
Topology Filters - Notes

Release 0.1

Carlos Caralps

Jul 25, 2021

CONTENTS

1	Filter definition and algebraic structure	1
----------	--	----------

FILTER DEFINITION AND ALGEBRAIC STRUCTURE

We will start defining filters and, then the elementary filter propositions will be proved by the usual way and by Lean. The final aim of this chapter is to define an algebraic structure with filters, defining two operations.

1.1 Filter definition

Firstly, we will introduce the filter definition of a giving set.

Definition 1.1 (Filter). Let X be a set, a filter is a family of subsets of the power set $F \subseteq \mathcal{P}(X)$ satisfying the next properties

- (i) The universal set is in the filter $X \in F$.
- (ii) If $E \in F$, then $\forall A \in \mathcal{P}(X)$ such that $E \subseteq A$, we have $A \in F$.
- (iii) If $E, A \in F$, then $E \cap A \in F$.

The reader might have noticed we haven't included the empty axiom, this axiom states that the empty set cannot be in any filter, commonly used in filter definitions and also required for topology filter convergence. Assuming this axiom would make it impossible to define the neutral element in one of the operations we will use later.

Having the conceptual definition of filters, we can define this structure in Lean. The following code lines were published, in the mathlib repository, by Johannes Hölzl in August of 2018.

```
structure filter (X : Type) :=
  (sets          : set (set X))
  (univ_sets     : set.univ ∈ sets)
  (sets_of_superset {x y} : x ∈ sets → x ⊆ y → y ∈ sets)
  (inter_sets {x y}      : x ∈ sets → y ∈ sets → x ∩ y ∈ sets)
```