

1 A stochastic programming model for systemic financial resiliency

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6 I Problem

7 Consider a network $\mathcal{G} = (V, E)$ consisting on a set of n nodes, $V = \{1, \dots, n\}$, and a set of m undirected edges
 8 $\{e_{ij}\} \in E$. Each node i represents one of the institutions identity, and each edge e_{ij} represents the *correlation* or
 9 contagion factor between two entities. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space ...

10 Consider a two-stage model, where on the first stage there is a random shock happening on the nodes. Let
 11 ξ_i a Bernoulli random variable such that $\xi_i^0 \in \{0, 1\}$, $i = 1, \dots, n$ represents the *distress state* of node- i on the
 12 network. On the other hand, the second stage captures the behavior of the *shock's propagation* over the network.
 13 In order to define this, consider the stochastic process P modeling the probability of contagion of a node, given
 14 that one of its neighbor is distressed, i.e., if $\xi_i^1 \in \{0, 1\}$ represents the distress state of node i in the second stage,
 15 then

$$P_{ij} = \mathbb{P}\{\xi_i^1(\cdot) = 1 \mid \xi_j^0 = 1\} \quad e_{ij} \in E, \forall i, j \in V$$

16 The problem is now to minimize the total cost of the system under shocks on the network. For this, the regu-
 17 lator is set to solve the problem of minimizing an overall cost, consisting on implementation cost and contagion
 18 cost, by deciding an optimal capital requirement. Let x^0 be the decision policy, $x^0 \in [0, 1]^n$ such that x_i^0 represents
 19 the policy required at entity i , and $x^1(\cdot)$ be a decision policy regarding the second stage (not sure if needed or not).
 20 The optimization problem is given by

$$(\mathcal{P}) \quad \begin{array}{ll} \min_{\{x^0, x^1(\cdot)\}} & \varphi^0(x^0) + \mathbb{E}\{\varphi^1(\cdot, x^0, x^1(\cdot))\} \\ \text{such that} & f^0(x^1) \leq 0 \\ & f^1(x^0, x^1(\omega), \omega) \leq 0, \omega - \text{a.s.} \\ & x^0 \in [0, 1]^n, x^1 : \Omega \rightarrow \mathbb{R}^N \end{array}$$

21 where φ^0 is the total cost of implementing a capital requirement policy, and φ^1 is the total cost of the second stage
 22 (probably related to the contagion cost). Here, the network constraints are included in the constraints $\{f^0, f^1\}$

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and for a random realization ω and a given policy x^1 , the cost $\varphi(\omega, x^0, x^1(\omega))$ should reflect the cost of the contagion on the system. For example, one can be interested in minimizing the expected cost of the contagion, but it is easy to incorporate a risk-measure for minimizing, for example, a measure like $C - Var$ of the tail of the distribution of distressed nodes.

2 Tue, Feb 20th

We explore a model with the following features

1. Consider a graph $G = (N, E)$, where each node represents a financial institution, and each edge reflects financial transactions between two institutions.
2. Institution i faces a financial shock, represented as ε_i , which impacts its assets over liabilities ratio, defined as $r_i = \frac{A_i}{L_i}$,¹. Additionally, we consider that a financial institution is under *distress* if its ratio is under a (given) threshold $\lambda \in (0, 1)$. Thus,

$$i \text{ under distress} \iff r_i(1 - \varepsilon_i) < \lambda$$

3. There is a central decision maker, focus on the stability of the system. We discussed the information that is available to this regulator, and propose a mechanism to oversee the overall stability within the financial network thorough a constraint over the ratio, given by x_i .
4. The financial institutions decide their ratio by maximizing their profits², given by a function π_i , with the minimum level of A/L ratio, i.e.,

$$r_i(x_i; p) \in \operatorname{argmax}_r \{ \mathbb{E}^p \{ \pi(r) \} \mid r \geq x_i, r \in R_i \}$$

Additionally, assuming that the function π_i is nonincreasing on r (and no further restrictions are imposed), the individual solution to this problem is given by $r_i^* = x_i$, i.e., the financial institution sets its ratio at minimum possible level.

5. There is contagion on the network, described in its stationary state as follows: if institution i gets distressed, there is a probability p that it affects its immediate neighbor, p^2 by a 2-edge neighbors, and so on. Defining the set $\{j \rightarrow i\}$ as the set of all possible simple paths coming to node i , and $d(j, i)$ the distance between j and i (amount of edges between them), the expected shock [Julio: Assuming that there is no amplification of shocks](#) is given by

$$\varepsilon_i = \sum_{j \rightarrow i} p^{d(i,j)} \varepsilon_j$$

and by defining the matrix $A_{ij} = \sum_{j \rightarrow i} p^{d(i,j)}$, the acceptable shocks are the solution of the eigen problem for the matrix A . Moreover, we interpret A as an stochastic (transition) matrix by enlarging it with an extra

¹Capital?

²utility?

49 *not distressed* node as follows,

$$\tilde{A} = \left[\begin{array}{c|cccc|c} & 1 & 2 & \dots & n & ND \\ \hline 1 & 0 & \sum_{2 \rightarrow 1} p^{d(2,1)} & \dots & \sum_{n \rightarrow 1} p^{d(n,1)} & 1 - \sum \mathcal{A}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ n & \sum_{n \rightarrow 1} p^{d(n,1)} & \vdots & \dots & 0 & 1 - \sum \mathcal{A}_1 \\ \hline ND & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

50 This is a stochastic matrix, and thus, it has an eigenvalue with value 1, wich associated eigenvector ε^0 . Let's
51 consider the first n components as acceptable shocks ε^0 for the corresponding nodes.

- Finally, consider the optimization problem solved by the central planner: set the ratio level [Julio: sth about the condition previously stated](#), such that it minimizes the total amount of financial institutions under distress. Let $y_i \in \{0, 1\}$ a binary variable such that $y_i = 1$ if insititution i is under distress or $y_i = 0$ otherwise, and let $M > 0$ large enough such that

$$\min_{x,y} \sum_{i=1}^n y_i + \varphi(x, y) \quad (1)$$

$$\text{such that } r_i(x_i)(1 - \varepsilon_i^0) - \lambda \leq M(1 - y_i), \quad i = 1, \dots, N \quad (2)$$

52 where φ is a cost function associated to the policy x and the instituions on distress. Note that this formu-
53 lation depends on p and the topology of the network through the selection of the ε^0 .

- The optimization problem 1 can have a robust formulation by considering an ambiguity set for the parameter p , thus

$$\min_{x,y} \sup_{p \in \mathcal{A}(p_0)} \sum_{i=1}^n y_i + \varphi(x, y) \quad (3)$$

$$\text{such that } r_i(x_i)(1 - \varepsilon_i^0) - \lambda \leq M(1 - y_i), \quad i = 1, \dots, N \quad (4)$$

- Finally, we are looking for a representative agent formulation of the benevolent social planner problem, such that the solutions of both problems coincide. For example, one wild guess is to consider the formulation proposed in [?] [Julio: Citation needed](#), where ambiguity is considered as a family of possible models for the parameter p , along with a probability distribution over these models, α . Therefore, the central planner problem has the following form

$$\max_x \left\{ \mathbb{E}^p \left\{ \sum_i u_i(x_i) \right\} - \frac{\mu}{2} \text{Var}^p \left(\sum_i u_i(x_i) \right) - \frac{\vartheta}{2} \text{Var}_\alpha \mathbb{E} \left(\sum_i u_i(x_i) \right) \right\}$$

59 3 Wed, Feb 21st

60 Continuing the idea of a (benevolent) central planner, let's define the following modified utility functions for
61 each agents:

$$\tilde{\pi}_i = \begin{cases} \pi_i & \text{institution } i \text{ operates normally} \\ 0 & \text{institution is on distress} \end{cases}$$

Therefore, we expect that the Central Planner solves the following problem

$$\max_x \left\{ \mathbb{E}^p \left\{ \sum_i \tilde{\pi}_i(x_i) \right\} - \frac{\mu}{2} \text{Var}^p \left(\sum_i \tilde{\pi}_i(x_i) \right) - \frac{\vartheta}{2} \text{Var}_\alpha \mathbb{E} \left(\sum_i \tilde{\pi}_i(x_i) \right) \right\}$$

Additionally, we modify the assumptions over the spread of the distress condition

3.1 Assumption (initial shocks) *Initial shock only affects one institution (node)*

3.2 Assumption (propagation) *Propagation occurs over simple paths*

Under Assumptions (3.1,3.2), one can formulate the probability of distress of node k , $\mathbb{P}\{D_k\}$, by a simple recursion. If j and i are directly connected, the probability is given by

$$\begin{aligned} \mathbb{P}_{j|i} &= \mathbb{P}\{D_j|D_i\} \\ &= \mathbb{P}\{r_j(1 - \varepsilon_j) < \lambda|D_i\} \\ &= \mathbb{P}\{r_j(1 - p\varepsilon_i) < \lambda|D_i\} \\ &= 1 - F_{\varepsilon_i} \left(\frac{1}{p} \left(1 - \frac{\lambda}{r_j} \right) \right) \end{aligned}$$

where F_{ε_i} corresponds to the cdf of ε_i . Finally, for every institution (node) of the network, the contagion will depend only on all the possible simple paths connecting the corresponding node and the initially infested [Julio: ?](#).

Therefore,

$$\mathbb{P}\{D_k\} = \sum_{l \rightarrow k} \mathbb{P}\{D_k|D_l\} \cdot \mathbb{P}\{D_l\} \quad (5)$$

3.1 Revisiting the institutions' problem

Let's consider an stochastic problem, where each node i faces uncertainty on the final ratio. Denote as ε the random shock that the agent expects ($\mathbb{E}\varepsilon = \bar{\varepsilon} > 0$), and assume that every agent is risk neutral [Julio: focused on network effects](#) and their profits are homogeneous and linear: $\pi(r) = a^0 - a^1 r$, $a^0, a^1 > 0$. Thus, the agent maximization problem is given by

$$r_i(x_i; \bar{\varepsilon}) \in \arg\max_r \left\{ \mathbb{E}\{\pi(r(1 - \varepsilon))\} \mid r \geq x_i, r \in R_i \right\}$$

Additionally, we include explicitly the participation constraint, where agent- i only participates in the economy if its expected profits are nonnegative.

$$r_i(x_i; \bar{\varepsilon}) \in \arg\max_r \left\{ a^0 - a^1(1 - \bar{\varepsilon})r \mid r \geq x_i, r \in R_i, a^0 - a^1(1 - \bar{\varepsilon})r \geq 0 \right\}$$

Note that in this formulation we consider agents that proceed in a *naïve* fashion by only maximizing their profits, without considering its exposure within the network.

Following steps: Solve the problem considering

1. Risk neutral CP
2. Risk averse and Ambiguity neutral
3. Risk averse and Ambiguity averse
4. Numerical example

4 Toy Model

4.1 Parameters

 a^0
 a^1
 λ
 F_ε

5 Thu, Feb 22

We realize that the only important condition for distress is given by the interaction of each node with its immediate neighbors. Consider the institution n , and denote by $N(n)$ the set of its neighbors, i.e., nodes that directly connect to n , and let q_n be the probability that institution n faces an idiosyncratic shock. Therefore, the probability of n entering the distress condition, D_n , is given by

$$\mathbb{P}\{D_n\} = q_n p_n^0 + \sum_{m \in N(n)} \mathbb{P}\{D_n | D_m\} \mathbb{P}\{D_m\},$$

where the first component of the sum reflects the distress due to a idiosyncratic shock to the institution n , and the associated probability of entering distress, $p_n^0 = \mathbb{P}\{r_n(1 - \varepsilon) < \lambda\}$, and the second component is the network effect, i.e., the probability of the contagion through the connection to the network.

Using matrix notation, define the matrix $\Gamma_{ij} = \mathbb{P}\{D_i | D_j\}$ for each $(i, j) \in E$, and zero otherwise. Then, the probabilities of distress are the solution of the system

$$P = qp^0 + \Gamma P \quad \Rightarrow \quad P = (I - \Gamma)^{-1}(qp^0) \quad (6)$$

Additionally, this equation imposes implicitly conditions over the parameters of the problems such that the solution is a vector of probabilities. We need to focus our attention to set of parameters such that the matrix Γ satisfies the following condition

$$(I - \Gamma)^{-1}(qp^0) \in [0, 1]^N$$

6 Optimal value of profits

For the CP problem, the expected utility is given by

$$\mathbb{E}\{\tilde{\pi}_i^*(x_i) = \mathbb{E}\mathbb{E}\{\tilde{\pi}_i^*(x_i) | D_i\}$$

7 Fri, Feb 23rd

$$r_i(x_i; F_\varepsilon(p, N)) \in \operatorname{argmax}_r \left\{ \mathbb{E}\{\pi(r(1 - \varepsilon))\} \mid r \geq x_i, \mathbb{E}\{\pi(r(1 - \varepsilon))\} \geq 0 \right\}$$

7.1 Analyzing the shock

The shock have two possible sources: Idiosyncratic ε^I , and Non-Idiosyncratic (coming from the network), ε^{Nt} . For institution i , the probability of receiving the idiosyncratic shock is given by q_i , and we assume that $\varepsilon_i^I \sim U[0, UB]$, and that

$$\varepsilon = \begin{cases} \varepsilon^I & \mathbb{P}\{\cdot\} = q_i \\ \varepsilon^{Nt} & 1 - q_i \end{cases}$$

We realize that the contagion mechanism and the shock propagation are going to be analyzed with different perspective

- The idiosyncratic risk can be initiated at node i with probability q_i
- Once the institution i receives the shock, it propagates it with an intensity of p times the shock [Julio: note that here, we can consider \$p < 1\$ for mitigation effect, or \$p > 1\$ for an increasing effect.](#)
- The shock only propagates by simple paths between nodes (no revisiting allowed)

7.1 Assumption (shock propagation) *The shock always propagates, independently of the distress condition of the institution.*

- The final form for the shock faced by each institution is given by the equation

$$\varepsilon = \left(\sum_{n=1}^{|N|-1} p^n \mathcal{A}^n + I \right) q \varepsilon^I \quad (7)$$

[Julio: Check the information available for each agent: Is the node totally visible for each agent?](#) We will continue assuming perfect information wrt the network

7.2 i -maximization problem

Define the *modified agent maximization problem*

$$\begin{aligned} & \max_r \mathbb{E}\{\pi(r(1 - \varepsilon))\} \\ & \text{such that } r \geq x, \\ & \mathbb{E}\{\pi(r(1 - \varepsilon))\} \geq 0, \end{aligned}$$

where ε comes from Equation (7). Note that by the linearity of the propagation, ε^3 followed the distribution of ε^I , modified by a constant (easy to compute). Thus, define these coefficients as S_i^4 , and let's compute the expectation,

$$\pi(\tau) = \begin{cases} a^0 - a^1 \tau & \tau \geq \lambda \\ 0 & \text{o.w.} \end{cases}$$

³Simple paths?

⁴ $S = (\sum p^n \mathcal{A}^n + I)q$

125

$$\begin{aligned}
\mathbb{E}\{\pi(r(1-\varepsilon))\} &= \mathbb{E}\{\mathbb{E}\{\pi(r(1-\varepsilon))|r(1-\varepsilon) \geq \lambda\} + \mathbb{E}\{\pi(r(1-\varepsilon))|r(1-\varepsilon) < \lambda\}\} \\
&= \left(\int_{\{\varepsilon:r(1-\varepsilon) \geq \lambda\}} (a^0 - a^1 r(1-\tau)) \mathbb{P}(d\tau) \right) \mathbb{P}\{r(1-\varepsilon) \geq \lambda\} \\
&= \begin{cases} 0 & 1 - \frac{\lambda}{r} \leq 0 \\ a^0 - a^1 r(1 - \mathbb{E}\{\varepsilon\}) & \frac{1}{S \cdot UB} (1 - \frac{\lambda}{r}) \geq 1 \\ (1 - \frac{\lambda}{r}) \int_0^{1-\frac{\lambda}{r}} (a^0 - a^1 r(1-t)) \frac{dt}{S \cdot UB} & \text{o.w.} \end{cases} \\
&= \begin{cases} 0 & 1 - \frac{\lambda}{r} \leq 0 \\ a^0 - a^1 r(1 - \mathbb{E}\{\varepsilon\}) & \frac{1}{S \cdot UB} (1 - \frac{\lambda}{r}) \geq 1 \\ \frac{1}{S^2 \cdot UB^2} (1 - \frac{\lambda}{r})^2 (a^0 - a^1 \frac{r+\lambda}{2}) & \text{o.w.} \end{cases}
\end{aligned}$$

126 8 Wed, Feb 28th

127 The computation of the expectation is

$$\mathbb{E}\{\pi(r(1-\varepsilon))\} = \begin{cases} 0 & 1 - \frac{\lambda}{r} \leq 0 \\ a^0 - a^1 r(1 - \mathbb{E}\{\varepsilon\}) & \frac{1}{S \cdot UB} (1 - \frac{\lambda}{r}) \geq 1 \\ \frac{1}{(S \cdot UB)^2} (1 - \frac{\lambda}{r})^2 (a^0 - a^1 r (1 - \frac{1}{2} (1 - \frac{\lambda}{r}))) & \text{o.w.} \end{cases}$$

128 Note that this function is continuous. The optimal value for $x \geq 0$ is given by

$$r^*(x) \in \operatorname{argmax}_r \{\mathbb{E}\{\pi(r(1-\varepsilon))\} \mid r \geq x, \mathbb{E}\{\pi(r(1-\varepsilon))\} \geq 0\} = \begin{cases} \emptyset & x \geq \frac{a^0}{a^1} \frac{1}{1-\mathbb{E}\{\varepsilon\}} \\ \frac{\lambda}{1-S \cdot UB} & 0 \leq x < \frac{\lambda}{1-S \cdot UB} \\ x & \text{o.w} \end{cases} \quad (8)$$

129 The function $\mathbb{E}\{\pi(r(1-\varepsilon))\}$ is depicted in Figure (1). From here it is easy to see that the optimum of the opti-
 130 mization problem depends on the values of x .

131 Finally, the optimal expected utility is given by

$$\mathbb{E}\pi\{r^*(x)(1-\varepsilon)\} = \begin{cases} a^0 - a^1 x(1 - \mathbb{E}\{\varepsilon\}) & \frac{\lambda}{1-S \cdot UB} \leq x \leq \frac{a^0}{a^1} \frac{1}{1-\mathbb{E}\{\varepsilon\}} \\ a^0 - a^1 \frac{\lambda}{1-S \cdot UB} (1 - \mathbb{E}\{\varepsilon\}) & 0 \leq x \leq \frac{\lambda}{1-S \cdot UB} \\ 0 & \text{ow} \end{cases} \quad (9)$$

132 and it's depicted on Figure (2).

133 9 Thu, Mar 1st

134 9.1 Variance of the CP Problem

135 Assuming a risk-averse central planner, we can compute the individual variance by considering different cases

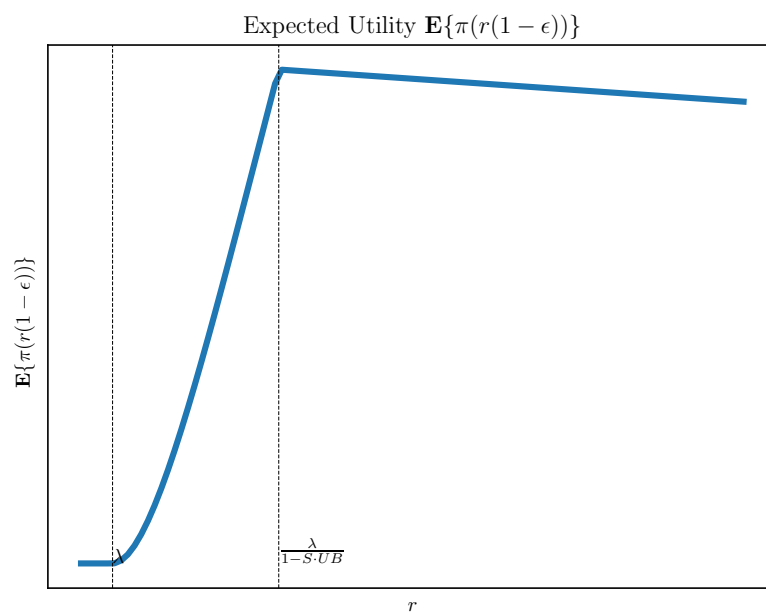


Figure 1: Expected utility

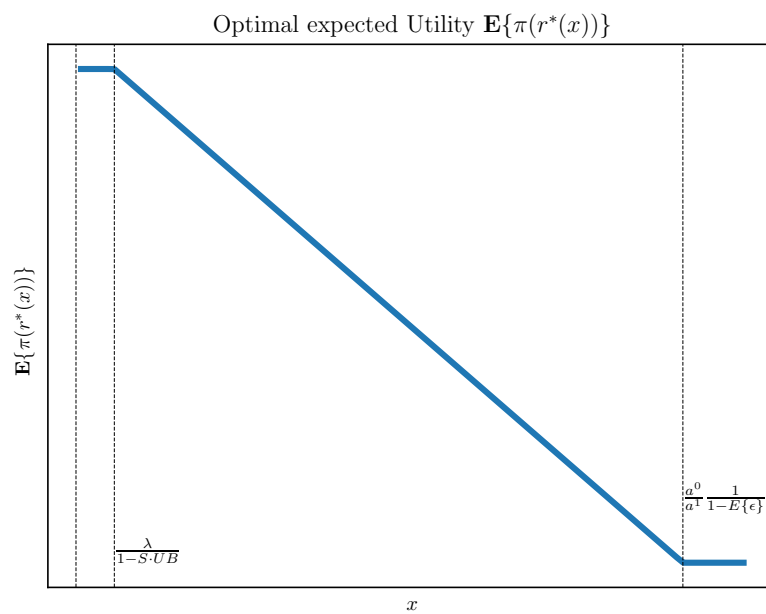


Figure 2: Optimal Expected utility

136 $x_i \in (0, \lambda_i = \frac{\lambda}{1-S_i \cdot UB})$ In this case, the optimal rule corresponds to $r_i^*(x_i) = \lambda_i$ and the final expression of the
 137 variance does not depend on x . Nevertheless, it is given by

$$\text{var}\{\pi_i(r_i^*(x_i)(1 - \varepsilon_i))\} = \begin{cases} (a_1 \lambda_i)^2 \frac{(S_i \cdot UB)^2}{12} & 1 - \frac{\lambda}{\lambda_i} > S_i \cdot UB \\ 0 & 1 - \frac{\lambda}{\lambda_i} < 0 \\ \left(1 - \frac{\lambda}{\lambda_i}\right) (a_1 \lambda_i)^2 \int_0^{1-\frac{\lambda}{\lambda_i}} (\tau - E\varepsilon)^2 \mathbb{P}(d\tau) & \text{o.w.} \end{cases}$$

138 $x_i \in (\lambda_i, \frac{a^0}{a^1} \frac{1}{1-E\varepsilon})$ Here, $r_i^*(x_i) = x_i$

$$\text{var}\{\pi_i(r_i^*(x_i)(1 - \varepsilon_i))\} = \begin{cases} (a_1 x)^2 \frac{(S_i \cdot UB)^2}{12} & x \geq \lambda_i \\ \left(1 - \frac{\lambda}{x}\right)^2 \frac{(a_1 x)^2}{3 S_i \cdot UB} \left(\left(1 - \frac{\lambda}{x} - \frac{S_i \cdot UB}{2}\right)^2 - \frac{S_i \cdot UB}{2} \left(1 - \frac{\lambda}{x} - \frac{S_i \cdot UB}{2}\right) + \left(\frac{S_i \cdot UB}{2}\right)^2 \right) & \text{o.w.} \end{cases}$$

139 [Julio: check the regions!](#)

140 9.2 Individual Variances

141 The individual variance is given by

$$\text{var}\{\pi_i(r_i^*(x_i)(1 - \varepsilon_i))\} = \begin{cases} (a_1 \lambda_i)^2 \frac{(S_i \cdot UB)^2}{12} & \lambda \leq x \leq \lambda_i \\ (a_1 x)^2 \frac{(S_i \cdot UB)^2}{12} & \lambda_i \leq x \leq 1 \\ 0 & \text{ow} \end{cases} \quad (10)$$

142 9.3 Pairwise Covariances

143 Consider the nodes i and j . The covariance of the profits between these two institutions can be decompsd ac-
 144 cording to the values of x_i and x_j . It's easier to see this on the Figure (3).

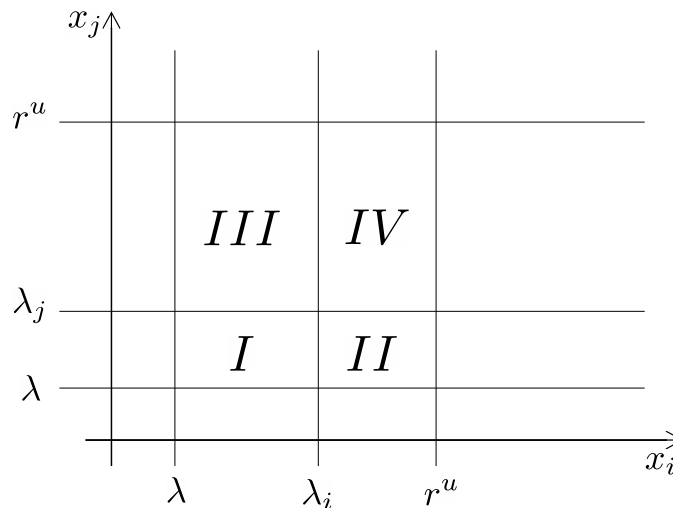


Figure 3: Areas for Covarianes

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After some algebra, the final form of the covariances is given by

$$\text{cov}(\pi_i(r_i^*(x_i(1 - \varepsilon_i))), \pi_j(r_j^*(x_j(1 - \varepsilon_j)))) = \begin{cases} (a_1)^2 S_i S_j \lambda_i \lambda_j \frac{(UB)^2}{12} & \lambda \leq x_i \leq \lambda_i, \lambda \leq x_j \leq \lambda_j, (I) \\ (a_1)^2 S_i S_j x_i \lambda_j \frac{(UB)^2}{12} & \lambda_i \leq x_i \leq r''', \lambda \leq x_j \leq \lambda_j, (I) \\ (a_1)^2 S_i S_j \lambda_i x_j \frac{(UB)^2}{12} & \lambda \leq x_i \leq \lambda, \lambda_j \leq x_j \leq r''', (III) \\ (a_1)^2 S_i S_j x_i x_j \frac{(UB)^2}{12} & \lambda_i \leq x_i \leq r''' \lambda, \lambda_j \leq x_j \leq r''', (IV) \\ 0 & \text{ow} \end{cases} \quad (\text{II})$$

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