Mathematics for Informatics

Carlos Areces and Patrick Blackburn

areces@loria.fr blackbur@loria.fr

http://www.loria.fr/~areces http://www.loria.fr/~blacbur

INRIA Lorraine
Nancy, France

2007/2008

Sets and computability theory

When we talk of a set A of natural numbers we can think of the characteristic function of that set.

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Hence, a set can be:

Sets and computability theory

When we talk of a set A of natural numbers we can think of the characteristic function of that set.

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Hence, a set can be:

- computable
- primitive recursive

Sets and computability theory

When we talk of a set A of natural numbers we can think of the characteristic function of that set.

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Hence, a set can be:

- computable
- primitive recursive

Theorem
Let A, B be sets in a PRC class $\mathcal{C}$. Then $A \cup B$, $A \cap B$ and $\overline{A}$ are $\mathcal{C}$.

Recursively enumerable sets

As with functions

- there are computable sets, for example
  - $\emptyset$
  - $\mathbb{N}$
  - $\{p : p \text{ is prime}\}$

Recursively enumerable sets

As with functions

- there are computable sets, for example
  - $\emptyset$
  - $\mathbb{N}$
  - $\{p : p \text{ is prime}\}$
- there are set which are not computable, as
  - $\{\langle x, y\rangle : \mathrm{HALT}(x, y)\}$
  - $\{\langle x, \langle y, z\rangle\rangle : \Phi_x(y) = z\}$

Recursively enumerable sets

As with functions

- there are computable sets, for example
  - $\emptyset$
  - $\mathbb{N}$
  - $\{p : p \text{ is prime}\}$
- there are set which are not computable, as
  - $\{\langle x, y\rangle : \mathrm{HALT}(x, y)\}$
  - $\{\langle x, \langle y, z\rangle\rangle : \Phi_x(y) = z\}$

A set A is recursively enumerable (r.e.) when there is a partially computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{x : g(x) \downarrow\}$$

Recursively enumerable sets

As with functions

- there are computable sets, for example
  - $\emptyset$
  - $\mathbb{N}$
  - $\{p : p \text{ is prime}\}$
- there are set which are not computable, as
  - $\{\langle x, y\rangle : \mathrm{HALT}(x, y)\}$
  - $\{\langle x, \langle y, z\rangle\rangle : \Phi_x(y) = z\}$

A set A is recursively enumerable (r.e.) when there is a partially computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{x : g(x)\downarrow\} = \mathrm{dom}\ g$$

Recursively enumerable sets

As with functions

- there are computable sets, for example
  - $\emptyset$
  - $\mathbb{N}$
  - $\{p : p \text{ is prime}\}$
- there are set which are not computable, as
  - $\{\langle x, y\rangle : \mathrm{HALT}(x, y)\}$
  - $\{\langle x, \langle y, z\rangle\rangle : \Phi_x(y) = z\}$

A set A is recursively enumerable (r.e.) when there is a partially computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{x : g(x) \downarrow\} = \text{dom } g$$

- we can algorithmically decide if an element does belongs to A, but when an element does not belongs to A, the algorithms is indefined
- these are usually call semi-decision algorithms: the solve an approximation to the problem of deciding membership of an element in a set A

Some properties of r.e. sets

Theorem

If A is computable then A is r.e.

Some properties of r.e. sets

Theorem

If A is computable then A is r.e.

Proof.

Let P_A be a program for [the characteristic function of] A. Let's consider the following program P:

$$[C] \qquad \text{IF } P_A(X) = 0 \text{ GOTO } C$$

Some properties of r.e. sets

Theorem

If A is computable then A is r.e.

Proof.

Let P_A be a program for [the characteristic function of] A. Let's consider the following program P:

$$[C] \qquad \text{IF } P_A(X) = 0 \text{ GOTO } C$$

Then

$$\Psi_P(x) = \begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

Some properties of r.e. sets

Theorem

If A is computable then A is r.e.

Proof.

Let P_A be a program for [the characteristic function of] A. Let's consider the following program P:

$$[C] \qquad \text{IF } P_A(X) = 0 \text{ GOTO } C$$

Then

$$\Psi_P(x) = \begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

and then

$$A = \{x : \Psi_P(x) \downarrow\}$$

$\square$

Some properties of r.e. sets

Theorem

If A and B are r.e. then $A \cup B$ and $A \cap B$ are also r.e.

Some properties of r.e. sets

Theorem

If A and B are r.e. then $A \cup B$ and $A \cap B$ are also r.e.

Proof.

Let $\quad A = \{x : \Phi_p(x) \downarrow\} \quad , \quad B = \{x : \Phi_q(x) \downarrow\}$

Some properties of r.e. sets

Theorem

If A and B are r.e. then $A \cup B$ and $A \cap B$ are also r.e.

Proof.

Let $A = \{x : \Phi_p(x) \downarrow\}$, $B = \{x : \Phi_q(x) \downarrow\}$

$(A \cap B)$ The following program R computes $A \cap B$:

$$Y \leftarrow \Phi_p(x)$$
$$Y \leftarrow \Phi_q(x)$$

Indeed, $\Psi_R(x) \downarrow$ iff $\Phi_p(x) \downarrow$ and $\Phi_q(x) \downarrow$.

Some properties of r.e. sets

Theorem

If A and B are r.e. then $A \cup B$ and $A \cap B$ are also r.e.

Proof.

Let $\quad A = \{x : \Phi_p(x)\downarrow\} \quad , \quad B = \{x : \Phi_q(x)\downarrow\}$

$(A \cap B)$ The following program R computes $A \cap B$:

$$
\begin{aligned}
&Y \leftarrow \Phi_p(x)\\
&Y \leftarrow \Phi_q(x)
\end{aligned}
$$

Indeed, $\Psi_R(x)\downarrow$ iff $\Phi_p(x)\downarrow$ and $\Phi_q(x)\downarrow$.

$(A \cup B)$ The following program R' computes $A \cup B$:

$$
\begin{aligned}
[C] \qquad &\text{IF } \mathrm{STP}^{(1)}(X, p, T) = 1 \text{ GOTO } E\\
&\text{IF } \mathrm{STP}^{(1)}(X, q, T) = 1 \text{ GOTO } E\\
&T \leftarrow T + 1\\
&\text{GOTO } C
\end{aligned}
$$

Indeed, $\Psi_{R'}(x)\downarrow$ iff $\Phi_p(x)\downarrow$ or $\Phi_q(x)\downarrow$.

Some properties of r.e. sets

Theorem

A is computable iff A and $\overline{A}$ are r.e.

Some properties of r.e. sets

Theorem

A is computable iff A and $\overline{A}$ are r.e.

Proof.

($\Rightarrow$) if A is computable then $\overline{A}$ is computable

Some properties of r.e. sets

Theorem

A is computable iff A and $\overline{A}$ are r.e.

Proof.

($\Rightarrow$) if A is computable then $\overline{A}$ is computable

($\Leftarrow$) suppose that A and $\overline{A}$ are r.e.

$$A = \{x : \Phi_p(x) \downarrow\} \qquad , \qquad \overline{A} = \{x : \Phi_q(x) \downarrow\}$$

Some properties of r.e. sets

Theorem

A is computable iff A and $\overline{A}$ are r.e.

Proof.

($\Rightarrow$) if A is computable then $\overline{A}$ is computable

($\Leftarrow$) suppose that A and $\overline{A}$ are r.e.

$$A = \{x : \Phi_p(x)\downarrow\} \qquad , \qquad \overline{A} = \{x : \Phi_q(x)\downarrow\}$$

Consider P:

$$\begin{array}{ll}
[C] & \text{IF } \mathrm{STP}^{(1)}(X, p, T) = 1 \text{ GOTO } F \\
 & \text{IF } \mathrm{STP}^{(1)}(X, q, T) = 1 \text{ GOTO } E \\
 & T \leftarrow T + 1 \\
 & \text{GOTO } C \\
[F] & Y \leftarrow 1
\end{array}$$

For each x, $x \in A$ or $x \in \overline{A}$. Then Ψ_P computes A.

Enumeration theorem

Let's define

$$W_n = \{x : \Phi_n(x) \downarrow\} = \text{the domaine of the } n\text{-th program}$$

Theorem

A set A is r.e. iff there is an n such that $A = W_n$.

There is an enumeration of all the r.e. sets

$$W_0, W_1, W_2, \ldots$$

The halting problem (seen as a set)

Remember that

$$W_n = \{x : \Phi_n(x) \downarrow\}$$

Let's define

$$K = \{n : n \in W_n\}$$

Observe that

$$n \in W_n \qquad \text{iff} \qquad \Phi_n(n) \downarrow \qquad \text{iff} \qquad \mathrm{HALT}(n, n)$$

Theorem

K is r.e. but it is not computable.

The halting problem (seen as a set)

Remember that

$$W_n = \{x : \Phi_n(x) \downarrow\}$$

Let's define

$$K = \{n : n \in W_n\}$$

Observe that

$$n \in W_n \qquad \text{iff} \qquad \Phi_n(n) \downarrow \qquad \text{iff} \qquad \mathrm{HALT}(n, n)$$

Theorem

K is r.e. but it is not computable.

Proof.

- the function $\Phi(n, n)$ is partially computable, and hence K is r.e.

The halting problem (seen as a set)

Remember that

$$W_n = \{x : \Phi_n(x) \downarrow\}$$

Let's define

$$K = \{n : n \in W_n\}$$

Observe that

$$n \in W_n \qquad \text{iff} \qquad \Phi_n(n) \downarrow \qquad \text{iff} \qquad \mathrm{HALT}(n, n)$$

Theorem

K is r.e. but it is not computable.

Proof.

- the function $\Phi(n, n)$ is partially computable, and hence K is r.e.
- suppose that K is computable.

The halting problem (seen as a set)

Remember that

$$W_n = \{x : \Phi_n(x) \downarrow\}$$

Let's define

$$K = \{n : n \in W_n\}$$

Observe that

$$n \in W_n \qquad \text{iff} \qquad \Phi_n(n) \downarrow \qquad \text{iff} \qquad \mathrm{HALT}(n, n)$$

Theorem

K is r.e. but it is not computable.

Proof.

- the function $\Phi(n, n)$ is partially computable, and hence K is r.e.
- suppose that K is computable. Then $\overline{K}$ would also be.

The halting problem (seen as a set)

Remember that

$$W_n = \{x : \Phi_n(x) \downarrow\}$$

Let's define

$$K = \{n : n \in W_n\}$$

Observe that

$$n \in W_n \quad \text{iff} \quad \Phi_n(n) \downarrow \quad \text{iff} \quad \text{HALT}(n, n)$$

Theorem

K is r.e. but it is not computable.

Proof.

- the function $\Phi(n, n)$ is partially computable, and hence K is r.e.
- suppose that K is computable. Then $\overline{K}$ would also be. Hence there is e such that $\overline{K} = W_e$.

The halting problem (seen as a set)

Remember that

$$W_n = \{x : \Phi_n(x) \downarrow\}$$

Let's define

$$K = \{n : n \in W_n\}$$

Observe that

$$n \in W_n \qquad \text{iff} \qquad \Phi_n(n) \downarrow \qquad \text{iff} \qquad \mathrm{HALT}(n, n)$$

Theorem

K is r.e. but it is not computable.

Proof.

- the function $\Phi(n, n)$ is partially computable, and hence K is r.e.
- suppose that K is computable. Then $\overline{K}$ would also be. Hence there is e such that $\overline{K} = W_e$. But then

$$e \in K \qquad \text{iff} \qquad e \in W_e \qquad \text{iff} \qquad e \in \overline{K}$$

More properties of r.e. sets

Theorem

If A is r.e., then there is a p.r. predicate $R : \mathbb{N}^2 \to \mathbb{N}$ such that

$$A = \{x : (\exists t)\ R(x, t)\}$$

More properties of r.e. sets

Theorem

If A is r.e., then there is a p.r. predicate $R : \mathbb{N}^2 \to \mathbb{N}$ such that

$$A = \{x : (\exists t)\ R(x, t)\}$$

Proof.

Let $A = W_e$. I.e.,

$$A = \{x : \Phi_e(x) \downarrow\}.$$

More properties of r.e. sets

Theorem

If A is r.e., then there is a p.r. predicate $R : \mathbb{N}^2 \to \mathbb{N}$ such that

$$A = \{x : (\exists t)\ R(x, t)\}$$

Proof.

Let $A = W_e$. I.e.,

$$A = \{x : \Phi_e(x) \downarrow\}.$$

Then $x \in A$ when for some t, the program e with input x halts, i.e.,

$$A = \{x : (\exists t)\ \underbrace{\mathrm{STP}^{(1)}(x, e, t)}_{R(x,t)}\}$$

$\square$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

Let $a \in A$ and define

$$f(u) = \begin{cases} l(u) & \text{if } P(l(u), r(u)) \\ a & \text{otherwise} \end{cases}$$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

Let $a \in A$ and define

$$f(u) = \begin{cases} l(u) & \text{if } P(l(u), r(u)) \\ a & \text{otherwise} \end{cases}$$

- $x \in A$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

Let $a \in A$ and define

$$f(u) = \begin{cases} l(u) & \text{if } P(l(u), r(u)) \\ a & \text{otherwise} \end{cases}$$

- $x \in A \Rightarrow$ there is t such that $P(x, t)$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

Let $a \in A$ and define

$$f(u) = \begin{cases} l(u) & \text{if } P(l(u), r(u)) \\ a & \text{otherwise} \end{cases}$$

- $x \in A \Rightarrow$ there is t such that $P(x, t) \Rightarrow f(\langle x, t \rangle) = x$

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

Let $a \in A$ and define

$$f(u) = \begin{cases} l(u) & \text{if } P(l(u), r(u)) \\ a & \text{otherwise} \end{cases}$$

- $x \in A \Rightarrow$ there is t such that $P(x, t) \Rightarrow f(\langle x, t \rangle) = x$
- let x be such that $f(u) = x$ for some u.

More properties of r.e. sets

Theorem

If $A \neq \emptyset$ is r.e., there is a p.r. function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{f(0), f(1), f(2), \dots\}$$

Proof.

By the previous theorem, there is P p.r. such that

$$A = \{x : (\exists t)\ P(x, t)\}.$$

Let $a \in A$ and define

$$f(u) = \begin{cases} l(u) & \text{if } P(l(u), r(u)) \\ a & \text{otherwise} \end{cases}$$

- $x \in A \Rightarrow$ there is t such that $P(x, t) \Rightarrow f(\langle x, t\rangle) = x$
- let x be such that $f(u) = x$ for some u. Then $x = a$ or u is of the form $u = \langle x, t\rangle$, with $P(x, t)$. Then $x \in A$.

More properties of r.e. sets

Theorem

If $f : \mathbb{N} \to \mathbb{N}$ *is partially computable,* $A = \{f(x) : f(x) \downarrow\}$ *is r.e.*

More properties of r.e. sets

Theorem

If $f : \mathbb{N} \to \mathbb{N}$ is partially computable, $A = \{f(x) : f(x)\downarrow\}$ is r.e.

Proof.

Let $\Phi_p = f$. Let's define P'

$$
\begin{array}{ll}
[A] & \text{IF } \mathrm{STP}^{(1)}(Z, p, T) = 0 \text{ GOTO } B \\
 & \text{IF } \Phi_p(Z) = X \text{ GOTO } E \\
[B] & Z \leftarrow Z + 1 \\
 & \text{IF } Z \leq T \text{ GOTO } A \\
 & T \leftarrow T + 1 \\
 & Z \leftarrow 0 \\
 & \text{GOTO } A
\end{array}
$$

More properties of r.e. sets

Theorem

If $f : \mathbb{N} \to \mathbb{N}$ is partially computable, $A = \{f(x) : f(x)\downarrow\}$ is r.e.

Proof.

Let $\Phi_p = f$. Let's define P'

$$
\begin{array}{ll}
[A] & \text{IF } \mathrm{STP}^{(1)}(Z, p, T) = 0 \text{ GOTO } B \\
 & \text{IF } \Phi_p(Z) = X \text{ GOTO } E \\
[B] & Z \leftarrow Z + 1 \\
 & \text{IF } Z \leq T \text{ GOTO } A \\
 & T \leftarrow T + 1 \\
 & Z \leftarrow 0 \\
 & \text{GOTO } A
\end{array}
$$

Notice that $\Psi_{P'}(X)\downarrow$ if there are Z, T such that

- $Z \leq T$
- $\mathrm{STP}^{(1)}(Z, p, T)$ is true (i.e., the program for f halts in T or less steps with input Z)
- $X = f(Z)$

$$\Psi_{P'}(x) = \begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

Hence A is r.e. □

Characterization of r.e. sets

Theorem

If $A \neq \emptyset$, then the following are equivalents:

1. *A is r.e.*
2. *A is the range of a primitive recursive function*
3. *A is the range of a computable function*
4. *A is the range of a partially computable function*

Proof.

$(1 \Rightarrow 2)$ *theorem in slide 10*

$(2 \Rightarrow 3)$ *Trivial*

$(3 \Rightarrow 4)$ *Trivial*

$(4 \Rightarrow 1)$ *theorem in slide 11*

$\square$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable.

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e, x)$.

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e, x)$.

- $e \in A$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e, x)$.

- $e \in A \Rightarrow \Phi_e = g$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t,x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e,x)$.

- $e \in A \Rightarrow \Phi_e = g \Rightarrow \Phi_e \notin \mathcal{C}$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e, x)$.

- $e \in A \Rightarrow \Phi_e = g \Rightarrow \Phi_e \notin \mathcal{C} \Rightarrow e \notin A$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e, x)$.

- $e \in A \Rightarrow \Phi_e = g \Rightarrow \Phi_e \notin \mathcal{C} \Rightarrow e \notin A$
- $e \notin A$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t, x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e, x)$.

- $e \in A \Rightarrow \Phi_e = g \Rightarrow \Phi_e \notin \mathcal{C} \Rightarrow e \notin A$
- $e \notin A \Rightarrow \Phi_e = f$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t,x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e,x)$.

- $e \in A \Rightarrow \Phi_e = g \Rightarrow \Phi_e \notin \mathcal{C} \Rightarrow e \notin A$
- $e \notin A \Rightarrow \Phi_e = f \Rightarrow \Phi_e \in \mathcal{C}$

Rice theorem

$A \subseteq \mathbb{N}$ is an index set if there is a class of partially computable functions $\mathcal{C}$ such that $A = \{x : \Phi_x \in \mathcal{C}\}$

Theorem

If A is an index set such that $\emptyset \neq A \neq \mathbb{N}$, A is not computable.

Proof.

Suppose that $\mathcal{C}$ is such that $A = \{x : \Phi_x \in \mathcal{C}\}$ is computable. Let $f \in \mathcal{C}$ and $g \notin \mathcal{C}$ be partially computable functions.
Let $h : \mathbb{N}^2 \to \mathbb{N}$ be the partially computable function:

$$h(t,x) = \begin{cases} g(x) & \text{if } t \in A \\ f(x) & \text{otherwise} \end{cases}$$

By the Recursion theorem, there is e such that $\Phi_e(x) = h(e,x)$.

- $e \in A \Rightarrow \Phi_e = g \Rightarrow \Phi_e \notin \mathcal{C} \Rightarrow e \notin A$
- $e \notin A \Rightarrow \Phi_e = f \Rightarrow \Phi_e \in \mathcal{C} \Rightarrow e \in A$

$\square$

Applications of Rice theorem

The theorem is a source of non computable sets:

- $\{x : \Phi_x \text{ is total}\}$
- $\{x : \Phi_x \text{ is increasing}\}$
- $\{x : \Phi_x \text{ has an infinite domain}\}$
- $\{x : \Phi_x \text{ is primitive recursive}\}$

They are all non computable, as they are all non trivial index sets.