HW5: Continuous models 3

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Exercise 1

Let us consider a sample of n random variables R_i with $\mathbb{F}_n$ the distribution function, we say the random variables $S_i = \sum_{j=1}^{i} R_j$ are a *Brownian motion* if they fulfill the following properties:

- Markov property: $\mathbb{E}(S_i|\ S_0, S_1, \ldots, S_{j-1}) = \mathbb{E}(S_i|\ S_{j-1})$
- Martingale property: $\mathbb{E}(S_i|\ S_j) = S_{j-1}$
- Quadratic variation: $\mathbb{E}(\sum_{i=1}^{j}(S_i - S_{i-1})^2) = j$
- Normality: $\lim\limits_{n\to\infty} \mathbb{F}_n \sim N(\mu, \sigma)$
- Continuity

Taking that into account, with a coin tossing game with same profit and loss, we've an expected value of 0 when doing n arbitrary amount of bets. Let's see now which paths do we get when betting $n = 1, 5, 10, 50, 100, 1000$ times:

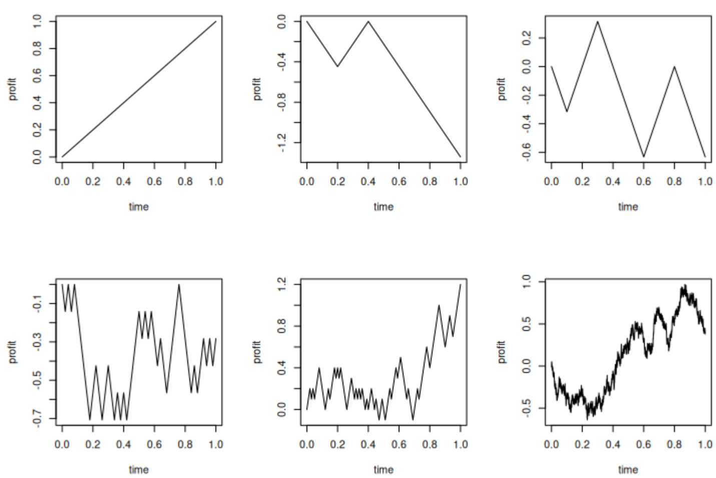


Figure 1: Coin tossing sample path

We've got our expected results, our path is always between -1 and 1 of accumulated profit, not depending on N. Let's see now the distribution of the accumulated profit at the end of the bets for $N = 100, 1000, 10000$:

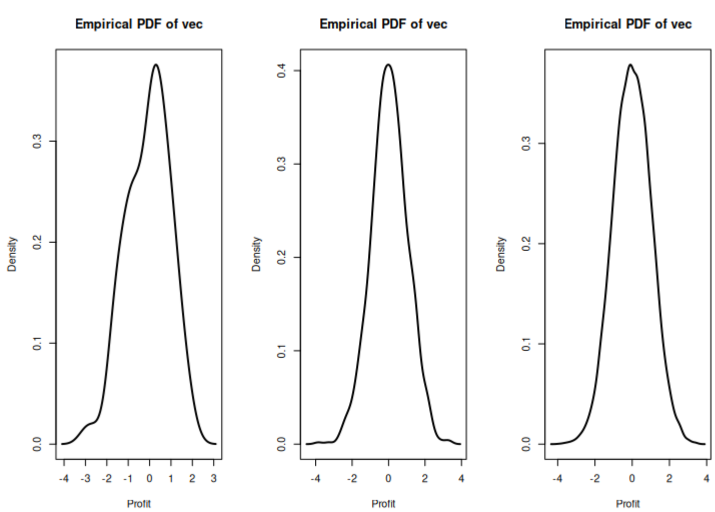


Figure 2: Coin tossing accumulated profit

We, as expected, see that all three are centered in 0 (it's our expected value). Performing normality tests, we see that, most of the times, we are facing a normal distribution. I say most of the times because, as we are playing with small amount of samples, we can get different situations in different rounds.
We conclude that the distribution of the accumulated profit at the end of the round is normally distributed, as we were expecting before checking it. Even more, this distribution can be approximated as the binomial distribution, which (for big N) can be approximated as a normal distribution.

Exercise 2

Given the simple way to price *option* derivatives, we want to complete it using an approximation of the continuous method. For instance, considering the differential model for the stock value evolution over time

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

we know that the pay-off function $V(S_t, t)$ that gives the price of an option derivative, should fulfill the EDP of *Itô Lemma*:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

However, solving it for $V(S,t)$ is indeed complex, feature many EDP share. Commonly one tries to find a numeric answer, and here is no different. Financially, one uses the *Risk-Neutral Probability*, which consists on computing the expectation value of the pay-off function at expiry (T). Therefore

$$V(S_t, t) = e^{-r(T-t)}\mathbb{E}[V(T, S_T)]$$

while giving a numeric value for the latter expectancy. For this exact reason, we proceed to give the *Euler discretization* of the *Black-Scholes model* ($\mu = r$) for the value evolution of stock over time.
In particular, let us consider the general stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

in a time interval $[t_0, t_n]$, the Euler discretization model consists in

$$\begin{cases} X_{t_i+\Delta t_n} = X_{t_i} + a(X_{t_i})\Delta t_n + b(X_{t_i})\sqrt{\Delta t_n}\varepsilon_{t_i} \\ X_{t_0} = X(t_0) \end{cases}$$

Where we have divided the interval $[t_0, t_n]$ in $N \in \mathbb{N}$ equal sub-intervals of size $\Delta t_n = \frac{t_n - t_0}{N}$, each one of them $t_i = i\Delta t_n$. In particular for the Black-Scholes stochastic differential equation

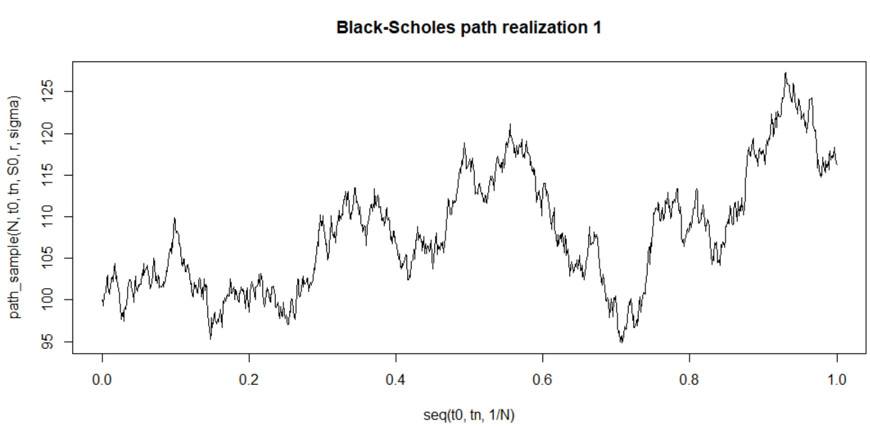
$$dS_t = rS_t dt + \sigma S_t dW_t$$

where r is the annualized risk free rate and σ is the annualized volatility, get

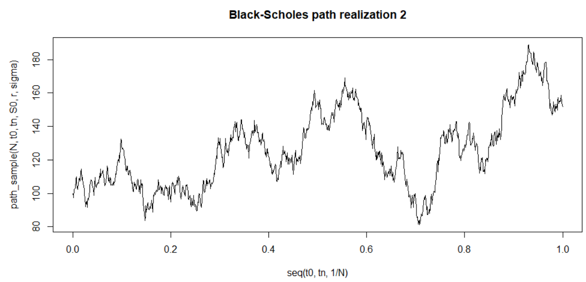
$$\begin{cases} S_{t_i+\Delta t_n} = S_{t_i} + rS_{t_i}\Delta t_n + \sigma S_{t_i}\sqrt{\Delta t_n}\varepsilon_{t_i} \\ S_{t_0} = S(t_0) \end{cases}$$

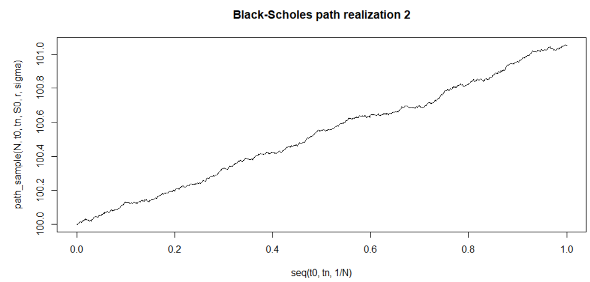
and $S(t_0)$ is the initial price of the stock. Finally, depending on the type of derivative we would implement the pay-off function $f(S_N)$ to the last value of the path. Doing this process π times, i.e using the method of *Monte Carlo*, we would get the mentioned "numeric mean" of $\mathbb{E}[V(T, S_T)] \simeq \frac{1}{\pi}\sum_{i=1}^{\pi} f(S_N^i)$.

Having this in mind, let us start by creating plots for several kinds of path realizations for different values of the pair (r, σ). We see that while is clear that the value σ controls the volatility of the time series, the r parameter transforms the sample path into an almost lineal function with increasing behaviour when $0 < r \sim 1$. For example, with the initial values of $N = 1000,\ t_0 = 0,\ t_n = 1,\ S_0 = 100,\ r = 0.01,\ \sigma = 0.30$

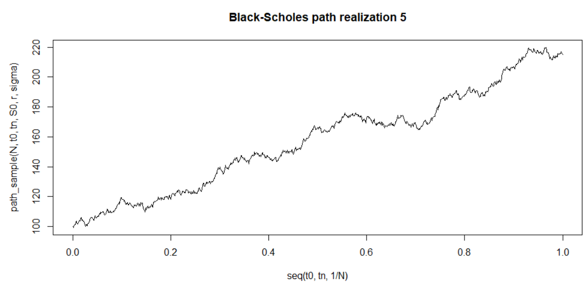


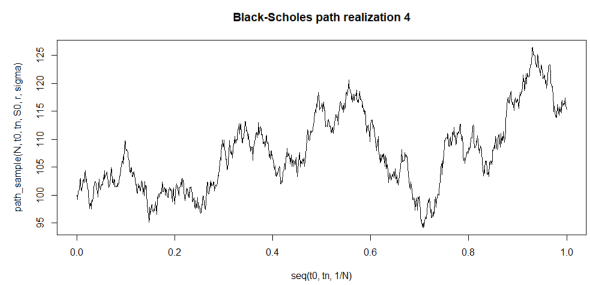
However, on one hand fixing only the parameter $r = 0.01$ and moving σ, we have for near one values a huge volatility and for near zero a lineal path.





On the other hand, doing the exactly opposite, we see that for near one values of r the series is almost lineal while it develops some volatility clusters when r lowers.





Having controlled the path behaviour, let us consider that we are pricing a call option with strike k. We know that the pay off function in this case was $f(S) = max(S - k, 0)$ at time $T = t_n$. As mentioned previously, using the method of Monte Carlo with $\pi = M$ we can now compute the actual price of the call which is

$$\frac{1}{M}\left[\sum_{i=1}^{M} max(S_n^i - k, 0)\right] e^{-r(t_n - t_0)}$$

For the last previous 5 samples, we can compute their price with $k = 140$ using the method of Monte Carlo. Summarising the information in the next table

	$r = 0.01$, $\sigma = 0.30$	$r = 0.01$, $\sigma = 0.999$	$r = 0.01$, $\sigma = 0.001$	$r = 0.999$, $\sigma = 0.30$	$r = 0.001$, $\sigma = 0.30$
Call price at time t_0 (EUR)	1.01819	23.36391	0	22.39878	0.6982642

Table 1: Call option price for several values of r and σ

Imagine we want to price a Put option instead. The method is actually the same, but we now have to consider a different pay-off function which is

$$g(S_t) = max(k - S_t, 0)$$

Consider the following situation, where the time lapse is $T = 1$ year, the interest rate is $r = 5\%$ and $\sigma = 40\%$ volatility. The stock currently trades at 90 EUR and the strike of the Put option is $k = 75$. Then using the method of Monte Carlo with $\pi = 100$ we can compute the option price and we have

$$\frac{1}{100}\left[\sum_{i=1}^{100} max(75 - S_n^i, 0)\right] e^{-0.05} = 6.059914 \text{ EUR}$$