

Hybrid Logics

Completeness

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Bibliography

- ▶ Modal Logic Book, Chapter 4, by Blackburn, Venema and de Rijke.

Introduction

- ▶ A logic can be specified semantically or syntactically.
- ▶ The semantic approach corresponds to the notion of **satisfiability** of a formula (or validity $\models \varphi$).
- ▶ The syntactic approach corresponds to the notion of **theorem** $\vdash \varphi$ (usually defined in terms of **axiomatic systems**).
- ▶ The question of when these two alternatives define the same logic arises.
- ▶ More precisely, we want to see if the set of valid formulas is exactly the same as the set of theorems.
- ▶ The answer is found by results of **correctness** and **completeness** that show whether \models and \vdash are equal.

Modal Logics are Old

- ▶ Aristotle (384BC – 322BC) was already a modal logician.
- ▶ In addition to “classical syllogism,” Aristotle discusses **modal syllogism** that results from adding the qualifications “necessarily” and “possibly” to premises and conclusions, in various ways.
- ▶ He actually already discusses different modal logics, as he considers two possible definitions of “possibly P”:
 - ▶ “possibly P” as equivalent to “not necessarily not P”.
 - ▶ “possibly P” as equivalent to “not necessarily P and not necessarily not P”.



Rini, Adriane.

Aristotle's Modal Proofs: Prior Analytics

A8–22 in *Predicate Logic*, Dordrecht: Springer, 2011.

Modal Logics are Simple

- Take the language of Propositional Logic and add the modal operators (this is the basic modal language \mathcal{K})

$$\varphi, \psi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi$$

- Classically
 - $\Diamond\varphi$ stands for “ φ is possible”
 - $\Box\varphi$ stands for “ φ is necessary”
- Now, argue away!
 - Should $\Box\varphi \rightarrow \varphi$ be valid?
 - What about $\Box\varphi \rightarrow \Box\Box\varphi$?
 - ...

Modal Logics are Flexible

- “Possibility” and “Necessity” are just two of the many possible options ...
 - Deontic Logic
 - $O\varphi$ It is **obligatory** that φ
 - $P\varphi$ It is **permitted** that φ
 - $F\varphi$ It is **forbidden** that φ
 - Temporal Logic
 - $G\varphi$ It **will always be** the case that φ
 - $F\varphi$ It **will be** the case that φ
 - $H\varphi$ It **has always been** the case that φ
 - $P\varphi$ It **has been** the case that φ
 - Doxastic Logic
 - $B_x\varphi$ x **believes** that φ
 - ...

Then Things got Complicated

- Without a semantics it was difficult to argue, e.g., when two axioms were independent.
- Enters Kripke (and Carnap; McKinsey, Jónsson & Tarski; Prior & Meredith; Kanger; Hintikka; Montague; and Beth circa 1960)
 - $\Box\varphi$ is true (i.e., “it is true that φ is necessary”) if φ is true in “all possible worlds”.
 - Consider a structure $\langle W, V \rangle$ where W is a set of “possible worlds” and V a function that assigns a propositional valuation to each world (i.e., $V(w) \subseteq \text{PROP}$).



With this semantics, both $\Box\varphi \rightarrow \varphi$ and $\Box\varphi \rightarrow \Box\Box\varphi$ are valid.

The Accessibility Relation

By introducing an **accessibility relation** between possible worlds it was possible to define weaker logics.

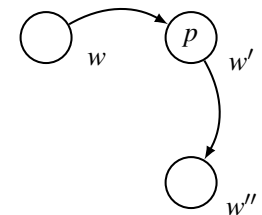
Nowadays, a **Kripke model** is a structure $\mathcal{M} = \langle W, R, V \rangle$ where W is a non-empty set, $R \subseteq W^2$ and $V : W \rightarrow 2^{\text{PROP}}$.

- I.e., A directed, labeled graph, or simply a relational model.

$$\mathcal{M}, w \models \Box\varphi \text{ iff } \forall w'. (Rww' \rightarrow \mathcal{M}, w' \models \varphi)$$

Examples

- $\mathcal{M}, w \not\models \Box p \rightarrow p$
- $\mathcal{M}, w \not\models \Box p \rightarrow \Box\Box p$



Completeness

- ▶ It was discovered that some axioms correspond to properties of the accessibility relation
 - ▶ $\Box p \rightarrow p \iff R$ is reflexive
 - ▶ $\Box p \rightarrow \Box \Box p \iff R$ is transitive
- ▶ ... and the first completeness results were proved.



Kripke, Saul.

A completeness theorem in modal logic.

The Journal of Symbolic Logic, 24, 1959.



Kripke, Saul.

Semantical analysis of modal logic I. Normal modal propositional calculi.

Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 9:67–96, 1963.

More Completeness

- ▶ Kripke's completeness proof employed a generalization of Beth's tableaux method.
- ▶ Completeness was established by showing how to derive a proof from a failed attempt to find a counter model.
- ▶ Kaplan (1966) criticized Kripke's proof as lacking in rigor and as making excessive use of “intuitive” arguments on the geometry of tableau proofs.
- ▶ He suggested a different, arguably, more elegant approach based on an adaptation of [Henkin's model theoretic completeness proof](#) for first-order logic.
(Kaplan was not the first neither the only: Bayart 1959, Makinson 1966, Cresswell 1967)

Henkin's Completeness

- ▶ Henkin's completeness proof for first-order logic uses (at least) two important ideas
 1. A consistent set of formulas can be extended to a [maximally consistent set of formulas](#)
 2. Existential quantifiers can be [witnessed using constants](#), which can then be used to form the domain of the model.
- ▶ (1) is key in modern completeness proofs for propositional modal logics which build a canonical model (satisfying all consistent formulas) that has as domain the (uncountable) set of all maximally consistent sets of formulas.
- ▶ (2) seemed less useful in a propositional setting.

Basic Definitions

- ▶ A [Modal Logic](#) Δ is a set of modal formulas that contain [all propositional tautologies](#) and is closed under:
 - i. [Modus Ponens](#): If $\varphi \in \Delta$ y $\varphi \rightarrow \psi \in \Delta$ then $\psi \in \Delta$.
 - ii. [Uniform Substitution](#): If $\varphi \in \Delta$ then any substitution also belongs. A substitution consists in uniformly replacing any proposition with an arbitrary formula
- ▶ If $\varphi \in \Delta$ we say that φ is a [theorem](#) of Δ , and we write $\vdash_{\Delta} \varphi$.
- ▶ If Δ_1 and Δ_2 are modal logics such that $\Delta_1 \subseteq \Delta_2$ we say that Δ_2 is an [extension](#) of Δ_1

Some Examples

- i) FORM the set of all the formulas is a logic. The **inconsistent** logic.
- ii) If $\{\Delta_i \mid i \in I\}$ is a collection of logics, then $\bigcap_{i \in I} \Delta_i$ is a logic.
- iii) If \mathcal{M} is a class of models, then the set of valid formulas $\Delta_{\mathcal{M}}$ in \mathcal{M} is **not necessarily** a logic. Think a counterexample!

There is a **minimal** logic that contains any set of formulas Γ . We call it the logic **generated** by Γ .

- For example, the logic generated by the empty set contains all the instances of propositional tautologies and nothing else. We call it PC.

Syntactic Consequence

- If $\Gamma \cup \{\varphi\}$ is a set of formulas, then φ is **deductible** in Δ starting from Γ (notation $\Gamma \vdash_{\Delta} \varphi$) if there are formulas $\psi_1, \dots, \psi_n \in \Gamma$ such that $\vdash_{\Delta} (\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \varphi$.
- A set of formulas Γ is Δ -consistent if $\Gamma \not\vdash_{\Delta} \perp$. A formula φ is Δ -consistent if $\{\varphi\}$ is.

Normal modal logic

- The previous definitions are generalizations of concepts from propositional calculus into modal languages.
- Now we will see a concept that is exclusively modal: normal modal logics.
- A modal logic Δ is **normal** if it contains the formula:

$$(K) \quad \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q).$$

and is closed under the **necesisitation** (or **generalization**) rule:

$$(Nec) \text{ If } \vdash_{\Delta} \varphi \text{ then } \vdash_{\Delta} \Box \varphi.$$

Examples

- i) The inconsistent logic is a normal logic.
- ii) If $\{\Delta_i \mid i \in I\}$ is a collection of normal logics, then $\bigcap_{i \in I} \Delta_i$ is a normal logic.
- iii) PC **is not** a normal logic.

There is a **minimal** normal modal logic that contains any set of formulas Γ . We call it the normal modal logic **generated** by Γ .

- The normal modal logic generated by the empty set is called **K**, and is the minimal normal modal logic. If Γ is a set of formulas, the normal modal logic generated by Γ is called **KT**.

Also, it is usual to say that Γ are the **axioms**. And to say that the logic is generated from Γ using the **inference rules** modus ponens, uniform substitution and generalization.

Semantic consequence

Given a class of models S and a set of formulas $\Gamma \cup \{\varphi\}$ there are different ways of defining the notion of **semantic consequence** in S :

- ▶ **Local Consequence:** $\Gamma \models_S^l \varphi$ iff

$$\forall M \in S \forall w (M, w \models \Gamma \implies M, w \models \varphi)$$

- ▶ **Global Consequence:** $\Gamma \models_S^g \varphi$ iff

$$\forall M \in S (M \models \Gamma \implies M \models \varphi)$$

The notion of local consequence is the strongest (i.e., $\Gamma \models_S^l \varphi$ implies $\Gamma \models_S^g \varphi$). **Exercise.**

From now on, $\Gamma \models_S \varphi$ es $\Gamma \models_S^l \varphi$.

Soundness & Completeness

Let's recap the definitions:

- ▶ A logic Δ is **correct** with respect to a class of models S if for all formula φ and for all model $\mathcal{M} \in S$, if $\vdash_\Delta \varphi$ then $\mathcal{M} \models \varphi$.
- ▶ A logic Δ is **strongly complete** with respect to a class of models S if for any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \models_S \varphi$ then $\Gamma \vdash_\Delta \varphi$.

Soundness & Completeness

The notion of completeness can be reformulated in terms of **consistency**.

- ▶ If $\Gamma \models_S \varphi$ then $\Gamma \vdash_\Delta \varphi$
- ▶ If $\Gamma \not\vdash_\Delta \varphi$ then $\Gamma \not\models_S \varphi$
- ▶ If $\Gamma \not\vdash_\Delta \neg\varphi \rightarrow \perp$ then $\Gamma \not\models_S \varphi$
- ▶ If $\Gamma \cup \{\neg\varphi\} \not\vdash_\Delta \perp$ then $\Gamma \not\models_S \varphi$
- ▶ If $\Gamma \cup \{\neg\varphi\}$ is Δ -consistent then $\Gamma \not\models_S \varphi$
- ▶ If $\Gamma \cup \{\neg\varphi\}$ is Δ -consistent then $\exists \mathcal{M} \in S$ s.t. $\mathcal{M}, w \models \Gamma \cup \{\neg\varphi\}$

That is:

- ▶ A normal modal logic Δ is **strongly complete** with respect to a class S iff each set of Δ -consistent formulas is satisfiable in some model $\mathcal{M} \in S$.

Canonical model

- ▶ Usually, to prove correctness with respect to a class S is easy.
 - ▶ Axioms need to be universally valid in S
 - ▶ The inference rules need to preserve truth in S
- ▶ We will focus in proving completeness. To this end, we need to find a model that satisfies every consistent set of formulas.
- ▶ And the “pieces” that we will use are maximally consistent sets.

A set of formulas Γ is **maximally Δ -consistent** if Γ is Δ -consistent and any set of formulas that properly contains Γ is Δ -inconsistent.

- ▶ If Γ is a maximally Δ -consistent set we say that it is an Δ -MCS.

Canonical Model

What is the intuition behind using MCSs in a completeness proof for modal logics?

- ▶ Observe that each point w in each model \mathcal{M} for a logic Δ is associated with a set of formulas $\{\varphi \mid \mathcal{M}, w \models \varphi\}$.
- ▶ It is not difficult to see that such set of formulas is actually a Δ -MCS. And this means that if φ is true in a model for a logic Δ , then φ belongs to a Δ -MCS.
- ▶ Also, if w is related with w' in some model \mathcal{M} , then the information in each of the MCS associated to w and w' needs to be “coherently related”.

Canonical Model

The idea is to work with collections of MCSs that are coherently related in order to construct the model we are looking for. The goal is to prove the **truth lemma**, that tell us that ‘ φ belongs to a MCS’ is equivalent to ‘ φ is true in a model’.

- ▶ The worlds of the canonical model are all the MCSs of the logic we are working with.
- ▶ We will see what it means for the information in the MCSs to be ‘coherently related’, and we will use this notion in order to define the accessibility relation.

Properties of MCSs

Let’s begin by seeing some properties of the MCSs. If Δ is a logic and Γ is a Δ -MCS then:

- ▶ $\Delta \subseteq \Gamma$.
- ▶ Γ is closed under modus ponens.
- ▶ For all formulas φ , $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
- ▶ For all formulas φ, ψ , $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lindenbaum Lemma

Since the MCSs are pieces we need in order to construct the canonical model, we have to make sure that we have enough of them. Any consistent set of formulas can be extended as a maximally consistent one.

- ▶ If Σ is a Δ -consistent set of formulas, then there is a Δ -MCS Σ^+ such that $\Sigma \subseteq \Sigma^+$.

Proof:

- i) We numerate the formulas in our language $\varphi_1, \varphi_2, \dots$
- ii) We define the sequence of sets:

$$\begin{aligned}\Sigma_0 &= \Sigma \\ \Sigma_{n+1} &= \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if the set is } \Delta\text{-consistent} \\ \Sigma_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases} \\ \Sigma^+ &= \bigcup_{n \geq 0} \Sigma_n\end{aligned}$$

Construction of the canonical model

The canonical model \mathcal{M}^Δ for a normal modal logic Δ (in the basic language) is $\langle W^\Delta, R^\Delta, v^\Delta \rangle$ where:

- ▶ W^Δ is the set of all Δ -MCSs.
- ▶ R^Δ is the binary relation defined as

$$R^\Delta_{wv} \text{ iff } \forall \varphi, (\varphi \in v \Rightarrow \Diamond \varphi \in w).$$



- ▶ $V^\Delta(p) = \{w \in W^\Delta \mid p \in w\}$
- ▶ Notice that:
 - ▶ The domain is the set of all Δ -MCS (uncountable).
 - ▶ As a corollary of Lindenbaum's Lemma, any consistent set of formulas appears as a subset of some state in the model.
 - ▶ The canonical valuation sets **truth** of a propositional symbol in w to **membership** in w .

Completeness

Now we can prove the following lemma. It says that there are 'enough' MCSs in our model for our goal:

- ▶ **Existence lemma:** For any normal modal logic Δ , and any state $w \in W^\Delta$, if $\Diamond \varphi \in w$ then there exists a state $v \in W^\Delta$ such that R^Δ_{wv} and $\varphi \in v$.

Proof: Suppose that $\Diamond \varphi \in w$. We will construct v such that R^Δ_{wv} and $\varphi \in v$. Let $v^- = \{\varphi\} \cup \{\psi \mid \Box \psi \in w\}$.

- ▶ v^- is consistent. **Exercise!**

Then for Lindenbaum there exists a Δ -MCS v that extends v^- . By construction $\varphi \in v$, and for all formula ψ , $\Box \psi \in w$ implies $\psi \in v$.

- ▶ This last step implies that R^Δ_{wv} . **Exercise!**

Completeness

Now we can take "truth = membership" to the level of arbitrary formulas.

- ▶ **Truth lemma:** For any normal modal logic Δ and any formula φ , $\mathcal{M}^\Delta, w \models \varphi$ iff $\varphi \in w$.

The proof is easy by induction on φ . The only interesting case is the modal one, and that is covered by the **existence lemma** (the hard direction).

Finally:

- ▶ **Theorem of the Canonical Model:** Any normal modal logic is strongly complete with respect to its canonical model.

Proof: Suppose that Σ is a Δ -consistent set. By the Lindenbaum lemma there exists a Δ -MCS Σ^+ that extends Σ . By the **truth lemma**, $\mathcal{M}^\Delta, \Sigma^+ \models \Sigma$.

Completeness

Corollary:

- ▶ The normal modal logic **K** is strongly complete with respect to the class of all the models.

By the previous theorem, given that **K** is a normal modal logic, it is strongly complete with respect to its model \mathcal{M}^K . We just need to check that \mathcal{M}^K belongs to the class of all the models, but this is trivial.

Hybrid Logic Completeness

- ▶ Remember the hybrid operators i (nominals) and $@$ (at).
- ▶ We will prove that by adding these operators to the language of the classical modal logic we can prove a **general** result of completeness.

Pure Formulas

- ▶ We say that a modal formula is **pure** if its only propositional symbols are nominals.
- ▶ Pure formulas let us define several properties of the accessibility relation.

Definibles en LMB

reflexivity $i \rightarrow \Diamond i$
symmetry $i \rightarrow \Box \Diamond i$
transitivity $\Diamond \Diamond i \rightarrow \Diamond i$
density $\Diamond i \rightarrow \Diamond \Diamond i$
determinism $\Diamond i \rightarrow \Box i$

No definibles en LMB

irreflexivity $i \rightarrow \neg \Diamond i$
asymmetry $i \rightarrow \neg \Diamond \Diamond i$
intransitivity $\Diamond \Diamond i \rightarrow \neg \Diamond i$
universality $\Diamond i$
trichotomy $@_j \Diamond i \vee @_j i \vee @_j \neg i$
at most 2 states $@_i (\neg j \wedge \neg k) \rightarrow @_j k$

- ▶ **Claim:** When a pure formula is valid in a frame, it defines a **first order** property over its accessibility relation (**Exercise**).

Axiomatization of Pure Formulas

- ▶ When a pure formula is used as an **axiom**, the axiomatization is automatically complete over the class of frames defined by that formula
Theorem: If P is the hybrid normal logic obtained by adding a set Π of pure formulas over the axiomatization of the minimal hybrid logic $\mathbf{K}_h + R$ (that we will introduce in a minute), then P is complete respect the class of frames defined by Π .
- ▶ We will prove this general completeness using a constructon proposed by Henkin for first order logic.

From Valid in a Model to Valid in a Frame

- ▶ Before starting with the completeness proof, let's see why we can, in this case, obtain such a general result.
- ▶ We say that a model \mathcal{M} is **named** if each state of \mathcal{M} belongs to the extension of at least one nominal.
- ▶ If φ is a pure formula, we say that ψ is a **pure instance** of φ if it is obtained from φ substituting nominals by nominals.
- ▶ **Proposition:** Let $\mathcal{M} = \langle W, R, V \rangle$ be a named model and φ a pure formula. Suppose that for every pure instance ψ of φ we have that $\mathcal{M} \models \psi$. Then $\langle W, R \rangle \models \varphi$.

Normal Hybrid Logics

Δ is a normal hybrid logic if it is a normal modal logic and in addition it includes the following axioms and is closed under the following rules:

Axioms:

$K_{@}$	$@_i(p \rightarrow q) \rightarrow (@_i p \rightarrow @_i q)$
self-dual	$@_i p \leftrightarrow \neg @_i \neg p$
introduction	$i \wedge p \rightarrow @_i p$
ref	$@_i i$
sym	$@_i j \leftrightarrow @_j i$
nom	$@_i j \wedge @_j p \rightarrow @_i p$
agree	$@_j @_i p \leftrightarrow @_i p$
back	$\Diamond @_i p \rightarrow @_i p$

Rules:

@-Gen Si $\vdash_{\Delta} \varphi$ implica $\vdash_{\Delta} @_i \varphi$

Soundness and Completeness

- ▶ As in the case of the normal modal logics, the set of all the hybrid formulas is a logic and the definition is closed under intersection. Hence, there is a hybrid logic known as \mathbf{K}_h .
- ▶ As usual, proving soundness is easy: you just need to check that all the axioms are valid and that the rules preserve validity. (Exercise).
- ▶ The proof of completeness also uses MCS. But in this case we use **named** MCSs. We say that a **is named** Γ by a nominal i if $i \in \Gamma$.

Properties of hybrid MCS

Proposition: Let Γ be a \mathbf{K}_h -MCS. For each nominal i we define $\Delta_i = \{\varphi \mid @_i \varphi \in \Gamma\}$.

1. Δ_i is a \mathbf{K}_h -MCS that contains i .
2. For all pair of nominals i, j , if $i \in \Delta_j$ then $\Delta_i = \Delta_j$.
3. For all pair of nominals i, j , $@_i \varphi \in \Delta_j$ iff $@_i \varphi \in \Gamma$.
4. If $k \in \Gamma$, then $\Delta_k = \Gamma$.

Proof Let's prove 1:

By (ref) $@_i i \in \Gamma$, then $i \in \Delta_i$.

Δ_i is maximally consistent. Suppose that it is not consistent: Then, there exists $\delta_k \in \Delta_i$ such that $\vdash \neg(\delta_1 \wedge \dots \wedge \delta_n)$. By @-Gen, $\vdash @_i \neg(\delta_1 \wedge \dots \wedge \delta_n)$ and by self-dual $\vdash \neg @_i(\delta_1 \wedge \dots \wedge \delta_n)$ and then it is in Γ . Since $\delta_k \in \Delta_i$, $@_i \delta_k \in \Gamma$. Since $@_i$ is a normal modality, $\vdash \bigwedge @_i \delta_k \rightarrow @_i(\bigwedge \delta_k)$. Hence $@_i(\delta_1 \wedge \dots \wedge \delta_n) \in \Gamma$ which contradicts the fact that Γ is consistent.

Maximality and the other points are **exercises**.

Canonical Model

At this point we could continue as we did for the case of \mathbf{K} .

- ▶ Use the Lindenbaum lemma to show that every consistent set Σ can be extended as a maximally consistent set Σ^+ .
- ▶ Consider the model whose domain is the set of all the MCS. (In fact we have to consider the submodel of the canonical model generated by $\Sigma^+ \cup \{\Delta_i \mid \Delta_i = \{\varphi \mid @_i \varphi \in \Sigma^+\}\}$, in order to make sure that we obtain a hybrid model).
- ▶ Prove the **Truth** and **Existence Lemmas**.

But the model obtained in this way is not a named model, and then if a pure formula is valid in this model does not imply that it is valid in its frame.

Admissible Rules

- ▶ As we mentioned, we can prove that the logic \mathbf{K}_h is complete in a similar way as we did with K .
- ▶ But from that result we cannot prove that $\mathbf{K}_h + \Pi$ is complete for the class defined by Π , where Π is an arbitrary set of pure formulas.
- ▶ In order to construct a named model, we need two additional rules.

Name $\vdash j \rightarrow \theta$ then $\vdash \theta$

Paste $\vdash @_i \Diamond j \wedge @_j \varphi \rightarrow \theta$ then $\vdash @_i \Diamond \varphi \rightarrow \theta$

j is a nominal different from i which does not appear in θ .

- ▶ Let $\mathbf{K}_h + R$ be the logic obtained by adding these two rules to \mathbf{K}_h .

Named and Pasted

Let Δ be a normal hybrid logic, a Δ -MCS Σ is called

- ▶ **named** if for some nominal $i, i \in \Sigma$
- ▶ **pasted** if $@_i \Diamond \varphi \in \Sigma$ implies $\exists j \in \mathbf{NOM}$ s.t. $@_i \Diamond j \wedge @_j \varphi \in \Sigma$

Lemma (Named and Pasted Lindenbaum's)

If Σ is Δ -consistent then there is a named and pasted Δ -MCS Σ^+ such that $\Sigma \subseteq \Sigma^+$ (in a language extended with additional nominals).

$$\Sigma^0 = \Sigma \cup \{k\}, k \text{ not in } \Sigma$$

$$\Sigma^{m+1} = \Sigma^m \text{ si } \Sigma^m \cup \{\varphi_{m+1}\} \text{ is inconsistent}$$

$$\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \text{ if consistent and } \varphi_{m+1} \text{ is not } @_i \Diamond \psi.$$

$$\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}, @_i \Diamond j \wedge @_j \psi\} \text{ if consistent and } \varphi_{m+1} \text{ is } @_i \Diamond \psi.$$

$$\Sigma^+ = \bigcup_{n \geq 0} \Sigma^n$$

Each Δ -MCS is a Full Model Description

Let Δ be a normal hybrid logic, then every Δ -MCS give rise to a collection of Δ -MCSs.

Lemma (Gossip Lemma)

Let Σ be a Δ -MCS. For any nominal $i \in \Sigma$, define

$\Gamma_i = \{\varphi \mid @_i \varphi \in \Sigma\}$. Then Γ_i is a Δ -MCS that contains i .

The Hybrid Canonical Model: Let Σ be a named and pasted Δ -MCS.

Define $\mathcal{M}^c = \langle W^c, R^c, V^c \rangle$ where R^c and V^c are defined as before, and

$$W^c = \{\Gamma_i \mid \Gamma_i = \{\varphi \mid @_i \varphi \in \Sigma\}\}.$$

Facts

- ▶ \mathcal{M}^c depends on Σ (we should actually talk about \mathcal{M}_{Σ}^c).
- ▶ \mathcal{M}^c is countable.
- ▶ \mathcal{M}^c is **named**: each state in W^c makes (at least) a nominal true.

Hybrid/Henkin Completeness

- ▶ The Truth and Existence Lemmas can be proved for \mathcal{M}^c in a similar way as before.
- ▶ As a result, every normal hybrid logic is complete w.r.t. the class of models obtained from the set of its MCS.

And now for the magic. Remember that:

- ▶ **Fact:** Let $\mathcal{M} = \langle W, R, V \rangle$ be named and φ pure. If for all pure instances ψ of φ , $\mathcal{M} \models \psi$, then for all $V', \langle W, R, V' \rangle \models \varphi$.
- ▶ **Fact:** If φ is pure and for all $V', \langle W, R, V' \rangle \models \varphi$ then φ defines a first-order property on R .
- ▶ Hence, if $\mathbf{K}_h + R$ is extended by a set of pure axioms Π then the resulting logic is **automatically strongly complete** w.r.t. the class of models defined by Π .

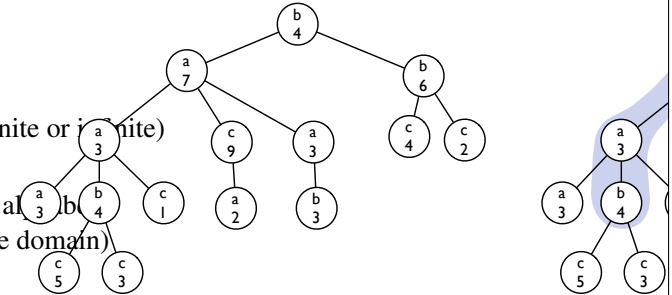
An Application to Semi-Structured Databases

- ▶ Many data-intensive applications use complex data that is not naturally encoded in a relational database.
 - ▶ Web data or biological data are better described using **semi-structured data models**, that organize information as labeled trees or graphs.
 - ▶ They contain labels from a finite alphabet (the structural information), and from an infinite alphabet (the actual data).
- ▶ A well known example is XML (eXtensible Markup Language), the most successful data model of this kind.
- ▶ **XPath** is the most widely used XML query language.

XPath with data tests

Data tree

- ▶ ordered tree (finite or infinite)
- ▶ nodes have
 - a **label** (finite alphabet)
 - a **data** (infinite domain)



Path expressions

- ▶ go to ancestor, check a , go to child, check c , go right sibling, go to descendant, check b
- ▶ $\uparrow^* [a] \downarrow [c] \rightarrow \downarrow_* [b]$
- ▶ $\mathcal{T}, x, y \models \uparrow^* [a] \downarrow [c] \rightarrow \downarrow_* [b]$

Node expressions

- ▶ $\mathcal{T}, x \models \langle \uparrow^* [a] \downarrow [c] \rightarrow \downarrow_* [b] \rangle$
- ▶ $\mathcal{T}, x \models \langle \downarrow [c] = \uparrow^* [a] \downarrow [c] \rightarrow \downarrow_* [b] \rangle$
- ▶ $\mathcal{T}, x \models \langle \downarrow [c] \neq \uparrow^* [a] \downarrow [c] \rightarrow \downarrow_* [b] \rangle$

Types of Diamonds in XPath₌

- ▶ $\mathcal{T}, x \models \langle \alpha \rangle$ iff there is a node y such that
 - ▶ $\mathcal{T}, x, y \models \alpha$
- ▶ $\mathcal{T}, x \models \langle \alpha = \beta \rangle$ iff there are nodes v and w such that
 - ▶ $\mathcal{T}, x, v \models \alpha$
 - ▶ $\mathcal{T}, x, w \models \beta$
 - ▶ $data(v) = data(w)$
- ▶ $\mathcal{T}, x \models \langle \alpha \neq \beta \rangle$ iff there are nodes v and w such that
 - ▶ $\mathcal{T}, x, v \models \alpha$
 - ▶ $\mathcal{T}, x, w \models \beta$
 - ▶ $data(v) \neq data(w)$

Hybrid XPath₌

Add nominals and @ to XPath₌, and call the resulting language **HXPath₌**.

Some HXPath₌ expressions together with their intuitive meaning:

- $\alpha[i]$ There is an α path from the current point to the node named i .
- $@_i \alpha$ There is an α path from node named i and some other node.
- $\langle @_i = @_j \rangle$ The node named i has the same data than the node named j .
- $\langle \alpha = @_i \beta \rangle$ There is a node accessible from the current point by an α path that has the same data than a node accessible from the point named i by a β path.

Axiomatizing XPath

- ▶ An axiomatization for $\text{XPath}_=(\downarrow)$ was first given by Abriola et al. but the completeness proof is non trivial
- ▶ We then proposed $\text{HXPath}_=(\downarrow\uparrow)$ where a Henkin completeness proof was possible.



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