HW3: Continuous models

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Financial Engineering

Exercise 1

In finance, derivatives are a product in which their value is based on the price of another asset. Examples of them are the forwards, futures, options, etc. Therefore, one must compute differently the price for each one of them, based on the law of "there is no such thing as free lunch" and arguments of *no-arbitrage*. Let us introduce the *call* and *put* option. Each one of them represents a contract in where the consumer or seller has the right, but not the obligation, to buy or sell respectively a underlying at a reasoned quantity at some future time T. Precisely, let $S(t)$ be the underlying value over the time t, and strike k the amount settled at the beginning:

- Buying for p_k the $k-$call option is buying the contract that allows, but does not require, a purchaser to buy an asset $S(t)$ at time T for a price k, without taking into account the return of $S(T)$. It is logical to think that the function that models the benefit of this idea is

$$max\,(S(T)-k,0)-p_k$$

  due its lack of obligation. For the dealer, the function plots not the benefits but the loss with $-max\,(S(T)-k,0)+p_k$.

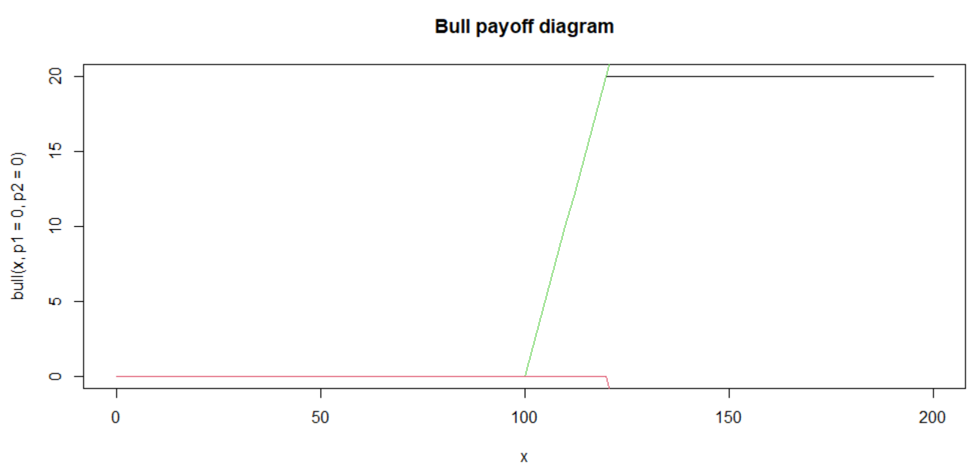
- Purchasing for p_k the $k-$put option is purchasing the contract that allows, but does not require, a seller to vend an asset $S(t)$ at time T for a price k, without taking into account the the market status of the asset when $t=T$. This time the function that provides the benefit is

$$max\,(k-S(T),0)-p_k$$

  while the loss for a purchaser is $-max\,(k-S(T),0)+p_k$.

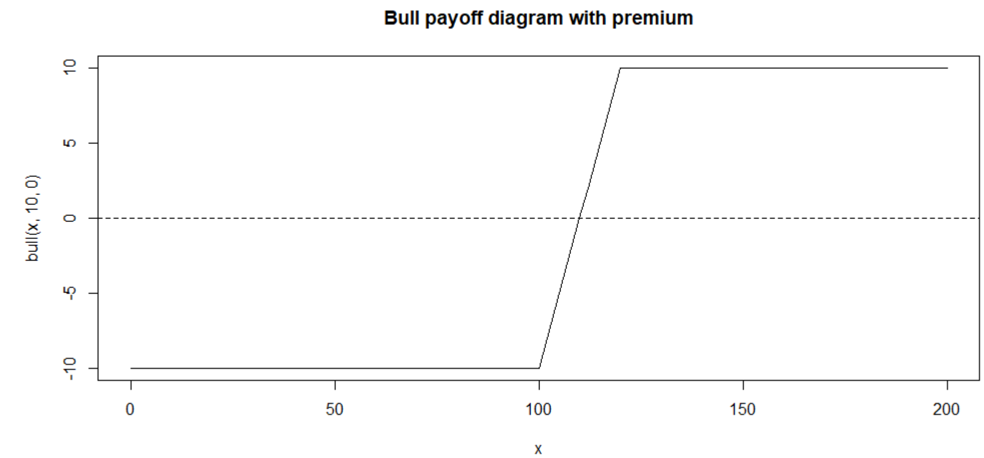
Where in both cases the value p_k is called the *premium*, and gives the price of the contract, in order to be fair while doing trading operations. Having this in mind, one can compute the payoff function in formulas and plotting them for each one of the following strategies:
Bull Spread: Buy a 100 Call option and sell a 120 Call option. The payoff formula is clearly easy to compute,

$$max(S(T)-100,0)-max(S(T)-120,0)$$



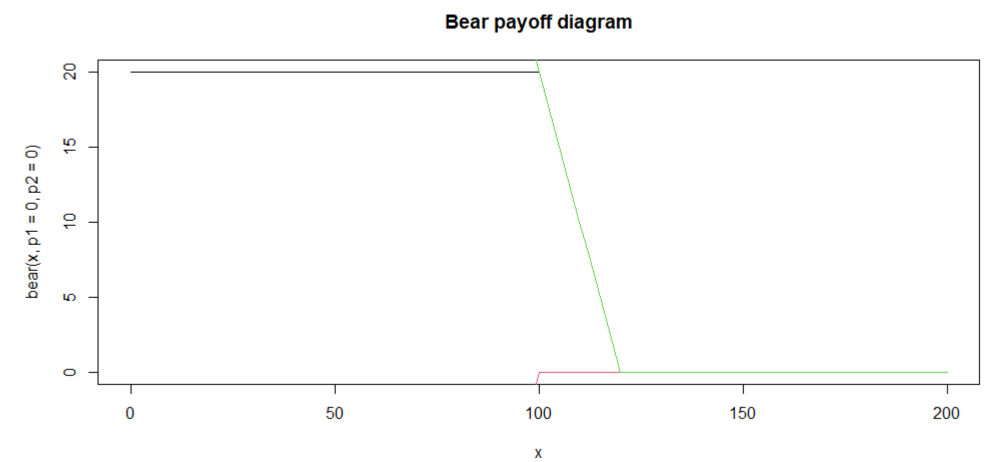
The general idea among all strategies is to buy an asset at a low price and sell it higher, which is the fundamentals of trading. However each one of them is much worth it depending if there are better chances of the price $S(t)$ increasing or decreasing. Beginning with the *bear spread* strategy, which is a clear example of a better use when the value of $S(T)>S(t)$. Since the price of $S(t)$ probably rises, one will never loose founds while buying the 100 no premium Call, and for the same reason the 120 Call probably sells fast due the expectancy of rising the value. There is never risk of loosing money and the profit is higher if one expects that $S(T)>S(t)$.

With every strategy one could measure the price of premiums with the lowest negative and highest positive value the strategy could hold. In particular, since the *bull spread* has no risk at all but secured profit one could put a premium of $p_{120} - p_{100} = -\$20/2 = -\$10$ that makes this strategy more fair following the lemma of "there is no such thing as free lunch".

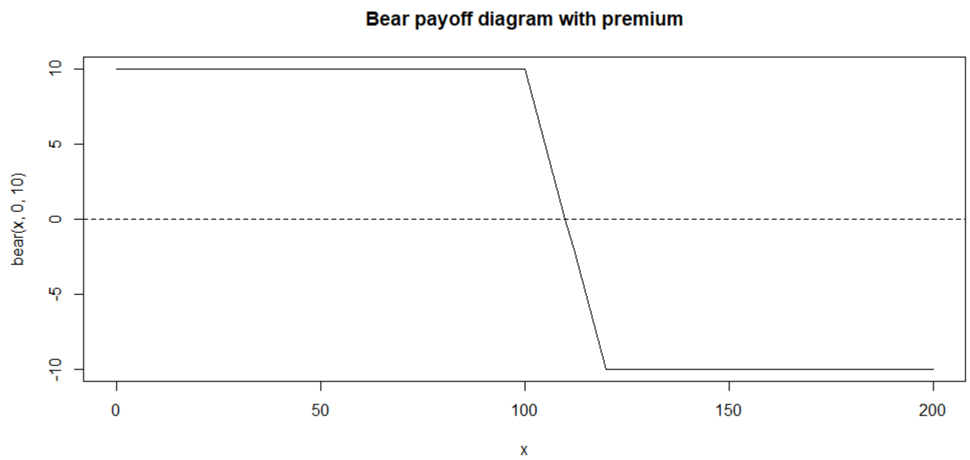


Bear Spread: Sell a 100 Put option and buy a 120 Put option. Similarly to the previous one, we compute the payoff function

$$-max(100 - S(T), 0) + max(120 - S(T), 0)$$

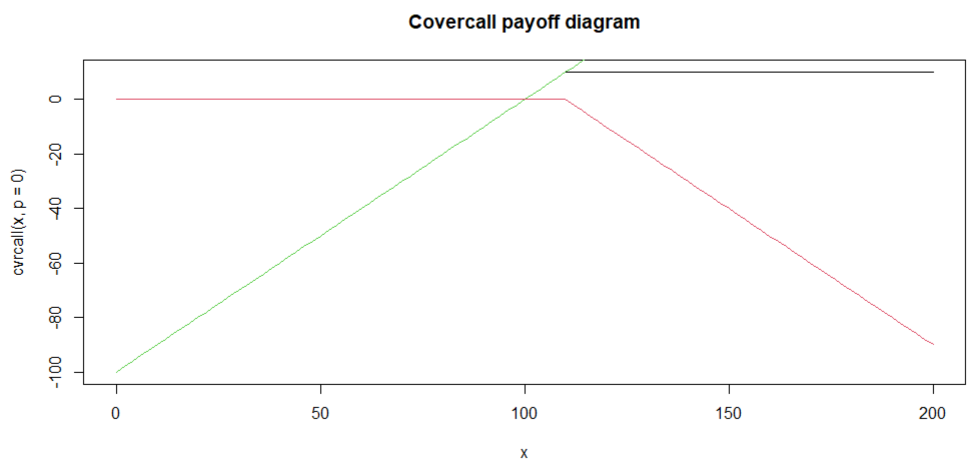


This time the strategy works better if $S(T) < K$. Firstly one sells the 100 put option, meaning having the right to sell the asset at \$100, indeed there are chances of wasting money if $S(T) < K_{100}$, since one has sold the offer. However, instantly buys the 120 put option that guarantees benefit thanks to the expected low price of $S(T) < K_{100} < K_{120}$ that makes more appealing to a customer.
However, if there is no one that buys the 120 put option, it is a risky strategy that could develop a huge loss. Nevertheless, considering that the options are sold for sure, in the end the strategy has no risk and secured benefit, again asking for a premium of $p_{100} - p_{120} = -\$20/2 = -\$10$ is the best option to stabilise the strategy.

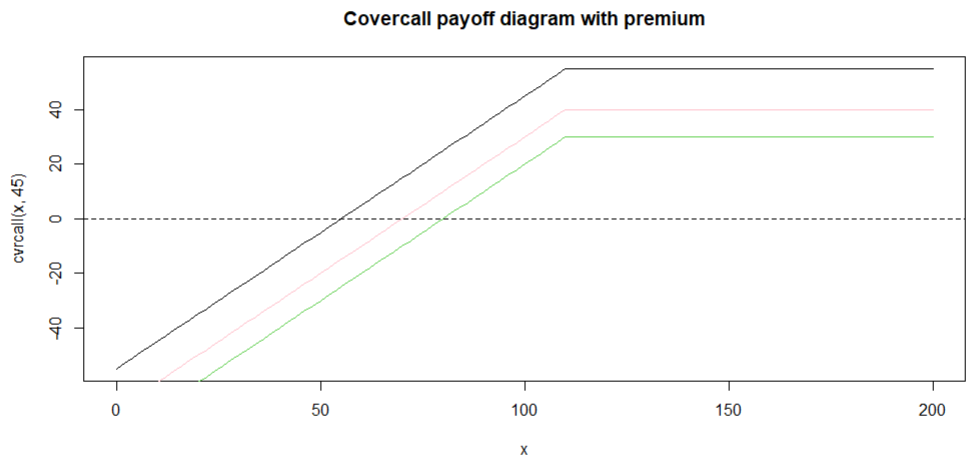


Covered Call: Being long on the underlying stock, then sell a 110 Call. Being long on the underlying means, one simply can buy the underlying today and the return will be $S(T) - S(t)$. Then the payoff is

$$S(T) - 100 - max(S(T) - 110, 0)$$

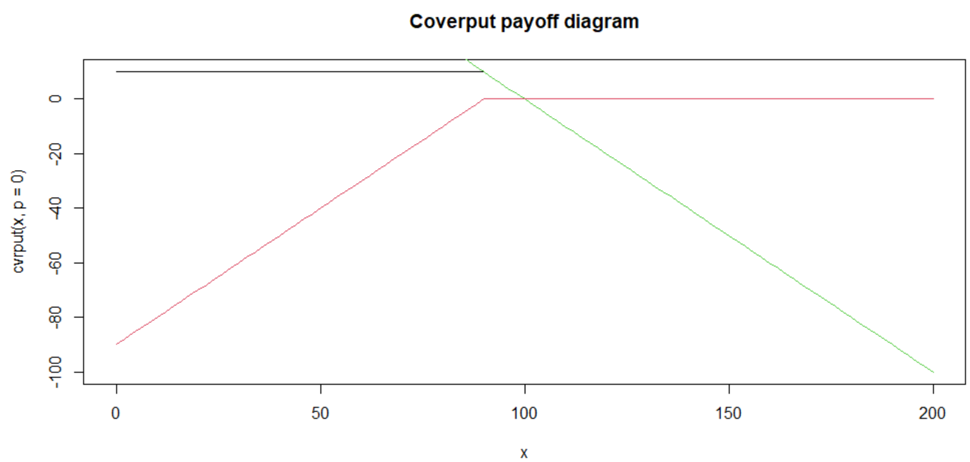


This strategy again is well suited for expected values of $S(t)$ to grow. Being long on the underlying means that one already has the asset and in order to sell it while gaining some profit, one sells the 110 call, that obviously will be bought, since everyone expects the asset to grow its value. This way the trader has a constant benefit of \$10. On the other hand, it is also a truth that if the value decreases, the trader may loose a huge quantity.
Due to the huge risk it involves, it would be reasonable to expect the premium to be $p_{110} = \$90/2$ at least. However since it depends on how much probable is the value $S(T)$ going to increase, or how much demand there is for the premium itself, there is not a certain value, therefore we've plotted several functions of possible Covered Calls with different premiums

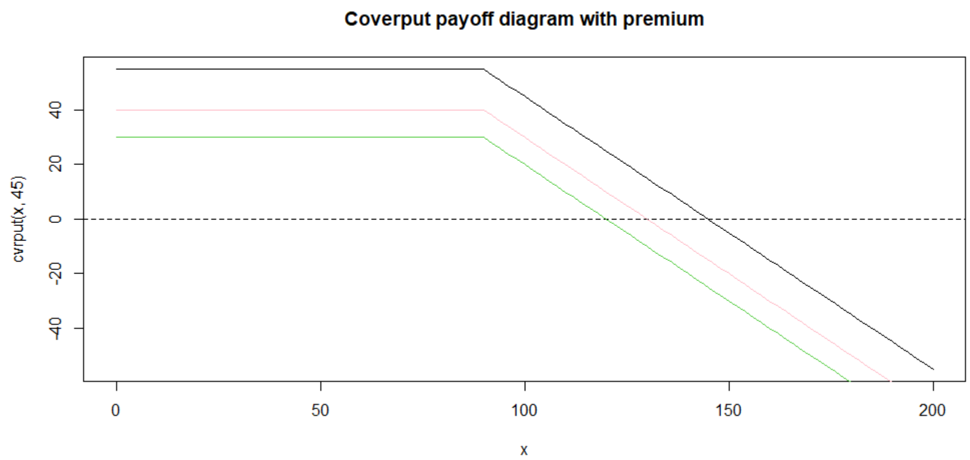


Covered Put: Assume being short on the underlying stock, then sell a 90 Put. Similarly to the previous one, but this time being short an asset means the sell of the stock, which the return is measured with $-(S(T) - S(t))$. Finally:

$$100 - S(T) - max(90 - S(T), 0)$$



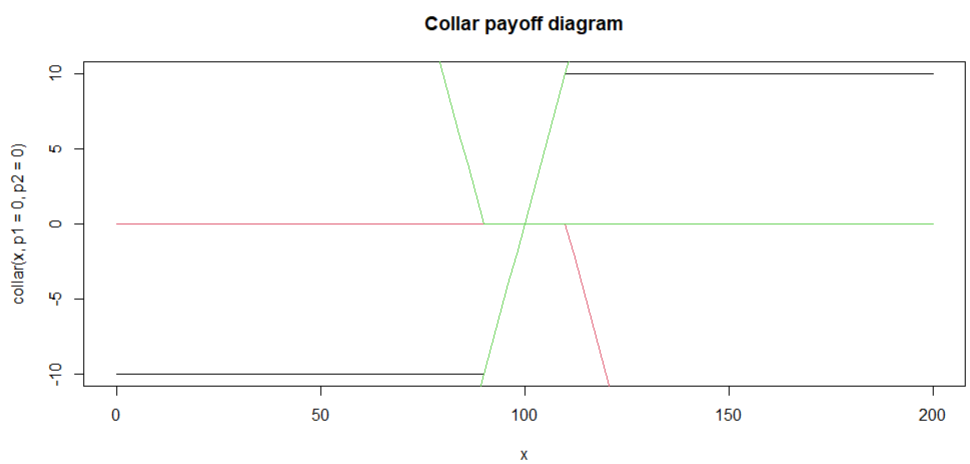
As in the *Bear Spread* strategy, the *Covered Put* aims when the $S(t)$ has negative tendencies. Imagine that today costs \$100, but one knows that in the future may cost less. One could just wait for the asset to lower its price and buy it. However, selling the opportunity (at price p) to *sell* the asset at \$90 when could be cheaper will make everyone wanting to buy the 90 put option, this way one only wastes \$90 when it is currently at \$100. Of course, due to the high demands the premium could aim high and make this a more beneficial strategy.
There is again one disadvantage again, if the stock increases its value, then the trader again looses high quantities. Again, for this main reason, one would expect to price the premium with \$90/2. But as reasoned before the chances of the asset lowering its value could be pretty high. As done with the *covered call* plotting some payoff functions with their respective premiums is the right option.



Collar: Assume being long on the underlying stock, then sell a 110 Call option and buy a 90 Put option. The payoff formula this time is

$$S(T) - 100 - max(S(T) - 110, 0) + max(90 - S(T), 0)$$

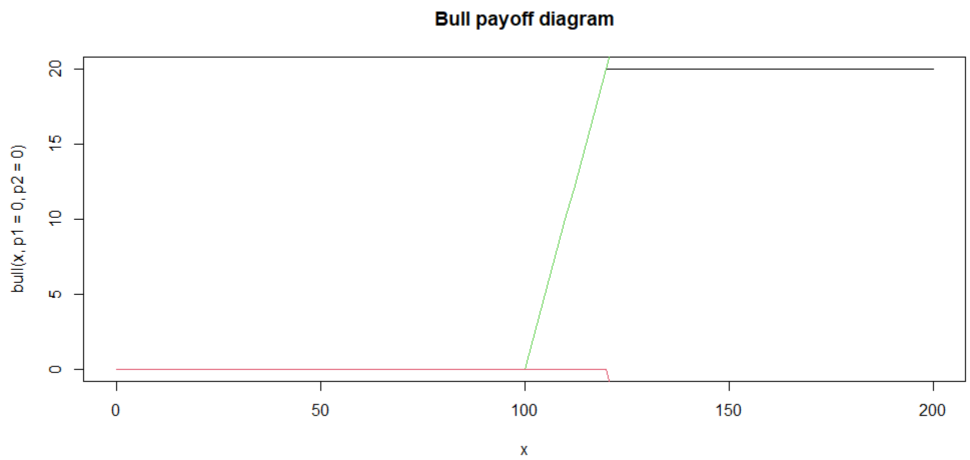
And the plot is



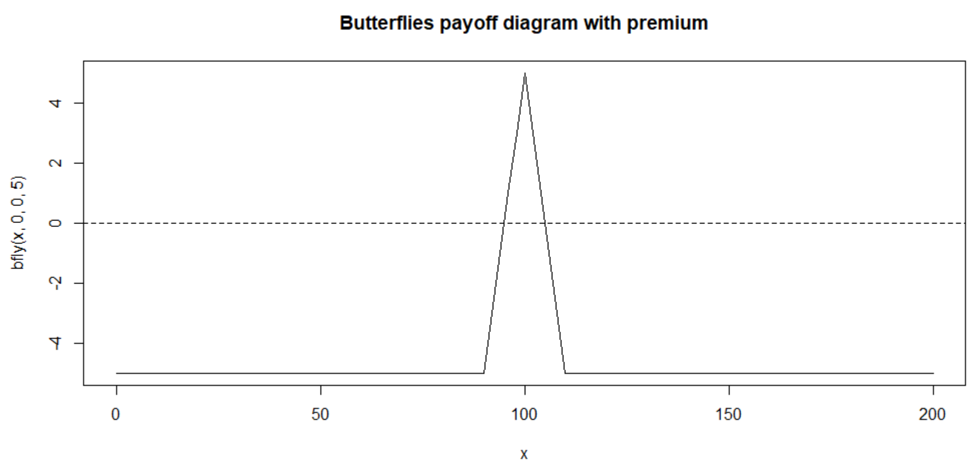
This complex strategy has no similarities with the previous ones. It is mostly used when the trader cannot expect the future value of $S(T)$, while owning it, and tries to have a minimum profit and loss. In case the $S(T) > S(t)$ one sells the opportunity to buy it at 110, having a positive income of \$10, while if the value $S(T) < S(t)$ the trader buys the opportunity to sell it for 90 when it could be cheaper, with a loss of \$10. Even if this strategy involves expected negatives incomes, it is good because it tries to lower them.
In addition, since there is the same maximum positive and negative return, it seems clear that the difference of premiums stay at $p_{110} - p_{90} = 0$, leaving the payoff function as it was initially.

Butterflies: Buy a 90 Call, sell two 100 Call and buy a 110 Call.

$$max(S(T) - 90, 0) - 2 \cdot max(S(T) - 100, 0) + max(S(T) - 110, 0)$$

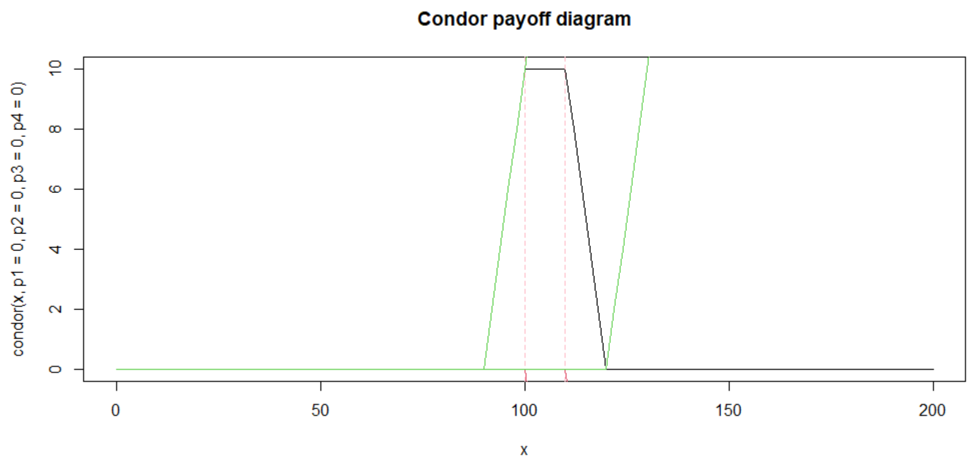


This strategy tries to generate most income from stock that the expected return does not change a lot from the initial value, and eliminates the possibilities of loosing money. For instance, if one buys a \$90 call, or a \$110 call just in case the actual price goes a little up or down respectively, and immediately sells the two engaging offers to buy it at \$100 (which is the initial value of the asset) one will make profit when the stock does not change its initial value.
To put a premium price, if the asset behaves as expected, one may say it is $2p_{100} - p_{90} - p_{110} = -\$10/2 = -\$5$, in order to fulfill the non-arbitrage lemma.

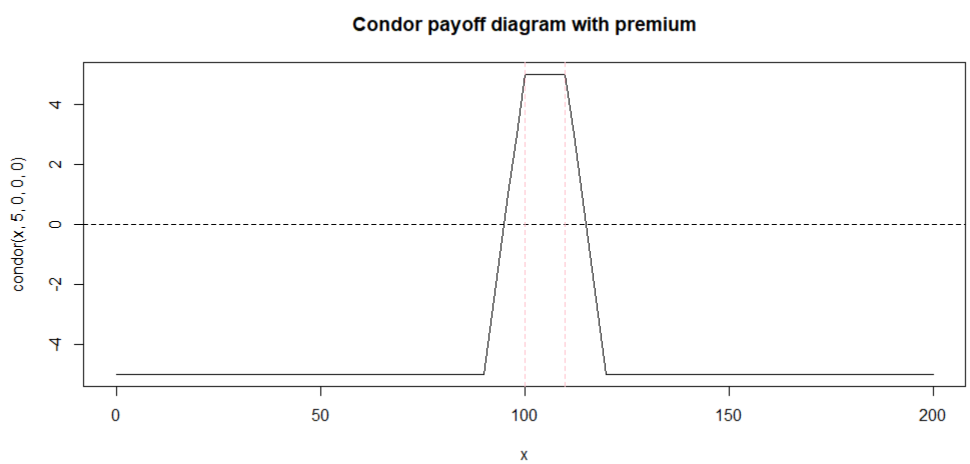


And finally
Condor: Buy a 90 Call, sell a 100 Call, sell 110 Call and buy a 120 Call

$$max(S(T) - 90, 0) - max(S(T) - 100, 0) - max(S(T) - 110, 0) + max(S(T) - 120, 0)$$



The last strategy follows the example of the previous one, therefore one will make profit using it with an asset that does not have huge volatility in its value. Its rationale is just as the last strategy, however the *Condor* tries to have the most income of \$10 even when the value increases a little, hence in an interval of $100 \leq S(T) \leq 110$.
As done in the latter strategy the premium would be $p_{110} + p_{100} - p_{120} - p_{90} = -\$10/2 = -\$5$



Exercise 2

Assume that a stock currently trades at \$60. Also assume that a Call with strike price \$58 and expiry 12 months trades at \$3, and a Put with same strike and maturity trades at \$2, and the interest rate at 12 months is 10%.
We need to check if there's arbitrage opportunities, taking care of the put-call parity. Having C as call price, P as put price, S as spot price today, and F as forward price. Let's see if there's parity:

$$C - P = S - F \iff 3 - 2 = 60 - 58 \cdot e^{-0.1} \iff 1 \approx 7.51$$

We get that there's arbitrage opportunities. Strategy in this case is really obvious. Selling our stock today (at spot price), buying a call option and selling a put option. Let's see the plot with the three options (call, put and spot) and the payoff, where x axis is the price after 12 months of the stock:

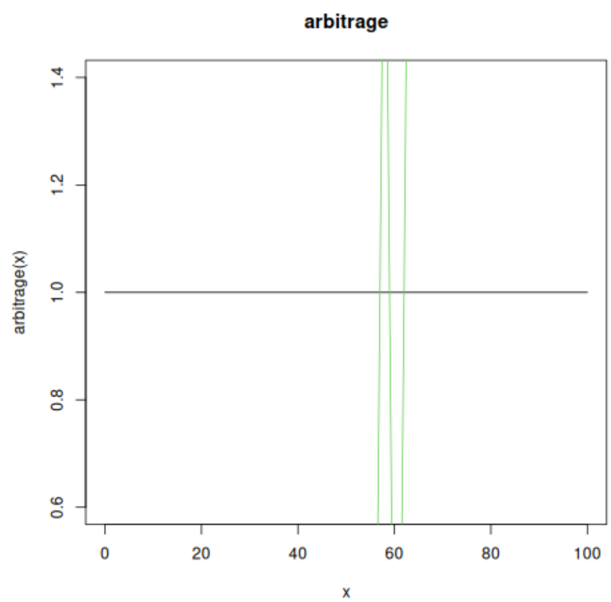


Figure 1: Payoff view

We see that, no matter which the price is, there's always a +1.0 in the payoff, so indeed there are arbitrage opportunities.
Let's calculate the payoff manually. We even don't need to use the interest rate for this, it's just a matter of spot and strike prices. Adding the benefits of a call, and subtracting spot and put benefits:

$$(max(s(t) - k, 0) - 3) - (max(k - s(t), 0) - 2) - (s(t) - s(t_0)) = s(t) - k - 1 - s(t) + s(t_0) = s(t_0) - k = 2$$

But we need to subtract 1 to this 2 value, as our initial state was -1, due to the price of call and put options. Then $2 - 1 = 1$, our payoff.
This would never happen in a real market, as we assumed (and, in practice, it is) that market give no arbitrage options and, concluding, these cannot be real prices of a real stock in the market.