Mathematics for Political Scientists

Master Political Science

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University of Mannheim

August 29 - September 2, 2022

Introduction

Course Objectives

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- Recap of your high-school / Abitur knowledge in mathematics.
- Introduction to the fundamentals in math that are necessary for your understanding of statistics and game theory.
- Overcome possible reservations against the use of mathematics.
- A refresher and starting point for future individual learning.

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What is this course not about?

- It is not a mathematical freak show!
- It does not introduce into advanced mathematical techniques.

Math and applications in political science

Why is math important to social scientists?

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- Mathematics allows for orderly and systematic communication. Ideas expressed mathematically can be more carefully defined and more directly communicated than narrative language, which is susceptible to vagueness and misinterpretation.
- Mathematics is an effective way to describe and model our world.

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- Mathematics is an effective way to describe and model our world.

Applications

- Game Theory, Decision Theory
- Computer Simulation, Agent-Based Modeling
- Statistics, Econometrics
- Empirical Analyses in any field

Syllabus

Course Website[1]: Syllabus, slides, exercises and extra materials.

I Set Theory

[1] https://math-refresher-22.netlify.app/

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Organization

- Lecture + Slides
- Exercises
  - on the board
  - independent work and self-study
  - group work
- Active participation

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This course is voluntary!

Schedule

Date	Day	Time	Room
29.08.2022	Monday	14:00 - 17:15	B 317 Seminarraum
30.08.2022	Tuesday	9:15 - 17:15	B 317 Seminarraum
31.08.2022	Wednesday	9:15 - 13:15	B 317 Seminarraum
01.09.2022	Thursday	9:15 - 17:15	B 317 Seminarraum
02.09.2022	Friday	9:15 - 12:30	B 317 Seminarraum

General Readings

Recommended:

General

- Gill (2006): Essential Mathematics for Political and Social Research.
- Moore/Siegel (2013): A Mathematics Course for Political and Social Research. *An introductory mathematics course aimed at social scientists, provides good intuitions for basic concepts and applications. It has accompanying video lectures on Youtube.*
- Simon/Blume (1994) *A comprehensive treatment of mathematics for students of economics for both undergraduate and more advanced level.*
- Sydsaeter/Hammond (2008) *Another standard mathematics textbook for economics undergraduates.*

Specific Readings

- Calculus
  - Spivak (2006) *A classic standard textbook for a first class in Calculus for mathematics students at undergraduate level.*
- Probability Theory
  - DeGroot/Schervish (2011) *A comprehensive standard treatment of probability and statistics for mathematics undergraduate students. Intuitive and (relatively) rigorous at the same time with lots of exercises.*
- Linear Algebra
  - Lay (2011) *A standard introduction for mathematics undergraduates.*
  - The Matrix Cookbook[2]
    An overview over some more advanced matrix calculus.

[2] http://www2.imm.dtu.dk/pubdb/views/edoc_download.php/3274/pdf/imm3274.pdf

Set Theory

Motivation

Explanations of political outcomes often begin with the presumption that such outcomes are the result of purposive decisions made by relevant individuals (e.g. voters, legislators) or groups of individuals (e.g. political parties, interest groups, nation states)

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Set Theory is fundamental to the formalization of these concepts.

Set Theory is fundamental to the understanding of many other fields of mathematics, e.g. the concept of 'functions'.

What Is a Set?

Definition (Set)

A **set** is a collection of distinct objects, where the objects therein are called **elements** or **members**.

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A **set** is a collection of distinct objects, where the objects therein are called **elements** or **members**.

For example $A = \{1, 2, 3\}$ is a set, and 1 is an element of A (write $1 \in A$), whereas 4 is not an element of A ($4 \notin A$).

If a set does not contain any elements, we call it an **empty set**. The shorthand for an empty set is $\emptyset$ or $\{\}$.

Example: Sets of Numbers

Symbol	Explanation	Example
$\mathbb{N}$	set of natural numbers	$1, 2, 3, 4, \ldots$
$\mathbb{Z}$	set of integers	$-2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	set of rational numbers (fractions)	$-\frac{9}{7}, -1, 0, \frac{1}{2}, 1, \ldots$
$\mathbb{R}$	set of real numbers	fractions plus e.g. π or e
$\mathbb{R}^+$	set of positive real numbers	
$\mathbb{C}$	set of complex numbers	$\sqrt{-1}$

Relations of Sets

A set itself can, furthermore, be part of another set. E.g. $A = \{1, 2, 3\}$ is part of $B = \{1, 2, 3, 4\}$. We then say that A is a **subset** of B and write $A \subseteq B$.

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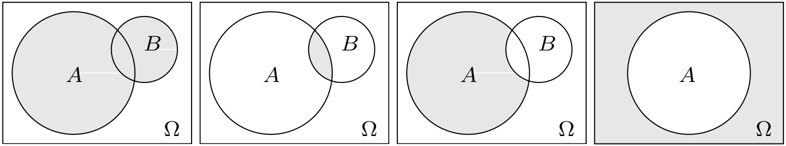
If two sets do not have any element in common, these sets are said to be **disjoint**. E.g. $A = \{1, 2, 3\}$ and $C = \{4, 5\}$ are disjoint.

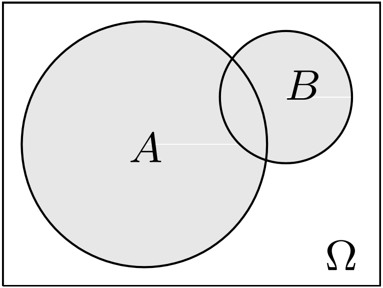
Operations on Sets I

We can visualize operations on sets using so called **Venn diagrams**.



Operations on Sets II

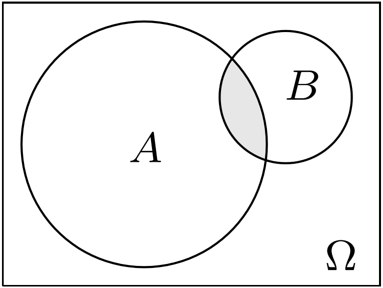
A **union** contains all elements that are either in A or B or in both. Formally, this is $A \cup B = \{x | x \in A \text{ or } x \in B \text{ or both}\}$.



If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

Operations on Sets III

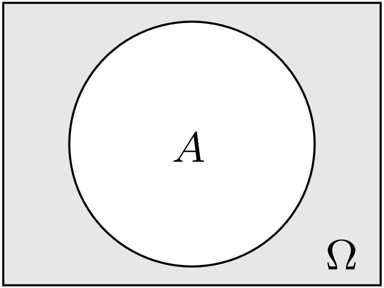
An **intersection** contains all elements that are both in A and B. Formally, this is $A \cap B = \{x | x \in A \text{ and } x \in B\}$.



If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then $A \cap B = \{3\}$.

Operations on Sets IV

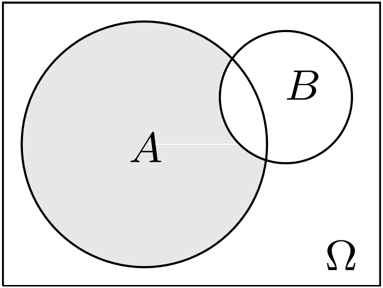
Let there be a **universal set** Ω with the subset A. The **complement** of A is every element of Ω that is not an element of A.
Formally, this is $A^C = \{x | x \notin A \text{ (and } x \in \Omega)\}$.



If $A = \{1, 2, 3\}$ and $\Omega = \{1, 2, 3, 4, 5\}$, then $A^C = \{4, 5\}$.

Operations on Sets V

We can also form **differences** of sets.
$A \setminus B = \{x | x \in A \text{ and } x \notin B\}$.



If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2\}$, then $A \setminus B = \{3, 4, 5\}$.

Cardinality

The **cardinality** of a set is a measure of the number of elements in the set.
Usually denoted with $|A|$ (alternatives: $n(A)$, $card(A)$ or $\#A$).

If $A = \{1, 2, 3, 4, 5\}$, then $|A| = 5$.

Summary of definitions

$\emptyset$	empty set
$\cup$	union of two sets
$\cap$	intersection of two sets
$\subseteq$	is a subset of
$\subset$	is a strict subset of
$\supseteq$	is a superset of
$\supset$	is a strict superset of

Useful Notation

$\in$	is an element of
$\forall$	for all
$\exists$	there exists
$\Rightarrow$	implies
$\Leftrightarrow$, iff	if and only if
: or $s.t.$	such that
$\equiv$	equivalent to
$\sim$ or $\neg$	not
$\setminus$	without

Laws of Set Theory

Commutative

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Laws of Set Theory

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$A \cup B = B \cup A$ and $A \cap B = B \cap A$

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$(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$

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Distributive
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$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

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$(A \cup B)^C = A^C \cap B^C$ and $(A \cap B)^C = A^C \cup B^C$

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De Morgan's Laws
$(A \cup B)^C = A^C \cap B^C$ and $(A \cap B)^C = A^C \cup B^C$
$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Spaces

Remember: $\mathbb{R}^1$ is the set of real numbers extending from $-\infty$ to ∞, the real number line.

$\mathbb{R}^n$ is an n-dimensional space **("Euclidean space")**, where each of the n axes extends from $-\infty$ to ∞.

Examples:

- $\mathbb{R}^1$ ($\mathbb{R}$) is a line.
- $\mathbb{R}^2$ is a plane.
- $\mathbb{R}^3$ is a 3D-space.

Points in $\mathbb{R}^n$ are ordered n-tuples, where each element of the n-tuple represents the coordinate along that dimension.

Interval Notation for $\mathbb{R}^1$

Open interval: $(a, b) \equiv \{x \in \mathbb{R}^1 : a < x < b\}$

Closed interval: $[a, b] \equiv \{x \in \mathbb{R}^1 : a \leq x \leq b\}$

Half open, half closed interval: $(a, b] \equiv \{x \in \mathbb{R}^1 : a < x \leq b\}$

Neighborhoods: Intervals, Disks, and Balls

We need a formal construct for what it means to be "near" a point **c** in $\mathbb{R}^n$. We call this the **neighborhood** of **c** and represent it by an open interval, disk, or ball, depending on whether n is one, two, or more dimensions, respectively. Given the point **c**, these are defined as

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  The open interior of the circle centered at **c** with radius ϵ.

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- ϵ-**ball** in $\mathbb{R}^n$: $\{x : ||x - c|| < \epsilon\}$
  The open interior of the sphere centered at **c** with radius ϵ.

Interior and Boundary Points

Definition (Interior Point)

The point $\mathbf{x}$ is an interior point of the set S if $\mathbf{x}$ is in S and if there is some ϵ-ball around $\mathbf{x}$ that contains only points in S. The **interior** of S is the collection of all interior points in S.

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Definition (Boundedness)

A set $A \subset \mathbb{R}^n$ is **bounded** if it can be contained within an ϵ-ball. That is, there will always be a real-valued number or vector that is outside the set.
Example: any interval that does not have ∞ or $-\infty$ as endpoints; any disk in a plane with finite radius.

Compact Set

Definition (Compact Set)

A set $A \subset \mathbb{R}^n$ is **compact** if it is closed and bounded.

Convexity

Definition (Convex Set)

A set A in $\mathbb{R}^n$ is said to be **convex** iff for each $x, y \in A$, the line segment $\lambda x + (1-\lambda)y$ for $\lambda \in (0,1)$ belongs to A. That is, all points on a line connecting two points in the set are in the set.

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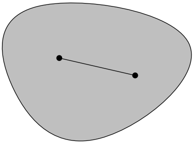
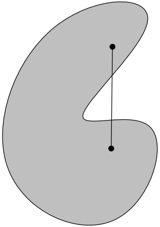
Definition (Convex Set)

A set A in $\mathbb{R}^n$ is said to be **convex** iff for each $x, y \in A$, the line segment $\lambda x + (1 - \lambda)y$ for $\lambda \in (0, 1)$ belongs to A. That is, all points on a line connecting two points in the set are in the set.



set is convex



set is not convex

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Beyond this example there are many other applications in political science that use the notion of compact sets.

Set Theory

Functions

What is a function?

Definition (Function)

A **function** or **map**, denoted by $f : X \mapsto Y$, has 3 parts:

- A set X to map from. This set is called the domain of f.
- A set Y to map to. This set is called the co-domain of f.
- A rule for every element $x \in X$, assigning it to some element $y \in Y$. This is written $f(x) = y$

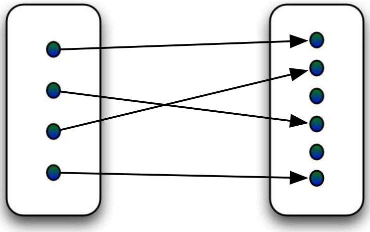
Examples:

- $$\begin{aligned} f &: \{1,2,3\} \to \{3,4,5\} \\ &: x \mapsto x + 2 \end{aligned}$$
- $$\begin{aligned} &f : \{1,2\} \to \{1,3\} \\ &f(1) = 1, f(2) = 3 \end{aligned}$$

Linking Sets: Injection, Bijection, and Surjection

Definition (Injection)

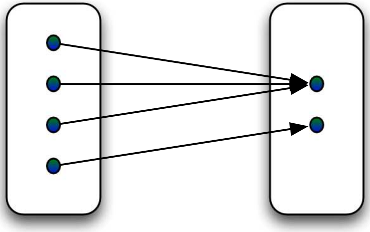
A function f is called **injective** if for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Verbally, every element of the codomain Y is linked to at most one element of the domain X.



Linking Sets: Injection, Bijection, and Surjection

Definition (Surjection)

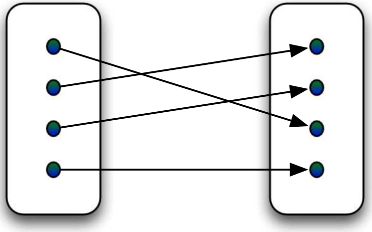
A function f is called **surjective** if for every $y \in Y$ there is an $x \in X$ with $f(x) = y$. Verbally, every element of the codomain Y is linked to at least one element of the domain X.



Linking Sets: Injection, Bijection, and Surjection

Definition (Bijection)

A function f is called **bijective** if it is injective and surjective, i.e. every element of the domain X is linked to one and only one element of the codomain Y and vice versa.



Set Theory

Binary Relations

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Binary relations are essential tools to formalize concepts like 'preferences' and 'choice'.

Definition

Definition (Binary Relation)

A binary relation R is a subset of $S \times S$ of **ordered pairs** of elements of S. R compares two elements of S, x and y, with each other. Write xRy.

Examples

1. Party Members
   $S = \{\text{CDU,SPD,Greens,FDP}\}$
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- **Antisymmetric:** $\forall x, y, \in S, [(xRy) \wedge (yRx)] \implies (x = y)$

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Preferences

Definition (Preferences)

A relation R is called a **preference relation** if and only if R is reflexive, transitive and complete (a weak order).

Maximal Elements

Definition (Maximal Elements)

Let R be a weak or partial order on Ω and $S \subseteq \Omega$. Then, the set of "R-maximal elements of S is"

$$M(S, R) = \{s \in S : \forall t \in S, sRt\}$$

Choice function

Definition (Choice function)

Let $\mathcal{X}$ be the family of all nonempty subsets of Ω. A choice function is a map $c\colon \mathcal{X} \to \mathcal{X}$ such that for all $S \in \mathcal{X}$, $c(S) \subseteq S$.

Defining Rationality

Definition (Rationality)

Given a choice function c, a choice is **rational** if and only if there $\exists$ a weak order R on Ω such that $\forall S \subseteq \Omega$, $c(S) = M(S, R)$.
R is said to be a preference relation that **rationalizes** the choice function $c()$.

Defining Utility

Definition (Utility Function)

A **utility function** for an individual is a function that maps every element in S into the reals, $u : S \to \mathbb{R}$ such that $\forall a, b \in S, aRb \Leftrightarrow u(a) \geq u(b)$.

Analysis I

Rules for Exponentials and Fractions

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- and universally stated: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$; $n \in \mathbb{N}$

Logarithms

- For those with a German background: Please note that in English texts the expression *log* without specification of a base is equal to *ln*, i.e. the natural logarithm!

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- Read $\log_a b$ as "the logarithm of b to the base a" or "the base-a logarithm of b"

Quadratic Expressions

Equations of the form $ax^2 + bx + c = 0$ can be solved using the quadratic formula (in German the so-called "Mitternachtsformel")

$$x_{1|2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equations with one variable

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Equations with several variables

In political science applications solving for one variable oftentimes is not enough. So let us now consider the solution of two simultaneous equations with two variables.

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\begin{aligned}
2x + 3y &= 4 \quad (1) \\
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This gives $y = -\frac{6}{7}$.

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This gives $y = -\frac{6}{7}$. Inserting this into (2)' gives $x = \frac{23}{7}$.

Analysis I

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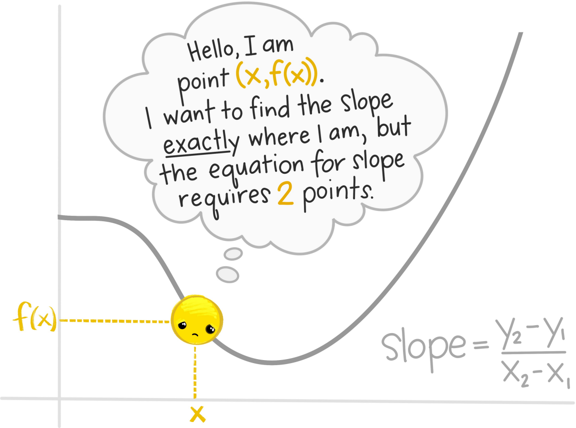
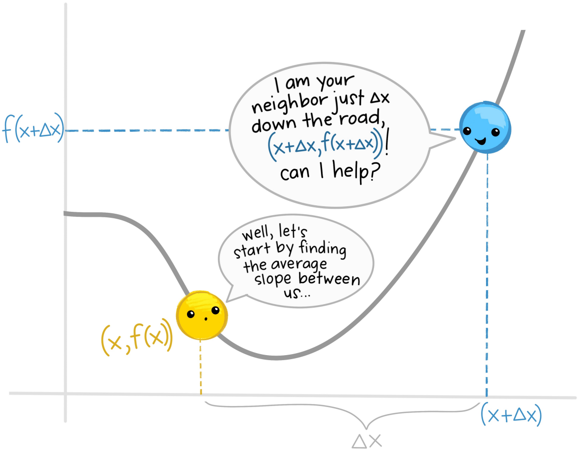
- for linear relationships, the information is directly available from the equation - the slope m
- What do we do when we have a non-linear function?
- What is the slope m at some point x_0?

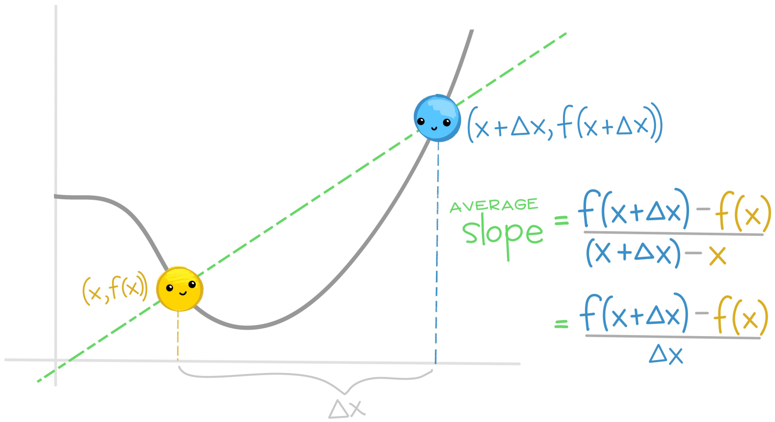
What is a derivative? I



What is a derivative? II



What is a derivative? III

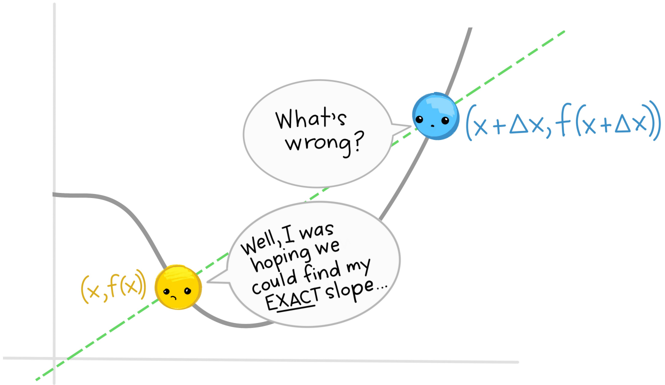


What is a derivative? IV

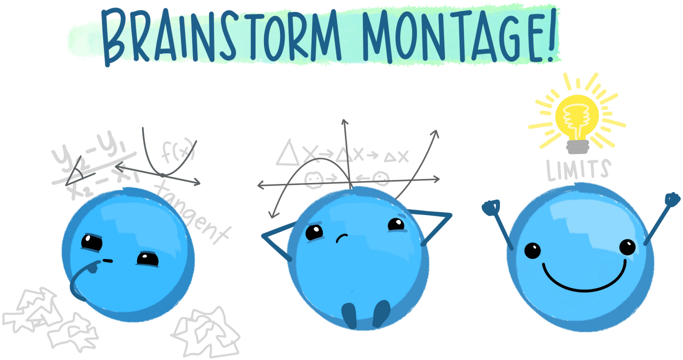
So: the average slope between ANY 2 POINTS on function $f(x)$ separated by Δx is

$$\boxed{m = \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

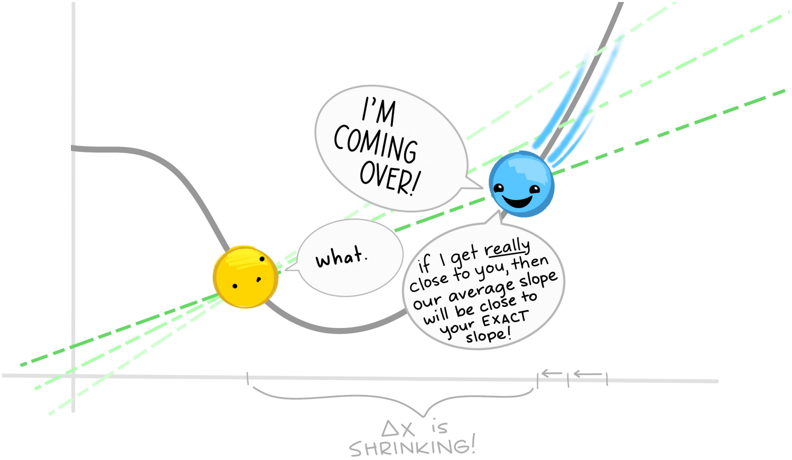
What is a derivative? V



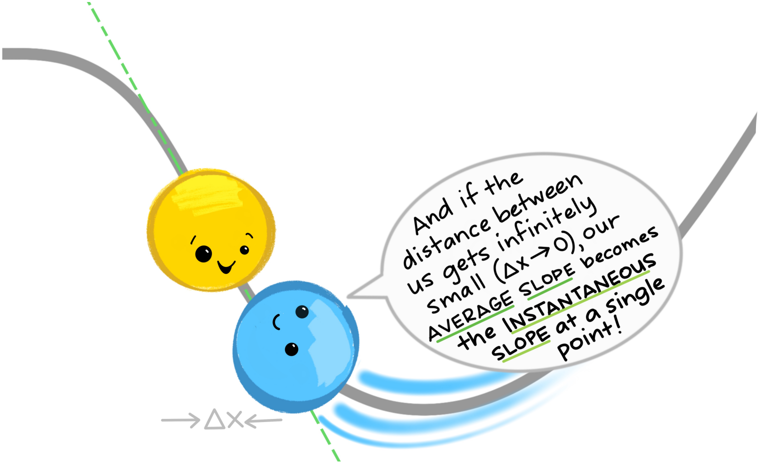
What is a derivative? VI



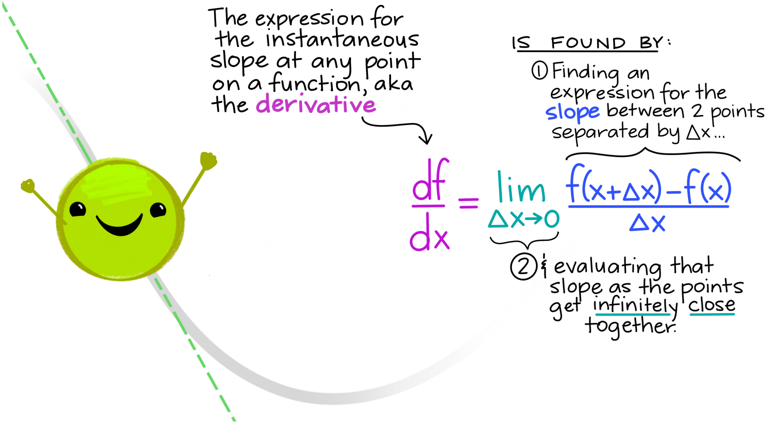
What is a derivative? VII



What is a derivative? VIII



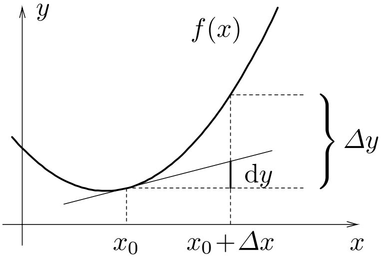
What is a derivative? IX



What is a derivative? X

We want to estimate the slope of a function at point x_0.

- As a rough estimate we can form the difference quotient $\frac{\Delta y}{\Delta x}$.
- Decreasing Δx continuously brings us closer and closer to the true slope...
- In limit we approach the **derivative** at point x_0.



Illustrations by Allison Horst

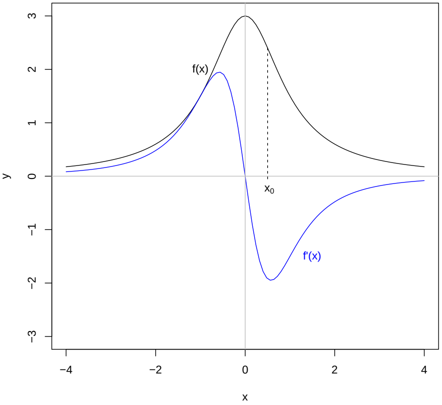
Intuition I

The derivative:

- is a measure of how a function changes as its input changes
- of a function at a chosen input value describes the best linear approximation of the function near that input value
- at a point equals the slope of the tangent line to the graph of the function at that point (linearization of a function for the multivariate case)

Intuition II

- $f(x) = \frac{3}{1+x^2}$
- $f'(x) = -\frac{6x}{(x^2+1)^2}$
- Observations:
  - slope is not a number anymore, but a function (it varies with x)
  - for any x, $f'(x)$ gives us the slope (a value)
  - e.g. $f'(x_0 = 0.5) =$ -1.92



Definition

Definition (Limit of a Function)

Assuming $x, p, c, L \in \mathbb{R}$, the limit of a real valued function f when x approaches p, denoted as $\lim_{x \to p} f(x) = L$, is L if $\forall \epsilon > 0 \exists c > 0, s.t. \forall x, 0 < |x - p| < c \implies |f(x) - L| < \epsilon$.

Note, that if $p = +\infty$ or $p = -\infty$, L is called the asymptote of the function.

Definition

Definition (Derivative)

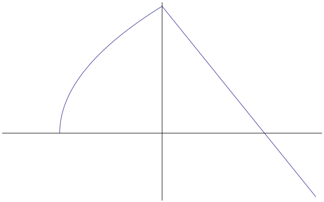
Let $(x_0, f(x_0))$ be a point on the graph of $y = f(x)$. The **derivative** of f at x_0, written $f'(x_0)$, $\frac{df}{dx}(x_0)$, $\frac{dy}{dx}(x_0)$ is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

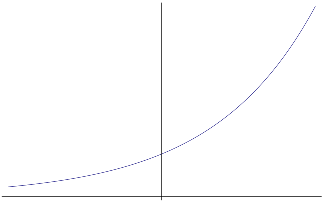
if this limit exists. If this limit exists for every point x in the domain of f, the function is differentiable.

Differentiability

- graph has to be 'smooth' (no gaps, holes, ...)
- if f is differentiable, it must be continuos (converse does not hold)



function is not differentiable



function is differentiable

Continuity

Definition (Continuity)

A function f is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

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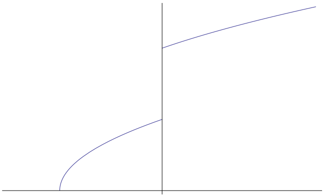
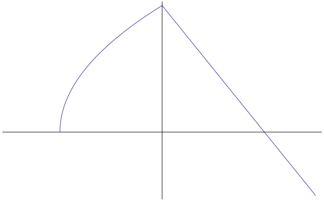
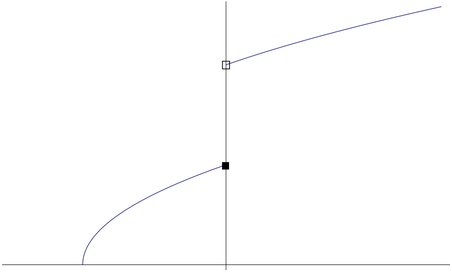


function is discontinuous



function is continuous

Semi-Continuity



function is lower
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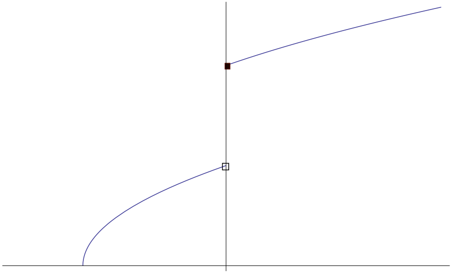

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Analysis I

Rules of Differentiation

Rules of Differentiation I

Rules for Common Functions

- $f(x) = x^a$, then $f'(x) = ax^{a-1}$

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- $f(x) = \frac{1}{x} = x^{-1},$ then $f'(x) = -\frac{1}{x^2}$

Rules of Differentiation II

Sum Rule

- $[f(x) + g(x)]' = f'(x) + g'(x)$
- Example:

$$\begin{aligned} h(x) &= 2x + x^2 \\ h'(x) &= 2 + 2x \end{aligned}$$

Rules of Differentiation II

Product Rule

- $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Example:

$$
\begin{aligned}
h(x) &= 2x \cdot \sqrt{x} \\
h'(x) &= 2 \cdot \sqrt{x} + 2x \cdot \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Rules of Differentiation III

Quotient Rule

- $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
- Example:

$$\begin{aligned} h(x) &= \frac{3x}{2-x^2} \\ h'(x) &= \frac{3 \cdot (2-x^2) - 3x \cdot (-2x)}{(2-x^2)^2} \end{aligned}$$

Rules of Differentiation III

Chain Rule

- $[f(g(x))]' = f'(g(x)) \cdot g'(x)$
- Example:

$$\begin{aligned} h(x) &= (5x-2)^3 \\ h'(x) &= 3(5x-2)^2 \cdot 5 \end{aligned}$$

Analysis I

Partial Derivatives

Motivation

- What if the relationship between the level of democracy does not only dependent on economic growth, but also on the political institutions?

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- We can generalize the concept of a derivative to the multivariate case
- Partial derivates say something about the changes in y given a change in x_i holding all other arguments at some level

Partial Derivatives I

Definition (Partial Derivatives)

Let f be a multivariate function. Then for each variable x_i at each set of points $(x_1^0, ..., x_n^0)$ in the domain of f:

$$\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h}$$

is called the partial derivative, if the limit exists.

Note, that we usually write $\frac{\partial f}{\partial x}$ for partial derivatives and $\frac{df}{dy}$ for derivatives.

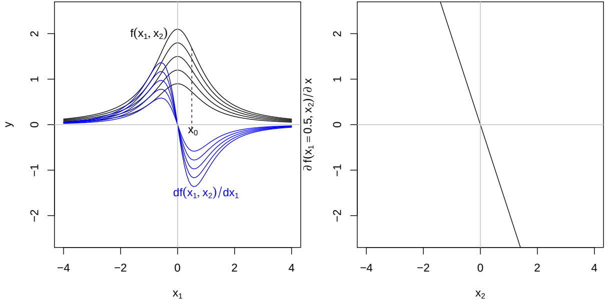
Partial Derivatives II

Example:

$$
\begin{aligned}
f(x_1, x_2) &= x_1^2 \cdot \ln x_2 \\
\frac{\partial f}{\partial x_1} &= 2x_1 \cdot \ln x_2 \\
\frac{\partial f}{\partial x_2} &= x_1^2 \cdot \frac{1}{x_2}
\end{aligned}
$$

Intuition

- $f(x_1, x_2) = \frac{3x_2}{1+x_1^2}$
- $\frac{\partial f(x_1,x_2)}{\partial x_1} = \frac{-6x_1x_2}{(x_1^2+1)^2}$
- Observations:
  - slope varies not only with x_1, but also with x_2
  - e.g. $\frac{\partial f(x_1=0.5,x_2)}{\partial x_1} = -1.92x_2$



Second-order Partial Derivatives

Reconsider the example from the last slide

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\frac{\partial^2 f}{\partial x_2^2} &= -x_1^2 \cdot \frac{1}{x_2^2}
\end{aligned}
$$

Second-order derivatives describe how the slope of the first derivative changes given changes in x.

Mixed Partial Derivatives I

Reconsider the example from the last slide

$$
\begin{aligned}
f(x_1, x_2) &= x_1^2 \cdot \ln x_2 \\
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\frac{\partial f}{\partial x_2} &= x_1^2 \cdot \frac{1}{x_2} \\
\frac{\partial^2 f}{\partial x_1 \partial x_2} &= 2x_1 \cdot \frac{1}{x_2}
\end{aligned}
$$

Mixed Partial Derivatives II

Theorem (Young's Theorem)

Suppose that all the m^{th}-order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ are continuous. If any of them involve differentiating with respect to each of the variables the same number of times, then they are necessarily equal.

In the case of $f(x_1, x_2)$, that implies for example:

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

Hessian Matrix I

Because of the importance of the second-order partial derivatives for constrained optimization there does exist a special of collecting them, the so-called **Hessian Matrix**

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n} \end{pmatrix}$$

Application

- Estimation of covariance matrix
- Optimization in maximum likelihood
- ...

Analysis II

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Optimization

Motivation for Optimization

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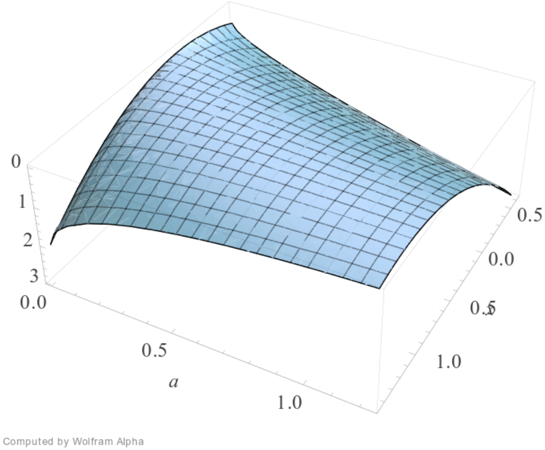
Let us assume, we have a specified utility function of a person $u(x) = -(x + \sqrt{a})^2$.

We want to know the optimal choice the person can take. How do we do this?

Motivation for Optimization

$$u(x) = -(x + \sqrt{a})^2.$$



Single Variable Optimization - FOC

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- So now we know that at this point the function either has a (local) maximum/minimum (or a saddle point).

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In our example $\frac{d^2 f}{dx^2}(-\sqrt{a}) = -2$. So we have a local maximum.

Controlling for the other parts of the function, we find that this is also a global maximum.

Convex, Concave, and Inflection Point

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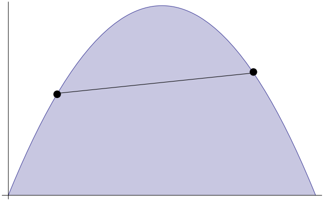
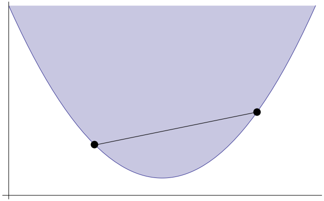
- If a is an inflection point and $\frac{df}{dx} = 0$, then it is a **saddle point**.

More General Definition of Concavity/Convexity

A function is called concave (convex) if the line segment joining any two points on the graph is below (above) the graph, or on the graph.



concave



convex

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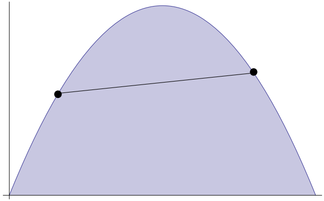
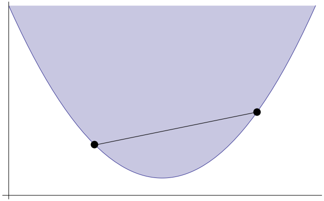


concave



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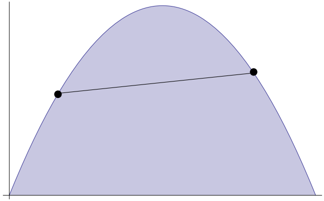
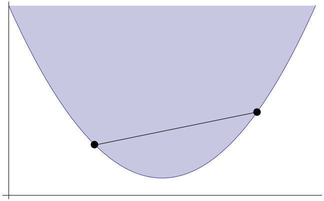


concave



convex

We can derive the concavity/convexity of functions from the concept of convex sets. A function is called **convex** if the set of all points which are on or above its graph is a convex set.

More General Definition of Concavity/Convexity

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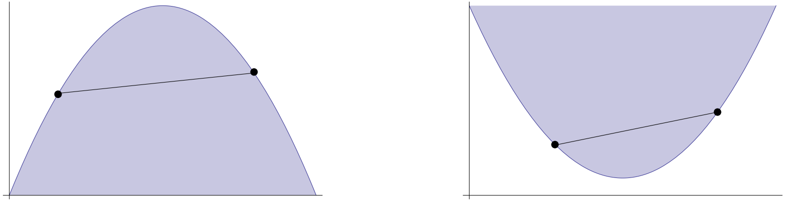

concave — convex

We can derive the concavity/convexity of functions from the concept of convex sets. A function is called **convex** if the set of all points which are on or above its graph is a convex set. Conversely, a function is called **concave** if the set of all points which are on or below its graph is a convex set.

Bivariate Optimization I

Consider a C^2 function (i.e. a function that is both continuous and twice differentiable) $f(x, y)$ in a convex set S.

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Second-order condition

- If for all (x, y) in S, $\frac{\partial^2 f}{\partial x^2} \leq 0$, $\frac{\partial^2 f}{\partial y^2} \leq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ then $(x^\star, y^\star)$ is a maximum point for $f(x, y)$ in S.
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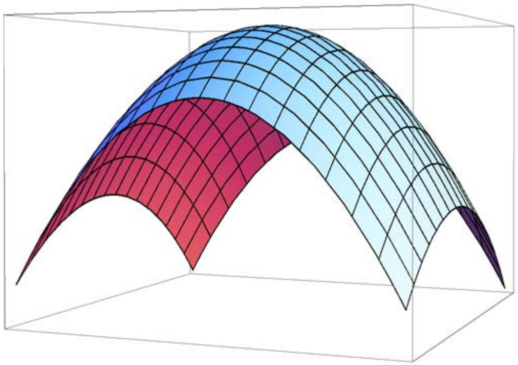
Bivariate Optimization II

Consider the function $f(x, y) = -0.5(x-1)^2 - y^2$.



Bivariate Optimization III

Function $f(x, y) = -0.5(x - 1)^2 - y^2$.

Bivariate Optimization III

Function $f(x, y) = -0.5(x-1)^2 - y^2$.

The first order condition

$$\begin{aligned} \frac{\partial f}{\partial x} &= -x + 1 \equiv 0 \\ \frac{\partial f}{\partial y} &= -2y \equiv 0 \end{aligned}$$

gives us a stationary point at $x = 1, y = 0$.

Bivariate Optimization IV

The second order condition

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -1 < 0 \\
\frac{\partial^2 f}{\partial y^2} &= -2 < 0 \\
\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 &= (-1) \cdot (-2) - 0 \geq 0
\end{aligned}$$

tells us that we have a maximum at $x = 1, y = 0$.

Extreme Value Theorem/Weierstrass Theorem

Theorem (Extreme Value Theorem/Weierstrass Theorem)

Suppose the function $f(\mathbf{x})$ *is continuous throughout a nonempty, closed and bounded set* S *in* $\mathbb{R}^n$. *Then there exists a point* $\mathbf{d}$ *in* S *where* f *has a minimum and a point* $\mathbf{c}$ *in* S *where* f *has a maximum. That is,*

$$f(\mathbf{d}) \leq f(\mathbf{x}) \leq f(\mathbf{c}) \text{ for all } \mathbf{x} \in S.$$

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You will find the Weierstrass Theorem on page 20 of McCarty and Meirowitz (2007).

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More generally: How do changes in the parameters of a model affect the model's solution?

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An increase of one unit a increases $u(x)$ by $\frac{1}{2\sqrt{a}}$ units, **ceteris paribus**.

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Let us consider the following example:
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A method to solve such problems is the so-called **Lagrangian multiplier method**.

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2. Differentiate $\mathcal{L}$ with respect to x and y, and equate the partial derivatives to 0.
3. Solve the system of equations that the two partials form together with the constraint.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \equiv 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \equiv 0 \\ g(x, y) &= c \end{aligned}$$

Application to our problem

The Lagrangian

$$\mathcal{L}(s, c, \lambda) = \frac{1}{2}s^2 + \left(c - \frac{1}{3}\right)^2 - \lambda(s + c - 2)$$

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If we solve the system of equations, we get $c = \frac{8}{9}$ and $s = \frac{10}{9}$.

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You find the formulation in Sysdsæter/Hammond (2008) on pp. 506-507.

Advanced Constrained Optimization

There is much more to constrained optimization!

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See Sysdsæter/Hammond (2008), Chapter 14.

Analysis II

Integration

Motivation I

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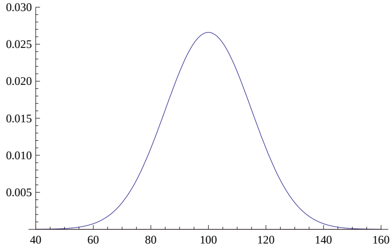
- p.d.f. relate a particular event (x) to a probability (y)
- when we are interested in calculating the probability for a range of events, we need to calculate the area under the curve

Motivation II

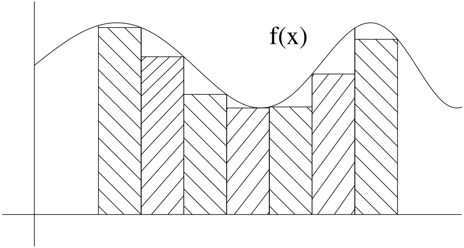
- We know that IQ test scores amongst people of the same age are distributed normally with mean 100 and standard deviation 15.
- What is the probability that a person has a score of more than 120?



It is the area below the normal p.d.f. for $x > 120$ (p$\approx$ 9.12%)

Intuition

- The **indefinite integral** $F(x)$ of a function $f(x)$ is the area between the function and the x-axis.



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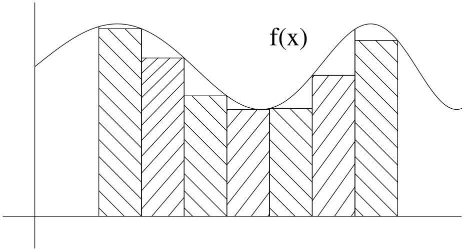

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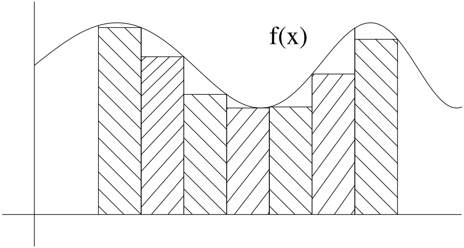
- Calculating an integral is the reverse process of taking a derivative. For this we sometimes refer to an integral as **antiderivative**.



Definition Integral

Definition (Riemann Integral)

Let f be a continuos function on a closed interval $[a, b]$. Let there be N equal subintervals, each of length $\delta = (b - a)/N$. Let $x_0, x_1, ..., x_N$ be the endpoints of these subintervals, e.i $x_0 = a, x_1 = a + \delta, x_2 = a + 2\delta, ...$ The sum

$$f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + ... + f(x_N)(x_N - x_{N-1}) = \sum_{i=1}^{N} f(x_i)\delta$$

is the Riemann sum. Taking the limit gives the Riemann integral:

$$\lim_{\delta \to 0} \sum_{i=1}^{N} f(x_i)\delta = \int_a^b f(x)dx$$

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus (Part I))

Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function for all $x \in [a, b]$, by

$$F(x) = \int_a^x f(t)dt$$

Then, F is continuous on $[a, b]$, differentiable on the open interval (a, b), and

$$F'(x) = f(x)$$

for all $x \in (a, b)$.

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus (Part II))

Let f and F be real-valued functions defined on a closed interval $[a, b]$, such that the derivative of F is f. If f is (riemann) integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Note, that there are infinitely many functions F that have f as their derivative, obtained by adding to F an arbitrary constant. So, we write $\int f(x)dx = F(x) + c$, where c is an arbitrary constant.

Example

$$\int_1^4 x dx \quad =$$

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Definite and Indefinite Integral

The difference between an indefinite and a definite integral is the interval of integration.

$$\begin{array}{ll} \int f(x)dx & \text{indefinite integral} \\ \int_a^b f(x)dx & \text{definite integral} \end{array}$$

The numbers a and b are called, respectively, the lower and upper limit of integration.

Properties

Properties (I)

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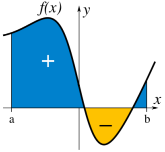

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- $\int a^x dx = \frac{1}{\ln a} a^x + c$, where $a > 0$ and $a \neq 1$

Linear Algebra I

Motivation I

- A statistical model describes how some variables ($x_0 ... x_k$) generate another variable y given some parameters ($\beta_0 ... \beta_k$) and an error term ($\epsilon_1 ... \epsilon_n$), e.g. the linear regression model

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$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \epsilon_1 \\
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&\vdots \\
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  - ...
- Matrix notation is a very efficient way to manipulate (simplify) systems of equations

Linear Algebra I

Vectors

Vector Spaces and Vectors

Definition (Vector Space)

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Definition (Vector Space)

A vector space V is a nonempty set of objects, called **vectors** denoted with lower case bold letters, on which are defined two operations (addition, multiplication by real scalars), subject to eight axioms:

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ Commutativity

$\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V \wedge c, d \in \mathcal{R}$

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- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ Associativity of vector addition

$\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V \wedge c, d \in \mathcal{R}$

Vector Spaces and Vectors

Definition (Vector Space)

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- $1\mathbf{a} = \mathbf{a}$ — Multiplication identity

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Euclidean Space

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$$\mathbf{a} = (a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}'$$

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- Terminology: a_i is an **element** or **component**; the vector's **dimension** is the equal to the number of components

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Vector Operations

Vector addition of vectors with the same dimension is defined as:

$$\begin{aligned}(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\ &= \mathbf{a} + \mathbf{b} = \mathbf{c}\end{aligned}$$

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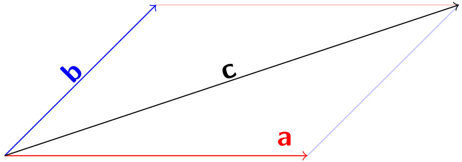
Graphically ($\mathbb{R}^2$):



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Vector Norm and Distance

The **norm** (length) of a vector $\mathbf{a} = (a_1, a_2, \dots a_n)$ is defined as:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 \dots a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

A **normalized vector** has a norm of 1. A **zero vector** has a norm of 0 (note: $\|\mathbf{a}\| = 0 \iff a_i = 0 \forall i$).

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Generalized to n-dimensions:

$$\|\mathbf{a} - \mathbf{b}\| = \sqrt{\sum_{i \in n} (a_i - b_i)^2}$$

Dot product

The **inner product** (dot product) of two vectors of equal dimension is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 \ldots a_n \cdot b_n = \sum_{i=1}^{n} a_i b_i$$

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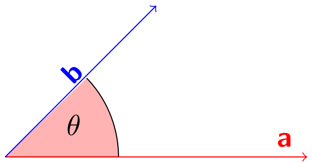

$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$, where θ is the **angle** between the vectors.

Properties

Properties of the Dot Product

If **a**, **b**, and **c** are *n*-vectors and α is a scalar, then

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- $(\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a}(\alpha\mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot \mathbf{a} > 0 \iff \mathbf{a} \neq \mathbf{0}$

Linear Algebra I

Matrices

Matrix

A **matrix A**, denoted with bold capital letters, is structured into I **rows** and J **columns**. It is said to have the **size** (dimension) $I \times J$. The cells in the matrix are called **elements**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{pmatrix}$$

Matrix Operations

Matrix Addition for two matrices **A** and **B** with the same dimension corresponds to vector addition for each column (or row).

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} =$$

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Matrix Operations

Scalar Multiplication for a matrix $\mathbf{A}$ with scalar α corresponds to scalar multiplication of a vector for each column (or row).

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Matrix Product

Matrix Product of two matrices **A** and **B** with dimension $w \times x$ and $y \times z$ is defined if the number of columns in **A** is equal to the number of rows in **B**, that is, $x = y$. The new matrix has dimension $w \times z$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1x} \\ a_{21} & a_{22} & \cdots & a_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{w1} & a_{w2} & \cdots & a_{wx} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1z} \\ b_{21} & b_{22} & \cdots & b_{2z} \\ \vdots & \vdots & \ddots & \vdots \\ b_{y1} & b_{y2} & \cdots & b_{yz} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{y} a_{1i} b_{i1} & \sum_{i=1}^{y} a_{1i} b_{i2} & \cdots & \sum_{i=1}^{y} a_{1i} b_{iz} \\ \sum_{i=1}^{y} a_{2i} b_{i1} & \sum_{i=1}^{y} a_{2i} b_{i2} & \cdots & \sum_{i=1}^{y} a_{2i} b_{iz} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{y} a_{wi} b_{i1} & \sum_{i=1}^{y} a_{wi} b_{i2} & \cdots & \sum_{i=1}^{y} a_{wi} b_{iz} \end{pmatrix}$$

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$$= \begin{pmatrix}27 & 30 & 33\\61 & 68 & 75\\95 & 106 & 117\end{pmatrix}$$

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Kronecker Product

If $\mathbf{A}$ is an $w \times x$ matrix and $\mathbf{B}$ is a $y \times z$ matrix, then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $wy \times xz$ block matrix.

$$\begin{aligned}
\mathbf{A} \otimes \mathbf{B} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1x} \\ a_{21} & a_{22} & \cdots & a_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{w1} & a_{w2} & \cdots & a_{wx} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1z} \\ b_{21} & b_{22} & \cdots & b_{2z} \\ \vdots & \vdots & \ddots & \vdots \\ b_{y1} & b_{y2} & \cdots & b_{yz} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1x}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2x}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{w1}\mathbf{B} & a_{w2}\mathbf{B} & \cdots & a_{wx}\mathbf{B} \end{pmatrix}
\end{aligned}$$

Matrix Transposition

The **Transpose** is defined as a matrix where rows and columns are "interchanged". We denote the transpose of a matrix $\mathbf{A}$ by $\mathbf{A}^T$ or $\mathbf{A}'$.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T =$$

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Properties

Properties of Matrices (III)

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3. $(\alpha\mathbf{A})' = \alpha\mathbf{A}'$
4. $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

Square Matrix

An $i \times j$ matrix $\mathbf{A}$ is called **square matrix** if $i = j$, that is, the numbers of rows and columns are the same.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Symmetric Matrix

A square matrix $\mathbf{A}$ is called **symmetric** if $\mathbf{A} = \mathbf{A}'$. That is, $\mathbf{A}$ is symmetric about its main diagonal. Another way to express this is $a_{ij} = a_{ji} \forall i, j$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Diagonal Matrix

A square symmetric matrix **A** is called **diagonal matrix** if $a_{ij} = 0 \; \forall \; i \neq j$. That is, every element is zero except for the elements on the main diagonal.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Identity Matrix

A square diagonal matrix **A** is called **identity matrix I** if the elements on the main diagonal are all equal to one.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Triangular Matrix

A square matrix $\mathbf{A}$ is called upper (lower) **triangular matrix** if $a_{ij} = 0$ for all $i > j$ ($i < j$), that is, a matrix in which all entries below (above) the main diagonal are 0.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

Idempotent Matrix

A square matrix $\mathbf{A}$ for which $\mathbf{A} \cdot \mathbf{A} = \mathbf{A}$ is called **idempotent**.

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} \times \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$$

The Hessian

Because of the importance of the second-order partial derivatives for constrained optimization there does exist a special way of collecting them, the so-called **Hessian matrix**.

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Trace

The **trace** of a matrix is the sum of the elements on the main diagonal.

$$\operatorname{tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 15$$

Linear Algebra II

Linear Algebra II

Systems of Equations

Linear Systems of Equations I

Definition (Linear Equation)

Linear Systems of Equations I

Definition (Linear Equation)

A linear equation in the variables $x_1, ..., x_k$ is an equation that can be written in the form

$$b = a_1 x_1 + a_2 x_2 + ... + a_k x_k,$$

where b and the coefficients $a_1, ..., a_k$ are known, real (or complex) numbers. k is an integer.

Note, in statistics the 'coefficients' are usually the unknowns and the x are known (the data). Just a matter of notation.

Linear Systems of Equations II

Definition (Systems of linear equations)

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A system of linear equations is a collection of n linear equations of the form:

$$
\begin{aligned}
b_1 &= a_{11}x_1 + a_{12}x_2 + ... + a_{1k}x_k \\
b_2 &= a_{21}x_1 + a_{22}x_2 + ... + a_{2k}x_k \\
\vdots &= \vdots \\
b_n &= a_{n1}x_1 + a_{n2}x_2 + ... + a_{nk}x_k
\end{aligned}
$$

Solving Systems of Linear Equations

1. Equation-by-equation substitution

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1. Equation-by-equation substitution
2. Gaussian elimination
3. Inverting the coefficient matrix
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Substitution

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Solve the following system of equations for $\{x_1, x_2\}$.

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Solve equation (2) for x_1 and insert this into (1):

$$\begin{aligned} x_1 &= 2x_2 + 5 && \text{(2)'} \\ 4x_2 + 10 + 3x_2 &= 4 && \text{(2)' in (1)} \end{aligned}$$

This gives $x_2 = -\frac{6}{7}$.

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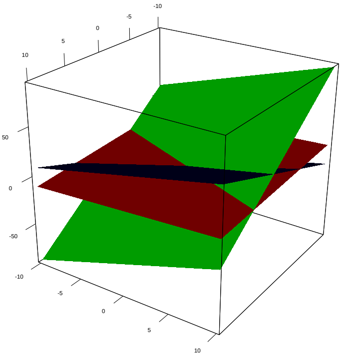
Inserting this into (2)' gives $x_1 = \frac{23}{7}$.

Geometric Interpretation

- Example:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 1 \text{(blue plane)} \\ 2x_1 - 2x_2 + 4x_3 &= -2 \text{(red plane)} \\ -x_1 + \frac{1}{2}x_2 - \frac{1}{6}x_3 &= 0 \text{(green plane)} \end{aligned}$$



- Solution:

$$\begin{aligned} x_1 &\approx 0.12 \\ x_2 &\approx 0.06 \\ x_3 &\approx -0.53 \end{aligned}$$

- intersection of the planes

Matrix Equations

The system of equations

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - 2x_2 + 4x_3 &= -2 \\ -x_1 + \frac{1}{2}x_2 - \frac{1}{6}x_3 &= 0 \end{aligned}$$

can be written as a matrix equation:

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & +4 \\ -1 & \frac{1}{2} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

Is there a solution to a system of linear equations?

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The Determinant

Determinant I

Consider the following system of linear equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
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or $\mathbf{Ax} = \mathbf{b}$. Solving the system of linear equations yields

$$
\begin{aligned}
x_1 &= \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}} \\
x_2 &= \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{21} a_{12}}
\end{aligned}
$$

Note that the denominators are the same. These have to be nonzero for a unique solution to exist. The system would have none or an infinite number of solutions otherwise.

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In this sense, the value of the denominator determines whether there is a unique solution to the equation system.

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In fact, $a_{11}a_{22} - a_{21}a_{12}$ is called the **determinant** of the matrix $\mathbf{A}$.

$$|\mathbf{A}| = det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12}$$

Determinant III

Sarrus's rule is a simple rule for calculating the determinant of 3×3 matrices.

$$\det(\mathbf{A}) \quad = \quad \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

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\end{aligned}
$$

Determinants for **square matrices** of dimension larger then three

are not that easy to determine. However, there are procedures to calculate them. See Sydsæter/Hammond (2008), 580-582.

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- If two rows (or columns) of $\mathbf{A}$ are interchanged, the determinant changes sign, but the absolute value remains unchanged.

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Note, $\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$.

Solving Systems of Linear Equations I

Solving Systems of Linear Equations I
Gaussian Elimination

Gaussian Elimination

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a_{11}x_1 + a_{12}x_2 ... a_{1k}x_k &= b_1 \\
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Rewrite the system in an **augmented matrix $\mathbf{A}|\mathbf{b}$**

Gaussian Elimination

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Rewrite the system in an **augmented matrix $\mathbf{A}|\mathbf{b}$**

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1k} & b_1 \\ a_{21} & a_{22} & \dots & a_{2k} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} & b_n \end{array}\right)$$

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obtain a **row echelon form** of the augmented matrix:

$$\left(\begin{array}{ccccc|c} a_{11}^* & a_{12}^* & a_{13}^* & \dots & a_{1n}^* & b_1^* \\ 0 & a_{22}^* & a_{23}^* & \dots & a_{2n}^* & b_2^* \\ 0 & 0 & a_{33}^* & \dots & a_{3n}^* & b_3^* \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{3n}^* & b_n^* \end{array}\right)$$

Iterated substitution gives you the solution vector $\mathbf{x}$ if it exists. Or..

Gaussian Elimination

continue to obtain a **reduced row echelon form** of the augmented matrix (Gauss-Jordan elimination):

Gaussian Elimination

continue to obtain a **reduced row echelon form** of the augmented matrix (Gauss-Jordan elimination):

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & \dots & 0 & \tilde{b}_1 \\ 0 & 1 & 0 & \dots & 0 & \tilde{b}_2 \\ 0 & 0 & 1 & \dots & 0 & \tilde{b}_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \tilde{b}_n \end{array}\right)$$

where $\tilde{\mathbf{b}}$ is the solution vector for $\mathbf{x}$.

Gaussian Elimination

Example (from Wikipedia)

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right)$$

Gaussian Elimination

Example (from Wikipedia)

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Is there a solution to a system of linear equations?

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The Matrix Rank

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Note, the concept also applies to non-square matrices.

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Solving Systems of Linear Equations II

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Inverting the coefficient matrix

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A matrix is invertible if and only if $\det(\mathbf{A}) \neq 0$. $\mathbf{A}$ is said to be **nonsingular** in this case. In the opposite case of $\det(\mathbf{A}) = 0$ we call **A singular**.

Inverse II

We can invert a matrix using the Gauss-Jordan algorithm for systems of linear equations.

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If we multiply both sides of the equation with the inverse of $\mathbf{X}$ from the left, we solve the system for $\mathbf{b}$.

$$\mathbf{b} = \mathbf{X}^{-1}\mathbf{y}$$

Other Ways of Calculating the Inverse

The inverse of every 2×2 matrix $\mathbf{A}$ can be derived the following way.

$$\text{Let } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \det(\mathbf{A}) = ad - bc \neq 0, \text{ then}$$

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An additional way for finding the inverse of an $n \times n$ matrix $\mathbf{A}$ that does not employ Gaussian elimination uses the so-called adjoint of $\mathbf{A}$ (see Sydsæter/Hammond 2008, 597).

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Probability Theory

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- **Event**: $A \subseteq S$, a subset from the sample space

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- note: classical probability $\neq$ empirical probability $\neq$ subjective probability

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- $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$, where $P(A \cap B)$ is the joint probability of A and B

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Basic Theorems

Let $A, B \subseteq S$:

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Note: $Pr(A \cap B)$ is also denoted $Pr(AB)$ or $P(A, B)$

Probability Theory

Combinatorics

Permutation and Combination

	with replacement	without replacement
Permutation (considering sequence)	n^k	$\binom{n}{k}k! = \frac{n!}{(n-k)!}$
Combination (disregarding sequence)	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Coefficient

- "n choose k"

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$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3! \times 4!} = \frac{7 \times 6 \times 5}{3 \times 2} = 35.$$

Examples I

$$k = 2,\ S = \{A, B, C\} \implies n = 3$$

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Examples II

Let there be 4 train passengers waiting for tickets. How many sequences are there to sell them their train tickets?

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$k = n = 4 \implies \binom{n}{k} k! = 24$

Probability Theory

Conditional Probability

Definition

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Let A,B be two events with probability larger than zero. The conditional probability of A given B is:
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Interpretation: Given that B occurred, what is the probability for A?

Corollaries

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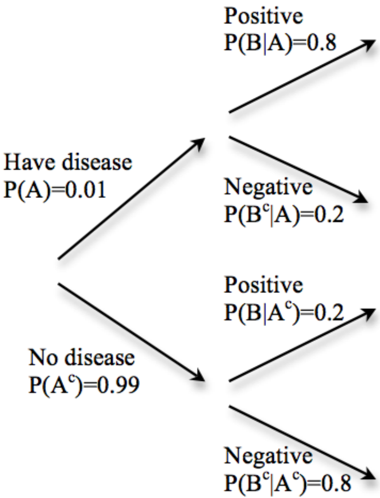
Example II


Positive
P(B|A)=0.8
Have disease
P(A)=0.01
Negative
P(B^c|A)=0.2
Positive
P(B|A^c)=0.2
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- foundation of Bayesian Statistics, formal modeling of learning, philosophy of scientific progress, ...

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- suppose you have no reason to belief more in either of the two hypothesis a-priori
- What is the probability of hypothesis F and $\neg F$ after you tossed the coin and you saw a head?

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- for three heads in a row $p(F|H_3) = 1/9$ ...
- this process is called **Bayesian Updating**

Applied Bayes: Statistical Models

- let θ denote a parameter and y the data

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- impossible from a frequentist point of view

Probability Theory

Probability Distributions

Random Variable I

Definition (Random Variable)

Let Ω be the sample space for an experiment. A real-valued function that is defined on Ω is called a **random variable**. The set of values the variable might take is the **distribution** of the random variable.

Random Variable II

Definition (Discrete Random Variable)

We say that a random variable X is a **discrete random variable** or that it has a **discrete distribution**, if X can take only a finite number k of different values or, at most, an infinite sequence of different values.

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Note, that a random variable is usually denoted with a capital letter, while its realizations are denoted with lowercase letters.

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Consider the sequence $q = HHTTTHTTTH$, then $X(q) = 4$.
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Probability Mass Function

Definition (Probability Mass Function, p.m.f.)

For a discrete random variable X the **probability mass function** of X is defined as a function $f(\cdot)$ such that for every real number x,

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- $Pr(C \subset \Omega) = \sum_{x_i \in C} f(x_i)$

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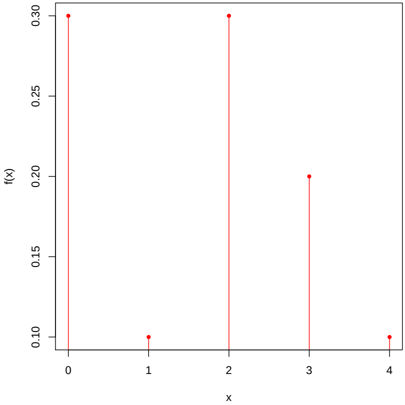
Example I

A p.m.f. defined as:

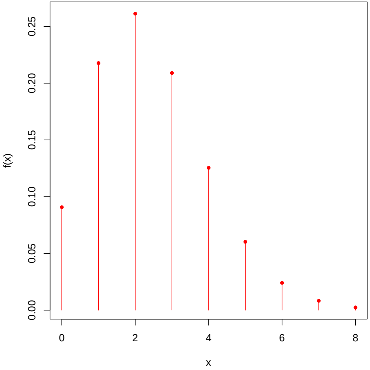
$$f(x) = \begin{cases} 0.3 & if \quad x = 0 \\ 0.1 & if \quad x = 1 \\ 0.3 & if \quad x = 2 \\ 0.2 & if \quad x = 3 \\ 0.1 & if \quad x = 4 \end{cases}$$



Example II

Let $\lambda \in \mathbb{R}_{>0}$ (intensity), the Poisson p.m.f. is defined as

$$f(x;\lambda) = \begin{cases} \frac{\lambda^x exp(-\lambda)}{x!} & \forall x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$



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- some authors use $f(X = x)$ instead of $f(x)$ only.

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Definition (Cumulative Distribution Function, c.d.f.)

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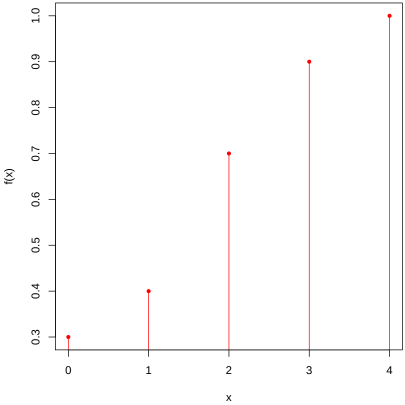
- c.d.f. is always continuous from the right, i.e. $F(x) = F(x^+)$ at every point x.

Example I

A c.d.f. defined as:

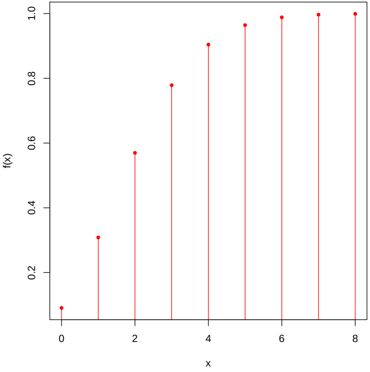
$$F(x) = \begin{cases} 0.3 & \textit{if} \quad x = 0 \\ 0.4 & \textit{if} \quad x = 1 \\ 0.7 & \textit{if} \quad x = 2 \\ 0.9 & \textit{if} \quad x = 3 \\ 1.0 & \textit{if} \quad x = 4 \end{cases}$$



Example II

Let $\lambda \in \mathbb{R}_{>0}$ (intensity), the Poisson c.d.f. is defined as

$$F(x) = exp(-\lambda) \sum_{i=0}^{|k|} \frac{\lambda^i}{i!}, \forall k \leq 0$$



Determining Probabilities from the c.d.f.

Let $F(x^-) = \lim_{y \to x} F(y) \forall y < x$ and $F(x^+) = \lim_{y \to x} F(y) \forall y > x$.

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Probability Density Function, p.d.f.

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Let x be a continuous random variable. A p.d.f. is a nonnegative function $f(\cdot)$, defined on the real line, such that:

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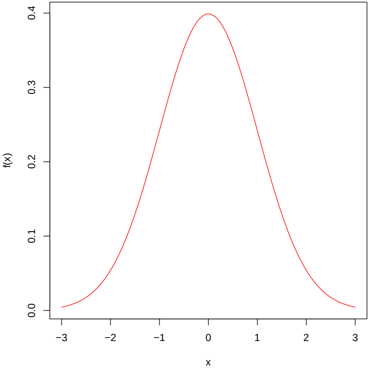
- $\int_a^b f(x)dx = 1$ where a, b are the bounds of the suppport for x

Example I

The p.d.f. of a normal (or Gaussian) distribution is defined as $f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$ where $\mu \in \mathbb{R}$ (mean) and $\sigma^2 \in \mathbb{R}_{>0}$ (variance). For the standard normal (picture) $\mu = 0$ and $\sigma^2 = 1$.



Probability Theory

Properties of Distributions

Expectation I

Definition (Expectation)

Let X be a discrete random variable with a p.m.f. $f(\cdot)$. The **expectation** (also: expected value, mean) of X, denoted $E(X)$ is a scalar defined as $E(X) = \sum_x xf(x)$. Similarly, if X is a continuous random variable, the **expectation** is a scalar defined as $E(X) = \int_{-\infty}^{+\infty} xf(x)\, dx$.

Variance

Definition (Variance)

Let X be a random variable with mean $\mu = E(X)$. The variance of X denoted by $Var(x)$ is defined as: $Var(x) = E\big((X-\mu)^2\big)$.

Properties:

- $Var(aX + b) = a^2 Var(X)$

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- $Var(X + Y) = Var(X) + Var(Y)$ iff (X, Y) are independent

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