Math 214

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Contents

§1 Topology

Definition 1.1 (Topological Space)**.** $(X, O_X \subset \mathcal{P}(X))$, where $A \in O_x$ are the open sets which satisfy the following:

1. $\emptyset, X \in O_X$.
2. $A, B \in O_X$ implies $A \cap B \in O_X$
3. $A_i \in O_X$, $i \in I$, then $\bigcup_{i \in I} A_i \in O_X$.

We say that $A \subset X$ is closed if $X \setminus A$ is open. $U \subset X$ is a neighborhood of $p \in X$ if $\exists A$ such that $p \in A \subset U$.

Definition 1.2. $\mathcal{B} \subset \mathcal{P}(X)$ is called a **basis** for the topology on X if for every subset $A \subset X$, A is open if and only if A is a union of elements of $\mathcal{B}$.

Let (X, O_X), (Y, O_Y) be topological spaces.

Definition 1.3. A function $\varphi : X \to Y$ is continuous if for any open subset $B \subset Y$, $\varphi^{-1}(B) \subset X$ is open.

Definition 1.4. $\varphi : X \to Y$ is a homeomorphism if it is a continuous bijection whose inverse is continuous.

Definition 1.5. Let $Y \subset X$ a topological space. We set $O_Y = \{A \cap Y : A \in O_X\}$.

Definition 1.6. Given a topological space X, X is called locally Euclidean (of dimension n) at $p \in X$ if there is an open neighborhood about $p \in U \subset X$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 1.7. A space X is **Hausdorff** if for any $p, q \in X$, $p \neq q$ there exists open subsets U, V with $p \in U$, $q \in V$ so that $U \cap V = \emptyset$.

Definition 1.8. $K \subset X$ is compact if every open cover of K has a finite subcover.

Given a topological space X, we have the following definitions:

Definition 1.9. X is connected if the only subsets that are open and closed are $\emptyset, X$.

Definition 1.10. A space is path-connected if for any $p, q \in X$ there is a continuous path between them.

Definition 1.11. An exhaustion by compact subsets is an increasing sequence of subsets $K_1 \subset K_2 \subset \cdots \subset X$ such that K_i is compact and $K_i \subset \mathrm{Int}(K_{i+1})$ and $\bigcup_i K_i = X$.

Definition 1.12. Take $\mathcal{U} \subset \mathcal{P}(X)$. This is a cover of X if $X = \bigcup_{U \in \mathcal{U}} U$. A collection is called locally finite if every $p \in X$ has a neighborhood $p \in W \subset X$ such that W only intersects finitely many $U \in \mathcal{U}$.

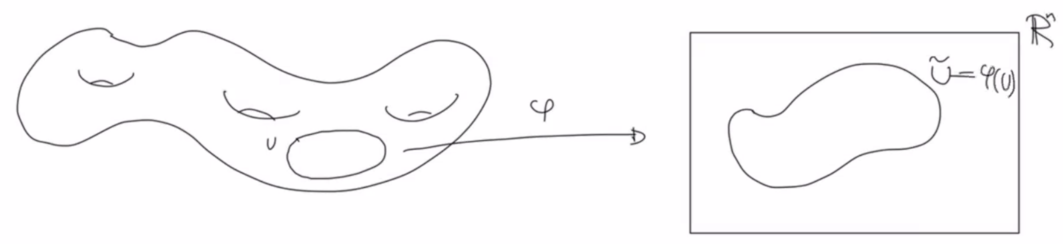
Definition 1.13. A collection of subsets $\mathcal{V}$ is called a refinement of some other collection $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.

§2 Topological Manifolds

Definition 2.1. A topological space M is called an n-dimensional **topological manifold** if M satisfies the following:

- M is locally Euclidean at any point,
- M is Hausdorff,
- M is second countable.

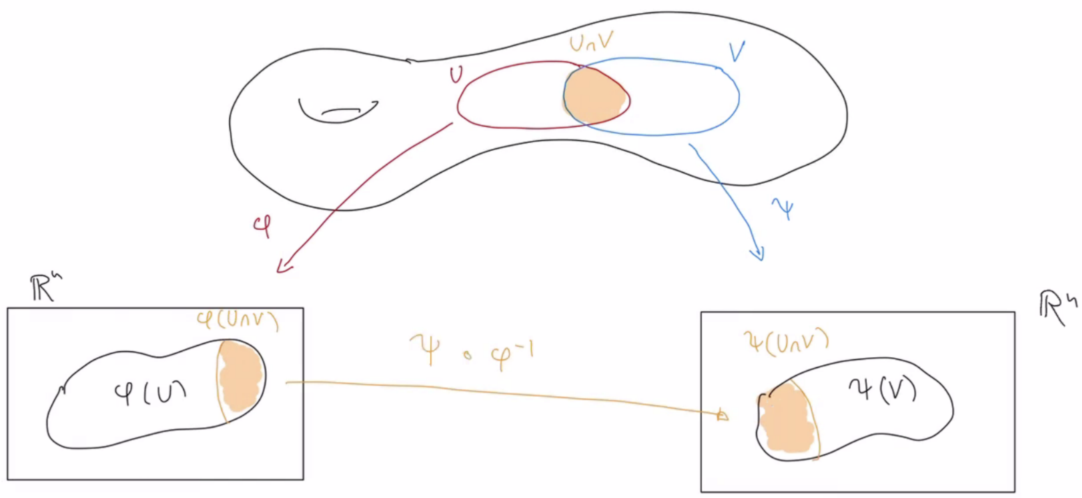
Definition 2.2. A **coordinate chart** on M is a pair (U, φ) where $U \subset M$ is open and $\varphi : U \to \widehat{U}$ is a homeomorphism to an open subset $\widehat{U} \subset \mathbb{R}^n$.



Definition 2.3. X is called paracompact if every open cover has a locally finite refinement.

§3 Smooth Structures

Definition 3.1. Let M^n be a topological manifold. Two charts $(U, \varphi), (V, \psi)$ of M have a transition map: $\psi \circ \varphi^{-1}$. This map is a homeomorphism.



Definition 3.2. Two charts are smoothly compatible if the transition maps in both directions are smooth.

Definition 3.3. An atlas $\mathcal{A}$ of M is a collection of charts such that the domains of the charts cover M. An atlas $\mathcal{A}$ is smooth if any two charts in $\mathcal{A}$ are smoothly compatible. An atlas $\mathcal{A}$ is called a maximal smooth atlas on M if there is no smooth atlas containing $\mathcal{A}'$ that contains $\mathcal{A}$.

Definition 3.4. A maximal smooth atlas $\mathcal{A}$ on a topological manifold M is called a smooth structure on M.

Definition 3.5. A smooth manifold is a pair $(M^n, \mathcal{A})$, where M^n is a topological manifold and $\mathcal{A}$ is a smooth structure.

§4 Manifolds with Boundary

Definition 4.1. We denote $H^n = \{x^n \geq 0\} \subset \mathbb{R}^n$, the upper half space, the most basic example. Note that $\partial H^n = \{x^n = 0\} \cong \mathbb{R}^{n-1}$. The interior $\text{Int } H^n = \{x^n > 0\}$.

Definition 4.2. A topological manifold with boundary M^n is a topological space such that is Hausdorff, second countable, and every point $p \in H^n$ has an open neighborhood $p \in U \subset M$ that is homeomorphic to some (relatively) open subset $\widehat{U} \subset H$.

§5 Smooth Maps

Definition 5.1. $f : M \to \mathbb{R}^m$ is smooth if for every $p \in M$, there is a smooth chart (U, φ), $\widehat{U} = \varphi(U)$ such that $p \in U$ and $\widehat{f} = f \circ \varphi^{-1} : \widehat{U} \to \mathbb{R}^n$ is smooth. We denote $C^\infty(M) : \{f : M \to \mathbb{R}^m \text{ smooth}\}$.

Definition 5.2. Suppose we have M^m, N^n smooth manifolds (with boundary) and take $F : M \to N$. F is called smooth if for any $p \in M$ there are smooth charts (U, φ) of M and (V, ψ) of N such that $p \in U$, $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1}$ is smooth.

Other equivalent definitions:

- For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.
- F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N respectively so that for each α, β $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Definition 5.3. Let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an open cover of some topological space X. A partition of unity subordinate to $\mathcal{X}$ is a family $(\psi_\alpha)_{\alpha \in A}$ of continuous maps on $\psi_\alpha : X \to \mathbb{R}$ such that $0 \leq \psi_\alpha \leq 1$, $\text{supp } \psi_\alpha \subset X_\alpha$, $(\text{supp } \psi_\alpha)_{\alpha \in A}$ is locally finite, and $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in X$.

Definition 5.4. An open subset $B \subset m$ is called a regular coordinate ball if there is a smooth chart (U, φ) such that $\varphi(U) = B_{r'}(0)$, $\varphi(B) = B_r(0)$ where $0 < r < r'$.

Definition 5.5. If M is a topological space, $A \subset M$ is a closed subset, and $U \subset M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a bump function for A supported in U if $0 \leq \psi \leq 1$ on M and $\psi \equiv 1$ on A, $\text{supp } \psi \subset U$.

§6 Tangent Vectors

Definition 6.1. For $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : C_\infty(\mathbb{R}^n) \to \mathbb{R}$ is defined by

$$D_v|_a f = D_v f(a) = \frac{d}{dt}\bigg|_{t=0} f(a+tv).$$

Definition 6.2. If $a \in \mathbb{R}^n$, $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a derivation if it is linear over $\mathbb{R}$ and

$$w(fg) = f(a)wg + g(a)wf.$$

$T_a\mathbb{R}^n$ denotes the set of derivations you gotat a.

Definition 6.3. If $p \in M$, $v : C^\infty(M) \to \mathbb{R}$ is called a derivation at p if it is linear and

$$v(fg) = f(p)vg + g(p)vf.$$

T_pM denotes the set of derivations at p, called the Tangent Space to M at p.

Definition 6.4. If $F : M \to N$ is a smooth map, for each $p \in M$, we define $dF_p : T_pM \to T_{F(p)}N$, the differential of F at p as follows: Given $v \in T_pM$, $dF_p(v)$ is the derivation at $F(p)$ acting on $f \in C^\infty(N)$ by

$$dF_p(v)(f) = v(f \circ F).$$

Some properties:

- dF_p is a derivation at $F(p)$.
- $dF_p : T_pM \to T_{F(p)}M$ is linear.
- If $F : M \to N$, $G : N \to P$ smooth, $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
- $d(id_M)_p = id_{T_pM}$.
- If F is a diffeomorphism, dF_p is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Definition 6.5. The tangent bundle $TM = \bigsqcup_{p\in M} T_pM$. We have a map $\pi : TM \to M$ given by $v \in T_pM \mapsto p$.

§7 Rank

Definition 7.1 (Rank)**.** Suppose M and N are smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$, we define the rank of F at p to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$; i. e. the rank of the Jacobian matrix of F in any smooth chart, or the dimension of $\operatorname{Im} dF_p \subset T_{F(p)}N$. If F has the same rank at every point , we say that it has constant rank.

Definition 7.2 (Local Diffeomorphism)**.** If M and N are smooth manifolds with or without boundary, a map $F : M \to N$ is called a local diffeomorphism if every point $P \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Definition 7.3 (Submersion)**.** A smooth map $F : M \to N$ is called a smooth submersion if $\operatorname{rank} F = \dim N$; dF_p is surjective for all $p \in M$.

Definition 7.4 (Immersion)**.** A smooth map $F : M \to N$ is called a smooth immersion if $\operatorname{rank} F = \dim M$; dF_p is injective for all $p \in M$.

Theorem 7.5

If dF_p has full rank, then there exists a neighborhood $p \in U$ such that $F|_U$ has full rank.

Theorem 7.6 (Rank Theorem)

Let M^m, N^n smooth manifolds, $p \in M$. Assume $F : M \to N$ smooth, constant rank r. Then, there are smooth charts (U, φ) of M and (V, psi) of N such that $p \in U$, $F(p) \subset V$ and such that F has a coordinate representation of the form

$$\widehat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

§8 Embeddings

§9 Submanifolds

§9.1 Level Sets

§9.2 Immersed Submanifolds

§9.3 Tangent Space of Submanifolds

§10 Sard's Theorem

§11 Whitney Embedding Theorem

§12 Whitney Approximation Theorems

§13 Transversality

Definition 13.1. Suppose M is a smooth manifold. Two embedded submanifolds $S, S' \subset M$ are said to intersect transversely if for each $p \in S \cap S'$, the tangent spaces T_pS and T_pS' together span T_pM.

Definition 13.2. If $F : N \to M$ is a smooth map and $S \subset M$ is an embedded submanifold, we say that F is transverse to S if for every $x \in F^{-1}(S)$, the spaces $T_{F(x)}S$ and $dF_x(T_xN)$ together span $T_{F(x)}M$.

> **Remark 13.3.** If F is a smooth submersion, then it is automatically tranverse to every embedded submanifold of M.

Theorem 13.4

Suppose N and M are smooth manifolds and $S \subset M$ is an embedded submanifold.

- If $F : N \to M$ is a smooth map that is tranverse to S, then $F^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S in M.
- If $S' \subset M$ is an embedded submanifold that intersects S transversely, then $S \cap S'$ is an embedded submanifold of M whose codimension is equal to the sum of the codimensions of S and S'.

Theorem 13.5 (Parameteric Transversality Theorem)

Suppose N and M are smooth manifolds, $X \subset M$ is an embedded submanifold, and $\{F_s : s \in S\}$ is a smooth family of maps from N to M. If the map $F : N \times S \to M$ is transverse to X, then for almost every $s \in S$, the map $F_s : N \to M$ is transverse to X.