

Worksheet 1

Modal Logics: A Modern Perspective

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1. Review

Exercise 1.1. Given a formula φ in the basic modal language, we define the set of positive, negative and all propositional symbols ($Pos(\varphi)$, $Neg(\varphi)$ y $Prop(\varphi)$ respectively) as follows:

φ	$Prop(\varphi)$	$Pos(\varphi)$	$Neg(\varphi)$
p	$\{p\}$	$\{p\}$	\emptyset
$\neg\varphi$	$Prop(\varphi)$	$Neg(\varphi)$	$Pos(\varphi)$
$\varphi \wedge \psi$	$Prop(\varphi) \cup Prop(\psi)$	$Pos(\varphi) \cup Pos(\psi)$	$Neg(\varphi) \cup Neg(\psi)$
$\langle R \rangle \varphi$	$Prop(\varphi)$	$Pos(\varphi)$	$Neg(\varphi)$

- (a) Prove (by induction) that for any formula φ

$$Prop(\varphi) = Pos(\varphi) \cup Neg(\varphi)$$

and give an example where $Pos(\varphi) \cap Neg(\varphi) \neq \emptyset$.

Exercise 1.2. Given a formula φ , we say that $Sub(\varphi)$ is the set of all subformulas of φ .

- (a) Give a recursive definition of Sub for the basic modal language and for PDL with tests.
 (b) Write down the set $Sub([R]([R]p \rightarrow p) \rightarrow [R]p)$.
 (c) Prove that if $\psi \in Sub(\varphi)$ then $Sub(\psi) \subseteq Sub(\varphi)$ for any ψ and φ .

2. Modal Language

Exercise 2.1. If $K\varphi$ means “the agent knows that φ ” and $M\varphi$ means “it is consistent with what the agent knows that φ ”, write down formulas that represent the following statements:

- (a) If φ is true, then it is consistent with the knowledge of the agent that (s)he knows φ .
 (b) If it is consistent with the knowledge of the agent that φ , and it is consistent with his/her knowledge that ψ , then it is consistent with his/her knowledge that $\varphi \wedge \psi$.
 (c) If the agent knows φ , then it is consistent with what (s)he knows that φ .
 (d) If it is consistent with what the agent knows that it is consistent with what (s)he knows that φ , then it is consistent with what (s)he knows that φ .

Which of this statements “make sense”?

Exercise 2.2. Suppose that $\Diamond\varphi$ is interpreted as “ φ is permitted. How should $\Box\varphi$ be interpreted? List some formulas which seems “plausible” under this interpretation.

Exercise 2.3. Consider PDL with tests. Write down formulas that could represent:

- (a) **while** φ **do** ψ
- (b) **repeat** φ **until** ψ

Exercise 2.4. The basic temporal language has two modalities, $\langle F \rangle$ and $\langle P \rangle$. The interpretation of $\langle F \rangle \varphi$ is “ φ is going to be true at some future time”, while $\langle P \rangle \varphi$ means “ φ was true at some past time”. Decide if the following formulas should be valid under this interpretation¹:

- (a) $\langle F \rangle \varphi \rightarrow \langle F \rangle \langle F \rangle \varphi$
- (b) $\langle F \rangle \varphi \rightarrow \langle P \rangle \langle F \rangle \varphi$
- (c) $\neg \langle F \rangle \neg \varphi \rightarrow \langle P \rangle \langle F \rangle \varphi$
- (d) $\langle P \rangle \varphi \rightarrow \langle F \rangle \neg \langle P \rangle \neg \varphi$

3. Satisfiability - Kripke Models

Exercise 3.1. Show that when evaluating a formula φ in a model, only the propositional symbols appearing in φ are relevant. I.e., show that given two models $\mathcal{M} = \langle W, \{R_i\}, V \rangle$ and $\mathcal{M}' = \langle W, \{R_i\}, V' \rangle$ s.t. $V(p) = V'(p)$ for all propositional symbols in φ , then for any w , $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w \models \varphi$.

Exercise 3.2. Let $\mathcal{N} = \langle \mathbb{N}, \{S_1, S_2\}, V_{\mathcal{N}} \rangle$ and $\mathcal{B} = \langle \mathbb{B}, \{R_1, R_2\}, V_{\mathcal{B}} \rangle$ be two models for a modal language with two modalities \Diamond_1 and \Diamond_2 . \mathbb{N} is the set of natural numbers, and \mathbb{B} is the set of all strings containing only 0s and 1s. The accessibility relations are defined as:

$$\begin{aligned}
 mS_1n & \text{ iff } n = m + 1 \\
 mS_2n & \text{ iff } m > n \\
 sR_1t & \text{ iff } t = s0 \text{ or } t = s1 \\
 sR_2t & \text{ iff } t \text{ is a proper initial segment of } s
 \end{aligned}$$

Which of the following formulas are true over \mathcal{N} and \mathcal{B} with respect to an arbitrary valuation?

- (a) $(\Diamond_1 p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge q)$
- (b) $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$
- (c) $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$
- (d) $p \rightarrow \Diamond_1 \Box_2 p$
- (e) $p \rightarrow \Diamond_2 \Box_1 p$
- (f) $p \rightarrow \Box_1 \Diamond_2 p$
- (g) $p \rightarrow \Box_2 \Diamond_1 p$

Exercise 3.3. Consider the basic temporal language (see Exercise 2.4, including footnote) and the models $(\mathbb{Z}, <, V_1)$, $(\mathbb{Q}, <, V_2)$ and $(\mathbb{R}, <, V_3)$. Define $E\varphi$ as an abbreviation of $P\varphi \vee \varphi \vee F\varphi$ and $A\varphi$ to represent $H\varphi \wedge \varphi \wedge G\varphi$. Which of the following formulas are true for an arbitrary valuation in each of the models?

- (a) $GGp \rightarrow p$
- (b) $(p \wedge Hp) \rightarrow FHp$
- (c) $(Ep \wedge E\neg p \wedge A(p \rightarrow Hp) \wedge A(\neg p \rightarrow G\neg p)) \rightarrow E(Hp \wedge G\neg p)$

Exercise 3.4. Prove that the following formulas of the basic modal language are not valid, showing a model where they are false.

- (a) $\Box \perp$
- (b) $\Diamond p \rightarrow \Box p$
- (c) $p \rightarrow \Box \Diamond p$

¹Historically, these modalities were written as $F\varphi$ and $P\varphi$ and its duals as $G\varphi$ and $H\varphi$.

(d) $\Diamond \Box p \rightarrow \Box \Diamond p$

Exercise 3.5. Demostrar que en todo modelo donde la relación de accesibilidad es transitiva, las siguientes fórmulas son verdaderas para cualquier valuación arbitraria:

(a) $\Box \Diamond \Diamond p \rightarrow \Box \Diamond p$

(b) $\Box \Diamond \Box \Diamond p \rightarrow \Box \Diamond p$

Exercise 3.6. Prove that the formula

$$\Diamond(i \wedge q) \wedge \Diamond(i \wedge p) \rightarrow \Diamond(p \wedge q)$$

in the basic hybrid logic is valid. Show a counter-example when i is replaced by a propositional symbol.

Exercise 3.7. Prove that in the hybrid logic $\mathcal{HL}(@)$ the $@$ operator defines a congruence relation. I.e., that the following formulas are valid:

- $\models @_i i.$
- If $\models @_i j$ then $\models @_j i.$
- If $\models @_i j$ and $\models @_j k$ then $\models @_i k.$
- If $\models @_i j$ then $\models @_i \varphi \leftrightarrow @_j \varphi.$

Exercise 3.8. Consider the logic $\mathcal{HL}(A)$, which is the basic modal logic extended with nominals and the universal operator A . Give a definition of $@$ in $\mathcal{HL}(A)$.