Math 258 Lecture Notes, Fall 2020

Harmonic Analysis

Professor: Michael Christ
Vishal Raman

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§1.1 Introduction

We begin by considering the problem of conduction of heat in a circle. We use the map $x \mapsto e^{ix}, x \in [0, 2\pi)$. Where u is the temperature, t is the time, we believed that $u_t = \gamma u_{xx}$, where subscripts denote partial derivatives. We also have an initial condition, $f(x) = u(x, 0)$.

There are some simple solutions $e^{inx}e^{-\gamma n^2 t}|_{t=0} = e^{inx}$. The product of solutions, the sum of solutions, and scalar multiple of solutions are all solutions, so he wrote the solution as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, u(x,t) = \sum_n a_n e^{-\gamma n^2 t} e^{inx}.$$

§1.2 Fourier Analysis

We take a circle $\{z \in \mathbb{C} : |z = 1|\}$, which can also be thought of as $\mathbb{R}/(2\pi\mathbb{Z})$, with the map $x \mapsto e^{ix}$. Suppose we have G a finite abelian group, and $\widehat{G} = \{\hom \varphi : G \to \mathbb{R}/\mathbb{Z}\}$, the dual group. $\widehat{G}$ is also a group, formally known as the set of characters.

Example 1.1

If we take $G = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, with the map $x \mapsto e^{2\pi i x n/N}$, for $n \in Z_n$.

Similarly, taking $G \cong Z_{N_1} \times \mathbb{Z}_{N_2} \times \dots$, we take $x \mapsto \prod e^{2\pi i x n/N_i}$.

Take $e_\xi(x) = e^{2\pi i \xi(x)}$, where $\xi : G \mapsto \mathbb{R}/\mathbb{Z}$. Working in $L^2(G)$, we note the following:

Fact 1.2. If $\xi \neq \varphi$, then $\langle e_\xi, e_\varphi \rangle = 0$.

Proof.

$$\sum_{x \in G} \xi(x)\overline{\varphi(x)} = \sum_u \xi(u+y)\overline{\varphi(u+y)} - \left(\sum_u \xi(u)\overline{\varphi(u)}\right) \xi(y)\overline{\varphi(u)}.$$

Hence, either $\langle \xi, \varphi \rangle = 0$ or $\xi(y)\overline{\varphi}(y) = 1$ for all $y \in G$, which implies $\xi = \varphi$. □

If follows that $\{e_f : f \in \widehat{G}\}$ is an orthonormal set in $L^2(G)$ Then, the dimension is $|\widehat{G}| = |G| = \dim(L^2(G))$. Hence, the set forms an orthonormal basis for $L^2(G)$.

Then, for all $f \in L^2(G)$, we have

$$\|f\|^2_{L^2(G)} = \sum_{\varphi \in \widehat{G}} |\langle f, e_\xi \rangle|^2,$$

$$f = \sum_{e_\xi \in \widehat{G}} \langle f, e_\xi \rangle e_\varphi.$$

§1.3 On Tori of Arbitrary Dimension

We define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, from $[0, 2\pi]$. We then work on $\mathbb{T}^d$, $d \geq 1$.

For $f \in L^2(\mathbb{T}^d)$, we define

$$\widehat{f}(n) = (2\pi)^{-d} \int f(x) e^{-inx} dx.$$

We have an inner product $\langle f, g \rangle = \int_{\mathbb{T}^d} f(x)\overline{g(x)} d\mu(x)$ defined over a Lebesgue measure or Euclidean measure on $\mathbb{T}^d$.

Theorem 1 (Parseval's Theorem)
For all $f \in L^2(\Pi^d)$,

$$\|f\|_{L^2}^2 = (2\pi)^d \sum_{n \in \mathbb{Z}^d} |\widehat{f}(n)|^2,$$

and we have

$$f = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx},$$

in the sense that

$$\|f - \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx}\|_L^2 \to 0.$$

Note: you can usually figure out the constant with the simplest example, $f = 1$.

Proof. Take $\mathbb{T}^d, e_n(x) = e^{in \cdot x}$. The $\{(2\pi)^{-d/2} e^n : n \in \mathbb{Z}^d\}$ is orthonormal(left as an exercise). Then, for all f, $\sum_n \langle f, (2\pi)^{-d/2} e_n \rangle \leq \|f\|_{L^2}^2$, with equality if the set is a basis(Bessel's inequality).

It suffices to show that $\text{span}\{e_n\}$ is dense in L^2. Take $P = \text{span}\{e_n\}$, and note that P is an algebra of continuous functions on Π^d, closed under conjugation, contains 1, and separates points. Hence, the Stone-Weierstrass theorem implies that P is dense in $C^o(\Pi^d)$ with respect to $\| \cdot \|_{C^o}$. Then $C^o \subset L^2$ is dense(general theory about Compact Hausdorff spaces, Radon Measures).

The statement $\|f - \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx}\|_L^2 \to 0$ follows from the general theory of orthonormal systems. □

§1.4 Euclidean Spaces

We work in $\mathbb{R}^d, (d \geq 1)$. Take $\xi \in R^d$, and $x \mapsto x\xi \in \mathbb{R}$ is a homomorphism from $\mathbb{R}^d \to \mathbb{R}$, but if we take $x \mapsto e^{ix\xi}$, we have a homomorphism from $\mathbb{R}^d \mapsto \Gamma$. We try to define the following:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx = \langle f, e_\xi \rangle_{L^2(\mathbb{R}^d)},$$

where $e_{xi}(x) = e^{ix\xi}$.

Some problems:

1. $e_\xi \notin L^2(\mathbb{R}^d)$
2. $f(x)e^{-ix\xi}$ need not be in L^1 if $f \in L^2$.

We fix this by imposing extra conditions.

Definition 1.3. For $f \in L^1(\mathbb{R}^d)$, we define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx.$$

Note that $f \in L^1$ implies that $\widehat{f}$ is bounded, continuous. We see this as follows: $\widehat{f}(\xi + u) - \widehat{f}(\xi) = \int f(x) e^{-ix\xi}(e^{-ixu} - 1) dx$. If we let $u \to 0$, the right goes to 0 pointwise, and $(2|f|) \in L^1$ dominates the integral, it goes to 0.

Proposition 1.4

If $f \in L^1 \cap L^2(\mathbb{R}^d)$, $\widehat{f} \in L^2(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2.$$

Theorem 2 (Plancherel's Theorem)

$\pi : L^1 \cap L^2 \to L^2$ extends uniquely to $\widehat{\pi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, linear, bounded, $\|\widehat{\pi} f\|_{L^2}^2 = (2\pi)^d \|f\|_{L^2}^2$, and for all $f \in L^2$, we have an inverse Fourier Transform, $\check{f}(y) = \int f(\xi) e^{iy\xi} d\xi$ for $f \in L^1 \cap L^2$, and $\check{\cdot}$ also extends.

Finally,

$$\|f - (2\pi)^{-d} \int_{|\xi| \leq R} \widehat{f}(\xi) e^{ix\xi} d\xi\|_{L^2} \to 0.$$

Note that $\check{f}(y) = \widehat{f}(-y)$.

Proof. We first prove that $\|f\|_{L^2}^2 = (2\pi)^{-d} \|\widehat{f}\|_{L^2}^2$ for all $f \in L^1 \cap L^2$. We prove this for a dense subspace $\mathscr{P}$ of L^2. We will show later that there exists a subspace $V \subset L^2(\mathbb{R}^d)$ so that V is dense in L^2, $V \subset L^1$, $\forall f \in V$, there exists $C_f < \infty$, so for all $\xi \in \mathbb{R}^d$, $|\widehat{f}(\xi)| \leq C_f (f(\xi))^{-d}$ and f is continuous with compact support.

We are given $f : \mathbb{R}^d \to \mathbb{C}$ supported where $|x| \leq R = R_f < \infty$. For large $t \geq 0$, define $f_t(x) = f(tx)$(this shrinks the support of f), supported where $|x| \leq R/t < \pi$. We can then think of $f_t : \mathbb{T}^d \to \mathbb{C}$.

Now, we calculate

$$\begin{aligned} \widehat{f_t}(n) &= (2\pi)^d \int_{\mathbb{T}^d} f_t(x) e^{-inx} dx \\ &= t^{-d} (2\pi)^d \int_{R^d} f(x) e^{-in/ty} dy \\ &= t^{-d} (2\pi)^{-d} \widehat{f}(t^{-1} n), \end{aligned}$$

where the first hat is on $\mathbb{T}^d$ and the second is on $\mathbb{R}^d$, so the Fourier coeficients in the euclidean case are scalar multiples of the Fourier coefficients in the Tori case.

Thus,

$$\|f_t\|_{L^2(\mathbb{T}^d)}^2 = t^{-d} \|f\|_{L^2(\mathbb{R}^d)}^2 = c_d \sum_{n \in \mathbb{Z}^d} |\widehat{f_t}(n)|^2 = c_d' t^{-2d} \sum_n |\widehat{f}(t^{-1} n)|^2$$

Hence,

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n |\widehat{f}(t^{-1} n)|^2.$$

This has a nice tiling Riemann sum interpretation: if we take $\mathbb{R}^d$ and tile it with cubes of sidelength $1/t$ where one corner is at $t^{-1} n$ for $n \in \mathbb{Z}^d$, then

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = c_d' t^{-d} \sum_n \left| \widehat{f}(t^{-1} n) \right|^2 = \int_{\mathbb{R}^d} |g_t|^2 dx,$$

where $g(x) = \widehat{f}(t^{-1} n)$.

We claim

$$\int_{\mathbb{R}^d} |g_t|^2 \to \int_{\mathbb{R}^d} |\widehat{f}|^2,$$

which follows from the dominated convergence theorem: where we take a sequence over t going to infinity, with dominator $C_f^2(1+|\xi|)^{-2d}$ in L^1 and $|\widehat{f}(\xi)| \le C_f^2(1+|\xi|)^{-2d}$. Furthermore, we have $g_t(\xi) \to \widehat{f}(\xi)$ as $t \to 0$, and $\widehat{f}$ is continuous so g_t is pointwise convergent, and we have

$$|g_t(\xi)| = |\widehat{f}(t^{-1}n)| \le C_f(1+|t^{-1}n|)^{-d} \le C'(1+|\xi|)^{-d}.$$

$\square$

§2 September 1st, 2020

§2.1 Proof of Plancherel's Theorem

Last time

- $\mathbb{R}^d$,
$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$
- $V = (f \in L_1 \cap L_2(\mathbb{R}^d)) : |\widehat{f}(\xi)| \langle \xi \rangle^d$ is a bounded linear function, $\langle x \rangle = (1+|x|^2)^{1/2} \geq 1, = |x|$ for x large.
- Claim: V is dense in $L^2(\mathbb{R}^d)$. Then $\|\widehat{f}\|_{L^2} = (2\pi)^{d/2}\|f\|_{L^2}$ for all $f \in V$ so there exists a unique bounded linear operator $\mathscr{F}$ on $L^2(\mathbb{R}^d)$, where $\mathscr{F}$ takes a function to it's fourier transform.
- We discussed some properties of $\mathscr{F}$.
  - $\|\mathscr{F} f\|_2 = (2\pi)^{d/2}\|f\|_2$
  - $\mathscr{F}$ is onto.
  - For all $f \in L^2$,
  $$\left\| f - (2\pi)^{-d} \int_{|\xi|\leq R} e^{ix\cdot\xi} \mathscr{F}(f)(\xi)d\xi \right\|_{L^2} \to 0,$$
  in the limit where $R \to \infty$.

First note that $\mathscr{F}$ has closed range(this was an exercise). It suffices to show: If $g \in L^2, g \perp \mathscr{F}(f)$ for all $f \in V$, then $g = 0$.

Proof. First, note that
$$0 = \langle g, \mathscr{F}(f) \rangle = \langle \mathscr{F}^*(g), f \rangle,$$
and for all $g \in V$,
$$\mathscr{F}^* g(x) = \int g(\xi) e^{ix\cdot\xi} d\xi$$
Therefore, $\mathscr{F}^*(g)(x) = (\mathscr{F} g)(-x)$ for all $g \in V$, which is dense in L^2. Hence, $\mathscr{F} g = 0$, and the Fourier transform preserves norms, so $g = 0$. □

We also claimed the following: Let $f \in L^2$:
$$\|f(x) - (2\pi)^{-d} \int_{|\xi|\leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi\|_2^2 \to 0.$$

Proof. Let $g_r = (2\pi)^{-d} \int_{|\xi|\leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi$.We have to show $\langle f, g_r \rangle \to \|f\|_2^2$. Then
$$\|f - g_r\|_2^2 = \|f\|_2^2 + \|g_r\|_2^2 - 2\text{Re}\langle f, g_r \rangle \to \|f\|_2^2 + \|f\|_2^2 - 2\|f\|_2^2.$$

$$\begin{aligned}
\langle f, g_R \rangle &= (2\pi)^{-d} \int f(x) \overline{\int_{|\xi|\leq R} (\mathscr{F} f)(\xi) e^{ix\cdot\xi} d\xi dx} \\
&= (2\pi)^{-d} \int_{|\xi|\leq R} \left(\int f(x) e^{-ix\cdot\xi} dx \right) \overline{(\mathscr{F} f)(\xi) d\xi} \\
&= (2\pi)^{-d} \int_{|\xi|\leq R} |\mathscr{F} f(\xi)|^2 d\xi \to (2\pi)^{-d} \|\mathscr{F} f\|_2^2 = \|f\|_2^2.
\end{aligned}$$

However, it's not clear that we can use Fubini's theorem. We would need $f \in L^1 \cap L^2$. But this is not an issue as $L^1 \cap L^2 \subset L^2$ is dense, so if we let $\epsilon > 0, f = G + h, \|h\|_2 \leq \epsilon$ and $G \in L^1 \cap L^2$. Showing the convergence from here is an exercise. $\square$

We still need $V = (f \in L^1 \cap L^2 : \langle \xi \rangle^d (\widehat{f}(\xi))$ is bounded) is dense in L^2. We'll discuss this in the future.

§2.2 Introduction to Convolution

Our meta definition is $f * g(x) = \int f(x-y)g(y)dy$, but it will depend on the conditions of the function for the integral to be defined.

Convolution is generally associated to a group, where

$$\int_G f(xy^{-1}g(y)d\mu(y)),$$

with the Haar measure(done in 202b).

If we substitute $y = x - u$, then

$$f * g(x) = \int f(u)g(x-u)du = g * f(x).$$

It is also associative: $(f * g) * g = f * (g * h)$ for all f, g, h(involves Fubini's theorem).

We can formally write

$$f * g(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u)g(v)d\lambda_x(u,v),$$

where λ_x is supported on $\Lambda = \{(u,v) : u + v = \lambda\}$(an affline subspace). If we have a subset $E \subset \Lambda$, $\lambda_x(E) = |\pi_1(E)| = |\pi_2(E)|$, where π_i are Lebesgue measure s of projections on the i-th factor. Note the following: suppose that f, g are continuous with compact support. Then $\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)$, where $A + B = \{a + b : (a,b) \in A \times B\}$.

Let $T : C_0^0(\mathbb{R}^d) \to C_b^0(\mathbb{R}^d)$ be bounded, linear and $T \circ \tau_y = \tau_y \circ T$ for all $x \in \mathbb{R}^d$ ($\tau_y f(x) = f(x+y)$, a translation). Then, there exists a Complex Radon measure μ on $\mathbb{R}^d$ so that for all $f \in C_0^0$, $T(f) = f * \mu$, where

$$f * \mu(x) = \int f(x-y)d\mu(y).$$

In the case of $\mathbb{T}^1$, $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}$ for all $f \in L^2$. Suppose we wanted to consider the partial sums,

$$\sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = S_N(f)(x).$$

In what sense does $S_N f \to f$, and for which functions f do we have convergence?

$$S_N(f)(x) = \sum_{n=-N}^{N} e^{inx}(2\pi)^{-1}\int_{-\pi}^{\pi} f(y)e^{-iny}dy = (2\pi)^{-1}\int f(y)\sum_{n=-N}^{N} e^{in(x-y)}dy$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi} f(y)D_n(x-y)dy.$$

The Dirichlet Kernels, $D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin{(N+1/2)x}}{\sin{(x/2)}}$ if $x \neq 0$ or $D_N(x) = 2N+1$ if $x = 0$.

§2.3 General Convolution

Theorem 3

Let $f, g \in L^1(\mathbb{R}^d)$. Then, the following are true:

- $y \mapsto f(x-y)g(y) \in L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- If $f, g \geq 0$, then $\|f * g\|_1 = \int f * g = \int f \int g$.
- The operation commutative and associative, so L^1 is an algebra, but it no multiplicative identity, so no inverses.
- For $f, g \in L^1$, $\widehat{(f \star g)} = \widehat{f} \cdot \widehat{g}$.

In other words, convolution is a nice bilinear operation.

Proof. Let $F(x,y) = f(x-y)g(y)$, $F : \mathbb{R}^{d+d} \to \mathbb{C}$ is Lebesgue measurable. We claim that $F \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. It follows from

$$\int |F(x,y)|dxdy = \int |f(x-y)||g(y)|dxdy = \int |g(y)|dy \int |f(x)|dx = \|g\|_1 \|f\|_1 < \infty.$$

Now, $F \in L^1$, so by Fubini's theorem, for almost every $x, y \to f(x-y)g(y) \in L^1$ and $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.

$$\|f*g\|_1 = \int |f*g(x)|dx = \int \left| \int f(x-y)g(y)dy \right| dx \leq \int \int |f(x-y)||g(y)|dydx = \|f\|_1 \|g\|_1.$$

Note that $\int (f*g)(x)dx = \|f\|_1 \|g\|_1$, for non-negative functions.
Finally,

$$\begin{aligned}
(f*g)^\wedge(\xi) &= \int e^{-ix\cdot\xi} \left(\int f(x-y)g(y)dy \right) dx \\
&= \int \left(\int e^{-ix\cdot\xi} f(x-y)dx \right) dy, x = u+y \\
&= \int \left(e^{-i(u+y)\cdot\xi} f(u)du \right) g(y)dy \\
&= \int e^{-iy\cdot\xi} \widehat{f}(u)g(y)dy \\
&= \widehat{f}(\xi) \cdot \widehat{g}(\xi).
\end{aligned}$$

□

Example 2.1 (A Warning)

In $\mathbb{R}^1$, $f(x) = |x|^{-2/3} 1_{|x| \leq 1}$, which has an asymptote at 0. $f \in L^1$, and

$$(f*f)(0) = \int_{-1}^{1} |u|^{-4/3} dy = +\infty.$$

Proposition 2.2

Let $p \in [1, \infty]$. Let $f \in L^1, g \in L^p$. Then,

- $y \mapsto f(x-y)g(y) \in L^1$ for almost every $x \in \mathbb{R}^d$.
- $x \mapsto \int f(x-y)g(y)dy$ is Lebesgue measurable.
- $f * g \in L^p(\mathbb{R}^d)$, $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

Proof. For $p = \infty$, $\int f(x-y)g(y)dy \in C_0(\mathbb{R}^d)$.
If $1 < p < \infty$, $L^P \subset L^1 + L^\infty$, as follows:

$$f(x) = f(x)1_{|f(x)| \le 1} + f(x)1_{f(x) > 1}.$$

We can prove the rest with Minkowski's inequality, or a simpler way. Let $q = p' = \frac{p}{p-1}$ (hence $\frac{1}{q} + \frac{1}{p} = 1$). We use the norm definition,

$$\|f * g\|_p = \sup_{\|h\|_q \le 1} \int |g * f| \cdot |h|.$$

$$\int |g * f| \cdot h \le \int (|g| * |f|) \cdot h = \int \int |g(x-y)||f(y)|dyh(x)dx$$

$$= \int |f(y)| \int |g(x-y)|h(x)dxdy \le \int |f(y)| \|g\|_p * 1dy = \|f\|_1 \|g\|_p.$$

□

§3 September 3rd, 2020

§3.1 Convolution and Continuity

Recall convolution:

$$f * g(x) = \int f(x-y)g(y)dy, f * \mu(x) = \int_{\mathbb{R}^d} f(x-y)d\mu(y),$$

where f is continuous, bounded, μ is a complex Radon measure($|\mu|$ is finite)

Proposition 3.1

Let $T : C_0^0 \to C_b^0$. Suppose T is translation invariant: $T \circ \tau_y = \tau_y \circ T$ for all $y \in \mathbb{R}^d$. [There exists $A < \infty : \|Tf\|_{C_0} \leq A\|f\|_{C_0}$ for all f. Recall $\|f\|_{C_0} = \sup_x |f(x)|$, and C_0^0, C_b^0 are Banach spaces.] There exists a complex radon measure μ such that $Tf = f * \mu$ for all f.

Proof. Given $T : C_0^0 \to C_b^0$, consider the map $\ell : \mathbb{C}_0^0 \to \mathbb{C}$ given by $f \mapsto (Tf)(0)$. It is clear that ℓ is linear. Furthermore, ℓ is bounded, since

$$|Tf(0)| \leq \|Tf\|_{C_0} \leq A\|f\|_{C_0},$$

so $\ell \in (C_0^0)^*$. Recall the Riesz Representation Theorem, there exists ν, a complex Radon measure, such that for all $f \in C_0^0$

$$\ell(f) = \int f d\nu.$$

Let $y \in \mathbb{R}^d$. We have

$$Tf(-y) = Tf(0-y) = (\tau_y Tf)(0) = T(\tau_y f)(0) = \int \tau_y f(x) d\nu(x) = \int f(x-y) d\nu(x).$$

Similarly, for all x, $(Tf)(-x) = \int f(y-x)d\nu(y)$. [See lecture notes for correct algebra, sad]. □

§3.2 Convolution and Differentiation

Informally,

$$\frac{\partial}{\partial x_j} \int f(x-y)g(y)dy = \int \frac{\partial f}{\partial x_j} f(x-y)g(y)dy.$$

Proposition 3.2

Assume $f \in C^1(\mathbb{R}^d), g \in L^1$ and $f, \nabla f$ is bounded. Then

$$f * g \in C^1, \frac{\partial}{\partial x_j}(f \star g) = \left(\frac{\partial f}{\partial x_j}\right) * g.$$

Proof. We assume $d = 1$ for clarity.

$$\frac{(f*g)(x+t) - (f*g)(x)}{t} = \int \frac{f(x+t-y) - f(x-y)}{t} g(y)dy.$$

Let $t \to 0$. Use DCT, with dominator

$$|g(y)| \cdot \sup_{t,u} \frac{|f(u+t) - f(u)|}{|t|}.$$

The supremum is finite by the mean value theorem. $\square$

Example 3.3

Take $g \in L^\infty$, $f \in C_1$, and there exists $a < \infty$ such that for all x,

$$|f(x)| + |\nabla f(x)| \le A\langle x\rangle^{-\gamma}.$$

Hence, $f, \nabla f \in L^1$. Then $f * g \in C^1, \nabla(f*g) = (\nabla f) * g$.

We can iterate this: Under appropriate conditions

$$\frac{\partial^\alpha (f*g)}{\partial x^\alpha} = \frac{\partial^\alpha f}{\partial x^\alpha} * g,$$

$$\frac{\partial^{\alpha+\beta} (f*g)}{\partial x^{\alpha_\beta}} = \frac{\partial^\alpha f}{\partial x^\alpha} * \frac{\partial^\beta g}{\partial x^\beta}.$$

Proposition 3.4

If $f \in L^1$ and $g \in L^\infty$, then $f * g \in C^0_b$.

Proof. Recall: If $f \in L^1(\mathbb{R}^d)$, then $y \mapsto \tau_y f \in L^1$ is continuous: As $y \to 0$,

$$\|\tau_y f - f\|_1 \to 0.$$

Then,

$$(f*g)(x) - (f*g)(x') = \int (f(x-y) - f(x'-y))g(y)dy = \int [f(x-y) - (\tau_u f)(x-y)]g(y)dy,$$

where $u = x' - x$. As $u \to 0$, $\|f - \tau_u f\|_1 \to 0$, and $g \in Ł^\infty$, so the integral approaches 0, as desired. $\square$

§3.3 Approximation

Definition 3.5 (Approximate Identity Sequence)**.** An approximate identity sequence for $\mathbb{R}^d$ is $(\varphi_n)_{n\in\mathbb{N}}, \varphi_n \in L^1(\mathbb{R}^d)$ with the following conditions:

- $\int_{\mathbb{R}^d} \varphi_n = 1$.
- For all $\delta > 0$, $\int_{|x|\ge\delta} |\varphi_n| dx \to 0$ as $n \to \infty$.

Theorem 4

Let (φ_n) be an approximate identity sequence in $\mathbb{R}^d$.

1. Let $f \in C_b^0$ be uniformly continuous. Then $f * \varphi_n \to f$ uniformly.
2. Let $f \in C_b^0$. Then $f * \varphi_n \to f$ uniformly on every compact set.
3. If $1 \leq p \leq \infty$, then for all $f \in L^p$, $\|f * \varphi_n - f\|_p \to 0$.

[All the above limits are taken for $n \to \infty$.]

Proof.

$$\begin{aligned} f * \varphi_n(x) - f(x) &= \int f(x-y)\varphi_n(y)dy - f(x) \\ &= \int (f(x-y) - f(x))\varphi_n(y)dy \end{aligned}$$

Then,

$$|f * \varphi_n(x) - f(x)| \leq \int |f(x-y) - f(x)||\varphi_n(y)|dy.$$

Let $\delta > 0$. Then,

$$\int |f(x-y)-f(x)||\varphi_n(y)|dy = \int_{|y \leq \delta|} |f(x-y)-f(x)||\varphi_n(y)|dy + \int_{|y \geq \delta|} |f(x-y)-f(x)||\varphi_n(y)|dy.$$

$$\begin{aligned} \int_{|y \leq \delta|} |f(x-y) - f(x)||\varphi_n(y)|dy &\leq \|\varphi_n\|_1 \cdot \sup_{x, |y| \leq \delta} |f(x-y) - f(x)| \\ &= \|\varphi_n\|_1 \cdot \omega_f(\delta) \\ &\leq A \cdot \omega_f(\delta). \end{aligned}$$

Then

$$\begin{aligned} \int_{|y \geq \delta|} |f(x-y) - f(x)||\varphi_n(y)|dy &\leq \int_{|y| \geq \delta} 2\|f\|_{C^0} \cdot |\varphi_n(y)|dy \\ &\leq 2\|f\|_{C^0} \int_{|y| \geq \delta} |\varphi_n|dy. \end{aligned}$$

Hence

$$|f * \varphi_n(x) - f(y)| \leq A\omega_f(\delta) + 2\|f\|_{C^0} \int_{|y| \geq \delta} |\varphi_n|dy.$$

Taking the lim sup, the second term goes to 0, so for all $\delta > 0$,

$$\lim_{n \to \infty} \sup \|f * \varphi_n - f\|_{C^0} \leq A\omega_f(\delta).$$

Since f is uniformly continuous, $\lim_{\delta \to 0} \omega_f(\delta) = 0$, which proves the claim. $\square$

Corollary 3.6

$C^\infty \cap L^p$ is dense in L^p for all $1 \le p \le \infty$.

Proof. We want to construct (φ_n) with $\varphi_n \in C_0^\infty$.

We claim there exists a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\int \varphi = 1$ and $\varphi \ge 0$. In $d = 1$, take $h(x) = 1x > 0e^{-\|x\|}$. Then, define $\varphi(x) = h(x)h(1-x) \in C_0^\infty$. Then, we normalize φ.

Now, take $\varphi_n(x) = n^d \varphi(nx)$. □

Example 3.7

Let $\varphi \ge 0$, $\int \varphi = 1$. Define $\varphi_n(x) = n^d\varphi(nx)$. Then $\int \varphi_n = 1$.

Furthermore,

$$\int_{|x|\ge\delta} n^d\varphi(nx)dx = \int_{|y|\ge n\delta} \varphi(y)dy \to 0.$$

Example 3.8

Let $\varphi(x) = (2\pi)^{-d/2}e^{-|x^2|/2}, x \in \mathbb{R}^d$. Let $t > 0$ and $\varphi_t(x) = (2\pi)^{-d/2}t^{-d/2}e^{-|x|^2/(2t)}$. Now $t \to 0^+$ and

$$\int_{|x|\ge\delta} \varphi_t(x)dx \to 0.$$

This is an approximate identity family.

Example 3.9 (Interpretation of $f * g$)

$$f * g = \int \tau_y f(x) \cdot g(y)dy.$$

If $g \ge 0$ and $\int g = 1$, then we have an **average** of translates of f.

As $n \to \infty$, $g = \varphi_n$ so the weight concentrates asymptotically at the origin.