University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2022
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Solutions Set Theory I

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{6, 8\}$. Find following:
   (a) $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$
   (b) $A \cap B = \{2, 4\}$
   (c) $A \cap B^C = \{1, 3, 5\}$
   (d) $B - A = \{6, 8\}$
   (e) $C - B = \emptyset$
   (f) $A \cap C = \emptyset$

2. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and $C = \{a, b, 1, 2\}$. Show that:
   (a) Distributivity: $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$

$$\begin{aligned} \{a, b\} \cup \{1, 2\} &= \{a, b, c, d, 1, 2, 3, 4\} \cap \{a, b, 1, 2\} \\ \{a, b, 1, 2\} &= \{a, b, 1, 2\} \end{aligned}$$

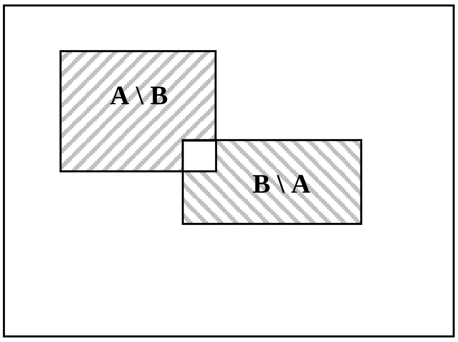
   (b) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$

$$\begin{aligned} \emptyset \cap \{a, b, 1, 2\} &= \{a, b, c, d\} \cap \{1, 2\} \\ \emptyset &= \emptyset \end{aligned}$$

   (c) De Morgan Laws: $C - (A \cup B) = (C - A) \cap (C - B)$

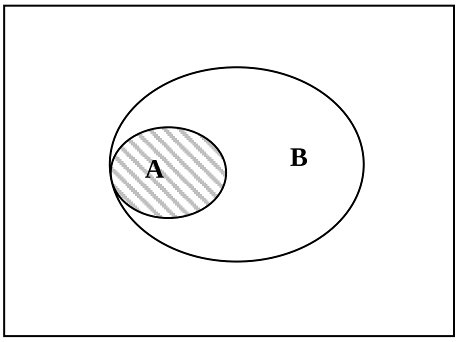
$$\begin{aligned} \{a, b, 1, 2\} - \{a, b, c, d, 1, 2, 3, 4\} &= \{1, 2\} \cap \{a, b\} \\ \emptyset &= \emptyset \end{aligned}$$

3. Determine which of the following formulas are true. If any formula is false, find a counterexample to demonstrate this using a Venn diagram.
   (a) $A \setminus B = B \setminus A$
   false



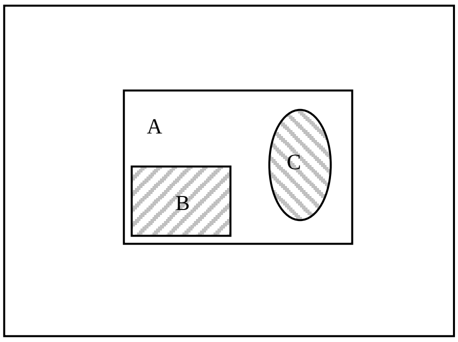
(b) $A \subseteq B \iff A \cap B = A$

true



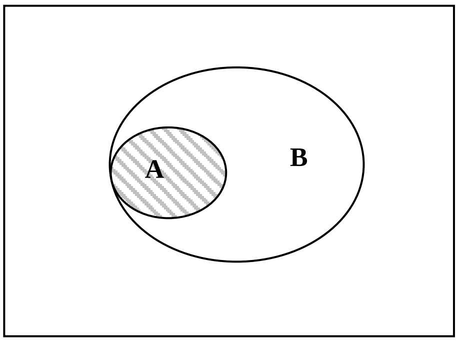
(c) $A \cup B = A \cup C \Longrightarrow B = C$

false



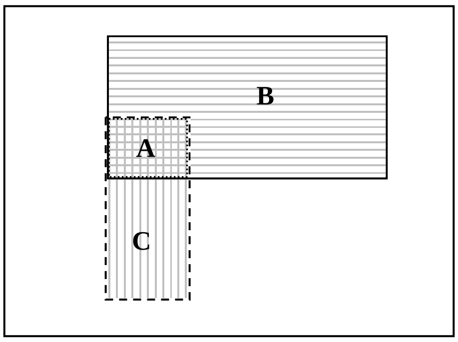
(d) $A \subseteq B \Longleftrightarrow A \cup B = B$

true



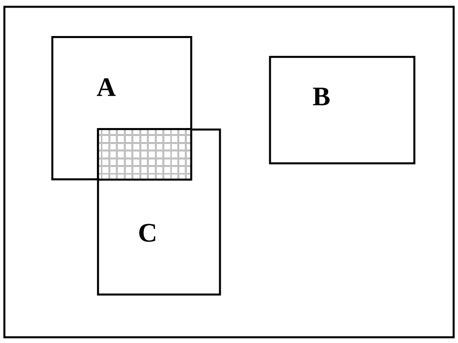
(e) $A \cap B = A \cap C \Longrightarrow B = C$

false



(f) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$

false



4. Explain in words why it is true that for any sets A, B, C:

(a) $(A \cup B) \cup C = A \cup (B \cup C)$
This is true since the union of two sets contains all elements included in either set.

(b) $(A \cap B) \cap C = A \cap (B \cap C)$
This is true since an intersection only includes those elements that are included in both sets.

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Let us think of B and C as a joint set. If we intersect this set with A, we receive $A \cap (B \cup C)$. If we now partition the joint set into two distinct sets and intersect these with A, we have partitioned $A \cap (B \cup C)$ into its two constituent elements $(A \cap B) \cup (A \cap C)$.

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Since A is included in either bracket on the right-hand side of the equation, it is also included in their intersection. Thus, "factor it out" and form a union of it with the intersection of B and C.

5. Find the interior point(s) and the boundary points(s) of the set $\{x : 1 \leq x \leq 5\}$.

   (a) Interior points: $\{x : 1 < x < 5\}$

   (b) Boundary points: $\{x : x = 1 \vee x = 5\}$

6. Why does every set in $\mathbb{R}$ that is nonempty, closed, and bounded have a greatest member?

   Denoting such a set by S, $\sup S$ is a boundary point. Since S is closed, $\sup S \in S$ and so S has a greatest member.

7. Which of the following sets in $\mathbb{R}$ and $\mathbb{R}^2$ are open, closed, or neither?

   (a) $A = \{x \in \mathbb{R}^1 : x = 2 \text{ or } 3 < x < 4\}$ Neither since it contains one but not all of its boundary points.

   (b) In each of the following three cases, the boundary points are the points on the parabola $y = x^2$ with $-1 \leq x \leq 1$, and the points on the line $y = 1$ with $-1 \leq x \leq 1$.
   $B = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq 1\}$
   Closed since it contains all its boundary points.

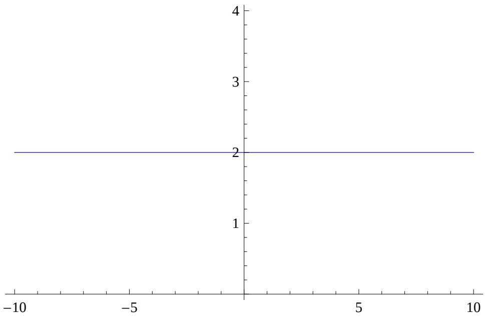
   (c) $C = \{(x, y) \in \mathbb{R}^2 : x^2 < y < 1\}$
   Open since it contains none of its boundary points.

   (d) $D = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y < 1\}$
   Neither since it contains some but not all its boundary points.

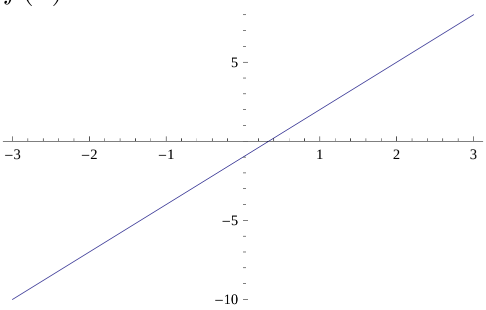
   (e) Universal set: both open and closed: "clopen".

8. Sketch the following functions:

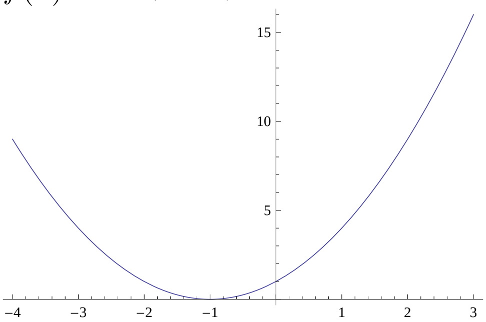
(a) $f(x) = 2$



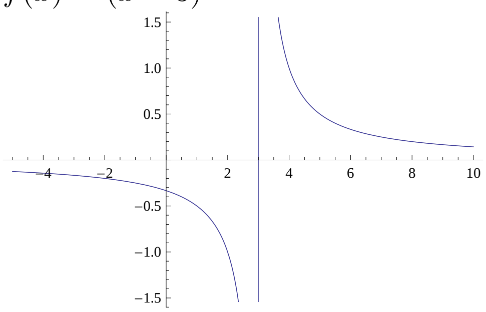
(b) $f(x) = 3x - 1$



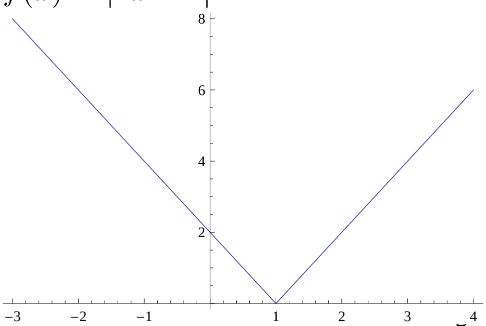
(c) $f(x) = x^2 + 2x + 1$



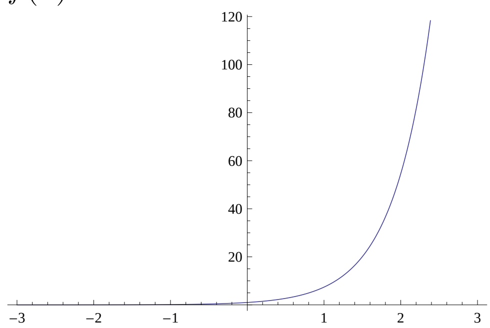
(d) $f(x) = (x - 3)^{-1}$



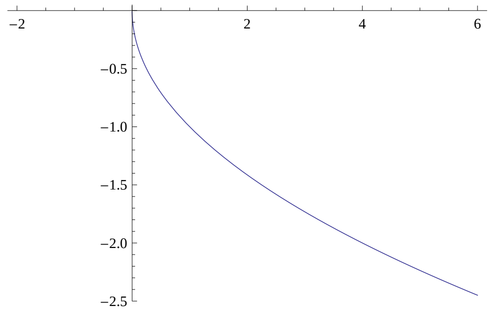
(e) $f(x) = |2x - 2|$



(f) $f(x) = e^{2x}$



(g) $f(x) = -\sqrt{x}$



9. Which of the following functions is injective, bijective, or surjective?

   (a) $a(x) = 2x + 1$
   $a(x)$ is both injective (every element of the domain is linked to at most one element in the co-domain) and surjective (since for every element in the co-domain there is at least one element in the domain) and, thus, bijective.

   (b) $b(x) = x^2$
   $b(x)$ is not injective since $b(x) = b(-x)$. It is also not surjective since there are no negative values for $b(x)$. However, if we would specify the range of $b(x) \in \mathbb{R}^+$, then it would be surjective.

   (c) $c(x) = \ln x$ for $(0, \infty) \mapsto \mathbb{R}$
   $c(x)$ is bijective.

   (d) $d(x) = e^x$ for $\mathbb{R} \mapsto \mathbb{R}$
   $d(x)$ is injective, but not surjective as there are no negative values for $d(x)$.