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Solutions for Problem Set 1

Mathematics for Social Scientists

Exercise 1. .

a. i.
- $A \cap B = \{3, 10\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $A \backslash B = \{1, 2, 5, 6, 7, 8\}$

ii.
-
$$\begin{aligned} A \cap B &= \{\text{"All odd prime numbers"}\} \\ &= \{n \in \mathbb{N} : \text{n is a prime number}\} \backslash \{2\} \\ &= \{n \in \mathbb{N} : \text{n is an odd prime number}\} \end{aligned}$$
- $A \cup B = \{n \in \mathbb{N} : \text{n is odd}\} \cup \{2\}$
- $A \backslash B = \{2\}$

iii.
- $A \cap B = (0, 1]$
- $A \cup B = [0, 2)$
- $A \backslash B = \{0\}$

b. i. $P(A) = \{\{\}, \{a\}, \{b\}, \{5\}, \{a, b\}, \{a, 5\}, \{b, 5\}, \{a, b, 5\}\}$

ii. $A \times B = \{(a, 1), (a, 2), (a, f), (b, 1), (b, 2), (b, f), (c, 1), (c, 2), (c, f)\}$

iii. $P(B) \backslash P(A) = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
$|P(A)| = 4, |P(B)| = 8$

iv. 2^n

Exercise 2. Fill in the blanks.

$\boxed{\Rightarrow}$ $\boxed{\not\Rightarrow}$ $\boxed{\Leftrightarrow}$ $\boxed{\exists}$ $\boxed{\exists!}$ $\boxed{\in}$ $\boxed{\notin}$ $\boxed{\forall}$

(i) $n \in \mathbb{Z}$ is even $\boxed{\Longleftrightarrow}$ $\frac{n}{2} \in \mathbb{Z}$.

(ii) $n \in \mathbb{N}$ is not prime $\Leftrightarrow$ $\boxed{\exists}$ $m \in \mathbb{N}$ such that m $\boxed{\notin}$ $\{1, n\}$, and $\frac{n}{m}$ $\boxed{\in}$ $\mathbb{N}$.

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(iii) $\boxed{\forall}$ $q \in \mathbb{Q}$, $\boxed{\exists!}$ $n \in \mathbb{Z}$ such that $n \leq q < n+1$.

(iv) $x \in [0,1]$ $\boxed{\not\Rightarrow}$ $x \in (0,1)$.

(v) $y \in (0,1)$ $\boxed{\Rightarrow}$ $y \in [0,1]$.

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$.

(i) Let $g : \mathbb{R} \to \mathbb{R} : x \mapsto x/2$. What are $g \circ f$ and $f \circ g$?
$g \circ f : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{x^2}{2}$
$f \circ g : \mathbb{R} \to \mathbb{R} : x \mapsto (\frac{x}{2})^2$

(ii) Suppose that $g \circ f : \mathbb{R} \to \mathbb{R} : x \mapsto x$. What is g?
No solution. But we can get something similar:
Let $g : \mathbb{R} \to \mathbb{R} : x \mapsto \sqrt{x}$
Then, $g \circ f : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$

(iii) Define a function g such that $f \circ g(2) = 2$, $f \circ g(0) = 2$.
We need $g(0), g(2) \in \{-\sqrt{2}, \sqrt{2}\}$. For example, $g(x) = \sqrt{2}$

(iv) In which of the following cases is $f \circ g$ a well-defined function?

(a) $g : \mathbb{N} \to \mathbb{R} : x \mapsto x+1$
Well defined. $f \circ g : \mathbb{N} \to \mathbb{R} : x \mapsto (x+1)^2$ is well defined.

(b) $g : \mathbb{R} \to \mathbb{R} : x \mapsto \sqrt{x}$
Not well defined.
g is not well defined, because $\sqrt{x} \notin \mathbb{R}$ for $x < 0$. Therefore, $f \circ g$ is not well defined.

(c) $g : \mathbb{R} \to [0, \infty) : x \to |x|$, the *modulus (absolute value)* function.
Well defined. $f \circ g : \mathbb{R} \to [0, \infty) : x \mapsto x^2$ is well defined.

Exercise 4. Let X and Y be sets, $f : X \to Y$ a function. Define the **image** of f as

$$\{y \in Y : \exists x \in X \text{ such that } f(x) = y\}.$$

Write down the images of the following functions.

(i) $X = Y = \mathbb{Q}$, $f(x) = x^3$.
$\mathbb{Q}$

(ii) $X = (0,1]$, $Y = \mathbb{R}$, $f(x) = \frac{1}{x}$.
$[1, \infty]$

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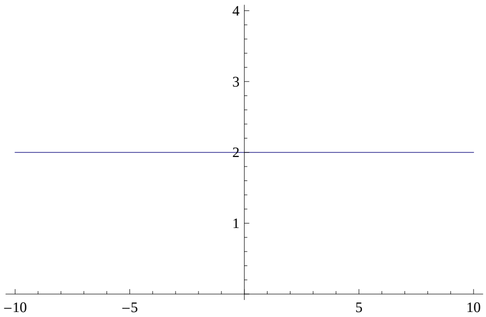
(iii) $X = Y = \mathbb{R}$, $f(x) = \sin(x)$.
$[-1.1]$

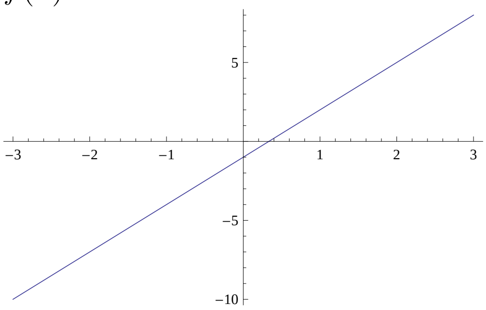
(iv) $X = Y = \mathbb{R}$, $f(x) = 1$.
$\{1\}$

Exercise 5. Sketch the following functions:

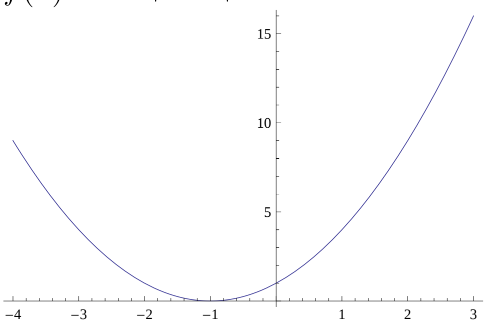
(i) $f(x) = 2$



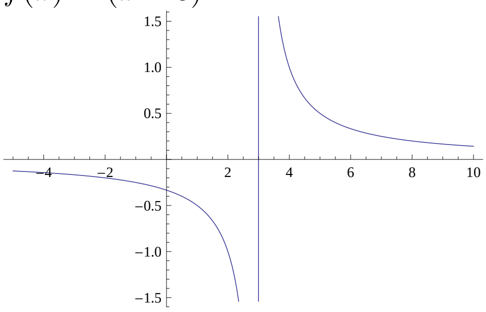
(ii) $f(x) = 3x - 1$



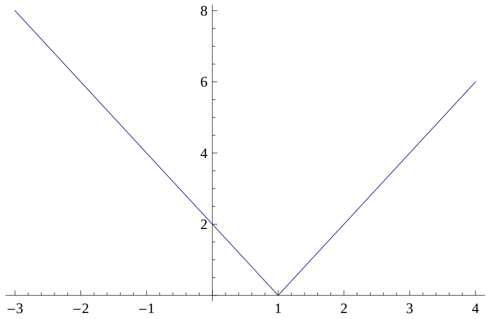
(iii) $f(x) = x^2 + 2x + 1$



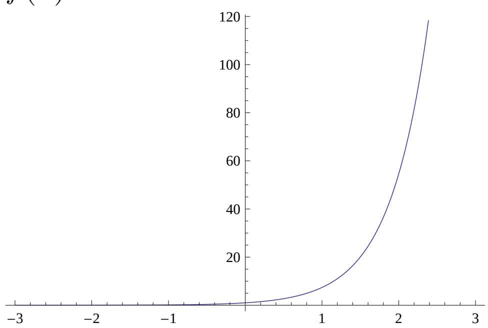
(iv) $f(x) = (x - 3)^{-1}$



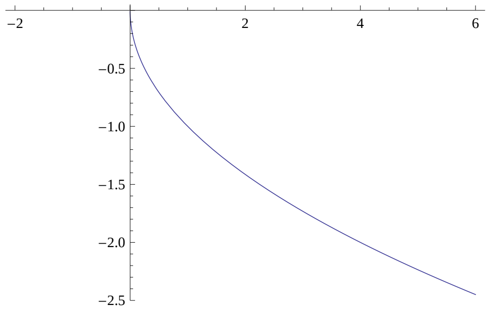
(v) $f(x) = |2x - 2|$



(vi) $f(x) = e^{2x}$



(vii) $f(x) = -\sqrt{x}$



Exercise 6. Which of the following functions is injective, bijective, or surjective?

(i) $a(x) = 2x + 1$

$a(x)$ is both injective (every element of the domain is linked to at most one element in the co-domain) and surjective (since for every element in the co-domain there is at least one element in the domain) and, thus, bijective.

(ii) $b(x) = x^2$

$b(x)$ is not injective since $b(x) = b(-x)$. It is also not surjective since there are no negative values for $b(x)$. However, if we would specify the range of $b(x) \in \mathbb{R}^+$, then it would be surjective.

(iii) $c(x) = \ln x$ for $(0, \infty) \mapsto \mathbb{R}$

$c(x)$ is bijective.

(iv) $d(x) = e^x$ for $\mathbb{R} \mapsto \mathbb{R}$

$d(x)$ is injective, but not surjective as there are no negative values for $d(x)$.