
Topology Filters - Notes

Release 0.1

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INTRODUCTION

1.1 What is Lean?

Lean is an open source proof-checker and a proof-assistant. One can *explain* mathematical proofs to it and it can check their correctness. It also simplifies the proof writing process by providing *goals* and *tactics*.

Lean is built on top of a formal system called type theory. In type theory, the basic notions are “terms” and “types” — compare to “elements” and “sets” in set theory. Every term has a type, and types are just a special kind of term. Terms can be interpreted as mathematical objects, functions, propositions, or proofs. The only two things Lean can do is *create* terms and *check* their types. By iterating these two operations, we can teach Lean to verify complex mathematical proofs.

```
def x := 2 + 2 -- a natural number
def f (x : ℕ) := x + 3 -- a function
def easy_theorem_statement := 2 + 2 = 4 -- a proposition
def fermats_last_theorem_statement -- another proposition
  :=
  ∀ n : ℕ,
  n > 2
  →
  ¬ (∃ x y z : ℕ, (x^n + y^n = z^n) ∧ (x ≠ 0) ∧ (y ≠ 0) ∧ (z ≠ 0))

theorem
easy_proof : easy_theorem_statement -- proof of easy_theorem
:=
begin
  exact rfl,
end

theorem
my_hard_proof : fermats_last_theorem_statement -- cheating!
:=
begin
  sorry,
end

#check x
#check f
#check easy_theorem_statement
#check fermats_last_theorem_statement
#check easy_proof
#check my_hard_proof
```

1.2 How to use these notes

Every once in a while, you will see a code snippet like this:

```
#eval "Hello, World!"
```

Clicking on the `try it!` button in the upper right corner will open a copy in a window so that you can edit it, and Lean provides feedback in the `Lean Infoview` window. We use this feature to provide exercises inline in the notes. We recommend attempting each exercise as you go along.

These notes are based a 5-day Lean crash course at Mathcamp 2020. We have adapted them to BIYSC 2021.

These notes provide a sneak-peek into the world of theorem proving in Lean and are by no means comprehensive. It is recommended that you simultaneously attempt the [Natural Number Game](#). It is a fun (and highly addictive!) game that proves some basic properties of natural numbers in Lean.

1.3 Acknowledgments.

These notes are based on work of [Apurva Nakade](#) and [Jalex Stark](#). Large chunks of these notes are taken directly from https://apurvanakade.github.io/courses/lean_at_MC2020/ and [___](#).

1.4 Useful Links.

1. [Formalizing 100 theorems](#)
2. [Formalizing 100 theorems in Lean](#)
3. **Articles, videos, blog posts, etc.**
 1. [The Xena Project](#)
 2. [The Mechanization of Mathematics](#)
 3. [The Future of Mathematics](#)
4. [Lean Zulip chat group](#)

PRETTY SYMBOLS IN LEAN

To produce a pretty symbol in Lean, type the *editor shortcut* followed by space or tab.

Unicode	Editor Shortcut	Definition
\rightarrow	<code>\to</code>	function or implies
\leftrightarrow	<code>\iff</code>	if and only if
\leftarrow	<code>\l</code>	used by the <code>rw</code> tactic
\neg	<code>\not</code>	negation operator
\wedge	<code>\and</code>	and operator
\vee	<code>\or</code>	or operator
\exists	<code>\exists</code>	there exists quantifier
\forall	<code>\forall</code>	for all quantifier
\mathbb{N}	<code>\nat</code>	type of natural numbers
\mathbb{Z}	<code>\int</code>	type of integers
\circ	<code>\circ</code>	composition of functions
\neq	<code>\ne</code>	not equal to
\in	<code>\in</code>	belongs to
\notin	<code>\notin</code>	does not belong to
\angle	<code>\angle</code>	angle
\triangle	<code>\triangle</code>	triangle
\cong	<code>\cong</code>	congruence of segments
\simeq	<code>\simeq</code>	congruence of angles

GLOSSARY OF TACTICS

3.1 Implications in Lean

<code>exact</code>	<p>If P is the target of the current goal and hp is a term of type P, then <code>exact hp</code>, will close the goal.</p> <p>Mathematically, this saying “this is <i>exactly</i> what we were required to prove”.</p>
<code>intro</code>	<p>If the target of the current goal is a function $P \rightarrow Q$, then <code>intro hp</code>, will produce a hypothesis $hp : P$ and change the target to Q.</p> <p>Mathematically, this is saying that in order to define a function from P to Q, we first need to choose an arbitrary element of P.</p>

<code>have</code>	<p><code>have</code> is used to create intermediate variables.</p> <p>If f is a term of type $P \rightarrow Q$ and hp is a term of type P, then <code>have hq := f hp</code>, creates the hypothesis $hq : Q$.</p>
<code>apply</code>	<p><code>apply</code> is used for backward reasoning.</p> <p>If the target of the current goal is Q and f is a term of type $P \rightarrow Q$, then <code>apply f</code>, changes target to P.</p> <p>Mathematically, this is equivalent to saying “because P implies Q, to prove Q it suffices to prove P”.</p>

3.2 Proof by contradiction

ex-falso	Changes the target of the current goal to false. The name derives from “ <i>ex falso, quodlibet</i> ” which translates to “from contradiction, anything”. You should use this tactic when there are contradictory hypotheses present.
by_cases	If $P : \text{Prop}$, then <code>by_cases P</code> , creates two goals, the first with a hypothesis $hp : P$ and second with a hypothesis $hp : \neg P$. Mathematically, this is saying either P is true or P is false. <code>by_cases</code> is the most direct application of the law of excluded middle.
by_contradiction	If the target of the current goal is Q , then <code>by_contradiction</code> , changes the target to false and adds $hnq : \neg Q$ as a hypothesis. Mathematically, this is proof by contradiction.
push_neg	<code>push_neg</code> , simplifies negations in the target. For example, if the target of the current goal is $\neg \neg P$, then <code>push_neg</code> , simplifies it to P . You can also push negations across a hypothesis $hp : P$ using <code>push_neg at hp</code> .
contrapose!	If the target of the current goal is $P \rightarrow Q$, then <code>contrapose!</code> , changes the target to $\neg Q \rightarrow \neg P$. If the target of the current goal is Q and one of the hypotheses is $hp : P$, then <code>contrapose! hp</code> , changes the target to $\neg P$ and changes the hypothesis to $hp : \neg Q$. Mathematically, this is replacing the target by its contrapositive.

3.3 And / Or

cases	<code>cases</code> is a general tactic that breaks a complicated term into simpler ones. If hpq is a term of type $P \wedge Q$, then <code>cases hpq with hp hq</code> , breaks it into $hp : P$ and $hq : Q$. If hpq is a term of type $P \times Q$, then <code>cases hpq with hp hq</code> , breaks it into $hp : P$ and $hq : Q$. If fg is a term of type $P \leftrightarrow Q$, then <code>cases fg with f g</code> , breaks it into $f : P \rightarrow Q$ and $g : Q \rightarrow P$. If hpq is a term of type $P \vee Q$, then <code>cases hpq with hp hq</code> , creates two goals and adds the hypotheses $hp : P$ and $hq : Q$ to one each.
split	<code>split</code> is a general tactic that breaks a complicated goal into simpler ones. If the target of the current goal is $P \wedge Q$, then <code>split</code> , breaks up the goal into two goals with targets P and Q . If the target of the current goal is $P \times Q$, then <code>split</code> , breaks up the goal into two goals with targets P and Q . If the target of the current goal is $P \leftrightarrow Q$, then <code>split</code> , breaks up the goal into two goals with targets $P \rightarrow Q$ and $Q \rightarrow P$.
left	If the target of the current goal is $P \vee Q$, then <code>left</code> , changes the target to P .
right	If the target of the current goal is $P \vee Q$, then <code>right</code> , changes the target to Q .

3.4 Quantifiers

have	If hp is a term of type $\forall x : X, P\ x$ and y is a term of type Y then <code>have hpy := hp (y)</code> creates a hypothesis $hpy : P\ y$.
intro	If the target of the current goal is $\forall x : X, P\ x$, then <code>intro x</code> , creates a hypothesis $x : X$ and changes the target to $P\ x$.

cases	If hp is a term of type $\exists x : X, P\ x$, then <code>cases hp with x key</code> , breaks it into $x : X$ and $key : P\ x$.
use	If the target of the current goal is $\exists x : X, P\ x$ and y is a term of type X , then <code>use y</code> , changes the target to $P\ y$ and tries to close the goal.

3.5 Proving “trivial” statements

norm_num	<code>norm_num</code> is Lean’s calculator. If the target has a proof that involves <i>only</i> numbers and arithmetic operations, then <code>norm_num</code> will close this goal. If $hp : P$ is an assumption then <code>norm_num at hp</code> , tries to use <code>simplify hp</code> using basic arithmetic operations.
ring	<code>ring</code> , is Lean’s symbolic manipulator. If the target has a proof that involves <i>only</i> algebraic operations, then <code>ring</code> , will close the goal. If $hp : P$ is an assumption then <code>ring at hp</code> , tries to use <code>simplify hp</code> using basic algebraic operations.
linarith	<code>linarith</code> , is Lean’s inequality solver.
simp	<code>simp</code> , is a very complex tactic that tries to use theorems from the <code>mathlib</code> library to close the goal. You should only ever use <code>simp</code> , to <i>close a goal</i> because its behavior changes as more theorems get added to the library.

3.6 Equality

refl	If the current goal is of the form $X = X$ or $P \leftrightarrow P$, then <code>refl</code> will finish the proof. As long as both sides are defined to be equal, this will work. For example, it will work with the goal $3 = 2 + 1$ because <i>by definition</i> the number 3 is defined to be 2 plus one. Mathematically, this says “check that both sides are equal <i>by definition</i> ”.
rw	If f is a term of type $P = Q$ (or $P \leftrightarrow Q$), then <code>rw f</code> , searches for P in the target and replaces it with Q . <code>rw <f</code> , searches for Q in the target and replaces it with P . If additionally, $hr : R$ is a hypothesis, then <code>rw f at hr</code> , searches for P in the expression R and replaces it with Q . <code>rw <f at hr</code> , searches for Q in the expression R and replaces it with P . Mathematically, this is saying because $P = Q$, we can replace P with Q (or the other way around).