Chapter 1: Preliminaries

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My solutions to the problems and exercises from the first chapter of Stein/Shakarchi, *Complex Analysis*, "Preliminaries to Complex Analysis". Any typos or errors found are my own - kindly direct any concerns to my inbox.

Contents

§1 Chapter 1:

§1.1 Exercise 1

Describe geometrically the sets of points z in the complex plane defined by the following relations:

- $|z - z_1| = |z - z_2|$, where $z_1, z_2 \in \mathbb{C}$.
- $1/z = \overline{z}$.
- $\mathrm{Re}(z) = 3$.
- $\mathrm{Re}(z) > c$ where $c \in \mathbb{R}$.
- $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
- $|z| = \mathrm{Re}(z) + 1$.
- $\mathrm{Im}(z) = c$.

Proof.
- This describes the perpendicular bisector between z_1, z_2, the set of points that are equidistant from both points.
- Equivalently, $|z| = 1$, the circle of radius 1.
- A vertical line through 3.
- The half plane to the right of c(excluding the boundary).
- This the half plane below a given line from the components of a, b.
- A horizontal parabola with vertex at $-i$.
- A horizontal line through c.

□

§1.2 Exercise 2

Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^2$. We have a Hermitian inner product in $\mathbb{C}$ by $(z, w) = z\overline{w}$. Show that

$$\langle z, w \rangle = \frac{1}{2}\left((z, w) + (w, z)\right) = \mathrm{Re}(z, w).$$

Proof. Let $z = a + bi, w = c + di$.

$$\begin{aligned}(z, w) + (w, z) &= z\overline{w} + w\overline{z} \\ &= (a + bi)(c - di) + (a - bi)(c + di) \\ &= (ac + bd) + (bc - ad)i + (ac + bd) + (bc - ad)i \qquad = 2(ac + bd) \\ &= 2\langle z, w \rangle = 2\mathrm{Re}(z, w).\end{aligned}$$

□

§1.3 Exercise 3

With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where $n \in \mathbb{N}$. How many solutions are there?

Proof. We have

$$z^n = \omega \implies z = (\omega)^{1/n} e^{\frac{2\pi i m}{n}},$$

where $m \in \mathbb{Z}/n\mathbb{Z}$.

$$(\omega^{1/n})^n = se^{i\varphi} \implies \omega^{1/n} = s^{1/n} e^{i\varphi/n + \frac{2\pi i k}{n}},$$

where $k \in \mathbb{Z}/n\mathbb{Z}$. It follows that

$$z = s^{1/n} e^{i\varphi/n} e^{\frac{2\pi i(k+m)}{n}}$$

and $k+m$ is uniformly distributed in $\mathbb{Z}/n\mathbb{Z}$, so we have n possible solutions if $s \neq 0$. Otherwise, we have one solution, namely 0. $\square$

§1.4 Exercise 4

Show that it is impossible to define a total ordering on $\mathbb{C}$.

Proof. Suppose $i \succ 0$. Then, we have $i^2 = -1 \succ 0$, which is impossible. Similarly, if $0 \succ i$, then

$$0 + (-i) = -i \succ i + (-i) = 0.$$

But then,

$$0 \cdot (-i) = 0 \succ i \cdot (-i) = 1,$$

a contradiction.

$\square$

§1.5 Exercise 5

A set Ω is said to be **pathwise connected** if any two points in Ω can be joined by a curve entirely contained in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

(a) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1, w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parameterization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$.

Let

$$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \forall 0 \leq s < t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$.

(b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

Proof. (Part A) Consider the point $z(t^*)$. Suppose $t^* < 1$. We cannot have $z(t^*) \in \Omega_1$, since this implies there is an open ball B containing $z(t^*)$ in Ω_1. It follows that $z^{-1}(B)$ is an open subset of $[0, 1]$ since z is continuous, so contains points to the right of t^*, a contradiction. If $t^* = 1$, then, there is a sequence of points in Ω_1 converging to $z(1) \in \Omega_2$, contradicting the assumption that $\Omega \setminus \Omega_2$ is closed.

If $z(t^*) \in \Omega_2$, then $z(t) \in \Omega_2$ if and only if $t > t^*$. Hence, t^* is the infumum of values of t with $z(t) \in \Omega_2$ and we repeat the argument from above. □

Proof. (Part B) It is clear that $\Omega_1 \cup \Omega_2 = \Omega$ and that they are disjoint.

Since Ω is open, we can find an open ball B around $v \in \Omega_1 \subset \Omega$ which is contained in Ω. If $x \in B$ then there is a path from v to x. There is also a path from w to v so by the gluing lemma, we can find a path from w to x. This implies that $B \subset \Omega_1$, which shows that Ω_1 is open.

Take $y \in \Omega_2$. There exists an open ball C around y contained in Ω. For any $t \in C$, if a path exists from w to C, then we can find a path from w to y by the gluing lemma. It follows that $C \subset \Omega_2$ which shows that Ω_2 is open.

Since Ω is connected, we must have either $\Omega = \Omega_1$ or $\Omega = \Omega_2$. However, $w \in \Omega_1$ implies that $\Omega = \Omega_1$. □

§1.6 Exercise 6

Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The connected component of Ω containing z is the set of points $\mathcal{C}_z$ of all points w in Ω that can be joined to a curve entirely contained in Ω.

(a) Check that $\mathcal{C}_z$ is open and connected. Then, show that $w \in C_z$ defines an equivalence relation.

(b) Show that Ω can have only countably many distinct connected components.

(c) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

Proof. (Part A) Note that C_z is a pathwise connected set which is open if and only if it is connected. For any $x \in C_z$, there exists a ball $B \subset \Omega$ containing x. Then, gluing the path from z to x and x to $y \in B$ shows that $B \subset C_z$, which implies that C_z is open. By Exercise 5, C_z is connected.

We now show that $w \in C_z$ is an equivalence relation. It is clear that $z \in C_z$. If $w \in C_z$, then there is a path from z to w, the reverse of which is a path from w to z, so $z \in C_w$. Finally, If $a \in B_b$ and $b \in B_c$, then gluing the paths from a to b and b to c gives a path from a to c, so $a \in B_c$. □

Proof. (Part B) Suppose not. Then there is an uncountable collection of disjoint open balls of Ω. From the density of $\mathbb{Q}$ in $\mathbb{R}$, each of these balls contains a unique rational, which is a contradiction since there are only countably many rationals. □

Proof. (Part C) Let $C \subset \Omega$ be the unbounded component. Since $\overline{\Omega}$ is compact, there exists an open ball $B \supset \overline{\Omega}$. Then $B^c \subset \Omega$, and note that B^c is unbounded and connected. Since $C \cap B = \emptyset$, we must have $C \cap B^c \neq \emptyset$ so it follows that $B^c = C$. Hence, we have exactly one unbounded component. □

§1.7 Exercise 7

We introduce mappings called **Blaschke factors**.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$ and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

if $|z| = 1$ or $|w| = 1$.

(b) Prove that for a fixed w in the unit disc $\mathcal{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following conditions

- F maps the unit disk to itself and is holomorphic.
- F interchanges 0 and w.
- $|F(z)| = 1$ if $|z| = 1$.
- $F : \mathcal{D} \to \mathcal{D}$ is bijective.

Proof. (Part A) We have

$$\begin{aligned}
\left|\frac{w-z}{1-\overline{w}z}\right| < 1 &\Leftrightarrow (w-z)\overline{(w-z)} \leq (1-\overline{w}z)\overline{(1-\overline{w}z)} \\
&\Leftrightarrow (w-z)(\overline{w}-\overline{z}) \leq (1-z\overline{w})(1-w\overline{z}) \\
&\Leftrightarrow |w|^2 + |z|^2 - z\overline{w} - w\overline{z} \leq 1 - z\overline{w} - w\overline{z} + |z|^2|w|^2 \\
&\Leftrightarrow (1-|w|^2)(1-|z|^2) \geq 0,
\end{aligned}$$

which gives both results.

□

Proof. (Part B) From the result above if $|z| \leq 1$ then since $|w| \leq 1$, we have $|F(z)| \leq 1$, which implies that $F(\mathcal{D}) \subset \mathcal{D}$. Then, for any $y \in \mathcal{D}$, note that $F(F(y)) = y$(this can be easily verified), so it follows that $\mathcal{D} \subset F(\mathcal{D})$, which shows that F maps $\mathcal{D}$ to $\mathcal{D}$, as desired. The function is holomorphic by the quotient rule. It is easy to see that $F(0) = w$ and $F(w) = 0$. Then, $|F(z)| = 1$ if $|z| = 1$ by Part A. From $F(F(y)) = y$ it is clear that F is surjective. Then, if $F(x) = y$ then $x = F(F(x)) = F(y)$, so $x = y$ implies that $F(x) = F(y)$. Therefore, F is bijective, as desired. □

§1.8 Exercise 8

Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable(in the real sense), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

Proof. Recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x}$$

and

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial y}$$

so it follows that

$$\frac{\partial h}{\partial z} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial z}$$

□

§1.9 Exercise 9

Show that in polar coordinates, the Cauchy-Riemann Equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta$$

where $z = re^{i\theta}$ with $-\pi < \theta < \pi$ is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. If we let $z = x + iy = r(\cos\theta + i\sin\theta)$, it follows that

$$\begin{aligned} u_r &= u_x\cos\theta + u_y\sin\theta \\ &= \frac{1}{r}\left(rv_y\cos\theta - rv_x\sin\theta\right) \\ &= \frac{1}{r}v_\theta. \end{aligned}$$

Similarly,

$$\begin{aligned} v_r &= v_x\cos\theta + v_y\sin\theta \\ &= -\frac{1}{r}\left(ru_y\cos\theta - ru_x\sin\theta\right) \\ &= -\frac{1}{r}u_\theta. \end{aligned}$$

For $\log z = \log r + i\theta$, we have $u(r,\theta) = \log r$ and $v(r,\theta) = \theta$, so

$$u_r = \frac{1}{r} = \frac{1}{r}v_\theta$$

and

$$v_r = 0 = -\frac{1}{r}u_\theta$$

so it follows that the function is holomorphic as desired. □

§1.10 Exercise 10

Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. Trivially follows from the definitions of the operators. □

§1.11 Exercise 11

Show that if f is holomorhpic in the open set Ω, then the real and imaginary parts of f are harmonic.

Proof. If f is holomorphic on Ω then $\frac{\partial}{\partial \bar{z}} f = 0$, and the result follows from applying the previous exercise. □

§1.12 Exercise 12

Consider the function defined by $f(x+iy) = \sqrt{|x||y|}$ whenever $x, y \in \mathbb{R}$. Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Proof. Note that $v_x, v_y = 0$, since f is a real-valued function.

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - h(0,0)}{h} = 0,$$

and

$$u_y(0,0) = \lim_{h \to 0} \frac{u(0,h) - h(0,0)}{h} = 0$$

but

$$\frac{f(t(1+i)) - f(0)}{t(1+i)} = \frac{|t|}{t(1+i)},$$

which has no limit. □

§1.13 Exercise 13

Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases

- $\mathrm{Re}(f)$ is constant;
- $\mathrm{Im}(f)$ is constant;
- $|f|$ is constant;

one can conclude that f is constant.

Proof. The first and second cases are equivalent, following from the C-R equations. It follows that $Re(f) + iIm(f) = f$ is constant in these cases.

Let $f = u + iv$. If $|f|$ is constant, then $|f|^2 = u^2 + v^2$ is constant, so it follows that

$$\frac{\partial}{\partial x}(u^2 + v^2) = \frac{\partial}{\partial y}(u^2 + v^2) = 0.$$

Therefore,

$$uu_x + vv_x = uu_y + vv_y = 0.$$

By C-R, we have

$$uv_y + vv_x = -uv_x - vv_y = 0.$$

□