I. Basic Idea

Consider an environment in which an infection spreads over n agents organized in a network economy, $\mathcal{G}_n$. How can we modify the network so that the infection is contained? In particular, how can we modify the network so the infection never affects a large fraction of the economy?

To make the problem more tractable, suppose the propagation of the infection among agents only depends on three features of the economy:

(a) the topology of $\mathcal{G}_n$

(b) the propensity of edges among agents to transmit the infection from one infected agent to a non-infected agent

(c) agent-specific characteristics that makes them less likely to acquire the infection

In the above setting, suppose (a) and (b) are known and cannot be changed. The problem is how we can change (c) so that the infection is contained.

II. Implementation of Basic Idea in a Simple Model

Consider a one-period economy populated by n productive units, henceforth entities, indexed by $i = \{1, \cdots, n\}$, with n being potentially large. Entities are in one of two states at the end of the period, either distressed and non-distressed. Let $\widetilde{s}_i$ denote a Bernoulli random variable, which equals one if entity i is distressed and zero otherwise, $i \in \{1, \cdots, n\}$.

The states of entities are related via a network of inter-entity relationships, denoted by $\mathcal{G}_n$, as distress spreads through probabilistic contagion via such relationships. In particular, the distribution of $\widetilde{s}_i$ is determined by the following stochastic process. At the beginning of the period, entity i faces distress, independently of other entities, with probability q_i, $i \in \{1, \cdots, n\}$. Immediately after, distress spreads via inter-entity relationships. Specifically, if entity i faces distress then entity j also faces distress, and, thus, $\widetilde{s}_i = \widetilde{s}_j = 1$ if two things happen: (1) there exists a sequence of inter-entity relationships in the economy that connects entities i and j and (2) each relationship in that sequence is active in transmitting distress. For simplicity, each relationship is in one of two states, either active in transmitting distress or not—independently of all other relationships. The relationship between entity i and j is active in transmitting distress with probability p_{ij}.[1]

Given how distress propagates, the joint distribution of the sequence $\{\widetilde{s}_i\}_{i=1}^n$ is determined by $\mathcal{G}_n$, the sequence $q \equiv \{q_i\}_{i=1}^n$, and the propensity matrix $p \equiv [p_{ij}]_{(i,j)\,\in\,\mathcal{G}_n}$. Moreover, the marginal distribution of $\widetilde{s}_i$, conditional on p, depends on q, the network $\mathcal{G}_n$, and the location of entity i in $\mathcal{G}_n$. In other words,

$$\mathbb{P}\left(\widetilde{s}_i = 1 \middle| p\right) \quad = \quad f\left(q, \mathcal{G}_n, \text{location of entity } i \text{ in } \mathcal{G}_n\right) \ , \tag{1}$$

where $\mathbb{P}\left(\widetilde{s}_i = 0 \middle| p\right) = 1 - \mathbb{P}\left(\widetilde{s}_i = 1 \middle| p\right)$, and $f(\cdot)$ is a mapping characterized by the stochastic process described above.

Despite the fact that the mapping $f(\cdot)$ is hard to characterize for large n, some of its properties are easy to describe given the formulation of the stochastic process that generates it. First, in the absence of relationships, $\mathbb{P}\left(\widetilde{s}_i = 1 \middle| p\right) \ = \ \mathbb{P}\left(\widetilde{s}_i = 1\right) \ = \ q_i \ , \ \forall \ i$, so the states of entities are independent. Second, if only one sequence of relationships exists between two entities, the longer the sequence, the smaller the correlation between their states. Thus, in networks in which there is at most one sequence of relationships between any two entities, the more distant the two entities are, the less related their states.

Let $\widetilde{X} \equiv \sum_{i=1}^n \widetilde{s}_i$ denote the (random) number of distressed entities in the economy at the end of the period. Given the network $\mathcal{G}_n$ and the propensity matrix p, we want to select the sequence q such that the the number of distressed entities is as small as we want it to be. In other words, I want to solve

$$\begin{aligned} \min_{q} \quad & \mathbb{P}\left(\widetilde{X} \geq t\right) \\ st. \quad & C(q) \leq 0 \end{aligned} \tag{2}$$

[1] This stochastic process can be thought of as a variation of either a reliability network or a bond percolation model. In a typical reliability network model, the edges of a given network are independently removed with some probability. The remaining edges are assumed to transmit a message. A message from node i to j is transmitted as long as there is at least one path from i to j after edge removal—see Colbourn (1987) for more details. Similarly, in a bond percolation model, edges of a given network are removed at random with some probability. Edges that are not removed are assumed to percolate a liquid. The question in percolation is whether or not the liquid percolates from one node to another in the network—which is similar to the problem of transmitting a message in a reliability context. For more details see Grimmett (1989), Stauffer and Aharony (1994), and Newman (2010, Chapter 16.1). The propagation mechanism analyzed here is an extension of the mechanism analyzed in Ramírez (2017).

where t is given and $C(\cdot)$ is a known function that captures how costly is to decrease vector q.

III. Some ideas on how to solve the problem

PROPOSITION 1: *Let d_n denote the number of relationships of the entity with the largest number of relationships in the economy. Let $\alpha \equiv \mathbb{P}\left(\widetilde{X} - \mathbb{E}\widetilde{X} \geq t_\alpha\right)$ be the upper tail probability. Then $q = q_\alpha + \epsilon_n$, where $q_\alpha = \{q_i(\alpha)\}_{i=1}^n$ is chosen such that the following equation is satisfied*

$$t_\alpha^2 + 4(d_n+1)\left(\mathbb{E}\widetilde{X} + \frac{t_\alpha}{3}\right)\log\left(\frac{\alpha}{d_n+1}\right) \quad = \quad 0, \tag{3}$$

and ϵ_n is a vector that tends towards zero as n grows large.

Proof. Follows from (Janson, 2002, Section 2.3.2) and Janson (2004) □

The following corollary applies the idea in the above proposition to the simplest case in which there are no relationships in the economy and all entities have the same propensity to face distress.

COROLLARY 1 (I.I.D. Case): *Suppose there are no relationships in the economy and $q_i = \bar{q}$, $i \in \{1, \cdots, n\}$. Namely, random variables $\widetilde{s}_i$ are independent and identically distributed. In this case, $d_n = 0$, $\mathbb{E}\widetilde{X} = n\bar{q}$, and $\bar{q} = q_\alpha + \epsilon_n$, where*

$$q_\alpha \quad = \quad -\frac{t_\alpha}{n}\left[\frac{t_\alpha}{4\log(\alpha)} + \frac{1}{3}\right], \tag{4}$$

and ϵ_n tends towards zero as n grows large.

Continue considering that there are no relationships in the economy but allow for differences across q_i. Namely, random variables $\widetilde{s}_i$ are independent but with different distributions. In this case, $\widetilde{X}$ follows a Poisson Binomial distribution. The following proposition determines $\{q_i\}_{i=1}^n$

COROLLARY 2 (Independent Case): *Suppose there are no relationships in the economy. Then $q = q_\alpha + \epsilon_n$, where $q_\alpha = \{q_i(\alpha)\}_{i=1}^n$ is chosen such that the following equation is satisfied*

$$\sum_{i=1}^{n} q_i(\alpha) \quad = \quad -t_\alpha\left[\frac{t_\alpha}{4\log(\alpha)} + \frac{1}{3}\right], \tag{5}$$

and ϵ_n is a vector that tends towards zero as n grows large.

REFERENCES

Colbourn, Charles J. 1987. *The Combinatorics of Network Reliability.* Oxford University Press.

Grimmett, Geoffrey. 1989. *Percolation.* Springer-Verlag.

Janson, Svante. 2002. "The infamous upper tail." *Random Structures and Algorithms* 20:317–342.

———. 2004. "Large deviation for sums of partly dependent random variables." *Random Structures and Algorithms* 24:234–248.

Newman, M.E.J. 2010. *Networks: An Introduction.* Oxford University Press.

Ramírez, Carlos. 2017. "Inter-firm Relationships and Asset Prices." *Finance and Economics Discussion Series. Board of Governors of the Federal Reserve System.* 14.

Stauffer, Dietrich and Amnon Aharony. 1994. *Introduction to Percolation Theory.* Taylor and Francis, second ed.