Topology Filters - Notes

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CONTENTS

CHAPTER

ONE

FILTER DEFINITION AND ALGEBRAIC STRUCTURE

We will start defining filters and, then the elementary filter propositions will be proved by the usual way and by Lean. This chapter aims to define an algebraic structure with filters using two operations.

1.1 Filter definition

Firstly, we will introduce the filter definition of a giving set.

Definition 1.1 *(Filter). Let* X *be a set, a filter is a family of subsets of the power ser* $F \subseteq \mathcal{P}(X)$ *satisfying the next properties*

(i) *The universal set is in the filter* $X \in F$.

(ii) *If* $E \in F$, *then* $\forall A \in \mathcal{P}(X)$ *such that* $E \subseteq A$, *we have* $A \in F$.

(iii) *If* $E, A \in F$, *then* $E \cap A \in F$.

The reader might have noticed we have not included the empty axiom (states that the empty set cannot be in any filter) commonly used in filter definitions and required for topology filter convergence. Assuming it, would make it impossible to define the neutral element in one of the operations we will use later.

Having the conceptual definition of filters, we can define this structure in Lean. The following code lines were published, in the mathlib repository, by Johannes Hölzl in August 2018.

```
structure filter (X : Type) :=
(sets                    : set (set X))
(univ_sets               : set.univ ∈ sets)
(sets_of_superset {x y} : x ∈ sets → x ⊆ y → y ∈ sets)
(inter_sets {x y}        : x ∈ sets → y ∈ sets → x ∩ y ∈ sets)
```

Having introduced the definition of filters, we will proceed with defining the principal filters. Those are essential to lots of topological structures as the open neighbourhood of a point.

Definition 1.2 *(Principal Filter). Let* X *a set and* $A \subseteq X$ *a subset. We define the principal filter as the subset* $\{t \in \mathcal{P}(X) \mid s \subseteq t\}$, *and from now onwards, it will be denoted as* $P(A)$.

We have introduced a definition of what we have supposed to be a particular type of filter. Now, we should prove that it fulfils the conditions for being a filter.

Proposition 1.3 *Let* X *a set. For all* $A \subseteq X$ *subsets, the principal filter of* A *is a filter.*

Proof.

```
def principal {X : Type} (s : set X) : filter X :=
{ sets              := {t | s ⊆ t},
  univ_sets         := subset_univ s,
  sets_of_superset  := assume x y hx hy, subset.trans hx hy,
  inter_sets        := assume x y, subset_inter }

localized "notation `P` := filter.principal" in filter
```

1.2 Filter Order