F

CHAPTER VIII

Poisson Brackets

Definition

Let $F = F(q_i, p_i, t)$ be any dynamical variable of a system represented by the conjugate variables q_i, p_i. Then:

$$\dot{F} \equiv \frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F}{\partial p_i}\dot{p}_i + \frac{\partial F}{\partial t} \tag{8.1}$$

From Hamilton's canonical equations (5.16) this becomes:

$$\dot{F} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial F}{\partial t} \tag{8.2}$$

The quantity $\sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$ turns out to be a very significant one in the formal development of mechanics and is called the Poisson bracket of F and H. In general, the Poisson bracket of any two dynamical variables X and Y is defined as:

$$\Big[X, Y\Big]_{q,p} = \sum_i \left(\frac{\partial X}{\partial q_i}\frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial Y}{\partial q_i}\right) \tag{8.3}$$

The concept does not assist materially in the complete solution of the equations of motion of a system, but is of use in discussing the constants of motion, as will be seen. It leads to a formalism which, when re-interpreted according to a simple recipe, forms a convenient way of introducing quantum rules in the Heisenberg development of quantum mechanics.

The following identities follow immediately from the definition:

$$\left.\begin{aligned}
\left[X, Y\right] &= -\left[Y, X\right] \\
\left[X, X\right] &= 0 \\
\left[X, Y+Z\right] &= \left[X, Y\right] + \left[X, Z\right] \\
\left[X, YZ\right] &= Y\left[X, Z\right] + \left[X, Y\right]Z
\end{aligned}\right\} \qquad (8.4)$$

also:

$$\left.\begin{aligned}
\left[q_i, q_j\right]_{q,p} &= 0 = \left[p_i, p_j\right]_{q,p} \\
\left[q_i, p_j\right]_{q,p} &= \delta_{ij}
\end{aligned}\right\} \qquad (8.5)$$

where δ_{ij} is the usual delta symbol with the property:

$$\begin{aligned}
\delta_{ij} &= 0 \qquad i \neq j \\
&= 1 \qquad i = j
\end{aligned}$$

The quantities (8.5) are known as the *fundamental* or *basic* Poisson brackets.

respect to which it is evaluated the subscript on the bracket is unnecessary and will now be omitted.

Angular Momentum

Angular momentum components have been identified with generalized momentum components in particular cases. In general the momentum conjugate to any angular co-ordinate can be identified in this way in a simple mechanical system where, for instance, electromagnetic effects are not present. It is of interest to investigate the Poisson bracket of two components of angular momentum. For simplicity consider a particle referred to a cartesian co-ordinate system, the angular momentum components are then given by:

$$l_1 = x_2p_3 - x_3p_2 \qquad l_2 = x_3p_1 - x_1p_3 \qquad l_3 = x_1p_2 - x_2p_1 \qquad (8.12)$$

where $p_1 = m\dot{x}_1$, etc.

Evaluating the Poisson bracket of l_1 and l_2 gives:

$$\left[l_1, l_2\right] = (p_2x_1 - p_1x_2) = l_3$$

Similar results are obtained for other combinations and they may be summarized in:

$$\left[l_i, l_j\right] = \sum_k \varepsilon_{ijk} l_k \tag{8.13}$$

where ε_{ijk} = 1 if (i, j, k) is an even permutation of (1, 2, 3)
= − 1 „ „ „ „ odd „ „ (1, 2, 3)
= 0 otherwise.

The implication of (8.13) is that no two components of angular momentum can simultaneously act as conjugate momenta since all conjugate variables must obey the laws relating to the fundamental brackets given in (8.7). Any one angular momentum component can, of course, be chosen as a generalized momentum co-ordinate but not more than one in any particular system of reference.

Consider now $\left[l_i, l^2\right]$, where l^2 is the square of the total angular momentum. Using the identities (8.4) and the result (8.13):

$$\left[l_i, l^2\right] = \left[l_i, \sum_j l_j^2\right] = \sum_j \left[l_i, l_j^2\right] = \sum_j \left\{2l_j\left[l_i, l_j\right]\right\}$$

$$= \sum_{j,k} 2l_j \varepsilon_{ijk} l_k \equiv 0 \tag{8.14}$$

i.e., l^2 and any one component of l can simultaneously be regarded as conjugate momenta. The results (8.13) and (8.14) are highly important in the extension of the formalism to quantum mechanics.

Other results with similar significance are:

$$\left[x_i, l_j\right] = \sum_k \varepsilon_{ijk} x_k \qquad \left[p_i, l_j\right] = \sum_k \varepsilon_{ijk} p_k \tag{8.15}$$

where the p's in this instance still denote linear cartesian momentum components.

Constants of Motion

It has already been emphasized that for some purposes the solution of a problem may be considered achieved by identifying the constants of motion. Rewriting (8.2) in Poisson bracket notation shows that the time variation of any dynamical variable F is given by:

$$\dot{F} = [F, H] + \frac{\partial F}{\partial t} \tag{8.2'}$$

This shows that if the variable does not contain the time explicitly it is sufficient for its Poisson bracket with H to vanish in order that it be a constant of motion. This result is independent of whether H itself is a constant of motion and provides a useful means for identifying constants of motion.

Special cases of (8.2′) are:

$$\dot{q}_i = [q_i, H] \qquad \dot{p}_i = [p_i, H] \tag{8.16}$$

these are identical with Hamilton's canonical equations and may be referred to as the equations of motion in Poisson bracket form.

Another special case is:

$$\frac{dH}{dt} = [H, H] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{8.17}$$

This relation has also appeared previously.

Jacobi's Identity

Consider the expression:

$$[X, [Y, Z]] - [Y, [X, Z]]$$

$$= \left[X, \sum_i \left(\frac{\partial Y}{\partial q_i}\frac{\partial Z}{\partial p_i} - \frac{\partial Y}{\partial p_i}\frac{\partial Z}{\partial q_i}\right)\right] - \left[Y, \sum_i \left(\frac{\partial X}{\partial q_i}\frac{\partial Z}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial Z}{\partial q_i}\right)\right]$$

using the identities (8.4) and regrouping gives:

$$\sum_i \left\{-\frac{\partial Z}{\partial q_i}\left(\left[\frac{\partial X}{\partial p_i}, Y\right] + \left[X, \frac{\partial Y}{\partial p_i}\right]\right) + \frac{\partial Z}{\partial p_i}\left(\left[\frac{\partial X}{\partial q_i}, Y\right] + \left[X, \frac{\partial Y}{\partial q_i}\right]\right)\right\}$$

$$+ \sum_i \left\{\frac{\partial Y}{\partial q_i}\left[X, \frac{\partial Z}{\partial p_i}\right] - \frac{\partial Y}{\partial p_i}\left[X, \frac{\partial Z}{\partial q_i}\right] - \frac{\partial X}{\partial q_i}\left[Y, \frac{\partial Z}{\partial p_i}\right] + \frac{\partial X}{\partial p_i}\left[Y, \frac{\partial Z}{\partial q_i}\right]\right\}$$

7

Using the identity:

$$\frac{\partial}{\partial x}\Big[X, Y\Big] \equiv \left[\frac{\partial X}{\partial x}, Y\right] + \left[X, \frac{\partial Y}{\partial x}\right] \tag{8.18}$$

the first expression reduces to:

$$\sum_i \left\{-\frac{\partial Z}{\partial q_i}\frac{\partial}{\partial p_i}\Big[X, Y\Big] + \frac{\partial Z}{\partial p_i}\frac{\partial}{\partial q_i}\Big[X, Y\Big]\right\} = -\Big[Z, \Big[X, Y\Big]\Big]$$

the second expression may be shown to vanish. Hence:

$$\Big[X, \Big[Y, Z\Big]\Big] - \Big[Y, \Big[X, Z\Big]\Big] = -\Big[Z, \Big[X, Y\Big]\Big]$$

which may be written in symmetrical form:

$$\Big[X, \Big[Y, Z\Big]\Big] + \Big[Y, \Big[Z, X\Big]\Big] + \Big[Z, \Big[X, Y\Big]\Big] = 0 \tag{8.19}$$

The result is known as Jacobi's identity. It may be applied in the following way. Let $Z = H$, then:

$$\Big[X, \Big[Y, H\Big]\Big] + \Big[Y, \Big[H, X\Big]\Big] + \Big[H, \Big[X, Y\Big]\Big] = 0 \tag{8.20}$$

if now X and Y are both constants of motion then:

$$\Big[Y, H\Big] = 0 \qquad \Big[X, H\Big] = 0 \tag{8.21}$$

hence:

$$\Big[H, \Big[X, Y\Big]\Big] = 0 \tag{8.22}$$

i.e., the dynamical variable $\Big[X, Y\Big]$ is also a constant of motion. The usefulness of this result lies in the possibility of constructing new constants of motion from known ones. It will not, however, always be the case that the new constants are other than trivialities (e.g., using p_i and p_j = const. merely leads to 0 = const.).

Poisson Brackets and Commutators

In quantum mechanics dynamical variables are represented by operators which do not obey the commutation rules of ordinary algebra. It is not possible to define Poisson brackets for these operators, but the universal nature and general usefulness of the brackets in classical mechanics suggests that there might be analogous quantities associated with the operators.

Equations (8.4) can be held to represent basic properties of Poisson brackets. Assume that these also represent properties of quantities associated with the corresponding quantum mechanical operators. (Note that this is not a necessary assumption and that, in fact, it is not true of all types of operator.) For any three operators X, Y, Z we have, on this assumption:

$$\left[X, YZ\right] = Y\left[X, Z\right] + \left[X, Y\right]Z$$

and

$$\left[XY, Z\right] = X\left[Y, Z\right] + \left[X, Z\right]Y$$

where for initial convenience we employ the same symbol to denote this unknown quantum analogue of the Poisson bracket. Care has been taken to preserve the order of the operators in view of their non-commutability.

It follows that, for any four operators W, X, Y, Z:

$$\begin{aligned}\left[WX, YZ\right] &= W\left[X, YZ\right] + \left[W, YZ\right]X \\ &= W\left[X, Y\right]Z + WY\left[X, Z\right] + Y\left[W, Z\right]X + \left[W, Y\right]ZX\end{aligned}$$

also, expanding in a different order:

$$\begin{aligned}\left[WX, YZ\right] &= \left[WX, Y\right]Z + Y\left[WX, Z\right] \\ &= W\left[X, Y\right]Z + \left[W, Y\right]XZ + YW\left[X, Z\right] + Y\left[W, Z\right]X\end{aligned}$$

Combining these results gives:

$$(WY - YW)\left[X, Z\right] = \left[W, Y\right](XZ - ZX) \qquad (8.23)$$

The four operators were assumed arbitrary. It follows that the identity (8.23) can only be satisfied if, for any two operators A, B:

$$(AB - BA) = \alpha\left[A, B\right] \qquad (8.24)$$

where α is some constant, i.e., the quantum analogue of the Poisson bracket is identified as a multiple of the commutator of the two operators concerned. Assuming that the operators corresponding to the conjugate variables q_i, p_i play the same fundamental role as the classical variables, then:

$$(q_i p_j - p_j q_i) = \alpha \delta_{ij} \tag{8.25}$$

It is by further postulating that $\alpha = \dfrac{ih}{2\pi}$ that the quantum rules are introduced in the Heisenberg formulation of quantum mechanics.