

Mathematics for Social Scientists

Part 1: Logic, Sets, Functions

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Outline Applied Logic

Truth Table

Implication and argument forms

Quantifiers

Axioms, Definitions, Theorems, and Proofs

Sets

Set Definition

Set Operations

Functions

Binary Operations, Neutral and Inverse elements

Properties of Functions and Inverse/Composite Functions

Propositions and predicates

- A **proposition** has a definite truth value (true or false, never both). Examples: $1 + 1 = 2$ (true); $1 = 2$ (false).
- A **conjecture** is a proposition whose truth value is currently unknown (e.g., Goldbach's conjecture).
- A **predicate** has free variables, e.g. $P(x) : x > 4$. It becomes a proposition once x (and its universe) are fixed.

Connectives and precedence

- $X \wedge Y$ is true iff both X and Y are true.
- $X \vee Y$ is false iff both are false (mathematical “or” is *inclusive*).
- $\neg X$ is true iff X is false.
- Parentheses avoid ambiguity: $X \wedge (Y \vee Z)$ vs. $(X \wedge Y) \vee Z$.

Logical equivalence (De Morgan)

Theorem 1.

$$\neg(X \vee Y) \equiv \neg X \wedge \neg Y \quad \text{and} \quad \neg(X \wedge Y) \equiv \neg X \vee \neg Y.$$

- Two propositions are *equivalent* when their truth tables match.
- Beware: $\neg X \vee Y$ means $(\neg X) \vee Y$, not $\neg(X \vee Y)$.

Some notions from logic

Implication and equivalence:

$A \Rightarrow B$ (A **implies** B)

$A \Leftrightarrow B$ (A holds **if and only if** B holds)
iff

And / Or:

$A \wedge B$ (A holds **and** B holds)

$A \vee B$ (A holds **or** B holds)

Quantifiers:

\forall **for all**

\exists **there exists**

$\exists!$ **there exists a unique** (exactly one)

Truth Table

- T stands for *True* and F stands for *False*.
- Here, p and q are some statements. The first row shows the outcome of the logical computations if both of these statements are true, the second row shows the outcome when p is true but q is false, and so on...

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Longleftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Examples

Example:

p = "Mensa is always completely empty" (False)

q = "The sky is blue" (True)

Then;

- $p \wedge q$ = "Mensa is always completely empty **and** the sky is blue" is a false statement,
- $p \vee q$ = "Mensa is always completely empty **or** the sky is blue" is a true statement,
- $p \Rightarrow q$ = "If Mensa is always completely empty, **then** the sky is blue" is a true statement.¹

¹This is a little counter-intuitive. The "if {something false}, then ..." format is always true. The reason behind this is, "if {something false}," puts us in a hypothetical world that doesn't exist. For example, no one can show the following statement to be false: "If my grandmother had wheels, then she would be a bike." My grandma doesn't have wheels, therefore it doesn't matter what comes next. Statement is true.

Examples on Reading the Logical symbols

- $(p \vee q) \Rightarrow q$: If p or q , then q
- $p \vee (q \Rightarrow r)$: p or " q implies r "²
- Let $e \in S$.
 $e * s = s * e = s \quad \forall s \in S \implies e$ is the neutral element of S :
*Let e be an element of S .
If $e * s = s * e = s$ for all s in S , then e is the neutral element of S .*
- f is called injective if $\forall s_1, s_2 \in S : f(s_1) = f(s_2) \implies s_1 = s_2$:
 f is called injective if for all s_1, s_2 in S , we have that "if $f(s_1) = f(s_2)$, then $s_1 = s_2$."

²"If... then..." and "implies" have the same meaning.

More examples

- $A = \{2 \cdot n : n \in \mathbb{Z}\}$:

A is the set consisting of all numbers that are in the form of "2 times n", where n are elements of \mathbb{Z} .

- $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z} \wedge 0 \neq q \in \mathbb{Z}\}$:

\mathbb{Q} is the set of numbers in the form $\frac{p}{q}$, such that p is an element of the integers and q is an element of the integers that does not equal 0.

- $A \subset B \iff \forall a \in A, a \in B$:

A is a subset of B, if and only if, for all a in A, we have that a is also in B.

Implication, contrapositive, and language

- $P \Rightarrow Q$ is **false only** when P is true and Q is false.
- **Contrapositive:** $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.
- **Converse** ($Q \Rightarrow P$) and **inverse** ($\neg P \Rightarrow \neg Q$) are generally *not* equivalent to $P \Rightarrow Q$.
- Language mapping:
 - “X only if Y” means $X \Rightarrow Y$ (Y necessary for X).
 - “X if Y” means $Y \Rightarrow X$ (Y sufficient for X).
 - “X iff Y” means $(X \Rightarrow Y) \wedge (Y \Rightarrow X)$.

Quantifiers and negation

- Fix a universe U . $\forall x \in U : P(x)$ is true iff $P(x)$ holds for every $x \in U$; a single counterexample makes it false.
- $\exists x \in U : P(x)$ is true iff $P(x)$ holds for at least one $x \in U$.
- Negations: $\neg \forall x P(x) \equiv \exists x \neg P(x)$, $\neg \exists x P(x) \equiv \forall x \neg P(x)$.
- Example: negate $1 \leq x < 5$:

$$\neg \left((1 \leq x) \wedge (x < 5) \right) \equiv (x < 1) \vee (x \geq 5).$$

Axioms

- Mathematics starts from **axioms**: basic premises taken as true.
- Axioms should be few and independent; changing them changes what can be proved.
- Example (empty set axiom): *There exists a set with no elements.*

Definitions, propositions, lemmas, theorems, corollaries

- **Definition** fixes meaning (not proved); it introduces objects/notation. Examples:
 $\mathbb{N} = \{1, 2, 3, \dots\}$; define $f : \mathbb{Z} \rightarrow \mathbb{R}$ by $f(x) = x^2$.
- **Result types (true statements with stated conditions and a conclusion):**
 - **Proposition:** general result.
 - **Lemma:** auxiliary result used to prove a theorem.
 - **Theorem:** central result.
 - **Corollary:** immediate consequence of a theorem.
- **Examples**
 - *Implication:* If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
 - *Equivalence:* $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$.

What is a proof?

- A **proof** is a finite, logically valid chain of statements taking the **assumptions** to the **conclusion**.
- Each step must be justified by an axiom, a definition, a previously proved result, or a valid rule of inference.
- Be explicit about scope: declare fixed objects ("Let $x \in S$ be arbitrary") and where each assumption is used.

Proof methods

- **Direct** (implication): assume A_1, \dots, A_k ; derive C step by step.
- **Contrapositive**: to prove $\forall x : P(x) \Rightarrow Q(x)$, prove $\forall x : \neg Q(x) \Rightarrow \neg P(x)$.
- **Contradiction**: assume A_1, \dots, A_k and $\neg C$; derive an impossibility (e.g. $R \wedge \neg R$).
- **By cases**: if $P \vee R$, prove C from P and also from R .
- **Existence/Uniqueness** ($\exists!$). Show existence: $\exists x P(x)$; then uniqueness:
 $\forall x, y \left(P(x) \wedge P(y) \Rightarrow x = y \right)$.
- **Induction on n** . Base case $P(n_0)$; inductive step $P(k) \Rightarrow P(k + 1)$; conclude
 $\forall n \geq n_0 : P(n)$.

Example (contradiction)

Theorem 2.

For any set A , $\emptyset \subseteq A$.

Proof.

Suppose not. Then there exists $x \in \emptyset$ with $x \notin A$. But \emptyset has no elements. Contradiction.
Hence $\emptyset \subseteq A$. ■

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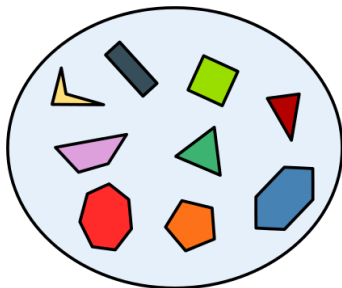
What is a set?

- A set is a collection, denoted by $\{\dots\}$, of distinct objects called **elements**.

Example.

The set of things I had for breakfast is $\{\text{muesli, milk, orange juice}\}$.

- Two sets A and B are equal if and only if they have precisely the same elements.
- Sets ignore repetition and order.
So $\{1, 1, 2, 3\} = \{1, 3, 2\}$.
- Sets are conventionally denoted by capital letters (e.g. A, B), and elements by lower-case letters (e.g. a, b, c).



A set of polygons

There are two ways of defining a set:

- A **descriptive definition**, for example
 - The set of colours in the British flag.
 - The set of prime numbers less than 10.
- An **explicit definition**, for example
 - $A = \{\text{blue, red, white}\}$
 - $B = \{2, 3, 5, 7\}$.
- *Notation:* $a \in A$ means “ a is an element of A ”. Otherwise $a \notin A$.

Example.

Let $B = \{2, 3, 5, 7\}$ be the set of prime numbers less than 10. Then

- $2 \in B, 3 \in B, \dots$
- $4 \notin B, 6 \notin B, \dots$

Definition 1.

N is called a **subset** of M if for all $n \in N$ it also holds that $n \in M$. This is denoted $N \subseteq M$.

Definition 2.

We can restate our definition of a subset as follows:

$$N \subseteq M \quad \Leftrightarrow \quad \forall a \in N, a \in M.$$

N is a subset of M if and only if for all a in N , a is in M .

Examples of sets

- $\emptyset = \{\}$
(the *empty set*)
- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
(the *natural numbers*)
- $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$
(the *natural numbers, including 0*)
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
(the *integers*)
- $\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } 0 \neq q \in \mathbb{Z} \right\}$
(the *rational numbers*)
- \mathbb{R}
(the *real numbers*)
- $\mathbb{R}^n = \{(r_1, \dots, r_n) : r_i \in \mathbb{R} \text{ for } i = 1, \dots, n\}$
(*n-tuples of real numbers*)
- \mathbb{C}
(the *complex numbers*)

More examples of sets

- Logical notation allows us to write descriptive definitions of sets using mathematics. For example,
 - $\{1, 2, 3\} = \{n \in \mathbb{N} : n < 4\}$
 - $2\mathbb{Z} = \{2 \cdot n : n \in \mathbb{Z}\}$, the set of even numbers.
- **Intervals:** for $a, b \in \mathbb{R}$.
 - $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ - open interval
 - $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ - closed interval

More examples of sets

- Let A be a set. The **power set** $\mathcal{P}(A)$ of A is the set of all subsets of A .

Example.

$$A = \{1, 2\} \Rightarrow \mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

- The power set always contains the set itself and the empty set.
- Let A, B be sets. The **cartesian product** $A \times B$ of A and B is defined as the set containing all pairs (a, b) such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Example.

Let $A = \{1, 2, 3\}$, $B = \{1, 4\}$. Then
 $A \times B = \{(1, 1), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4)\}.$

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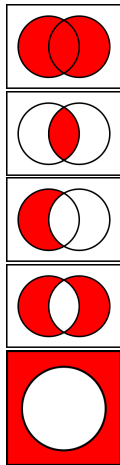
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Set operations

Let A and B be sets. The following *set operations* allow us to build new sets from A and B :

- **Union:** $A \cup B := \{x : x \in A \text{ or } x \in B\}$
- **Intersection:** $A \cap B := \{x : x \in A \text{ and } x \in B\}$
- **Difference** or *relative complement*
 $A \setminus B := \{x : x \in A \text{ and } x \notin B\}$
- **Symmetric difference:**
 $A \triangle B := \{x : x \in A \setminus B \text{ or } x \in B \setminus A\}$
- **Complement** with respect to a universal set Ω :
 $A^c := \{x \in \Omega : x \notin A\}$



Laws for set operators

Let A , B and C be sets. Then, the following properties hold:

- $A \cap B = B \cap A$, $A \cup B = B \cup A$ (Commutative law)
- $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
- $A \cup A = A$ and $A \cap A = A$ (Idempotence)
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive laws)
- $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$ (De Morgan's laws)

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Defining functions

Definition 3.

A **function** or **map**, denoted by $f : S \rightarrow T$, from a set S to a set T is a rule which assigns to each element $s \in S$ exactly one element $t \in T$. $f : S \rightarrow T, f(s) = t$.

A Function has 3 parts:

- A set S to map *from*. This set is called the **domain** of f .
- A set T to map *to*. This set is called the **codomain** of f .
- A rule for *every* element $a \in S$, assigning it to some element $b \in T$. This is written $f(a) = b$.

Domain, codomain, image (and the word “range”)

Let $f : S \rightarrow T$.

- **Domain** S : the allowable inputs.
- **Codomain** T : the declared target set.
- **Image (set of values)**: $f[S] = \{f(s) : s \in S\} \subseteq T$.
- The word “range” is used *ambiguously* in books: sometimes it means the codomain T , sometimes the image $f[S]$.
We will avoid “range” and say “codomain” or “image” explicitly.

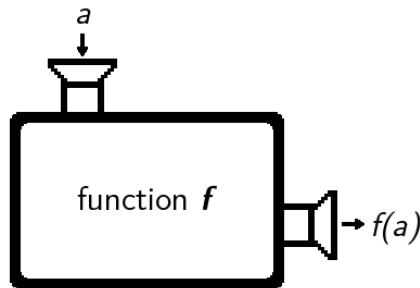
Examples.

- $x \mapsto x^2$ as $f : \mathbb{R} \rightarrow \mathbb{R}$: codomain = \mathbb{R} , image = $\{y \in \mathbb{R} : y \geq 0\}$ (not onto).
- $x \mapsto 2x$ as $g : \mathbb{R} \rightarrow \mathbb{R}$: image = \mathbb{R} (onto).
- $n \mapsto 2n$ as $h : \mathbb{Z} \rightarrow \mathbb{Z}$: image = $2\mathbb{Z}$; redefining the codomain to $2\mathbb{Z}$ makes h surjective.

Functions

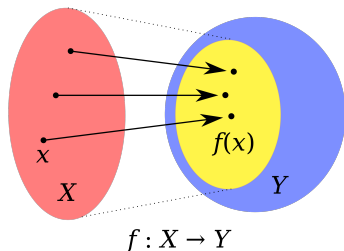
Examples:

- $f : \{1, 2, 3\} \rightarrow \{3, 4, 5\},$
 $: x \mapsto x + 2.$
- $f : \{1, 2\} \rightarrow \{1, 3\},$
 $f(1) = 1, f(2) = 3.$



- A common notation for functions is $y = f(x)$, which tells us that a function f is performed upon the values of $x \in A$ (the domain) to obtain the values of $y \in B$ (the codomain).

Domain, codomain, image



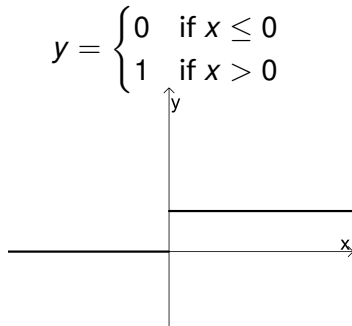
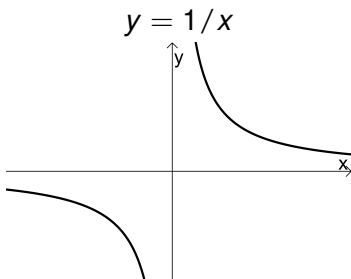
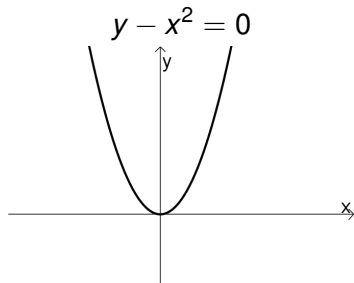
Setup. $f: X \rightarrow Y$ is a function. The yellow oval inside Y is the **image** $f[X] = \{f(x) : x \in X\}$.

Terminology.

- **Domain** = X **Codomain** = Y **Image** = $f[X] \subseteq Y$.
- The word “**range**” is *ambiguous*: some books use it for the image $f[X]$, others for the codomain Y . We will say “codomain” or “image” explicitly.

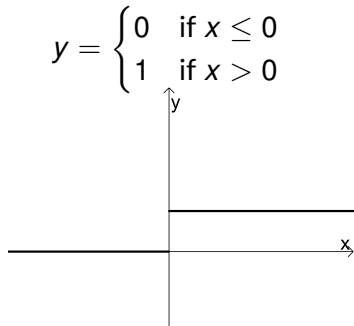
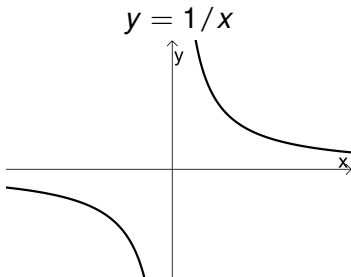
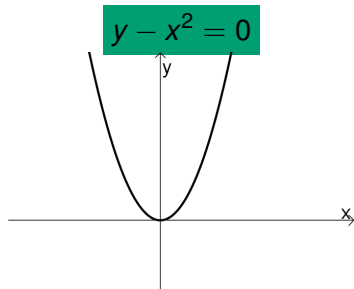
Visualising functions

Q: Which of the following are functions : $\mathbb{R} \rightarrow \mathbb{R}$?



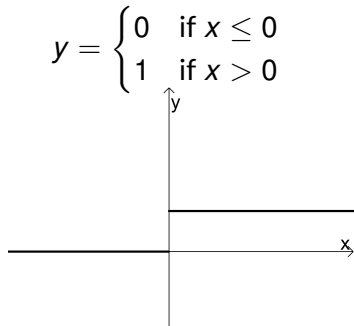
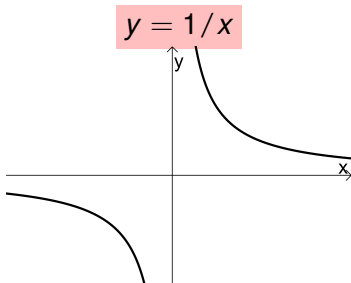
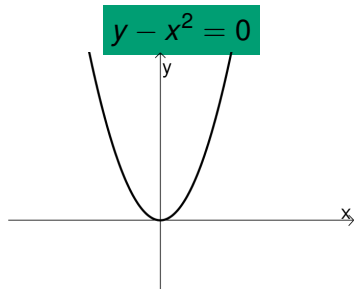
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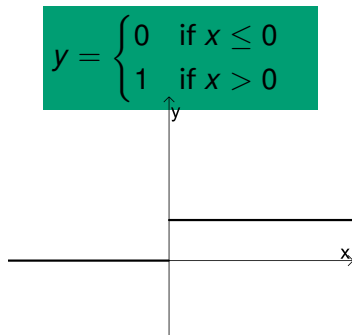
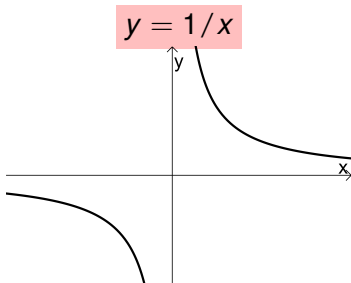
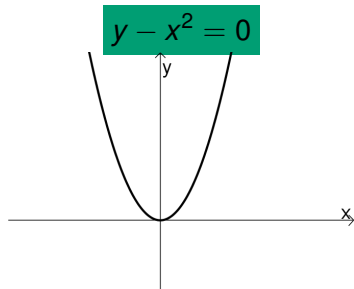
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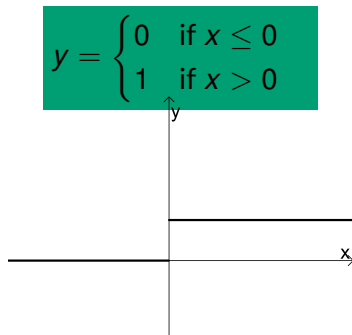
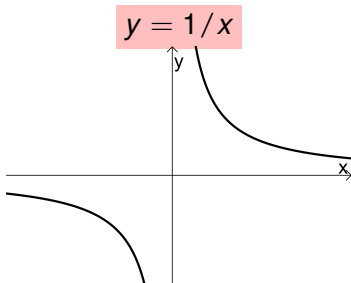
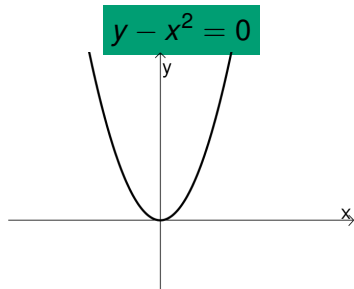
Visualising functions

Q: Which of the following are functions : $\mathbb{R} \rightarrow \mathbb{R}$?



Visualising functions

Q: Which of the following are functions : $\mathbb{R} \rightarrow \mathbb{R}$?



Functions are valuable to social scientists because most relationships can be modelled by a function.

More examples

- *Constant function:* $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = c \in \mathbb{R}$ for all $x \in \mathbb{R}$.
- *Sine function:* $f : \mathbb{R} \rightarrow [-1, 1], f(x) = \sin(x)$.
- *Polynomial in x :* $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$, where $a_i \in \mathbb{R}$.
- Let A be a finite set. $f : \mathcal{P}(A) \rightarrow \mathbb{N}, f(B) = |B|$ for all $B \subseteq A$.

More examples

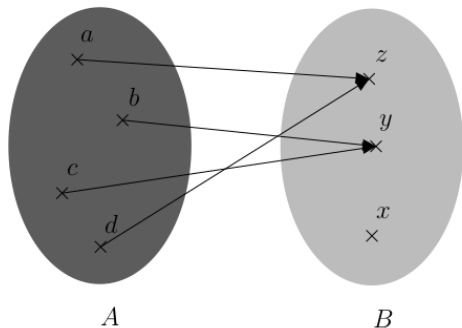


Figure: A function!

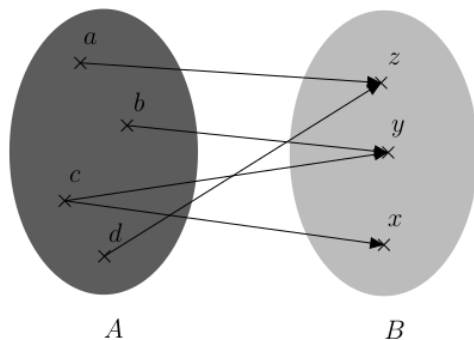


Figure: Not a function!

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Binary operations

Definition 4.

A **binary operation** $*$ on a set S is a function

$$* : S \times S \rightarrow S, \quad (s, t) \mapsto s * t.$$

The set-operation pair is denoted by $(S, *)$.

Examples:

- $(\mathbb{N}, +)$; (\mathbb{Q}, \times) ; $(\mathcal{P}(A), \cup)$, for some set A .

Further notions:

- An operation $*$ is **commutative** if $s * t = t * s$ for all $s, t \in S$.
- An operation $*$ is **associative** if $(r * s) * t = r * (s * t)$ for all $r, s, t \in S$.

Exercise: Find an example of a non-commutative operation.

Neutral element

Let $(S, *)$ be an operation on a set S .

Definition 5.

An element $e \in S$ is called the **neutral element** of S if

$$e * s = s * e = s \quad \text{for all} \quad s \in S.$$

Examples:

- $(\mathbb{N}, +)$: neutral element is $e = 0$.
- (\mathbb{Z}, \cdot) : neutral element is $e = 1$.
- Let A be a set. $(\mathcal{P}(A), \cup)$: neutral element is \emptyset .
Q: What is the neutral element of $(\mathcal{P}(A), \cap)$?
- **Exercise:** Find a pair $(S, *)$ with no neutral element.

Inverse elements

Let $(S, *)$ be an operation on a set S with neutral element e .

Definition 6.

An element $t \in S$ is called the **inverse** of $s \in S$ if

$$s * t = t * s = e.$$

Examples:

- $(\mathbb{Z}, +)$: $e = 0$ and the inverse of $n \in \mathbb{Z}$ is given by $-n$ for all $n \in \mathbb{Z}$.
- **Q:** What is the neutral element of (\mathbb{Q}, \times) ? Which elements have an inverse?

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Basic properties of functions

Let S, T be sets and $f : S \rightarrow T$ a function.

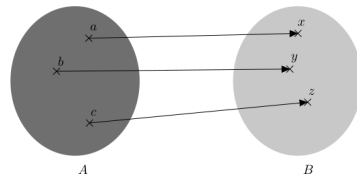
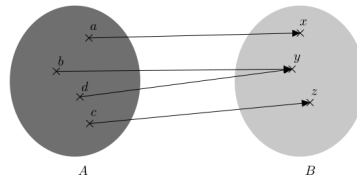
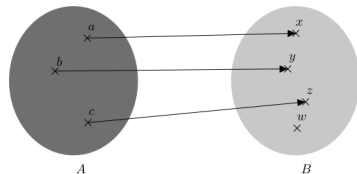
- f is called **injective** (one to one) if

$$\forall s_1, s_2 \in S : f(s_1) = f(s_2) \Rightarrow s_1 = s_2.$$

- f is called **surjective** (onto) if

$$\forall t \in T, \exists s \in S : f(s) = t.$$

- f is called **bijective** if it is injective *and* surjective.



Examples

Examples:

- The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin(x)$ is neither injective nor surjective.
- The function $f : \mathbb{R} \rightarrow [-1, 1]$, $f(x) = \sin(x)$ is surjective but not injective.
- **Exercise:** Check whether the functions $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = x^3$ are injective/surjective/bijective.

Questions

- Q: Let S, T be finite sets and let $f : S \rightarrow T$ be bijective. What restrictions does this impose on the sets S and T ?
- Q: Is this still true if the sets S and T are not finite?

Inverse function

Let $f : S \rightarrow T$ be a *bijective* function. Then there exists a function $f^{-1} : T \rightarrow S$ which is the **inverse function** of f in the sense that:

$$\begin{aligned}f^{-1}(f(s)) &= s \quad \text{for all } s \in S, \\f(f^{-1}(t)) &= t \quad \text{for all } t \in T.\end{aligned}$$

Example.

The inverse of $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ : x \mapsto x^2$ is $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ : x \mapsto \sqrt{x}$.

- Q: Why doesn't $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2$ have an inverse?

Composition of functions

Let X, Y, Z be sets and $f : X \rightarrow Y, g : Y \rightarrow Z$ be functions. Then the **composition** of f and g is the function $g \circ f$ from X to Z defined by

$$g \circ f : X \rightarrow Z, \quad (g \circ f)(x) = g(f(x)).$$

Example.

Let $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2, g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \sin(x)$. Then

$$(g \circ f) : \mathbb{R} \rightarrow \mathbb{R}, \quad (g \circ f)(x) = \sin(x^2).$$

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The composition of functions is an operation on the set

$$Fun_X := \{\text{functions } f : X \rightarrow X\}.$$

- Q: What is the neutral element?
- Q: For which $f \in Fun_X$ does an inverse element exist, and how is it defined?