HW6: Portfolio Optimization

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Exercise 1

The main objective of the whole exercise is to knowledge the functionality the tools of portfolio optimization holds, which consist on maximizing the return with the lowest risk. Let us begin with some definitions.
Consider $\{r_i\}_{i=1}^N$ returns of N different assets with respective expected return $\{\hat{r}_i\}_{i=1}^N$ and respective weights $\{\omega_i\}_{i=1}^N$, where $\sum_{i=1}^N \omega_i = 1$. Since a portfolio P fulfills their return $r_P = \sum_{i=1}^N \omega_i r_i$, then we define the pair (σ_P^2, μ_P) the *Feasible Set of portfolios P* where

$$\begin{cases} \mu_P = \mathbb{E}[r_P] = \displaystyle\sum_{i=1}^N \omega_i \hat{r}_i \\ \sigma_P^2 = \displaystyle\sum_{i=1}^N \sum_{j=1}^N \omega_i \omega_j cov(r_i, r_j) \end{cases}$$

Matricially if $r = (r_1, \ldots, r_N)$, $\mu = (\mu_1, \ldots, \mu_N)$ and $\omega = (\omega_1, \ldots, \omega_N)$ are each of them the vectors of returns, their respective expectancy and weights and $C = [cov(r_i, r_j)]_{1 \leq i,j \leq N}$ the matrix of co-variances we can re-write the feasible set as

$$\begin{cases} \mu_P = \omega^t \mu \\ \sigma_P^2 = \omega^t C \omega \end{cases} \tag{1}$$

In particular, let us consider the several adjusted prices for Google (GOOG), FaceBook (FB), Apple (AAPL) and Microsoft (MSFT) between 01/01/2015 and 31/12/2015. From the data we will compute their respective returns where

$$r_i = \frac{data_i}{data_{i-1}} - 1$$

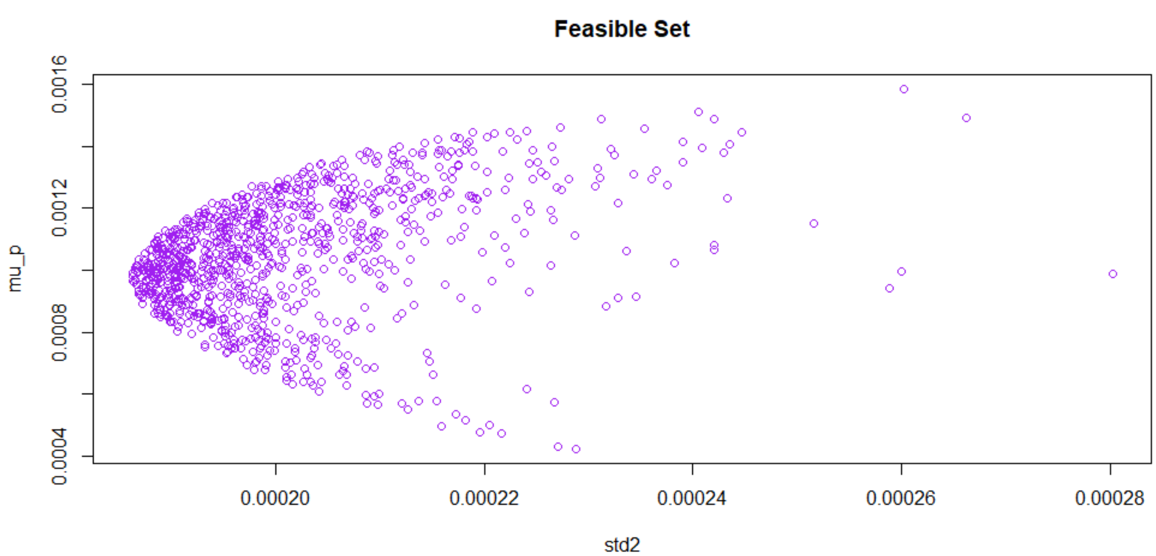
then the vector of expected returns is

$$\begin{aligned} \mu &= (\overline{r}_{goog}, \overline{r}_{fb}, \overline{r}_{apl}, \overline{r}_{msft}) \\ &= (0.001717564, 0.001343775, 0.000134906, 0.001007764) \end{aligned}$$

and matrix of co-variances

$$C = \begin{pmatrix} 0.0003468938 & 0.0001714771 & 0.0001187137 & 0.0001722509 \\ 0.0001714771 & 0.0002626470 & 0.0001494275 & 0.0001363390 \\ 0.0001187137 & 0.0001494275 & 0.0002841017 & 0.0001565954 \\ 0.0001722509 & 0.0001363390 & 0.0001565954 & 0.0003171199 \end{pmatrix}$$

In order to have simulations of several portfolios P_k, to plot the feasible set, let us consider 1000 random vectors ω where each component $\omega_i = \frac{W_i}{\sum_{j=1}^4 W_j}$ and $W_j \sim U[0, 1]$. Using (1) we can plot the feasible set



Exercise 2

This exercise's objective is also get more knowledge about portfolio optimisation. In this case, we are going to use Lagrangian multipliers to find the optimal weights of each item in our theoretical portfolio. After building up the linear model and solving it, we get the following plot of the frontier (taking $N = 500$ as proposed):

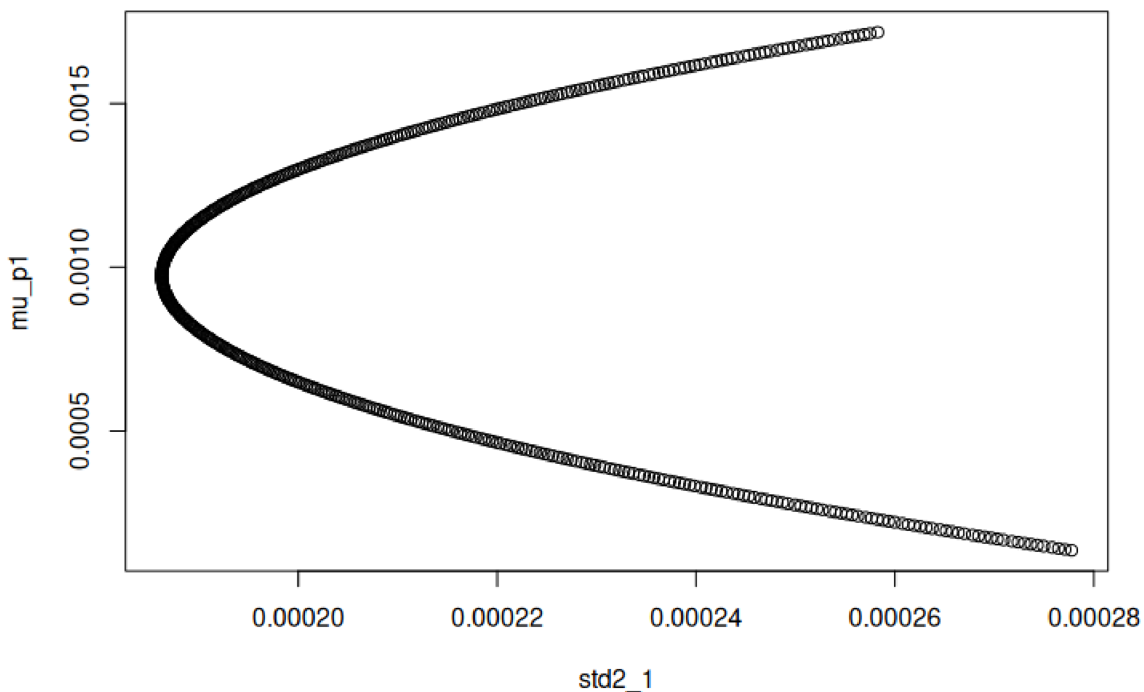


Figure 1: Frontier for each ω_r

Exercise 3

In Exercise 2, we plotted the frontier of the feasible set (plotted in Exercise 1). Observe that, in that frontier, we still have portfolios with same variance. It is crystal clear that, given such portfolios, we are only interested on the ones with maximum return. This frontier subset is what we call the *Efficient Frontier*.

Lastly, the frontier also shows a single point where we have minimum variance. This point contains the *minimum variance portfolio*.

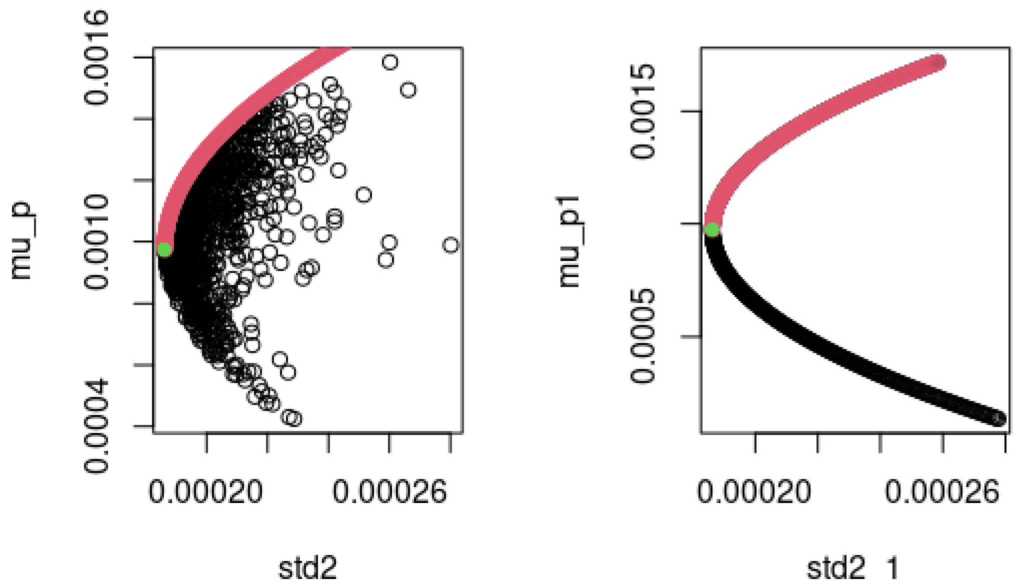


Figure 2: Previous plots with efficient frontier in red and minimum variance portfolio in green. Feasible set is the region inside the frontier.