

# On the minimum strength of (unobserved) covariates to overturn an insignificant result

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*Abstract.* We study conditions under which the addition of variables to a regression equation can turn a previously statistically insignificant result into a significant one. Specifically, we characterize the minimum strength of association required for these variables—both with the dependent and independent variables, or with the dependent variable alone—to elevate the observed t-statistic above a specified significance threshold. Interestingly, we show that it is considerably difficult to overturn a statistically insignificant result solely by reducing the standard error. Instead, included variables must also alter the point estimate to achieve such reversals in practice. Our results can be used to conduct sensitivity analyses against unobserved variables and to bound the maximum t-value one can obtain given different subsets of observed covariates, and may also offer algebraic explanations for patterns of reversals seen in empirical research, such as those documented by [Lenz and Sahn \(2021\)](#).

*Key words and phrases:* ordinary least squares, linear regression, omitted variable bias, robustness values, p-hacking.

## 1. INTRODUCTION

There are many legitimate reasons for including covariates in a regression equation. These reasons arise both in experimental and observational settings. Consider, for example, a randomized controlled trial (RCT) in which a researcher uses ordinary least squares (OLS) to estimate the average effect of a treatment on an outcome. Although randomization eliminates confounding by design, including pre-treatment covariates can still improve the precision of estimated effects. In observational studies, beyond precision gains, accounting for covariates is often necessary to block spurious pathways, or mitigate other sources of bias ([Cinelli, Forney and Pearl, 2022](#)). Such concerns not only motivate the inclusion of observed covariates in a regression equation, but also raise the question of how results might have differed had certain unmeasured variables been included.

Despite these valid motivations, applied researchers often fail to adequately justify their choice of control variables. And it is not uncommon for the statistical significance of main findings to depend on such choices. For example, in the American Journal of Political Science, [Lenz and Sahn \(2021\)](#) found that over 30% of ar-

ticles relied on the inclusion of covariates to turn previously statistically insignificant findings into significant ones. According to [Lenz and Sahn \(2021\)](#), none of the articles justified this choice, nor did they disclose these reversals. More generally, the practice of testing various model specifications until one obtains a statistically significant result—whether intentionally or not—appears to be prevalent across disciplines ([Brodeur et al., 2016](#); [Vivalt, 2019](#)). These significance reversals raise several questions. Under what conditions can they occur? Are there natural limits to the t-statistic one can obtain via different choices of covariates? What observable patterns in the data should emerge when these reversals take place?

In this paper, we provide simple algebraic answers to these questions in the context of OLS. Building on recent results from [Cinelli and Hazlett \(2020, 2025\)](#), we first characterize the maximum change in the t-statistic that covariates with bounded strength can produce. We then derive the minimum strength of association that such covariates must have—whether with both the dependent and independent variables or with the dependent variable alone—to elevate the observed t-statistic above a given threshold. Lastly, we provide an empirical example. Our results can be used to conduct sensitivity analyses regarding unobserved “suppressors” and to bound the maximum attainable t-value resulting from the choice of observed control variables. They also offer algebraic explanations

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for patterns of significance reversals observed in empirical research.<sup>1</sup>

## 2. PRELIMINARIES

### 2.1 Problem setup

Let  $Y$  be an  $(n \times 1)$  vector containing the dependent variable for  $n$  observations;  $D$  be an  $(n \times 1)$  vector of the independent variable of interest and  $\mathbf{X}$  be an  $(n \times p)$  matrix of observed covariates including a constant. Consider the regression equation

$$(1) \quad Y = \hat{\lambda}_r D + \mathbf{X} \hat{\beta}_r + \hat{\epsilon}_r,$$

where  $\hat{\lambda}_r$ ,  $\hat{\beta}_r$  are the OLS estimates of the regression coefficients of  $Y$  on  $D$  and  $\mathbf{X}$ , and  $\hat{\epsilon}_r$  is the corresponding  $(n \times 1)$  vector of residuals.

Let  $\widehat{\text{se}}(\hat{\lambda}_r)$  be the estimated classical (homoskedastic) standard error of  $\hat{\lambda}_r$ . Under the classical linear regression model, the t-statistic for testing the null hypothesis  $H_0 : \lambda_r = \lambda_0$ , i.e.,

$$(2) \quad t_r := \frac{\hat{\lambda}_r - \lambda_0}{\widehat{\text{se}}(\hat{\lambda}_r)},$$

follows a t-distribution with  $df := n - p - 1$  degrees of freedom. Denoting by  $t_{\alpha, df}^*$  the  $(1 - \alpha/2)$  quantile of this distribution, the t-statistic (2) is considered “statistically significant with significance level  $\alpha$ ” if the absolute value of  $t_r$  exceeds that of  $t_{\alpha, df}^*$ . Note that the t-statistic depends on the choice of  $\lambda_0$ . For simplicity, we use the notation  $t_r$  with the understanding that a particular  $\lambda_0$  has been chosen.

Now suppose the t-statistic (2) is insignificant. Let  $Z$  be an  $(n \times 1)$  vector of a (potentially unobserved) covariate whose inclusion in the regression equation we wish to assess. In contrast to the “restricted” regression in (1), we now consider the long regression equation of  $Y$  on  $D$  after adjusting for both  $\mathbf{X}$  and  $Z$ ,

$$(3) \quad Y = \hat{\lambda} D + \mathbf{X} \hat{\beta} + \hat{\gamma} Z + \hat{\epsilon}.$$

Here, the t-statistic for testing  $H_0 : \lambda = \lambda_0$  is

$$(4) \quad t := \frac{\hat{\lambda} - \lambda_0}{\widehat{\text{se}}(\hat{\lambda})}$$

where  $\hat{\lambda}$  and  $\widehat{\text{se}}(\hat{\lambda})$  have the same interpretation as before, just now with an additional adjustment for  $Z$ . We wish to

<sup>1</sup>Note that post-hoc selection of variables based on the data usually invalidates standard distributional results of test statistics—see, e.g., Berk et al. (2013) for a proposal of valid post-selection inference in the context of linear regression. We clarify that such results are orthogonal to the main results of our paper. Here we answer the following question: *when can the addition of variables to a regression equation elevate the observed t-statistic above a pre-specified threshold?* These results pertain to the algebra of least squares alone, and do not depend on assumptions on the underlying sampling distribution.

quantify the properties that  $Z$  needs to have such that the t-statistic in (4) will exceed the critical threshold  $t_{\alpha, df-1}^*$ .

REMARK 1. Throughout the paper we illustrate results using the critical threshold  $t_{\alpha, df-1}^*$ , which is usually justified by classical linear model theory. However, we remind the reader that researchers may choose the critical threshold as they see fit, possibly justified by different distributional assumptions. For example, if a researcher is testing  $k$  individual hypotheses, they may consider the Bonferroni-corrected threshold  $t_{\alpha/k, df-1}^*$ .

### 2.2 Omitted variable bias formulas

Comparing (2) with (4), observe that the relative change in the absolute value of the t-statistic can be decomposed as the product of the relative change in the bias and the relative change in the standard error:

$$(5) \quad \left| \frac{t}{t_r} \right| = \left| \frac{\hat{\lambda} - \lambda_0}{\hat{\lambda}_r - \lambda_0} \right| \times \left( \frac{\widehat{\text{se}}(\hat{\lambda}_r)}{\widehat{\text{se}}(\hat{\lambda})} \right).$$

Concretely, for  $Z$  to double the t-statistic, it must either double the absolute difference between the point estimate and  $\lambda_0$ , halve the standard errors, or achieve some combination of both.

To characterize these changes in terms of how much residual variation  $Z$  explains of  $D$  and  $Y$ , we refer to the following result from Cinelli and Hazlett (2020). In what follows, we define the sample partial  $R^2$  of  $A$  with  $B$  given  $C$  as  $R_{A \sim B|C}^2 := 1 - \frac{\widehat{\text{var}}(A^{\perp B, C})}{\widehat{\text{var}}(A^{\perp C})}$ , where  $\widehat{\text{var}}(\cdot)$  denotes the sample variance, and  $A^{\perp B, C}$  denotes the OLS residual from regressing  $A$  on  $B$  and  $C$ .

THEOREM 1 (Cinelli and Hazlett, 2020). *Let  $R_{Y \sim Z|D\mathbf{X}}^2$  denote the sample partial  $R^2$  of  $Y$  with  $Z$  after adjusting for  $D$  and  $\mathbf{X}$ , and let  $R_{D \sim Z|\mathbf{X}}^2 < 1$  denote the sample partial  $R^2$  of  $D$  with  $Z$  after adjusting for  $\mathbf{X}$ . Then,*

$$(6) \quad |\hat{\lambda}_r - \hat{\lambda}| = \underbrace{\sqrt{\frac{R_{Y \sim Z|D\mathbf{X}}^2 R_{D \sim Z|\mathbf{X}}^2}{1 - R_{D \sim Z|\mathbf{X}}^2}}}_{BF} \times \widehat{\text{se}}(\hat{\lambda}_r) \times \sqrt{df}$$

and

$$(7) \quad \widehat{\text{se}}(\hat{\lambda}) = \underbrace{\sqrt{\frac{1 - R_{Y \sim Z|D\mathbf{X}}^2}{1 - R_{D \sim Z|\mathbf{X}}^2}}}_{SEF} \times \widehat{\text{se}}(\hat{\lambda}_r) \times \sqrt{\frac{df}{df-1}}.$$

To aid interpretation, we call the terms  $BF$  in (6) and  $SEF$  in (7) the “bias factor” and the “standard error factor” of  $Z$ , respectively.

We can use Theorem 1 to write the absolute value of the t-statistic (4) as a function of  $R_{Y \sim Z|D\mathbf{X}}^2$  and  $R_{D \sim Z|\mathbf{X}}^2$ , i.e.,

$$(8) \quad t(R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2) = \frac{|\hat{\lambda}_r - \lambda_0| \pm \text{BF} \times \widehat{\text{se}}(\hat{\lambda}_r) \times \sqrt{df}}{\widehat{\text{se}}(\hat{\lambda}_r) \times \text{SEF} \times \sqrt{\frac{df}{df-1}}},$$

where the sign of the bias term, denoted by  $\pm$ , depends on whether  $\hat{\lambda}_r > \lambda_0$  or vice-versa. This re-formulation allows us to assess how  $Z$  affects inferences for any postulated pair of partial  $R^2$  values  $\{R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2\}$ , and it will help us determine the conditions under which the addition of  $Z$  turns a previously statistically insignificant result into a significant one.

To set the stage for upcoming results, we note an immediate but important corollary of Theorem 1: for a fixed observed t-statistic and fixed partial  $R^2$  values of  $Z$ , the impact that  $Z$  has on the relative bias depends on the sample size, whereas the impact it has on the relative change in standard errors does not. That is, the relative change in the bias is given by

$$\frac{\hat{\lambda} - \lambda_0}{\hat{\lambda}_r - \lambda_0} = 1 \pm \frac{\text{BF}}{t_r} \times \sqrt{df}.$$

For fixed  $t_r$  and fixed  $\{R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2\}$  (which sets BF), larger sample sizes yield larger relative changes in the distance of the estimate from the null hypothesis. Whereas, the relative change in the standard error is given by

$$(9) \quad \frac{\widehat{\text{se}}(\hat{\lambda}_r)}{\widehat{\text{se}}(\hat{\lambda})} = \frac{1}{\text{SEF}} \times \sqrt{\frac{df-1}{df}} \approx \frac{1}{\text{SEF}}$$

and is thus unaffected by the sample size. As an example, halving standard errors is equally challenging in a sample of 100 as in a sample of 1,000,000; in contrast, given a fixed  $t_r$ , doubling the point estimate becomes much easier as the sample size grows. This distinction will become clearer as we continue with our analysis.

### 3. RESULTS

In this section we present two main results. First, given upper-bounds on  $R_{Y \sim Z|D\mathbf{X}}^2$  and  $R_{D \sim Z|\mathbf{X}}^2$ , we derive the “maximum adjusted t-statistic” which quantifies the maximum possible value that  $t(R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2)$  can attain after including  $Z$  in the regression equation. Then we solve for the minimum upper bound on  $\{R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2\}$ , hereby referred to as the “strength of  $Z$ ,” such that it guarantees that the maximum adjusted t-statistic exceeds the desired threshold.

In what follows, it is useful to define the quantities

$$f_r := |t_r| / \sqrt{df} \quad \text{and} \quad f_{\alpha, df}^* := t_{\alpha, df}^* / \sqrt{df},$$

which normalize the observed t-statistic and the critical threshold by the degrees of freedom. These definitions greatly simplify formulas and derivations.

#### 3.1 On the maximum adjusted t-statistic

We start by defining the maximum value that the t-statistic (4) could attain given  $Z$  with bounded strength.

**DEFINITION 1** (Maximum adjusted t-statistic). For a fixed null hypothesis  $H_0 : \lambda = \lambda_0$ , and upper bounds on  $R_{Y \sim Z|D\mathbf{X}}^2$  and  $R_{D \sim Z|\mathbf{X}}^2$ , denoted by  $\mathbf{R}^2 = \{R_Y^{\max}, R_D^{\max}\}$ , we define the maximum adjusted t-statistic as

$$t_{\mathbf{R}^2}^{\max} := \max_{R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2} t(R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2) \\ \text{s.t. } R_{Y \sim Z|D\mathbf{X}}^2 \leq R_Y^{\max}, R_{D \sim Z|\mathbf{X}}^2 \leq R_D^{\max}.$$

The solution to the above problem has a simple closed-form characterization.

**THEOREM 2** (Closed-form solution to  $t_{\mathbf{R}^2}^{\max}$ ). Let  $R_Y^{\max} < 1$ . Then,

$$t_{\mathbf{R}^2}^{\max} = \frac{f_r \sqrt{1 - R_{D \sim Z|\mathbf{X}}^2} + \sqrt{R_{Y \sim Z|D\mathbf{X}}^2 R_{D \sim Z|\mathbf{X}}^2}}{\sqrt{(1 - R_{Y \sim Z|D\mathbf{X}}^2)/(df-1)}}$$

where

$$\{R_{Y \sim Z|D\mathbf{X}}^{*2}, R_{D \sim Z|\mathbf{X}}^{*2}\} = \{R_Y^{\max}, R_D^{\max}\}$$

if  $f_r^2 < R_Y^{\max}(1 - R_D^{\max})/R_D^{\max}$  and

$$\{R_{Y \sim Z|D\mathbf{X}}^{*2}, R_{D \sim Z|\mathbf{X}}^{*2}\} = \left\{R_Y^{\max}, \frac{R_Y^{\max}}{f_r^2 + R_Y^{\max}}\right\}$$

otherwise.

If  $t_{\mathbf{R}^2}^{\max} < t_{\alpha, df-1}^*$ , then we can be assured that no  $Z$  with the specified maximum strength would be able to overturn an insignificant result. On the other hand, if  $t_{\mathbf{R}^2}^{\max} > t_{\alpha, df-1}^*$ , we know that there exists at least one  $Z$  with strength no greater than  $\mathbf{R}^2$  that is capable of bringing the t-statistic above the specified threshold. We note that the sign of the bias term in definition (8) of  $t(R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2)$  is resolved by pushing  $t_r$  further in the direction of  $\text{sign}(t_r)$ .

**REMARK 2.** It is always necessary to constrain the strength of  $Z$  with respect to  $Y$  in order to obtain a finite solution for  $t_{\mathbf{R}^2}^{\max}$ . If  $R_Y^{\max} = 1$ , then as  $R_{Y \sim Z|D\mathbf{X}}^2$  approaches one, the standard error approaches zero and the t-statistic grows without bounds.

**REMARK 3.** In contrast, it is possible to leave  $R_{D \sim Z|\mathbf{X}}^2$  unconstrained. While the optimal value of  $R_{Y \sim Z|D\mathbf{X}}^2$  always reaches the upper bound  $R_Y^{\max}$ ,  $R_{D \sim Z|\mathbf{X}}^2$  may either reach its upper bound  $R_D^{\max}$  or result in the interior point solution  $\frac{R_Y^{\max}}{f_r^2 + R_Y^{\max}}$ . This happens because increasing  $R_{D \sim Z|\mathbf{X}}^2$  has two counterbalancing effects on the t-statistic. On one hand, it changes the point estimate, as

described by (6), in a direction that is favorable for rejecting  $H_0 : \lambda = \lambda_0$ . On the other hand, it increases standard errors due to the variance inflation factor  $\frac{1}{1-R_{D \sim Z|X}^2}$  in (7). Intuitively, the more variation  $Z$  explains of  $D$ , the less residual variation of  $D$  is used to estimate  $\hat{\lambda}$ , which becomes noisier. Eventually, the increase in standard error is so large that it exceeds the benefit of the increase in point estimate, thus reducing the t-statistic.

Of course, there naturally could be multiple latent variables instead of a single one, and so one might wonder about the case when  $Z$  takes the form of a matrix rather than a vector. The following theorem demonstrates that it is sufficient to consider a single unmeasured latent variable, up to a correction in the degrees of freedom.

**THEOREM 3** ( $t_{R^2}^{\max}$  for matrix  $\mathbf{Z}$ ). *Let  $\mathbf{Z}$  be an  $(n \times m)$  matrix of covariates. Then the solution to  $t_{R^2}^{\max}$  is the same as that of Theorem 2, save for the adjustment in the degrees of freedom, which now is  $df - m$ .*

In what follows, for simplicity we keep  $Z$  as a vector with the understanding that all results still hold for matrix  $\mathbf{Z}$ . But before moving forward, there is an interesting corollary of the previous result. It characterizes the maximum extent to which one can p-hack results by testing different subsets of a fixed set of observed covariates  $\mathbf{X}$ .

**COROLLARY 1** (Upper bound on the t-statistic for any subset of covariates). *For observed covariates  $\mathbf{X}$ , let  $R_{Y \sim X|D}^2, R_{D \sim X}^2$  denote the strengths of the associations of  $\mathbf{X}$  with  $Y$  and  $D$ , respectively. Let  $t_{Y \sim D|X_S}$  denote the t-statistic for the coefficient of the regression of  $Y$  on  $D$  when adjusting for a subset of covariates  $\mathbf{X}_S$  (consisting of a subset of the columns of  $\mathbf{X}$ ). Then for any  $\mathbf{X}_S$ ,*

$$|t_{Y \sim D|X_S}| \leq t_{R^2}^{\max},$$

where  $t_{R^2}^{\max}$  is the solution of Theorem 2 using  $t_r = t_{Y \sim D}$  and  $R^2 = \{R_{Y \sim X|D}^2, R_{D \sim X}^2\}$ .

When the number of covariates is small, it is feasible to run all possible regressions to identify the exact maximum t-statistic across all specifications. However, when the number of covariates is large, this exhaustive approach becomes impractical. For example, with  $p = 40$ , there are  $2^{40}$  (approximately 1 trillion) possible specifications. In such a case,  $t_{R^2}^{\max}$  offers a simple upper bound on the maximum t-value that one could obtain via specification searches, without the need to actually run all 1 trillion regressions.

**EXAMPLE 1.** Let  $p = 40$ ,  $df = 100$  and the t-statistic of the regression of  $Y$  on  $D$  be equal to 1. Then, if  $\{R_{Y \sim X|D}^2, R_{D \sim X}^2\} = \{0.08, 0.08\}$ , Corollary 1 assures us that none of the 1 trillion specifications can yield a t-statistic greater than 1.83.

**REMARK 4.** Note that the set of covariates  $\mathbf{X}$  is arbitrary and can thus accommodate any nonlinear transformations of the original variables. For example, if we again consider an original set of  $p = 40$  covariates, but also logarithms, squares, and pairwise interactions, Corollary 1 can be applied to characterize the maximum adjusted t-statistic of the  $2^{3p+p(p-1)/2}$  possible specifications, including these nonlinear transformations.

**REMARK 5.** Note that  $t_{R^2}^{\max}$  is achievable when the only constraint on the variables to be included is their maximum explanatory power. For any given set of observed covariates,  $t_{R^2}^{\max}$  is a potentially loose upper bound. It is possible to tighten this bound by applying the corollary iteratively within subsets of regressions. Obtaining tight bounds without running all of the  $2^p$  possible regressions remains an open problem.

### 3.2 On the minimal strength of $Z$ to reverse statistical insignificance

Equipped with the notion of the maximum adjusted t-statistic, we can now characterize the minimum strength of  $Z$  to obtain a statistically significant result. Following the convention in Cinelli and Hazlett (2020, 2025) we call our metrics “robustness values for insignificance.” They quantify how “robust” an insignificant result is to the inclusion of covariates in the regression equation.<sup>2</sup>

**3.2.1 Extreme robustness value for insignificance.** As highlighted in Remark 2, the parameter  $R_{Y \sim Z|D\mathbf{X}}^2$  is essential for assessing the potential of  $Z$  to bring about a significant result, as it always needs to be bounded. Thus we begin by characterizing the minimal strength of association of  $Z$  with  $Y$  alone in order to achieve significance.

**DEFINITION 2** (Extreme Robustness Value for Insignificance). For fixed  $R_D^{\max} \in [0, 1]$ , the extreme robustness value for insignificance,  $\text{XRVI}_{\alpha}^{R_D^{\max}}$ , is the minimum upper bound on  $R_{Y \sim Z|D\mathbf{X}}^2$  such that  $t_{R^2}^{\max}$  is large enough to reject null hypothesis  $H_0 : \lambda = \lambda_0$  at specified significance level  $\alpha$ , i.e.,

$$\text{XRVI}_{\alpha}^{R_D^{\max}} := \min\{\text{XRVI} : t_{\text{XRVI}, R_D^{\max}}^{\max} \geq t_{\alpha, df-1}^*\}.$$

For a fixed bound on  $R_{D \sim Z|X}^2$ , the  $\text{XRVI}_{\alpha}^{R_D^{\max}}$  describes how robust an insignificant result is in terms of the minimum explanatory power that  $Z$  needs to have with  $Y$

<sup>2</sup>Our metrics complement the “robustness value” (RV) and “extreme robustness value” (XRV) introduced in Cinelli and Hazlett (2020, 2025), which deal with the opposite problem and quantify the minimum strength of  $Z$  to turn a *significant result insignificant*. See the supplementary material for further discussion.



in order to overturn it. Theorem 7 in the appendix provides an analytical expression for  $\text{XRVI}_\alpha^{R_D^{\max}}$  given arbitrary  $R_D^{\max} \in [0, 1]$ . Here we focus on two important cases:  $R_D^{\max} = 0$  and  $R_D^{\max} = 1$ .

Starting with  $R_D^{\max} = 0$ , we first consider the scenario where the point estimate remains unchanged, and any increase in the t-statistic occurs solely due to a reduction in the standard error. That is,  $\text{XRVI}_\alpha^0$  quantifies how much variation a control variable  $Z$  that is uncorrelated with  $D$  must explain of the dependent variable  $Y$  in order to overturn a previously insignificant result. This turns out to have a remarkably simple and insightful characterization.

**THEOREM 4** (Closed-form expression for  $\text{XRVI}_\alpha^0$ ). *Let  $f_r > 0$ , then the analytical solution for  $\text{XRVI}_\alpha^0$  is*

$$\text{XRVI}_\alpha^0 = \begin{cases} 0, & \text{if } f_{\alpha, df-1}^* \leq f_r, \\ 1 - \left( \frac{f_r}{f_{\alpha, df-1}^*} \right)^2, & \text{otherwise.} \end{cases}$$

*If  $f_r = 0$ , there is no value of  $R_{Y \sim Z|D\mathbf{X}}^2$  capable of overturning an insignificant result.*

**REMARK 6.** It is useful to understand how  $\text{XRVI}_\alpha^0$  changes as the sample size grows, when keeping the observed t-statistic and the significance level fixed.

$$\text{XRVI}_\alpha^0 \approx 1 - \left( \frac{t_r}{t_{\alpha, df-1}^*} \right)^2 \xrightarrow[df \rightarrow \infty]{} 1 - \left( \frac{t_r}{z_\alpha^*} \right)^2,$$

where here  $\xrightarrow[df \rightarrow \infty]{} \cdot$  denotes the limit as  $df$  goes to infinity and  $z_\alpha^*$  denotes the  $(1 - \alpha/2)$  quantile of the standard normal distribution. In other words, when considering only a reduction in the standard error, the amount of residual variation that  $Z$  must explain of  $Y$  in order to overturn an insignificant result depends solely on the ratio of the observed t-statistic to the critical threshold. Apart from changes in the critical threshold due to degrees of freedom—which eventually converges to  $z_\alpha^*$ —this value remains constant regardless of sample size.

**EXAMPLE 2.** Consider testing the null hypothesis of zero effect with an observed t-statistic of 1 and 100 degrees of freedom. The percentage of residual variation of  $Y$  that  $Z$  needs to explain in order to bring a t-statistic of 1 to the critical threshold of 2, *only through a reduction in the standard error*, is

$$\text{XRVI}_\alpha^0 \approx 1 - \left( \frac{1}{2} \right)^2 = 1 - (1/4) = 3/4 = 75\%.$$

That is, if we are considering a reduction in the standard error alone,  $Z$  needs to explain at least 75% of the variation of  $Y$  in order to elevate the observed t-statistic to 2.

Notably, this number is (virtually) the same across sample sizes, be it  $df = 100$ ,  $df = 1,000$  or  $df = 1,000,000$ . As variables that explain 75% of the variation of  $Y$  are rare in most settings, this simple fact suggests that it should also be rare to see a reversal of significance driven by gains in precision when  $t_r = 1$ . We return to this point in the discussion.

The previous result describes how to achieve statistical significance via precision gains. We now move to the case with  $R_D^{\max} = 1$ . As noted in Remark 3, this is the scenario in which we *impose no constraints* on the strength of association between  $Z$  and  $D$ . In this sense,  $\text{XRVI}_\alpha^1$  computes the *bare minimum* amount of variation that  $Z$  needs to explain of  $Y$  in order to reverse an insignificant result. Any variable that does not explain at least  $(100 \times \text{XRVI}_\alpha^1)\%$  of the variation of  $Y$  is logically incapable of making the t-statistic significant.

**THEOREM 5** (Closed-form expression for  $\text{XRVI}_\alpha^1$ ). *The analytical solution for  $\text{XRVI}_\alpha^1$  is:*

$$\text{XRVI}_\alpha^1 = \begin{cases} 0, & \text{if } f_{\alpha, df-1}^* \leq f_r, \\ \frac{f_{\alpha, df-1}^{*2} - f_r^2}{1 + f_{\alpha, df-1}^{*2}}, & \text{otherwise.} \end{cases}$$

**REMARK 7.** Contrary to the previous case, we observe the following behaviour as the sample size grows,

$$\text{XRVI}_\alpha^1 \approx \left( \frac{t_{\alpha, df-1}^* - t_r}{df + t_{\alpha, df-1}^{*2}} \right) \xrightarrow[df \rightarrow \infty]{} 0.$$

Therefore, if we allow  $Z$  to change point estimates, then for a fixed observed t-statistic, the minimal strength of  $Z$  with  $Y$  to bring about a reversal tends to zero as the sample size grows to infinity.

**EXAMPLE 3.** Consider again testing the null hypothesis of zero effect with an observed t-statistic of 1 and 100 degrees of freedom. If we allow  $Z$  to be arbitrarily associated with  $D$ , it needs only to explain 2.9% of the residual variation of  $Y$  in order to bring the t-statistic to 2:

$$\text{XRVI}_\alpha^1 \approx \frac{2^2 - 1^2}{100 + 2^2} = \frac{3}{104} = 2.9\%.$$

Also note that any  $Z$  that explains less than 2.9% of the variation of  $Y$  is logically incapable of bringing about such change. Corroborating our previous analysis, the  $Z$  that achieves this must do so via an increase in point estimate, and not via a decrease in standard errors. As per Theorem 2, the optimal value of the association with  $D$  is  $R_{D \sim Z|\mathbf{X}}^2 \approx 74\%$ . Notice that  $\text{SEF} \approx 1.94$ , meaning that the inclusion of  $Z$  almost *doubles* the standard error, instead of reducing it. This, however, is compensated by the fact that  $Z$  increases the point estimate by a factor of  $1 + \text{BF} \times \sqrt{df} = 3.88$ , thus doubling the t-statistic despite the loss in precision.

EXAMPLE 4. For the same observed t-statistic of 1, consider a sample size that is an order of magnitude larger, say,  $df = 1,000$ . The minimum residual variation that  $Z$  needs to explain of  $Y$  then reduces to 0.29%:

$$\text{XRVI}_\alpha^1 \approx \frac{2^2 - 1^2}{1000 + 2^2} = \frac{3}{1004} = 0.29\%.$$

As per Theorem 2, this  $Z$  has an association with  $D$  of  $R_{D \sim Z|X}^2 \approx 75\%$ . Here, we have the same situation as before: the standard error doubles while the point estimate increases by a factor of four, thus doubling the t-statistic.

3.2.2 *Robustness value for insignificance.* While in the previous section we investigated the minimal bound on  $R_{Y \sim Z|DX}^2$  alone in order to reverse an insignificant result, here we investigate the minimal bound on both  $R_{Y \sim Z|DX}^2$  and  $R_{D \sim Z|X}^2$  simultaneously.

DEFINITION 3 (Robustness Value for Insignificance). The robustness value for insignificance,  $\text{RVI}_\alpha$ , is the minimum upper bound on both  $R_{Y \sim Z|DX}^2$  and  $R_{D \sim Z|X}^2$  such that  $t_{R^2}^{\max}$  is large enough to reject the null hypothesis  $H_0 : \lambda = \lambda_0$  at specified significance level  $\alpha$ . That is,

$$\text{RVI}_\alpha := \min\{\text{RVI} : t_{\text{RVI}, \text{RVI}}^{\max} \geq t_{\alpha, df-1}^*\}.$$

Note that  $\text{RVI}_\alpha$  provides a convenient summary of the minimum strength of association that  $Z$  needs to have, jointly with  $D$  and  $Y$ , in order to bring about a statistically significant result. Any  $Z$  that has both partial  $R^2$  values no stronger than  $\text{RVI}_\alpha$  cannot reverse a statistically insignificant finding. On the other hand, we can always find a  $Z$  with both partial  $R^2$  values at least as strong as  $\text{RVI}_\alpha$  that does so. The solution of this problem is given in the following result.

THEOREM 6 (Closed-form expression for  $\text{RVI}_\alpha$ ). The analytical solution for  $\text{RVI}_\alpha$  is

$$\text{RVI}_\alpha = \begin{cases} 0, & \text{if } f_{\alpha, df-1}^* \leq f_r, \\ \frac{1}{2}(\sqrt{f_\Delta^4 + 4f_\Delta^2 - f_\Delta^2}), & \text{if } f_r < f_{\alpha, df-1}^* < f_r^{-1}, \\ \text{XRVI}_\alpha^1, & \text{otherwise} \end{cases}$$

where  $f_\Delta := f_{\alpha, df-1}^* - f_r$ .

REMARK 8. The first case in Theorem 6 occurs when the t-statistic for  $H_0 : \lambda = \lambda_0$  is already statistically significant, even when losing one degree of freedom. The second case occurs when both constraints on  $R_{D \sim Z|X}^2$  and  $R_{Y \sim Z|DX}^2$  are binding. The third case is the interior point solution, as defined in Theorem 5, where only the constraint on  $R_{Y \sim Z|DX}^2$  is binding.

Notice that in the second solution of  $\text{RVI}_\alpha$ , we have  $R_{D \sim Z|X}^2 = R_{Y \sim Z|DX}^2$ ; thus,  $\text{SEF} = 1$  and standard errors remain unchanged. Therefore,  $\text{RVI}_\alpha$  represents the minimal strength of  $Z$  needed to achieve statistical significance via a *change in the point estimate alone*, usefully complementing  $\text{XRVI}_\alpha^0$ , which quantifies the minimal strength of  $Z$  needed solely through a reduction in standard errors.

REMARK 9. We recover  $\text{XRVI}_\alpha^1$  as the solution to  $\text{RVI}_\alpha$  if and only if the conditions (1)  $f_{\alpha, df-1}^* > f_r^{-1}$  and (2)  $f_{\alpha, df-1}^* > f_r$  both hold. This rarely occurs. To see this more clearly, note that condition (1) simplifies to  $\sqrt{df-1}\sqrt{df} < t_r \times t_{\alpha, df-1}^*$ , or, approximately,

$$df \lesssim t_r \times t_{\alpha, df-1}^*$$

which only occurs when there are few degrees of freedom, for typical critical thresholds (e.g. 1.96).

REMARK 10. As with  $\text{XRVI}_\alpha^1$ , for fixed  $t_r$  and significance level  $\alpha$ , we observe the same behaviour for  $\text{RVI}_\alpha$  as the sample size grows,

$$\text{RVI}_\alpha \approx \frac{1}{2} \left( \sqrt{\frac{t_\Delta^4}{df^2} + 4\frac{t_\Delta^2}{df} - \frac{t_\Delta^2}{df}} \right) \xrightarrow[\infty]{df} 0,$$

where  $t_\Delta = t_{\alpha, df-1}^* - t_r$ . Therefore, the larger the sample size, any change in the point estimate will eventually be sufficiently strong to bring about statistical significance.

REMARK 11. The statistics we introduced here obey the following ordering,

$$\text{XRVI}_\alpha^1 \leq \text{RVI}_\alpha \leq \text{XRVI}_\alpha^0.$$

This follows directly from their definitions, as each case represents a constrained minimization problem and the constraint becomes stricter as we move from  $R_D^{\max} = 1$  to  $R_D^{\max} = 0$ . Moreover,  $\text{XRVI}_\alpha^{\text{RVI}_\alpha} = \text{RVI}_\alpha$ .

EXAMPLE 5. Continuing with the case where the t-statistic is 1, we obtain  $\text{RVI}_\alpha \approx 9.5\%$  when  $df = 100$  and  $\text{RVI}_\alpha \approx 3\%$  when  $df = 1,000$ . In both cases, the inflation of the t-statistic by a  $Z$  that attains the optimal strength is driven solely by changes in the point estimate.

In the supplementary material we discuss the sampling distribution of these statistics, both under the null and the alternative hypothesis.

#### 4. EMPIRICAL EXAMPLE

We demonstrate the use of our metrics in an empirical example that estimates the effect of a vote-by-mail policy on various outcomes (Amlani and Collitt, 2022). This work includes an analysis of the effect of a US

Estimate	Std. Error	t-statistic	$\text{XRVI}_\alpha^1$	$\text{RVI}_\alpha$	$\text{XRVI}_\alpha^0$
0.103	0.873	0.118	0.089%	2.77%	99.6%

Note:  $df = 4307$ ,  $\lambda_0 = 0$ ,  $\alpha = 0.05$ .

TABLE 1

Robustness values for insignificance for the vote-by-mail policy study.

county's vote-by-mail (VBM) policy on the Republican presidential vote share (dependent variable  $Y$ ) in the 2020 election. There are 5 treatment conditions concerning the VBM policy change from 2016 to 2020, of which the authors are specifically interested in the condition: no-excuse-needed (in 2016) to ballots-sent-in (in 2020), which we will refer to as *condition-1*. The authors fit a differences-in-differences model using OLS, adjusting for various covariates including an indicator for battleground states and the median age and the median income of residents in the county. Notably, the interaction term for *condition-1*  $\times$  *year* (independent variable  $D$ ) is not statistically significant at the 5% level: the t-statistic is  $t_r = 0.12$  with 4,307 degrees of freedom.

#### 4.1 Robustness to unobserved suppressors

The authors were concerned that the lack of significance for the coefficient of interest could have been due to suppression effects of unobserved variables. To address this, they use the formulas of Theorem 1 to examine whether different hypothetical values for the strength of  $Z$  yield a statistically significant t-statistic. Here we complement their analysis by providing the three proposed metrics,  $\text{XRVI}_\alpha^1$ ,  $\text{RVI}_\alpha$  and  $\text{XRVI}_\alpha^0$ .

The results are displayed in Table 1. We find that any latent variable  $Z$  that explains less than 2.77% of the residual variation of both  $Y$  and  $D$  ( $\text{RVI}_\alpha = 2.77\%$ ) would not be sufficiently strong to make the estimate statistically significant. Moreover, if we impose no constraints on  $R_{D \sim Z|X}^2$ , then  $Z$  needs to explain at least 0.089% of the residual variation of  $Y$  in order to attain such a reversal ( $\text{XRVI}_\alpha^1 = 0.089\%$ ). Finally, our analysis shows that a reversal of significance solely due to gains in precision is virtually impossible: a  $Z$  orthogonal to  $D$  would need to explain a remarkable 99.6% of the residual variation of  $Y$  in order to overturn the insignificant result ( $\text{XRVI}_\alpha^0 = 99.6\%$ ).

To put these statistics in context, a latent variable with the same strength of association with  $D$  and  $Y$  as that of the battleground state indicator would only explain 0.6% of the residual variation in  $Y$  and 0.027% of the residual variation in  $D$ . Since both of these numbers are below the  $\text{RVI}_\alpha$  value of 2.77%, we can immediately conclude that adjusting for a latent variable  $Z$  of similar strength to this observed covariate would not be sufficient to overturn the insignificant result.

#### 4.2 Robustness to subsets of controls

We now illustrate how  $t_{R^2}^{\max}$  can be used to understand whether one can easily rule out the possibility of obtaining a statistically significant t-statistic when adjusting for different subsets of control variables. The authors present three model specifications, all of which found the effect of interest to be statistically insignificant. However, could there be a specification where the results turn out to be significant? Here we consider all possible variations between their base model and an expanded model that includes 12 additional control variables. This amounts to  $2^{12} = 4,096$  possible regressions. Applying the results of Corollary 1, we obtain  $t_{R^2}^{\max} \approx 20.7$ , meaning that we cannot rule out that there exists a specification where the interaction term becomes significant. Given the relatively small number of combinations, we can actually compute the ground truth to verify—and, indeed there are 510 models that yield a statistically significant result. As per Remark 1, one could also consider a similar exercise using a critical threshold that accounts for multiple testing.

### 5. DISCUSSION

The algebra of OLS both imposes strong limits on and reveals clear patterns in how reversals of statistical insignificance occur. The first lesson that emerges from our analysis is that reversals of low t-statistics are unlikely to occur through reductions in standard errors alone. As shown in the first row of Table 2, even with a t-statistic of 1.75, one would need to explain at least 20% of the residual variation in  $Y$  to achieve statistical significance at the 5% level solely through a reduction in standard errors. Elevating a t-statistic of .5 to statistical significance requires explaining a remarkable 93% of the residual variation of  $Y$ . Such strong associations with the response variable are not typically common in many empirical applications.

A second consequence of our findings is that, in RCTs, it should be difficult to observe reversals of low t-statistics due to covariate adjustments. Since covariates in such trials typically have zero association with the treatment by design (barring sampling errors), their inclusion is unlikely to significantly shift the point estimate. A back-of-the-envelope calculation illustrates this point: if  $D$  is randomized, then  $df R_{D \sim Z|X}^2$  asymptotically follows a chi-squared distribution with one degree of freedom (see sup-

$t_r$	0.25	0.50	0.75	1.00	1.25	1.50	1.75
$\text{XRVI}_{\alpha=0.05}^0$	0.98	0.93	0.85	0.74	0.59	0.41	0.20
$\text{XRVI}_{\alpha=0.05}^{9.95}$	0.41	0.32	0.24	0.16	0.10	0.05	0.01

Note:  $df = 1000$ ,  $t_{\alpha, df-1}^* \approx 1.96$ .

TABLE 2

Approximate values of  $\text{XRVI}$  for various values of  $t_r$ .

plement). Thus, letting  $q_{.95}$  denote the asymptotic approximation of the 95th percentile for  $R_{D \sim Z|X}^2$ , we can calculate  $\text{XRVI}_{0.05}^{q_{.95}}$  for various values of  $t_r$ . These values are recorded in the second line of Table 2. With the exception of  $t_r = 1.75$  and  $t_r = 1.5$ , the values of  $R_{Y \sim Z|DX}^2$  required to reverse statistical insignificance remain moderate to large, suggesting that such reversals should be uncommon in practice.

Finally, and perhaps counter-intuitively, even when included variables are highly predictive of the response, reversals of insignificance are still typically driven by shifts in the point estimate rather than by reductions in standard errors. To illustrate, consider the usual critical threshold of  $t_{\alpha, df-1}^* \approx 2$  and any observed t-statistic below 1. Then, if  $R_{Y \sim Z|DX}^2 \leq .5$ , it is *impossible* to obtain a reversal that is not mainly driven by changes in point estimate. In other words, any post-mortem analysis of such significance reversals, using decompositions like (5), must necessarily find that the relative change in bias is larger than the relative change in standard errors. Overall, these results closely mirror empirical patterns of reversals of statistical insignificance observed in applied research, such as those documented by Lenz and Sahn (2021), and may offer a purely algebraic explanation for at least some of these patterns.

## APPENDIX: DEFERRED PROOFS

PROOF OF THEOREM 2. From (6) and (7), the magnitude of the t-statistic for  $H_0 : \lambda = \lambda_0$  can be written as a function of  $R_{Y \sim Z|DX}^2$  and  $R_{D \sim Z|X}^2$ ,

$$(10) \quad t(R_{Y \sim Z|DX}^2, R_{D \sim Z|X}^2) = \frac{(\hat{\lambda}_r - \lambda_0) \pm \text{BF} \times \widehat{\text{se}}(\hat{\lambda}_r) \times \sqrt{df}}{\widehat{\text{se}}(\hat{\lambda}_r) \times \text{SEF} \times \sqrt{\frac{df}{df-1}}}.$$

We wish to maximize  $t(R_{Y \sim Z|DX}^2, R_{D \sim Z|X}^2)$  under the posited bounds  $R_{Y \sim Z|DX}^2 \leq R_Y^{\max}$  and  $R_{D \sim Z|X}^2 \leq R_D^{\max}$ . That is, we want to solve the constrained maximization problem,

$$(11) \quad t_{R^2}^{\max} = \max_{R_{Y \sim Z|DX}^2, R_{D \sim Z|X}^2} t(R_{Y \sim Z|DX}^2, R_{D \sim Z|X}^2)$$

such that  $R_{Y \sim Z|DX}^2 \leq R_Y^{\max}$ ,  $R_{D \sim Z|X}^2 \leq R_D^{\max}$ . First notice that we should choose the direction of the bias that increases the magnitude of the difference  $(\hat{\lambda} - \lambda_0)$ . If  $(\hat{\lambda}_r - \lambda_0) > 0$  then we should add the BF term in (10), whereas if  $(\hat{\lambda}_r - \lambda_0) < 0$  then the bias should be subtracted. Both cases yield the same (simplified) objective function as argued below.

Suppose that  $\hat{\lambda}_r > \lambda_0$ . Then the absolute value from (11) can be dropped and the objective function becomes

$$\frac{\hat{\lambda}_r - \lambda_0 + \text{BF} \times \widehat{\text{se}}(\hat{\lambda}_r) \times \sqrt{df}}{\widehat{\text{se}}(\hat{\lambda}_r) \times \text{SEF} \times \sqrt{\frac{df}{df-1}}} = \frac{f_r \times \widehat{\text{se}}(\hat{\lambda}_r) + \text{BF} \times \widehat{\text{se}}(\hat{\lambda}_r)}{\widehat{\text{se}}(\hat{\lambda}_r) \times \text{SEF} \times \sqrt{\frac{1}{df-1}}}.$$

Now suppose that  $\hat{\lambda}_r < \lambda_0$ . We again have,

$$\frac{\lambda_0 - \hat{\lambda}_r + \text{BF} \times \widehat{\text{se}}(\hat{\lambda}_r) \times \sqrt{df}}{\widehat{\text{se}}(\hat{\lambda}_r) \times \text{SEF} \times \sqrt{\frac{df}{df-1}}} = \frac{f_r \times \widehat{\text{se}}(\hat{\lambda}_r) + \text{BF} \times \widehat{\text{se}}(\hat{\lambda}_r)}{\widehat{\text{se}}(\hat{\lambda}_r) \times \text{SEF} \times \sqrt{\frac{1}{df-1}}}.$$

Therefore, after some algebraic manipulation, the maximum t-value in (11) will be of the form,

$$t_{R^2}^{\max} = \frac{f_r \sqrt{1 - R_{D \sim Z|X}^2} + \sqrt{R_{Y \sim Z|DX}^2 R_{D \sim Z|X}^2}}{\sqrt{(1 - R_{Y \sim Z|DX}^2)/(df-1)}}$$

where  $R_{Y \sim Z|DX}^2$ ,  $R_{D \sim Z|X}^2$  are the values of  $R_{Y \sim Z|DX}^2$  and  $R_{D \sim Z|X}^2$  that optimize (11).

We now find analytical expressions for the optimizers  $R_{Y \sim Z|DX}^2$ ,  $R_{D \sim Z|X}^2$ . In what follows we write  $t$  for the objective function with the understanding that it is written in its modified form above. The partial derivative of  $\frac{t}{\sqrt{df-1}}$  with respect to  $R_{Y \sim Z|DX}^2$  is

$$\frac{\partial t / \sqrt{df-1}}{\partial R_{Y \sim Z|DX}^2} = \frac{f_r \sqrt{(1 - R_{D \sim Z|X}^2) R_{Y \sim Z|DX}^2} + \sqrt{R_{D \sim Z|X}^2}}{2(1 - R_{Y \sim Z|DX}^2)^{\frac{3}{2}} \sqrt{R_{Y \sim Z|DX}^2}}.$$

Since the partial of  $t$  with respect to  $R_{Y \sim Z|DX}^2$  is always positive, we have  $R_{Y \sim Z|DX}^2 = R_Y^{\max}$  unconditionally, i.e.  $R_{Y \sim Z|DX}^2$  always lies on the boundary.

We now turn to the partial derivative of  $\frac{t}{\sqrt{df-1}}$  with respect to  $R_{D \sim Z|X}^2$ :

$$\frac{\partial t / \sqrt{df-1}}{\partial R_{D \sim Z|X}^2} = \frac{-f_r \sqrt{R_{D \sim Z|X}^2} + \sqrt{R_{Y \sim Z|DX}^2 (1 - R_{D \sim Z|X}^2)}}{2 \sqrt{(1 - R_{Y \sim Z|DX}^2) (1 - R_{D \sim Z|X}^2) R_{D \sim Z|X}^2}}.$$

It is straightforward to check that the second derivative of  $\frac{t}{\sqrt{df-1}}$  is negative with respect to  $R_{D \sim Z|X}^2$ . Thus, when attainable, the zero of the first partial derivative with respect to  $R_{D \sim Z|X}^2$  is a maximizer. Solving for the value that makes the first derivative zero yields:

$$(12) \quad R_{D \sim Z|X}^2 = \frac{R_Y^{\max}}{f_r^2 + R_Y^{\max}}.$$

This interior point solution is only feasible when  $f_r^2 \geq R_Y^{\max} (1 - R_D^{\max}) / R_D^{\max}$ . Otherwise, if

$$(13) \quad f_r^2 < R_Y^{\max} (1 - R_D^{\max}) / R_D^{\max},$$

then the partial with respect to  $R_{D \sim Z|X}^2$  is strictly positive for all  $R_{D \sim Z|X}^2 \leq R_D^{\max}$  and so we obtain the boundary solution  $R_{D \sim Z|X}^2 = R_D^{\max}$ .  $\square$

PROOF OF THEOREM 3. Let  $\mathbf{Z}$  denote an  $(n \times m)$  matrix of unobserved covariates and let  $\hat{\gamma}$  denote the coefficient vector of  $\mathbf{Z}$ . We are now interested in the long regression

$$(14) \quad Y = \hat{\lambda}D + \mathbf{X}\hat{\beta} + \mathbf{Z}\hat{\gamma} + \hat{\epsilon}.$$

Consider the  $(n \times 1)$  vector  $Z_L := \mathbf{Z}\hat{\gamma}$ . The regression

$$(15) \quad Y = \hat{\lambda}D + \mathbf{X}\hat{\beta} + Z_L + \hat{\epsilon}$$



yields the same value for  $\hat{\lambda}$ ; therefore, the bias induced by  $\mathbf{Z}$  is equal to that induced by  $Z_L$  and  $R_{Y \sim Z_L | D, \mathbf{X}}^2 = R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2$ . On the other hand, since  $\hat{\gamma}$  is chosen solely to maximize  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2$ , we also have that  $R_{D \sim Z_L | \mathbf{X}}^2 \leq R_{D \sim \mathbf{Z} | \mathbf{X}}^2$ . Now observe that the standard error formula from (7) holds for multivariate  $\mathbf{Z}$  if we correctly adjust for the degrees of freedom (Cinelli and Hazlett, 2020). Further note that the bias induced by  $Z_L$  is a strictly increasing function of  $R_{D \sim Z_L | \mathbf{X}}^2$ . Thus, the most adversarial choice of  $\mathbf{Z}$  is such that  $R_{D \sim Z_L | \mathbf{X}}^2 = R_{D \sim \mathbf{Z} | \mathbf{X}}^2$ . We can thus assess the maximum t-statistic of a matrix  $\mathbf{Z}$  by considering that of a single adversarial vector  $Z_L$  and further adjusting for the degrees of freedom.  $\square$

PROOF OF COROLLARY 1. For any subset  $\mathbf{X}_S$  of the columns of observed covariate matrix  $\mathbf{X}$ , recall that  $R_{Y \sim \mathbf{X}_S | D}^2 \leq R_{Y \sim \mathbf{X} | D}^2$  and  $R_{D \sim \mathbf{X}_S}^2 \leq R_{D \sim \mathbf{X}}^2$ . Now apply the proof of Theorem 3 with the alteration that the constraint  $R_{D \sim Z_L | \mathbf{X}}^2 \leq R_{D \sim \mathbf{Z} | \mathbf{X}}^2$  is not necessarily tight, since  $Z_L$  may not be adversarial for a specific dataset.  $\square$

For all cases below, consider the following condition for significance:

$$(16) \quad t_{\alpha, df-1}^* \leq t_{R^2}^{\max}.$$

PROOF OF THEOREM 4. First consider the case in which  $f_r = 0$ . This only happens if  $\hat{\lambda}_r = \lambda_0$ . Since here  $R_{D \sim \mathbf{Z} | \mathbf{X}}^2 = 0$ , the inclusion of  $Z$  does not alter the point estimate and we still obtain  $\hat{\lambda} = \lambda_0$  after adjusting for  $Z$ . Therefore, the adjusted t-statistic will be zero regardless of the value of the standard error.

If  $f_r > f_{\alpha, df-1}^*$  then we are already able to reject  $H_0$  even if  $Z$  has zero explanatory power. Otherwise, given the constraints  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2 \leq \text{XRVI}$  and  $R_{D \sim \mathbf{Z} | \mathbf{X}}^2 = 0$ , the expression for  $t_{R^2}^{\max}$  simplifies to

$$t_{\text{XRVI}, 0}^{\max} = \max_{R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2} \frac{f_r}{\sqrt{1 - R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2}} \sqrt{df-1}$$

such that  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2 \leq \text{XRVI}$ . This is a strictly increasing function of  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2$  and thus attains its maximum at  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2 = \text{XRVI}$ . Thus solving for the minimum value of XRVI that satisfies (16) is equivalent to solving for XRVI at the equality. That is,

$$f_{\alpha, df-1}^* = \frac{f_r}{\sqrt{1 - \text{XRVI}_{\alpha}^0}}.$$

Squaring and rearranging terms, we obtain

$$\text{XRVI}_{\alpha}^0 = 1 - \left( \frac{f_r}{f_{\alpha, df-1}^*} \right)^2,$$

as desired.  $\square$

PROOF OF THEOREM 5. If  $f_r > f_{\alpha, df-1}^*$  then we are already able to reject  $H_0$  even if  $Z$  has zero explanatory power, and thus the minimal strength to reject  $H_0$  is zero. Otherwise, consider constraints  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2 \leq \text{XRVI}$  and  $R_{D \sim \mathbf{Z} | \mathbf{X}}^2 \leq 1$ . From the proof of Theorem 2 we see that  $t_{R^2}^{\max}$  is an increasing function of XRVI. Thus solving for the minimum value of XRVI that satisfies (16) is equivalent to solving for XRVI at the equality. Notice that (13) is not satisfied here. We therefore plug in the interior point solution from Theorem 2 for  $t_{R^2}^{\max}$  and solve for  $\text{XRVI}_{\alpha}^1$ . This results in the equation

$$f_{\alpha, df-1}^* = \sqrt{\frac{f_r^2 + \text{XRVI}_{\alpha}^1}{1 - \text{XRVI}_{\alpha}^1}}$$

which yields the solution

$$\text{XRVI}_{\alpha}^1 = \frac{f_{\alpha, df-1}^{*2} - f_r^2}{1 + f_{\alpha, df-1}^{*2}},$$

as desired.  $\square$

THEOREM 7 (Closed-form expression for  $\text{XRVI}_{\alpha}^{R_D^{\max}}$ ). Let  $R_D^{\max} > 0$  or  $f_r > 0$ . Then the analytical expression for  $\text{XRVI}_{\alpha}^{R_D^{\max}}$  is

$$\text{XRVI}_{\alpha}^{R_D^{\max}} = \begin{cases} 0, & \text{if } q \leq 0, \\ \text{XRVI}_{\alpha}^1, & \text{if } 0 < q \leq R_D^{\max}, \\ \frac{-b + s\sqrt{b^2 - 4ac}}{2a}, & \text{otherwise,} \end{cases}$$

where

$$(17) \quad a = 1,$$

$$(18) \quad b = -2 \left[ \frac{f_{\alpha, df-1}^{*2} - (1 - R_D^{\max})f_r^2}{f_{\alpha, df-1}^{*2} + R_D^{\max}} + \frac{2f_r^2(1 - R_D^{\max})R_D^{\max}}{(f_{\alpha, df-1}^{*2} + R_D^{\max})^2} \right],$$

$$(19) \quad c = \left[ \frac{f_{\alpha, df-1}^{*2} - (1 - R_D^{\max})f_r^2}{f_{\alpha, df-1}^{*2} + R_D^{\max}} \right]^2,$$

$$(20) \quad q = \frac{f_{\alpha, df-1}^{*2} - f_r^2}{f_{\alpha, df-1}^{*2}(1 + f_r^2)},$$

and  $s \in \{-1, 1\}$  is chosen to yield the valid quadratic root, i.e.,  $t_{\text{XRVI}, R_D^{\max}}^{\max} = t_{\alpha, df-1}^*$ . If  $R_D^{\max} = 0$  and  $f_r = 0$ , there is no value of  $R_{Y \sim \mathbf{Z} | D, \mathbf{X}}^2$  capable of overturning an insignificant result.

PROOF OF THEOREM 7. As argued in the proofs of Theorems 4 and 5, if  $f_{\alpha, df-1}^* \leq f_r$  (which implies  $q \leq 0$ ), then the minimum strength of  $Z$  necessary to attain significance is zero. If  $f_r = 0$  and  $R_D^{\max} = 0$  then by Theorem 4, we cannot overturn an insignificant result. Otherwise, solving for the minimum value of XRVI that satisfies (16) is equivalent to solving for XRVI at the equality.

Now let  $f_r > 0$ . Here we have two cases. Either both  $R^2$  values reach the bound or the optimal value of  $R_{D \sim \mathbf{Z} | \mathbf{X}}^2$  is

an interior point. For the latter case, this means the constraint  $R_{D \sim Z|X}^2$  is not binding and thus  $XRVI_\alpha^{R_D^{\max}}$  should equal  $XRVI_\alpha^1$ . Recall that  $t$  is concave down with respect to  $R_D^{\max}$ ; therefore, the optimal value of  $R_{D \sim Z|X}^2$  is the interior point solution  $R_{D \sim Z|X}^{*2}$ , as defined in (12), if and only if  $R_{D \sim Z|X}^{*2} \leq R_D^{\max}$ . We can simplify this condition to be of the form,

$$q := \frac{f_{\alpha,df-1}^{*2} - f_r^2}{f_{\alpha,df-1}^{*2}(1 + f_r^2)} \leq R_D^{\max}.$$

It remains to solve for the case where both coordinates reach the bound. Here, the equality in (16) simplifies to

$$(21) \quad t_{\alpha,df-1}^* = \frac{f_r \sqrt{1 - R_D^{\max}} + \sqrt{R_D^{\max} XRVI}}{\sqrt{(1 - XRVI)/(df - 1)}}.$$

We now solve for  $XRVI$  in (21) by squaring both sides and taking the valid root of the quadratic equation. The simplified quadratic form is

$$(22) \quad XRVI^2 - \left[ \frac{2(f_{\alpha,df-1}^{*2} - (1 - R_D^{\max})f_r^2)}{f_{\alpha,df-1}^{*2} + R_D^{\max}} + \frac{4f_r^2(1 - R_D^{\max})R_D^{\max}}{(f_{\alpha,df-1}^{*2} + R_D^{\max})^2} \right] XRVI + \left[ \frac{f_{\alpha,df-1}^{*2} - (1 - R_D^{\max})f_r^2}{f_{\alpha,df-1}^{*2} + R_D^{\max}} \right]^2 = 0.$$

The expressions for  $a, b, c$  in (17)-(19) immediately follow from Sridharacharya-Bhaskara's formula for quadratic equations. Finally, we note that if  $f_r = 0$  and  $R_D^{\max} > 0$  then the  $XRVI^{R_D^{\max}}$  solution is valid.  $\square$

**PROOF OF THEOREM 6.** The derivation of  $RVI_\alpha$  follows that of  $XRVI_\alpha^{R_D^{\max}}$  very closely (see proof of Theorem 7). The only difference is that when  $f_r^2 < 1 - RVI$ , the equality in (16) simplifies to

$$t_{\alpha,df-1}^* = \frac{f_r \sqrt{1 - RVI} + RVI}{\sqrt{1 - RVI}} \times \sqrt{df - 1}$$

which is equivalent to

$$(23) \quad f_{\alpha,df-1}^* = f_r + \frac{RVI}{\sqrt{1 - RVI}}.$$

Let  $f_\Delta = f_{\alpha,df-1}^* - f_r$ . Then (23) further reduces to a quadratic function of  $RVI$  with positive root

$$RVI = \frac{1}{2}(\sqrt{f_\Delta^4 + 4f_\Delta^2} - f_\Delta^2).$$

It remains to show that (13), i.e.  $f_r^2 \geq 1 - RVI$ , is equivalent to

$$(24) \quad f_r > \frac{1}{f_{\alpha,df-1}^*}.$$

This is equivalent to showing that  $RVI_\alpha$  is given by the interior point solution if and only if (24) holds. To see why, recall that for the interior point solution,

$$(25) \quad R_Y^{\max} = R_D^{\max} = RVI_\alpha = \frac{f_{\alpha,df-1}^{*2} - f_r^2}{1 + f_{\alpha,df-1}^{*2}}.$$

Plugging (25) into (13), we have

$$f_r^2 \geq 1 - RVI_\alpha = 1 - \frac{f_{\alpha,df-1}^{*2} - f_r^2}{1 + f_{\alpha,df-1}^{*2}}$$

which reduces to  $f_{\alpha,df-1}^* \geq \frac{1}{f_r}$ .  $\square$

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## SUPPLEMENTARY MATERIAL

### Online Supplement

Section 1 provides a comparison of  $RVI_\alpha$  and  $XRVI_\alpha$  with  $RV_\alpha$ ,  $XR_V_\alpha$  from Cinelli and Hazlett (2020, 2025). Section 2 derives the sampling distributions of  $RVI_\alpha$  and  $XRVI_\alpha$  both under the null and alternative hypothesis. Section 3 derives the sampling distribution of  $R_{D \sim Z|X}^2$  when the treatment  $D$  is randomized.

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# Online Supplement

On the minimum strength of (unobserved) covariates to overturn an insignificant result

## 1 Comparison of $\text{RVI}_\alpha$ and $\text{XRVI}_\alpha$ with $\text{RV}_\alpha$ , $\text{XRV}_\alpha$ from Cinelli and Hazlett (2020, 2025)

In this section we show how  $\text{RVI}_\alpha$  and  $\text{XRVI}_\alpha^1$  are “mirrored” versions of the metrics  $\text{RV}_{q,\alpha}$  and  $\text{XRV}_{q,\alpha}$  that were introduced by Cinelli and Hazlett (2020, 2025); that is,  $\text{RV}_{q,\alpha}$  and  $\text{XRV}_{q,\alpha}$  address the mirrored problem of the current paper, where  $H_0 : \lambda = (1 - q)\hat{\lambda}_r$  (generalized here to  $H_0 : \lambda = \lambda_0$ ) is significant and the goal is to find the minimum strength of confounder  $Z$  that renders  $H_0$  insignificant. In fact, the idea for writing the current paper arose from readers’ inquiries to Cinelli and Hazlett (2020) about how to perform sensitivity analysis for insignificant, rather than significant, results. For completeness, here we provide a side-by-side comparison of the two sets of metrics.

In the setting of the current paper, our metrics are derived by first finding the general form of the maximum adjusted t-statistic with respect to arbitrary constraints on the strength of the confounder:

$$t_{\mathbf{R}^2}^{\max} := \max_{R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2} t(R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2) \quad \text{s.t. } R_{Y \sim Z|D\mathbf{X}}^2 \leq R_Y^{\max}, R_{D \sim Z|\mathbf{X}}^2 \leq R_D^{\max}.$$

The metrics  $\text{RVI}_\alpha$  and  $\text{XRVI}_\alpha^1$  follow from solving for the minimum partial  $R^2$  values such that  $H_0$  is rejected using the adjusted t-statistic:

$$\text{XRVI}_\alpha^1 := \min\{\text{XRVI} : t_{\text{XRVI},1}^{\max} \geq t_{\alpha,df-1}^*\} = \begin{cases} 0, & \text{if } f_{\alpha,df-1}^* \leq f_r, \\ \frac{f_{\alpha,df-1}^{*2} - f_r^2}{1 + f_{\alpha,df-1}^{*2}}, & \text{otherwise.} \end{cases}$$

and

$$\text{RVI}_\alpha := \min\{\text{RVI} : t_{\text{RVI},\text{RVI}}^{\max} \geq t_{\alpha,df-1}^*\} = \begin{cases} 0, & \text{if } f_{\alpha,df-1}^* \leq f_r, \\ \frac{1}{2}(\sqrt{f_\Delta^4 + 4f_\Delta^2 - f_\Delta^2}), & \text{if } f_r < f_{\alpha,df-1}^* < f_r^{-1}, \\ \text{XRVI}_\alpha^1, & \text{otherwise} \end{cases}$$

where  $f_\Delta := f_{\alpha,df-1}^* - f_r$ .

In contrast, the metrics provided by Cinelli and Hazlett (2020, 2025) can be derived by finding the general form of the *minimum* bias-adjusted t-statistic with respect to the same arbitrary constraints, where we replace  $f_q$  and  $H_0 : \lambda = (1 - q)\hat{\lambda}_r$  with the more general forms  $f_r$  and  $H_0 : \lambda = \lambda_0$ :

$$t_{\mathbf{R}^2}^{\min} := \min_{R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2} t(R_{Y \sim Z|D\mathbf{X}}^2, R_{D \sim Z|\mathbf{X}}^2) \quad \text{s.t. } R_{Y \sim Z|D\mathbf{X}}^2 \leq R_Y^{\max}, R_{D \sim Z|\mathbf{X}}^2 \leq R_D^{\max}.$$

$RV_{q,\alpha}$  and  $XR V_{q,\alpha}$  are similarly defined as above, where we henceforth omit the subscript  $q$  to maintain notational consistency with our general formulation with null hypothesis  $H_0 : \lambda = \lambda_0$ , and we include the 1 explicitly in  $XR V$ :

$$XR V_{\alpha}^1 := \min\{XR V : t_{1, XR V}^{\min} \leq t_{\alpha, df-1}^*\} = \begin{cases} 0, & \text{if } f_r \leq f_{\alpha, df-1}^*, \\ \frac{f_r^2 - f_{\alpha, df-1}^{*2}}{1 + f_r^2}, & \text{otherwise} \end{cases}$$

and

$$RV_{\alpha} := \min\{RV : t_{RV, RV}^{\min} \leq t_{\alpha, df-1}^*\} = \begin{cases} 0, & \text{if } f_r \leq f_{\alpha, df-1}^*, \\ \frac{1}{2}(\sqrt{f_{\Delta}^4 + 4f_{\Delta}^2} - f_{\Delta}^2), & \text{if } f_{\alpha, df-1}^* < f_r < f_{\alpha, df-1}^{*-1}, \\ XR V_{\alpha}, & \text{otherwise} \end{cases}$$

where again  $f_{\Delta} := f_{\alpha, df-1}^* - f_r$ . Note how  $RV I_{\alpha}$  and  $XR VI_{\alpha}^1$  “mirror”  $RV_{q,\alpha}$  and  $XR V_{q,\alpha}$  in the sense that their definitions and derivations are symmetric, and their solutions can be obtained by swapping the roles of  $f_r$  for  $f_{\alpha, df-1}^*$  and vice-versa.

For completeness, we also derive the analogous metric to  $XR VI_{\alpha}^0$  in the setting of Cinelli and Hazlett (2020, 2025) where  $H_0$  is significant. We obtain,

$$XR V_{\alpha}^0 = \min\{XR V : t_{0, XR V}^{\min} \leq t_{\alpha, df-1}^*\} = \begin{cases} 0, & \text{if } f_r \leq f_{\alpha, df-1}^*, \\ 1 - \left(\frac{f_{\alpha, df-1}^*}{f_r}\right)^2, & \text{otherwise.} \end{cases}$$

In words,  $XR V_{\alpha}^0$  is the minimum strength that a variable  $Z$  that solely acts on  $D$  must have such that adjusting for  $Z$  would yield an insignificant result. This is indeed the mirrored version of  $XR VI_{\alpha}^0$ , with the roles of  $f_r$  and  $f_{\alpha, df-1}^*$  reversed,

$$XR VI_{\alpha}^0 := \min\{XR VI : t_{XR VI, 0}^{\max} \geq t_{\alpha, df-1}^*\} = \begin{cases} 0, & \text{if } f_{\alpha, df-1}^* \leq f_r, \\ 1 - \left(\frac{f_r}{f_{\alpha, df-1}^*}\right)^2, & \text{otherwise.} \end{cases}$$

Note that the interpretation of the statistic is also very different, as  $XR V_{\alpha}^0$  quantifies the minimal strength of a  $Z$  that is solely related to  $D$  whereas  $XR VI_{\alpha}^0$  quantifies the minimal strength of a  $Z$  that is solely related to  $Y$ . Thus,  $XR V_{\alpha}^0$  seems like a less useful metric than  $XR VI_{\alpha}^0$ ; rarely would an investigator wish to control for a variable that is not related to the outcome.

## 2 Distribution of $RV I_{\alpha}$ and $XR VI_{\alpha}$ under the null and alternative

In what follows,  $\xrightarrow{d}$  denotes convergence in distribution,  $t(b)$  denotes Student’s t-distribution with  $b$  degrees of freedom,  $F(a, b)$  denotes the F-distribution with  $a$  and  $b$  degrees of freedom,  $\chi^2(b)$  denotes the chi-squared distribution with  $b$  degrees of freedom, and  $Beta(c, d)$  denotes the Beta distribution with shape parameters  $c$  and  $d$ .

*Exact distribution of  $XR VI_{\alpha}^1$  and  $XR VI_{\alpha}^0$  under  $H_0 : \lambda_r = \lambda_0$  and  $t_r \sim t(df)$ .* When  $t_r \sim t(df)$ , we then have  $t_r^2 \sim F(1, df)$ . It then immediately follows that

$$XR VI_{\alpha}^0 = \max\left(0, 1 - \left(\frac{f_r}{f_{\alpha, df-1}^*}\right)^2\right) = \max(0, 1 - U)$$



where  $U$  follows a scaled  $F(1, df)$  distribution. Similarly,

$$\text{XRVI}_\alpha^1 = \max \left( 0, \frac{f_{\alpha, df-1}^{*2} - f_r^2}{1 + f_{\alpha, df-1}^{*2}} \right) = \max(0, 1 - V)$$

where  $V$  follows a scaled and shifted  $F(1, df)$  distribution.  $\square$

*Asymptotic distributions of  $\text{XRVI}_\alpha^0$ ,  $\text{XRVI}_\alpha^1$  and  $\text{RVI}_\alpha$  under  $H_0 : \lambda_r = \lambda_0$  and  $t_r \xrightarrow{d} N(0, 1)$ .* When  $t_r \xrightarrow{d} N(0, 1)$ , we have

$$\text{XRVI}_\alpha^0 = \max \left( 0, 1 - \left( \frac{f_r}{f_{\alpha, df-1}^*} \right)^2 \right) \xrightarrow{d} \max \left( 0, 1 - \frac{C}{(z^*)^2} \right)$$

where  $t_{\alpha, df-1}^* \rightarrow z_\alpha^*$  and  $C \sim \chi^2(1)$ .

In contrast, the other two metrics converge in probability to zero. Asymptotically, the non-zero component of  $\text{XRVI}_\alpha^1$  behaves the same (up to a fractional difference of  $df/(df-1)$ ) as the following expression:

$$\frac{t_{\alpha, df-1}^{*2} - t_r^2}{df + t_{\alpha, df-1}^{*2}} = \frac{t_{\alpha, df-1}^{*2}}{df + t_{\alpha, df-1}^{*2}} - \frac{1}{df + t_{\alpha, df-1}^{*2}} \times t_r^2.$$

It follows that

$$\text{XRVI}_\alpha^1 = o_p(1) + o_p(1) \times O_p(1) = o_p(1).$$

Finally,  $\text{RVI}_\alpha$  simplifies for sufficiently large  $n$  as follows:

$$\text{RVI}_\alpha \approx \max \left( 0, \frac{1}{2} \sqrt{\frac{t_\Delta^4}{df^2} + 4 \frac{t_\Delta^2}{df}} - \frac{t_\Delta^2}{df} \right).$$

Since  $t_\Delta^2 = O_p(1)$  under  $H_0$ , the non-zero component of  $\text{RVI}_\alpha$  is  $O_p(1/\sqrt{df})$ .  $\square$

*Asymptotic distribution of  $\text{XRVI}_\alpha^0$ ,  $\text{XRVI}_\alpha^1$  and  $\text{RVI}_\alpha$  under  $H_1 : \lambda_r \neq \lambda_0$ .* When the test is consistent against a fixed alternative, the probability of obtaining a significant result converges to one. Thus  $\text{XRVI}_\alpha^0$ ,  $\text{XRVI}_\alpha^1$  and  $\text{RVI}_\alpha$  all converge in probability to zero.  $\square$

### 3 Distribution of $R_{D \sim \mathbf{Z} | \mathbf{X}}^2$ under randomization of $D$

Assume we have i.i.d data. Let  $\mathbf{Z}$  be an  $n \times m$  matrix. Recall that

$$R_{D \sim \mathbf{Z} | \mathbf{X}}^2 = \frac{mF}{df + mF}$$

where  $F$  denotes the F statistic to test whether the coefficients of  $\mathbf{Z}$  in the regression of  $D$  on  $\mathbf{Z}$  and  $\mathbf{X}$  are zero. Now note the randomization of  $D$  trivially implies both linearity of the conditional expectation function of  $D$ , since  $E[D_i | \mathbf{Z}_i, \mathbf{X}_i] = E[D_i]$ , and homoskedasticity, since  $\text{Var}(D_i | \mathbf{Z}_i, \mathbf{X}_i) = \text{Var}(D_i)$ . We can thus leverage standard textbook results on the exact and asymptotic distribution of  $F$ , under the classical linear regression model setting (e.g., [Hansen, 2022](#)), to derive the distribution of  $R_{D \sim \mathbf{Z} | \mathbf{X}}^2$ . First, if  $D_i$  is normally distributed then we have the well known exact result,  $F \sim F(m, df)$ . This yields the finite-sample result

$$R_{D \sim \mathbf{Z} | \mathbf{X}}^2 \sim \text{Beta}(m/2, df/2).$$

More generally, regardless of normality,  $mF \xrightarrow{d} \chi^2(m)$ . The asymptotic distribution of  $df R_{D \sim \mathbf{Z} | \mathbf{X}}^2$  then follows by Slutsky's theorem.