Convex Hulls

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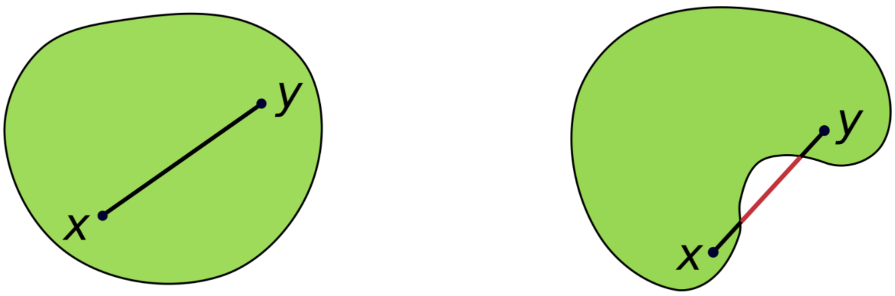
The following is the first of a series of handouts in combinatorial geometry. Convex hulls are a useful tool for understanding finite set systems. The following is an exposition with some worked examples. Any typos or mistakes found are my own - kindly direct them to my inbox.

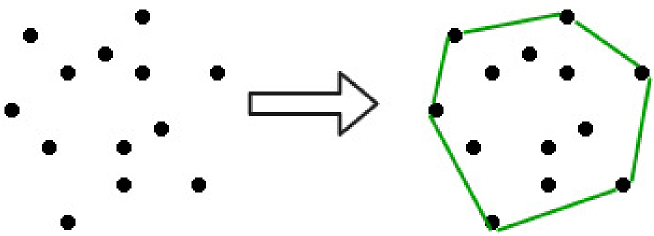
§1 Definitions



Definition 1.1. A set $C \subseteq \mathbb{R}^n$ is **convex** if for any two points $x, y \in C$, the line segment between x and y is contained in C. In other words, for $x, y \in C$, $\lambda \in [0, 1]$

$$\lambda x + (1 - \lambda)y \in C.$$

Definition 1.2. Given a subset S of the plane, the convex hull (denoted conv(S)) is the smallest convex set containing S.



§2 Examples

Problem 1 (Happy-Ending Problem)

Suppose we have five points in the plane with no three collinear. Show that we can find four points whose convex hull is a quadrilateral.

Proof. Take the convex hull of the five points. If it is a quadrilateral or pentagon, we are done(choose any 4 points in the latter case). Suppose the convex hull is a triangle. Label the points with A through E and without loss of generality, let the points A, B, C form the triangle and D, E, be the points inside the hull.

Extend the line DE. Note that two points must lie on one side of the line - if not then we have three collinear points. It is easy to show that these four points form a convex quadrilateral. □

Problem 2

There are $n > 3$ coplanar points, no three collinear and every four of them are the vertices of a convex quadrilateral. Prove that the n points are the vertices of a convex n-sided polygon.

Proof. Suppose that some point P is inside the convex hull of the n points. Let Q be some vertex of the convex hull. The diagonals from Q to the other vertices divide the convex hull into triangles and since no three points are collinear, P must lie inside some triangle $\triangle QRS$. But this is a contradiction since P, Q, R, S do not form a convex quadrilateral. □

Problem 3 (1985 IMO Longlist)

Let A, B be finite disjoint sets of points in the plane such that any three distinct points in $A \cup B$ are not collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices are contained in A or in B that does not contain in its interior any point from the other set.

Proof. Suppose A has at least five points. Take A_1A_2 on the boundary of the convex hull of A. For any other $A_i \in A$, define $\theta_i = \angle A_1A_2A_i$. Without loss of generality, $\theta_3 < \theta_4 < \cdots < 180°$. It follows that $\text{conv}(\{A_1, A_2, A_3, A_4, A_5\})$ contains no other points of A. □

§3 Problems

Problem 4 (Putnam 2001 B6)

Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-i} + a_{n+i} < 2a_n$$

for $i = 1, \ldots, n-1$?

Proof. We claim that such a subsequence exists. Let A be the convex hull of the points (n, a_n) for $n \geq 0$. We show that there are infinitely many points on the boundary of A which satisfy the condition.

If (n, a_n) is on the boundary of A, then there is a line through (n, a_n) with all other points of A below the line. In other words, there exists a slope $m > 0$ such that

$$a_k < a_n + m(k-n)$$

for all $k \geq 1$.

For $n = 1$, since $\frac{a_k}{k} \to 0$, it follows that

$$\frac{a_k - a_1}{k-1} \to 0,$$

so the sequence $\{(a_k - a_1)/(k-1)\}$ is bounded. If we let m be the upper bound, then

$$a_k \leq a_1 + m(k-1).$$

Now, suppose there is some n satisfying the condition. Since $\frac{a_k}{k} \to 0$ it follows that

$$\frac{a_k - a_n}{k-n} \to 0,$$

and $\{(a_k - a_n)/(k-n)\}$ is bounded, which implies that it has a maximum element. If we let $k = r$ be the largest value where the maximum is attained, set $m = \frac{a_r - a_n}{r-n}$. The line through (r, a_r) of slope m lies above (k, a_k) for $k > r$ and passes through or lies above (k, a_k) for $k < r$. It follows that the condition holds for $n = r$ with m replaced by $m - \epsilon$ for some small $\epsilon > 0$.

Hence, there are infinitely points on the boundary of A. For each n on the boundary, there exists $m > 0$ such that for $i = 1, \ldots, n-1$, the points $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ lie below the line through (n, a_n) of slope m. It follows that $a_{n+i} < a_n + mi$ and $a_{n-i} < a_n - m_i$, so $a_{n-i} + a_{n+i} < 2a_n$, as desired. $\square$