Theoretical Foundations of Semiclassical Gravity

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PREFACE

These notes aim to keep track of my studies for my undergraduate project, developed while I am persuing my Bachelor degree in Physics at the Institute of Physics, University of São Paulo (IFUSP).

First of all, we shall cover the basics aspects of the standard bottom-top approach to describe quantum gravity in low-energies, commonly known as quantum field theory in curved spacetimes. The main goal is to understand one prediction of this theory, named Unruh effect and develop the basic concepts of theory. After that, we shall procede to formulation of electromagnetism in Minkowski spacetime, to understand the radiation emitted by an accelerated particle, a result from classical physics. Finally, we explore the correspondence between the Unruh effect, a essentialy quantum effect, and Larmor radiation, a classical effect, when we move from the frame of a inertial observer to an accelerated one.

I must acknowledge the help of my advisor, André G. S. Landulfo (UFABC), that has always been extremely patient and found the right words to clear my mind so many times, without him the path would be much harder.

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PART I

QUANTUM FIELD THEORY IN CURVED SPACETIMES

CHAPTER 1

GEOMETRIC QUANTUM MECHANICS

Throught this chapter we shall develop the necessary aspects of classical and quantum theory in geometric picture. Moreover, we apply this construction to a famous linear system: N decoupled harmonic oscillators.

1.1 Classical theory

Geometric formulation

In Hamiltonian classical mechanics, the space of study is $2n$-dimensional manifold $\mathcal{P}$ called phase space whose points are determined by n generalized coordinates $(q^1, \ldots, q^n)$ and their conjugated momenta $(p_1, \ldots, p_n)$. Each system has an associated Hamiltonian function $H = H(q^1, \ldots, q^n; p_1, \ldots, p_n)$ that provides the time evolution of the system by Hamilton equations

$$\frac{\mathrm{d}q^\mu}{\mathrm{d}t} = \frac{\partial H}{\partial p_\mu} \quad \text{and} \quad \frac{\mathrm{d}p_\mu}{\mathrm{d}t} = -\frac{\partial H}{\partial q^\mu}. \tag{1.1}$$

To put the coordinates of phase space in the same level, we define the $2n$-component object

$$y \equiv \left(q^1, \ldots, q^n; p_1, \ldots, p_n\right). \tag{1.2}$$

Moreover, we also define a $2n \times 2n$ object, Ω^{ij}, with components given by

$$\Omega^{ij} = \begin{pmatrix} 0_n & \mathbb{I}_n \\ -\mathbb{I}_n & 0_n \end{pmatrix}, \tag{1.3}$$

then, we are able to write the Hamilton's equations as

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}t} = \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \frac{\partial H}{\partial y^\nu}. \tag{1.4}$$

Inspired by this manipulation, in a more abstract formulation, a system with n degrees of freedom is represented by points in the $2n$-dimensional manifold $\mathcal{P}$ with a sympletic form Ω_{ab}, i.e., a 2-form that is non-degenerated and closed in $\mathcal{P}$. Mathematically, this is statement is equivalent to

1. $\Omega_{ab} = \Omega_{[ab]}$.
2. $\nabla_{[a}\Omega_{bc]} = 0$.
3. For every tagent vector v^b on $\mathcal{P}$, we have

$$v^b\Omega_{ab} = 0 \iff v^b = 0. \tag{1.5}$$

Since the form is non-degenerated, it has an inverse, Ω^{ab}, defined by the relation

$$\Omega^{ab}\Omega_{bc} = \delta^a_c. \tag{1.6}$$

From the Hamiltonian function, we define the hamiltonian vector field, h^a, by

$$h^a \equiv \Omega^{ab}\nabla_b H. \tag{1.7}$$

Thereby, in the abstract formulation, the dynamics are such that the possible physical paths are the ones that follows the orbits of h^a.

To relate both formulations, we pick a n-dimensional manifold $\mathcal{Q}$ with local coodinates an take $\mathcal{P} = T_*(\mathcal{Q})$, i.e., the cotangent bundle of $\mathcal{Q}$ formed by the points of $\mathcal{Q}$ and the cotagent vectors at each point. Therefore, the sympletic form, expressed as a function of local coordinates, can be defined as

$$\Omega_{ab} = 2\sum_\mu \left(\nabla p_\mu\right)_{[a} \left(\nabla q^\mu\right)_{b]}\,, \tag{1.8}$$

or

$$\Omega = \sum_\mu \mathrm{d}p_\mu \wedge \mathrm{d}q_\mu. \tag{1.9}$$

Even though we constructed the form using a choice of coordinates, it is coordinate independent[1].

[1] *show it!*

In order to have equivalent constructions (conventional and abstract), they must lead the same dynamical equations for the system. Notice that it indeed hols[2].

[2] *show it!*

Thus, the abstract formulation provides coordinate independet equations of motion. Moreover, notice that we can identify the manifold $\mathcal{P}$ with the solution space $\mathcal{S}$ associating each $y \in \mathcal{P}$ with the solution whose initial conditions is given by that configuration.

Now, remember that physical observables in Hamiltonian mechanics is given by smooth function acting on the phase space, $f \in C^\infty : \mathcal{P} \to \mathbb{R}$. These observables have algebric structure given by the Poisson brackets. For $f, g : \mathcal{P} \to \mathbb{R}$, it is defined as

$$\{f, g\} \equiv \Omega^{ab} \nabla_a f \nabla_b g. \tag{1.10}$$

The calculation of the Poisson brackets for the fundamentals observables (q^μ, p_ν) follows:

$$\begin{aligned}
\{q^\mu, q^\nu\} &= \Omega^{ab} \nabla_a q^\mu \nabla_b q^\nu && (1.11a)\\
&= \Omega^{ab} \sum_\mu \nabla_a q^\mu \nabla_b q^\mu \delta^\nu_\mu && (1.11b)\\
&= \sum_\mu \Omega^{[ab]} \nabla_{(a} q^\mu \nabla_{b)} q^\mu \delta^\nu_\mu && (1.11c)\\
&= 0. && (1.11d)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\{p_\mu, p_\nu\} &= \Omega^{ab} \nabla_a p_\mu \nabla_b p_\nu && (1.12a)\\
&= \Omega^{ab} \sum_\mu \nabla_a q^\mu \nabla_b q_\mu \delta^\mu_\nu && (1.12b)\\
&= \sum_\mu \Omega^{[ab]} \nabla_{(a} p_\mu \nabla_{b)} p_\mu \delta^\mu_\nu && (1.12c)\\
&= 0. && (1.12d)
\end{aligned}$$

At last, we have

$$\{q^\mu, p_\nu\} = ? \tag{1.13}$$

Linear dynamical systems

Henceforth, we shall restrict our analysis to system that are linear in order to have additional structures in our construction. More specifically, we consider systems that satisfies the following conditions:

1. The manifold of the configuration space, $\mathcal{Q}$, has vector space structure, hence it's possible to choose a linear canonical coordinate system $(q^1, \ldots, q^n)$ globally in $\mathcal{Q}$. Consequently, there exists a global coordinate system induced in $\mathcal{P}$ that also has vector space structure.

2. The Hamiltonian is quadratic, thus the solution space, $\mathcal{S}$, has vector space structure.

As a consequence of Item 1, we can identify the tangent space of any point $p \in \mathcal{P}$ as $\mathcal{P}$ itself. Thereby, the sympletic form is a bilinear map $\Omega : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ in $\mathcal{P}$. Furthermore, since the components of Ω_{ab} are constant in the canonical global basis[3], the map independ of the chosen y used to make the identification. The action of the sympletic structure in $y_1, y_2 \in \mathcal{P}$ is

[3] *show it!*

$$
\begin{aligned}
\Omega(y_1, y_2) &= \Omega_{ab} y_1^a y_2^b && (1.14a)\\
&= \sum_\mu \left(\nabla_a p_\mu \nabla_b q^\mu - \nabla_a q^\mu \nabla_b p_\mu\right) y_1^a y_2^b && (1.14b)\\
&= \sum_\mu \left(\delta_a^{\mu+n}\delta_b^\mu - \delta_a^\mu \delta_b^{\mu+n}\right) y_1^a y_2^b && (1.14c)\\
&= \sum_\mu \left(p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}\right). && (1.14d)
\end{aligned}
$$

Now, we can use the sympletic structure to substitute the canonical coordiantes that describe the manifold of interest. Notice that for a fixed $y \in \mathcal{P}$, the quantity $\Omega(y, \cdot)$ is a linear function in $\mathcal{P}$. Therefore, we have

$$
\begin{cases}
y = (0, \ldots, 0, q^\mu = 1, 0, \ldots, 0) \implies \Omega(y, \cdot) = -p_\mu \\
y = (0, \ldots, 0, p_\mu = 1, 0, \ldots, 0) \implies \Omega(y, \cdot) = q^\mu
\end{cases}, \tag{1.15}
$$

i.e., we can replace the coordinate by coordinates independent structures. In this geometric description, the Poisson bracktes, that satisfy the canonical relations Equations (1.11) to (1.13), is given by the more general expression

$$
\{\Omega(y_1, \cdot), \Omega(y_2, \cdot)\} = -\Omega(y_1, y_2). \tag{1.16}
$$

From now on, we consider the implications of Item 2. First of all, the Hamiltonian has the general form

$$
H(t, y) = \frac{1}{2} \sum_{\mu,\nu} K_{\mu\nu}(t) y^\mu y^\nu, \tag{1.17}
$$

in which, without loss of generality, holds $K_{\mu\nu} = K_{\nu\mu}$. For this special case, the Hamilton equations are

$$
\frac{\mathrm{d}y^\mu}{\mathrm{d}t} = \sum_{\nu,\sigma} \Omega^{\mu\sigma} K_{\sigma\nu} y^\nu. \tag{1.18}
$$

Let $y_1, y_2 \in \mathcal{S}$ be two solutions of Hamilton equations, hence we define

$$
s(t) \equiv \Omega(y_1(t), y_2(t)) = \sum_{\mu\nu} \Omega_{\mu\nu} y_1^\mu y_2^\nu. \tag{1.19}
$$

Notice that

$$\begin{aligned}
\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= \sum_{\mu\nu} \Omega_{\mu\nu} \left[\frac{\mathrm{d}y_1^\mu}{\mathrm{d}t} y_2^\nu + y_1^\mu \frac{\mathrm{d}y_2^\nu}{\mathrm{d}t} \right] && (1.20a)\\
&= \sum_{\mu,\nu,\alpha,\beta} \Omega_{\mu\nu} \left[\Omega^{\mu\alpha} K_{\alpha\beta} y_1^\beta y_2^\nu + \Omega^{\nu\alpha} K_{\alpha\beta} y_1^\mu y_2^\beta \right] && (1.20b)\\
&= -\sum_{\nu,\beta} K_{\nu\beta} y_1^\beta y_2^\nu + \sum_{\mu\beta} K_{\mu\beta} y_1^\mu y_2^\beta && (1.20c)\\
&= 0. && (1.20d)
\end{aligned}$$

Therefore, Ω induces a sympletic structure on the solution space $\mathcal{S}$, since the identification between $\mathcal{P}$ and $\mathcal{S}$ is independent of the time at it is done. Thus, $(\mathcal{S}, \Omega)$ has a sympletic vector space structure.

1.2 Quantum theory

Quantization

The quantization process of a theory, consists in finding a Hilbert space $\mathcal{H}$ that represent all the possible states and transform the classical observers f in self-adjoint operators $\hat{f}$. The leading point is to identify the Poisson bracktes as the equivalent of the quantum commutator, i.e., a map $\hat{}: f \to \hat{f}$ that satisfies

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}}, \tag{1.21}$$

where we used $\hbar = 1$. It is possible to construct this map in a such a way that all the fundamental observers respect Equation (1.21), thus any observable that is linear q, p also satisfy. However, in the cases that Item 1 holds, we have

$$\left[\hat{\Omega}(y_1, \cdot), \hat{\Omega}(y_2, \cdot)\right] = -i\Omega(y_1, y_2)\mathbb{I}. \tag{1.22}$$

Weyl's relations

Besides that, in general, the operators in the coordinate independent description are not bounded (and only densely defined), in such a way that we must call on the exponential version of the commutation relation. To do it, we define

$$W(y) \equiv \exp\left[i\Omega(y, \cdot)\right] \tag{1.23}$$

Furthermore, we impose that $\hat{W}(y)$ be unitary, be strongly continuous with respect to y and satisfy

$$\hat{W}(y_1)\hat{W}(y_2) = \exp\left[\frac{i}{2}\Omega(y_1, y_2)\right]\hat{W}(y_1 + y_2), \tag{1.24}$$

and

$$\hat{W}^\dagger(y) = \hat{W}(-y). \tag{1.25}$$

The Equations (1.24) and (1.25) are known as Weyl's relations and they are enough the uniquely determine $\left(\mathcal{H}, \hat{W}(y)\right)$. Therefore, a choice of Hilbert space $\mathcal{H}$ and operators $\hat{W}(y)$ that satisfies the Weyl relations are called irredutible representation of Weyl's relations.

Equivalent theories

Moreover, a Hilbert space $\mathcal{H}$ and a set of indexed operatores $V_\alpha : \mathcal{H} \to \mathcal{H}$ is unitarily equivalente to $\mathcal{H}'$ and V'_α if exists and unitary transformation $U : \mathcal{H} \to \mathcal{H}'$ such that

$$V_\alpha = U^{-1} V'_\alpha U, \qquad \forall\alpha \tag{1.26}$$

After the discussion above, is convenient to present Stone-von Neumann's theorem:

Theorem 1.1 (Stone-von Neumann)**.** Let $(\mathcal{P}, \Omega)$ be a finite dimension symplетic vector space and $\left(\mathcal{H}, \hat{W}(y)\right)$, $\left(\mathcal{H}', \hat{W}'(y)\right)$ be two strongly continuous, irredutible and unitary representation of Weyl's relations. Then, both representations are unitarily equivalent.

Thereby, since these representations of Weyl's relations generate equivalent theories, both describe the same physical system. Furthermore, notice that the theorem requires that the space is finite dimensional, hence in the treatment of scalar fields, exists many choices of Weyl's relations that yields non equivalent theories.

1.3 Harmonic oscillators

Right now, we shall apply the develop formalism in a well known system: quantum harmonic oscillators. Our main goal is to study scalar fields and we can interpret them as a set of oscillators in each point in space, hence this construction is a premliminary result for our quantization.

Single oscillator

First, let us remind the standard formulation presented in most textbooks. The system of a harmonic oscillator is represented by the hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2, \tag{1.27}$$

The quantized system is attained by choosing the Hilbert space $L^2(\mathbb{R})$ and the quantized operator is obtained by putting $\hat{q}$ and $\hat{p}$ that satisfies the commutation relation

$$[\hat{q}, \hat{p}] = i\mathbb{I}. \tag{1.28}$$

The description of the system is easier if we introduce the annihilation operator

$$\hat{a} \equiv \sqrt{\frac{\omega}{2}}\left(\hat{q} + \frac{i}{\omega}\hat{p}\right), \tag{1.29}$$

whose adjoint, named creation operator, is given by

$$\hat{a}^\dagger = \sqrt{\frac{\omega}{2}}\left(\hat{q} - \frac{i}{\omega}\hat{p}\right). \tag{1.30}$$

The commutator of these operators is

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \frac{\omega}{2}\left[\hat{q} + \frac{i}{\omega}\hat{p}, \hat{q} - \frac{i}{\omega}p\right] && (1.31a)\\
&= \frac{\omega}{2}\left(-\frac{i}{\omega}[\hat{q}, \hat{p}] + \frac{i}{\omega}[\hat{p}, \hat{q}]\right) && (1.31b)\\
&= -i[\hat{q}, \hat{p}] && (1.31c)\\
&= \mathbb{I}. && (1.31d)
\end{aligned}$$

Furthermore, notice that

$$\begin{aligned}
\hat{a}^\dagger \hat{a} &= \frac{\omega}{2}\left(\hat{q} - \frac{i}{\omega}\hat{p}\right)\left(\hat{q} + \frac{i}{\omega}\hat{p}\right) && (1.32a)\\
&= \frac{\omega}{2}\left(\hat{q}^2 + \frac{1}{\omega^2}\hat{p}^2 + \frac{1}{\omega}[\hat{q}, \hat{p}]\right) && (1.32b)\\
&= \frac{\omega}{2}\hat{q}^2 + \frac{1}{2\omega}\hat{p}^2 + \frac{i}{2}[\hat{q}, \hat{p}] && (1.32c)\\
&= \frac{1}{\omega}\hat{H} - \frac{1}{2}\mathbb{I}, && (1.32d)
\end{aligned}$$

thus,

$$\hat{H} = \omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\mathbb{I}\right). \tag{1.33}$$

It is also convenient to compute the commutator

$$\begin{aligned}\left[\hat{H}, \hat{a}\right] &= \omega\left[\hat{a}^\dagger \hat{a} + \frac{1}{2}\mathbb{I}, \hat{a}\right] && (1.34a)\\ &= \omega\left[\hat{a}^\dagger, \hat{a}\right]\hat{a} && (1.34b)\\ &= -\omega\hat{a}. && (1.34c)\end{aligned}$$

Therefore, in the Heisenberg picture, we have

$$\frac{\mathrm{d}\hat{a}_H}{\mathrm{d}t} = i\left[\hat{H}, \hat{a}_H\right] = -i\omega\hat{a}_H \implies \hat{a}_H(t) = e^{-i\omega t}\hat{a}. \tag{1.35}$$

This leads to the fundamental operators to be

$$\hat{q}_H(t) = \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger\right), \tag{1.36}$$

and

$$\hat{p}_H(t) = \frac{\mathrm{d}\hat{q}_H(t)}{\mathrm{d}t} = -i\sqrt{\frac{\omega}{2}}\left(e^{-i\omega t}\hat{a} - e^{i\omega t}\hat{a}^\dagger\right). \tag{1.37}$$

One should be familiar with the result that the ground state, $|0\rangle$, satisfies

$$\hat{a}\,|0\rangle = 0, \tag{1.38}$$

and the n^{th} state is obtained applying the creation operator n times, i.e.,

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle, \tag{1.39}$$

that satisfies

$$\hat{H}\,|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle. \tag{1.40}$$

Multiple oscillators

To generalize for N decoupled oscillators, inspired by the previous construction, we take the Hilbert space to be $\mathcal{H} \equiv \otimes_{j=1}^{N}\mathcal{H}_j$ with fundamental operators whose commutators are given by

$$[\hat{q}_j, \hat{p}_k] = i\delta_{jk}\mathbb{I}. \tag{1.41}$$

The vacuum state is taken as the tensor product of each oscillator, so we guarantee that is annihilated by any annihilation operator, i.e.,

$$|0\rangle \equiv \otimes_{j=1}^{N}|0\rangle_j\,. \tag{1.42}$$

The excited states are obtained by a similar approach

$$|n_1, \ldots, n_N\rangle \equiv \frac{1}{\sqrt{n_1!}}\left(\hat{a}_1^\dagger\right)^{n_1}\cdots\frac{1}{\sqrt{n_N!}}\left(\hat{a}_N^\dagger\right)^{n_N}|0\rangle\,. \tag{1.43}$$

Abstract formulation

Now, we focus on applying the geometric approach to the N decoupled quantum harmonic oscillators. Remember that due to Stone-von Neumann's theorem, the previous formulations is unitarily equivalent, hence describe the same physical system.

First, we take the sympletic vector space of classical solutions $(\mathcal{S}, \Omega)$ and complexify it, i.e., $\mathcal{S} \to \mathcal{S}^{\mathbb{C}}$. It is possible to extend the action of the form Ω to the complexified space by linearity. Thereby, using the sympletic structure, we define a map in the solution space $\mathcal{S}^{\mathbb{C}}$, named Klein-Gordon inner product, as

$$\langle y_1, y_2 \rangle \equiv -i\Omega(\overline{y_1}, y_2). \tag{1.44}$$

Notice that a general (complex) solution for a set of oscillators is

$$q^j(t) = \alpha_j e^{-i\omega_j t} + \beta_j e^{i\omega_j t}. \tag{1.45}$$

Then, consider

$$\begin{aligned}
\langle y, y \rangle &= -i\Omega(\overline{y}, y) && \text{(1.46a)}\\
&= -i \sum_j \overline{p}_j q^j - p_j \overline{q}^j && \text{(1.46b)}\\
&= -i \sum_j \omega_j \left(\overline{\alpha}_j e^{i\omega_j t} - \beta_j e^{-i\omega_j t}\right) \left(\alpha_j e^{-i\omega_j t} + \beta_j e^{i\omega_j t}\right) && \text{(1.46c)}\\
&- i\omega_j \left(-\alpha_j e^{-i\omega_j t} + \beta_j e^{i\omega_j t}\right) \left(\overline{\alpha_j} e^{i\omega_j t} + \overline{\beta}_j e^{-i\omega_j t}\right) && \text{(1.46d)}\\
&= 2 \sum_j \omega_j \left(|\alpha_j|^2 - |\beta_j|^2\right). && \text{(1.46e)}
\end{aligned}$$

Thus, notice that, in general, it is not positive-definite. However, is we take the n-dimensional space $\mathcal{H}$ of the complex solutions with positive frequency, i.e., such that $q^j \in \mathcal{H}$, then

$$q^j(t) = \alpha_j e^{-i\omega_j t}. \tag{1.47}$$

If we impose, $\beta_j = 0$ in Equation (1.46), it is evident that the map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is positive-definite, hence an inner product. Then, $\mathcal{H}$ is a Hilbert space and it's symmetric Fock space, $\mathcal{F}_S(\mathcal{H})$, is the Hilbert sapce used in our alternative quantization process.

Let $\xi^j \in \mathcal{H}$ be a complex solution such that only the j^{th} mode is excited and suppose it is normalized, i.e.,

$$\|\xi^j\|^2 = \left\langle \xi^j, \xi^j \right\rangle = 1, \tag{1.48}$$

then the set $\{\xi^j\}_{j\in I_n}$ is an orthonormal basis of $\mathcal{H}$. Now, we define the position and momentum operators (in Heisenberg picture) to be

$$\hat{q}^j(t) = \xi^j(t)\hat{a}\left(\overline{\xi}\right) + \overline{\xi}^j\hat{a}^\dagger(\xi) \tag{1.49}$$

and

$$p_j(t) = \frac{\mathrm{d}q^j}{\mathrm{d}t}, \tag{1.50}$$

in which $\hat{a}_j$ is the annihilation operator of the Fock space. Using the Proposition B.1, we find[4]

[4] *calculate!*

$$[q^\mu, p_\nu] = ? \tag{1.51}$$

[5] *should I show it?*

As mentioned earlier, one can show[5] that the formulations developed previously and the latter one are unitarily equivalente, hence describe the same physical system. Moreover, due to Stone-von Neumman theorem, there exists a unitarily transformation $U : L^2(\mathbb{R}) \to \mathcal{F}_S(\mathcal{H})$.

In order to generalize our construction to a basis independet language, we take the solution space of negative frequencies, denoted by $\overline{\mathcal{H}}$, such that for any $y \in \mathcal{S}^\mathbb{C}$, exists an unique representation

$$y = y^+ + y^-, \tag{1.52}$$

where $y^+ \in \mathcal{H}$ and $y^- \in \overline{\mathcal{H}}$. Thus, we define a linear map $K : \mathcal{S}^\mathbb{C} \to \mathcal{H}$ that takes the positive frequency part of the complex solution. Similarly, we define the map $\overline{K} : \mathcal{S}^\mathbb{C} \to \overline{\mathcal{H}}$ that take the negative part. Thereby,

$$Ky = y^+ \qquad \text{and} \qquad \overline{K}y = y^-. \tag{1.53}$$

Let $y \in \mathcal{S} \subset \mathcal{S}^\mathbb{C}$, then

$$\overline{K}y = y^- = \overline{y}^+ \equiv \overline{Ky}. \tag{1.54}$$

Therefore, for $y \in \mathcal{S}$, the operator that represents the classical observable $\Omega(y, \cdot)$ is given by

$$\hat{\Omega}(y, \cdot) = ia\left(\overline{Ky}\right) - ia^\dagger\left(Ky\right), \tag{1.55}$$

because this operator satisfies the Equation (1.16) as proved below:

$$[\hat{\Omega}(y_1,\cdot),\hat{\Omega}(y_2,\cdot)] = \left[ia\left(\overline{Ky_1}\right) - ia^\dagger\left(Ky_1\right)\right]\left[ia\left(\overline{Ky_2}\right) - ia^\dagger\left(Ky_2\right)\right] \tag{1.56a}$$

$$- \left[ia\left(\overline{Ky_2}\right) - ia^\dagger\left(Ky_2\right)\right]\left[ia\left(\overline{Ky_1}\right) - ia^\dagger\left(Ky_1\right)\right] \tag{1.56b}$$

$$= \left[a\left(\overline{Ky_2}, a\left(\overline{Ky_1}\right)\right)\right] + \left[a^\dagger\left(Ky_2\right), a^\dagger\left(Ky_1\right)\right] \tag{1.56c}$$

$$+ \left[a\left(\overline{Ky_1}, a^\dagger(Ky_2)\right)\right] - \left[a\left(\overline{Ky_2}\right), a^\dagger\left(Ky_1\right)\right] \tag{1.56d}$$

$$= \langle Ky_1, Ky_2\rangle - \langle Ky_2, Ky_1\rangle \tag{1.56e}$$

$$= -i\Omega\left(\overline{Ky_1}, Ky_2\right) + i\Omega\left(\overline{Ky_2}, Ky_1\right) \tag{1.56f}$$

$$= -i\Omega\left(Ky_1 + \overline{Ky_1}, Ky_2 + \overline{Ky_2 s}\right) \tag{1.56g}$$

$$= -i\Omega(y_1, y_2). \tag{1.56h}$$

The operator in Heisenberg picture is obtained substituting y by $y(t)$ with the condition $y(0) = y$.

Notice that the previous construction relied on the fact that the Hamiltonian is time-independent, hence the solutions has purely positive or negative frenquencies, however we can generalize this result.

1.4 Generic linear system

For a time-dependet Hamiltonian the procedure is similar. One must choose a subspace, $\mathcal{H} \subset \mathcal{S}^{\mathbb{C}}$ that satisfy

1. The Klein-Gordon inner product is positive-definite in $\mathcal{H}$, hence transforming it in a Hilbert space.

2. The complexified space of solutions $\mathcal{S}^{\mathbb{C}}$ is generated by $\mathcal{H}$ and $\overline{\mathcal{H}}$.

3. Let $y^+ \in \mathcal{H}$ and $y^- \in \overline{\mathcal{H}}$, then

$$\left\langle y^+, y^-, = \right\rangle 0. \tag{1.57}$$

Therefore, every $y \in \mathcal{S}^{\mathbb{C}}$ can be uniquely written as a sum of an element of $\mathcal{H}$ and of it's conjugate. We define the projective maps the same way.

Then, notice that for $y_1, y_2 \in \mathcal{S}$, we have

$$\begin{aligned}
\mathrm{Im}\langle Ky_1, Ky_2\rangle &= -\mathrm{Re}\left[\Omega\left(\overline{Ky_1}, Ky_2\right)\right] &(1.58a)\\
&= -\frac{1}{2}\Omega\left(\overline{Ky_1}, Ky_2\right) - \frac{1}{2}\Omega\left(Ky_1, \overline{Ky_2}\right) &(1.58b)\\
&= -\frac{1}{2}\Omega\left(y_1, y_2\right). &(1.58c)
\end{aligned}$$

Thus, if we define

$$\mu(y_1, y_2) \equiv \mathrm{Re}\left\langle Ky_1, Ky_2, =\right\rangle \mathrm{Im}\Omega\left(\overline{Ky_1}, Ky_2\right), \tag{1.59}$$

we have

$$\left\langle Ky_1, Ky_2, =\right\rangle \mu(y_1, y_2) - \frac{i}{2}\Omega(y_1, y_2). \tag{1.60}$$

Choices of Hilbert space

From the Schwarz's inequality, for every $z_1, z_2 \in \mathcal{H}$, we have

$$\|z_1\|^2\|z_2\|^2 \geq |\langle z_1, z_2\rangle|^2 \geq |\mathrm{Im}\langle z_1, z_2\rangle|^2. \tag{1.61}$$

For the special case in which $z_1 = Ky_1$ and $z_2 = Ky_2$,

$$\begin{aligned}
\|Ky_1\|^2\|Ky_2\|^2 &= \langle Ky_1, Ky_1\rangle\langle Ky_2, Ky_2\rangle &(1.62a)\\
&= \mu(y_1, y_1)\mu(y_2, y_2) - \frac{1}{4}\Omega(y_1, y_1)\Omega(y_2, y_2)+ &(1.62b)\\
&- \frac{i}{2}\left[\mu(y_1, y_1)\Omega(y_2, y_2) + \mu(y_2, y_2)\Omega(y_1, y_1)\right] &(1.62c)\\
&\geq \frac{1}{4}\left[\Omega(y_1, y_2)\right]^2, &(1.62d)
\end{aligned}$$

hence,

$$\mu(y_1, y_1)\mu(y_2, y_2) \geq \frac{1}{4}\left[\Omega(y_1, y_2)\right]^2. \tag{1.63}$$

Since is always possivle to "saturate" Schwarz's inequality, we can impose

$$\mu(y_1, y_1) = \frac{1}{4}\max_{y_2 \neq 0}\frac{\left[\Omega(y_1, y_2)\right]^2}{\mu(y_2, y_2)}, \tag{1.64}$$

however, since μ appears in both sides, this condition does not completly determine the function. Conversely, suppose that exists a map $\mu : \mathcal{S}^{\mathbb{C}} \times \mathcal{S}^{\mathbb{C}} \to \mathbb{R}$ that is positive-definite, bilinear, symmetric and satisfies Equation (1.64).

Proposition 1.2. For $y_1 \in \mathcal{S}$, there exists a unique $y_2 \in \mathcal{S}$ such that

$$\frac{1}{2}\Omega(y_1, y_2) = \mu(y_1, y_1) \quad \text{and} \quad \mu(y_1, y_1) = \mu(y_2, y_2). \tag{1.65}$$

Proof. show it later! $\square$

One may verify[6]

$$y_1 \to \frac{1}{2}(y_1 + iy_2), \tag{1.66}$$

leads to a Hilbert space that also satisfy the desired conditions. Thus, notice that the choice $\mathcal{H}$ can be replaced by a specification of μ.

[6] *verify!*

CHAPTER 2

KLEIN-GORDON FIELD

The construction of a quantum field theory often requires an unitary time evolution to be well defined, hence we must require some features of the spacetime we are working on, we shall discuss such requirements. Moreover, we also describe the quantization of a real, linear and scalar field, known as Klein-Gordon field.

2.1 Globally hyperbolic spacetimes

Let $(\mathcal{M}, g_{ab})$ be an arbitrary spacetime. A real, linear and scalar Klein-Gordon field has action given by

$$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\nabla^a\phi\nabla_a\phi + m^2\phi^2\right), \tag{2.1}$$

in which ∇_a is the covariant derivative, with no torsion and compatible with the metric. If we minimize the action

$$\begin{aligned}
\delta S &= S[\phi + \delta\phi] - S[\phi] && (2.2a)\\
&= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\nabla^a(\phi+\delta\phi)\nabla_a(\phi+\delta\phi) + m^2(\phi+\delta\phi)\right] && (2.2b)\\
&+ \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\nabla^a\phi\nabla_a\phi + m^2\phi^2\right) && (2.2c)\\
&= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\nabla^a\phi\nabla_a\delta\phi + \nabla^a\delta\phi\nabla_a\phi + 2m^2\phi\delta\phi + \mathcal{O}(\delta\phi^2)\right) && (2.2d)\\
&= -\int \mathrm{d}^3x\sqrt{h}\delta\phi n^a\nabla_a\phi + \int \mathrm{d}x^4\sqrt{-g}\delta\phi\left[\nabla^a\nabla_a\phi - m^2\phi\right] && (2.2e)\\
&= 0. && (2.2f)
\end{aligned}$$

Therefore, ignoring the border terms, Klein-Gordon equation is

$$\left(-\nabla^a\nabla_a + m^2\right)\phi = 0. \tag{2.3}$$

To construct the quantum theory, we need to define precisely the classical system phase space $\mathcal{P}$ and, to do it, is necessary that the intial conditions determine uniquely a solution of Equation (2.3). Thus, suppose that the spacetime is orientable and $\Sigma \subset \mathcal{M}$ is a closed and achronal set, i.e., none two points of Σ can be connected by a timelike curve. The domain of dependence of Σ is defined as

$$D(\Sigma) = \{p \in \mathcal{M} | \text{ every inextensible causal curve through } p \text{ intersects } \Sigma\}.$$

If $D(\Sigma) = \mathcal{M}$, then Σ is a Cauchy surface. A spacetime that admits a Cauchy surface is called globally hyperbolic and it's imporance is expressed in two theorems, the first one states about the space topology:

Theorem 2.1. Let $(\mathcal{M}, g_{ab})$ be a globally spacetime with Cauchy surface Σ, then $\mathcal{M}$ has topology $\mathbb{R} \times \Sigma$. Furthermore, $\mathcal{M}$ can be decomposed in a one parameter family of smooth Cauchy surfaces Σ_t, i.e., parametrized by a "time function".

The second one, enable the one-to-one correspondence between solutions and initial conditions:

Theorem 2.2. Let $(\mathcal{M}, g_{ab})$ be a globally hyperbolic spacetime with Cauchy surface Σ. Given any pair of smooth function (ϕ_0, π_0) in Σ, there exists an unique solution of Equation (2.3) in $\mathcal{M}$ such that

$$\phi\big|_\Sigma = \phi_0 \quad \text{and} \quad n^a \nabla_a \phi\big|_\Sigma = \pi_0, \tag{2.4}$$

in which n^a is the unit vector perpendicular to Σ.

Henceforth, we shall work with globally hyperbolic spacetimes to be able to make the desired identification between the phase space and solution space as previously used.

2.2 Theory's structures

Let $(\mathcal{M}, g_{ab})$ be a orientable, globally hyperbolic spacetime with a foliation of $\mathcal{M}$ with smooth Cauchy surfaces Σ_t indexed by a smooth function t. We introduce the vector field t^a that represents the time evolution in $\mathcal{M}$ and satisfies $t^a \nabla_a t = 1$, then

$$t^a = N n^a + N^a, \tag{2.5}$$

in which n^a and N^a are the vectors perpendicular and parallel to Σ_t respectively. Take local coordinates t, x^1, x^2, x^3 such that $t^a \nabla_a x^i = 0$, i.e., such that $t^a = (\partial_t)^a$. Thereby, in this coordiantes, the Klein-Gordon action is

$$S = \int \mathcal{L} \mathrm{d}t, \tag{2.6}$$

with

$$\mathcal{L} = \frac{1}{2} \int \mathrm{d}^3 x \sqrt{h} N \left[(n^a \nabla_a \phi)^2 - h^{ab} \nabla_a \phi \nabla_b \phi - m^2 \phi^2 \right], \tag{2.7}$$

in which h_{ab} is the metric induced in Σ_t. Notice that

$$n^a \nabla_a \phi = \frac{1}{N} (t^a - N^a) \nabla_a \phi = \frac{1}{N} \dot{\phi} - \frac{1}{N} N^a \nabla_a \phi. \tag{2.8}$$

Then, substituting in Equation (2.7), we have

$$\mathcal{L} = \frac{1}{2} \int \mathrm{d}^3 x \sqrt{h} \left[\frac{1}{N} \dot{\phi}^2 - \frac{2}{N} \dot{\phi} N^a \nabla_a \phi - h^{ab} \nabla_a \phi \nabla_b \phi - m^2 \phi^2 \right]. \tag{2.9}$$

Thus, the conjugated momentum density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \sqrt{h} \left(\frac{1}{N} \dot{\phi} - \frac{1}{N} N^a \nabla_a \phi \right) = \sqrt{h} n^a \nabla_a \phi. \tag{2.10}$$

A point in the classical phase space is determined by functions $\phi(x)$ and $\pi(x)$ in Σ_0, the Cauchy surface that represents $t = 0$. The class of functions that allows all the necessary structures to be well defined is the smooth ones with compact support $C_0^\infty(\Sigma_0)$, i.e.,

$$\mathcal{P} = \{ [\phi, \pi] | \phi : \Sigma_0 \to \mathbb{R} \text{ and } \pi : \Sigma_0 \to \mathbb{R};\ \phi, \pi \in C_0^\infty (\Sigma_0) \}. \tag{2.11}$$

Therefore, we take the solution space $\mathcal{S}$ to be the set of all solutions of Equation (2.3) whose initial conditions lies in $\mathcal{P}$. Furthermore, we define the sympletic structure $\Omega : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ as

$$\Omega \left([\phi_1, \pi_1], [\phi_2, \pi_2] \right) \equiv \int_{\Sigma_0} \mathrm{d}^3 x (\pi_1 \phi_2 - \pi_2 \phi_1), \tag{2.12}$$

or, in a equivalently, identifying $\mathcal{P}$ with $\mathcal{S}$,

$$Omega(\phi_1, \phi_2) = \int_{\Sigma_0} \mathrm{d}^3 x \sqrt{h} \left(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2 \right). \tag{2.13}$$

In order to induce a sympletic structure in $\mathcal{S}$, we must show that the identification is time-indepent.

Proposition 2.3. The sympletic product of two solutions ϕ_1 and ϕ_2

$$s(t) = \Omega(\phi_1(t), \phi_2(t)) = \int_{\Sigma_t} \mathrm{d}^3x\sqrt{h}\,(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2)\,, \quad (2.14)$$

is constant in time, i.e., does not depend on t.

Proof. Notice that the statement is equivalent to show that for any t_1 and t_2 holds

$$\int_{\Sigma_{t_1}} \mathrm{d}^3x\sqrt{h}\,(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) = \int_{\Sigma_{t_2}} \mathrm{d}^3x\sqrt{h}\,(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2)\,. \quad (2.15)$$

To show it, consider the region of spacetimes Σ that has the Cauchy surfaces Σ_{t_1} and Σ_{t_2} as boundaries[7], then

$$\int_{\Sigma_{t_1}} \mathrm{d}^3x\sqrt{h}\,(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) - \int_{\Sigma_{t_2}} \mathrm{d}^3x\sqrt{h}\,(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) \quad (2.16a)$$

$$= \int_{\partial\Sigma} \mathrm{d}^3x\sqrt{h}\,(\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) \quad (2.16b)$$

$$= \int_{\Sigma} \mathrm{d}^4x\sqrt{-g}\,[\nabla^a(\phi_2 \nabla_a \phi_1) - \nabla^a(\phi_1 \nabla_a \phi_2)] \quad (2.16c)$$

$$= \int_{\Sigma} \mathrm{d}^4x\sqrt{-g}\,[\nabla^a\phi_2 \nabla_a \phi_1 + \phi_2 \nabla^a \nabla_a \phi_1 - \nabla^a \phi_1 \nabla_a \phi_2 - \phi_1 \nabla^a \nabla_a \phi_2] \quad (2.16d)$$

$$= \int_{\Sigma} \mathrm{d}^4x\sqrt{-g}\,\left(\phi_2 m^2 \phi_1 - \phi_1 m^2 \phi_2\right) = 0. \quad (2.16e)$$

$\square$

Again exchaging between $\mathcal{P}$ and $\mathcal{S}$, we want operators $\hat{\Omega}(\psi, \cdot)$, $\psi \in \mathcal{S}$, such that

$$\left[\hat{\Omega}(\psi_1, \cdot), \hat{\Omega}(\psi_2, \cdot)\right] = -i\Omega(\psi_1, \psi_2)\mathbb{I}. \quad (2.17)$$

Notice that it would be natural to choose $\mathcal{H}$ such that it is the positive frequency solution's set, however, not only none of these satisfy the initial condition compacity in Σ_0, but this notion is not extended to arbitrary spacetimes. Thus, the Hilbert space must be obtained by completing the complexified space $\mathcal{S}^{\mathbb{C}}$ with $\langle\cdot,\cdot\rangle$. One may notice that the choice of inner product also depend on $\mathcal{H}$, hence the construction becomes circular.

An alternative procedure to complete $\mathcal{S}^{\mathbb{C}}$ would be similar to what was done in Section 1.4. We can define a real inner product $\mu : \mathcal{S}^{\mathbb{C}} \times \mathcal{S}^{\mathbb{C}} \to \mathbb{R}$ such that, for all $\psi_1 \in \mathcal{S}$,

$$\mu(\psi_1, \psi_1) \equiv \frac{1}{4} \sup_{\psi_2 \neq 0} \frac{[\Omega(\psi_1, \psi_2)]^2}{\mu(\psi_2, \psi_2)}. \tag{2.18}$$

Now, we complete $\mathcal{S}$ to $\mathcal{S}_\mu$ using the inner product

$$\langle \psi_1, \psi_2 \rangle_\mu \equiv 2\mu(\psi_1, \psi_2). \tag{2.19}$$

Since Ω is bounded[8], by Equation (2.18), we can extend it's action to $\mathcal{S}_\mu$ by continuity. We deifne the operator $J : \mathcal{S}_\mu \to \mathcal{S}_\mu$ by means of

[7] *image!*

$$\Omega(\psi_1, \psi_2) = 2\mu(\psi_1, J\psi_2) = \langle \psi_1, J\psi_2 \rangle_\mu. \tag{2.20}$$

Since Ω is antisymmetric and μ is real, we have

$$\begin{aligned} \langle \psi_1, J\psi_2 \rangle_\mu &= -\langle \psi_2, J\psi_1 \rangle_\mu && (2.21a)\\ &= -\langle J\psi_1, \psi_2 \rangle_\mu && (2.21b)\\ &= -\left\langle \psi, J^\dagger \psi_2 \right\rangle_\mu, && (2.21c) \end{aligned}$$

Therefore, we comparing the second and last equalities,

$$J^\dagger = -J. \tag{2.22}$$

Furthermore, from Equation (2.18), J preserves the norm[9], then

[8] *why?*

$$J^\dagger J = \mathbb{I} \implies J^2 = -\mathbb{I}. \tag{2.23}$$

Then, μ naturally provides a complex structure in $\mathcal{S}_\mu$. Now, we complexify $\mathcal{S}_\mu$ and extend the action of Ω, μ and J by (complex) continuity. The Klein-Gordon product is, finally, defined as

$$\langle \psi_1, \psi_2 \rangle \equiv 2\mu(\overline{\psi}_1, \psi_2). \tag{2.24}$$

Notice that $iJ : \mathcal{S}_\mu^{\mathbb{C}} \to \mathcal{S}_\mu^{\mathbb{C}}$ is self-adjoint, with eigenvalues $\pm i$, then by the spectral theorem, the set $\mathcal{S}_\mu^{\mathbb{C}}$ is the direct sum of the orthogonal spaces correspondent to each eigenvalue. We take $\mathcal{H} \subset \mathcal{S}^{\mathbb{C}}$ to be the subspace of eigenvaleu i, hence from spectral theorem properties that $\mathcal{H}$ satisfies the necessary properties of quantization.

Finally, the quantum theory is obtained by choosing $\mathcal{F}_S(\mathcal{H})$ as the Hilbert space and defining the operators correspondent to the classical observables $\Omega(\psi, \cdot)$, $\psi \in \mathcal{S}$, as

$$\hat{\Omega}(\psi, \cdot) = ia\left(\overline{K\psi}\right) - ia^\dagger (K\psi), \tag{2.25}$$

in which K is the projector defined as usual.

2.3 Interpretation of field operator

Indepdent of the choices for the construction of the theory, the operators $\hat{\Omega}(\psi, \cdot)$ can be interpreted as a spacetime average of the operator (in the Heisenberg picture) that represents the field value.

Let $\mathfrak{F}(\mathcal{M}) = C_0^\infty(\mathcal{M})$ be the vector space of smooth functions with compact support in our spacetime of interest. Moreover, let $G_A(x, x')$ and $G_R(x, x')$ be the retarded and advanced Green functions of the Klein-Gordon operator, i.e.,

$$\left(-\nabla^a \nabla_a + m^2\right) G_A(x, x') = \frac{1}{\sqrt{-g}} \delta(x, x') = \delta_M(x, x') \tag{2.26a}$$

$$\left(-\nabla^a \nabla_a + m^2\right) G_R(x, x') = \frac{1}{\sqrt{-g}} \delta(x, x') = \delta_M(x, x'). \tag{2.26b}$$

Thereby, we can define the advanced and retarded solutions with source $f \in \mathfrak{F}(\mathcal{M})$ as

$$Rf(x) \equiv \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} G_R(x, x') f(x') \tag{2.27a}$$

$$Af(x) \equiv \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} G_A(x, x') f(x'). \tag{2.27b}$$

These solutions are the ones that, if $\mathrm{supp} f$ denotes the support of the function f, then $\mathrm{supp} Rf \subset J^+(\mathrm{supp} f)$ (i.e., it propagates f to the future) and $\mathrm{supp} Af \subset J^-(\mathrm{supp} f)$ (i.e., it proapgates f to the past)[10]. Furthermore, notice that, indeed, Rf is a solution of Equation (2.3) with source f,

[9] *why?*

$$\left(-\nabla^a \nabla_a + m^2\right) Rf(x) = \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} (-\nabla^a \nabla_a + m^2) G_R(x, x') f(x') \tag{2.28a}$$

$$= \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} \delta_M(x, x') f(x') \tag{2.28b}$$

$$= f(x), \tag{2.28c}$$

where the covariant derivatives acted only on the Green functions because they are taken with respect to x. It is evident that the same holds for Af. Then, we define

$$Ef \equiv Af - Rf. \tag{2.29}$$

It is immediate that Ef is a solution of the homogenous Klein-Gordon equation with initial condition $f \in \mathfrak{F}(\mathcal{M})$. Therefore, we obtain a linear map

$E : \mathfrak{F}(\mathcal{M}) \to \mathcal{S}$ whose important properties are discussed in the following proposition.

Proposition 2.4. The map $E : \mathfrak{F}(\mathcal{M}) \to \mathcal{S}$ is surjective, i.e., for every $\psi \in \mathcal{S}$, exists $f \in \mathfrak{F}(\mathcal{M})$ such that

$$\psi = Ef. \tag{2.30}$$

Proof. Let $\psi \in \mathcal{S}$ and $\xi \in C^{\infty}$ such that

$$\chi = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1 \end{cases} . \tag{2.31}$$

Now, we define

$$f \equiv - \langle (, \chi \rangle \, \psi), \tag{2.32}$$

notice that since ψ has compact support in Σ_0, it's time evolution also do. Putting together with the fact that $\chi\,(t \leq 0) = 0$, we conclude that f has compact support in $\mathcal{M}$. Furthermore, we have $Af = (1 - \chi)\psi$ and $Rf = -\chi\psi$, then

$$Ef = \psi. \tag{2.33}$$

□

Proposition 2.5. For the map $E : \mathfrak{F}(\mathcal{M}) \to \mathcal{S}$ holds that

$$Ef = 0 \iff f = \left(-\nabla^a \nabla_a + m^2\right) g, \quad g \in \mathfrak{F}(\mathcal{M}). \tag{2.34}$$

Proof. We separate the proof in two parts:

- $\implies$ If $Ef = 0$, then $Af = Rf$. Since both advanced and retarded solutions are the same, both has support equal to the support of f. Then, $Af, Rf \in \mathfrak{F}(\mathcal{M})$ and from Equation (2.28),

$$f = \left(-\nabla^a \nabla_a + m^2\right) g, \tag{2.35}$$

where $g = Rf \in \mathfrak{F}(\mathcal{M})$

- $\Longleftarrow$ Suppose that $f = \left(-\nabla'^a\nabla'_a + m^2\right) g$, then

$$
\begin{aligned}
Af(x) &= \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} G_A(x,x') \left(-\nabla'^a\nabla'_a + m^2\right) g(x') && \text{(2.36a)}\\
&= \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} \left[\nabla'^a \left(G_A \nabla_a g\right) - \nabla'^a G_A \nabla_a g - m^2 G_A g\right] && \text{(2.36b)}\\
&= \int_{\partial\mathcal{M}} \mathrm{d}^3x' \sqrt{h} G_A n^a \nabla'_a g && \text{(2.36c)}\\
&+ \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} \left[-\nabla'_a \left(g\nabla'^a G_A\right) + g\nabla'^a\nabla'_a G_A - m^2 G_A g\right] && \text{(2.36d)}\\
&= -\int_{\partial\mathcal{M}} \mathrm{d}^3x' \sqrt{h} g n^a \nabla'^a G_A && \text{(2.36e)}\\
&+ \int_{\mathcal{M}} \mathrm{d}^4x' \left[\left(-\nabla'^a\nabla'_a + m^2\right) G_A\right] g && \text{(2.36f)}\\
&= \int_{\mathcal{M}} \mathrm{d}^4x' \sqrt{-g} \delta_M(x,x') g(x') = g(x). && \text{(2.36g)}
\end{aligned}
$$

The border terms were removed because g has compact support. An identical procedure shows that $Rf(x) = g(x)$, then

$$
Ef(x) = Af(x) - Rf(x) = 0. \tag{2.37}
$$

□

Finally, the last proposition follows:

Proposition 2.6. For every $\psi \in \mathcal{S}$ and $f \in \mathfrak{F}(\mathcal{M})$, we have

$$
\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} \psi f = \Omega(Ef, \psi), \tag{2.38}
$$

where $E : \mathfrak{F}(\mathcal{M}) \to \mathcal{S}$ is the map of interest.

Proof. Take a Cauchy surface Σ that is outside the causal future of the support of f, i.e., $\Sigma \subset \mathcal{M} - J^{+}(\mathrm{supp} f)$, then

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\psi f = \int_{J^{+}(\Sigma)} \mathrm{d}^4x\sqrt{-g}\psi f \tag{2.39a}$$

$$= \int_{J^{+}(\Sigma)} \mathrm{d}^4x\sqrt{-g}\psi \left(-\nabla^a\nabla_a + m^2\right) Af \tag{2.39b}$$

$$= \int_{J^{+}(\Sigma)} \mathrm{d}^4x\sqrt{-g}\nabla^a \left(\psi\nabla_a Af - Af\nabla_a\psi\right) \tag{2.39c}$$

$$+ \int_{J^{+}(\Sigma)} \mathrm{d}^4x\sqrt{-g}Af \left(-\nabla^a\nabla_a + m^2\right)\psi \tag{2.39d}$$

$$= \int_{\Sigma} \mathrm{d}^3x\sqrt{h}\left(\psi n^a\nabla_a Af - Afn^a\nabla_a\right). \tag{2.39e}$$

Since in Σ $Rf = 0$, then $Af = Ef$, therefore

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\psi f = \Omega(Ef, \psi). \tag{2.40}$$

□

From Proposition 2.6, we conclude that the function $\Omega(Ef, \cdot)$ is equivalent to a spacetime average with weight f. Thus, the operator $\hat{\Omega}(Ef, \cdot)$ is interpreted as a spacetime average of the quantum field with weight f, hence we define

$$\hat{\phi}(f) \equiv \hat{\Omega}\left(Ef, \cdot\right) = ia\left(\overline{KFf}\right) - ia^{\dagger}\left(KEf\right). \tag{2.41}$$

Furthermore, notice that by Proposition 2.5, the observables related to $f \in \mathfrak{F}(\mathcal{M})$ always correspond to an observable $O\hat{mega}(\psi, \cdot)$ with $\psi \in \mathcal{S}$. Morevoer, it is possible to obtain a distributional form of the fact that ϕ satisfy Klein-Gordon equation with the aid of Proposition 2.4,

$$\hat{\phi}\left(\left(-\nabla^a\nabla_a + m^2\right)g\right) = \hat{\Omega}\left(E\left(-\nabla^a\nabla_a + m^2\right)g, \cdot\right) = 0. \tag{2.42}$$

Now, the commutation relations are given by

$$\left[\hat{\phi}(f), \hat{\phi}(g)\right] = \left[\hat{\Omega}\left(Ef, \cdot\right), \hat{\Omega}\left(Eg, \cdot\right)\right] = -i\Omega(Ef, Eg)\mathbb{I}. \tag{2.43}$$

If we define

$$E(f, g) \equiv \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}Ef(x)g(x), \tag{2.44}$$

then, by Proposition 2.6, we have

$$\left[\hat{\phi}(f), \hat{\phi}(g)\right] = -iE(f, g)\mathbb{I}. \tag{2.45}$$

Finally, with the developed theory, the two-point function of the vacuum is

$$\langle 0|\,\hat{\phi}(f)\hat{\phi}(g)\,|0\rangle = \langle 0|\, a\left(\overline{KEf}\right) a^{\dagger}\left(KEg\right)|0\rangle \tag{2.46a}$$
$$= \langle KEf, KEg\rangle \tag{2.46b}$$
$$= \mu(Ef, Eg) - \frac{i}{2}E(f,g). \tag{2.46c}$$

Thereby, is evident the dependence of the expected values of the theory with the choice of μ used in the quantization process.

Remark. Notice that we implicitly used that f is a real function in order to Ef be a real solution. However, if f is complex, we have

$$\begin{cases} \hat{\phi}(\mathrm{Re}f) = ia\left(\overline{KE\mathrm{Re}f}\right) - ia^{\dagger}\left(KE\mathrm{Re}f\right) \\ \hat{\phi}(\mathrm{Im}f) = ia\left(\overline{KE\mathrm{Im}f}\right) - ia^{\dagger}\left(KE\mathrm{Im}f\right) \end{cases} \tag{2.47}$$

Thus, since $f = \mathrm{Re}f + i\mathrm{Im}f$, then

$$\hat{\phi}(f) = ia\left(\overline{KE\overline{f}}\right) - ia^{\dagger}\left(KEf\right). \tag{2.48}$$

Now, the commutation relations are given by

$$\left[\hat{\phi}(f), \hat{\phi}(g)\right] = \left[\langle KE\overline{f}, KEg\rangle - \langle KE\overline{g}, KEf\rangle\right]\mathbb{I} \tag{2.49a}$$
$$= \left[\langle KE\overline{f}, KEg\rangle + \langle \overline{KE\overline{f}}, \overline{KEg}\rangle\right]\mathbb{I} \tag{2.49b}$$
$$= \langle E\overline{f}, Ef\rangle\mathbb{I} \tag{2.49c}$$
$$= -E(\overline{f}, g). \tag{2.49d}$$

With this tiny adjusts, we are able to cover a larger set of functions that will be important after.

CHAPTER 3

PARTICLE CONCEPT

Throughout this chapter we will show that when symmetries are present in the spacetime, we can use them to construct a "natural" choice of Hilbert space for the quantization and apply it to a static spacetimes. Often this construction allows us to gives rise to the concept of particles of the field, hence we also cover a two-level particle detector. Finally, we discuss Boguliubov transformations, that is a map between two constructions of the quantum theories.

3.1 Stationary spacetimes

Let $(\mathcal{M}, g_{ab})$ is a globally hyperbolic stationary spacetime, i.e., it admits a timelike Killing field ξ^a that is associated with a one parameter group of isometries $\phi_t^{\xi} : \mathcal{M} \to \mathcal{M}$ with timelike orbits. The choice of $\mathcal{H}$ will be the one of solutions that are positive frequency with respect to the Killing time t, a function such that

$$\xi^a \nabla_a t = 1, \tag{3.1}$$

i.e., ξ^a plays the role of t^a in the previous construction. To avoid technical difficulties, we impose

$$m > 0, \tag{3.2}$$

and thata exists a Cauchy surface Σ such that, $\exists \epsilon > 0$,

$$-\xi^a \xi_a \geq -\epsilon \xi^a n_a > \epsilon, \tag{3.3}$$

in this surface, where n^a is the vector normal to Σ. We define the "energy" inner product in $\mathcal{S}^{\mathbb{C}}$ as

$$\langle \psi_1, \psi_2 \rangle_\xi \equiv \int_\Sigma \mathrm{d}^3x \sqrt{h} \xi^a n^b T_{ab}, \tag{3.4}$$

where T_{ab} is classical stress-energy tensor given by

$$T_{ab}(\psi_1, \psi_2) = \nabla_{(a}\overline{\psi}_1 \nabla_{b)}\psi_2 - \frac{1}{2} g_{ab} \left(\nabla^c \overline{\psi}_1 \nabla_c \psi_2 + m^2 \overline{\psi}_1 \psi_2 \right). \tag{3.5}$$

By the definition, it is immediate that

$$\langle \psi_1, \psi_2 \rangle_\xi = \overline{\langle \psi_2, \psi_1 \rangle_\xi}. \tag{3.6}$$

Proposition 3.1. The inner product $\langle \cdot, \cdot \rangle_\xi$ is positive-definite in $\mathcal{S}^{\mathbb{C}}$.

Proof. First, we have the following definitions that help in the calculation. Let $\psi \in \mathcal{S}^{\mathbb{C}}$, then

$$\begin{cases} \nabla_a \psi = -(n^b \nabla_b \psi) n_a + h_a^b \nabla_b \psi \equiv -\psi_n n_a + D_a \psi \\ \xi^a = -(\xi^b n_b) n^a + h^{ab} \xi_b \equiv N_\xi n^a + N^a \end{cases}. \tag{3.7}$$

Therefore, if we write the stress-energy tensor as function of the new parameters, we have

$$\nabla_{(a}\overline{\psi} \nabla_{b)}\psi = \overline{\psi}_n \psi_n n_a n_b - \overline{\psi}_n n_{(a} D_{b)} \psi - \psi_n D_{(a} \overline{\psi} n_{b)} + D_{(a} \overline{\psi} D_{b)} \psi, \tag{3.8}$$

and also,

$$\begin{aligned} \nabla^c \overline{\psi} \nabla_c \psi &= \left(-\overline{\psi}_n n^c + D^c \overline{\psi} \right) \left(-\psi_n n_c + D_c \psi \right) && (3.9\text{a}) \\ &= -\overline{\psi_n} \psi_n + D^c \overline{\psi} D_c \psi. && (3.9\text{b}) \end{aligned}$$

Now, we can calculate the integrand of the inner product,

$$\begin{aligned} \xi^a n^b T_{ab} &= \frac{1}{2} \left[2N_\xi \overline{\psi}_n \psi_n + \overline{\psi}_n N^b D_b \psi + \psi_n N^a D_a \psi \right] && (3.10\text{a}) \\ &\quad - \frac{\xi^b n_b}{2} \left[-\overline{\psi}_n \psi_n + D^c \overline{\psi} D_c \psi + m^2 \overline{\psi} \psi \right] && (3.10\text{b}) \\ &= \frac{N_\xi}{2} \left[\overline{\psi}_n \psi_n D^c \overline{\psi} D_c \psi + m^2 \overline{\psi} \psi \right] && (3.10\text{c}) \\ &\quad + \frac{1}{2} \left[\overline{\psi}_n N^b D_b \psi + \psi_n N^a D_a \overline{\psi} \right]. && (3.10\text{d}) \end{aligned}$$

Remember that the momentum density is given by Equation (2.10), then

$$\langle\psi,\psi\rangle_\xi = \int_\Sigma \mathrm{d}^3x\sqrt{h}\xi^a n^b T_{ab}(\psi,\psi) \tag{3.11a}$$
$$= \frac{1}{2}\int_\Sigma \mathrm{d}^3x \left[N_\xi \frac{|\pi|^2}{\sqrt{h}} + \sqrt{h}N_\xi\|D\psi\|^2 + \sqrt{h}m^2|\psi|^2\right] \tag{3.11b}$$
$$+ \frac{1}{2}\int_\Sigma \mathrm{d}^3x \left[\frac{\pi}{N_\xi}N^a\nabla_a\overline{\psi} + \frac{\overline{\pi}}{N_\xi}N^a\nabla_a\psi\right] \tag{3.11c}$$
$$= \frac{1}{2}\int_\Sigma \mathrm{d}^3x N_\xi \left[\left(1 - \frac{N^a N_a}{N_\xi^2}\right)\frac{|\pi|^2}{\sqrt{h}} + \sqrt{h}m^2|\psi|^2\right] \tag{3.11d}$$
$$+ \frac{1}{2}\int_\Sigma \mathrm{d}^3x N_\xi\sqrt{h}h^{ab}\left(D_a\psi + \frac{\pi N_a}{\sqrt{h}N_\xi}\right)\left(D_b\overline{\psi} + \frac{\overline{\pi}N_b}{\sqrt{h}N_\xi}\right) \tag{3.11e}$$
$$= \frac{1}{2}\int_\Sigma \mathrm{d}^3x N_\xi \left[-\frac{\xi^a\xi_a}{N_\xi^2}\frac{|\pi|^2}{\sqrt{h}} + \sqrt{h}m^2|\psi|^2\right] \tag{3.11f}$$
$$+ \frac{1}{2}\int_\Sigma \mathrm{d}^3x N_\xi\sqrt{h}\left\|D_a\psi + \frac{\pi N_a}{\sqrt{h}N_\xi}\right\|^2. \tag{3.11g}$$

It is immediate that

$$\psi = 0 \iff \langle\psi,\psi\rangle_\xi = 0, \tag{3.12}$$

then take $\psi \neq 0$, using Equations (3.2) and (3.3),

$$\langle\psi,\psi\rangle_\xi \geq \epsilon\int_\Sigma \frac{\mathrm{d}^3x}{2}\left(\frac{|\pi|^2}{\sqrt{h}} + m^2|\psi|^2\right) \tag{3.13a}$$
$$\geq \frac{1}{2}\min\{\epsilon, m^2\epsilon\}\int_\Sigma \mathrm{d}^3x\sqrt{h}\left[|n^a\nabla_a|^2 + |\psi|^2\right] > 0. \tag{3.13b}$$

□

Now, we will perform auxiliary calculations to show that the inner product is independent of the Cauchy surface Σ we ue calculate it. First, notice that for $\psi_1, \psi_2 \in \mathcal{S}^{\mathbb{C}}$, as expected of a stress-energy tensor,

$$2\nabla^a T_{ab} = \nabla^a \left[\nabla_a\overline{\psi}_1\nabla_b\psi_2 + \nabla_b\overline{\psi}_1\nabla_a\psi_2\right] - g_{ab}\nabla^a\left[\nabla^c\overline{\psi}_1\nabla_c\psi_2 + m^2\overline{\psi}_1\psi_2\right] \tag{3.14a}$$

$$= \nabla^a\nabla_a\overline{\psi}_1\psi_2 + \nabla_a\overline{\psi}_1\nabla^a\nabla_b\psi_2 + \nabla^a\nabla_b\overline{\psi}_1\nabla_a\psi_2 + \nabla_b\overline{\psi}_1\nabla^a\nabla_a\psi_2 \tag{3.14b}$$

$$- g_{ab}\left[\nabla^a\nabla^c\overline{\psi}_1\nabla_c\psi_2 + \nabla^c\overline{\psi}_1\nabla^a\nabla_c\psi_2 + m^2\nabla^a\overline{\psi}_1\psi_2 + m^2\overline{\psi_1}\nabla^a\psi_2\right] \tag{3.14c}$$

$$= m^2\overline{\psi}_1\nabla_b\psi_2 + \nabla^a\overline{\psi}_1\nabla_a\nabla_b\psi_2 + \nabla^a\psi_2 + \nabla_a\nabla_b\overline{\psi}_1 + m^2\nabla_b\overline{\psi}_1\psi_2 \tag{3.14d}$$

$$- \nabla^c\psi_2\nabla_c\nabla_b\overline{\psi}_1 - \nabla^c\overline{\psi}_1\nabla_c\nabla_b\psi_2 - m^2\nabla_b\overline{\psi}_1\psi_2 - m^2\overline{\psi}_1\nabla_b\psi_2 \tag{3.14e}$$

$$= 0. \tag{3.14f}$$

From the previous result, follows that

$$\nabla^a\left(T_{ab}\xi^b\right) = \nabla^a T_{ab}\xi^b + T_{ab}\nabla^a\xi^b \tag{3.15a}$$

$$= T_{ab}\nabla^{(a}\xi^{b)} \tag{3.15b}$$

$$= 0. \tag{3.15c}$$

Thus, consider a region $\mathcal{V}$ confined by two Cauchy surfaces Σ_{t_1} and Σ_{t_2}, then

$$\int_{\Sigma_{t_2}} \mathrm{d}^3x\sqrt{h}\xi^a n^b T_{ab} - \int_{\Sigma_{t_1}} \mathrm{d}^3x\sqrt{h}\xi^a n^b T_{ab} = \int_{\partial\mathcal{V}} \mathrm{d}^3x\sqrt{h}\xi^a n^b T_{ab} \tag{3.16a}$$

$$= \int_{\mathcal{V}} \mathrm{d}^4\sqrt{-g}\nabla^b\left(\xi^a T_{ab}\right) \tag{3.16b}$$

$$= \int_{\mathcal{V}} \mathrm{d}^4\sqrt{-g}\nabla^b\left(T_{ba}\xi^a\right) \tag{3.16c}$$

$$= 0, \tag{3.16d}$$

i.e., the inner product is identical for any smooth Cauchy surfaces of the one parameters group associated with ξ^a. Now, we complete the solution space with this inner product to obtain the Hilbert space $\tilde{\mathcal{H}}$.

Define the operator $V_t : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ as

$$V_t\psi \equiv \psi \circ \phi^{\xi}_{-t}, \tag{3.17}$$

notice that the invariance under time translation of the inner product implies that this operator is unitary. As a consequence, vy Stone theorem,

$$V_t = e^{-i\tilde{h}t}, \tag{3.18}$$

where $\tilde{h}$ is a self-adjoit operator. From the definition, we have

$$\begin{aligned}
\tilde{h}\psi &= i\frac{\mathrm{d}}{\mathrm{d}t}V_t\psi\bigg|_{t=0} &(3.19a)\\
&= i\frac{\mathrm{d}}{\mathrm{d}t}\psi\circ\phi^{\xi}_{-t}\bigg|_{t=0} &(3.19b)\\
&= i\mathcal{L}_\xi\psi, &(3.19c)
\end{aligned}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field ξ^a.

Henceforth, we will use the identification between the solution space and the phase space. First, we define $B : \mathcal{S}^{\mathbb{C}} \times \mathcal{S}^{\mathbb{C}} \to \mathbb{C}$ as

$$B(\psi_1, \psi_2) \equiv \Omega\left(\overline{\psi}_1, \psi_2\right) \tag{3.20}$$

Remark. Notice that $\langle\psi_1, \psi_2\rangle = -iB(\psi_1, \psi_2)$.

We represent an element of the phase space by

$$\psi = \begin{pmatrix}\phi\\ \pi\end{pmatrix}, \tag{3.21}$$

with inner product given by

$$\langle\psi_1, \psi_2\rangle_{L^2} = \int_\Sigma \mathrm{d}^3x \begin{pmatrix}\overline{\phi}_1 & \overline{\pi}_1\end{pmatrix}\begin{pmatrix}\phi_2\\ \pi_2\end{pmatrix} = \int_\Sigma \mathrm{d}^3x \left(\overline{\phi}_1\phi_2 + \overline{\pi}_1\pi_2\right). \tag{3.22}$$

Notice that, given the matrix

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{3.23}$$

we have

$$\begin{aligned}
\langle\psi_1, J\psi_2\rangle_{L^2} &= \int_\Sigma \mathrm{d}^3x \begin{pmatrix}\overline{\phi}_1 & \overline{\pi}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\phi_2\\ \pi_2\end{pmatrix} &(3.24a)\\
&= \int_\Sigma \mathrm{d}^3x(\overline{\pi}_1\phi_2 - \pi_2\overline{\phi}_1) &(3.24b)\\
&= \Omega\left(\overline{\phi_1}, \phi_2\right). &(3.24c)
\end{aligned}$$

Furthermore, as shown in Proposition 3.1,

$$\langle \psi, \psi \rangle_\xi \geq C^{-1} \langle \psi, \psi \rangle_{L^2}, \tag{3.25}$$

with $C^{-1} = 1/2 \min \{\epsilon, m^2 \epsilon\}$, then

$$\|\psi\|_{L^2} \leq C^{1/2} \|\psi\|_\xi. \tag{3.26}$$

Moreover, notice that J is norm preserving in the phase space,

$$\|J\psi\|^2_{L^2} = \langle J\psi, J\psi \rangle_{L^2} \tag{3.27a}$$
$$= \int_\Sigma \mathrm{d}^3 x \begin{pmatrix} -\overline{\pi} & \overline{\phi} \end{pmatrix} \begin{pmatrix} -\pi \\ \psi \end{pmatrix} \tag{3.27b}$$
$$= \int_\Sigma \mathrm{d}^3 x \left(\|\phi\|^2 + \|\pi\|^2 \right) \tag{3.27c}$$
$$= \|\psi\|^2. \tag{3.27d}$$

Therefore, due to the Cauchy-Schwartz,

$$|B(\psi_1, \psi_2)| = |\Omega \left(\overline{\psi}_1, \psi_2 \right)| \tag{3.28a}$$
$$= |\langle \psi_1, J\psi_2 \rangle_{L^2}| \tag{3.28b}$$
$$\leq \|\psi_1\|_{L^2} \|\psi_2\|_{L^2} \tag{3.28c}$$
$$\leq C \|\psi_1\|_\xi \|\psi_2\|_\xi. \tag{3.28d}$$

Thus, B is bounded and defined in a dense domain of $\tilde{\mathcal{H}}$, hecnce we can extend it's action to this set as a bounded operator.

Proposition 3.2. For every $\psi_1, \psi_2 \in \mathcal{S}^{\mathbb{C}}$,

$$B(\psi_1, \tilde{h}\psi_2) = 2i \left\langle \psi_1, \psi_2 \right\rangle_\xi . \tag{3.29}$$

Proof. show it! □

Finally, notice that the Klein-Gordon, then

$$\langle \psi_1, \tilde{h}\psi_2 \rangle = 2 \left\langle \psi_1, \psi_2 \right\rangle_\xi . \tag{3.30}$$

Therefore, due to Equation (3.28), we have

$$2|\langle \psi_1, \psi_2 \rangle_\xi| = |\langle \psi_1, \tilde{h}\psi_2 \rangle| \leq C \|\psi_1\|_\xi \|\tilde{h}\psi_2\|_\xi. \tag{3.31}$$

In particular, for $\psi \neq 0$,

$$2\|\psi\|^2_\xi \leq C \|\psi\|_\xi \|\tilde{h}\psi\|_\xi \implies \frac{\|\tilde{h}\psi\|_\xi}{\|\psi\|_\xi} \geq \frac{2}{C} > 0. \tag{3.32}$$

We conclude that the infimum of the spectrum of $\tilde{h}$, σ, is greater than zero, hence $\tilde{h}^{-1}$ is well defined in a dense domain of $\tilde{\mathcal{H}}$. Therefore, from the spectral theorem, we have

$$\tilde{h} = \int_{\omega\in\sigma^+} \omega \mathrm{d}P_\omega + \int_{\omega\in\sigma^-} \omega \mathrm{d}P_\omega, \tag{3.33}$$

where

$$\begin{cases} \sigma = \sigma^+ \cup \sigma^- \\ \sigma^+ = -\sigma^- \\ \sigma^+ \cap \sigma^- = \varnothing \end{cases}. \tag{3.34}$$

Let $\tilde{\mathcal{H}}^+$ be the space associated with the solutions that lies in σ^+ completed with Klein-Gordon product. It is evident that the map that projects in this space is

$$K = \int_{\omega\in\sigma^+} \mathrm{d}P_\omega. \tag{3.35}$$

At last, for every $\psi_1, \psi_2 \in \mathcal{S}$, we define the map μ as

$$\mu(\psi_1,\psi_2) = \mathrm{Im}B(K\psi_1, K\psi_2) = 2\mathrm{Re}\left\langle K\psi_1, \tilde{h}^{-1}K\psi_2\right\rangle_\xi. \tag{3.36}$$

Proposition 3.3. The product μ satisfies the condition given in Equation (2.18).

Proof. show it! □

Finally, we introduce a convenient notation. Let Λ, μ be a measure sapce in which $j \in \Lambda$ represents a set of quantum numbers. Then, it is possivle to choose an orthonormal basis $\{u_j\}_{j\in\Lambda}$ of the solutions of Klein-Gordon equations such that

$$\tilde{h}u_j = \omega_j u_j \iff u_j = \frac{e^{-i\omega_j t}}{\sqrt{\omega_j}}\varphi_j, \tag{3.37}$$

and also

$$\langle u_j, u_{j'}\rangle = \delta_\mu(j,j'), \tag{3.38}$$

where

$$\int_{j\in\Lambda} \mathrm{d}\mu(j')\delta_\mu(j,j') = 1. \tag{3.39}$$

Thus, any solution $\psi \in \tilde{\mathcal{H}}^+$ can be written as

$$\psi(x) = \int_{j\in\Lambda} \mathrm{d}\mu(j)\tilde{\psi}(j)\frac{e^{-i\omega_j t}}{\sqrt{2\omega_j}}\varphi_j(x). \tag{3.40}$$

Now, it is evident that the space $\tilde{\mathcal{H}}^+$ concerns the positive frequency solutions with respect to the Killing field $\xi = \partial_t$.

3.2 Quantization is static spacetimes

Let $(\mathcal{M}, g_{ab})$ be globally hyperbolic and static spacetime, i.e., stationary that admits an hypersurface orthogonal to ξ. Under these conditions, exists a coordinate system in which the line element is expressed as

$$\mathrm{d}s^2 = -f(x)\mathrm{d}t^2 + \sum_{i,j} h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j, \tag{3.41}$$

where $f(x) > 0$ and x represents the coordiantes in the Cauchy surfaces Σ_t. The vector field $\xi = \partial_t$ is a Killing one and timelike. Furthermore, it's integral curves are given by $\phi_t(t_1, x_1) = (t + t_1,)$, i.e., they represent time translation. Since they are timelike, we can interpret them as the worldline of a congruence of observers.

In an arbitrary coordinate system, Klein-Gordon equation is

$$\frac{1}{\sqrt{-g}} \sum_{\mu,\nu} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) - m^2\phi = 0. \tag{3.42}$$

In the static coordinates, for $\mu = 0$, we have

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} \left(\sqrt{-g} g^{0\nu} \frac{\partial \phi}{\partial x^\nu} \right) = -f^{-1} \frac{\partial^2 \phi}{\partial t^2} \tag{3.43}$$

thus, the equation is

$$\left(-\frac{\partial^2}{\partial t^2} - \mathcal{K} \right) \phi = 0, \tag{3.44}$$

where

$$\mathcal{K}\phi \equiv f \left[-\frac{1}{\sqrt{-g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{-g} h^{ij} \frac{\partial \phi}{\partial x^j} \right) + m^2\phi \right], \tag{3.45}$$

denotes the purely spatial operator.

Proposition 3.4. The operator $\mathcal{K}$ is hermitian in $L^2\left(\Sigma_t, \sqrt{-g} f^{-1}\mathrm{d}x\right)$.

Proof. The statement comes from a direct computation

$$\langle \mathcal{K}\phi_1, \phi_2 \rangle_{L^2} = \int_{\Sigma_t} \mathrm{d}^3x \sqrt{-g} f^{-1} f \left[-\frac{1}{\sqrt{-g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{-g} h^{ij} \frac{\partial \overline{\phi}_1}{\partial x^j} \right) + m^2 \overline{\phi}_1 \right] \phi_2 \tag{3.46a}$$

$$= \int_{\Sigma_t} \mathrm{d}^3x \left[\sum_{i,j} \sqrt{-g} h^{ij} \frac{\partial \phi_2}{\partial x^i} \frac{\partial \overline{\phi}_1}{\partial x^j} + \sqrt{-g} m^2 \overline{\phi}_1 \phi_2 \right] \tag{3.46b}$$

$$- \int_{\partial\Sigma_t} \mathrm{d}^2x \sum_{i,j} \sqrt{-g} h^{ij} \phi_2 \frac{\partial \overline{\phi}_1}{\partial x^j} n_i \tag{3.46c}$$

$$= \int_{\Sigma_t} \mathrm{d}^3x \left[-\sum_{i,j} \overline{\phi}_1 \frac{\partial}{\partial x^j} \left(\sqrt{-g} h^{ij} \frac{\partial \phi_2}{\partial x^i} \right) + \sqrt{-g} m^2 \overline{\phi}_1 \phi_2 \right] \tag{3.46d}$$

$$+ \int_{\partial\Sigma_t} \mathrm{d}^2x \sum_{i,j} \sqrt{-g} h^{ij} \overline{\phi}_1 \frac{\partial \phi_2}{\partial x^i} n_j \tag{3.46e}$$

$$= \int_{\Sigma_t} \mathrm{d}^3x \sqrt{-g} f^{-1} \overline{\phi}_1 f \left[-\frac{1}{\sqrt{-g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{-g} h^{ij} \frac{\partial \phi_2}{\partial x^j} \right) + m^2 \phi_2 \right] \tag{3.46f}$$

$$= \langle \phi_1, \mathcal{K}\phi_2 \rangle_{L^2} \tag{3.46g}$$

□

In general, $\mathcal{K}$ is a function of a complete set o operators, $\mathcal{T}$, that commute. Let $\mathcal{J}$ be the spectrum of these set of operators, μ a measure in $\mathcal{J}$ and ψ_j, $j \in \mathcal{J}$, eigenfunctions of $\mathcal{K}$ with eigenvalues $\omega_j^2 > 0$, i.e.,

$$\mathcal{K}\psi_j = \omega_j^2 \psi_j, \tag{3.47}$$

that satisfy

$$\int_{\Sigma_t} \mathrm{d}^3x \sqrt{-g} f^{-1} \overline{\psi}_j(x) \psi_{j'}(x) = \delta_\mu(j, j'). \tag{3.48}$$

Finally, we define the Hilbert space

$$\mathcal{H}_\xi \equiv \left\{ \varphi(x) = \int \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}} \tilde{\varphi}(j) e^{-i\omega_j t} \psi_j(x) \middle| \tilde{\varphi} \in L^2\left(\mathcal{J}, \mathrm{d}\mu(j)\right) \right\}, \tag{3.49}$$

of the solutions of Klein-Gordon equation.

Proposition 3.5. The Hilbert space $\mathcal{H}_\xi$ satisfies the necessary conditions for quantization.

Proof. show it! $\square$

The quantum field theory constructed in this approach has a natural particle interpretation depending on the field state. We say that the state given by

$$\frac{1}{\sqrt{n!}}\left(a^\dagger(\chi)\right)^n |0\rangle\,, \tag{3.50}$$

represents n particles in the mode $\chi \in \mathcal{H}_\xi$. However, for non-stationary states, any intepretation of particles can be very problematic, hence we emphasize the remark that we are constructing a theory of fields, not particles.

3.3 Particle detector

Let $(\mathcal{M}, g_{ab})$ be a globally hyperbolic spacetime and static spacetime with Killing field ξ that will provide the background for the interaction of a two-level particle detector and a quantized scalar field ϕ. Let Ω be the energetic gap between the detector' levels, then it's the Hamiltonian is defined as

$$H_D = \Omega D^\dagger D, \tag{3.51}$$

where $D|0\rangle = D^\dagger|1\rangle = 0$, $D|1\rangle = |0\rangle$, $D^\dagger|0\rangle = |1\rangle$ and $|0\rangle$, $|1\rangle$ are the ground and excited energy eigenstates. Moreover, the coupling between the detector and the field are described by the interaction Hamiltonian

$$H_{\text{int}}(t) = \epsilon(t)\int_{\Sigma_t} \mathrm{d}^3x\sqrt{-g}\hat{\phi}(x)\left[\psi(t,x)D + \overline{\psi}(t,x)D^\dagger\right], \tag{3.52}$$

where $\hat{\phi}(x)$ is the Klein-Gordon free field operator. The function $\epsilon(t) \in C_0^\infty(\mathcal{M})$ is real and model the fact that the interaction happens in a finite proper time Δ (the time that the detector is on) and, for a fixed $t \in \mathbb{R}$, $\psi(x) \in C_0^\infty(\Sigma_t)$ represents that the coupling is relevant only in a neighborhood of the detector's worldline.

The total Hamiltonian of the system is

$$H = H_0 + H_{\text{int}}, \tag{3.53}$$

where $H_0 = H_{KG} + H_D$ is the free system Hamiltonian. In the interaction picture, the state $|\Psi_t\rangle$ that describes the system at an instant t can be written as

$$|\Psi_t\rangle = T\exp\left[-i\int_{-\infty}^t \mathrm{d}t' H^I_{\text{int}}(t')\right]|\Psi_{-\infty}\rangle\,, \tag{3.54}$$

where T is the time-ordering operator and

$$H^I_{\text{int}} = U_0^\dagger(t) H_{\text{int}} U_0, \tag{3.55}$$

where U_0 is the evolution operator of H_0. Therefore, we have, for $|\Psi_\infty\rangle \equiv |\Psi_{t>\Delta}\rangle$,

$$\begin{aligned}
|\Psi_\infty\rangle &= T\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x\sqrt{-g}\hat{\phi}(x)\epsilon(t)\left(\psi(t,x)U_0^\dagger D U_0 + \overline{\psi}(t,x)U_0^\dagger D U_0\right)\right]|\Psi_{-\infty}\rangle \qquad (3.56a)\\
&= T\exp\left[\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\hat{\phi}(x)\left(\epsilon(t)\psi(t,x)e^{-i\Omega t}D + \overline{\epsilon}(t)\overline{\psi}(t,x)e^{i\Omega t}D^\dagger\right)\right]|\Psi_{-\infty}\rangle \qquad (3.56b)\\
&= T\exp\left[-i\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\hat{\phi}(x)\left(f(t,x)D + \overline{f}(t,x)D^\dagger\right)\right]|\Psi_{-\infty}\rangle\,, \qquad (3.56c)
\end{aligned}$$

where

$$f(t,x) \equiv \epsilon(t)e^{-i\Omega t}\psi(t,x), \tag{3.57}$$

is a complex function with compact support. Henceforth, we assume that the detector is following one orbit of ξ, thus it follows the spatial parametrization, i.e., it has constant spatial coordiantes x in every Σ_t. As a consequence, our function $\psi(t,x)$ is time idenpedent, since the position of the detector is constant (with respect to the notion provided by the Killing field) along the spacetime.

Now, from first order in perturbation theory[11], we have

$$|\Psi_\infty\rangle = \left[\mathbb{I} - i\left(\hat{\phi}(f)D + \hat{\phi}^\dagger(f)D^\dagger\right)\right]|\Psi_{-\infty}\rangle\,. \tag{3.58}$$

[10] *add images on the supports*

Suppose the the function $\epsilon(t)$ vary slowly when compared to the frequency Ω and that $\Delta \gg \Omega^{-1}$. We claim that under this conditions, f is approximately a positive frequency function, i.e., $KEf \cong Ef$ and $KE\overline{f} \cong 0$. To show it, we decompose Ef in terms of positive and negative frequency modes, v_α and $\overline{v}_\alpha$,

$$Ef = \int \mathrm{d}\mu(\alpha)\left(\langle v_\alpha, Ef\rangle\, v_\alpha - \langle \overline{v}_\alpha, Ef\rangle\,\overline{v}_\alpha\right), \tag{3.59}$$

where v_α are solutions of Klein-Gordon equation that satisfies $\langle v_\alpha, v_{\alpha'}\rangle = \delta_\mu(\alpha,\alpha')$. From the fact that v_α is a positive frequency solution, it follows that is also eigenfunction of $i\partial_t$ with eigenvalue $\omega_\alpha > 0$. Thus we can write

$$v_\alpha = e^{-i\omega_\alpha t}\varphi_\alpha(x), \tag{3.60}$$

where

$$\mathcal{K}\varphi_\alpha = \omega_\alpha^2 \varphi_\alpha, \tag{3.61}$$

and $\mathcal{K}$ is the same operator defined in Equation (3.45). Furthermore, from Proposition 2.6, we have

$$\langle v_\alpha, Ef \rangle = i \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} f \overline{v}_\alpha \tag{3.62a}$$

$$\langle \overline{v}_\alpha, Ef \rangle = i \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} f v_\alpha. \tag{3.62b}$$

Using our approximation that $\epsilon(t) \cong \epsilon = \text{cte}$ when the detector is turned on (and $\epsilon(t) = 0$ when it's turned off), we have

$$\langle \overline{v}_\alpha, Ef \rangle = i \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} f v_\alpha \tag{3.63a}$$

$$= i \int_{-\infty}^{\infty} \mathrm{d}t \int_{\Sigma_t} \mathrm{d}^3x \sqrt{-g} \epsilon(t) e^{-i\Omega t} \psi(x) e^{-i\omega_\alpha t} \varphi_\alpha(x) \tag{3.63b}$$

$$= i\epsilon \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \mathrm{d}t e^{-i(\Omega+\omega_\alpha)t} \int_{\Sigma_t} \mathrm{d}^3x \sqrt{-g} \psi(x) \varphi_\alpha(x) \tag{3.63c}$$

$$= i\epsilon\gamma_\alpha \left(-\frac{1}{i(\Omega+\omega_\alpha)} e^{-i(\Omega+\omega_\alpha)t} \right) \Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \tag{3.63d}$$

$$= 2i\epsilon\gamma_\alpha \frac{\sin\left[(\Omega+\omega_\alpha)\frac{\Delta}{2}\right]}{\Omega+\omega_\alpha}, \tag{3.63e}$$

where

$$\gamma_\alpha \equiv \int_{\Sigma_t} \mathrm{d}^3x \sqrt{-g} \psi(x) \varphi_\alpha(x). \tag{3.64}$$

Now, we have, from Proposition A.1,

$$\frac{\sin\left[(\Omega+\omega_\alpha)\frac{\Delta}{2}\right]}{\Omega+\omega_\alpha} \cong \pi\delta(\Omega+\omega_\alpha), \tag{3.65}$$

when $\Delta \gg \Omega^{-1}$. This express the fact that only the mode with frequency $\omega_\alpha = -\Omega$ contributes for the negative frequency part, thus $\langle \overline{v}_\alpha, Ef \rangle \cong 0$. Therefore, we have $KEf \cong Ef$.

Similarly, consider,

$$\left\langle v_\alpha, E\overline{f}\right\rangle = i\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\overline{f}\overline{v}_\alpha \tag{3.66a}$$

$$= i\int_{-\infty}^{\infty}\mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x\sqrt{-g}\overline{\epsilon}(t)e^{i\Omega t}\overline{\psi}(x)e^{i\omega_\alpha t}\overline{\varphi}(x) \tag{3.66b}$$

$$= i\epsilon\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}\mathrm{d}te^{i(\Omega+\omega_\alpha)t}\int_{\Sigma_t}\mathrm{d}^3x\sqrt{-g}\overline{\psi}(x)\overline{\varphi}(x) \tag{3.66c}$$

$$= i\epsilon\overline{\gamma}_\alpha\left(\frac{1}{i(\Omega+\omega_\alpha)}e^{i(\Omega+\omega_\alpha)t}\right)\Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \tag{3.66d}$$

$$= 2i\epsilon\overline{\gamma}_\alpha\frac{\sin\left[(\Omega+\omega_\alpha)\frac{\Delta}{2}\right]}{\Omega+\omega_\alpha}. \tag{3.66e}$$

The same analysis shows that $KE\overline{f} \cong 0$, i.e., the solutions $E\overline{f}$ is approximately of negative frequency.

Now, if we define

$$\lambda \equiv -KEf, \tag{3.67}$$

we have,

$$\hat{\phi}(f) = ia\left(\overline{KE\overline{f}}\right) - ia^\dagger\left(KEf\right) \cong ia^\dagger(\lambda). \tag{3.68}$$

Substituting in Equation (3.58),

$$|\Psi_\infty\rangle = \left(\mathbb{I} + a^\dagger(\lambda)D - a(\overline{\lambda})D^\dagger\right)|\Psi_{-\infty}\rangle\,. \tag{3.69}$$

The expression above shows that the excitation and de-excitation of a detector following the orbits of temporal isometry is associated with the absorption and emission of particles naturally define by the observers comoving with the detector.

In order to illustrate it, we shall present some examples.

- Vacuum field:

  Consider the initial state

$$|\Psi_{-\infty}\rangle = |0\rangle_\xi \otimes \left(\alpha\,|0\rangle + \beta\,|1\rangle\right), \tag{3.70}$$

  where $|0\rangle_\xi$ is the vacuum associated with the temporal isometries. Substituting in Equation (3.69) and defining the normalized mode

$$\hat{\lambda} \equiv \frac{\lambda}{\|\lambda\|}, \tag{3.71}$$

we find

$$
\begin{aligned}
|\Psi_\infty\rangle &= |\Psi_{-\infty}\rangle + \beta a^\dagger(\lambda)\,|0\rangle_\xi \otimes D\,|1\rangle - \alpha a(\overline{\lambda})\,|0\rangle_\xi \otimes D^\dagger\,|0\rangle \qquad (3.72a)\\
&= |0\rangle_\xi \otimes (\alpha\,|0\rangle + \beta\,|1\rangle) + \beta\|\lambda\|\,|\hat{\lambda}\rangle \otimes |0\rangle\,. \qquad (3.72b)
\end{aligned}
$$

Let $\alpha = 0$ and $\beta = 1$, then the probability of the detector de-excitate emitting a mode $\hat{\lambda}$ is

$$
\begin{aligned}
P_{1\to 0} &= \langle \Psi_\infty|\,\mathbb{I}_\phi \otimes |0\rangle\,\langle 0|\,|\Psi_\infty\rangle \qquad (3.73a)\\
&= \|\lambda\|^2 \qquad (3.73b)\\
&= \|KEf\|^2. \qquad (3.73c)
\end{aligned}
$$

Similarly, if $\alpha = 1$ and $\beta = 0$, then

$$
|\Psi_\infty\rangle = |0\rangle_\xi \otimes |0\rangle\,, \qquad (3.74)
$$

i.e., the detector remains in the ground state.

- One particle field:

  Now, we take the initial state

$$
|\Psi_{-\infty}\rangle = |1\rangle_\chi \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)\,, \qquad (3.75)
$$

where $|1\rangle_\chi$ is the state with one particle of the mode $\chi \in \mathcal{H}_\xi$. Therefore,

$$
|\Psi_\infty\rangle = |1\rangle_\chi \otimes (\alpha\,|0\rangle + \beta\,|1\rangle) + \beta\|\lambda\|\,|1_\chi, 1_{\hat{\lambda}}\rangle \otimes |0\rangle - \alpha\|\lambda\|\left\langle \hat{\lambda}, \chi \right\rangle |0\rangle_\xi \otimes |1\rangle\,. \qquad (3.76)
$$

Let $\alpha = 1$ and $\beta = 0$, the probability of de-excitation, detecting χ, is

$$
P_{0\to 1} = \langle \Psi_\infty|\,\mathbb{I}_\phi \otimes |1\rangle\,\langle 1|\,|\Psi_\infty\rangle = |\langle KEf, \chi\rangle|^2. \qquad (3.77)
$$

In the approximation $KEf \cong Ef$, we have

$$
P_{0\to 1} = |\langle Ef, \chi\rangle|^2 = \left| \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\,\overline{\chi}(x) f(x) \right|^2. \qquad (3.78)
$$

3.4 Bogoliubov transformations

APPENDIX A

MATHEMATICAL IDENTITIES

This chapter intends to develop mathematical identities used throughout the text.

Proposition A.1. It holds

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x} f(x) \mathrm{d}x = 0, \tag{A.1}$$

i.e.,

$$\lim_{\epsilon \to 0^+} \frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x} = \delta(x). \tag{A.2}$$

APPENDIX B

FOCK SPACE

The symmetric Fock space is the Hilbert space used in the construction of our quantum field theory in curved spacetimes for a scalar field. However, it must be constructed from a existent Hilbert space, that physically can have the interpretation of the space of one particle.

Let $\mathcal{H}$ be a Hilbert space, then the Fock space associated to it, $\mathcal{F}(\mathcal{H})$, is defined as

$$\mathcal{F}(\mathcal{H}) \equiv \oplus_{n=0}^{\infty} \left(\otimes^{n} \mathcal{H}\right). \tag{B.1}$$

Therefore, an element $\Psi \in \mathcal{F}(\mathcal{H})$ is infinite dimension vector whose each entry is a tensor product of elements of $\mathcal{H}$, i.e.,

$$\Psi = \Big(\psi, \psi^{a}, \psi^{ab}, \dots\Big), \tag{B.2}$$

where

$$\psi^{a_1 \dots a_j} \in \otimes^{j} \mathcal{H}. \tag{B.3}$$

The symmetric Fock space, $\mathcal{F}_S(\mathcal{H})$, is defined as

$$\mathcal{F}_S(\mathcal{H}) \equiv \oplus_{n=0}^{\infty} \left(\otimes_S^{n} \mathcal{H}\right), \tag{B.4}$$

that is very similar to the ordinary Fock space, however, the possible entries must be symmetric, i.e.,

$$\psi^{a_1 \dots a_j} = \psi^{(a_1 \dots a_j)}. \tag{B.5}$$

In order to the Fock space be a Hilbert space, we equip it with inner product that act in two elements $\Psi, \Phi \in \mathcal{F}_S(\mathcal{H})$ as

$$\langle \Psi, \Phi \rangle_{\mathcal{F}_S} \equiv \overline{\psi}\phi + \overline{\psi}_a \phi^a + \dots, \tag{B.6}$$

in which the elements $\overline{\psi}_a$ are the dual vectors correspondent to the conjugated one in $\overline{\mathcal{H}}$.

Algebra of operators

Now, we define the annihilation and creation operators in the Fock space. Let $\xi^a \in \mathcal{H}$, then the annihilation operator $a(\overline{\xi}) : \mathcal{F}_S(\mathcal{H}) \to \mathcal{F}_S(\mathcal{H})$ is defined as

$$a(\overline{\xi})\Psi \equiv \left(\overline{\xi}_{a_1}\psi^{a_1}, \sqrt{2}\overline{\xi}_{a_1}\psi^{a_1a_2}, \sqrt{3}\overline{\xi}_{a_1}\psi^{a_1a_2a_3}, \dots\right). \tag{B.7}$$

Similarly, the creation operator $a^\dagger(\xi) : \mathcal{F}_S(\mathcal{H}) \to \mathcal{F}_S(\mathcal{H})$ is defined as

$$a^\dagger(\xi)\Psi \equiv \left(0, \psi\xi^a, \sqrt{(2)}\xi^{(a_1}\psi^{a_2)}, \sqrt{3}\xi^{(a_1}\psi^{a_2a_3)}, \dots\right). \tag{B.8}$$

Proposition B.1. Let $\xi, \eta \in \mathcal{H}$, then the annihilation and creation operators has commutation relation given by

$$[a(\overline{\xi}), a(\overline{\eta})] = [a^\dagger(\xi), a^\dagger(\eta)] = 0 \tag{B.9}$$

and

$$[a(\overline{\xi}), a^\dagger(\eta)] = \overline{\xi}_a\eta^a\mathbb{I}. \tag{B.10}$$

Proof. We shall restric our proof to the first two non-trivial termos. Using our definitions, we have

$$a(\overline{\xi})a(\overline{\eta})\Psi = \left(\sqrt{2}\overline{\xi}_{a_2}\overline{\eta}_{a_1}\psi^{a_1a_2}, \sqrt{6}\overline{\xi}_{a_2}\overline{\eta}_{a_1}\psi^{a_1a_2a_3}, \dots\right). \tag{B.11}$$

Similarly, we have

$$a(\overline{\eta})a(\overline{\xi})\Psi = \left(\sqrt{2}\overline{\eta}_{a_2}\overline{\xi}_{a_1}\psi_2^{a_1a_2}, \sqrt{6}\overline{\eta}_{a_2}\overline{\xi}_{a_1}\psi_3^{a_1a_2a_3}, \dots\right). \tag{B.12}$$

However, from the Fock space symmetry,

$$\overline{\xi}_{a_2}\overline{\eta}_{a_1}\psi_2^{a_1a_2} = \overline{\eta}_{a_2}\overline{\xi}_{a_1}\psi_2^{a_1a_2} \quad \text{and} \quad \overline{\xi}_{a_2}\overline{\eta}_{a_1}\psi_3^{a_1a_2a_3} = \overline{\eta}_{a_2}\overline{\xi}_{a_1}\psi_3^{a_1a_2a_3}. \tag{B.13}$$

Arguments exploring the symmetry also holds for the other components. Since it is applied in an arbitrary vector Ψ, is evident that the commutator vanishes. Similarly, from the definition of creation operator, we have

$$\begin{aligned} a^\dagger(\xi)a^\dagger(\eta)\Psi &= \left(0, 0, \sqrt{2}\psi\xi^{(a_1}\eta^{a_2)}, \sqrt{6}\xi^{(a_1}\eta^{a_2}\psi^{a_3)}, \dots\right) \\ a^\dagger(\eta)a^\dagger(\xi)\Psi &= \left(0, 0, \sqrt{2}\psi\eta^{(a_1}\xi^{a_2)}, \sqrt{6}\eta^{(a_1}\xi^{a_2}\psi^{a_3)}, \dots\right). \end{aligned} \tag{B.14}$$

Again, invoking the symmmetry of the Fock space, we have

$$\psi_0\xi^{(a_1}\eta^{a_2)} = \psi_0\eta^{(a_1}\xi^{a_2)} \quad \text{and} \quad \xi^{(a_1}\eta^{a_2}\psi_1^{a_3)} = \eta^{(a_1}\xi^{a_2}\psi_1^{a_3)}. \tag{B.15}$$

With the same strategies for the other components, we find that the commutator of creation operators are null.
Finally, the elements for the last commutator are

$$a(\bar{\xi})a^{\dagger}(\eta)\Psi = \Big(\psi\bar{\xi}_a\eta^a, 2\bar{\xi}_a\eta^{(a}\psi^{a_2)}, 3\bar{\xi}_a\eta^{(a}\psi^{a_1a_2),\cdots}\Big) \tag{B.16a}$$

$$a^{\dagger}(\eta)a(\bar{\xi})\Psi = \Big(0, \bar{\xi}_a\psi^a\eta^{a_1}, 2\bar{\xi}_a\psi^{a(a_1}\eta^{a_2)}, \dots\Big). \tag{B.16b}$$

Notice that

$$2\bar{\xi}_a\eta^{(a}\psi^{a_1)} = \bar{\xi}_a\eta^a\psi^{a_1} + \bar{\xi}_a\psi^a\eta^{a_1} \tag{B.17}$$

and

$$3\bar{\xi}_{a_1}\eta^{(a_1}\psi^{a_2a_3)} = \frac{1}{2}\bar{\xi}_a[\eta^{a_1}\psi^{a_2a_3} + \eta^{a_1}\psi^{a_3a_2} + \eta^{a_2}\psi^{a_1a_3} \tag{B.18a}$$

$$+ \eta^{a_2}\psi^{a_3a_1} + \eta^{a_3}\psi^{a_1a_2} + \eta^{a_3}\psi^{a_2a_1}] \tag{B.18b}$$

$$= \bar{\xi}_{a_1}\eta^{a_1}\psi^{(a_2a_3)} + \bar{\xi}_{a_1}\left[\psi^{a_1a_3}\eta^{a_2} + \psi^{a_1a_2}\eta^{a_3}\right] \tag{B.18c}$$

$$= \bar{\xi}_{a_1}\eta^{a_1}\psi_2^{(a_2a_3)} + 2\bar{\xi}_{a_1}\psi_2^{(a_1a_2}\eta^{a_3)}. \tag{B.18d}$$

Thus, if we procede with similar manipulations for the other components, we find that

$$[a(\bar{\xi}), a^{\dagger}(\eta)]\Psi = \bar{\xi}_a\eta^a\Psi, \tag{B.19}$$

that proves the desired relations. □