

# 1 A stochastic programming model for systemic financial resiliency

2 Julio Deride <sup>\*1</sup> and Carlos Ramírez <sup>†2</sup>

3 <sup>1</sup>Sandia National Laboratories, USA

4 <sup>2</sup>Federal Reserve Board, USA

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## 6 I Problem

7 Consider a network  $\mathcal{G} = (V, E)$  consisting on a set of  $n$  nodes,  $V = \{1, \dots, n\}$ , and a set of  $m$  undirected edges  
 8  $\{e_{ij}\} \in E$ . Each node  $i$  represents one of the institutions identity, and each edge  $e_{ij}$  represents the *correlation* or  
 9 contagion factor between two entities. Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space ...

10 Consider a two-stage model, where on the first stage there is a random shock happening on the nodes. Let  
 11  $\xi_i$  a Bernoulli random variable such that  $\xi_i^0 \in \{0, 1\}$ ,  $i = 1, \dots, n$  represents the *distress state* of node- $i$  on the  
 12 network. On the other hand, the second stage captures the behavior of the *shock's propagation* over the network.  
 13 In order to define this, consider the stochastic process  $P$  modeling the probability of contagion of a node, given  
 14 that one of its neighbor is distressed, i.e., if  $\xi_i^1 \in \{0, 1\}$  represents the distress state of node  $i$  in the second stage,  
 15 then

$$P_{ij} = \mathbb{P}\{\xi_i^1(\cdot) = 1 \mid \xi_j^0 = 1\} \quad e_{ij} \in E, \forall i, j \in V$$

16 The problem is now to minimize the total cost of the system under shocks on the network. For this, the regu-  
 17 lator is set to solve the problem of minimizing an overall cost, consisting on implementation cost and contagion  
 18 cost, by deciding an optimal capital requirement. Let  $x^0$  be the decision policy,  $x^0 \in [0, 1]^n$  such that  $x_i^0$  repre-  
 19 sents the policy required at entity  $i$ , and  $x^1(\cdot)$  be a decision policy regarding the second stage (not sure if needed  
 20 or not). The optimization problem is given by

$$(\mathcal{P}) \quad \begin{aligned} \min_{\{x^0, x^1(\cdot)\}} \quad & \varphi^0(x^0) + \mathbb{E}\{\varphi^1(\cdot, x^0, x^1(\cdot))\} \\ \text{such that} \quad & f^0(x^1) \leq 0 \\ & f^1(x^0, x^1(\omega), \omega) \leq 0, \omega - \text{a.s.} \text{ ,} \\ & x^0 \in [0, 1]^n, x^1 : \Omega \rightarrow \mathbb{R}^N \end{aligned}$$

21 where  $\varphi^0$  is the total cost of implementing a capital requirement policy, and  $\varphi^1$  is the total cost of the second  
 22 stage (probably related to the contagion cost). Here, the network constraints are included in the constraints

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<sup>\*</sup>jaderid@sandia.gov

<sup>†</sup>carlos.ramirez@frb.gov

$\{f^0, f^1\}$  and for a random realization  $\omega$  and a given policy  $x^1$ , the cost  $\varphi(\omega, x^0, x^1(\omega))$  should reflect the cost of the contagion on the system. For example, one can be interested in minimizing the expected cost of the contagion, but it is easy to incorporate a risk-measure for minimizing, for example, a measure like  $C - Var$  of the tail of the distribution of distressed nodes.

## 2 Tue, Feb 20th

We explore a model with the following features

1. Consider a graph  $G = (N, E)$ , where each node represents a financial institution, and each edge reflects financial transactions between two institutions.
2. Institution  $i$  faces a financial shock, represented as  $\varepsilon_i$ , which impacts its assets over liabilities ratio, defined as  $r_i = \frac{A_i}{L_i}$ ,<sup>1</sup>. Additionally, we consider that a financial institution is under *distress* if its ratio is under a (given) threshold  $\lambda \in (0, 1)$ . Thus,

$$i \text{ under distress} \iff r_i(1 - \varepsilon_i) < \lambda$$

3. There is a central decision maker, focus on the stability of the system. We discussed the information that is available to this regulator, and propose a mechanism to oversee the overall stability within the financial network thorough a constraint over the ratio, given by  $x_i$ .
4. The financial institutions decide their ratio by maximizing their profits<sup>2</sup>, given by a function  $\pi_i$ , with the minimum level of A/L ratio, i.e.,

$$r_i(x_i; p) \in \operatorname{argmax}_r \{ \mathbb{E}^p \{ \pi(r) \} \mid r \geq x_i, r \in R_i \}$$

Additionally, assuming that the function  $\pi_i$  is nonincreasing on  $r$  (and no further restrictions are imposed), the individual solution to this problem is given by  $r_i^* = x_i$ , i.e., the financial institution sets its ratio at minimum possible level.

5. There is contagion on the network, described in its stationary state as follows: if institution  $i$  gets distressed, there is a probability  $p$  that it affects its immediate neighbor,  $p^2$  by a 2-edge neighbors, and so on. Defining the set  $\{j \rightarrow i\}$  as the set of all possible simple paths coming to node  $i$ , and  $d(j, i)$  the distance between  $j$  and  $i$  (amount of edges between them), the expected shock [Julio: Assuming that there is no amplification of shocks](#) is given by

$$\varepsilon_i = \sum_{j \rightarrow i} p^{d(i,j)} \varepsilon_j$$

and by defining the matrix  $A_{ij} = \sum_{j \rightarrow i} p^{d(i,j)}$ , the acceptable shocks are the solution of the eigen problem for the matrix  $A$ . Moreover, we interpret  $A$  as an stochastic (transition) matrix by enlarging it with an extra

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<sup>1</sup>Capital?

<sup>2</sup>utility?

not distressed node as follows,

$$\tilde{A} = \left[ \begin{array}{c|cccc|c} & 1 & 2 & \dots & n & ND \\ \hline 1 & 0 & \sum_{2 \rightarrow 1} p^{d(2,1)} & \dots & \sum_{n \rightarrow 1} p^{d(n,1)} & 1 - \sum \mathcal{A}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ n & \sum_{n \rightarrow 1} p^{d(n,1)} & \vdots & \dots & 0 & 1 - \sum \mathcal{A}_1 \\ \hline ND & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

This is a stochastic matrix, and thus, it has an eigenvalue with value 1, with associated eigenvector  $\varepsilon^0$ . Let's consider the first  $n$  components as acceptable shocks  $\varepsilon^0$  for the corresponding nodes.

- Finally, consider the optimization problem solved by the central planner: set the ratio level [Julio: sth about the condition previously stated](#), such that it minimizes the total amount of financial institutions under distress. Let  $y_i \in \{0, 1\}$  a binary variable such that  $y_i = 1$  if institution  $i$  is under distress or  $y_i = 0$  otherwise, and let  $M > 0$  large enough such that

$$\min_{x, y} \sum_{i=1}^n y_i + \varphi(x, y) \quad (1)$$

$$\text{such that } r_i(x_i)(1 - \varepsilon_i^0) - \lambda \leq M(1 - y_i), \quad i = 1, \dots, N \quad (2)$$

where  $\varphi$  is a cost function associated to the policy  $x$  and the institutions on distress. Note that this formulation depends on  $p$  and the topology of the network through the selection of the  $\varepsilon^0$ .

- The optimization problem 1 can have a robust formulation by considering an ambiguity set for the parameter  $p$ , thus

$$\min_{x, y} \sup_{p \in \mathcal{A}(p_0)} \sum_{i=1}^n y_i + \varphi(x, y) \quad (3)$$

$$\text{such that } r_i(x_i)(1 - \varepsilon_i^0) - \lambda \leq M(1 - y_i), \quad i = 1, \dots, N \quad (4)$$

- Finally, we are looking for a representative agent formulation of the benevolent social planner problem, such that the solutions of both problems coincide. For example, one wild guess is to consider the formulation proposed in [?] [Julio: Citation needed](#), where ambiguity is considered as a family of possible models for the parameter  $p$ , along with a probability distribution over these models,  $\alpha$ . Therefore, the central planner problem has the following form

$$\max_x \left\{ \mathbb{E}^p \left\{ \sum_i u_i(x_i) \right\} - \frac{\mu}{2} \text{Var}^p \left( \sum_i u_i(x_i) \right) - \frac{\vartheta}{2} \text{Var}_\alpha \mathbb{E} \left( \sum_i u_i(x_i) \right) \right\}$$

### 3 Wed, Feb 21st

Continuing the idea of a (benevolent) central planner, let's define the following modified utility functions for each agents:

$$\tilde{\pi}_i = \begin{cases} \pi_i & \text{institution } i \text{ operates normally} \\ 0 & \text{institution is on distress} \end{cases}$$

Therefore, we expect that the Central Planner solves the following problem

$$\max_x \left\{ \mathbb{E}^P \left\{ \sum_i \tilde{\pi}_i(x_i) \right\} - \frac{\mu}{2} \text{Var}^P \left( \sum_i \tilde{\pi}_i(x_i) \right) - \frac{\vartheta}{2} \text{Var}_\alpha \mathbb{E} \left( \sum_i \tilde{\pi}_i(x_i) \right) \right\}$$

Additionally, we modify the assumptions over the spread of the distress condition

3.1 Assumption (initial shocks) *Initial shock only affects one institution (node)*

3.2 Assumption (propagation) *Propagation occurs over simple paths*

Under Assumptions (3.1,3.2), one can formulate the probability of distress of node  $k$ ,  $\mathbb{P}\{D_k\}$ , by a simple recursion. If  $j$  and  $i$  are directly connected, the probability is given by

$$\begin{aligned} \mathbb{P}_{j|i} &= \mathbb{P}\{D_j|D_i\} \\ &= \mathbb{P}\{r_j(1 - \varepsilon_j) < \lambda|D_i\} \\ &= \mathbb{P}\{r_j(1 - p\varepsilon_i) < \lambda|D_i\} \\ &= 1 - F_{\varepsilon_i} \left( \frac{1}{p} \left( 1 - \frac{\lambda}{r_j} \right) \right) \end{aligned}$$

where  $F_{\varepsilon_i}$  corresponds to the cdf of  $\varepsilon_i$ . Finally, for every institution (node) of the network, the contagion will depend only on all the possible simple paths connecting the corresponding node and the initially infested [Julio: ?](#).

Therefore,

$$\mathbb{P}\{D_k\} = \sum_{l \rightarrow k} \mathbb{P}\{D_k|D_l\} \cdot \mathbb{P}\{D_l\} \quad (5)$$

### 3.1 Revisiting the institutions' problem

Let's consider an stochastic problem, where each node  $i$  faces uncertainty on the final ratio. Denote as  $\varepsilon$  the random shock that the agent expects ( $\mathbb{E}\varepsilon = \bar{\varepsilon} > 0$ ), and assume that every agent is risk neutral [Julio: focused on network effects](#) and their profits are homogeneous and linear:  $\pi(r) = a^0 - a^1 r$ ,  $a^0, a^1 > 0$ . Thus, the agent maximization problem is given by

$$r_i(x_i; \bar{\varepsilon}) \in \underset{r}{\operatorname{argmax}} \{ \mathbb{E}\{\pi(r(1 - \varepsilon))\} \mid r \geq x_i, r \in R_i \}$$

Additionally, we include explicitly the participation constraint, where agent- $i$  only participates in the economy if its expected profits are nonnegative.

$$r_i(x_i; \bar{\varepsilon}) \in \underset{r}{\operatorname{argmax}} \{ a^0 - a^1(1 - \bar{\varepsilon})r \mid r \geq x_i, r \in R_i, a^0 - a^1(1 - \bar{\varepsilon})r \geq 0 \}$$

Note that in this formulation we consider agents that proceed in a *naïve* fashion by only maximizing their profits, without considering its exposure within the network.

Following steps: Solve the problem considering

1. Risk neutral CP
2. Risk averse and Ambiguity neutral
3. Risk averse and Ambiguity averse
4. Numerical example

## 4 Toy Model

### 4.1 Parameters

 $a^0$ 
 $a^1$ 
 $\lambda$ 
 $F_\varepsilon$ 

## 5 Thu, Feb 22

We realize that the only important condition for distress is given by the interaction of each node with its immediate neighbors. Consider the institution  $n$ , and denote by  $N(n)$  the set of its neighbors, i.e., nodes that directly connect to  $n$ , and let  $q_n$  be the probability that institution  $n$  faces an idiosyncratic shock. Therefore, the probability of  $n$  entering the distress condition,  $D_n$ , is given by

$$\mathbb{P}\{D_n\} = q_n p_n^0 + \sum_{m \in N(n)} \mathbb{P}\{D_n | D_m\} \mathbb{P}\{D_m\},$$

where the first component of the sum reflects the distress due to a idiosyncratic shock to the institution  $n$ , and the associated probability of entering distress,  $p_n^0 = \mathbb{P}\{r_n(1 - \varepsilon) < \lambda\}$ , and the second component is the network effect, i.e., the probability of the contagion through the connection to the network.

Using matrix notation, define the matrix  $\Gamma_{ij} = \mathbb{P}\{D_i | D_j\}$  for each  $(i, j) \in E$ , and zero otherwise. Then, the probabilities of distress are the solution of the system

$$P = qp^0 + \Gamma P \quad \Rightarrow \quad P = (I - \Gamma)^{-1}(qp^0) \quad (6)$$

Additionally, this equation imposes implicitly conditions over the parameters of the problems such that the solution is a vector of probabilities. We need to focus our attention to set of parameters such that the matrix  $\Gamma$  satisfies the following condition

$$(I - \Gamma)^{-1}(qp^0) \in [0, 1]^N$$

## 6 Optimal value of profits

For the CP problem, the expected utility is given by

$$\mathbb{E}\{\tilde{\pi}_i^*(x_i)\} = \mathbb{E}\mathbb{E}\{\tilde{\pi}_i^*(x_i) | D_i\}$$

## 7 Fri, Feb 23rd

$$r_i(x_i; F_\varepsilon(p, N)) \in \operatorname{argmax}_r \{\mathbb{E}\{\pi(r(1 - \varepsilon))\} \mid r \geq x_i, \mathbb{E}\{\pi(r(1 - \varepsilon))\} \geq 0\}$$

## 7.1 Analyzing the shock

The shock have two possible sources: Idiosyncratic  $\varepsilon^I$ , and Non-Idiosyncratic (coming from the network),  $\varepsilon^{Nt}$ . For institution  $i$ , the probability of receiving the idiosyncratic shock is given by  $q_i$ , and we assume that  $\varepsilon_i^I \sim U[0, UB]$ , and that

$$\varepsilon = \begin{cases} \varepsilon^I & P\{\cdot\} = q_i \\ \varepsilon^{Nt} & 1 - q_i \end{cases}$$

We realize that the contagion mechanism and the shock propagation are going to be analyzed with different perspective

- The idiosyncratic risk can be initiated at node  $i$  with probability  $q_i$
- Once the institution  $i$  receives the shock, it propagates it with an intensity of  $p$  times the shock [Julio: note that here, we can consider  \$p < 1\$  for mitigation effect, or  \$p > 1\$  for an increasing effect.](#)
- The shock only propagates by simple paths between nodes (no revisiting allowed)

**7.1 Assumption (shock propagation)** *The shock always propagates, independently of the distress condition of the institution.*

- The final form for the shock faced by each institution is given by the equation

$$\varepsilon = \left( \sum_{n=1}^{|N|-1} p^n \mathcal{A}^n + I \right) q \varepsilon^I \quad (7)$$

[Julio: Check the information available for each agent: Is the node totally visible for each agent?](#) We will continue assuming perfect information wrt the network

## 7.2 $i$ -maximization problem

Define the *modified agent maximization problem*

$$\begin{aligned} & \max_r \mathbb{E}\{\pi(r(1 - \varepsilon))\} \\ & \text{such that } r \geq x, \\ & \mathbb{E}\{\pi(r(1 - \varepsilon))\} \geq 0, \end{aligned}$$

where  $\varepsilon$  comes from Equation (7). Note that by the linearity of the propagation,  $\varepsilon^3$  followed the distribution of  $\varepsilon^I$ , modified by a constant (easy to compute). Thus, define these coefficients as  $S_i^4$ , and let's compute the expectation,

$$\pi(\tau) = \begin{cases} a^0 - a^1 \tau & \tau \geq \lambda \\ 0 & \text{o.w.} \end{cases}$$

<sup>3</sup>Simple paths?

<sup>4</sup> $S = (\sum p^n \mathcal{A}^n + I)q$

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$$\begin{aligned}
\mathbb{E}\{\pi(r(1-\varepsilon))\} &= \mathbb{E}\{\mathbb{E}\{\pi(r(1-\varepsilon))|r(1-\varepsilon) \geq \lambda\} + \mathbb{E}\{\pi(r(1-\varepsilon))|r(1-\varepsilon) < \lambda\}\} \\
&= \left( \int_{\{\varepsilon:r(1-\varepsilon) \geq \lambda\}} (a^0 - a^1 r(1-\tau)) P(d\tau) \right) P\{r(1-\varepsilon) \geq \lambda\} \\
&= \begin{cases} 0 & 1 - \frac{\lambda}{r} \leq 0 \\ a^0 - a^1 r(1 - \mathbb{E}\{\varepsilon\}) & \frac{1}{S \cdot UB} \left(1 - \frac{\lambda}{r}\right) \geq 1 \\ \left(1 - \frac{\lambda}{r}\right) \int_0^{1-\frac{\lambda}{r}} (a^0 - a^1 r(1-t)) \frac{dt}{S \cdot UB} & \text{o.w.} \end{cases} \\
&= \begin{cases} 0 & 1 - \frac{\lambda}{r} \leq 0 \\ a^0 - a^1 r(1 - \mathbb{E}\{\varepsilon\}) & \frac{1}{S \cdot UB} \left(1 - \frac{\lambda}{r}\right) \geq 1 \\ \frac{1}{S^2 \cdot UB^2} \left(1 - \frac{\lambda}{r}\right)^2 (a^0 - a^1 \frac{r+\lambda}{2}) & \text{o.w.} \end{cases}
\end{aligned}$$

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