Mathematics for Social Scientists

Part 1: Logic, Sets, Functions

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Outline

Propositions and predicates

- A **proposition** has a definite truth value (true or false, never both). Examples: $1 + 1 = 2$ (true); $1 = 2$ (false).
- A **conjecture** is a proposition whose truth value is currently unknown (e.g., Goldbach's conjecture).
- A **predicate** has free variables, e.g. $P(x) : x > 4$. It becomes a proposition once x (and its universe) are fixed.

Connectives and precedence

- $X \wedge Y$ is true iff both X and Y are true.
- $X \vee Y$ is false iff both are false (mathematical "or" is *inclusive*).
- $\neg X$ is true iff X is false.
- Parentheses avoid ambiguity: $X \wedge (Y \vee Z)$ vs. $(X \wedge Y) \vee Z$.

Logical equivalence (De Morgan)

Theorem 1.
$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$ and $\neg(X \wedge Y) \equiv \neg X \vee \neg Y$.

- Two propositions are *equivalent* when their truth tables match.
- Beware: $\neg X \vee Y$ means $(\neg X) \vee Y$, not $\neg(X \vee Y)$.

Some notions from logic

Implication and equivalence:

$A \Rightarrow B$ (A implies B)

$A \Leftrightarrow B$ (A holds $\underbrace{\text{if and only if}}_{\text{iff}}$ B holds)

And / Or:

$A \wedge B$ (A holds and B holds)

$A \vee B$ (A holds or B holds)

Quantifiers:

$\forall$ for all

$\exists$ there exists

$\exists!$ there exists a unique (exactly one)

Truth Table

- T stands for *True* and F stands for *False*.
- Here, p and q are some statements. The first row shows the outcome of the logical computations if both of these statements are true, the second row shows the outcome when p is true but q is false, and so on...

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \iff q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Examples

Example:
p = ”Mensa is always completely empty” (False)
q = ”The sky is blue” (True)

Then;

- $p \wedge q$ = ”Mensa is always completely empty **and** the sky is blue” is a false statement,
- $p \vee q$ = ”Mensa is always completely empty **or** the sky is blue” is a true statement,
- $p \Rightarrow q$ = ”**If** Mensa is always completely empty, **then** the sky is blue” is a true statement.[1]

[1] *This is a little counter-intuitive. The ”if {something false}, then ...” format is always true. The reason behind this is, ”if {something false},” puts us in a hypothetical world that doesn't exist. For example, no one can show the following statement to be false: ”If my grandmother had wheels, then she would be a bike.” My grandma doesn't have wheels, therefore it doesn't matter what comes next. Statement is true.*

Examples on Reading the Logical symbols

- $(p \vee q) \Rightarrow q$: *If p or q, then q*
- $p \vee (q \Rightarrow r)$: *p or "q implies r"*[2]
- Let $e \in S$.
  $e * s = s * e = s \quad \forall s \in S \implies e$ is the neutral element of S:
  Let e be an element of S.
  If $e * s = s * e = s$ for all s in S, then e is the neutral element of S.
- f is called injective if $\forall s_1, s_2 \in S : f(s_1) = f(s_2) \implies s_1 = s_2$:
  f is called injective if for all s_1, s_2 in S, we have that "if $f(s_1) = f(s_2)$, then $s_1 = s_2$."

[2] "If... then..." and "implies" have the same meaning.

More examples

- $A = \{2 \cdot n : n \in \mathbb{Z}\}$:
  A is the set consisting of all numbers that are in the form of "2 times n", where n are elements of $\mathbb{Z}$.

- $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z} \wedge 0 \neq q \in \mathbb{Z}\}$:
  $\mathbb{Q}$ is the set of numbers in the form $\frac{p}{q}$, such that p is an element of the integers and q is an element of the integers that does not equal 0.

- $A \subset B \iff \forall a \in A, a \in B$:
  A is a subset of B, if and only if, for all a in A, we have that a is also in B.

Implication, contrapositive, and language

- $P \Rightarrow Q$ is **false only** when P is true and Q is false.
- **Contrapositive:** $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.
- **Converse** ($Q \Rightarrow P$) and **inverse** ($\neg P \Rightarrow \neg Q$) are generally *not* equivalent to $P \Rightarrow Q$.
- Language mapping:
  - "X only if Y" means $X \Rightarrow Y$ (Y necessary for X).
  - "X if Y" means $Y \Rightarrow X$ (Y sufficient for X).
  - "X iff Y" means $(X \Rightarrow Y) \wedge (Y \Rightarrow X)$.

Quantifiers and negation

- Fix a universe U. $\forall x \in U: \; P(x)$ is true iff $P(x)$ holds for every $x \in U$; a single counterexample makes it false.

- $\exists x \in U: \; P(x)$ is true iff $P(x)$ holds for at least one $x \in U$.

- Negations: $\neg \forall x \, P(x) \; \equiv \; \exists x \, \neg P(x), \quad \neg \exists x \, P(x) \; \equiv \; \forall x \, \neg P(x)$.

- Example: negate $1 \leq x < 5$:

$$\neg\Big((1 \leq x) \wedge (x < 5)\Big) \; \equiv \; (x < 1) \; \vee \; (x \geq 5).$$

Axioms

- Mathematics starts from **axioms**: basic premises taken as true.
- Axioms should be few and independent; changing them changes what can be proved.
- Example (empty set axiom): *There exists a set with no elements*.

Definitions, propositions, lemmas, theorems, corollaries

- **Definition** fixes meaning (not proved); it introduces objects/notation. Examples: $\mathbb{N} = \{1, 2, 3, \ldots\}$; define $f : \mathbb{Z} \to \mathbb{R}$ by $f(x) = x^2$.

- **Result types (true statements with stated conditions and a conclusion):**
  - **Proposition**: general result.
  - **Lemma**: auxiliary result used to prove a theorem.
  - **Theorem**: central result.
  - **Corollary**: immediate consequence of a theorem.

- **Examples**
  - *Implication:* If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
  - *Equivalence:* $A = B \;\Leftrightarrow\; (A \subseteq B) \;\wedge\; (B \subseteq A)$.

What is a proof?

- A **proof** is a finite, logically valid chain of statements taking the **assumptions** to the **conclusion**.
- Each step must be justified by an axiom, a definition, a previously proved result, or a valid rule of inference.
- Be explicit about scope: declare fixed objects (“Let $x \in S$ be arbitrary”) and where each assumption is used.

Proof methods

- **Direct** (implication): assume $A_1, \ldots, A_k$; derive C step by step.
- **Contrapositive**: to prove $\forall x : \; P(x) \Rightarrow Q(x)$, prove $\forall x : \; \neg Q(x) \Rightarrow \neg P(x)$.
- **Contradiction**: assume $A_1, \ldots, A_k$ and $\neg C$; derive an impossibility (e.g. $R \wedge \neg R$).
- **By cases**: if $P \vee R$, prove C from P and also from R.
- **Existence/Uniqueness ($\exists!$).** Show existence: $\exists x \, P(x)$; then uniqueness: $\forall x, y \left(P(x) \wedge P(y) \Rightarrow x = y \right)$.
- **Induction on n.** Base case $P(n_0)$; inductive step $P(k) \Rightarrow P(k+1)$; conclude $\forall n \geq n_0 : \; P(n)$.

Example (contradiction)

Theorem 2.
For any set A, $\emptyset \subseteq A$.

Proof.
Suppose not. Then there exists $x \in \emptyset$ with $x \notin A$. But $\emptyset$ has no elements. Contradiction. Hence $\emptyset \subseteq A$. ∎

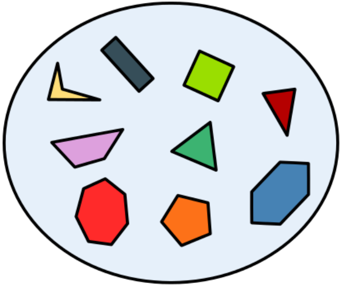
Outline

What is a set?

- A set is a collection, denoted by $\{\dots\}$, of distinct objects called **elements**.

Example.
The set of things I had for breakfast is {Müsli, milk, orange juice}.

- Two sets A and B are equal if and only if they have precisely the same elements.
- Sets ignore repetition and order. So $\{1, 1, 2, 3\} = \{1, 3, 2\}$.
- Sets are conventionally denoted by capital letters (eg. A, B), and elements by lower-case letters (eg. a, b, c).



A set of polygons

There are two ways of defining a set:

- A **descriptive definition**, for example
  - The set of colours in the British flag.
  - The set of prime numbers less than 10.
- An **explicit definition**, for example
  - $A = \{\text{blue, red, white}\}$
  - $B = \{2, 3, 5, 7\}$.
- *Notation*: $a \in A$ means 'a is an element of A'. Otherwise $a \notin A$.

Example.

Let $B = \{2, 3, 5, 7\}$ be the set of prime numbers less than 10. Then

- $2 \in B, 3 \in B, \dots$
- $4 \notin B, 6 \notin B, \dots$

Definition 1.

N is called a **subset** of M if for all $n \in N$ it also holds that $n \in M$. This is denoted $N \subseteq M$.

Definition 2.

We can restate our definition of a subset as follows:

$$N \subseteq M \qquad \Leftrightarrow \qquad \forall a \in N, a \in M$$

N is a of subset M if and only if for all *a* in N, *a* is in M.

Examples of sets

- $\emptyset = \{\}$ (the *empty set*)
- $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ (the *natural numbers*)
- $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$ (les *natural numbers*)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the *integers*)
- $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z} \text{ and } 0 \neq q \in \mathbb{Z}\}$ (the *rational numbers*)
- $\mathbb{R}$ (the *real numbers*)
- $\mathbb{R}^n = \{(r_1, \ldots, r_n) : r_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$ (*n-tuples* of real numbers)
- $\mathbb{C}$ (the *complex numbers*)

More examples of sets

- Logical notation allows us to write descriptive definitions of sets using mathematics. For example,
  - $\{1, 2, 3\} = \{n \in \mathbb{N} : n < 4\}$
  - $2\mathbb{Z} = \{2 \cdot n : n \in \mathbb{Z}\}$, the set of even numbers.

- Intervals: for $a, b \in \mathbb{R}$.
  - $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
  - $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

More examples of sets

- Let A be a set. The power set $\mathcal{P}(\boldsymbol{A})$ of A is the set of all subsets of A.

Example.
$A = \{1, 2\} \quad \Rightarrow \quad \mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}.$

  - The power set always contains the set itself and the empty set.

- Let A, B be sets. The cartesian product $\boldsymbol{A} \times \boldsymbol{B}$ of A and B is defined as the set containing all pairs (a, b) such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

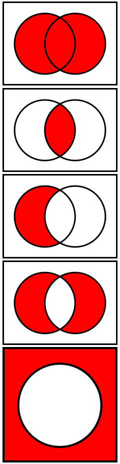
Example.
Let $A = \{1, 2, 3\}$, $B = \{1, 4\}$. Then
$A \times B = \{(1, 1), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4)\}$

Outline

Set operations

Let A and B be sets. The following *set operations* allow us to build new sets from A and B:

- *Union*: $$A \cup B := \{x : x \in A \text{ or } x \in B\}$$
- *Intersection*: $$A \cap B := \{x : x \in A \text{ and } x \in B\}$$
- *Difference* or *relative complement* $$A \backslash B := \{x : x \in A \text{ and } x \notin B\}$$
- *Symmetric difference*: $$A \triangle B := \{x : x \in A \backslash B \text{ or } x \in B \backslash A\}$$
- *Complement* with respect to a universal set Ω: $$A^c := \{x \in \Omega : x \notin A\}$$



Laws for set operators

Let A, B and C be sets. Then, the following properties hold:

- $A \cap B = B \cap A$, $A \cup B = B \cup A$ (Commutative law)
- $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
- $A \cup A = A$ and $A \cap A = A$ (Idempotenz)
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive laws)
- $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$ (De Morgan's laws)

Outline

Defining functions

Definition 3.
A **function** or **map**, denoted by $f : S \to T$, from a set S to a set T is a rule which assigns to each element $s \in S$ exactly one element $t \in T$. $f : S \to T$, $f(s) = t$

A Function has 3 parts:

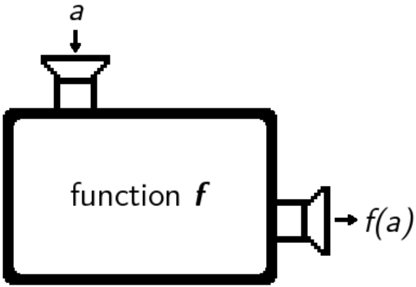
- A set S to map *from*. This set is called the **domain** of f.
- A set T to map *to*. This set is called the **co-domain** of f.
- A rule for *every* element $a \in S$, assigning it to some element $b \in T$. This is written $f(a) = b$.

Functions

Examples:

- $f : \{1, 2, 3\} \to \{3, 4, 5\}$,
  $: x \mapsto x + 2$.

- $f : \{1, 2\} \to \{1, 3\}$,
  $f(1) = 1$, $f(2) = 3$.

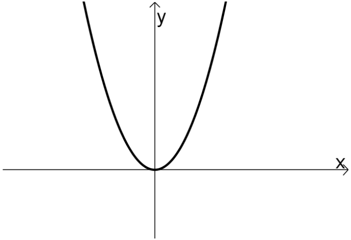


- A common notation for functions is $y = f(x)$, which tells us that a function f is performed upon the values of $x \in A$ (the domain) to obtain the values of $y \in B$ (the co-domain).

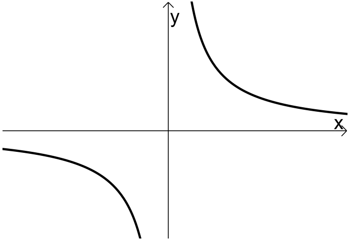
Visualising functions

Q: Which of the following are functions $: \mathbb{R} \to \mathbb{R}$?

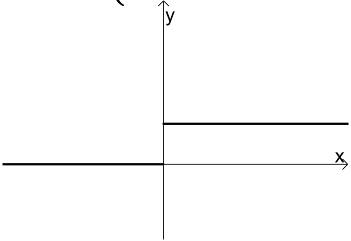
$$y - x^2 = 0$$



$$y = 1/x$$



$$y = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$



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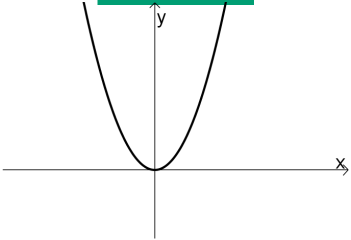

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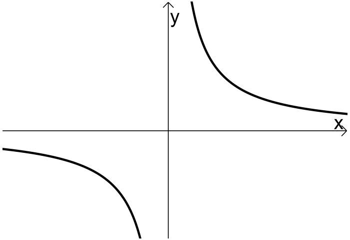

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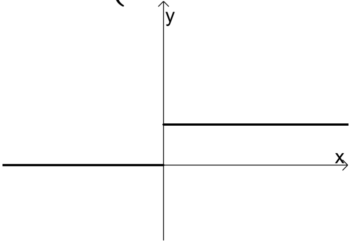

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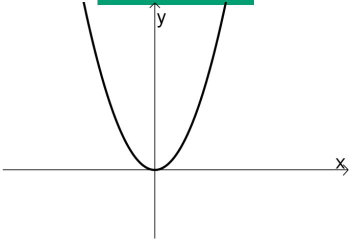

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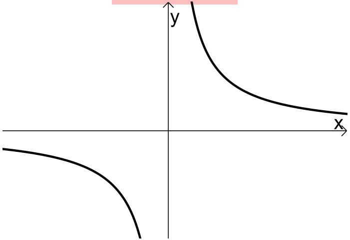

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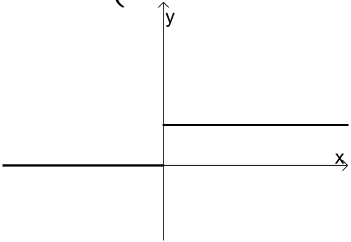
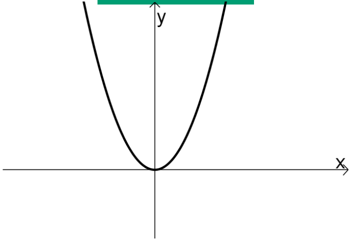


Visualising functions

Q: Which of the following are functions $: \mathbb{R} \to \mathbb{R}$?

$y - x^2 = 0$



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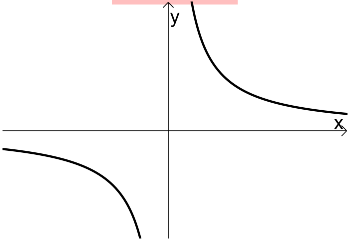

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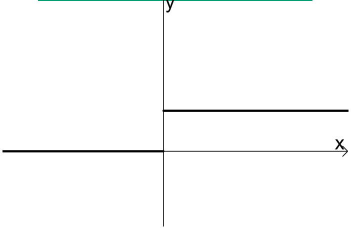

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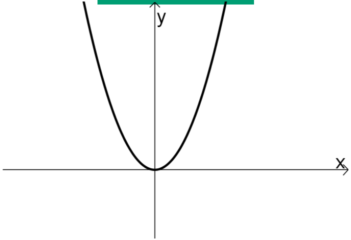

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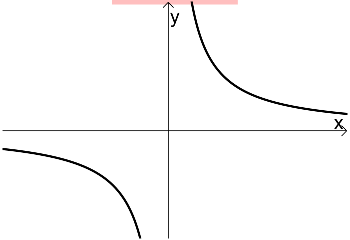

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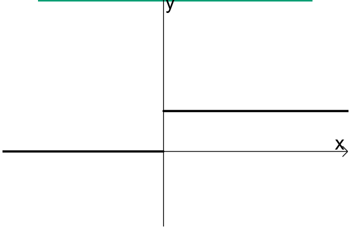


Functions are valuable to social scientists because most relationships can be modelled by a function.

More examples

- *Constant function*: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c \in \mathbb{R}$ for all $x \in \mathbb{R}$.
- *Sin function*: $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin(x)$.
- *Polynomial in x*: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0$, where $a_i \in \mathbb{R}$.
- Let A be a finite set. $f : \mathcal{P}(A) \to \mathbb{N}$, $f(B) = |B|$, where B is a subset of A.

More examples

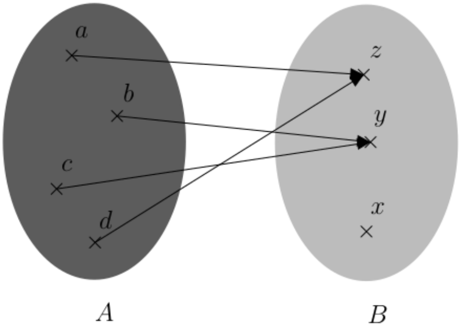


Figure: A function!

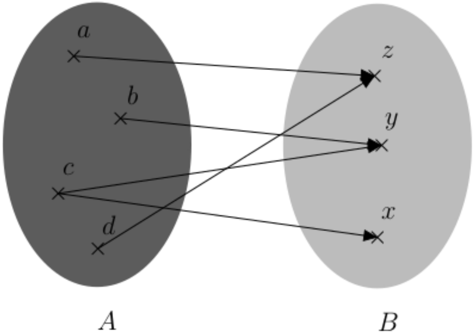


Figure: Not a function!

Outline

Binary operations

Definition 4.
A **binary operation** $*$ on a set S is a function

$$* : S \times S \to S, \quad (s,t) \mapsto s * t.$$

The set-operation pair is denoted by $(S, *)$.

Examples:

- $(\mathbb{N}, +)$; $(\mathbb{Q}, \times)$; $(\mathcal{P}(A), \cup)$, for some set A.

Further notions:

- An operation $*$ is **commutative** if $s * t = t * s$ for all $s, t \in S$.
- An operation $*$ is **associative** if $(r * s) * t = r * (s * t)$ for all $r, s, t \in S$.

Exercise: Find an example of a non-commutative operation.

Neutral element

Let $(S, *)$ be an operation on a set S.

Definition 5.
An element $e \in S$ is called the **neutral element** of S if

$$e * s = s * e = s \qquad \underline{\text{for all}} \qquad s \in S.$$

Examples:

- $(\mathbb{N}, +)$: neutral element is $e = 0$.
- $(\mathbb{Z}, \cdot)$: neutral element is $e = 1$.
- Let A be a set. $(\mathcal{P}(A), \cup)$: neutral element is $\emptyset$.
  Q: What is the neutral element of $(\mathcal{P}(A), \cap)$?
- **Exercise:** Find a pair $(S, *)$ with no neutral element.

Inverse elements

Let $(S, *)$ be an operation on a set S with neutral element e.

Definition 6.
An element $t \in S$ is called the **inverse** of $s \in S$ if

$$s * t = t * s = e.$$

Examples:

- $(\mathbb{Z}, +)$: $e = 0$ and the inverse of $n \in \mathbb{Z}$ is given by $-n$ for all $n \in \mathbb{Z}$.
- **Q:** What is the neutral element of $(\mathbb{Q}, \times)$? Which elements have an inverse?

Outline

Basic properties of functions

Let S, T be sets and $f : S \to T$ a function.

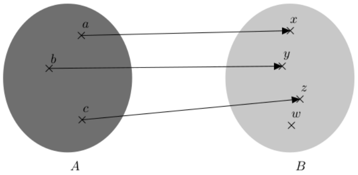
- f is called **injective** (one to one) if

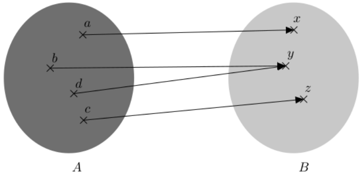
$$\forall s_1, s_2 \in S : f(s_1) = f(s_2) \Rightarrow s_1 = s_2.$$

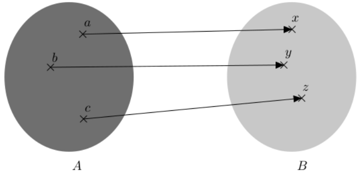
- f is called **surjective** (onto) if

$$\forall t \in T, \exists s \in S : f(s) = t$$

- f is called **bijective** if it is injective *and* surjective.







Examples

Examples:

- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ is neither injective nor surjective.

- The function $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin(x)$ is surjective but not injective.

- **Exercise:** Check whether the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$ are injective/surjective/bijective.

Questions

- **Q:** Let S, T be finite sets and let $f : S \to T$ be bijective. What restrictions does this impose on the sets S and T?

- **Q:** Is this still true if the sets S and T are not finite?

Inverse function

Let $f : S \to T$ be a *bijective* function. Then there exists a function $f^{-1} : T \to S$ which is the **inverse function** of f in the sense that:

$$f^{-1}(f(s)) = s \quad \text{for all } s \in S,$$
$$f(f^{-1}(t)) = t \quad \text{for all } t \in T.$$

Example.
The inverse of $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+ : x \mapsto x^2$ is $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+ : x \mapsto \sqrt{x}$.

- **Q:** Why doesn't $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ have an inverse?

Composition of functions

Let X, Y, Z be sets and $f : X \to Y$, $g : Y \to Z$ be functions. Then the **composition** of f and g is the function $g \circ f$ from X to Z defined by

$$g \circ f : X \to Z, \quad (g \circ f)(x) = g(f(x))$$

Example.

Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$, $g : \mathbb{R} \to \mathbb{R} : x \mapsto \sin(x)$. Then

$$(g \circ f)(x) : \mathbb{R} \to \mathbb{R} : x \mapsto \sin(x^2).$$

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The composition of functions is an operation on the set

$$Fun_X := \{\text{functions } f : X \to X\}.$$

- **Q:** What is the neutral element?
- **Q:** For which $f \in Fun_X$ does an inverse element exist, and how is it defined?