Math 212, Lecture Notes

Several Complex Variables

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§1 Lecture 1: 8/26/2021

§1.1 Review of 1D Complex Analysis

Definition 1.1 (Holomorphic)**.** Let $D \subset \mathbb{C}$ be an open connected domain and take $u \in C^1(D)$. The function u is **holomorphic** if $\partial_{\overline{z}} u = 0$ where $\partial_{\overline{z}} = (\partial_x + i\partial_y)$.

We also have the equivalent conditions that

$$u \in \mathrm{Hol}(D) \Leftrightarrow \partial_{\overline{z}} = 0 \Leftrightarrow \lim_{h \to 0} \frac{u(z+h) - u(z)}{h} \text{ exists and is continuous.}$$

Fact 1.2 (Green's Theorem)**.** For $\Omega \subset \mathbb{C}, \partial\Omega \in C^1$, we have

$$\int_{\partial\Omega} u \, dz = \iint_{\Omega} \partial_{\overline{z}} u \, d\overline{z} \wedge dz.$$

Theorem 1.3 (Cauchy-Pompieu Formula)
Let $u \in C^1(\overline{\Omega})$. For all $\zeta \in \Omega$,

$$u(\zeta) = \frac{1}{2\pi i} \left(\int_{\partial\Omega} \frac{u(z)}{z - \zeta} \, dz + \iint_{\Omega} \frac{\partial_{\overline{z}} u(z)}{z - \zeta} \, dz \wedge d\overline{z} \right)$$

Proof. Let $\Omega_\varepsilon = \Omega \setminus \overline{D(\zeta, \epsilon)}$, where $0 < \epsilon << 1$. Applying Green's Theorem to $w(z) = \frac{u(z)}{z-\zeta} \in C^1(\overline{\Omega_\varepsilon})$ and noting that $\partial_{\overline{z}} w = \frac{\partial_{\overline{z}} u(z)}{z-\zeta}$, we have

$$\iint_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} u(z)}{z - \zeta} d\overline{z} \wedge dz = \int_{\partial\Omega} \frac{u(z)}{z - \zeta} \, dz - \int_{\partial D(\zeta, \epsilon)} \frac{u(z)}{z - \zeta} \, dz.$$

The left-hand side converges to $\iint_\Omega \frac{\partial_{\overline{z}} u(z)}{z-\zeta} d\overline{z} \wedge dz$ by the dominated convergence theorem. Parameterizing the disc via polar coordinates, we can write

$$\int_{\partial D(\zeta,\epsilon)} \frac{u(z)}{z - \zeta} \, dz = \int_0^{2\pi} u(\zeta + \epsilon e^{i\theta}) \, d\theta \to 2\pi i u(\zeta).$$

The desired formula follows from rearranging the terms upon taking the limit as $\epsilon \to 0$. $\square$

Remark 1.4. We also have a partial converse: let $\varphi \in C_c^k(\mathbb{C})$ with $k \geq 1$ and $u(z) = \iint \frac{\varphi(z)}{z-\zeta} dz \wedge d\overline{z}$. Then $u \in C^k(\mathbb{C})$ and $\partial_{\overline{z}} u = \varphi$.

Some other notable corollaries that follow from Cauchy's Theorem:

- $u \in \mathrm{Hol}(D) \Rightarrow u \in C^\infty(D)$.
- For all $K \Subset \Omega \Subset D$, k, there exists C such that for all $u \in \mathrm{Hol}(D)$, we have
$$\sup_K |u^{(j)}(z)| \leq C \|u\|_{L^1(\Omega)}.$$
- $u_j \in \mathrm{Hol}(D)$, $u_j \to u$ uniformly on bounded sets, then $u \in \mathrm{Hol}(D)$.

§2 Lecture 2: 8/31/2021

We introduce the notation $u \in \mathcal{O}(D)$ to mean that u is holomorphic. We continue with corollaries following from Cauchy's Theorem:

- Let $\{u_j\} \subset \mathcal{O}(D)$. If for all $K \Subset D$, there exists C such that $|u_j| \leq C$, then there exists $u \in \mathcal{O}(D)$ and a subsequence u_{j_k} such that $u_{j_k} \to u$ uniformly on compact sets.

  Proof. Recall the Arzela-Ascoli Theorem:

  Theorem 2.1 (Arzela-Ascoli)
  Suppose $K \Subset \mathbb{C}$, $\{w_j\} \subset C(K)$ and there exists C such that $|w_j| \leq C$ and equicontinuous: for all $\epsilon > 0$, there exists δ such that for all $z, \zeta \in K$,

  $$\|z - \zeta\| < \delta \Rightarrow \|w_j(z) - w_j(\zeta)\| < \epsilon.$$

  Then, there exists j_k and $w \in C(K)$ such that $w_{j_k} \to w$ in $C(K)$.

  Let $D = \bigcup K_j$, $K_j \subset K_{j+1} \Subset D$. For example, we could take

  $$K_j = \{z \in D : d(zm\mathbb{C} \setminus D) \geq 1/j, |z| \leq j\}.$$

  Then, $|u_j| \leq C_{K_j}$ on K_j so it follows that $|u_j'| \leq C_k'$ on any K_j.

  By Arzela-Ascoli, we have a subsequence $\{u_k^{j+1}\} \subset \{u_k^j\}$ such that $u_{n_k^j} \to u^j$ uniformly on K_j. Then, since $u^{j+1}|_{K_j} = u^j$, and $u^j \to u$ uniformly on compact sets. □

- Maximum Principle: $u \in \mathcal{O}(D(z_0, r))$ and $|u(z)| \leq |u(z_0)|$, $z \in D(z_0, r)$, then u is identically a constant.

  Proof. Suppose $u(z_0) \neq 0$(otherwise the problem is trivial).

  $$\begin{aligned} u(z_0) &= \frac{1}{2\pi i} \int_{\partial D(z_0, \rho), \rho < r} \frac{u(z)}{z - z_0} \, dz \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \, dz \end{aligned}$$

  It follows that we have

  $$0 = \int_0^{2\pi} \left(1 - \frac{u(z_0 + \rho e^{i\theta})}{u(z_0)}\right)$$

  Taking real parts, it follows that

  $$1 = \frac{\operatorname{Re}\{u(z_0 + \rho e^{i\theta})\}\overline{u(z_0)}}{|u(z_0)|^2},$$

  which implies the result. □

- Maximum Principle for Bounded Domain: $\overline{D} \Subset \mathbb{C}$, $u \in \mathcal{O}(D) \cap C(\overline{D})$ then $\max_{\overline{D}} |u|$ is attained on the boundary.

Proof. Suppose $\operatorname{argmax}|u| = z_0$ with $z_0 \in D$. It follows that u is constant in $D(z_0, r)$. We will show later that this implies that u is constant on D. □

- Let $u \in \mathcal{O}(D(0,r))$ then $u(z) = \sum \frac{u^{(n)}(0)}{n!} z^n$ with the series converging uniformly in $\overline{D(0,\rho)}$ for $\rho < r$.
- $u \in \mathcal{O}(D(0,r)), u \not\equiv 0$, then there exists n, $v \in \mathcal{O}(D(0,r))$ with $u(z) = z^n v(z)$, $v(0) \neq 0$.
- If $\sum a_n z^n$ converges in $|z| \leq r$, it is holomorphic on the disc.
- If $u \in \mathcal{O}(D)$ and there exists $z_0 \in D$ such that $u^{(n)}(z_0) = 0$ for all n, then $u \equiv 0$.
- Liouville's Theorem: SUppose $u \in \mathcal{O}(\mathbb{C})$ and $|u(z)| \leq C + C|z|^n$ for all $z \in \mathbb{C}$. Then u is a polynomial of degree at most n.
- Suppose $u \in \mathcal{O}(\mathbb{C})$, $u \in L^p(\mathbb{C}, d\mu)$ for $p \in [1, \infty)$. Then $u \equiv 0$.

§3 Lecture 3: 9/2/2021

We now move to complex variables in $\mathbb{C}^n$.

Definition 3.1. f is complex differentiable at $z_0 \in D$ if there exists $D \in \mathbb{C}$ such that

$$\frac{|f(z+h) - f(z) - Dh|}{|h|} \xrightarrow{h \to 0} 0.$$

Then, f is holomorphic in D if it is complex differentiable at all points in D, denoted $f \in \mathcal{O}(D)$.

An alternative definition is as follows:

Definition 3.2. $f \in C^1(D)$ is holomorphic if $\partial_{\overline{z}_j} f = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})f = 0$ for all j.