Guia de estudio parcial 1

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Tensors

Let V be a vector space of dimension n over a field F, $\boldsymbol{x} \in V$, let $\boldsymbol{e_1, e_2, .., e_n}$ be a basis for V then:

$$x = x_i \boldsymbol{e_i}$$

Let $\boldsymbol{\tilde{e_1}, \tilde{e_2}, ..., \tilde{e_3}}$ be another basis for V, then:

$$x = \tilde{x}_i \boldsymbol{\tilde{e_i}}$$

where:

$$\boldsymbol{\tilde{e_i}} = S_{ij} \boldsymbol{e_j}$$

then we have the following relation:

$$x_i = S_{ij} \tilde{x_j}$$
$$\tilde{x}_i = T_{ij} x_j$$

from this relation we obtain:

$$\frac{\partial x_i}{\partial \tilde{x}_j} = S_{ij}$$

In this example $V = R^2$ and $F = R$.

$$\boldsymbol{e_1, e_2} = \boldsymbol{i, j}$$
$$\boldsymbol{\tilde{e_1}, \tilde{e_2}} = \boldsymbol{e_r, e_\theta}$$

let $\boldsymbol{x} \in V$:

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j}$$
$$\boldsymbol{x} = r\boldsymbol{e_r}$$

we know:

$$\boldsymbol{e_r} = cos\theta \boldsymbol{i} + sin\theta \boldsymbol{j}$$
$$\boldsymbol{e_\theta} = -sin\theta \boldsymbol{i} + cos\theta \boldsymbol{j}$$

hence we have:

$$S = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$
$$T = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

we know the following functions:

$$x(r, \theta) = rcos\theta$$
$$y(r, \theta) = rsin\theta$$

Movement in different coordinate systems

Let $\boldsymbol{x}$ be the function that describes the position of our object as a function of time.

Cylindrical coordinates

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$
$$\boldsymbol{r} = \rho\boldsymbol{e_r} + z\boldsymbol{k}$$

from where we obtain:

$$\dot{\boldsymbol{r}} = \dot{\rho}\boldsymbol{e_r} + \rho\dot{\theta}\boldsymbol{e_\theta} + \dot{z}\boldsymbol{k}$$
$$\ddot{\boldsymbol{r}} = (\ddot{\rho} - \rho\dot{\theta}^2)\boldsymbol{e_r} + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\boldsymbol{e_\theta} + \ddot{z}\boldsymbol{k}$$

Spherical coordinates

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$
$$\boldsymbol{r} =$$

Generalized coordinates

Let $\boldsymbol{r}$ be the function that returns the position of our object at a time t:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

define:

$$\boldsymbol{b_i} = \frac{\partial \boldsymbol{r}}{\partial q_i} = h_i\boldsymbol{e_i}$$
$$\boldsymbol{v_i} = \nabla x_i$$

we want to know:

$$\boldsymbol{r} = r_i\boldsymbol{e_i}$$
$$\dot{\boldsymbol{r}} = v_i\boldsymbol{e_i}$$
$$\ddot{\boldsymbol{r}} = a_i\boldsymbol{e_i}$$

Motion in other systems of reference

Translation

Let $\boldsymbol{r}(t)$ be the function that returns the position of an object at time t, where:

$$\boldsymbol{r}(t) = \boldsymbol{R}(t) + \boldsymbol{r'}(t)$$

where:

$$\boldsymbol{r'}(t) = x'(t)\boldsymbol{i} + y'(t)\boldsymbol{j} + z'(t)\boldsymbol{k}$$

then we have the relation:

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{R}}{dt} + \frac{d\boldsymbol{r'}}{dt}$$

Rotation

Let $\boldsymbol{r}(t)$ be the funcion that returns the position of an object at time t, where:

$$\begin{aligned}\boldsymbol{r}(t) &= x'(t)\boldsymbol{i}(t) + y'(t)\boldsymbol{j}(t) + z'(t)\boldsymbol{k}(t)\\ \frac{d\boldsymbol{r}}{dt}(t) &= \frac{dx'}{dt}(t)\boldsymbol{i}(t) + \frac{dy'}{dt}(t)\boldsymbol{j}(t) + \frac{dz'}{dt}(t)\boldsymbol{k}(t) + \boldsymbol{\Omega}(t)\times\boldsymbol{r}(t)\end{aligned}$$

let's define:

$$\boldsymbol{r'}(t) = x'(t)\boldsymbol{i} + y'(t)\boldsymbol{j} + z'(t)\boldsymbol{k}$$

then we can write the derivative as:

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r'}}{dt} + \boldsymbol{\Omega}\times\boldsymbol{r'}$$

we can find the acceleration:

$$\begin{aligned}\frac{d^2\boldsymbol{r}}{dt^2} &= \frac{d}{dt}\left(\frac{d\boldsymbol{r'}}{dt} + \boldsymbol{\Omega}\times\boldsymbol{r'}\right) + \boldsymbol{\Omega}\times\left(\frac{d\boldsymbol{r'}}{dt} + \boldsymbol{\Omega}\times\boldsymbol{r'}\right)\\ &= \frac{d^2\boldsymbol{r'}}{dt^2} + \frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r'} + \boldsymbol{\Omega}\times\frac{d\boldsymbol{r'}}{dt} + \boldsymbol{\Omega}\times\frac{d\boldsymbol{r'}}{dt} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r'})\\ &= \frac{d^2\boldsymbol{r'}}{dt^2} + \frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r'} + 2\boldsymbol{\Omega}\times\frac{d\boldsymbol{r'}}{dt} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r'})\end{aligned}$$

Translation plus rotation

Let $\boldsymbol{r}(t)$ the function that returns the position of an object at time t where:

$$\boldsymbol{r}(t) = \boldsymbol{R}(t) + x(t)\boldsymbol{i}(t) + y(t)\boldsymbol{j}(t) + z(t)\boldsymbol{k}(t)$$

thus we have:

$$\frac{d\boldsymbol{r}}{dt} = \frac{d(\boldsymbol{R'}+\boldsymbol{r'})}{dt} + \boldsymbol{\Omega}\times(\boldsymbol{R'}+\boldsymbol{r'})$$

Example: let's examinate a plomada, we have the conditions:

$$\begin{aligned}\boldsymbol{R}(t) &= R\boldsymbol{k}(t)\\ \frac{d^2\boldsymbol{r}}{dt^2}(t) &= -g\boldsymbol{k}(t)\end{aligned}$$

and we can ignore all the other terms, thus we have:

$$\frac{d^2\boldsymbol{r'}}{dt^2} = -\frac{d^2\boldsymbol{r}}{dt^2} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{R'})$$