HW2: Temporal Series 02

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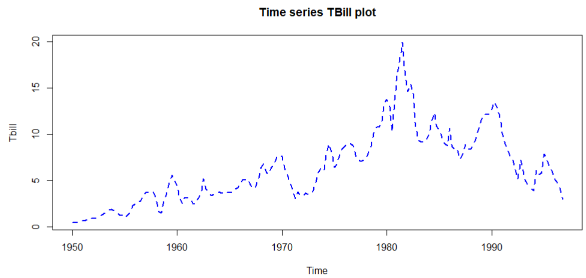
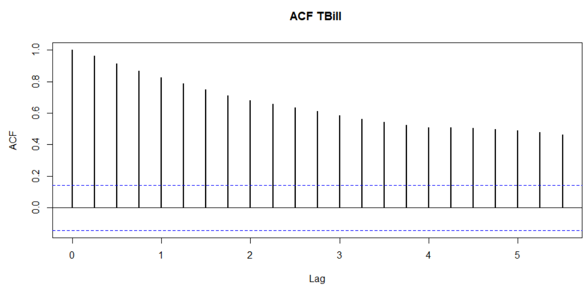
Financial Engineering

Exercise 1

In this exercise, we are looking at data from the 91-day Treasury Bill Rate:



First thing one can see from the plot is a non mean reversion. We notice that, on average, the rate increases until 1970, where it is cut down in half, just to grow again and much bigger in the next 10 years until it peaks and goes downhill from there. Given these monotonic long periods, we would think this series is not stationary. Performing an ACF plot confirms our thoughts:



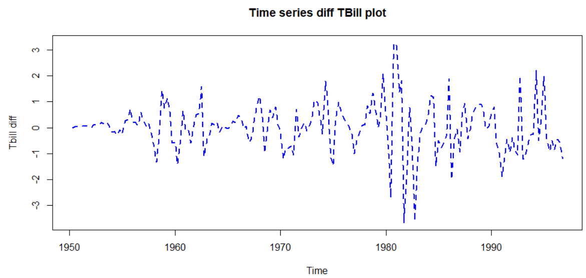
Look at how slow the function decreases and how far the lags are from zero, this is a clear signal that this series is not stationary.

But perhaps the reader is still not convinced of our claim. In that case, we will settle this once for all with some unit root tests:

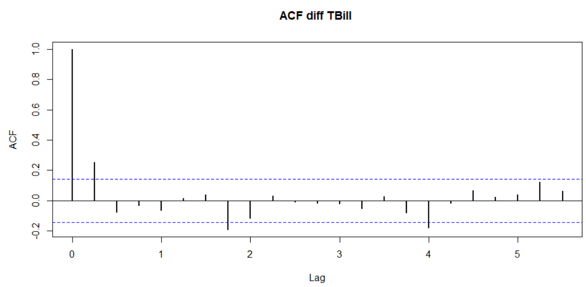
```
Augmented Dickey-Fuller Test

data:  Tbill
Dickey-Fuller = -1.925, Lag order = 5, p-value = 0.6075
alternative hypothesis: stationary

KPSS Test for Level Stationarity

data:  Tbill
KPSS Level = 2.2592, Truncation lag parameter = 4, p-value = 0.01
```

As it's shown above, the Augmented Dickey-Fuller test tells us to accept null hypothesis: the series is non stationary. On the other hand, KPSS test (strongly) encourages us to reject null hypothesis that claims stationarity on this series. So both performed tests support that we are working with a non stationary series.

But there's still some hope for us as this treasury bill rate seems to have a global linear upward trend. Here is the differenciated data plot:



Now we see apparent mean reversion, especially on the first half, therefore this series is likely to be stationary. In terms of heteroskedasticity, we notice that volatility increases in 1950-1965 and again in 1965-1980 resetting in between, and from 1980 on it appears to remain constant. Let's see the ACF for this data:



Look at that abrupt downhill at the very first lags, that's so much better to deal with! Let's see what do the tests have to say:

```
Augmented Dickey-Fuller Test

data:  Del.Tbill
Dickey-Fuller = -5.2979, Lag order = 5, p-value = 0.01
alternative hypothesis: stationary

KPSS Test for Level Stationarity

data:  Del.Tbill
KPSS Level = 0.1577, Truncation lag parameter = 4, p-value = 0.1
```

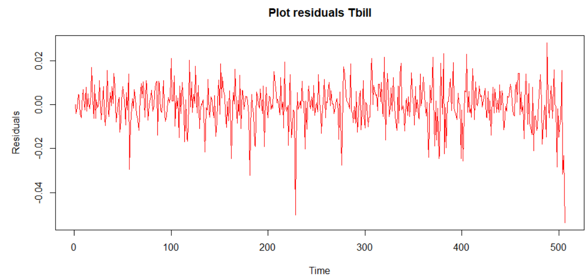
This time, the Augmented Dickey-Fuller test wants us to reject its null hypothesis and KPSS test tells us to the accept its null hypothesis: stationary. So both performed tests support that the differentiated series is stationary.

We are now asked to fit an ARMA(1,0)/GARCH(1,0) model to the differentiated series, which is believed to be stationary. Parameter estimates in the model are obtained with the `garch.model@fit$matcoef` command, that outputs the following table:

```
          Estimate   Std. Error  t value      Pr(>|t|)
mu      0.08349652  0.05390549 1.548943 1.213955e-01
ar1     0.24163450  0.07279700 3.319292 9.024598e-04
omega   0.33815662  0.06144724 5.503203 3.729535e-08
alpha1  0.83482811  0.24295106 3.436199 5.899382e-04
```

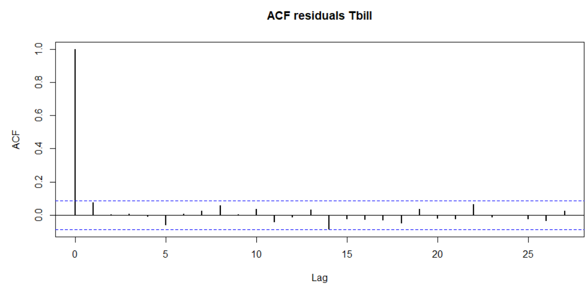
Note that this model does not give a good estimation of the mean mu.

Now, onto the model's residuals:

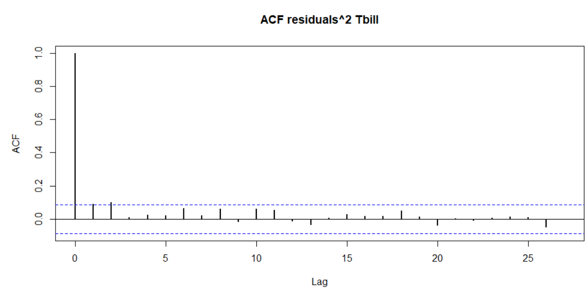


The plot shows mean reversion, as expected, but we observe some sheds of heteroskedasticity: volatility seems to be lower in 0-100 than in 100-200, then it lowers again until 350, etcetera. Even though changes in volatility are not huge, residuals don't seem to behave like a white noise and there might be some extra information we need to consider into the model.

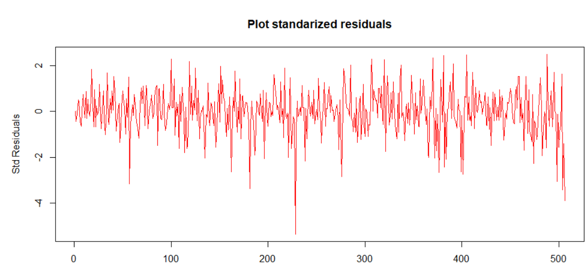
In order to check our hypothesis that residuals are not a white noise, we will do some more plots and estimate an adeaquate model for them:



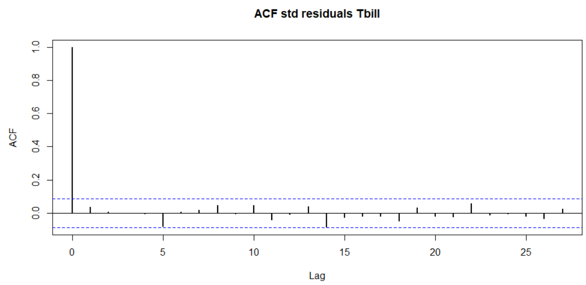
This part means how we can model the residuals of the data set. Firstly, the part of ARMA there is not any lag that cross the threshold, and we can say that it is not possible to fit the residuals with ARMA method. Now, we have to note if there can be GARCH part, with the auto correlation plot of the squared residuals.



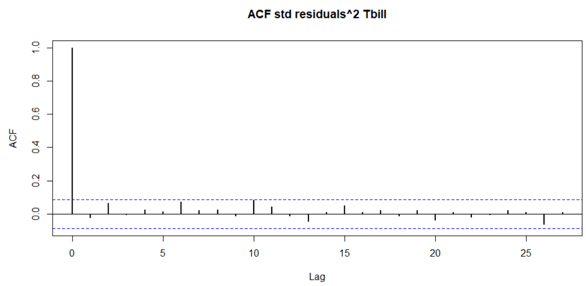
In this picture, we have to consider that in the lag 2 cross the threshold, and it will be model GARCH(2). The residuals can be fitted with this model. To correctly explain the residuals behaviour, it can be plotted the standardized residuals, and theirs auto correlations plots.



This figure has not trend, but high volatility. We remark one thing of the plot, we can say that the standardized residuals will be hard to fit it in any model. However, we need to view it with the ACF plot.

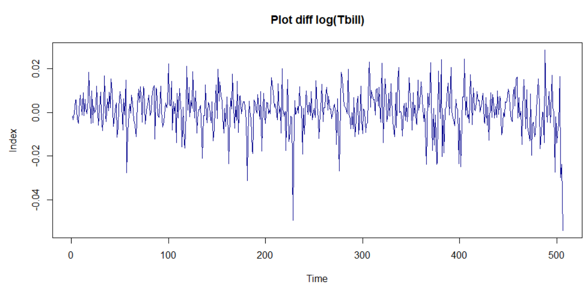


There is no possibility to fit the standardized residuals in any ARMA model. We have to continue searching the best way to fit the residuals, therefore in order to be able to model them with a GARCH process, we have to plot the squared standardized residuals.

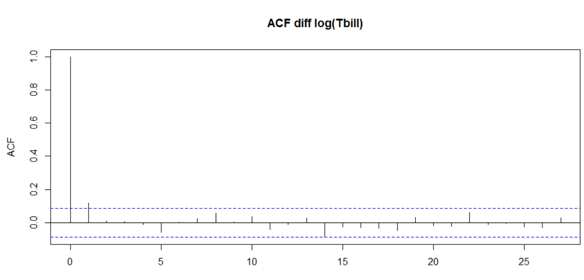


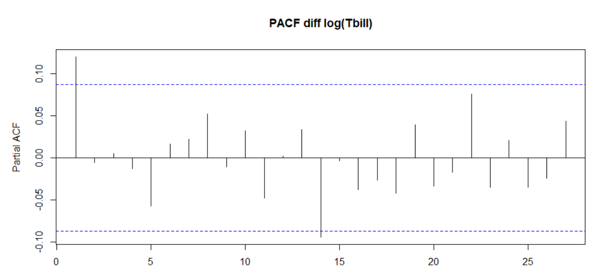
The standardized residuals can be fitted with a GARCH(2), there is the only possible model. The reason of this plot, is that we have to train the residuals because we have to predict the time series.

So, after all this plotting, of course we did not have enough and now we wish to study the differentiated log () rates, given below:

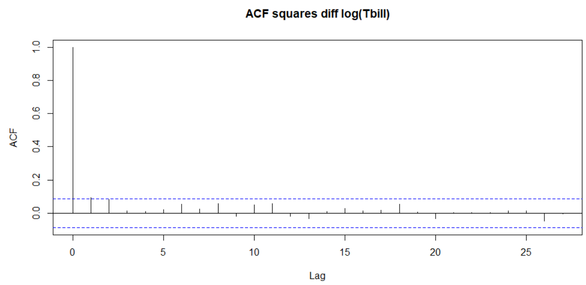


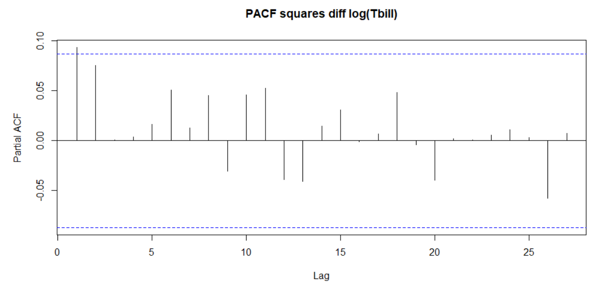
This series displays mean reversion so there is no need to differentiate it. Furthermore, we do not see much heterosketasticity as data moves approximately between -0.02 and 0.02 during all the period, without many pieces where we can shrink or need to enlarge the range.





Both ACF and PACF go down quickly to zero, so this suggests an ARMA part with low order as ARMA(0,1) or similar.





Same with squared data, suggesting a GARCH(1,1) part or similar.

So now we fit the series to a MA(1)/GARCH(1,1) model and `garchFit` agrees that this isn't a bad model. Here are the parameter estimates:

	Estimate	Std. Error	**t** value	Pr(>|**t**|)
mu	0.030593834	0.0113598696	2.693150	7.078039e−03
ma1	0.311708712	0.0775057183	4.021751	5.776707e−05
omega	0.001968961	0.0009939315	1.980982	4.759328e−02
alpha1	0.435561635	0.1450356336	3.003135	2.672136e−03
beta1	0.564412116	0.1011858613	5.577974	2.433358e−08

Exercise 2

The late 80's decade case of the huge drop of the American companies **S&P500 index**, is a clear example of how much the information shocks can affect drastically the global economy and how unpredictable they can be. In particular, the Black Monday event holds a small of -22.8% dated in 1987, and the aim of this exercise is to try to answer if this result was predictable or not.

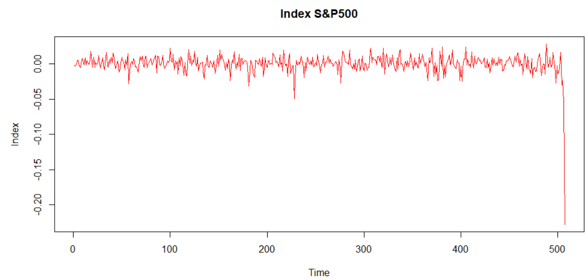


Figure 1: S&P500 index time series

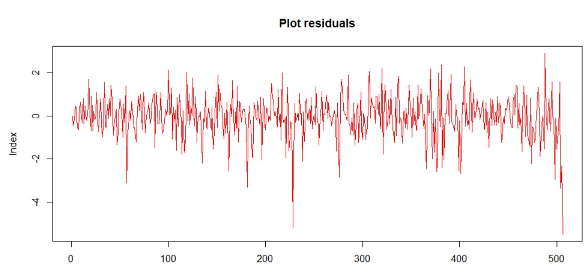
A first estimation of the model is using an $AR(1)/GARCH(1,1)$ model. Let us remember that given $\{\varepsilon_t\}_t$ a $WN(0,\sigma_\varepsilon^2)$, and the constants $\alpha, \beta \geq 0,\ \omega > 0,\ \mu$ and σ, we define $\{Y_t\}_t$ to be an $AR(1)/GARCH(1,1)$ process if the

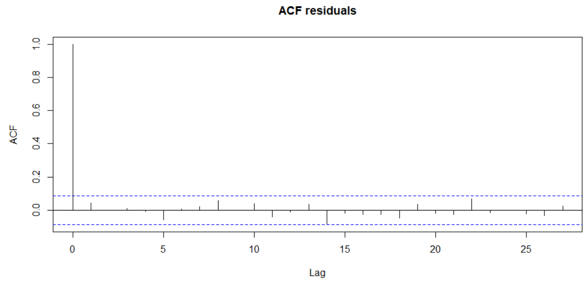
$$\begin{aligned} Y_t - \mu &= \phi(Y_{t-1} - \mu) + a_t \\ a_t &= \varepsilon_t \sigma_t \\ \sigma_t &= \sqrt{\omega + \alpha a_{t-1}^2 + \beta} \end{aligned}$$

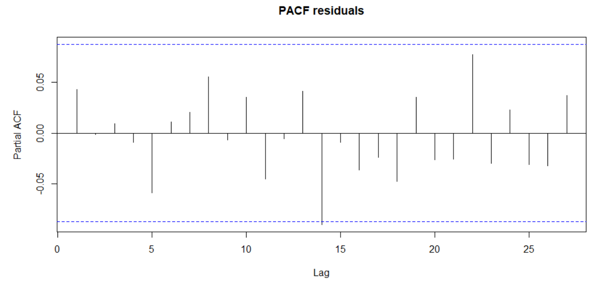
Where $\{a_t\}_t$ corresponds the $GARCH(1,1)$ part.
Considering Y_t the random variable from the model, we want to know the probability of Y_t to be $r = -0.228$ and know how much predictable it was. Let us consider given the past values Y_k for $k < t$, and the following parameters estimators $\hat{\mu} = \overline{Y_k},\ \hat{\phi},\ \hat{\omega},\ \hat{\alpha},\ \hat{\beta}$. Hence, the only random variable in $Y_t - \hat{\mu} = \hat{\phi}(Y_{t-1} - \hat{\mu}) + a_t$ is $a_t = \varepsilon_t \sigma_t$, where the White Noise $\{\varepsilon_t\}_t$ has a t-distribution with *df* degrees of freedom, in fact:

$$\mathbb{P}(Y_t \leq r) = \mathbb{P}(Y_t - \overline{Y_t} \leq r - \overline{Y_t}) = \mathbb{P}(\sigma_t \varepsilon_t \leq r - \overline{Y_t}) = \mathbb{P}(\varepsilon_t \leq \frac{r - \overline{Y_t}}{\sigma_t}) \quad \text{where } \varepsilon_t \sim t_{df}$$

finally, replacing the estimators for the proper values, the conditional probability is $8.132691e-05$, which means the event was totally unpredictable with an $AR(1)/GARCH(1,1)$ fit.
Let us try and see which part of the model was the one that could have been improved. In particular, the residuals of $\{Y_t\}_t$ display the following ACF and $PACF$ plots,



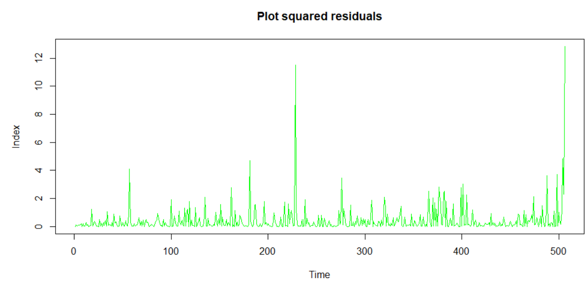


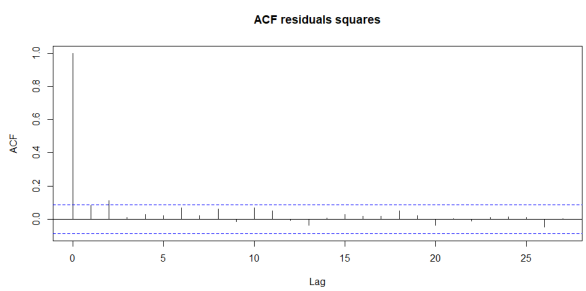


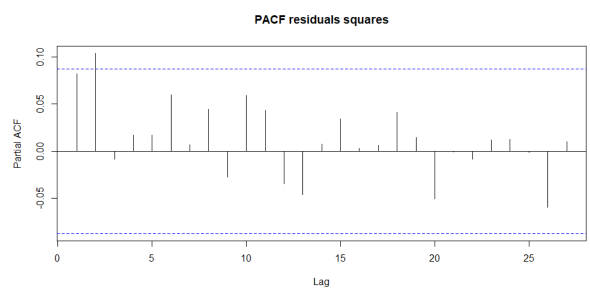
Where it is clear that the White Noise behaves like a normal distribution $WN(0, \sigma_{\varepsilon}^2)$ due its lack of correlation displayed on the ACF plots and the low values of the lags that do not exceed the threshold in both plots. Meaning, the $AR(1)$ part of the model is a great fit for the temporal series. In fact, computing a *Ljung-Box* test with R we have the following $p-$value:

```
Box.test(residuals)
=> p-value = 0.5898
```

where we have to accept the null hypotheses, and the residuals are a $WN(0, \sigma_{\varepsilon}^2)$. Nevertheless, proceeding the same way for the $GARCH$ part, we have that the residuals for $\{Y_t^2\}_t$ show the following graphics:



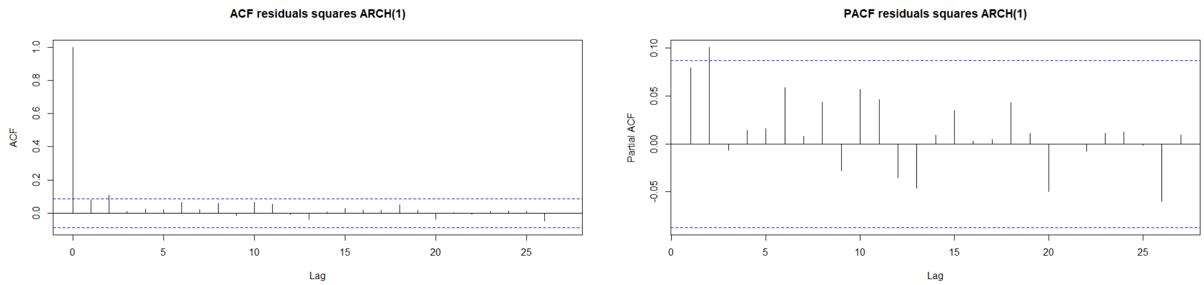




Unlike with the AR part, the $GARCH(1,1)$ does not seem to be a good enough process to model the time series due to the information left of Y_t^2 in the ACF plot, and displaying signs of correlation. As a matter of fact, the *Ljung-Box* test provides the same answer

```
adf.test(residuals2)
=> p-value = 0.03723
```

to decline H_0 of the squared residuals behaving like a $WN(0, \sigma_{\varepsilon}^2)$. This proves that try to search a better fit for the time series with another $GARCH(P, Q)$ part is the best option. For example, doing the same study with $P = 1$ and $Q = 0$, we have the following plots:



and result with the *Ljung-Box* test:

```
Box.test(resid2, lag=10, type="Ljung")
=> p-value = 0.04901
```

which means that $AR(1)/ARCH(1)$ is not a completely suitable fit, even if it has improved from $AR(1)/GARCH(1,1)$. Let us consider studying to fit the $S\&P500$ returns with an $AR(1)$, therefore the random noise is no longer a $GARCH$ variable. Again, it is enough to see if the residuals distribute like a $WN(0, \sigma_\varepsilon^2)$. Judging by the ACF plot

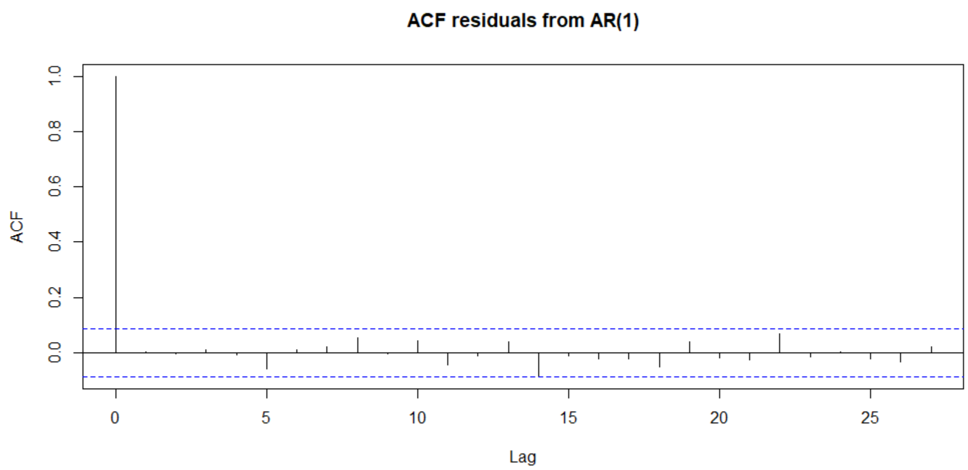


Figure 2: ACF plot of the residuals by modeling with $AR(1)$ process

One notices that this time, the residuals display no visible correlation with each other, which means simply $AR(1)$ model is a great fit. Finally, one admits it by reproducing the Ljung-Box test:

```
Box.test(residuals(fit1), lag=10, type="Ljung")
=> p-value = 0.9223
```

Exercise 3

To choose an adequate model for this data set, let us first start checking its values.

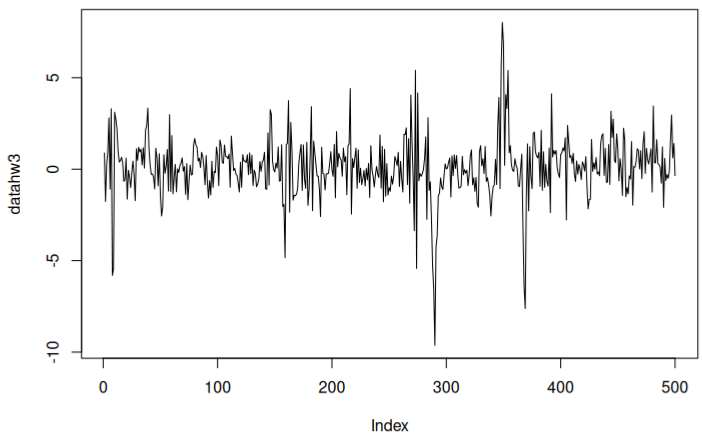


Figure 3: Data set values

We see that the mean is close to 0 and we experience periods of high volatility and periods of less volatility. Starting with the ARMA(p, q) component, let us consider plotting the ACF and $PACF$, in order to get the p, q degrees :

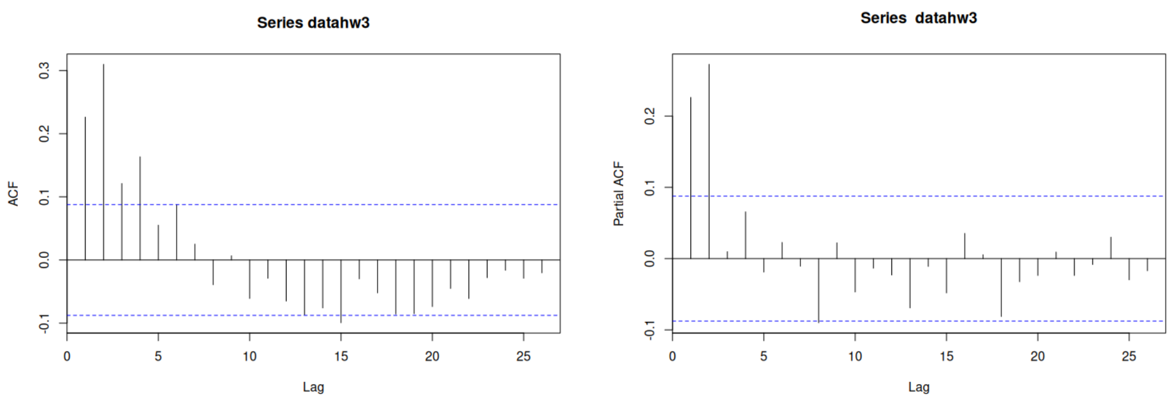


Figure 4: ACF and PACF plots of the data

With this plots we can clearly see that one parameter (the one related to MA) is really likely to be 0, and the other one to be close to 2. This is compliant with the hint we got, $q = 0$. To get the full model and get information about $GARCH(P, Q)$ component, let us proceed the study with the squares' plots:

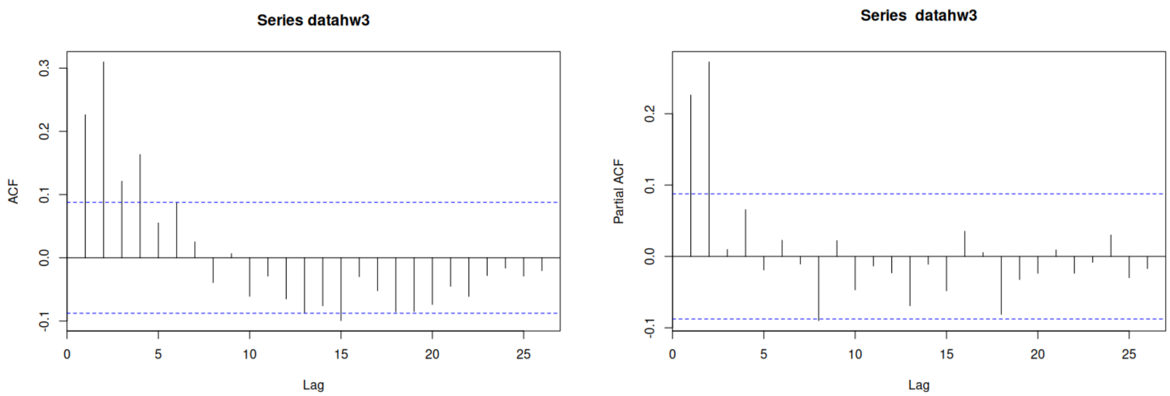


Figure 5: ACF and PACF plots of the squares

Taking this into account, one parameter will be 0 and the other 1 or 2.
After trying some combinations and using AICC criterion to choose, we decided to stick to ARMA$(2, 0)$ and GARCH$(1, 0)$. We get all the values that are significant. With this model fit, checking the residuals behaviour is required in order to prove it is a great fit and useful for predictions. Let us see the ACF and PACF plots of the residuals, and squared ones:

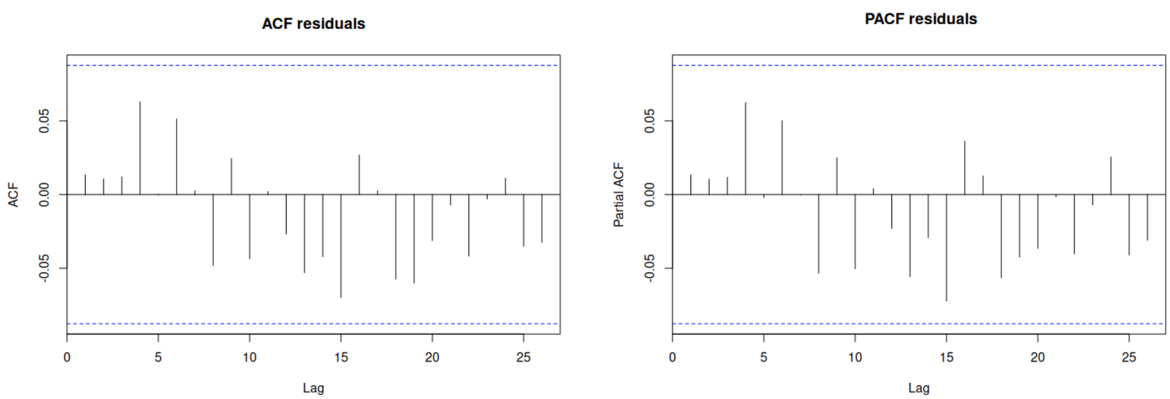


Figure 6: ACF and PACF plots of the residuals

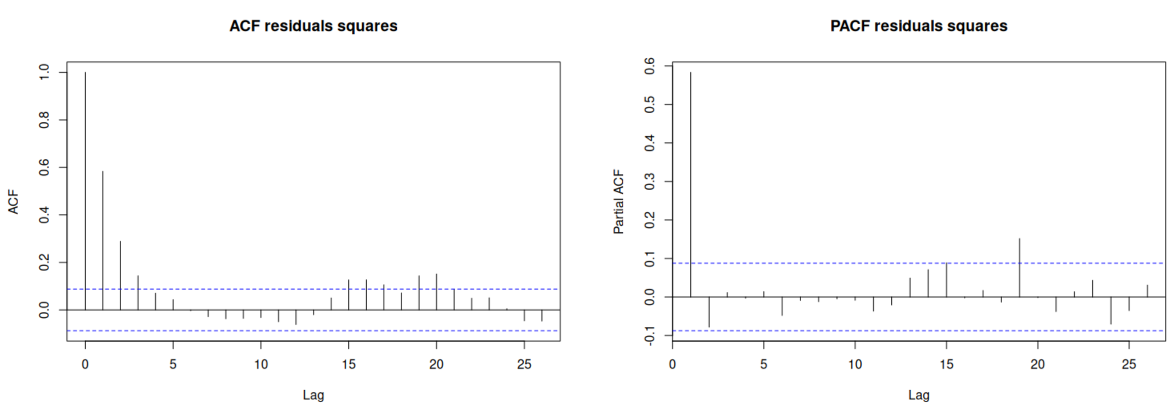


Figure 7: ACF and PACF plots of the square residuals

We notice that we are missing some information in the square residuals, as we see in the following plot, it's not white noise. We would still need to check further than GARCH, as this noise it's not related to GARCH component.

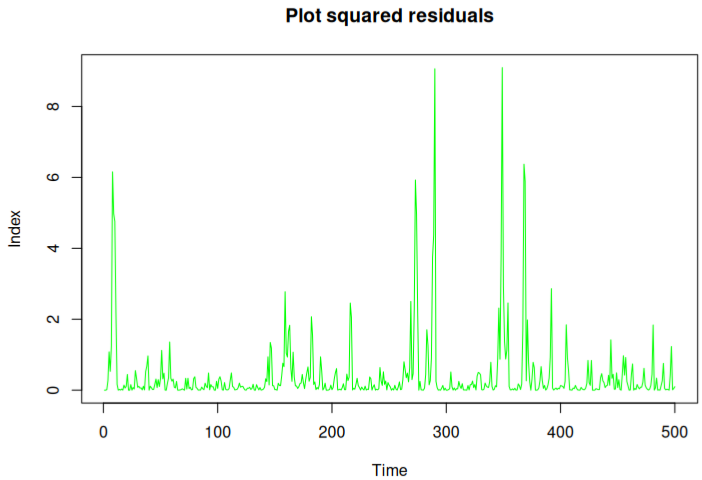


Figure 8: Square residuals

Performing a Box Ljung test on both regular and square residuals, we see that in the first one there's white noise and, in the second, there's more information there, as we were expecting.

```
Box.test(residuals, lag=10, type="Ljung")
=> p-value = 0.813

Box.test(residuals2, lag=10, type="Ljung")
=> p-value < 2.2e-16
```

As said, we would have needed to continue looking further in with the third moment to be more accurate.