces, Radon Measures).

The statement $\|f - \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{inx}\|_L^2 \to 0$ follows from the general theory of orthonormal systems. □

§1.4 Euclidean Spaces

We work in $\mathbb{R}^d, (d \ge 1)$. Take $\xi \in R^d$, and $x \mapsto x\xi \in \mathbb{R}$ is a homomorphism from $\mathbb{R}^d \to \mathbb{R}$, but if we take $x \mapsto e^{ix\xi}$, we have a homomorphism from $\mathbb{R}^d \mapsto \Gamma$. We try to define the following:
$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx = \langle f, e_\xi\rangle_{L^2(\mathbb{R}^d)},$$
where $e_{xi}(x) = e^{ix\xi}$.

Some problems:

1. $e_\xi \notin L^2(\mathbb{R}^d)$
2. $f(x)e^{-ix\xi}$ need not be in L^1 if $f \in L^2$.

We fix this by imposing extra conditions.

Definition 1.3. For $f \in L^1(\mathbb{R}^d)$, we define
$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx.$$

Note that $f \in L^1$ implies that $\hat{f}$ is bounded, continuous. We see this as follows: $\hat{f}(\xi+u) - \hat{f}(\xi) = \int f(x)e^{-ix\xi}(e^{-ixu} - 1)dx$. If we let $u \to 0$, the right goes to 0 pointwise, and $(2|f|) \in L^1$ dominates the integral, it goes to 0.

Proposition 1.4

If $f \in L^1 \cap L^2(\mathbb{R}^d)$, $\hat{f} \in L^2(\mathbb{R}^d)$,

$$\|\hat{f}\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2.$$

Theorem 2 (Plancherel's Theorem)

$\pi : L^1 \cap L^2 \to L^2$ extends uniquely to $\hat{\pi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, linear, bounded, $\|\hat{\pi} f\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2$, and for all $f \in L^2$, we have an inverse Fourier Transform, $\check{f}(y) = \int f(\xi) e^{iy\xi} d\xi$ for $f \in L^1 \cap L^2$, and $\check{\cdot}$ also extends.

Finally,

$$\|f - (2\pi)^{-d} \int_{|\xi| \le R} \hat{f}(\xi) e^{ix\xi} d\xi\|_{L^2} \to 0.$$

Note that $\check{f}(y) = \hat{f}(-y)$.

Proof. We first prove that $\|f\|_{L^2}^2 = (2\pi)^{-d} \|\hat{f}\|_{L^2}^2$ for all $f \in L^1 \cap L^2$. We prove this for a dense subspace $\mathscr{P}$ of L^2. We will show later that there exists a subspace $V \subset L^2(\mathbb{R}^d)$ so that V is dense in L^2, $V \subset L^1$, $\forall f \in V$, there exists $C_f < \infty$, so for all $\xi \in \mathbb{R}^d$, $|\hat{f}(\xi)| \le C_f (f(\xi))^{-d}$ and f is continuous with compact support.

We are given $f : \mathbb{R}^d \to \mathbb{C}$ supported where $|x| \le R = R_f < \infty$. For large $t \ge 0$, define $f_t(x) = f(tx)$(this shrinks the support of f), supported where $|x| \le R/t < \pi$. We can then think of $f_t : \mathbb{T}^d \to \mathbb{C}$.

Now, we calculate

$$\begin{aligned} \hat{f}_t(n) &= (2\pi)^d \int_{\mathbb{T}^d} f_t(x) e^{-inx} dx \\ &= t^{-d} (2\pi)^d \int_{R^d} f(x) e^{-in/ty} dy \\ &= t^{-d} (2\pi)^{-d} \hat{f}(t^{-1} n), \end{aligned}$$

where the first hat is on $\mathbb{T}^d$ and the second is on $\mathbb{R}^d$, so the Fourier coeficients in the euclidean case are scalar multiples of the Fourier coefficients in the Tori case.

Thus,

$$\|f_t\|_{L^2(\mathbb{T}^d)}^2 = t^{-d} \|f\|_{L^2(\mathbb{R}^d)}^2 = c_d \sum_{n \in \mathbb{Z}^d} |\hat{f}_t(n)|^2 = c_d' t^{-2d} \sum_n |\hat{f}(t^{-1} n)|^2$$

Hence,

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n |\hat{f}(t^{-1} n)|^2.$$

This has a nice tiling Riemann sum interpretation: if we take $\mathbb{R}^d$ and tile it with cubes of sidelength $1/t$ where one corner is at $t^{-1} n$ for $n \in \mathbb{Z}^d$, then

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n \left| \hat{f}(t^{-1} n) \right|^2 = \int_{\mathbb{R}^d} |g_t|^2 dx,$$

where $g(x) = \hat{f}(t^{-1} n)$.

We claim

$$\int_{\mathbb{R}^d} |g_t|^2 \to \int_{\mathbb{R}^d} |\hat{f}|^2,$$

which follows from the dominated convergence theorem: where we take a sequence over t going to infinity, with dominator $C_f^2(1+|\xi|)^{-2d}$ in L^1 and $|\hat{f}(\xi)| \le C_f^2(1+|\xi|)^{-2d}$. Furthermore, we have $g_t(\xi) \to \hat{f}(\xi)$ as $t \to 0$, and $\hat{f}$ is continuous so g_t is pointwise convergent, and we have

$$|g_t(\xi)| = |\hat{f}(t^{-1}n)| \le C_f(1+|t^{-1}n|)^{-d} \le C'(1+|\xi|)^{-d}.$$

$\square$