Lógicas modales

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Part I

Memory logics

Changing the model

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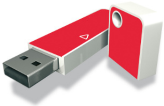
> A modal formula is a little automaton standing at some state in a relational structure, and only permitted to explore the structure by making journeys to neighbouring states.

- What about granting our automaton the additional power to **modify the model** during its exploratory trips?
- There may be many ways to modify a model (changing the domain, the edges, the valuation, . . .)
- We want to restrict our atention to a specific way of modifying a model: **adding a memory** to the model, and **performing changes** on it

Changing the model

- We are going to add a storage structure to standard Kripke models:

$$\mathcal{M} = \langle W, (R_r)_{r \in \mathsf{rel}}, V \rangle \quad + $$



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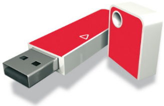

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Memory Kripke model

Given a set $S \subseteq W$, a memory Kripke model is

$$\mathcal{M} = \langle W, (R_r)_{r \in \mathsf{rel}}, V, S \rangle$$

Changing the model

We have to add suitable operators to manipulate the memory

- Since we are using a set S as the container, there are two "natural" operators to use:
  - An operator $\texttt{r}$ to *remember* the current point, storing it in S.
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Some notation

Given $\mathcal{M} = \langle W, (R_r)_{r \in \mathsf{rel}}, V, S \rangle$, $w \in W$, we define

$$\mathcal{M}[w] = \langle W, (R_r)_{r \in \mathsf{rel}}, V, S \cup \{w\} \rangle$$

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Now, more formally

Semantics of ⓡ and ⓚ

$$\begin{array}{lll} \mathcal{M}, w \models ⓡ\varphi & \text{iff} & \mathcal{M}[w], w \models \varphi \\ \mathcal{M}, w \models ⓚ & \text{iff} & w \in S \end{array}$$

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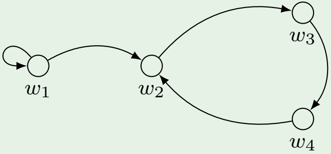
Let's see the use of ⓡ and ⓚ with an example. Suppose we start with the following model:

A model with an initially empty memory



- $V(p) = \emptyset$ for all $p \in$ prop
- $S = \emptyset$

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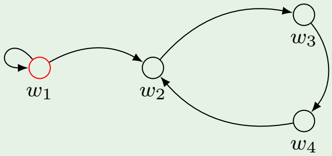

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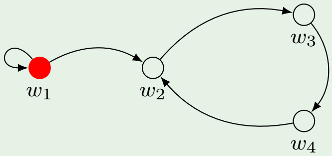

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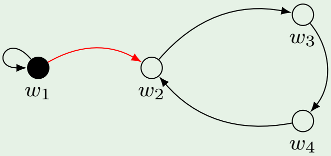

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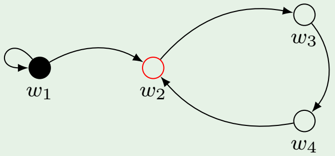

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Memory logics

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- The family of languages with these characteristics are sometimes called **dynamic logics**
- For example:
  - Dynamic epistemic logics
  - Real time logics
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- For example:
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- Memory logics can be seen as dynamic languages that
  - Do not add any domain-specific behaviour in the evolution of the model
  - Analyze dynamic behaviour from a very simple perspective
  - Can be thought of as a 'weak' version of the standard $\downarrow$ modal binder
- Can be combined with other modal and hybrid operators (A, nominals, $@$, etc.)

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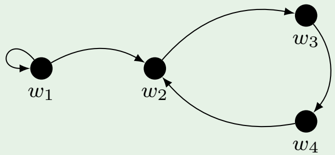

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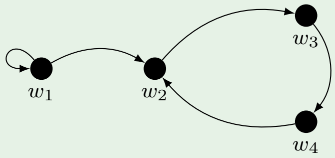

- How can we check whether w_1 has a successor different from itself?
- There doesn't seem to be a way...

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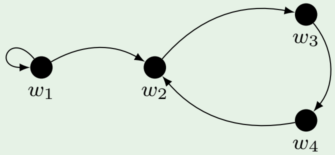

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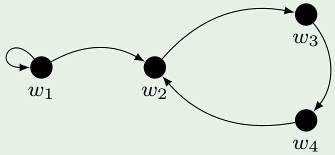

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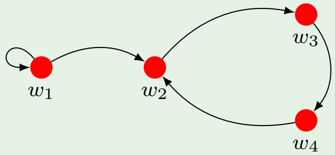

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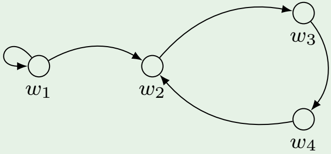

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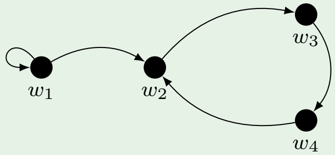

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This formula works independently of the initial state of the memory

Other ingredients

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- Class of models: for example, it is quite natural to consider the class of models whose memory is initially empty

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- Memorizing policies: we can try to impose some restrictions on the interplay between memory and modal operators
  - These restrictions are going to help us find decidable fragments
- Other memory operators and containers: are there other memory operators? What happens if we change a set by other type of structure?
  - We can define ⓕ, a local version of ⓔ
  - We can try using a stack instead of a set as the memory container

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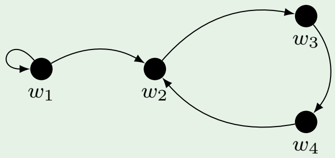

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Other ingredients: classes of models

Observe that when the memory of $\mathcal{M}$ is initially empty,

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But this formula is also true at

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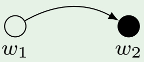

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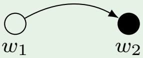

Taking this into consideration, it is natural to consider memory logics restricted to

$$\mathcal{C}_\emptyset = \{\mathcal{M} \mid \mathcal{M} = \langle W, (R_r)_{r \in \mathsf{rel}}, V, \emptyset \rangle\}$$

the class of models with an empty memory.

Other ingredients: memorizing policies

- Until now memory and modal operators were working ‘in parallel’
- Restricting expressivity sometimes can be helpful to reduce computational cost
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$\langle r \rangle$ and ⓡ working together

$$\mathcal{M}, w \models \langle\!\langle r \rangle\!\rangle \varphi \quad \text{iff} \quad \begin{array}{l} \exists w' \in W, R_r(w, w') \text{ and} \\ \mathcal{M}[w], w' \models \varphi. \end{array}$$

We are going to see later that this operator helps us to find **decidable** memory fragments

Notation

We are going to work with several memory logic fragments

Notational convention

- We call $\mathcal{ML}$ the basic modal logic, and $\mathcal{HL}$ the extension of $\mathcal{ML}$ with nominals
- When we add a set S and the operators ⓡ and ⓚ we add m as a superscript, e.g. $\mathcal{ML}^m(\ldots$
- We add $\emptyset$ as a subscript when we work with $\mathcal{C}_\emptyset$ (otherwise is the class of all models), e.g. $\mathcal{ML}^m_\emptyset(\ldots$
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For example

- $\mathcal{ML}^m_\emptyset(\langle r\rangle, ⓔ)$: the modal memory logic with ⓡ, ⓚ, ⓔ and the usual diamond $\langle r\rangle$ over the class $\mathcal{C}_\emptyset$
- $\mathcal{HL}^m(@, \langle r\rangle)$: the hybrid memory logic with ⓡ, ⓚ, $\langle r\rangle$, $@$ over the class of all models

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Disclaimer: we are not going to see all these topics during this talk

Expressivity results

We compare the expressive power of the different fragments via the existence of *equivalence preserving translations*

$\mathcal{L}'$ is as least as expressive as $\mathcal{L}$ ($\mathcal{L} \leq \mathcal{L}'$) if there is a Tr such that

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Theorem

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To see that $\mathcal{ML}^m_\emptyset(\langle r \rangle) \leq \mathcal{HL}(\downarrow)$ we define a translation Tr that maps formulas of $\mathcal{ML}^m_\emptyset(\langle r \rangle)$ into sentences of $\mathcal{HL}(\downarrow)$.

- We use $\downarrow$ to simulate ⓡ.
- We use a finite set N to simulate that ⓚ does not distinguish between different memorized states.

$$\begin{array}{rcll} \mathsf{Tr}_N(\text{ⓡ}\varphi) & = & \downarrow i.\mathsf{Tr}_{N\cup\{i\}}(\varphi) & \text{(for } i \text{ a new nominal)} \\ \mathsf{Tr}_N(\text{ⓚ}) & = & \bigvee_{i \in N} i & \end{array}$$

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- We developed a notion of *bisimulation* for each fragment. Intuitively, two models are bisimilar for a logic $\mathcal{L}$ when they cannot be distinguished by $\mathcal{L}$-formulas

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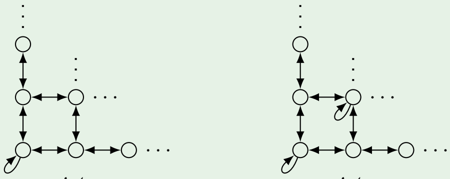
$\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{ML}^m_\emptyset(\langle r\rangle)$-bisimilar



$\mathcal{M}_1$ $\quad$ $\mathcal{M}_2$

But there is a formula $\varphi \in \mathcal{HL}(\downarrow)$ such that

$$\mathcal{M}_1, w \models_{\mathcal{HL}(\downarrow)} \varphi \text{ and } \mathcal{M}_2, v \not\models_{\mathcal{HL}(\downarrow)} \varphi$$

So a translation from $\mathcal{HL}(\downarrow)$ to $\mathcal{ML}^m_\emptyset(\langle r\rangle)$ cannot exist

Expressivity results

We establish in this way an "expressivity map" for many memory logic fragments:

$$
\begin{tikzcd}
 & & & \mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{f}) \arrow[dd, "\not\leq"] \arrow[dr, "\leq"] & & \\
\mathcal{ML} \arrow[r] & \mathcal{ML}^m_\emptyset(\langle\langle r\rangle\rangle) \arrow[r] & \mathcal{ML}^m_\emptyset(\langle r\rangle) \arrow[ur, bend left] \arrow[dr, bend right] & & \mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e}, \textcircled{f}) \arrow[r, "\leq"] & \mathcal{HL}(\downarrow) \\
 & & & \mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e}) \arrow[ur, bend right] & &
\end{tikzcd}
$$

$\boxed{\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} < \mathcal{L}'}$

- All the memory logic fragments are between the basic modal logic and the logic $\mathcal{HL}(\downarrow)$ (and therefore below first order logic)

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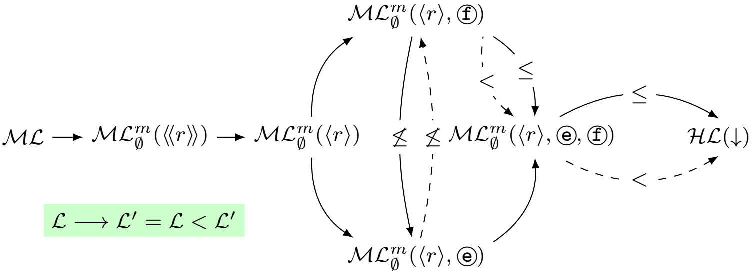

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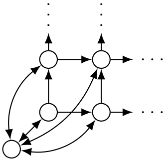
Decidability results

- We have encoded the tiling problem for several memory fragments using a *spy point*: a point that sees every other point in the model



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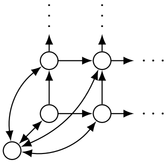

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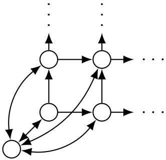

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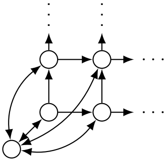

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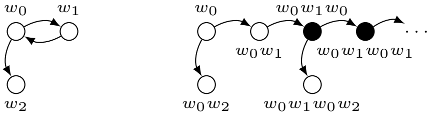
$\langle r \rangle$ and ⓡ working together

$$\mathcal{M}, w \models \langle\!\langle r \rangle\!\rangle \varphi \quad \text{iff} \quad \exists w' \in W, R_r(w, w') \text{ and } \mathcal{M}[w], w' \models \varphi.$$

Decidability results

- We proved that some fragments are PSPACE-complete showing that they enjoy the bounded tree-model property: every satisfiable formula can be satisfied in a bounded tree
- We showed that there is a procedure to transform an arbitrary model into a tree-like model, preserving equivalence



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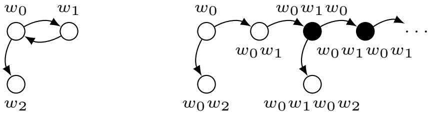

- We also built a "decidability map" for the different memory fragments

PSPACE-complete	Undecidable
$\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle)$	$\mathcal{ML}^m_\emptyset(\langle\!\langle r \rangle\!\rangle)$, $\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle) + i$
$\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle, ⓕ)$	$\mathcal{ML}^m(\langle r \rangle)$, ...

Axiomatizations

- We characterized many memory logics fragments in terms of axiomatic systems *à la Hilbert*
- **Nominals** proved to be a very useful device to find sound and complete axiomatizations

Axiomatization for $\mathcal{HL}^m(@, \langle r \rangle)$

All axioms and rules for $\mathcal{HL}(@)$

$$+$$

$$\vdash @_i(\textcircled{\texttt{r}}\varphi \leftrightarrow \varphi[\textcircled{\texttt{k}}/(\textcircled{\texttt{k}} \vee i)])$$

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All axioms and rules for $\mathcal{HL}(@)$

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$$\vdash @_i(ⓡ\varphi \leftrightarrow \varphi[ⓚ/(ⓚ \vee i)])$$

- We found sound and complete axiomatizations for all the *hybrid* memory fragments (and establish automatic completeness for pure extensions)
- We could provide axiomatizations for some cases even in the absence of nominals (i.e., $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$ and $\mathcal{ML}^m(\langle\langle r \rangle\rangle, ⓕ)$)
- The tree-model property was a key feature to use when nominals were not present

Tableau systems

- We presented a sound and complete tableau system for $\mathcal{ML}^m(\langle r \rangle, ⓔ, ⓕ)$, $\mathcal{ML}^m_\emptyset(\langle r \rangle, ⓔ, ⓕ)$, and its sublanguages
- It is a *prefixed* tableau where we use prefixed formulas with the shape

$$\langle w, R, F \rangle^{\mathcal{C}} : \varphi$$

- w: point of evaluation
- R: set of memorized labels
- F: set of forgotten labels
- $\mathcal{C}$: either $\mathcal{C}_\emptyset$ or the class of all models
- φ: current formula

- The rules for propositional and modal operators are standard

Tableau systems

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$$(\textcircled{k}) \qquad \frac{\langle w, \{v_1, \ldots v_k\}, F\rangle^C : \textcircled{k}}{w \approx v_1 \mid \cdots \mid w \approx v_k \mid \langle w, \emptyset, \emptyset\rangle^C : \textcircled{k}}$$

$$(\text{repl}) \qquad \frac{\begin{array}{c}\langle w, R, F\rangle^C : \varphi \\ w \approx^* w'\end{array}}{\langle w', R[w \mapsto w'], F[w \mapsto w']\rangle^C : \varphi}$$

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- Since this fragment in undecidable, the tableau is non-terminating
- We also provided a sound, complete and terminating tableau for the decidable fragments

Open questions

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- Beth definability needs further research, we would like some general result
- We want to explore the relation between memory logics and other dynamic logics (DEL is a good candidate). This could also lead to decidable fragments
- Can we find suitable axiomatizations in the absence of nominals. We still don't have one for $\mathcal{ML}^m(\langle r \rangle)$!

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- Areces, C., Figueira, S., and Mera, S.. Completeness results for memory logics. Annals of Pure and Applied Logic, 163(7):961–972, 2012.
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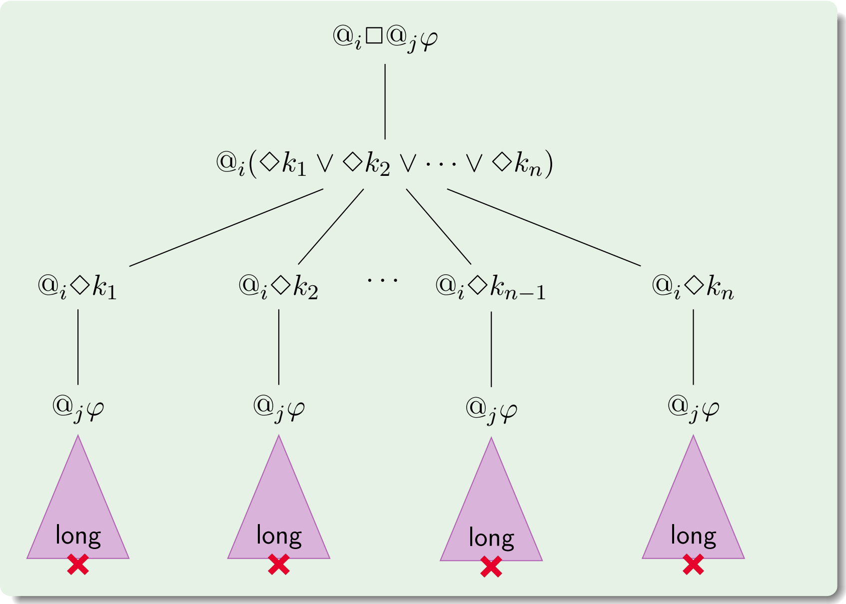
Part III

Coinduction, extractability, normal forms

Global modalities should be "extracted"



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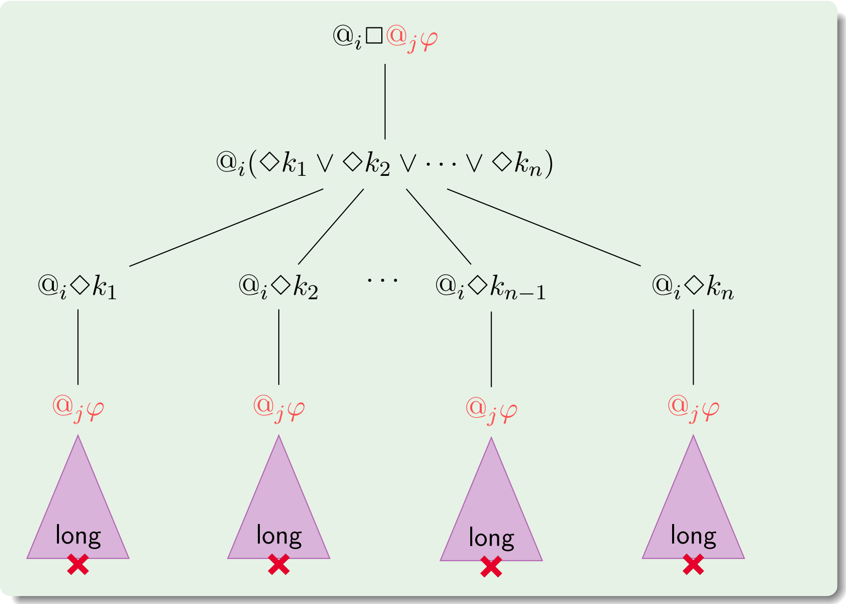

Global modalities should be "extracted"

$$@_j\varphi \vee @_i\Box\bot$$

$$@_i(\Diamond k_1 \vee \Diamond k_2 \vee \cdots \vee \Diamond k_n)$$

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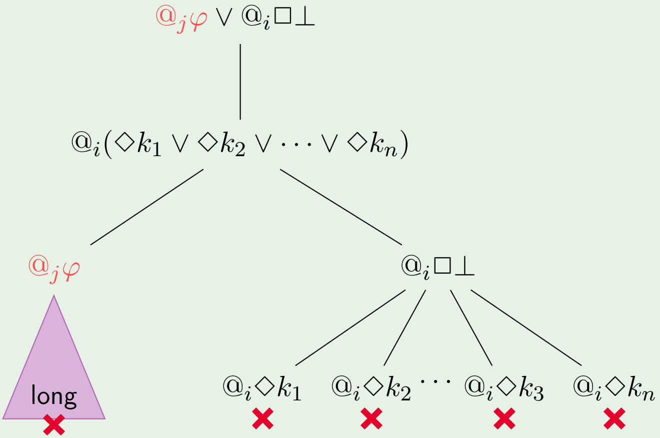

Globality $\sim$ extractability?

Global modalities are extractable from other modalities...

$$[r]@_i\varphi \equiv [r]\bot \vee @_i\varphi \qquad [r]\mathsf{A}\varphi \equiv [r]\bot \vee \mathsf{A}\varphi$$

$$@_j@_i\varphi \equiv @_j\bot \vee @_i\varphi \qquad @_j\mathsf{A}\varphi \equiv @_j\bot \vee \mathsf{A}\varphi$$

$$\mathsf{A}@_i\varphi \equiv \mathsf{A}\bot \vee @_i\varphi \qquad \mathsf{A}\mathsf{A}\varphi \equiv \mathsf{A}\bot \vee \mathsf{A}\varphi$$

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$$\vdots \qquad\qquad \vdots$$

...but some modalities are more equal than others

$$\downarrow i.@_i\varphi \not\equiv \downarrow i.\bot \vee @_i\varphi$$

$$\textcircled{r}\mathsf{A}\varphi \not\equiv \textcircled{r}\bot \vee \mathsf{A}\varphi$$

Coinductive models – a unifying framework

The class of all (rooted) Kripke models with domain W

- $\mathrm{Kripke}_W \stackrel{def}{=}$ all the tuples $\langle W, w_0, V, R\rangle$ such that
  - $w_0 \in W$
  - $V(p) \subseteq W$
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- More modal operators can be defined as classes of models

Defining Conditions

Defining condition
$\mathcal{P}_{\mathsf{A}}(\mathcal{M}) \iff R^{\mathcal{M}}(\mathsf{A}, w) = \{\langle v, |\mathcal{M}|, V^{\mathcal{M}}, R^{\mathcal{M}}\rangle \mid v \in |\mathcal{M}|\}$

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$\mathcal{P}_{\downarrow i}(\mathcal{M}) \iff R^{\mathcal{M}}(\downarrow i, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[i \mapsto \{w\}], R^{\mathcal{M}}\rangle\}, i \in \mathsf{Nom}$
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$\mathcal{P}_{\texttt{f}}(\mathcal{M}) \iff R^{\mathcal{M}}(\texttt{f}, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[\texttt{k} \mapsto V^{\mathcal{M}}(\texttt{k}) \setminus \{w\}], R^{\mathcal{M}}\rangle\}$
$\mathcal{P}_{\texttt{f}}(\mathcal{M}) \iff R^{\mathcal{M}}(\texttt{e}, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[\texttt{k} \mapsto \emptyset], R^{\mathcal{M}}\rangle\}$

Some initial results using the coinductive framework

- The basic modal logic is complete wrt coinductive models
- *Bisimulations*: one size fits all
- General conditions that guarantee extractability
- Extractability is preserved when new operators are added

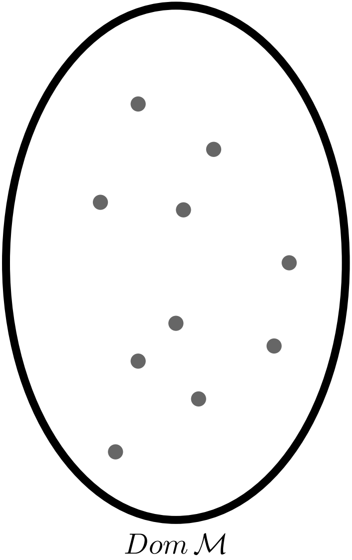
References

- Areces, C. and Gorín, D.. Coinductive models and normal forms for modal logics (or how we learned to stop worrying and love coinduction). Journal of Applied Logic, 8(4):305–318, Elsevier, 2010.

Part IV

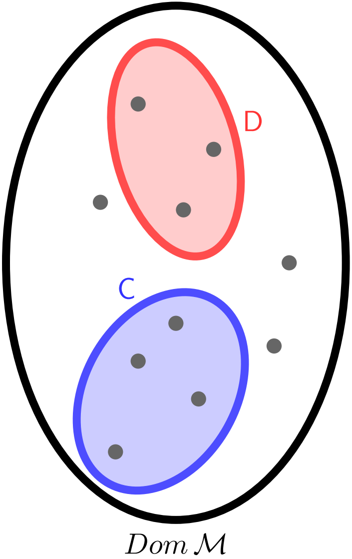
Logical methods in the generation de referring expressions

Separations and descriptions



- Let $\mathcal{M}$ be a Kripke model

Separations and descriptions



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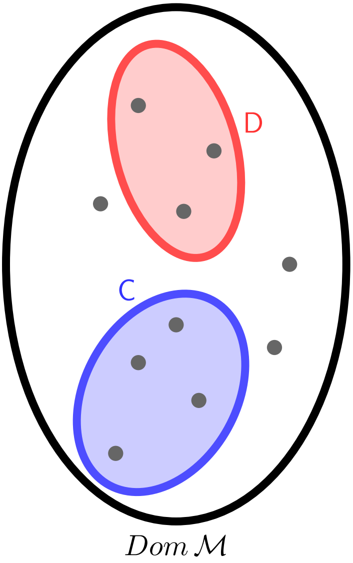

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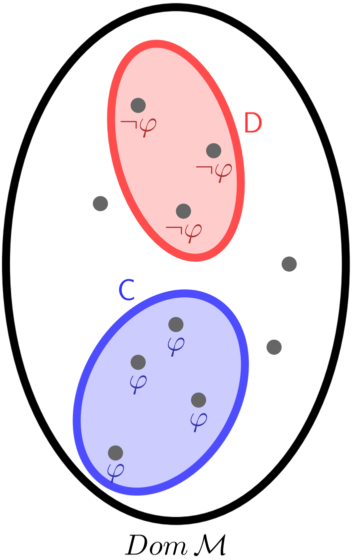

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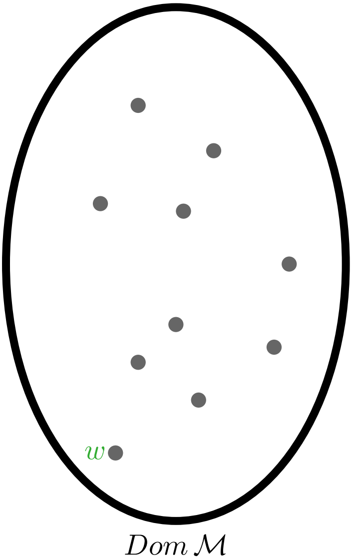

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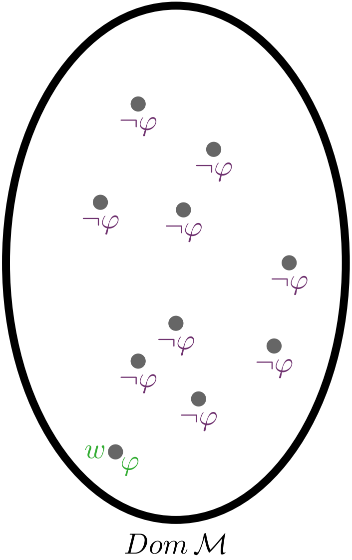

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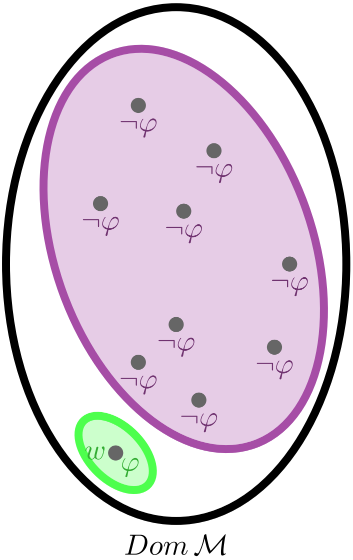

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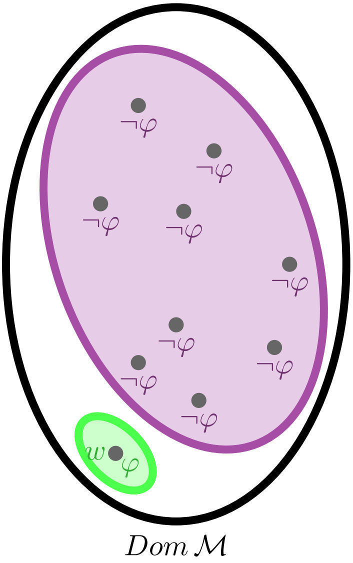
Separations and descriptions



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- Description is a form of separation
- φ could be of any suitable logic

Separation and description problems

The separation problem

Given a finite model $\mathcal{M}$ and sets $\mathsf{C}, \mathsf{D} \subset Dom\,\mathcal{M}$, find a φ that separates C and D, if possible.

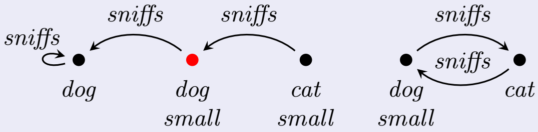
The description problem

Given a finite model $\mathcal{M}$ and a world $w \in Dom\,\mathcal{M}$, find a φ that describes w, if possible.

- They can be seen as another kind of inference task
- But they didn't receive much attention so far
- We are interested in their computational properties

Motivation: Generation of Referring Expressions

An application of logics in Natural Language Generation



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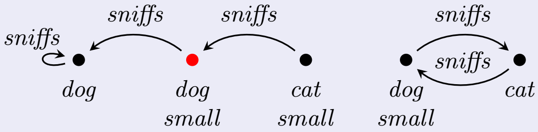

- "the dog that is sniffing another dog"

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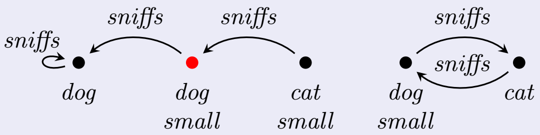

content determination

(e.g., as a FOL formula)

- $dog(x) \wedge \exists y.(x \not\approx y \wedge dog(y) \wedge sniffs(x, y))$

surface realization

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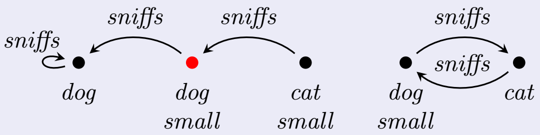

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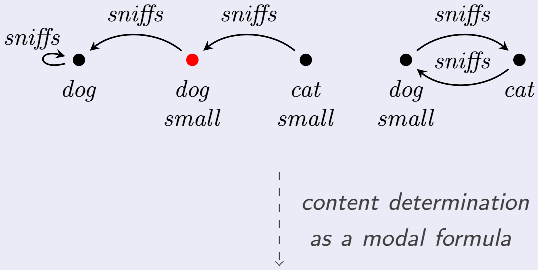

$$dog \wedge small \wedge \langle sniffs \rangle dog$$

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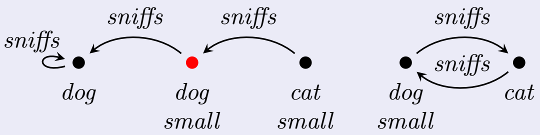


content determination as a modal formula

• $dog \wedge small \wedge \langle sniffs \rangle dog$

surface realization

• "the small dog that is sniffing a dog"

(logical) content determination $\approx$ description problem

Motivation: Modal logics in the GRE

Areces, Koller & Striegnitz (2008)

- We propose modal logics for content determination:

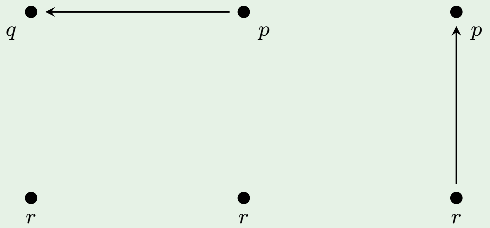
  $\mathcal{ML}$ – the basic modal language $(\neg, \wedge, \Diamond)$

  $\mathcal{EL}$ – the existential positive fragment of $\mathcal{ML}$ $(\wedge, \Diamond)$
- *Rationale*:
  - Good expressive power
  - Simple surface realization algorithms
  - Relatively low computational complexity for inference tasks
- In particular, we show that:

  "The modal description problem needs polynomial time"

The modal description problem in polynomial time

A variation of Tarjan's bisimulation contraction algorithm



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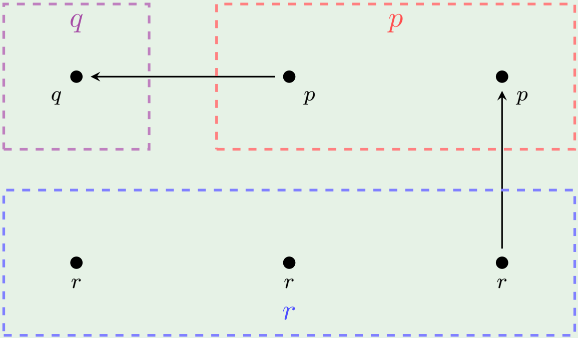

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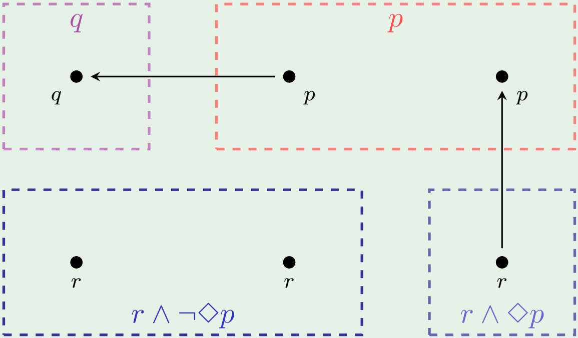

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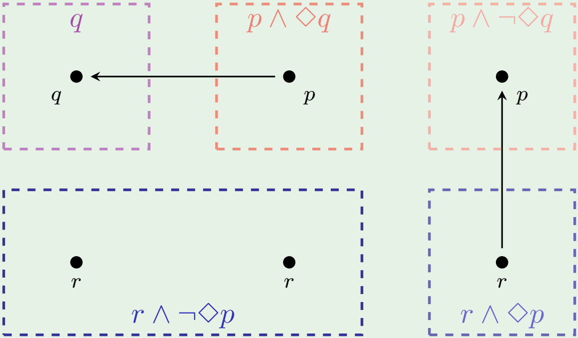

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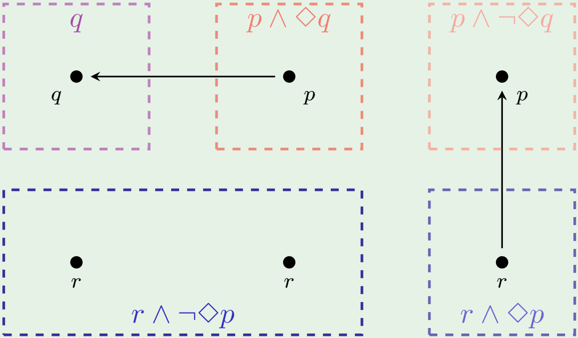

- Tarjan's algorithm runs in polynomial time
- Hence, the modal description problem is polynomial

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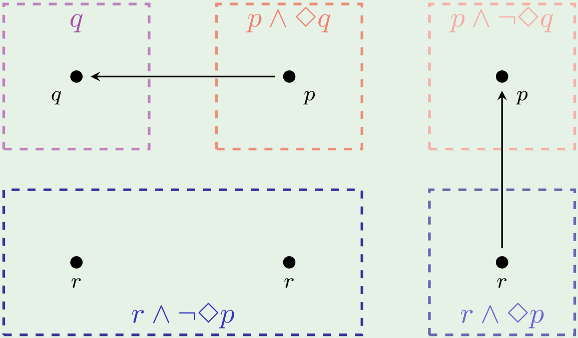

- Tarjan's algorithm runs in polynomial time
- Hence, the modal description problem is polynomial
- But this is assuming that $\wedge$ takes constant time!

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for DAG representation!

- This algorithm produces a formula represented as a DAG

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- Is the *tree* representation of this formula also polynomial?
- If not, "modal content determination" can't be said to take polynomial time

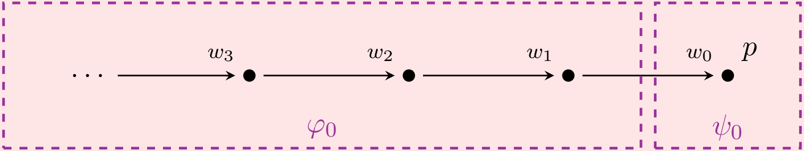
The modal description problem in polynomial time

also for tree representation?

$$\cdots \longrightarrow w_3 \longrightarrow w_2 \longrightarrow w_1 \longrightarrow w_0 \quad p$$

The modal description problem in polynomial time

also for tree representation?



$\varphi_0 = \neg p$ $\psi_0 = p$

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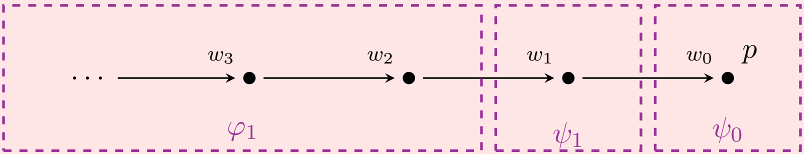

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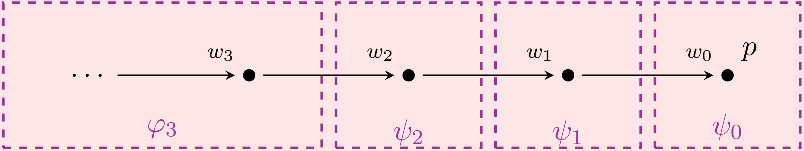

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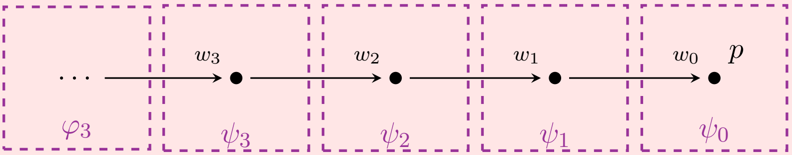

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$$\vdots \qquad \vdots$$

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- Each ψ_i is description for w_i with size exponential in i

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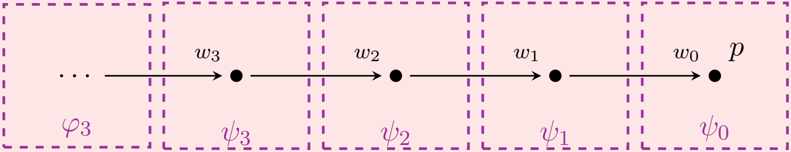

$$
\begin{aligned}
\varphi_0 &= \neg p & \psi_0 &= p \\
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&\vdots & &\vdots \\
\varphi_{i+1} &= \varphi_i \wedge \neg\Diamond\psi_i & \psi_{i+1} &= \varphi_i \wedge \Diamond\psi_i
\end{aligned}
$$

- Each ψ_i is description for w_i with size exponential in i
- Observe that w_i admits a linear description: $\underbrace{\Diamond\Diamond\ldots\Diamond}_{i \text{ times}}p$

Where do we go from here?

- The example shows that this algorithm is not polynomial
- Can we *fix* it?
- Can we find *another* one that is indeed polynomial?

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- The example shows that this algorithm is not polynomial
- Can we *fix* it?
- Can we find *another* one that is indeed polynomial?
- We show that no such algorithm exists!

Bounds for the separation / description problems

Basic modal language $\mathcal{ML}$

Theorem (Lower bound)

Any upper bound for the size of a solution for the separation or description problem for $\mathcal{ML}$ *is at least exponential.*

Corollary

No polynomial time algorithm exists that solves the description or separation problem returning the formula as a tree.

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Theorem (Upper bound)

If $\varphi \in \mathcal{ML}$ is a minimum description for v in $\mathcal{M} = \langle W, R, V \rangle$, then $|\varphi| \in O(2^{\frac{1}{2}|W|^2} \cdot |V|)$.