Math 205: Complex Variables

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§1 January 20th, 2021

§1.1 Intro to Riemann Mapping Theorem

Our first goal is to proof a fundamental theorem of Riemann on conformal mappings. We start with several preparations, including some detours. The theorem essentially says that lots of open sets in $\mathbb{C}$ are holomorphically isomorhpic, given that they satisfy some simple topological conditions.

§1.2 Cauchy's Integral Formula

Recall Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz$$

where Γ is a simple closed curve, piecewise differentiable, $z_0 \in \mathrm{Int}(\Gamma)$, and $f : \Omega \to \mathbb{C}$ is a holomorphic function, with Ω is open, $\Omega \supset \Gamma \cup \mathrm{Int}(\Gamma)$.

If Γ is the circle $|z - z_0| = R$, we parameterize with $z = Re^{i\theta} + z_0$ with $\theta \in [0, 2\pi)$. This gives

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta,$$

which represents the average of f on the circle.

It follows that

$$|f(z_0)| \le \max_{\partial B_R(z_0)} |f(z)|,$$

with equality if and only if f is constant.

If $f : \Omega \to \mathbb{C}$ is holomorphic for Ω connected, open and $z_0 \in \Omega$, then

$$|f(z_0)| \le \sup_{z\in\Omega} |f(z)|$$

with equality if and only if f is constant.

§1.3 Schwarz Lemma

Theorem 1 (Schwarz Lemma)
For $f : B_1(0) \to \mathbb{C}$ holomorphic with $|f(z)| \le 1$ for all z and $f(0) = 0$. Then

$$|f(z)| \le |z|, |f'(0)| \le 1.$$

If for some $z_0 \ne 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = cz$ for some $|c| = 1$.

Proof. Define a function

$$g(z) = \begin{cases} f(z)/z, \text{if } 0 \le |z| \le 1 \\ f'(0), \text{if } z = 0 \end{cases}.$$

Note that $g(z)$ is continuous since at zero,

$$\lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Hence, $|g(z)| \leq C < \infty$ using the Weierstrass Extreme Value theorem. If $0 < \epsilon < |w| < r < 1$, note that taking a Keyhole Contour, we have

$$g(w) = \frac{1}{2\pi i}\left(\int_{|z|=r} - \int_{|z|=\epsilon}\right) \frac{g(z)}{z-w}\, dz.$$

Note that

$$\left|\int_{|z|=\epsilon} \frac{g(z)}{z-w}\, dz\right| \leq (2\pi\epsilon)\cdot C\frac{1}{|w|-\epsilon} \xrightarrow{\epsilon\to 0} 0.$$

It follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)}{z-w}\, dz$$

for $0 < |w| < r$. The right side is holomorphic in w if $|w| < r$, so it follows that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)}{z-w}\, dz$$

is holomorphic in $|z| < 1$.

This can also be proved by taking a Taylor series about the origin. Since there is no constant term, we can divide by z to still have a convergent Taylor series.

If $r < 1$,

$$\sup_{|z|\leq r} |g(z)| = \sup_{|z|=r} |g(z)| \leq \sup_{|z|=r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

If we let $r \uparrow 1$, then we get $\sup_{|z|<1} |g(z)| \leq 1$. It follows that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

If $|f(z_0)| = z_0$ for some $0 < |z_0| < 1$ then $|g(z_0)| = 1$ and g is constant by the maximum principle so $g(z) = c$, $f(z) = cz$. If $|f'(0)| = 1$, then $|g(0)| = 1$ so g is constant and $f = cz$. □

§1.4 Maximum Principles

In the above proof, we used the maximum principle. Some other versions we will use are the following:

If $K \subset \mathbb{C}$ compact and $f : K \to \mathbb{C}$ continuous, and the restriction of f to the interior of K is holomorphic, then

$$\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|.$$

If Ω is open and connected, $f : \Omega \to \mathbb{C}$, $z_0 \in \Omega$, and $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$, then f is constant. Applying this to e^f and using that $|e^f| = e^{\mathrm{Re}\, f}$, we find that

$$\mathrm{Re}\, f(z_0) = \sup_{z\in\Omega} \mathrm{Re}\, f(z),$$

implies that f is constant. We have the same result for $\mathrm{Im}\, f$ by replacing f with $-if$.

§2 January 25th, 2021

§2.1 Uniform Convergence

> **Remark 2.1.** They sometimes call open connected sets "regions".

Definition 2.2 (Uniform Convergence)**.** Let $\Omega \subset \mathbb{C}$ be open. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic and $f : \Omega \to \mathbb{C}$ a function so that $\lim_{n\to\infty} \sup_{z\in K} |f(z) - f_n(z)| = 0$ for all $K \subset \Omega$ compact(also denoted $K \subset\subset \Omega$).

> **Remark 2.3.** Recall from real analysis that f is a continuous function.

Some further remarks:

- It suffices to check the result for a sequence of compact subsets K_m so that $\bigcup_m K_m^\circ = \Omega$, the it suffices to check those. If $K \subset\subset \Omega$, then K is compact and covered by the union of the subsets so there exists a finite subcovering, and uniform convergence on the subcovering implies uniform convergence on K.

- It is often convenient to introduce $\|g\|_K = \sup_{z\in K} |g(z)|$. Uniform convergence can be restated as $\|f_n - f\|_K \to 0$ for all $K \subset\subset \Omega$.

- If $\|f_n - f\|_K \to 0$ for all $K \subset\subset \Omega$, then f is also holomorphic. It follows by passing to the limit in the Cauchy Integral formula. Namely, take $\{z : |z - z_0| \le R\} \subset \Omega$ and consider the points in $|z_0 - \zeta| < R$.

$$\begin{aligned}\left|f_n(\zeta) - \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{z-\zeta}\,dz\right| &= \left|\frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f_n(z)}{z-\zeta}\,dz - \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{z-\zeta}\,dz\right| \\ &\le \frac{1}{2\pi}\frac{1}{R - |z_0 - \zeta|} \cdot (2\pi R)\|f_n - f\|_{|z-z_0|=R} \to 0.\end{aligned}$$

  So it follows that

$$f(\zeta) = \lim_{n\to\infty} f_n(\zeta) = \frac{1}{2\pi i}\int_{|z-z_0|} \frac{f(z)}{z-\zeta}\,dz.$$

  It follows that f continuous on $|z - z_0| = R$ is holomorphic in $\zeta \in \{|z - z_0| < R\}$, so it follows that f is holomorphic.

- We can similarly show that

$$f_n^(j)(\zeta) = \frac{n!}{2\pi i}\int_{|z-z_0|=R} \frac{f_n(z)}{(z-\zeta)^{n+1}}\,dz$$

  and $\|f_n^{(j)} - f^(j)\|_K \to 0$.

From the last item, we have the following theorem.

> **Theorem 2**
>
> If $f_n \to f$ on compact subsets of Ω, the if f_n is holomorphic we find that f is holomorphic and $f_n^(j) \to f^{(j)}$ uniformly on compact subsets of Ω.

Theorem 3 (Hurwitz)
Let Ω be a region, $f : \Omega \to \mathbb{C}$ and $f_n : \Omega \to \mathbb{C}$ holomorphic with $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$ and $\|f_n - f\|_K \to 0$ for all compact subsets. Then either $f \equiv 0$ or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

Proof. If f is not identically zero on ω, then since f is holomorphic, its zeros are isolated. If $z_0 \in \Omega$, $f(z_0) = 0$, then there is $\epsilon > 0$ so that when $0 < |z - z_0| < \epsilon$, $f(z) \neq 0$.

Since $f(z) \neq 0$ for $|z - z_0| = \epsilon/2$, by the Weierstrass theorem applied to $|f|$ on $|z - z_0| = \epsilon$, we have $|f(z)| \geq m > 0$ on $\{|z - z_0| = \epsilon/2\} = \Gamma$. If $\|f_n - f\|_\Gamma \leq m/2$ for $n \geq N$, then

$$|f_n(z)| \geq |f(z)| - m/2 \geq m - m/2 = m/2$$

for $z \in \Gamma$. Hence, it follows that $\|1/f_n - 1/f\|_\Gamma \to 0$(we leave this as an exercise).

Since $\|f_n' - f'\|_\Gamma \to 0$, we find that $\|f_n'/f_n - f'/f\| \to 0$(another exercise) and hence

$$\frac{1}{2\pi i} \int_\Gamma \frac{f_n'}{f_n}\, dz \to \frac{1}{2\pi i} \int_\Gamma \frac{f'}{f}\, dz.$$

The integrand of the left hand side is $(\log f_n)'$, whose integral is 0, and the right side is the order of the zero of f at z_0 by the argument principle. It follows that the order of z_0 as a possible zero is 0, so $f(z_0) \neq 0$. $\square$

Theorem 4
For $\Omega \subset \mathbb{C}$ open, $\mathcal{F}$ a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ with $n_1 < n_2 < \ldots$ so that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

Proof. We first show 2 implies 1. If $\sup_{f \in \mathcal{F}} \|f\|_K = \infty$, then we can find for each $n \in \mathbb{N}$ $f_n \in \mathcal{F}$ so that $\|f_n\|_K \geq n$. If we abstract a convergence subsequence, then $\|f_{n_j} - f\|_K \leq C < \infty$ and $\|f_{n_j}\|_K \leq \|f\|_K + C$, while $\|f_{n_j}\|_K \to \infty$, a contradiction. $\square$

§3 January 27th, 2021

§3.1 Uniform Convergence, continued

Theorem 5

For $\Omega \subset \mathbb{C}$ open, $\mathcal{F}$ a set of holomorphic functions, the following are equivalent:

- for every $K \subset\subset \Omega$ $\sup_{f\in\mathcal{F}} \|f\|_K < \infty$
- for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, there is a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ with $n_1 < n_2 < \dots$ so that $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

I missed the beginning of the class, but I will add the proof of the theorem once notes are posted.

§3.2 Metric Convergence

One can put a metric on holomorphic functions so that convergence in the metric is uniform convergence on compact sets. For $f : \Omega \to \mathbb{C}$, but $K_n \Subset \Omega$ so that $\bigcup_n K_n^\circ = \Omega$ and take

$$d(f,g) = \sum_{n=1}^{\infty} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}} 2^{-n}.$$

§3.3 Riemann Sphere

On the set $\mathbb{C} \cup \{\infty\}$, we consider the topology which makes it the Alexandroff(one-point) compactification of $\mathbb{C}$. If $z \in \mathbb{C}$, a neighborhood is one that contains a neighborhood in $\mathbb{C}$ and a neighborhood of ∞ is of the form $\{\infty\} \cup (\mathbb{C} \setminus K)$ for $K \Subset \mathbb{C}$.

Let $U_+ = \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ and $U_- = (C \setminus \{0\}) \cup \{\infty\}$. Note that the union of the two sets covers the Riemann Sphere. Define $\psi_+ : U_+ \to \mathbb{C}$ by $\psi_+(z) = z$ and $\psi_i : U_- \to \mathbb{C}$ is given by $\psi_-(w) = 1/w$ if $w \in \mathbb{C} \setminus \{\infty\}$ and 0 if $w = \infty$. Notice that these two functions are bijections.

If $V \subset \mathbb{C} \cup \{\infty\}$ is open, a function $f : V \to \mathbb{C}$ is holomorphic if

$$f|_{V\cup U_\pm} \circ (\psi_\pm|_{V\cup U_\pm})^{-1} : \psi_\pm(V \cup U_\pm) \to \mathbb{C}$$

is holomorphic. In this way, we know what holomorphic functions are on open sets of $\mathbb{C} \cup \{\infty\}$.

More generally, we can describe a Riemann surface in the following way - Let X be a topological space. Take $\{(U_\alpha, z_\alpha)\}_{\alpha\in I}$ where $U_\alpha \subset X$ is open, and $\bigcup_{\alpha\in I} U_\alpha = X$ and $z_\alpha : U_\alpha \to \mathbb{C}$ is continuous, $z_\alpha(U_\alpha)$ is open and z_α is a homeomorphism. The key requirement is that the maps $z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cup U_\beta) \to z_\alpha(U_\alpha \cup U_\beta)$ are holomorphic.

Then, if $U \subset X$ is open, $f : U \to \mathbb{C}$ is holomorphic if for all $\alpha \in I$,

$$f|_{U\cup U_\alpha} \circ (z_\alpha|_{u\cup U_\alpha})^{-1}$$

is holomorphic. Two such atlases give the same Riemann surface if put together, we get an atlas.

§4 February 1st, 2021

§4.1 Connectivity

Definition 4.1. $\Omega \subset \mathbb{C}$ open is connected if $\Omega = \Omega_1 \cup \Omega_2$ open with $\Omega_1 \cap \Omega_2 = \emptyset$ implies that one of the two is empty. For open sets, this is equivalent to arcwise connected.

Definition 4.2. An set is arcwise connected if for every $z_1, z_2 \in \Omega$, there is a path $\varphi : [0,1] \to \Omega$ which is continuous and $\varphi(0) = z_1$, $\varphi(1) = z_2$.

Definition 4.3. Ω is simply connected if for $z_0 \in \Omega$, $\Gamma : [0,1] \to \Omega$ continuous and $\Gamma(0) = \Gamma(1) = z_0$, then there is $G : [0,1] \times [0,1] \to \Omega$ continuous with $G(t,0) = \Gamma(t)$ for $t \in [0,1]$ and $G(t,1) = z_0$, for $t \in [0,1]$.

Simply connected corresponds to the idea of being able to continuously deform the set to a point for each point.

In $\mathbb{R}^2 \cong \mathbb{C}$, Ω-open simply connected is equivalent to $(C \cup \{\infty\}) \setminus \Omega$ is connected in $\mathbb{C} \cup \{\infty\}$. That is, if $F = \mathbb{C} \cup \{\infty\} \setminus \Omega$, which is closed in $\mathbb{C} \cup \{\infty\}$, with $F \cap V_1 \cap V_2 = \emptyset$, then at least one of the $F \cap V_k = \emptyset$. If $0 \in \Omega$, then Ω is simply connected if and only if $\{0\} \cup \{1/z : z \in \mathbb{C} \setminus \Omega\}$ is connected(this is a local representation).

- Take $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^m \{tz_j : t \in [1, \infty)\}$ for $z_1, \ldots, z_n \in \mathbb{C} \setminus \{0\}$.
- $\mathbb{C} \setminus$ spirals.

Theorem 6 (Riemann Mapping Theorem)
If $\Omega \subset \mathbb{C}$ open, connected, simply connected, $\emptyset \neq \Omega \neq \mathbb{C}$, then Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic isomorphisms.

§4.2 Fractional Linear Transformations

Recall that if $f \in \text{Aut}(\mathbb{D})$ then $f(z) = \frac{az+b}{xz+d}$, which was proved using the Schwarz lemma. We view the fractional linear maps from a different context.

We define a map $p : \mathbb{C}^2 \setminus \{\binom{0}{0}\} \to \mathbb{C} \cup \{\infty\}$ given by

$$p\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{cases} z_1/z_2 \text{ if } z_2 \neq 0 \\ \infty \text{ if } z_2 = 0 \end{cases}.$$

Then $p(\xi) = p(\eta)$ if and only if $\xi = \lambda\eta$ for $\lambda \in C^\times = C \setminus \{0\}$.

There is a larger group acting on $C^2 \setminus \{\binom{0}{0}\}$ given by $GL(2, \mathbb{C})$ the invertible 2×2 matrices in the natural way so that

$$A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \frac{A_{11}p(\xi) + A_{12}}{A_{21}p(\xi) + A_{22}}.$$

Define $T_g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by

$$T_g z = \frac{az+b}{cz+d},$$

with $T_g(\infty) = \frac{a}{c}$. We have the action $T_g p(\xi) = p(g\xi)$ for $g \in GL(2, \mathbb{C})$.

This gives

$$T_{g_1} \circ T_{g_2} = T_{g_1 g_2},$$
$$(T_g)^{-1} = T_{g^{-1}}.$$

We can also ask about the fixed point:

$$T_g p(\xi) = p(\xi) \leftrightarrow p(\xi) = p(g\xi) \Leftrightarrow g\xi = \lambda\xi \ , \lambda \in C^\times$$

It follows that the fixed points of T_g correspond to the eigenvectors of $GL(2, \mathbb{C})$.

§4.3 Fractional Linear Transformations, Unit Disk

If we have $\xi = \binom{z_1}{z_2}$, then $p(\xi) \in \mathbb{D}$ if and only if $|z_1| < |z_2|$ if and only if $z_1\overline{z_1} - z_2\overline{z_2} < 0$. If we let

$$J = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix},$$

we consider the sesquilinear form $\langle J\binom{\xi_1}{\xi_2}, \binom{\eta_1}{\eta_2}\rangle$, where it is linear in the first coordinate and conjugate linear in the second coordinate. Note that

$$\langle J\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\rangle = \xi_1\overline{\eta_1} - \xi_2\overline{\eta_2}.$$

When does $g \in GL(2, \mathbb{C})$ preserve $\langle J\xi, \xi\rangle$?

This means that

$$\langle Jg\xi, g\xi\rangle = \langle J\xi, \xi\rangle$$

for all $\xi \in C^2 \setminus \{0\}$. Then,

$$\langle g^* Jg\xi, \xi\rangle = \langle J\xi, \xi\rangle$$

so it follows that $g^*Jg = J$. (We prove this by transforming ξ in polar coordinates, $\xi = x + i^k y$, and considering $k = 0, 1, 2, 3$. These four equations allow us to determine the equality). Note that $U(1, 1) = \{g : g^*Jg = J\}$ forms a group structure where J has eigenvalues ± 1 for this reason, we denote $U(1, 1) \subset GL_2(\mathbb{C})$.

We claim the following: $T_g \in \text{Aut}(\mathbb{D}) \Leftrightarrow g \in C^\times \cdot U(1, 1)$.

§5 February 3rd, 2021

§5.1 Remark on the Zeta Function

Theorem 5.1 (S.M. Voronin 1975)
For $D = \{\frac{1}{2} < \text{Re}(z) < 1\}$, $f : D \to \mathbb{C} \setminus \{0\}$. If $K \subset\subset D$ and $\epsilon > 0$, then there exists $t \in \mathbb{R}$ such that
$$\|f(\cdot) - \zeta(\cdot + it)\|_K < \epsilon.$$

This theorem essentially says that if I slide around the zeta function in the strip D, I can uniformly approximate pretty much any function I want.

§5.2 Fractional Linear Transformations, continued

Note that $\text{Ker}(g \mapsto T_g) = C^\times I_2$. We define $SL(2; C) = \{g \in GL(2; \mathbb{C}) : \det g = 1\}$, the special linear group.

Theorem 5.2
For $g \in SL(2; C)$, $T_g \in \text{Aut}(\mathbb{D})$ if and only if $g \in U(1, 1)$.

Proof. We start with the forward direction. From the first homework, we showed that $f \in \text{Aut}(\mathbb{D})$ implies that $f(z) = T_g z$ where g is the composition of a rotation g_1 and $g_2 = \begin{pmatrix} 1 & z_0 \\ \overline{z_0} & 1 \end{pmatrix}$ for $z_0 \in \mathbb{D}$. It suffices to check that $g_1, g_2 \in U(1, 1) \times \mathbb{C}^\times I_2$. This is easy to check.

Now, we show the converse. If $g \in U(1, 1)$, then $g^{-1} \in U(1, 1)$. If $z \in \mathbb{D}$, then $z = p(\xi), \langle J\xi, \xi\rangle < 0$. We have $T_g z = p(g\ xi)$ and $\langle J\xi, \xi\rangle < 0$ implies that $\langle g^* jg\xi, \xi\rangle < 0$, which implies that $\langle Jg\xi, g\xi\rangle < 0$, which shows that $T_g z = p(g\xi) \in \mathbb{D}$. Hence $T_g\mathbb{D} \subset \mathbb{D}$. The same argument holds for $T_g^{-1}\mathbb{D} \subset \mathbb{D}$ so we have $T_g\mathbb{D} = \mathbb{D}$ exactly, so $T_g = \text{Aut}(\mathbb{D})$. $\square$

§5.3 Automorphisms of the Half Plane

There is a conformal map from $\mathbb{H}_+ \to \mathbb{D}$ given by $f : z \mapsto \frac{z-i}{z+i}$. This corresponds to
$$f = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Note that
$$f^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Now, $\text{Aut}(\mathbb{H}_+) = \{(T_f)^{-1} T_g T_f | T_g \in \text{Aut}(\mathbb{D})\} = \{T_{f^{-1}gf} | g \in SU(1, 1)\}$. it follows that $\text{Aut}(\mathbb{H}_+) = \{T_h | fhf^{-1} \in SU(1, 1)\}$(assuming $h \in SL(2, \mathbb{C})$, $fhf^{-1} \in SL(2, \mathbb{C})$). It follows that $(fhf^{-1})^* J(fhf^{-1}) = J$, so $h^*(f^*Jf)h = f^*Jf$. We can compute
$$f^*Jf = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}.$$

It follows that
$$h^*\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h = I_2.$$

If we check the computation, we find that $a, b, c, d \in \mathbb{R}$, so it follows that $h \in SL(2, \mathbb{R})$.

§5.4 The Cross Ratio

Note that T_g is completely determined by $T_g 0, T_g 1, T_g \infty$. Suppose $T_g 0 = T_h 0$, $T_g 1 = T_h 1$, $T_g \infty = T_h \infty$. If we let $r = g^{-1}h$, we have $T_r 0 = 0, T_r 1 = 1, T_r \infty = \infty$, so it follows that $r \in C^\times I_2$ (carry out the matrix multiplication for an arbitrary matrix).

if we look at g^{-1} instead of g, we find that T_g is completely determined by $a, b, c \in C \cup \infty$ so that $Ta = 1, Tb = 0, Tc = \infty$. Given, a, b, c, such a T_g is the map

$$z \mapsto \frac{z-b}{z-c} : \frac{a-b}{a-c}.$$

We denote the RHS by (z, a, b, c), which is a fractional linear map taking a, b, c to $1, 0, \infty$. This is called the cross ratio of z, a, b, c.

Theorem 5.3

If T_g is a fractional linear transformation and z_1, z_2, z_3, z_4 are distinct points in $\mathbb{C} \cup \infty$, then

$$(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4).$$

Remark 5.4. The above theorem shows that cross ratios are invariant under fractional linear transformations.

§6 February 8th, 2021

§6.1 Mappings of Circles and Lines

Lemma 6.1

For $g \in GL_2(\mathbb{C})$, $\{w \in \mathbb{C} \cup \{\infty\} : T_g w \in \mathbb{R} \cup \{\infty\}\}$ is a circle or a straight line with a point at infinity.

Proof.

$$\frac{aw+b}{cw+d} = \overline{\frac{\overline{aw+b}}{cw+d}},$$

Then $(a\overline{c} - c\overline{a})|w|^2 + (a\overline{d} - c\overline{b})w + (b\overline{c} - d\overline{a})\overline{w} + b\overline{d} - d\overline{b} = 0$. If $a\overline{c} - c\overline{a} = 0$, then we have a straight line. If $a\overline{c} - c\overline{a} \neq 0$, we have

$$\left| w + \frac{\overline{a}d - \overline{c}b}{\overline{a}c - \overline{c}a} \right| = \left| \frac{ad - bc}{\overline{a}c - \overline{c}a} \right|,$$

a circle. $\square$

§6.2 Revisiting the Schwarz Lemma

Recall we have $f \in \operatorname{Aut}(\mathbb{D})$, with $f(0) = 0$. We will use the fractional linear transformations so that $0 \in \mathbb{D}$ no longer has a special role.

Given $f : \mathbb{D} \to \mathbb{D}$ holomorphic with $z_0 \in \mathbb{D}$. Take an automorphism mapping $0 \to z_0$ given by $\frac{\cdot + z_0}{1 + \overline{z}_0(\cdot)}$. Then, applying f and applying $(\frac{\cdot + f(z_0)}{1 + \overline{f}(z_0)(\cdot)})^{-1}$, which sends $f(z_0) \to 0$. These are all holomorphic, so it follows that the composition is a holomoprhism from $\mathbb{D} \to \mathbb{D}$ mapping $0 \to 0$. Now, we can apply the Schwarz Lemma as usual: For the derivatives, we use the chain rule:

$$\left(\frac{\cdot + z_0}{1 + \overline{z}_0(\cdot)} \right)' |_{z=0} = 1 - |a|^2.$$

Composing the derivatives along the composition, we find the derivative evaluated at 0 which we require to be ≤ 1.

It follows that

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} \leq \frac{1}{1 - |z_0|^2}.$$

Moreover, by the Schwarz Lemma, we have equality if and only if $f \in \operatorname{Aut}(\mathbb{D})$. if we put $w = f(z)$, then $dw = f'dz$ and the inequality is

$$\frac{|dw|}{1 - |w|^2} \leq \frac{dz}{1 - |z|^2}.$$

This can be interpreted as having on $\mathbb{D}$ the Riemannian metric

$$\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

and $f : \mathbb{D} \to \mathbb{D}$, contracting the metric.

§6.3 Functions on Simply Connected Regions

Recall the following properties of holomorphic functions in simply connected regions:

- For $f : \Omega \to \mathbb{C}$ holomoprhic, then there is $F : \Omega \to \mathbb{C}$ holomorphic so that $F' = f$.
- $f : \Omega \to \mathbb{C} \setminus \{0\}$, then there exists $g : \Omega \to \mathbb{C}$ holomorphic so that $e^g = f$.
- $f : \Omega \to \mathbb{C} \setminus \{0\}$ holomorphic, then there exists $g : \Omega \to \mathbb{C}$ so that $h^n = f$.
- $f : \Omega \to \mathbb{C}$ holomorphic and non-constant, Ω a region, then $f(V)$ is open if $V \subset \Omega$, V is open.

§6.4 Injective Functions

Let $f : \Omega \to G$ be a holomorphic function with Ω open and connected. If f is injective, then $f'(z) \neq 0$. If so, then $f(z) - f(z_0) = u(z)^n$ if $0 = f'(z_0) = \ldots, f^{n-1}(z_0)$ and $f^{(n)}(z_0) \neq 0$, with $u(z_0) = 0$. Then $u(\{|z - z_0| < \epsilon\})$ is open for some $\epsilon > 0$ so it contains $\{|\zeta| < \delta\}$ for some $\delta > 0$. It follows that $U(z_k) = \frac{\delta}{z} e^{2\pi i k/n}$ for $1 \leq k \leq n$ and $f(z_1) = \cdots = f(z_n)$. We could also use the argument principle to show that $f'(z) \neq 0$.

Then, $f(\Omega)$ is open and f has local inverses: for each $z \in \Omega$, there is a neighborhood V_z, where f is a holomorphic isomorphism in the region. It follows that $f : \Omega \to G$ is holomoprhic, injective, then $f|f(\Omega) : \Omega \to f(\Omega)$ is a holomorphic isomorphism.

If Ω is an open region so that $f : \Omega \to \mathbb{D}$ is a holomorphic isomorphism, then if fix $z_0 \in \Omega$, we have $g \in \mathrm{Iso}(\Omega, \mathbb{D}) \to (g(z_0), \frac{g'(z_0)}{|g'(z_0)|}) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

§7 February 10th, 2021

Lemma 7.1

If Ω is an open region so that $f : \Omega \to \mathbb{D}$ is a holomorphic isomorphism, then if fix $z_0 \in \Omega$, we have $g \in \text{Iso}(\Omega, \mathbb{D}) \to (g(z_0), \frac{g'(z_0)}{|g'(z_0)|}) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

Proof. We provide a sketch of the proof. Replace f with

$$\left(\frac{\cdot - f(z_0}{1 - \overline{f(z_0)}\cdot}\right) \circ f$$

so that $f(z_0) = 0$.Then, $Iso(\Omega, \mathbb{D}) \ni g \to g \circ f^{-1} \in Aut(\mathbb{D})$ is a bijection and

$$\left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) = \left((g \circ f^{-1})(0), \frac{(g \circ f^{-1})'(0)}{|(g \circ f^{-1})'(0)|} \frac{f'(z_0)}{|f'(z_0)|}\right)$$

so the proof reduces to the case where $\Omega = \mathbb{D}$ and $z_0 = 0$. It is easy to show that the map is onto and 1-1. □

§7.1 Riemann Mapping Theorem

Theorem 7 (Riemann Mapping Theorem)

Suppose Ω is simply connected and $\Omega \neq \mathbb{C}$. Then, there exists $f : \Omega \to \mathbb{D}$ a holomorphic isomorphism.

> **Remark 7.2.** There is no holomorphic isomorphism from $\mathbb{D} \to \mathbb{C}$ because of Liouville's Theorem.

Proof. (Kobe) Let $z_0 \in \Omega$ and $\mathcal{F} = \{f : \Omega \to \mathbb{D} : f \text{ injective}, f(z_0) = 0, f'(z_0) > 0\}$. The steps are as follows:

- $\mathcal{F} \neq \emptyset$.

  Proof. If $\Omega \neq \mathbb{C}$, there is a point $a \in \mathbb{C} \setminus \Omega$. If Ω is simply connected, there exists $h : \Omega \to \mathbb{C}$ holomorphic with $h^2(z) = z - a$. Then $h(\Omega)$ is open and there exists r such that $B_r(h(z_0)) \subset h(\Omega)$. Then $h^2(\cdot) = \cdot - a$ is injective, so h is injective. Then $-B(h(z_0), r) \cap h(\omega) = \emptyset$. Otherwise, there are z_1, z_2 with $h(z_1) = -h(z_2) \neq 0$. Then, we have $z_1 \neq z_2$ and $h(z_1) = -h(z_2)$ which implies that $h^2(z_1) = h^2(z_2)$.

  Hence, $|h(z) - h(z_0)| \geq r$ for all $z \in \Omega$. It we take $p = r/2 > 0$, then we have $|h(z) + h(z_0)| \geq p$. Then, we find $c \in \mathbb{C}^\times$ so that

  $$c\frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathbb{D}.$$

  Rotating by a sufficient $\theta \in \mathbb{R}$, we have

  $$z \mapsto ce^{i\theta}\frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathcal{F}$$

  □

- Show there is f which maximizes $f'(z_0)$ in $\mathcal{F}$.

  Proof. Let $g_n \in \mathcal{F}$ so that $\lim_{n\to\infty} g_n'(z_0) = \sup_{f\in\mathcal{F}} f'(z_0)$. Since $\|g_n\|_\Omega \le 1$, $n \in \mathbb{N}$, we can pass to a subsequence so that $g_n \to g$ uniformly on compact subsets of Ω for some holomorphic $g : \Omega \to \mathbb{C}$ and $g_n' \to g'$ uniformly on compact sets in Ω. Hence $\lim_{n\to\infty} g_n'(z_0) = g'(z_0)$ and $\sup_{f\in\mathcal{F}} f'(z_0) = g'(z_0) < \infty$ and $g'(z_0) > 0$.

  We still need to show g is injective. Let $z_1 \neq z_2$, $z_1, z_2 \in \Omega$, $g(z_1) = g(z_2)$. Then in $\Omega \setminus \{z_1\}$, $g_n(\cdot) - g_n(z_1) \neq 0$ for all points in $\Omega \setminus \{z_1\}$. By the Hurwitz theorem, $g(\cdot) - g(z_1)$ is either 0 or never vanishes. But $g(\cdot)$ is not a constant function since $g'(z_0) > 0$, so we have $g(\cdot) - g(z_1)$ never vanishes on $\Omega \setminus \{z_1\}$, so $g(z_2) \neq g(z_1)$, a contradiction.

  Moreoever, $\|g\|_\Omega \le 1$ gives that $g(\Omega) \subset \overline{\mathbb{D}}$, but by the maximum principle, we have $g(\Omega) \subset \mathbb{D}$.

  □

- If $f'(z_0)$ maximal, then f is an isomorphism.

  Proof. It suffices to show that $g(\Omega) = \mathbb{D}$. Suppose there is $w_0 \in \mathbb{D} \setminus g(\Omega)$. We perform several modifications of g.

  First, let $F(z) = \sqrt{\frac{g(z)-w_0}{1-\overline{w_0}g(z)}}$. This is well-defined since Ω is simply connected. Note that $F(\Omega) \subset \mathbb{D}$ and F is injective with $0 \notin F(\Omega)$.

  Second, we make z_0 go to 0. Define $G(z) = \frac{F(z)-F(z_0)}{1-\overline{F}(z_0)F(z)}$. Then, G is injective from $\Omega \to \mathbb{D}$ and $G(z_0) = 0$.

  We now show that $G'(z_0) > g'(z_0)$, a contradiction. We will show that $g = k \circ G$, where $k : \mathbb{D} \to \mathbb{D}$, holomorphic. The inverse of G is a fractional linear transformation given by $\begin{pmatrix} 1 & F(z_0) \\ \overline{F}(z_0) & 1 \end{pmatrix}$.

  From F to g, we take the $T_w \circ (z \mapsto z^2)$, where w is the corresponding matrix from the initial FLT. So we have $k = T_w \circ (z \mapsto z^2) \circ T_h$. Note that $k(\mathbb{D}) \subset \mathbb{D}$ and $k(0) = \frac{F(z_0)^2+w_0}{1+\overline{w_0}F(z_0)^2}$, so since we have $F(z_0)^2 = -w_0$, we get $k(0) = 0$.

  Since $k \notin \mathrm{Aut}(\mathbb{D})$, so we must have $|k'(0)| < 1$ by the Schwarz Lemma. It follows that
  $$|G'(z_0)| > |k'(0)||G'(z_0)| = |(k \circ G)'(z_0)| = |g'(z_0)|,$$
  a contradiction. □

□

§8 February 17th, 2021

§8.1 Caratheodory Extension Theorem

Definition 8.1. A Jordan curve is given by a map $[0,1] \ni t \to C(t) \in \mathbb{C}$ which is continuous, 1-1 on $[0,1]$ and $C(0) = C(1)$.

Theorem 8 (Jordan Curve Theorem)
If $C : [0,1] \to \mathbb{C}$ is a Jordan curve, then $\mathbb{C} \setminus C([0,1])$ has 2 connected components, one if which is bounded and the other is unbounded.

We refer to the bounded component as the interior region, or the Jordan region.
We denote $C([0,1])$ as $|C|$ when $C : [0,1] \to \mathbb{C}$.

Theorem 9 (Caratheodory)
Let Γ be a Jordan curve and Ω the bounded region determined by Γ(then $\partial\Omega = |\Gamma|$). if $f : \mathbb{D} \to \Omega$ is a holomorphic isomorphism, then f extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$ where $\partial\mathbb{D}$ is mapped to $\partial\Omega = |\Gamma|$.

Some remarks:

- Note that the winding of the boundary around interior points is preserved so correspondence $\partial\mathbb{D} \to \partial\Omega$ preserves clockwise orientation(see Ahlfors for more detail).
- It is easy to derive a more general statement for Ω_1, Ω_2 of Jordan curves Γ_1, Γ_2. So we have homeomorphisms giving $\Omega_1 \cup |\Gamma_1| = \overline{\Omega_1}$ and $\Omega_2 \cup |\Gamma_2| = \overline{\Omega_2}$.
- It also tells us things about regions with slits. For instance, take $\mathbb{D} \to \mathbb{D} \setminus [0,1)$. By the Riemann Mapping Theorem, we have a holomorphic isomorphism between this set and the unit disk. The boundary behaves as if $[0,1)$ would infinitesimally be a double line, but we can still factor a map $g :\mapsto \mathbb{D} \cap \{Im(z) > 0\}$. Then the map $z \mapsto z^2$ sends this set to $\mathbb{D} \setminus [0,1)$. Then, the homeomorphism $\partial\mathbb{D} \to \partial(\mathbb{D} \cap \{Im(z) > 0\})$ is given by Caratheodory.

§8.2 Rectifiable Arcs

Definition 8.2. An arc $\varphi : [a,b] \to \mathbb{C}$ is a 1-1, continuous map is rectifiable if it has "length"(bounded variation) that is finite:

$$\sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} |\varphi(t_{j-1}) - \varphi(t_j)| < \infty.$$

If this definition is bothersome, we can make stronger assumptions about the arc being piecewise differentiable.

First, we present an analytic continuation theorem. Here the rectificable arc will be without endpoints $\varphi : (a,b) \to \mathbb{C}$.

Theorem 10

If Ω, ω are disjoint regions and Γ a rectifiable arc, so that $|\Gamma| = \partial\Omega \cap \partial\omega$ and $|\Gamma| \cap \Omega \cap \omega$ is open. Assume $f : |\Gamma| \cup \Omega \to \mathbb{C}$, $g; |\Omega| \cup \omega \to \mathbb{C}$ is continuous and $f|_\Omega$, $g|_\omega$ holomorphic and $f|_{|\Gamma|} = g|_{|\Gamma|}$. Then $F : \Omega \cup |\Gamma| \cup \omega \to \mathbb{C}$ defined by $F|_{\Omega \ cup|\Gamma|} = f$, $F|_{|\Gamma| \cup \omega} = g$ is holomorphic.

Proof. We sketch the proof. Analyticity is a local property, so we only need to show that for a point on $|\Gamma|$, there is a neighborhood where F is holomorphic. While F had no endpoints, we take γ, a small portion of the arc. Then, for an open ball containing the arc, we split into regions C_1,C_2. On this, we define

$$f^*(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \Omega_1 \cup \omega_1,$$

going counterclockwise. Similarly, we define $g^*(z)$ over the lower part. Intersection over γ is a Stieltjes integral.

When we add the two, we get $F(z) = \frac{1}{2\pi i} \oint \frac{F(\zeta)}{\zeta - z} \, d\zeta$. This shows that F is holomorphic. □

§9 February 22nd, 2021

§9.1 Schwarz Reflection and Variants

Let $\Omega = \Omega^* = \{\overline{z} | z \in \Omega\}$ an open region. Suppose that $\Omega \cap \mathbb{R} \subset (a, b)$. Then, $\Omega_{\pm} = \omega \cap \{\pm Im(z) > 0\}$. If $f : \Omega_+ \cup (a, b) \to \mathbb{C}$ continuous and $f|_{(a,b)} \subset \mathbb{R}$, $f|_{\Omega_+}$ holomorphic, then

$$F(z) = \begin{cases} f(z), & z \in \Omega_+ \cup (a, b) \\ \overline{f(\overline{z})}, & z \in \Omega_- \end{cases}$$

is holomorphic in $\Omega_+ \cup (a, b) \cup \Omega_-$.

Proof. Use the previous result with $\Omega = \Omega_+$, $\omega = \Omega_-$, $|\Gamma| = (a, b)$ with $f = f$, $\overline{f(\overline{\cdot})} = g(\cdot)$. □

Variants:

- Suppose we set $\Omega_+ \subset \mathbb{D}$, γ, an arc in $\{|z| = 1\} \cap \partial\Omega_+$. We have $|\gamma| \cup \Omega_+$ open, and $f : |\gamma| \cup \Omega_+ \to \mathbb{C}$ continuous, $f|_{\Omega_+}$ holomorphic and $f|_{|\gamma|} \subset \mathbb{R}$.

  We set
  $$F(z) = \begin{cases} f(z), & z \in \Omega_+ \cup |\gamma| \\ \overline{f(1/\overline{z})}, & z \in \{1/\overline{w} : w \in \Omega_+ \setminus \{0\}\} \end{cases}$$

  If we work on the Riemann sphere, we don't need to remove 0, as it gets mapped to ∞. For circles, we have $OA \cdot OB = R^2$.

- Let $\varphi : (a, b) \to \mathbb{C}$ be an Analytic arc - that there is $f : \omega \to \mathbb{C}$ univalent so that $\omega \supset (a, b)$, $f|_{(a,b)} = \varphi$, a holomorphic extension. (this definition avoids the discussion of real analytic functions).

  Let Ω be a region, γ an analytic arc, $|\gamma| \supset \partial\Omega$ from univalent $f : \omega \to \mathbb{C}$ and we assume ω is chosen so that
  $$f(\omega \cap \{Im(z) > 0\}) \subset \Omega, \quad f(\omega \cap \{Im(z) < 0\}) \cap \Omega = \emptyset.$$

  Let $F; \Omega \cup |\gamma| \to \mathbb{C}$ continuous. $F|_{\Omega}$ holomorphic, where $F(|\gamma|) \subset |\Gamma|$, where Γ is another analytic arc. Then, there is Ω_1 open with $\Omega_1 \supset \Omega \cup |\gamma|$ so that it has F has a holomorphic extension to Ω_1 with $|\gamma|$ mapping to another analytic arc.

  First, after a suitable restriction, we take $g^{-1} \circ F \circ f$, reducing the result where we have a segment on the real axis mapped to $\mathbb{R}$. We then apply Schwarz reflection to the segment.

- Let Ω be an inner region of a polygon(not necessarily convex). Suppose $z_1, \ldots, z_n$ appear counterclockwise and $\alpha_k\pi$, $1 \le k \le n$ inner angles $0 < \alpha_k < z$ and $\beta_k\pi$ the outer angles, $\pi - \alpha_k\pi = \beta_k\pi$ or $1 - \alpha_k = \beta_k$. Then $\sum_k \beta_2 = 2$(the sum of exterior angles is 2π). A function $f : \Omega \to \mathbb{D}$ a holomorphic isomorphism has continuous extension to $\widetilde{f} : \overline{\Omega} \to \overline{\mathbb{D}}$ by Caratheodory with $\widetilde{f}(\partial\Omega) = \partial\mathbb{D}$. We let $F : \mathbb{D} \to \Omega$ be the inverse map. We choose f so that $f(z_j) = w_j$, preserving the counterclockwise orientation.

  By the Schwarz Reflection, since $f((z_k, z_{k+1})) = (w_k, w_{k+1})$, f has an analytic extension across (z_k, z_{k+1}) and some neighborhood of (z_k, z_{k+1}) is mapped injectively into a neighborhood of (w_k, w_{k+1}). Note that F has holomorphic extension into a neighborhood of (w_k, w_{k+1}) and etc.

§9.2 Schwarz-Christoffel Formula

$F : \overline{\mathbb{D}} \to \overline{\Omega}$ is a homeomorphism which extends the inverse map ad $F(w_k) = z_k$. $\overline{\Omega}$ iis a polygon with angles $\alpha_k\pi, \beta_k = 1 - \alpha_k$. Then

$$F(w) = C \int_0^w \prod_{i=1}^k (w - w_k)^{-\beta_k}\, dw + C'.$$

> **Remark 9.1.** This is not an explicit formula. The constants C, C' need to be found and $w_1, \dots, w_n$ are not known. We can fix w_1, w_2, w_3, but not more.

Proof. Consider a map $\varphi(\zeta) = \zeta^{\alpha_k} e^{i\omega_k} + z_k$, which maps a semicircle to the angle $\alpha_k\pi$. Note that φ extends to $\{|\zeta| < \epsilon : Im(\zeta) \geq 0\}$ and maps $(-\epsilon, \epsilon)$ to the corner at z_k. Then $\widetilde{f} \circ \varphi$ maps $(-\epsilon, \epsilon)$ to an arc of the circle containing w_k.

Applying the reflection principle to the segment, $\widetilde{f} \circ \varphi$ has an analytic extension to the open disc of radius ϵ. Moreover, this extension has nonzero derivative at 0, so it has a local inverse at w_k.

So, take $(\widetilde{f} \circ \varphi)^{-1}(w) = (w - w_k)K(w)$ with $K(w_k) \neq 0$ in a neighborhood of w_k. But then, in a neighborhood of w_k, if $w \in \overline{\mathbb{D}}$, we have

$$F(w) = \varphi \circ (\widetilde{f} \circ \varphi)^{-1}(w) = (w - w_k)^{\alpha_k} \cdot e^{i\omega_k} K(w)^{\alpha_k} + z_k.$$

But $K(w)^{\alpha_k}$ is holomorphic near w_k since $K(w_k) \neq 0$ so we can define (the branch of) this power in a small disc around w_k. Thus, locally near $w_k \in \overline{\mathbb{D}}$, we have

$$F(w) - z_k = (w - w_k)^{alpha_k} \cdot G_k(w)$$

where $G_k(w_k) \neq 0$ and holomorphic in a neighborhood of w_k.

Computing the derivative, we have

$$F'(w) = (w - w_k)^{-\beta_k}(\alpha_k G_k(w) + (w - w_k)G_k'(w))$$

or $(w - w_k)^{\beta_k} F'(w)$ is holomorphic and nonzero near w_k so $F'(w)\prod_{k=1}^n (w - w_k)^{\beta_k}$ is are holomorphic near $\overline{\mathbb{D}}$. $\square$