


Theoretical Foundations of Semiclassical Gravity



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CH

PREFACE

These notes aim to keep track of my studies for my undergraduate project, developed while I am persuing my Bachelor degree in Physics at the Institute of Physics, University of São Paulo (IFUSP).

First of all, we shall cover the basics aspects of the standard bottom-top approach to describe quantum gravity in low-energies, commonly known as quantum field theory in curved spacetimes. The main goal is to understand one prediction of this theory, named Unruh effect and develop the basic concepts of theory. After that, we shall procede to formulation of electromagnetism in Minkowski spacetime, to understand the radiation emitted by an accelerated particle, a result from classical physics. Finally, we explore the correspondence between the Unruh effect, a essentialy quantum effect, and Larmor radiation, a classical effect, when we move from the frame of a inertial observer to an accelerated one.

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PART I



QUANTUM FIELD THEORY IN CURVED SPACETIMES

CHAPTER 1

GEOMETRIC QUANTUM MECHANICS

Through this chapter we shall develop the necessary aspects of classical and quantum theory in geometric picture. Moreover, we apply this construction to a famous linear system: N decoupled harmonic oscillators.

1.1 Classical theory

Geometric formulation

In Hamiltonian classical mechanics, the space of study is $2n$ -dimensional manifold \mathcal{P} called phase space whose points are determined by n generalized coordinates (q^1, \dots, q^n) and their conjugated momenta (p_1, \dots, p_n) . Each system has an associated Hamiltonian function $H = H(q^1, \dots, q^n; p_1, \dots, p_n)$ that provides the time evolution of the system by Hamilton equations

$$\frac{dq^\mu}{dt} = \frac{\partial H}{\partial p_\mu} \quad \text{and} \quad \frac{dp_\mu}{dt} = -\frac{\partial H}{\partial q^\mu}. \quad (1.1)$$

To put the coordinates of phase space in the same level, we define the $2n$ -component object

$$y \equiv (q^1, \dots, q^n; p_1, \dots, p_n). \quad (1.2)$$

Moreover, we also define a $2n \times 2n$ object, Ω^{ij} , with components given by

$$\Omega^{ij} = \begin{pmatrix} 0_n & \mathbb{I}_n \\ -\mathbb{I}_n & 0_n \end{pmatrix}, \quad (1.3)$$

then, we are able to write the Hamilton's equations as

$$\frac{dy^\mu}{dt} = \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \frac{\partial H}{\partial y^\nu}. \quad (1.4)$$

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Inspired by this manipulation, in a more abstract formulation, a system with n degrees of freedom is represented by points in the $2n$ -dimensional manifold \mathcal{P} with a symplectic form Ω_{ab} , i.e., a 2-form that is non-degenerated and closed in \mathcal{P} . Mathematically, this is statement is equivalent to

1. $\Omega_{ab} = \Omega_{[ab]}$.
2. $\nabla_{[a}\Omega_{bc]} = 0$.
3. For every tagent vector v^b on \mathcal{P} , we have

$$v^b \Omega_{ab} = 0 \iff v^b = 0. \quad (1.5)$$

Since the form is non-degenerated, it has an inverse, Ω^{ab} , defined by the relation

$$\Omega^{ab}\Omega_{bc} = \delta_c^a. \quad (1.6)$$

From the Hamiltonian function, we define the hamiltonian vector field, h^a , by

$$h^a \equiv \Omega^{ab}\nabla_b H. \quad (1.7)$$

Thereby, in the abstract formulation, the dynamics are such that the possible physical paths are the ones that follows the orbits of h^a .

To relate both formulations, we pick a n -dimensional manifold \mathcal{Q} with local coodinates an take $\mathcal{P} = T_*(\mathcal{Q})$, i.e., the cotangent bundle of \mathcal{Q} formed by the points of \mathcal{Q} and the cotangent vectors at each point. Therefore, the symplectic form, expressed as a function of local coordinates, can be defined as

$$\Omega_{ab} = 2 \sum_{\mu} (\nabla p_{\mu})_{[a} (\nabla q^{\mu})_{b]}, \quad (1.8)$$

or

$$\Omega = \sum_{\mu} dp_{\mu} \wedge dq_{\mu}. \quad (1.9)$$

Even though we constructed the form using a choice of coordinates, it is coordinate independent¹.

In order to have equivalent constructions (conventional and abstract), they must lead the same dynamical equations for the system. Notice that it indeed hols².

Thus, the abstract formulation provides coordinate independet equations of motion. Moreover, notice that we can identify the manifold \mathcal{P} with the solution space \mathcal{S} associating each $y \in \mathcal{P}$ with the solution whose initial conditions is given by that configuration.

¹ *show it!*

² *show it!*

Now, remember that physical observables in Hamiltonian mechanics is given by smooth function acting on the phase space, $f \in C^\infty : \mathcal{P} \rightarrow \mathbb{R}$. These observables have algebraic structure given by the Poisson brackets. For $f, g : \mathcal{P} \rightarrow \mathbb{R}$, it is defined as

$$\{f, g\} \equiv \Omega^{ab} \nabla_a f \nabla_b g. \quad (1.10)$$

The calculation of the Poisson brackets for the fundamentals observables (q^μ, p_ν) follows:

$$\{q^\mu, q^\nu\} = \Omega^{ab} \nabla_a q^\mu \nabla_b q^\nu \quad (1.11a)$$

$$= \Omega^{ab} \sum_\mu \nabla_a q^\mu \nabla_b q^\mu \delta_\mu^\nu \quad (1.11b)$$

$$= \sum_\mu \Omega^{[ab]} \nabla_{(a} q^\mu \nabla_{b)} q^\mu \delta_\mu^\nu \quad (1.11c)$$

$$= 0. \quad (1.11d)$$

Similarly,

$$\{p_\mu, p_\nu\} = \Omega^{ab} \nabla_a p_\mu \nabla_b p_\nu \quad (1.12a)$$

$$= \Omega^{ab} \sum_\mu \nabla_a q^\mu \nabla_b q_\mu \delta_\nu^\mu \quad (1.12b)$$

$$= \sum_\mu \Omega^{[ab]} \nabla_{(a} p_\mu \nabla_{b)} p_\mu \delta_\nu^\mu \quad (1.12c)$$

$$= 0. \quad (1.12d)$$

At last, we have

$$\{q^\mu, p_\nu\} = ? \quad (1.13)$$

Linear dynamical systems

Henceforth, we shall restrict our analysis to system that are linear in order to have additional structures in our construction. More specifically, we consider systems that satisfies the following conditions:

1. The manifold of the configuration space, \mathcal{Q} , has vector space structure, hence it's possible to choose a linear canonical coordinate system (q^1, \dots, q^n) globally in \mathcal{Q} . Consequently, there exists a global coordinate system induced in \mathcal{P} that also has vector space structure.
2. The Hamiltonian is quadratic, thus the solution space, \mathcal{S} , has vector space structure.

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As a consequence of [Item 1](#), we can identify the tangent space of any point $p \in \mathcal{P}$ as \mathcal{P} itself. Thereby, the symplectic form is a bilinear map $\Omega : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ in \mathcal{P} . Furthermore, since the components of Ω_{ab} are constant in the canonical global basis³, the map independ of the chosen y used to make the identification. The action of the symplectic structure in $y_1, y_2 \in \mathcal{P}$ is

$$\Omega(y_1, y_2) = \Omega_{ab} y_1^a y_2^b \quad (1.14a)$$

$$= \sum_{\mu} (\nabla_a p_{\mu} \nabla_b q^{\mu} - \nabla_a q^{\mu} \nabla_b p_{\mu}) y_1^a y_2^b \quad (1.14b)$$

$$= \sum_{\mu} \left(\delta_a^{\mu+n} \delta_b^{\mu} - \delta_a^{\mu} \delta_b^{\mu+n} \right) y_1^a y_2^b \quad (1.14c)$$

$$= \sum_{\mu} (p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}). \quad (1.14d)$$

Now, we can use the symplectic structure to substitute the canonical coordiantes that describe the manifold of interest. Notice that for a fixed $y \in \mathcal{P}$, the quantity $\Omega(y, \cdot)$ is a linear function in \mathcal{P} . Therefore, we have

$$\begin{cases} y = (0, \dots, 0, q^{\mu} = 1, 0, \dots, 0) \implies \Omega(y, \cdot) = -p_{\mu} \\ y = (0, \dots, 0, p_{\mu} = 1, 0, \dots, 0) \implies \Omega(y, \cdot) = q^{\mu} \end{cases}, \quad (1.15)$$

i.e., we can replace the coordinate by coordinates independent structures. In this geometric description, the Poisson bracktes, that satisfy the canonical relations [Equations \(1.11\) to \(1.13\)](#), is given by the more general expression

$$\{\Omega(y_1, \cdot), \Omega(y_2, \cdot)\} = -\Omega(y_1, y_2). \quad (1.16)$$

From now on, we consider the implications of [Item 2](#). First of all, the Hamiltonian has the general form

$$H(t, y) = \frac{1}{2} \sum_{\mu, \nu} K_{\mu\nu}(t) y^{\mu} y^{\nu}, \quad (1.17)$$

in which, without loss of generality, holds $K_{\mu\nu} = K_{\nu\mu}$. For this special case, the Hamilton equations are

$$\frac{dy^{\mu}}{dt} = \sum_{\nu, \sigma} \Omega^{\mu\sigma} K_{\sigma\nu} y^{\nu}. \quad (1.18)$$

Let $y_1, y_2 \in \mathcal{S}$ be two solutions of Hamilton equations, hence we define

$$s(t) \equiv \Omega(y_1(t), y_2(t)) = \sum_{\mu\nu} \Omega_{\mu\nu} y_1^{\mu} y_2^{\nu}. \quad (1.19)$$

Notice that

$$\frac{ds(t)}{dt} = \sum_{\mu\nu} \Omega_{\mu\nu} \left[\frac{dy_1^\mu}{dt} y_2^\nu + y_1^\mu \frac{dy_2^\nu}{dt} \right] \quad (1.20a)$$

$$= \sum_{\mu,\nu,\alpha,\beta} \Omega_{\mu\nu} \left[\Omega^{\mu\alpha} K_{\alpha\beta} y_1^\beta y_2^\nu + \Omega^{\nu\alpha} K_{\alpha\beta} y_1^\mu y_2^\beta \right] \quad (1.20b)$$

$$= - \sum_{\nu,\beta} K_{\nu\beta} y_1^\beta y_2^\nu + \sum_{\mu,\beta} K_{\mu\beta} y_1^\mu y_2^\beta \quad (1.20c)$$

$$= 0. \quad (1.20d)$$

Therefore, Ω induces a symplectic structure on the solution space \mathcal{S} , since the identification between \mathcal{P} and \mathcal{S} is independent of the time at it is done. Thus, (\mathcal{S}, Ω) has a symplectic vector space structure.

1.2 Quantum theory

Quantization

The quantization process of a theory, consists in finding a Hilbert space \mathcal{H} that represent all the possible states and transform the classical observers f in self-adjoint operators \hat{f} . The leading point is to identify the Poisson bracktes as the equivalent of the quantum commutator, i.e., a map $\hat{\cdot}: f \rightarrow \hat{f}$ that satisfies

$$[\hat{f}, \hat{g}] = i\{f, g\}, \quad (1.21)$$

where we used $\hbar = 1$. It is possible to construct this map in a such a way that all the fundamental observers respect [Equation \(1.21\)](#), thus any observable that is linear q, p also satisfy. However, in the cases that [Item 1](#) holds, we have

$$[\hat{\Omega}(y_1, \cdot), \hat{\Omega}(y_2, \cdot)] = -i\Omega(y_1, y_2)\mathbb{I}. \quad (1.22)$$

Weyl's relations

Besides that, in general, the operators in the coordinate independent description are not bounded (and only densely defined), in such a way that we must call on the exponential version of the commutation relation. To do it, we define

$$W(y) \equiv \exp[i\Omega(y, \cdot)] \quad (1.23)$$

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Furthermore, we impose that $\hat{W}(y)$ be unitary, be strongly continuous with respect to y and satisfy

$$\hat{W}(y_1)\hat{W}(y_2) = \exp\left[\frac{i}{2}\Omega(y_1, y_2)\right]\hat{W}(y_1 + y_2), \quad (1.24)$$

and

$$\hat{W}^\dagger(y) = \hat{W}(-y). \quad (1.25)$$

The [Equations \(1.24\)](#) and [\(1.25\)](#) are known as Weyl's relations and they are enough to uniquely determine $(\mathcal{H}, \hat{W}(y))$. Therefore, a choice of Hilbert space \mathcal{H} and operators $\hat{W}(y)$ that satisfies the Weyl relations are called irreducible representation of Weyl's relations.

Equivalent theories

Moreover, a Hilbert space \mathcal{H} and a set of indexed operators $V_\alpha : \mathcal{H} \rightarrow \mathcal{H}$ is unitarily equivalent to \mathcal{H}' and V'_α if exists and unitary transformation $U : \mathcal{H} \rightarrow \mathcal{H}'$ such that

$$V_\alpha = U^{-1}V'_\alpha U, \quad \forall \alpha \quad (1.26)$$

After the discussion above, is convenient to present Stone-von Neumann's theorem:

Theorem 1.1 (Stone-von Neumann). Let (\mathcal{P}, Ω) be a finite dimension symplectic vector space and $(\mathcal{H}, \hat{W}(y))$, $(\mathcal{H}', \hat{W}'(y))$ be two strongly continuous, irreducible and unitary representation of Weyl's relations. Then, both representations are unitarily equivalent.

Thereby, since these representations of Weyl's relations generate equivalent theories, both describe the same physical system. Furthermore, notice that the theorem requires that the space is finite dimensional, hence in the treatment of scalar fields, exists many choices of Weyl's relations that yields non equivalent theories.

1.3 Harmonic oscillators

Right now, we shall apply the develop formalism in a well known system: quantum harmonic oscillators. Our main goal is to study scalar fields and we can interpret them as a set of oscillators in each point in space, hence this construction is a preliminary result for our quantization.

Single oscillator

First, let us remind the standard formulation presented in most textbooks. The system of a harmonic oscillator is represented by the hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2, \quad (1.27)$$

The quantized system is attained by choosing the Hilbert space $L^2(\mathbb{R})$ and the quantized operator is obtained by putting \hat{q} and \hat{p} that satisfies the commutation relation

$$[\hat{q}, \hat{p}] = i\mathbb{I}. \quad (1.28)$$

The description of the system is easier if we introduce the annihilation operator

$$\hat{a} \equiv \sqrt{\frac{\omega}{2}} \left(\hat{q} + \frac{i}{\omega} \hat{p} \right), \quad (1.29)$$

whose adjoint, named creation operator, is given by

$$\hat{a}^\dagger = \sqrt{\frac{\omega}{2}} \left(\hat{q} - \frac{i}{\omega} \hat{p} \right). \quad (1.30)$$

The commutator of these operators is

$$[\hat{a}, \hat{a}^\dagger] = \frac{\omega}{2} \left[\hat{q} + \frac{i}{\omega} \hat{p}, \hat{q} - \frac{i}{\omega} \hat{p} \right] \quad (1.31a)$$

$$= \frac{\omega}{2} \left(-\frac{i}{\omega} [\hat{q}, \hat{p}] + \frac{i}{\omega} [\hat{p}, \hat{q}] \right) \quad (1.31b)$$

$$= -i[\hat{q}, \hat{p}] \quad (1.31c)$$

$$= \mathbb{I}. \quad (1.31d)$$

Furthermore, notice that

$$\hat{a}^\dagger \hat{a} = \frac{\omega}{2} \left(\hat{q} - \frac{i}{\omega} \hat{p} \right) \left(\hat{q} + \frac{i}{\omega} \hat{p} \right) \quad (1.32a)$$

$$= \frac{\omega}{2} \left(\hat{q}^2 + \frac{1}{\omega^2} \hat{p}^2 + \frac{1}{\omega} [\hat{q}, \hat{p}] \right) \quad (1.32b)$$

$$= \frac{\omega}{2} \hat{q}^2 + \frac{1}{2\omega} \hat{p}^2 + \frac{i}{2} [\hat{q}, \hat{p}] \quad (1.32c)$$

$$= \frac{1}{\omega} \hat{H} - \frac{1}{2} \mathbb{I}, \quad (1.32d)$$

thus,

$$\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \mathbb{I} \right). \quad (1.33)$$

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It is also convenient to compute the commutator

$$[\hat{H}, \hat{a}] = \omega \left[\hat{a}^\dagger \hat{a} + \frac{1}{2} \mathbb{I}, \hat{a} \right] \quad (1.34a)$$

$$= \omega [\hat{a}^\dagger, \hat{a}] \hat{a} \quad (1.34b)$$

$$= -\omega \hat{a}. \quad (1.34c)$$

Therefore, in the Heisenberg picture, we have

$$\frac{d\hat{a}_H}{dt} = i [\hat{H}, \hat{a}_H] = -i\omega \hat{a}_H \implies \hat{a}_H(t) = e^{-i\omega t} \hat{a}. \quad (1.35)$$

This leads to the fundamental operators to be

$$\hat{q}_H(t) = \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega t} \hat{a} + e^{i\omega t} \hat{a}^\dagger \right), \quad (1.36)$$

and

$$\hat{p}_H(t) = \frac{d\hat{q}_H(t)}{dt} = -i\sqrt{\frac{\omega}{2}} \left(e^{-i\omega t} \hat{a} - e^{i\omega t} \hat{a}^\dagger \right). \quad (1.37)$$

One should be familiar with the result that the ground state, $|0\rangle$, satisfies

$$\hat{a} |0\rangle = 0, \quad (1.38)$$

and the n^{th} state is obtained applying the creation operator n times, i.e.,

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad (1.39)$$

that satisfies

$$\hat{H} |n\rangle = \omega \left(n + \frac{1}{2} \right) |n\rangle. \quad (1.40)$$

Multiple oscillators

To generalize for N decoupled oscillators, inspired by the previous construction, we take the Hilbert space to be $\mathcal{H} \equiv \otimes_{j=1}^N \mathcal{H}_j$ with fundamental operators whose commutators are given by

$$[\hat{q}_j, \hat{p}_k] = i\delta_{jk} \mathbb{I}. \quad (1.41)$$

The vacuum state is taken as the tensor product of each oscillator, so we guarantee that is annihilated by any annihilation operator, i.e.,

$$|0\rangle \equiv \otimes_{j=1}^N |0\rangle_j. \quad (1.42)$$

The excited states are obtained by a similar approach

$$|n_1, \dots, n_N\rangle \equiv \frac{1}{\sqrt{n_1!}} (\hat{a}_1^\dagger)^{n_1} \dots \frac{1}{\sqrt{n_N!}} (\hat{a}_N^\dagger)^{n_N} |0\rangle. \quad (1.43)$$

Abstract formulation

Now, we focus on applying the geometric approach to the N decoupled quantum harmonic oscillators. Remember that due to Stone-von Neumann's theorem, the previous formulations is unitarily equivalent, hence describe the same physical system.

First, we take the symplectic vector space of classical solutions (\mathcal{S}, Ω) and complexify it, i.e., $\mathcal{S} \rightarrow \mathcal{S}^{\mathbb{C}}$. It is possible to extend the action of the form Ω to the complexified space by linearity. Thereby, using the symplectic structure, we define a map in the solution space $\mathcal{S}^{\mathbb{C}}$, named Klein-Gordon inner product, as

$$\langle y_1, y_2 \rangle \equiv -i\Omega(\overline{y_1}, y_2). \quad (1.44)$$

Notice that a general (complex) solution for a set of oscillators is

$$q^j(t) = \alpha_j e^{-i\omega_j t} + \beta_j e^{i\omega_j t}. \quad (1.45)$$

Then, consider

$$\langle y, y \rangle = -i\Omega(\overline{y}, y) \quad (1.46a)$$

$$= -i \sum_j \overline{p_j} q^j - p_j \overline{q^j} \quad (1.46b)$$

$$= -i \sum_j \omega_j \left(\overline{\alpha_j} e^{i\omega_j t} - \beta_j e^{-i\omega_j t} \right) \left(\alpha_j e^{-i\omega_j t} + \beta_j e^{i\omega_j t} \right) \quad (1.46c)$$

$$- i\omega_j \left(-\alpha_j e^{-i\omega_j t} + \beta_j e^{i\omega_j t} \right) \left(\overline{\alpha_j} e^{i\omega_j t} + \overline{\beta_j} e^{-i\omega_j t} \right) \quad (1.46d)$$

$$= 2 \sum_j \omega_j \left(|\alpha_j|^2 - |\beta_j|^2 \right). \quad (1.46e)$$

Thus, notice that, in general, it is not positive-definite. However, if we take the n -dimensional space \mathcal{H} of the complex solutions with positive frequency, i.e., such that $q^j \in \mathcal{H}$, then

$$q^j(t) = \alpha_j e^{-i\omega_j t}. \quad (1.47)$$

If we impose, $\beta_j = 0$ in Equation (1.46), it is evident that the map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is positive-definite, hence an inner product. Then, \mathcal{H} is a Hilbert space and its symmetric Fock space, $\mathcal{F}_S(\mathcal{H})$, is the Hilbert space used in our alternative quantization process.

Let $\xi^j \in \mathcal{H}$ be a complex solution such that only the j^{th} mode is excited and suppose it is normalized, i.e.,

$$\|\xi^j\|^2 = \langle \xi^j, \xi^j \rangle = 1, \quad (1.48)$$

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then the set $\{\xi^j\}_{j \in I_n}$ is an orthonormal basis of \mathcal{H} . Now, we define the position and momentum operators (in Heisenberg picture) to be

$$\hat{q}^j(t) = \xi^j(t) \hat{a}(\bar{\xi}) + \bar{\xi}^j \hat{a}^\dagger(\xi) \quad (1.49)$$

and

$$p_j(t) = \frac{dq^j}{dt}, \quad (1.50)$$

in which \hat{a}_j is the annihilation operator of the Fock space. Using the [Proposition B.1](#), we find⁴

$$[q^\mu, p_\nu] = ? \quad (1.51)$$

⁴ calculate!

⁵ should I show it?

As mentioned earlier, one can show⁵ that the formulations developed previously and the latter one are unitarily equivalent, hence describe the same physical system. Moreover, due to Stone-von Neumann theorem, there exists a unitary transformation $U : L^2(\mathbb{R}) \rightarrow \mathcal{F}_S(\mathcal{H})$.

In order to generalize our construction to a basis independent language, we take the solution space of negative frequencies, denoted by $\bar{\mathcal{H}}$, such that for any $y \in \mathcal{S}^\mathbb{C}$, exists a unique representation

$$y = y^+ + y^-, \quad (1.52)$$

where $y^+ \in \mathcal{H}$ and $y^- \in \bar{\mathcal{H}}$. Thus, we define a linear map $K : \mathcal{S}^\mathbb{C} \rightarrow \mathcal{H}$ that takes the positive frequency part of the complex solution. Similarly, we define the map $\bar{K} : \mathcal{S}^\mathbb{C} \rightarrow \bar{\mathcal{H}}$ that take the negative part. Thereby,

$$Ky = y^+ \quad \text{and} \quad \bar{K}y = y^-. \quad (1.53)$$

Let $y \in \mathcal{S} \subset \mathcal{S}^\mathbb{C}$, then

$$\bar{K}y = y^- = \bar{y}^+ \equiv \overline{Ky}. \quad (1.54)$$

Therefore, for $y \in \mathcal{S}$, the operator that represents the classical observable $\Omega(y, \cdot)$ is given by

$$\hat{\Omega}(y, \cdot) = ia(\bar{Ky}) - ia^\dagger(Ky), \quad (1.55)$$

because this operator satisfies the Equation (1.16) as proved below:

$$[\hat{\Omega}(y_1, \cdot), \hat{\Omega}(y_2, \cdot)] = \left[ia \left(\overline{Ky_1} \right) - ia^\dagger (Ky_1) \right] \left[ia \left(\overline{Ky_2} \right) - ia^\dagger (Ky_2) \right] \quad (1.56a)$$

$$- \left[ia \left(\overline{Ky_2} \right) - ia^\dagger (Ky_2) \right] \left[ia \left(\overline{Ky_1} \right) - ia^\dagger (Ky_1) \right] \quad (1.56b)$$

$$= \left[a \left(\overline{Ky_2}, a \left(\overline{Ky_1} \right) \right) \right] + \left[a^\dagger (Ky_2), a^\dagger (Ky_1) \right] \quad (1.56c)$$

$$+ \left[a \left(\overline{Ky_1}, a^\dagger (Ky_2) \right) \right] - \left[a \left(\overline{Ky_2} \right), a^\dagger (Ky_1) \right] \quad (1.56d)$$

$$= \langle Ky_1, Ky_2 \rangle - \langle Ky_2, Ky_1 \rangle \quad (1.56e)$$

$$= -i\Omega \left(\overline{Ky_1}, Ky_2 \right) + i\Omega \left(\overline{Ky_2}, Ky_1 \right) \quad (1.56f)$$

$$= -i\Omega \left(Ky_1 + \overline{Ky_1}, Ky_2 + \overline{Ky_2} \right) \quad (1.56g)$$

$$= -i\Omega(y_1, y_2). \quad (1.56h)$$

The operator in Heisenberg picture is obtained substituting y by $y(t)$ with the condition $y(0) = y$.

Notice that the previous construction relied on the fact that the Hamiltonian is time-independent, hence the solutions has purely positive or negative frequencies, however we can generalize this result.

1.4 Generic linear system

For a time-dependet Hamiltonian the procedure is similar. One must choose a subspace, $\mathcal{H} \subset \mathcal{S}^\mathbb{C}$ that satisfy

1. The Klein-Gordon inner product is positive-definite in \mathcal{H} , hence transforming it in a Hilbert space.
2. The complexified space of solutions $\mathcal{S}^\mathbb{C}$ is generated by \mathcal{H} and $\overline{\mathcal{H}}$.
3. Let $y^+ \in \mathcal{H}$ and $y^- \in \overline{\mathcal{H}}$, then

$$\langle y^+, y^-, \rangle = 0. \quad (1.57)$$

Therefore, every $y \in \mathcal{S}^\mathbb{C}$ can be uniquely written as a sum of an element of \mathcal{H} and of it's conjugate. We define the projective maps the same way.

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Then, notice that for $y_1, y_2 \in \mathcal{S}$, we have

$$\text{Im}\langle Ky_1, Ky_2 \rangle = -\text{Re} \left[\Omega \left(\overline{Ky_1}, Ky_2 \right) \right] \quad (1.58a)$$

$$= -\frac{1}{2} \Omega \left(\overline{Ky_1}, Ky_2 \right) - \frac{1}{2} \Omega \left(Ky_1, \overline{Ky_2} \right) \quad (1.58b)$$

$$= -\frac{1}{2} \Omega(y_1, y_2). \quad (1.58c)$$

Thus, if we define

$$\mu(y_1, y_2) \equiv \text{Re} \langle Ky_1, Ky_2 \rangle = \text{Im} \Omega \left(\overline{Ky_1}, Ky_2 \right), \quad (1.59)$$

we have

$$\langle Ky_1, Ky_2 \rangle = \mu(y_1, y_2) - \frac{i}{2} \Omega(y_1, y_2). \quad (1.60)$$

Choices of Hilbert space

From the Schwarz's inequality, for every $z_1, z_2 \in \mathcal{H}$, we have

$$\|z_1\|^2 \|z_2\|^2 \geq |\langle z_1, z_2 \rangle|^2 \geq |\text{Im} \langle z_1, z_2 \rangle|^2. \quad (1.61)$$

For the special case in which $z_1 = Ky_1$ and $z_2 = Ky_2$,

$$\|Ky_1\|^2 \|Ky_2\|^2 = \langle Ky_1, Ky_1 \rangle \langle Ky_2, Ky_2 \rangle \quad (1.62a)$$

$$= \mu(y_1, y_1) \mu(y_2, y_2) - \frac{1}{4} \Omega(y_1, y_1) \Omega(y_2, y_2) + \quad (1.62b)$$

$$- \frac{i}{2} [\mu(y_1, y_1) \Omega(y_2, y_2) + \mu(y_2, y_2) \Omega(y_1, y_1)] \quad (1.62c)$$

$$\geq \frac{1}{4} [\Omega(y_1, y_2)]^2, \quad (1.62d)$$

hence,

$$\mu(y_1, y_1) \mu(y_2, y_2) \geq \frac{1}{4} [\Omega(y_1, y_2)]^2. \quad (1.63)$$

Since is always possible to "saturate" Schwarz's inequality, we can impose

$$\mu(y_1, y_1) = \frac{1}{4} \max_{y_2 \neq 0} \frac{[\Omega(y_1, y_2)]^2}{\mu(y_2, y_2)}, \quad (1.64)$$

however, since μ appears in both sides, this condition does not completely determine the function. Conversely, suppose that exists a map $\mu : \mathcal{S}^{\mathbb{C}} \times \mathcal{S}^{\mathbb{C}} \rightarrow \mathbb{R}$ that is positive-definite, bilinear, symmetric and satisfies [Equation \(1.64\)](#).

Proposition 1.2. For $y_1 \in \mathcal{S}$, there exists a unique $y_2 \in \mathcal{S}$ such that

$$\frac{1}{2}\Omega(y_1,y_2)=\mu(y_1,y_1)\quad\text{and}\quad \mu(y_1,y_1)=\mu(y_2,y_2). \tag{1.65}$$

Proof. show it later! □

One may verify⁶ ⁶ *verify!*

$$y_1 \rightarrow \frac{1}{2}(y_1 + iy_2), \tag{1.66}$$

leads to a Hilbert space that also satisfy the desired conditions. Thus, notice that the choice \mathcal{H} can be replaced by a specification of μ .



CHAPTER 2

KLEIN-GORDON FIELD

The construction of a quantum field theory often requires an unitary time evolution to be well defined, hence we must require some features of the spacetime we are working on, we shall discuss such requirements. Moreover, we also describe the quantization of a real, linear and scalar field, known as Klein-Gordon field.

2.1 Globally hyperbolic spacetimes

Let (\mathcal{M}, g_{ab}) be an arbitrary spacetime. A real, linear and scalar Klein-Gordon field has action given by

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left(\nabla^a \phi \nabla_a \phi + m^2 \phi^2 \right), \quad (2.1)$$

in which ∇_a is the covariant derivative, with no torsion and compatible with the metric. If we minimize the action

$$\delta S = S[\phi + \delta\phi] - S[\phi] \quad (2.2a)$$

$$= -\frac{1}{2} \int d^4x \sqrt{-g} \left[\nabla^a (\phi + \delta\phi) \nabla_a (\phi + \delta\phi) + m^2 (\phi + \delta\phi)^2 \right] \quad (2.2b)$$

$$+ \frac{1}{2} \int d^4x \sqrt{-g} \left(\nabla^a \phi \nabla_a \phi + m^2 \phi^2 \right) \quad (2.2c)$$

$$= -\frac{1}{2} \int d^4x \sqrt{-g} \left(\nabla^a \phi \nabla_a \delta\phi + \nabla^a \delta\phi \nabla_a \phi + 2m^2 \phi \delta\phi + \mathcal{O}(\delta\phi^2) \right) \quad (2.2d)$$

$$= - \int d^3x \sqrt{h} \delta\phi n^a \nabla_a \phi + \int d^4x \sqrt{-g} \delta\phi \left[\nabla^a \nabla_a \phi - m^2 \phi \right] \quad (2.2e)$$

$$= 0. \quad (2.2f)$$

Therefore, ignoring the border terms, Klein-Gordon equation is

$$\left(-\nabla^a \nabla_a + m^2 \right) \phi = 0. \quad (2.3)$$

To construct the quantum theory, we need to define precisely the classical system phase space \mathcal{P} and, to do it, is necessary that the initial conditions determine uniquely a solution of Equation (2.3). Thus, suppose that the spacetime is orientable and $\Sigma \subset \mathcal{M}$ is a closed and achronal set, i.e., none two points of Σ can be connected by a timelike curve. The domain of dependence of Σ is defined as

$$D(\Sigma) = \{p \in \mathcal{M} \mid \text{every inextendible causal curve through } p \text{ intersects } \Sigma\}.$$

If $D(\Sigma) = \mathcal{M}$, then Σ is a Cauchy surface. A spacetime that admits a Cauchy surface is called globally hyperbolic and its importance is expressed in two theorems, the first one states about the space topology:

Theorem 2.1. Let (\mathcal{M}, g_{ab}) be a globally spacetime with Cauchy surface Σ , then \mathcal{M} has topology $\mathbb{R} \times \Sigma$. Furthermore, \mathcal{M} can be decomposed in a one parameter family of smooth Cauchy surfaces Σ_t , i.e., parametrized by a "time function".

The second one, enable the one-to-one correspondence between solutions and initial conditions:

Theorem 2.2. Let (\mathcal{M}, g_{ab}) be a globally hyperbolic spacetime with Cauchy surface Σ . Given any pair of smooth function (ϕ_0, π_0) in Σ , there exists a unique solution of Equation (2.3) in \mathcal{M} such that

$$\phi|_{\Sigma} = \phi_0 \quad \text{and} \quad n^a \nabla_a \phi|_{\Sigma} = \pi_0, \quad (2.4)$$

in which n^a is the unit vector perpendicular to Σ .

Henceforth, we shall work with globally hyperbolic spacetimes to be able to make the desired identification between the phase space and solution space as previously used.

2.2 Theory's structures

Let (\mathcal{M}, g_{ab}) be a orientable, globally hyperbolic spacetime with a foliation of \mathcal{M} with smooth Cauchy surfaces Σ_t indexed by a smooth function t . We introduce the vector field t^a that represents the time evolution in \mathcal{M} and satisfies $t^a \nabla_a t = 1$, then

$$t^a = N n^a + N^a, \quad (2.5)$$

in which n^a and N^a are the vectors perpendicular and parallel to Σ_t respectively. Take local coordinates t, x^1, x^2, x^3 such that $t^a \nabla_a x^i = 0$, i.e., such that $t^a = (\partial_t)^a$. Thereby, in this coordiantes, the Klein-Gordon action is

$$S = \int \mathcal{L} dt, \quad (2.6)$$

with

$$\mathcal{L} = \frac{1}{2} \int d^3x \sqrt{h} N \left[(n^a \nabla_a \phi)^2 - h^{ab} \nabla_a \phi \nabla_b \phi - m^2 \phi^2 \right], \quad (2.7)$$

in which h_{ab} is the metric induced in Σ_t . Notice that

$$n^a \nabla_a \phi = \frac{1}{N} (t^a - N^a) \nabla_a \phi = \frac{1}{N} \dot{\phi} - \frac{1}{N} N^a \nabla_a \phi. \quad (2.8)$$

Then, substituting in [Equation \(2.7\)](#), we have

$$\mathcal{L} = \frac{1}{2} \int d^3x \sqrt{h} \left[\frac{1}{N} \dot{\phi}^2 - \frac{2}{N} \dot{\phi} N^a \nabla_a \phi - h^{ab} \nabla_a \phi \nabla_b \phi - m^2 \phi^2 \right]. \quad (2.9)$$

Thus, the conjugated momentum density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \sqrt{h} \left(\frac{1}{N} \dot{\phi} - \frac{1}{N} N^a \nabla_a \phi \right) = \sqrt{h} n^a \nabla_a \phi. \quad (2.10)$$

A point in the classical phase space is determined by functions $\phi(x)$ and $\pi(x)$ in Σ_0 , the Cauchy surface that represents $t = 0$. The class of functions that allows all the necessary structures to be well defined is the smooth ones with compact support $C_0^\infty(\Sigma_0)$, i.e.,

$$\mathcal{P} = \{[\phi, \pi] | \phi : \Sigma_0 \rightarrow \mathbb{R} \text{ and } \pi : \Sigma_0 \rightarrow \mathbb{R}; \phi, \pi \in C_0^\infty(\Sigma_0)\}. \quad (2.11)$$

Therefore, we take the solution space \mathcal{S} to be the set of all solutions of [Equation \(2.3\)](#) whose initial conditions lies in \mathcal{P} . Furthermore, we define the symplectic structure $\Omega : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ as

$$\Omega([\phi_1, \pi_1], [\phi_2, \pi_2]) \equiv \int_{\Sigma_0} d^3x (\pi_1 \phi_2 - \pi_2 \phi_1), \quad (2.12)$$

or, in a equivalently, identifying \mathcal{P} with \mathcal{S} ,

$$\Omega(\phi_1, \phi_2) = \int_{\Sigma_0} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2). \quad (2.13)$$

In order to induce a symplectic structure in \mathcal{S} , we must show that the identification is time-indepent.

Proposition 2.3. The symplectic product of two solutions ϕ_1 and ϕ_2

$$s(t) = \Omega(\phi_1(t), \phi_2(t)) = \int_{\Sigma_t} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2), \quad (2.14)$$

is constant in time, i.e., does not depend on t .

Proof. Notice that the statement is equivalent to show that for any t_1 and t_2 holds

$$\int_{\Sigma_{t_1}} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) = \int_{\Sigma_{t_2}} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2). \quad (2.15)$$

To show it, consider the region of spacetimes Σ that has the Cauchy surfaces Σ_{t_1} and Σ_{t_2} as boundaries⁷, then

$$\int_{\Sigma_{t_1}} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) - \int_{\Sigma_{t_2}} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) \quad (2.16a)$$

$$= \int_{\partial \Sigma} d^3x \sqrt{h} (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) \quad (2.16b)$$

$$= \int_{\Sigma} d^4x \sqrt{-g} [\nabla^a (\phi_2 \nabla_a \phi_1) - \nabla^a (\phi_1 \nabla_a \phi_2)] \quad (2.16c)$$

$$= \int_{\Sigma} d^4x \sqrt{-g} [\nabla^a \phi_2 \nabla_a \phi_1 + \phi_2 \nabla^a \nabla_a \phi_1 - \nabla^a \phi_1 \nabla_a \phi_2 - \phi_1 \nabla^a \nabla_a \phi_2] \quad (2.16d)$$

$$= \int_{\Sigma} d^4x \sqrt{-g} (\phi_2 m^2 \phi_1 - \phi_1 m^2 \phi_2) = 0. \quad (2.16e)$$

□

Again exchanging between \mathcal{P} and \mathcal{S} , we want operators $\hat{\Omega}(\psi, \cdot)$, $\psi \in \mathcal{S}$, such that

$$[\hat{\Omega}(\psi_1, \cdot), \hat{\Omega}(\psi_2, \cdot)] = -i\Omega(\psi_1, \psi_2)\mathbb{I}. \quad (2.17)$$

Notice that it would be natural to choose \mathcal{H} such that it is the positive frequency solution's set, however, not only none of these satisfy the initial condition compacity in Σ_0 , but this notion is not extended to arbitrary spacetimes. Thus, the Hilbert space must be obtained by completing the complexified space $\mathcal{S}^{\mathbb{C}}$ with $\langle \cdot, \cdot \rangle$. One may notice that the choice of inner product also depend on \mathcal{H} , hence the construction becomes circular.

An alternative procedure to complete $\mathcal{S}^{\mathbb{C}}$ would be similar to what was

done in [Section 1.4](#). We can define a real inner product $\mu : \mathcal{S}^{\mathbb{C}} \times \mathcal{S}^{\mathbb{C}} \rightarrow \mathbb{R}$ such that, for all $\psi_1 \in \mathcal{S}$,

$$\mu(\psi_1, \psi_1) \equiv \frac{1}{4} \sup_{\psi_2 \neq 0} \frac{[\Omega(\psi_1, \psi_2)]^2}{\mu(\psi_2, \psi_2)}. \quad (2.18)$$

Now, we complete \mathcal{S} to \mathcal{S}_μ using the inner product

$$\langle \psi_1, \psi_2 \rangle_\mu \equiv 2\mu(\psi_1, \psi_2). \quad (2.19)$$

Since Ω is bounded⁸, by [Equation \(2.18\)](#), we can extend it's action to \mathcal{S}_μ ^{7 image!} by continuity. We define the operator $J : \mathcal{S}_\mu \rightarrow \mathcal{S}_\mu$ by means of

$$\Omega(\psi_1, \psi_2) = 2\mu(\psi_1, J\psi_2) = \langle \psi_1, J\psi_2 \rangle_\mu. \quad (2.20)$$

Since Ω is antisymmetric and μ is real, we have

$$\langle \psi_1, J\psi_2 \rangle_\mu = -\langle \psi_2, J\psi_1 \rangle_\mu \quad (2.21a)$$

$$= -\langle J\psi_1, \psi_2 \rangle_\mu \quad (2.21b)$$

$$= -\langle \psi, J^\dagger \psi_2 \rangle_\mu, \quad (2.21c)$$

Therefore, we comparing the second and last equalities,

$$J^\dagger = -J. \quad (2.22)$$

Furthermore, from [Equation \(2.18\)](#), J preserves the norm⁹, then ^{8 why?}

$$J^\dagger J = \mathbb{I} \implies J^2 = -\mathbb{I}. \quad (2.23)$$

Then, μ naturally provides a complex structure in \mathcal{S}_μ . Now, we complexify \mathcal{S}_μ and extend the action of Ω , μ and J by (complex) continuity. The Klein-Gordon product is, finally, defined as

$$\langle \psi_1, \psi_2 \rangle \equiv 2\mu(\bar{\psi}_1, \psi_2). \quad (2.24)$$

Notice that $iJ : \mathcal{S}_\mu^{\mathbb{C}} \rightarrow \mathcal{S}_\mu^{\mathbb{C}}$ is self-adjoint, with eigenvalues $\pm i$, then by the spectral theorem, the set $\mathcal{S}_\mu^{\mathbb{C}}$ is the direct sum of the orthogonal spaces correspondent to each eigenvalue. We take $\mathcal{H} \subset \mathcal{S}^{\mathbb{C}}$ to be the subspace of eigenvalue i , hence from spectral theorem properties that \mathcal{H} satisfies the necessary properties of quantization.

Finally, the quantum theory is obtained by choosing $\mathcal{F}_S(\mathcal{H})$ as the Hilbert space and defining the operators correspondent to the classical observables $\Omega(\psi, \cdot)$, $\psi \in \mathcal{S}$, as

$$\hat{\Omega}(\psi, \cdot) = ia \left(\overline{K\psi} \right) - ia^\dagger (K\psi), \quad (2.25)$$

in which K is the projector defined as usual.

2.3 Interpretation of field operator

Indepdent of the choices for the construction of the theory, the operators $\hat{\Omega}(\psi, \cdot)$ can be interpreted as a spacetime average of the operator (in the Heisenberg picture) that represents the field value.

Let $\mathfrak{F}(\mathcal{M}) = C_0^\infty(\mathcal{M})$ be the vector space of smooth functions with compact support in our spacetime of interest. Moreover, let $G_A(x, x')$ and $G_R(x, x')$ be the retarded and advanced Green functions of the Klein-Gordon operator, i.e.,

$$(-\nabla^a \nabla_a + m^2) G_A(x, x') = \frac{1}{\sqrt{-g}} \delta(x, x') = \delta_M(x, x') \quad (2.26a)$$

$$(-\nabla^a \nabla_a + m^2) G_R(x, x') = \frac{1}{\sqrt{-g}} \delta(x, x') = \delta_M(x, x'). \quad (2.26b)$$

Thereby, we can define the advanced and retarded solutions with source $f \in \mathfrak{F}(\mathcal{M})$ as

$$Rf(x) \equiv \int_{\mathcal{M}} d^4x' \sqrt{-g} G_R(x, x') f(x') \quad (2.27a)$$

$$Af(x) \equiv \int_{\mathcal{M}} d^4x' \sqrt{-g} G_A(x, x') f(x'). \quad (2.27b)$$

These solutions are the ones that, if $\text{supp} f$ denotes the support of the function f , then $\text{supp} Rf \subset J^+(\text{supp} f)$ (i.e., it propagates f to the future) and $\text{supp} Af \subset J^-(\text{supp} f)$ (i.e., it proapgates f to the past)¹⁰. Furthermore, notice that, indeed, Rf is a solution of [Equation \(2.3\)](#) with source f ,

$$(-\nabla^a \nabla_a + m^2) Rf(x) = \int_{\mathcal{M}} d^4x' \sqrt{-g} (-\nabla^a \nabla_a + m^2) G_R(x, x') f(x') \quad (2.28a)$$

$$= \int_{\mathcal{M}} d^4x' \sqrt{-g} \delta_M(x, x') f(x') \quad (2.28b)$$

$$= f(x), \quad (2.28c)$$

where the covariant derivatives acted only on the Green functions because they are taken with respect to x . It is evident that the same holds for Af . Then, we define

$$Ef \equiv Af - Rf. \quad (2.29)$$

It is immediate that Ef is a solution of the homogenous Klein-Gordon equation with initial condition $f \in \mathfrak{F}(\mathcal{M})$. Therefore, we obtain a linear map

⁹ why?

$E : \mathfrak{F}(\mathcal{M}) \rightarrow \mathcal{S}$ whose important properties are discussed in the following proposition.

Proposition 2.4. The map $E : \mathfrak{F}(\mathcal{M}) \rightarrow \mathcal{S}$ is surjective, i.e., for every $\psi \in \mathcal{S}$, exists $f \in \mathfrak{F}(\mathcal{M})$ such that

$$\psi = Ef. \quad (2.30)$$

Proof. Let $\psi \in \mathcal{S}$ and $\xi \in C^\infty$ such that

$$\chi = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1 \end{cases}. \quad (2.31)$$

Now, we define

$$f \equiv -\langle (\cdot, \chi) \psi \rangle, \quad (2.32)$$

notice that since ψ has compact support in Σ_0 , it's time evolution also do. Putting together with the fact that $\chi(t \leq 0) = 0$, we conclude that f has compact support in \mathcal{M} . Furthermore, we have $Af = (1 - \chi)\psi$ and $Rf = -\chi\psi$, then

$$Ef = \psi. \quad (2.33)$$

□

Proposition 2.5. For the map $E : \mathfrak{F}(\mathcal{M}) \rightarrow \mathcal{S}$ holds that

$$Ef = 0 \iff f = \left(-\nabla^a \nabla_a + m^2\right) g, \quad g \in \mathfrak{F}(\mathcal{M}). \quad (2.34)$$

Proof. We separate the proof in two parts:

- \implies If $Ef = 0$, then $Af = Rf$. Since both advanced and retarded solutions are the same, both has support equal to the support of f . Then, $Af, Rf \in \mathfrak{F}(\mathcal{M})$ and from [Equation \(2.28\)](#),

$$f = \left(-\nabla^a \nabla_a + m^2\right) g, \quad (2.35)$$

where $g = Rf \in \mathfrak{F}(\mathcal{M})$

- \Leftarrow Suppose that $f = (-\nabla'^a \nabla'_a + m^2) g$, then

$$Af(x) = \int_{\mathcal{M}} d^4x' \sqrt{-g} G_A(x, x') (-\nabla'^a \nabla'_a + m^2) g(x') \quad (2.36a)$$

$$= \int_{\mathcal{M}} d^4x' \sqrt{-g} \left[\nabla'^a (G_A \nabla_a g) - \nabla'^a G_A \nabla_a g - m^2 G_A g \right] \quad (2.36b)$$

$$= \int_{\partial\mathcal{M}} d^3x' \sqrt{h} G_A n^a \nabla'_a g \quad (2.36c)$$

$$+ \int_{\mathcal{M}} d^4x' \sqrt{-g} \left[-\nabla'_a (g \nabla'^a G_A) + g \nabla'^a \nabla'_a G_A - m^2 G_A g \right] \quad (2.36d)$$

$$= - \int_{\partial\mathcal{M}} d^3x' \sqrt{h} g n^a \nabla'^a G_A \quad (2.36e)$$

$$+ \int_{\mathcal{M}} d^4x' \left[(-\nabla'^a \nabla'_a + m^2) G_A \right] g \quad (2.36f)$$

$$= \int_{\mathcal{M}} d^4x' \sqrt{-g} \delta_M(x, x') g(x') = g(x). \quad (2.36g)$$

The border terms were removed because g has compact support. An identical procedure shows that $Rf(x) = g(x)$, then

$$Ef(x) = Af(x) - Rf(x) = 0. \quad (2.37)$$

□

Finally, the last proposition follows:

Proposition 2.6. For every $\psi \in \mathcal{S}$ and $f \in \mathfrak{F}(\mathcal{M})$, we have

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \psi f = \Omega(Ef, \psi), \quad (2.38)$$

where $E : \mathfrak{F}(\mathcal{M}) \rightarrow \mathcal{S}$ is the map of interest.

Proof. Take a Cauchy surface Σ that is outside the causal future of

the support of f , i.e., $\Sigma \subset \mathcal{M} - J^+(\text{supp} f)$, then

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \psi f = \int_{J^+(\Sigma)} d^4x \sqrt{-g} \psi f \quad (2.39a)$$

$$= \int_{J^+(\Sigma)} d^4x \sqrt{-g} \psi \left(-\nabla^a \nabla_a + m^2 \right) A f \quad (2.39b)$$

$$= \int_{J^+(\Sigma)} d^4x \sqrt{-g} \nabla^a (\psi \nabla_a A f - A f \nabla_a \psi) \quad (2.39c)$$

$$+ \int_{J^+(\Sigma)} d^4x \sqrt{-g} A f \left(-\nabla^a \nabla_a + m^2 \right) \psi \quad (2.39d)$$

$$= \int_{\Sigma} d^3x \sqrt{h} (\psi n^a \nabla_a A f - A f n^a \nabla_a \psi). \quad (2.39e)$$

Since in Σ $Rf = 0$, then $Af = Ef$, therefore

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \psi f = \Omega(Ef, \psi). \quad (2.40)$$

□

From [Proposition 2.6](#), we conclude that the function $\Omega(Ef, \cdot)$ is equivalent to a spacetime average with weight f . Thus, the operator $\hat{\Omega}(Ef, \cdot)$ is interpreted as a spacetime average of the quantum field with weight f , hence we define

$$\hat{\phi}(f) \equiv \hat{\Omega}(Ef, \cdot) = ia \left(\overline{KFf} \right) - ia^\dagger (KEf). \quad (2.41)$$

Furthermore, notice that by [Proposition 2.5](#), the observables related to $f \in \mathfrak{F}(\mathcal{M})$ always correspond to an observable $\hat{\Omega}(\psi, \cdot)$ with $\psi \in \mathcal{S}$. Moreover, it is possible to obtain a distributional form of the fact that ϕ satisfy Klein-Gordon equation with the aid of [Proposition 2.4](#),

$$\hat{\phi} \left(\left(-\nabla^a \nabla_a + m^2 \right) g \right) = \hat{\Omega} \left(E \left(-\nabla^a \nabla_a + m^2 \right) g, \cdot \right) = 0. \quad (2.42)$$

Now, the commutation relations are given by

$$\left[\hat{\phi}(f), \hat{\phi}(g) \right] = \left[\hat{\Omega}(Ef, \cdot), \hat{\Omega}(Eg, \cdot) \right] = -i\Omega(Ef, Eg)\mathbb{I}. \quad (2.43)$$

If we define

$$E(f, g) \equiv \int_{\mathcal{M}} d^4x \sqrt{-g} Ef(x)g(x), \quad (2.44)$$

then, by [Proposition 2.6](#), we have

$$\left[\hat{\phi}(f), \hat{\phi}(g) \right] = -iE(f, g)\mathbb{I}. \quad (2.45)$$

2. Klein-Gordon field

Finally, with the developed theory, the two-point function of the vacuum is

$$\langle 0 | \hat{\phi}(f) \hat{\phi}(g) | 0 \rangle = \langle 0 | a \left(\overline{KEf} \right) a^\dagger (KEg) | 0 \rangle \quad (2.46a)$$

$$= \langle KEf, KEg \rangle \quad (2.46b)$$

$$= \mu(Ef, Eg) - \frac{i}{2} E(f, g). \quad (2.46c)$$

Thereby, is evident the dependence of the expected values of the theory with the choice of μ used in the quantization process.

Remark. Notice that we implicitly used that f is a real function in order to Ef be a real solution. However, if f is complex, we have

$$\begin{cases} \hat{\phi}(\text{Re}f) = ia \left(\overline{KE\text{Re}f} \right) - ia^\dagger (KE\text{Re}f) \\ \hat{\phi}(\text{Im}f) = ia \left(\overline{KE\text{Im}f} \right) - ia^\dagger (KE\text{Im}f) \end{cases} \quad (2.47)$$

Thus, since $f = \text{Re}f + i\text{Im}f$, then

$$\hat{\phi}(f) = ia \left(\overline{KEf} \right) - ia^\dagger (KEf). \quad (2.48)$$

Now, the commutation relations are given by

$$\left[\hat{\phi}(f), \hat{\phi}(g) \right] = \left[\langle KE\overline{f}, KEg \rangle - \langle KE\overline{g}, KEf \rangle \right] \mathbb{I} \quad (2.49a)$$

$$= \left[\langle KE\overline{f}, KEg \rangle + \langle \overline{KEf}, \overline{KEg} \rangle \right] \mathbb{I} \quad (2.49b)$$

$$= \langle E\overline{f}, Ef \rangle \mathbb{I} \quad (2.49c)$$

$$= -E(\overline{f}, g). \quad (2.49d)$$

With this tiny adjusts, we are able to cover a larger set of functions that will be important after.



CHAPTER 3

PARTICLE CONCEPT

Throughout this chapter we will show that when symmetries are present in the spacetime, we can use them to construct a "natural" choice of Hilbert space for the quantization and apply it to a static spacetimes. Often this construction allows us to give rise to the concept of particles of the field, hence we also cover a two-level particle detector. Finally, we discuss Bogoliubov transformations, that is a map between two constructions of the quantum theories.

3.1 Stationary spacetimes

Let (\mathcal{M}, g_{ab}) is a globally hyperbolic stationary spacetime, i.e., it admits a timelike Killing field ξ^a that is associated with a one parameter group of isometries $\phi_t^\xi : \mathcal{M} \rightarrow \mathcal{M}$ with timelike orbits. The choice of \mathcal{H} will be the one of solutions that are positive frequency with respect to the Killing time t , a function such that

$$\xi^a \nabla_a t = 1, \quad (3.1)$$

i.e., ξ^a plays the role of t^a in the previous construction. To avoid technical difficulties, we impose

$$m > 0, \quad (3.2)$$

and that there exists a Cauchy surface Σ such that, $\exists \epsilon > 0$,

$$-\xi^a \xi_a \geq -\epsilon \xi^a n_a > \epsilon, \quad (3.3)$$

in this surface, where n^a is the vector normal to Σ . We define the "energy" inner product in $\mathcal{S}^{\mathbb{C}}$ as

$$\langle \psi_1, \psi_2 \rangle_\xi \equiv \int_{\Sigma} d^3x \sqrt{h} \xi^a n^b T_{ab}, \quad (3.4)$$

3. Particle concept

where T_{ab} is classical stress-energy tensor given by

$$T_{ab}(\psi_1, \psi_2) = \nabla_{(a} \bar{\psi}_1 \nabla_{b)} \psi_2 - \frac{1}{2} g_{ab} \left(\nabla^c \bar{\psi}_1 \nabla_c \psi_2 + m^2 \bar{\psi}_1 \psi_2 \right). \quad (3.5)$$

By the definition, it is immediate that

$$\langle \psi_1, \psi_2 \rangle_\xi = \overline{\langle \psi_2, \psi_1 \rangle_\xi}. \quad (3.6)$$

Proposition 3.1. The inner product $\langle \cdot, \cdot \rangle_\xi$ is positive-definite in $\mathcal{S}^\mathbb{C}$.

Proof. First, we have the following definitions that help in the calculation. Let $\psi \in \mathcal{S}^\mathbb{C}$, then

$$\begin{cases} \nabla_a \psi = -(n^b \nabla_b \psi) n_a + h_a^b \nabla_b \psi \equiv -\psi_n n_a + D_a \psi \\ \xi^a = -(\xi^b n_b) n^a + h^{ab} \xi_b \equiv N_\xi n^a + N^a \end{cases}. \quad (3.7)$$

Therefore, if we write the stress-energy tensor as function of the new parameters, we have

$$\nabla_{(a} \bar{\psi} \nabla_{b)} \psi = \bar{\psi}_n \psi_n n_a n_b - \bar{\psi}_n n_{(a} D_{b)} \psi - \psi_n D_{(a} \bar{\psi} n_{b)} + D_{(a} \bar{\psi} D_{b)} \psi, \quad (3.8)$$

and also,

$$\nabla^c \bar{\psi} \nabla_c \psi = \left(-\bar{\psi}_n n^c + D^c \bar{\psi} \right) \left(-\psi_n n_c + D_c \psi \right) \quad (3.9a)$$

$$= -\bar{\psi}_n \psi_n + D^c \bar{\psi} D_c \psi. \quad (3.9b)$$

Now, we can calculate the integrand of the inner product,

$$\xi^a n^b T_{ab} = \frac{1}{2} \left[2N_\xi \bar{\psi}_n \psi_n + \bar{\psi}_n N^b D_b \psi + \psi_n N^a D_a \bar{\psi} \right] \quad (3.10a)$$

$$- \frac{\xi^b n_b}{2} \left[-\bar{\psi}_n \psi_n + D^c \bar{\psi} D_c \psi + m^2 \bar{\psi} \psi \right] \quad (3.10b)$$

$$= \frac{N_\xi}{2} \left[\bar{\psi}_n \psi_n D^c \bar{\psi} D_c \psi + m^2 \bar{\psi} \psi \right] \quad (3.10c)$$

$$+ \frac{1}{2} \left[\bar{\psi}_n N^b D_b \psi + \psi_n N^a D_a \bar{\psi} \right]. \quad (3.10d)$$

Remember that the momentum density is given by [Equation \(2.10\)](#),

then

$$\langle \psi, \psi \rangle_\xi = \int_\Sigma d^3x \sqrt{h} \xi^a n^b T_{ab}(\psi, \psi) \quad (3.11a)$$

$$= \frac{1}{2} \int_\Sigma d^3x \left[N_\xi \frac{|\pi|^2}{\sqrt{h}} + \sqrt{h} N_\xi \|D\psi\|^2 + \sqrt{h} m^2 |\psi|^2 \right] \quad (3.11b)$$

$$+ \frac{1}{2} \int_\Sigma d^3x \left[\frac{\pi}{N_\xi} N^a \nabla_a \bar{\psi} + \frac{\bar{\pi}}{N_\xi} N^a \nabla_a \psi \right] \quad (3.11c)$$

$$= \frac{1}{2} \int_\Sigma d^3x N_\xi \left[\left(1 - \frac{N^a N_a}{N_\xi^2} \right) \frac{|\pi|^2}{\sqrt{h}} + \sqrt{h} m^2 |\psi|^2 \right] \quad (3.11d)$$

$$+ \frac{1}{2} \int_\Sigma d^3x N_\xi \sqrt{h} h^{ab} \left(D_a \psi + \frac{\pi N_a}{\sqrt{h} N_\xi} \right) \left(D_b \bar{\psi} + \frac{\bar{\pi} N_b}{\sqrt{h} N_\xi} \right) \quad (3.11e)$$

$$= \frac{1}{2} \int_\Sigma d^3x N_\xi \left[-\frac{\xi^a \xi_a}{N_\xi^2} \frac{|\pi|^2}{\sqrt{h}} + \sqrt{h} m^2 |\psi|^2 \right] \quad (3.11f)$$

$$+ \frac{1}{2} \int_\Sigma d^3x N_\xi \sqrt{h} \left\| D_a \psi + \frac{\pi N_a}{\sqrt{h} N_\xi} \right\|^2. \quad (3.11g)$$

It is immediate that

$$\psi = 0 \iff \langle \psi, \psi \rangle_\xi = 0, \quad (3.12)$$

then take $\psi \neq 0$, using [Equations \(3.2\)](#) and [\(3.3\)](#),

$$\langle \psi, \psi \rangle_\xi \geq \epsilon \int_\Sigma \frac{d^3x}{2} \left(\frac{|\pi|^2}{\sqrt{h}} + m^2 |\psi|^2 \right) \quad (3.13a)$$

$$\geq \frac{1}{2} \min \{ \epsilon, m^2 \epsilon \} \int_\Sigma d^3x \sqrt{h} \left[|n^a \nabla_a|^2 + |\psi|^2 \right] > 0. \quad (3.13b)$$

□

Now, we will perform auxiliary calculations to show that the inner product is independent of the Cauchy surface Σ we use to calculate it. First, notice

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that for $\psi_1, \psi_2 \in \mathcal{S}^{\mathbb{C}}$, as expected of a stress-energy tensor,

$$2\nabla^a T_{ab} = \nabla^a \left[\nabla_a \bar{\psi}_1 \nabla_b \psi_2 + \nabla_b \bar{\psi}_1 \nabla_a \psi_2 \right] - g_{ab} \nabla^a \left[\nabla^c \bar{\psi}_1 \nabla_c \psi_2 + m^2 \bar{\psi}_1 \psi_2 \right] \quad (3.14a)$$

$$= \nabla^a \nabla_a \bar{\psi}_1 \psi_2 + \nabla_a \bar{\psi}_1 \nabla^a \nabla_b \psi_2 + \nabla^a \nabla_b \bar{\psi}_1 \nabla_a \psi_2 + \nabla_b \bar{\psi}_1 \nabla^a \nabla_a \psi_2 \quad (3.14b)$$

$$- g_{ab} \left[\nabla^a \nabla^c \bar{\psi}_1 \nabla_c \psi_2 + \nabla^c \bar{\psi}_1 \nabla^a \nabla_c \psi_2 + m^2 \nabla^a \bar{\psi}_1 \psi_2 + m^2 \bar{\psi}_1 \nabla^a \psi_2 \right] \quad (3.14c)$$

$$= m^2 \bar{\psi}_1 \nabla_b \psi_2 + \nabla^a \bar{\psi}_1 \nabla_a \nabla_b \psi_2 + \nabla^a \psi_2 + \nabla_a \nabla_b \bar{\psi}_1 + m^2 \nabla_b \bar{\psi}_1 \psi_2 \quad (3.14d)$$

$$- \nabla^c \psi_2 \nabla_c \nabla_b \bar{\psi}_1 - \nabla^c \bar{\psi}_1 \nabla_c \nabla_b \psi_2 - m^2 \nabla_b \bar{\psi}_1 \psi_2 - m^2 \bar{\psi}_1 \nabla_b \psi_2 \quad (3.14e)$$

$$= 0. \quad (3.14f)$$

From the previous result, follows that

$$\nabla^a (T_{ab} \xi^b) = \nabla^a T_{ab} \xi^b + T_{ab} \nabla^a \xi^b \quad (3.15a)$$

$$= T_{ab} \nabla^{(a} \xi^{b)} \quad (3.15b)$$

$$= 0. \quad (3.15c)$$

Thus, consider a region \mathcal{V} confined by two Cauchy surfaces Σ_{t_1} and Σ_{t_2} , then

$$\int_{\Sigma_{t_2}} d^3x \sqrt{h} \xi^a n^b T_{ab} - \int_{\Sigma_{t_1}} d^3x \sqrt{h} \xi^a n^b T_{ab} = \int_{\partial\mathcal{V}} d^3x \sqrt{h} \xi^a n^b T_{ab} \quad (3.16a)$$

$$= \int_{\mathcal{V}} d^4x \sqrt{-g} \nabla^b (\xi^a T_{ab}) \quad (3.16b)$$

$$= \int_{\mathcal{V}} d^4x \sqrt{-g} \nabla^b (T_{ba} \xi^a) \quad (3.16c)$$

$$= 0, \quad (3.16d)$$

i.e., the inner product is identical for any smooth Cauchy surfaces of the one parameters group associated with ξ^a . Now, we complete the solution space with this inner product to obtain the Hilbert space $\tilde{\mathcal{H}}$.

Define the operator $V_t : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ as

$$V_t \psi \equiv \psi \circ \phi_{-t}^\xi, \quad (3.17)$$

notice that the invariance under time translation of the inner product implies that this operator is unitary. As a consequence, by Stone theorem,

$$V_t = e^{-i\tilde{h}t}, \quad (3.18)$$

where \tilde{h} is a self-adjoint operator. From the definition, we have

$$\tilde{h}\psi = i \frac{d}{dt} V_t \psi \Big|_{t=0} \quad (3.19a)$$

$$= i \frac{d}{dt} \psi \circ \phi_{-t}^\xi \Big|_{t=0} \quad (3.19b)$$

$$= i \mathcal{L}_\xi \psi, \quad (3.19c)$$

where \mathcal{L}_ξ is the Lie derivative with respect to the vector field ξ^a .

Henceforth, we will use the identification between the solution space and the phase space. First, we define $B : \mathcal{S}^\mathbb{C} \times \mathcal{S}^\mathbb{C} \rightarrow \mathbb{C}$ as

$$B(\psi_1, \psi_2) \equiv \Omega(\bar{\psi}_1, \psi_2) \quad (3.20)$$

Remark. Notice that $\langle \psi_1, \psi_2 \rangle = -iB(\psi_1, \psi_2)$.

We represent an element of the phase space by

$$\psi = \begin{pmatrix} \phi \\ \pi \end{pmatrix}, \quad (3.21)$$

with inner product given by

$$\langle \psi_1, \psi_2 \rangle_{L^2} = \int_\Sigma d^3x \begin{pmatrix} \bar{\phi}_1 & \bar{\pi}_1 \end{pmatrix} \begin{pmatrix} \phi_2 \\ \pi_2 \end{pmatrix} = \int_\Sigma d^3x (\bar{\phi}_1 \phi_2 + \bar{\pi}_1 \pi_2). \quad (3.22)$$

Notice that, given the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.23)$$

we have

$$\langle \psi_1, J\psi_2 \rangle_{L^2} = \int_\Sigma d^3x \begin{pmatrix} \bar{\phi}_1 & \bar{\pi}_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_2 \\ \pi_2 \end{pmatrix} \quad (3.24a)$$

$$= \int_\Sigma d^3x (\bar{\pi}_1 \phi_2 - \pi_2 \bar{\phi}_1) \quad (3.24b)$$

$$= \Omega(\bar{\phi}_1, \phi_2). \quad (3.24c)$$

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Furthermore, as shown in [Proposition 3.1](#),

$$\langle \psi, \psi \rangle_\xi \geq C^{-1} \langle \psi, \psi \rangle_{L^2}, \quad (3.25)$$

with $C^{-1} = 1/2 \min \{\epsilon, m^2 \epsilon\}$, then

$$\|\psi\|_{L^2} \leq C^{1/2} \|\psi\|_\xi. \quad (3.26)$$

Moreover, notice that J is norm preserving in the phase space,

$$\|J\psi\|_{L^2}^2 = \langle J\psi, J\psi \rangle_{L^2} \quad (3.27a)$$

$$= \int_\Sigma d^3x \begin{pmatrix} -\bar{\pi} & \bar{\phi} \end{pmatrix} \begin{pmatrix} -\pi \\ \psi \end{pmatrix} \quad (3.27b)$$

$$= \int_\Sigma d^3x (\|\phi\|^2 + \|\pi\|^2) \quad (3.27c)$$

$$= \|\psi\|^2. \quad (3.27d)$$

Therefore, due to the Cauchy-Schwartz,

$$|B(\psi_1, \psi_2)| = |\Omega(\bar{\psi}_1, \psi_2)| \quad (3.28a)$$

$$= |\langle \psi_1, J\psi_2 \rangle_{L^2}| \quad (3.28b)$$

$$\leq \|\psi_1\|_{L^2} \|\psi_2\|_{L^2} \quad (3.28c)$$

$$\leq C \|\psi_1\|_\xi \|\psi_2\|_\xi. \quad (3.28d)$$

Thus, B is bounded and defined in a dense domain of $\tilde{\mathcal{H}}$, hence we can extend its action to this set as a bounded operator.

Proposition 3.2. For every $\psi_1, \psi_2 \in \mathcal{S}^\mathbb{C}$,

$$B(\psi_1, \tilde{h}\psi_2) = 2i \langle \psi_1, \psi_2 \rangle_\xi. \quad (3.29)$$

Proof. show it! □

Finally, notice that the Klein-Gordon, then

$$\langle \psi_1, \tilde{h}\psi_2 \rangle = 2 \langle \psi_1, \psi_2 \rangle_\xi. \quad (3.30)$$

Therefore, due to [Equation \(3.28\)](#), we have

$$2|\langle \psi_1, \psi_2 \rangle_\xi| = |\langle \psi_1, \tilde{h}\psi_2 \rangle| \leq C \|\psi_1\|_\xi \|\tilde{h}\psi_2\|_\xi. \quad (3.31)$$

In particular, for $\psi \neq 0$,

$$2\|\psi\|_\xi^2 \leq C \|\psi\|_\xi \|\tilde{h}\psi\|_\xi \implies \frac{\|\tilde{h}\psi\|_\xi}{\|\psi\|_\xi} \geq \frac{2}{C} > 0. \quad (3.32)$$

We conclude that the infimum of the spectrum of \tilde{h} , σ , is greater than zero, hence \tilde{h}^{-1} is well defined in a dense domain of $\tilde{\mathcal{H}}$. Therefore, from the spectral theorem, we have

$$\tilde{h} = \int_{\omega \in \sigma^+} \omega dP_\omega + \int_{\omega \in \sigma^-} \omega dP_\omega, \quad (3.33)$$

where

$$\begin{cases} \sigma = \sigma^+ \cup \sigma^- \\ \sigma^+ = -\sigma^- \\ \sigma^+ \cap \sigma^- = \emptyset \end{cases}. \quad (3.34)$$

Let $\tilde{\mathcal{H}}^+$ be the space associated with the solutions that lies in σ^+ completed with Klein-Gordon product. It is evident that the map that projects in this space is

$$K = \int_{\omega \in \sigma^+} dP_\omega. \quad (3.35)$$

At last, for every $\psi_1, \psi_2 \in \mathcal{S}$, we define the map μ as

$$\mu(\psi_1, \psi_2) = \text{Im} B(K\psi_1, K\psi_2) = 2\text{Re} \left\langle K\psi_1, \tilde{h}^{-1} K\psi_2 \right\rangle_\xi. \quad (3.36)$$

Proposition 3.3. The product μ satisfies the condition given in [Equation \(2.18\)](#).

Proof. show it! □

Finally, we introduce a convenient notation. Let Λ, μ be a measure sapce in which $j \in \Lambda$ represents a set of quantum numbers. Then, it is possible to choose an orthonormal basis $\{u_j\}_{j \in \Lambda}$ of the solutions of Klein-Gordon equations such that

$$\tilde{h}u_j = \omega_j u_j \iff u_j = \frac{e^{-i\omega_j t}}{\sqrt{\omega_j}} \varphi_j, \quad (3.37)$$

and also

$$\langle u_j, u_{j'} \rangle = \delta_\mu(j, j'), \quad (3.38)$$

where

$$\int_{j \in \Lambda} d\mu(j') \delta_\mu(j, j') = 1. \quad (3.39)$$

Thus, any solution $\psi \in \tilde{\mathcal{H}}^+$ can be written as

$$\psi(x) = \int_{j \in \Lambda} d\mu(j) \tilde{\psi}(j) \frac{e^{-i\omega_j t}}{\sqrt{2\omega_j}} \varphi_j(x). \quad (3.40)$$

Now, it is evident that the space $\tilde{\mathcal{H}}^+$ concerns the positive frequency solutions with respect to the Killing field $\xi = \partial_t$.

3.2 Quantization in static spacetimes

Let (\mathcal{M}, g_{ab}) be globally hyperbolic and static spacetime, i.e., stationary that admits an hypersurface orthogonal to ξ . Under these conditions, exists a coordinate system in which the line element is expressed as

$$ds^2 = -f(x)dt^2 + \sum_{i,j} h_{ij}(x)dx^i dx^j, \quad (3.41)$$

where $f(x) > 0$ and x represents the coordinates in the Cauchy surfaces Σ_t . The vector field $\xi = \partial_t$ is a Killing one and timelike. Furthermore, it's integral curves are given by $\phi_t(t_1, x_1) = (t + t_1, \cdot)$, i.e., they represent time translation. Since they are timelike, we can interpret them as the worldline of a congruence of observers.

In an arbitrary coordinate system, Klein-Gordon equation is

$$\frac{1}{\sqrt{-g}} \sum_{\mu,\nu} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) - m^2 \phi = 0. \quad (3.42)$$

In the static coordinates, for $\mu = 0$, we have

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} \left(\sqrt{-g} g^{0\nu} \frac{\partial \phi}{\partial x^\nu} \right) = -f^{-1} \frac{\partial^2 \phi}{\partial t^2} \quad (3.43)$$

thus, the equation is

$$\left(-\frac{\partial^2}{\partial t^2} - \mathcal{K} \right) \phi = 0, \quad (3.44)$$

where

$$\mathcal{K} \phi \equiv f \left[-\frac{1}{\sqrt{-g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{-g} h^{ij} \frac{\partial \phi}{\partial x^j} \right) + m^2 \phi \right], \quad (3.45)$$

denotes the purely spatial operator.

Proposition 3.4. The operator \mathcal{K} is hermitian in $L^2(\Sigma_t, \sqrt{-g} f^{-1} dx)$.

Proof. The statement comes from a direct computation

$$\langle \mathcal{K}\phi_1, \phi_2 \rangle_{L^2} = \int_{\Sigma_t} d^3x \sqrt{-g} f^{-1} f \left[-\frac{1}{\sqrt{-g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{-g} h^{ij} \frac{\partial \bar{\phi}_1}{\partial x^j} \right) + m^2 \bar{\phi}_1 \right] \phi_2 \quad (3.46a)$$

$$= \int_{\Sigma_t} d^3x \left[\sum_{i,j} \sqrt{-g} h^{ij} \frac{\partial \phi_2}{\partial x^i} \frac{\partial \bar{\phi}_1}{\partial x^j} + \sqrt{-g} m^2 \bar{\phi}_1 \phi_2 \right] \quad (3.46b)$$

$$- \int_{\partial \Sigma_t} d^2x \sum_{i,j} \sqrt{-g} h^{ij} \phi_2 \frac{\partial \bar{\phi}_1}{\partial x^j} n_i \quad (3.46c)$$

$$= \int_{\Sigma_t} d^3x \left[-\sum_{i,j} \bar{\phi}_1 \frac{\partial}{\partial x^j} \left(\sqrt{-g} h^{ij} \frac{\partial \phi_2}{\partial x^i} \right) + \sqrt{-g} m^2 \bar{\phi}_1 \phi_2 \right] \quad (3.46d)$$

$$+ \int_{\partial \Sigma_t} d^2x \sum_{i,j} \sqrt{-g} h^{ij} \bar{\phi}_1 \frac{\partial \phi_2}{\partial x^i} n_j \quad (3.46e)$$

$$= \int_{\Sigma_t} d^3x \sqrt{-g} f^{-1} \bar{\phi}_1 f \left[-\frac{1}{\sqrt{-g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{-g} h^{ij} \frac{\partial \phi_2}{\partial x^j} \right) + m^2 \phi_2 \right] \quad (3.46f)$$

$$= \langle \phi_1, \mathcal{K}\phi_2 \rangle_{L^2} \quad (3.46g)$$

□

In general, \mathcal{K} is a function of a complete set of operators, \mathcal{T} , that commute. Let \mathcal{J} be the spectrum of these set of operators, μ a measure in \mathcal{J} and ψ_j , $j \in \mathcal{J}$, eigenfunctions of \mathcal{K} with eigenvalues $\omega_j^2 > 0$, i.e.,

$$\mathcal{K}\psi_j = \omega_j^2 \psi_j, \quad (3.47)$$

that satisfy

$$\int_{\Sigma_t} d^3x \sqrt{-g} f^{-1} \bar{\psi}_j(x) \psi_{j'}(x) = \delta_\mu(j, j'). \quad (3.48)$$

Finally, we define the Hilbert space

$$\mathcal{H}_\xi \equiv \left\{ \varphi(x) = \int \frac{d\mu(j)}{\sqrt{2\omega_j}} \tilde{\varphi}(j) e^{-i\omega_j t} \psi_j(x) \mid \tilde{\varphi} \in L^2(\mathcal{J}, d\mu(j)) \right\}, \quad (3.49)$$

of the solutions of Klein-Gordon equation.

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Proposition 3.5. The Hilbert space \mathcal{H}_ξ satisfies the necessary conditions for quantization.

Proof. show it! □

The quantum field theory constructed in this approach has a natural particle interpretation depending on the field state. We say that the state given by

$$\frac{1}{\sqrt{n!}} \left(a^\dagger(\chi) \right)^n |0\rangle, \quad (3.50)$$

represents n particles in the mode $\chi \in \mathcal{H}_\xi$. However, for non-stationary states, any interpretation of particles can be very problematic, hence we emphasize the remark that we are constructing a theory of fields, not particles.

3.3 Particle detector

Let (\mathcal{M}, g_{ab}) be a globally hyperbolic spacetime and static spacetime with Killing field ξ that will provide the background for the interaction of a two-level particle detector and a quantized scalar field ϕ . Let Ω be the energetic gap between the detector's levels, then it's the Hamiltonian is defined as

$$H_D = \Omega D^\dagger D, \quad (3.51)$$

where $D|0\rangle = D^\dagger|1\rangle = 0$, $D|1\rangle = |0\rangle$, $D^\dagger|0\rangle = |1\rangle$ and $|0\rangle$, $|1\rangle$ are the ground and excited energy eigenstates. Moreover, the coupling between the detector and the field are described by the interaction Hamiltonian

$$H_{\text{int}}(t) = \epsilon(t) \int_{\Sigma_t} d^3x \sqrt{-g} \hat{\phi}(x) \left[\psi(t, x) D + \bar{\psi}(t, x) D^\dagger \right], \quad (3.52)$$

where $\hat{\phi}(x)$ is the Klein-Gordon free field operator. The function $\epsilon(t) \in C_0^\infty(\mathcal{M})$ is real and model the fact that the interaction happens in a finite proper time Δ (the time that the detector is on) and, for a fixed $t \in \mathbb{R}$, $\psi(x) \in C_0^\infty(\Sigma_t)$ represents that the coupling is relevant only in a neighborhood of the detector's worldline.

The total Hamiltonian of the system is

$$H = H_0 + H_{\text{int}}, \quad (3.53)$$

where $H_0 = H_{KG} + H_D$ is the free system Hamiltonian. In the interaction picture, the state $|\Psi_t\rangle$ that describes the system at an instant t can be written as

$$|\Psi_t\rangle = T \exp \left[-i \int_{-\infty}^t dt' H_{\text{int}}^I(t') \right] |\Psi_{-\infty}\rangle, \quad (3.54)$$

where T is the time-ordering operator and

$$H_{\text{int}}^I = U_0^\dagger(t) H_{\text{int}} U_0, \quad (3.55)$$

where U_0 is the evolution operator of H_0 . Therefore, we have, for $|\Psi_\infty\rangle \equiv |\Psi_{t>\Delta}\rangle$,

$$|\Psi_\infty\rangle = T \exp \left[-i \int_{-\infty}^{\infty} dt \int_{\Sigma_t} d^3x \sqrt{-g} \hat{\phi}(x) \epsilon(t) \left(\psi(t, x) U_0^\dagger D U_0 + \bar{\psi}(t, x) U_0^\dagger D U_0 \right) \right] |\Psi_{-\infty}\rangle \quad (3.56a)$$

$$= T \exp \left[\int_{\mathcal{M}} d^4x \sqrt{-g} \hat{\phi}(x) \left(\epsilon(t) \psi(t, x) e^{-i\Omega t} D + \bar{\epsilon}(t) \bar{\psi}(t, x) e^{i\Omega t} D^\dagger \right) \right] |\Psi_{-\infty}\rangle \quad (3.56b)$$

$$= T \exp \left[-i \int_{\mathcal{M}} d^4x \sqrt{-g} \hat{\phi}(x) \left(f(t, x) D + \bar{f}(t, x) D^\dagger \right) \right] |\Psi_{-\infty}\rangle, \quad (3.56c)$$

where

$$f(t, x) \equiv \epsilon(t) e^{-i\Omega t} \psi(t, x), \quad (3.57)$$

is a complex function with compact support. Henceforth, we assume that the detector is following one orbit of ξ , thus it follows the spatial parametrization, i.e., it has constant spatial coordinates x in every Σ_t . As a consequence, our function $\psi(t, x)$ is time independent, since the position of the detector is constant (with respect to the notion provided by the Killing field) along the spacetime.

Now, from first order in perturbation theory¹¹, we have

$$|\Psi_\infty\rangle = \left[\mathbb{I} - i \left(\hat{\phi}(f) D + \hat{\phi}^\dagger(f) D^\dagger \right) \right] |\Psi_{-\infty}\rangle. \quad (3.58)$$

Suppose the the function $\epsilon(t)$ vary slowly when compared to the frequency Ω and that $\Delta \gg \Omega^{-1}$. We claim that under this conditions, f is approximately a positive frequency function, i.e., $KEf \cong Ef$ and $KE\bar{f} \cong 0$. To show it, we decompose Ef in terms of positive and negative frequency modes, v_α and \bar{v}_α ,

$$Ef = \int d\mu(\alpha) (\langle v_\alpha, Ef \rangle v_\alpha - \langle \bar{v}_\alpha, Ef \rangle \bar{v}_\alpha), \quad (3.59)$$

where v_α are solutions of Klein-Gordon equation that satisfies $\langle v_\alpha, v_{\alpha'} \rangle = \delta_\mu(\alpha, \alpha')$. From the fact that v_α is a positive frequency solution, it follows that is also eigenfunction of $i\partial_t$ with eigenvalue $\omega_\alpha > 0$. Thus we can write

$$v_\alpha = e^{-i\omega_\alpha t} \varphi_\alpha(x), \quad (3.60)$$

¹⁰ add images on the supports

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where

$$\mathcal{K}\varphi_\alpha = \omega_\alpha^2 \varphi_\alpha, \quad (3.61)$$

and \mathcal{K} is the same operator defined in Equation (3.45). Furthermore, from Proposition 2.6, we have

$$\langle v_\alpha, Ef \rangle = i \int_{\mathcal{M}} d^4x \sqrt{-g} f \bar{v}_\alpha \quad (3.62a)$$

$$\langle \bar{v}_\alpha, Ef \rangle = i \int_{\mathcal{M}} d^4x \sqrt{-g} f v_\alpha. \quad (3.62b)$$

Using our approximation that $\epsilon(t) \cong \epsilon = \text{cte}$ when the detector is turned on (and $\epsilon(t) = 0$ when it's turned off), we have

$$\langle \bar{v}_\alpha, Ef \rangle = i \int_{\mathcal{M}} d^4x \sqrt{-g} f v_\alpha \quad (3.63a)$$

$$= i \int_{-\infty}^{\infty} dt \int_{\Sigma_t} d^3x \sqrt{-g} \epsilon(t) e^{-i\Omega t} \psi(x) e^{-i\omega_\alpha t} \varphi_\alpha(x) \quad (3.63b)$$

$$= i\epsilon \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} dt e^{-i(\Omega + \omega_\alpha)t} \int_{\Sigma_t} d^3x \sqrt{-g} \psi(x) \varphi_\alpha(x) \quad (3.63c)$$

$$= i\epsilon\gamma_\alpha \left(-\frac{1}{i(\Omega + \omega_\alpha)} e^{-i(\Omega + \omega_\alpha)t} \right) \Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \quad (3.63d)$$

$$= 2i\epsilon\gamma_\alpha \frac{\sin \left[(\Omega + \omega_\alpha) \frac{\Delta}{2} \right]}{\Omega + \omega_\alpha}, \quad (3.63e)$$

where

$$\gamma_\alpha \equiv \int_{\Sigma_t} d^3x \sqrt{-g} \psi(x) \varphi_\alpha(x). \quad (3.64)$$

Now, we have, from Proposition A.1,

$$\frac{\sin \left[(\Omega + \omega_\alpha) \frac{\Delta}{2} \right]}{\Omega + \omega_\alpha} \cong \pi \delta(\Omega + \omega_\alpha), \quad (3.65)$$

when $\Delta \gg \Omega^{-1}$. This express the fact that only the mode with frequency $\omega_\alpha = -\Omega$ contributes for the negative frequency part, thus $\langle \bar{v}_\alpha, Ef \rangle \cong 0$. Therefore, we have $KEf \cong Ef$.

Similarly, consider,

$$\langle v_\alpha, E\bar{f} \rangle = i \int_{\mathcal{M}} d^4x \sqrt{-g} \bar{f} \bar{v}_\alpha \quad (3.66a)$$

$$= i \int_{-\infty}^{\infty} dt \int_{\Sigma_t} d^3x \sqrt{-g} \bar{\epsilon}(t) e^{i\Omega t} \bar{\psi}(x) e^{i\omega_\alpha t} \bar{\varphi}(x) \quad (3.66b)$$

$$= i\epsilon \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} dt e^{i(\Omega+\omega_\alpha)t} \int_{\Sigma_t} d^3x \sqrt{-g} \bar{\psi}(x) \bar{\varphi}(x) \quad (3.66c)$$

$$= i\epsilon \bar{\gamma}_\alpha \left(\frac{1}{i(\Omega + \omega_\alpha)} e^{i(\Omega+\omega_\alpha)t} \right) \Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \quad (3.66d)$$

$$= 2i\epsilon \bar{\gamma}_\alpha \frac{\sin \left[(\Omega + \omega_\alpha) \frac{\Delta}{2} \right]}{\Omega + \omega_\alpha}. \quad (3.66e)$$

The same analysis shows that $KE\bar{f} \cong 0$, i.e., the solutions $E\bar{f}$ is approximately of negative frequency.

Now, if we define

$$\lambda \equiv -KEf, \quad (3.67)$$

we have,

$$\hat{\phi}(f) = ia \left(\overline{KEf} \right) - ia^\dagger (KEf) \cong ia^\dagger(\lambda). \quad (3.68)$$

Substituting in [Equation \(3.58\)](#),

$$|\Psi_\infty\rangle = \left(\mathbb{I} + a^\dagger(\lambda)D - a(\bar{\lambda})D^\dagger \right) |\Psi_{-\infty}\rangle. \quad (3.69)$$

The expression above shows that the excitation and de-excitation of a detector following the orbits of temporal isometry is associated with the absorption and emission of particles naturally define by the observers comoving with the detector.

In order to illustrate it, we shall present some examples.

- Vacuum field:

Consider the initial state

$$|\Psi_{-\infty}\rangle = |0\rangle_\xi \otimes (\alpha |0\rangle + \beta |1\rangle), \quad (3.70)$$

where $|0\rangle_\xi$ is the vacuum associated with the temporal isometries. Substituting in [Equation \(3.69\)](#) and defining the normalized mode

$$\hat{\lambda} \equiv \frac{\lambda}{\|\lambda\|}, \quad (3.71)$$

3. Particle concept

we find

$$|\Psi_\infty\rangle = |\Psi_{-\infty}\rangle + \beta a^\dagger(\lambda) |0\rangle_\xi \otimes D |1\rangle - \alpha a(\bar{\lambda}) |0\rangle_\xi \otimes D^\dagger |0\rangle \quad (3.72a)$$

$$= |0\rangle_\xi \otimes (\alpha |0\rangle + \beta |1\rangle) + \beta \|\lambda\| |\hat{\lambda}\rangle \otimes |0\rangle. \quad (3.72b)$$

Let $\alpha = 0$ and $\beta = 1$, then the probability of the detector de-excite emitting a mode $\hat{\lambda}$ is

$$P_{1 \rightarrow 0} = \langle \Psi_\infty | \mathbb{I}_\phi \otimes |0\rangle \langle 0| | \Psi_\infty \rangle \quad (3.73a)$$

$$= \|\lambda\|^2 \quad (3.73b)$$

$$= \|KEf\|^2. \quad (3.73c)$$

Similarly, if $\alpha = 1$ and $\beta = 0$, then

$$|\Psi_\infty\rangle = |0\rangle_\xi \otimes |0\rangle, \quad (3.74)$$

i.e., the detector remains in the ground state.

- One particle field:

Now, we take the initial state

$$|\Psi_{-\infty}\rangle = |1\rangle_\chi \otimes (\alpha |0\rangle + \beta |1\rangle), \quad (3.75)$$

where $|1\rangle_\chi$ is the state with one particle of the mode $\chi \in \mathcal{H}_\xi$. Therefore,

$$|\Psi_\infty\rangle = |1\rangle_\chi \otimes (\alpha |0\rangle + \beta |1\rangle) + \beta \|\lambda\| |1_\chi, 1_{\hat{\lambda}}\rangle \otimes |0\rangle - \alpha \|\lambda\| \langle \hat{\lambda}, \chi \rangle |0\rangle_\xi \otimes |1\rangle. \quad (3.76)$$

Let $\alpha = 1$ and $\beta = 0$, the probability of de-excitation, detecting χ , is

$$P_{0 \rightarrow 1} = \langle \Psi_\infty | \mathbb{I}_\phi \otimes |1\rangle \langle 1| | \Psi_\infty \rangle = |\langle KEf, \chi \rangle|^2. \quad (3.77)$$

In the approximation $KEf \cong Ef$, we have

$$P_{0 \rightarrow 1} = |\langle Ef, \chi \rangle|^2 = \left| \int_{\mathcal{M}} d^4x \sqrt{-g} \bar{\chi}(x) f(x) \right|^2. \quad (3.78)$$

3.4 Bogoliubov transformations

Suppose we have a spacetime that is stationary in both past and future, hence it allows two constructions of quantum theories with natural particle interpretation. If we can find a transformation that relate this two constructions, we would have all the information about scattering, creation and annihilation of particles in this spacetime. Motivated by this, we shall study when such map exists and it's properties.

On the existence of the map

Let μ_1 and μ_2 be two inner products used to construct the quantum field theory with operators $\hat{\Omega}_1(\psi, \cdot)$ and $\hat{\Omega}_2(\psi, \cdot)$. We wish to know if exists a unitary map $S : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that

$$S\hat{\Omega}_1(\psi, \cdot)S^{-1} = \hat{\Omega}_2(\psi, \cdot) \quad \forall \psi \in \mathcal{S}, \quad (3.79)$$

and if it exists, to obtain an explicit expression for it. We separate the analysis in two cases¹².

¹¹ *come back to this!*

1. Exist $c, c' > 0$ such that

$$c\mu_1(\psi, \psi) \leq \mu_2(\psi, \psi) \leq c'\mu_1(\psi, \psi) \quad \psi \in \mathcal{S}. \quad (3.80)$$

2. Does not exist such c and c' .

Analysing the latter case, suppose, without loss of generality, that does not exist $c > 0$ such that $c\mu_1(\psi, \psi) \leq \mu_2(\psi, \psi)$ for every $\psi \in \mathcal{S}$. In other words, for every $c > 0$, exists $\psi \in \mathcal{S}$ such that $c\mu_1(\psi, \psi) > \mu_2(\psi, \psi)$. Notice that we can always take ψ with unity μ_1 -norm:

$$c\mu_1(\psi, \psi) > \mu_2(\psi, \psi) \implies c \frac{\mu_1(\psi, \psi)}{\mu_1(\psi, \psi)} > \frac{\mu_2(\psi, \psi)}{\mu_1(\psi, \psi)}, \quad (3.81)$$

then,

$$c = c\mu_1 \left(\frac{\psi}{\sqrt{\mu_1(\psi, \psi)}}, \frac{\psi}{\sqrt{\mu_1(\psi, \psi)}} \right) > \mu_2 \left(\frac{\psi}{\sqrt{\mu_1(\psi, \psi)}}, \frac{\psi}{\sqrt{\mu_1(\psi, \psi)}} \right). \quad (3.82)$$

For $c = \frac{1}{n}$, we have a sequence $(\psi_n)_{n \in \mathbb{N}}$ such that $\mu_1(\psi_n, \psi_n) = 1$, but $\mu_2(\psi_n, \psi_n) \rightarrow 0$. As a consequence, in \mathcal{F}_2 , $e^{i\hat{\Omega}_2(\psi_n, \cdot)} - \mathbb{I}_2$ strongly converges to zero, i.e., $\forall |\varphi\rangle_2 \in \mathcal{F}_2$,

$$\left\| e^{i\hat{\Omega}_2(\psi_n, \cdot)} |\varphi\rangle_2 - |\varphi\rangle_2 \right\| \rightarrow 0. \quad (3.83)$$

Meanwhile, $e^{i\hat{\Omega}_1(\psi_n, \cdot)} |0\rangle_1 - |0\rangle_1$ does not converge. Therefore, any map S , takes $S|0\rangle_1$ to a state that does not respect the convergence, hence S cannot exist¹³.

Returning to the first case, we conclude that both norms are equivalent¹⁴. Thereby, completing \mathcal{S} with respect to any of the inner products result in the same completed space \mathcal{S}_μ .

¹² *from now on, there will be a lot of functional analysis. get back later!*

¹³ *the functional analysis crap ended!*

3. Particle concept

Let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{S}_\mu^{\mathbb{C}}$ be the one-particle state correspondent to each of the constructions. We equip $\mathcal{S}_\mu^{\mathbb{C}}$ with the Klein-Gordon's inner product, $\langle \psi_1, \psi_2 \rangle_1 \equiv 2\mu_1 \langle \bar{\psi}_1, \psi_2 \rangle$ and define the induced projections $K_1 : \mathcal{S}_\mu^{\mathbb{C}} \rightarrow \mathcal{H}_1$ and $\bar{K}_1 : \mathcal{S}_\mu^{\mathbb{C}} \rightarrow \bar{\mathcal{H}}_1$. The same procedure follows to define the projections $K_2 : \mathcal{S}_\mu^{\mathbb{C}} \rightarrow \mathcal{H}_2$ and $\bar{K}_2 : \mathcal{S}_\mu^{\mathbb{C}} \rightarrow \bar{\mathcal{H}}_2$. Finally, we have equivalent theories if exists a unitary map $S : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that

$$S \left[ia_1 \left(\bar{K}_1 \psi \right) - ia_1^\dagger (K_1 \psi) \right] S^{-1} = ia_2 \left(\bar{K}_2 \psi \right) - ia_2^\dagger (K_2 \psi), \quad \forall \psi \in \mathcal{S}. \quad (3.84)$$

The equation extends to $\mathcal{S}^{\mathbb{C}}$ by (complex) linearity and to $\mathcal{S}_\mu^{\mathbb{C}}$ by continuity. Now, we define the maps $A : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ and $B : \mathcal{H}_2 \rightarrow \bar{\mathcal{H}}_1$ as the restrictions of K_1 and \bar{K}_1 to \mathcal{H}_2 . Similarly, we have the restrictions of K_2 and \bar{K}_2 to \mathcal{H}_1 defined as the operators $C : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $D : \mathcal{H}_1 \rightarrow \bar{\mathcal{H}}_2$ ¹⁵.

Now, take $\psi \in \bar{\mathcal{H}}_1$ and define $\chi \equiv \bar{\psi} \in \mathcal{H}_1$. Notice that $K_1 \psi = 0$ and $\bar{K}_1 \psi = \psi$. Then, from Equation (3.84), we have

$$Sa_1(\psi)S^{-1} = a_2 \left(\bar{K}_2 \bar{\psi} \right) - a_2^\dagger \left(\bar{K}_2 \bar{\psi} \right) \quad (3.85a)$$

$$Sa_1(\bar{\chi})S^{-1} = a_2 \left(\bar{C} \chi \right) - a_2^\dagger \left(\bar{D} \chi \right). \quad (3.85b)$$

Properties of the operators

Since the norms in \mathcal{H}_1 and \mathcal{H}_2 are equivalent, we have the the operators A, B, C and D are bounded¹⁶. We have many properties concerning these operators. For example, take $\psi, \chi \in \mathcal{H}_2$, then

$$\langle \psi, \chi \rangle_{\mathcal{H}_2} = -i\Omega(\bar{\psi}, \chi) \quad (3.86a)$$

$$= -i\Omega \left(\overline{K_1 \psi + \bar{K}_1 \psi}, K_1 \chi + \bar{K}_1 \chi \right) \quad (3.86b)$$

$$= -i\Omega \left(\overline{K_1 \psi}, K_1 \chi \right) - i\Omega \left(\overline{\bar{K}_1 \psi}, \bar{K}_1 \chi \right) \quad (3.86c)$$

$$= \langle K_1 \psi, K_1 \chi \rangle_{\mathcal{H}_1} + i\Omega \left(\bar{K}_1 \chi, \overline{\bar{K}_1 \psi} \right) \quad (3.86d)$$

$$= \langle K_1 \psi, K_1 \chi \rangle_{\mathcal{H}_1} + i\Omega \left(\bar{K}_1 \bar{\chi}, K_1 \bar{\psi} \right) \quad (3.86e)$$

$$= \langle K_1 \psi, K_1 \chi \rangle_{\mathcal{H}_1} - \left\langle K_1 \bar{\chi}, K_1 \bar{\psi} \right\rangle_{\mathcal{H}_1} \quad (3.86f)$$

$$= \langle K_1 \psi, K_1 \chi \rangle_{\mathcal{H}_1} - \left\langle \bar{K}_1 \psi, \bar{K}_1 \chi \right\rangle_{\bar{\mathcal{H}}_1} \quad (3.86g)$$

$$= \langle A\psi, A\chi \rangle_{\mathcal{H}_1} - \langle B\psi, B\chi \rangle_{\bar{\mathcal{H}}_1} \quad (3.86h)$$

$$= \left\langle \psi, A^\dagger A \chi \right\rangle_{\mathcal{H}_2} - \left\langle \psi, B^\dagger B \chi \right\rangle_{\mathcal{H}_2}, \quad (3.86i)$$

¹⁴ why?

¹⁵ add a diagram!

therefore,

$$A^\dagger A - B^\dagger B = \mathbb{I}. \quad (3.87)$$

Now, for any operator $\mathcal{O} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, we define its conjugated $\overline{\mathcal{O}} : \overline{\mathcal{H}}_1 \rightarrow \overline{\mathcal{H}}_2$ as

$$\overline{\mathcal{O}}v = \overline{\mathcal{O}\overline{v}}, \quad \forall v \in \overline{\mathcal{H}}_1. \quad (3.88)$$

With this in mind, if we take $\psi \in \overline{\mathcal{H}}_2$ and $\chi \in \mathcal{H}_2$, then

$$0 = -i\Omega(\overline{\psi}, \chi) \quad (3.89a)$$

$$= \langle K_1\psi, K_1\chi \rangle_{\mathcal{H}_1} - \langle \overline{K}_1\psi, \overline{K}_1\chi \rangle_{\overline{\mathcal{H}}_1} \quad (3.89b)$$

$$= \langle \overline{B}\psi, A\chi \rangle_{\mathcal{H}_1} - \langle \overline{A}\psi, B\chi \rangle_{\overline{\mathcal{H}}_1} \quad (3.89c)$$

$$= \langle A^\dagger \overline{B}\psi, \chi \rangle_{\mathcal{H}_2} - \langle B^\dagger \overline{A}\psi, \chi \rangle_{\mathcal{H}_2}, \quad (3.89d)$$

hence,

$$A^\dagger \overline{B} = B^\dagger \overline{A}. \quad (3.90)$$

The last both calculations with 1 exchanged by 2, leads to

$$C^\dagger C - D^\dagger D = \mathbb{I}, \quad (3.91)$$

and

$$C^\dagger \overline{D} = D^\dagger \overline{C}. \quad (3.92)$$

Finally, for $\psi \in \mathcal{H}_1$, $\chi \in \mathcal{H}_2$, we have

$$\langle \psi, A\chi \rangle_{\mathcal{H}_1} = -i\Omega(\overline{\psi}, A\chi) \quad (3.93a)$$

$$= -i\Omega(\overline{\psi}, K_1\chi) \quad (3.93b)$$

$$= -i\Omega(\overline{\psi}, K_1\chi + \overline{K}_1\chi) \quad (3.93c)$$

$$= -i\Omega(\overline{\psi}, \chi) \quad (3.93d)$$

$$= -i\Omega(\overline{K_2\psi + \overline{K}_2\psi}, \chi) \quad (3.93e)$$

$$= -i\Omega(\overline{K_2\psi}, \chi) \quad (3.93f)$$

$$= \langle K_2\psi, \chi \rangle_{\mathcal{H}_2} \quad (3.93g)$$

$$= \langle C\psi, \chi \rangle_{\mathcal{H}_2} \quad (3.93h)$$

$$= \langle \psi, C^\dagger \chi \rangle_{\mathcal{H}_1}, \quad (3.93i)$$

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thus,

$$A = C^\dagger. \quad (3.94)$$

Similarly, if we take $\psi \in \overline{\mathcal{H}}_1$ and $\chi \in \mathcal{H}_2$, then

$$\langle \psi, B\chi \rangle_{\overline{\mathcal{H}}_1} = -i\Omega(\overline{\psi}, B\chi) \quad (3.95a)$$

$$= -i\Omega(\overline{\psi}, \overline{K}_1\chi) \quad (3.95b)$$

$$= -i\Omega(\overline{\psi}, K_1\chi + \overline{K}_1\chi) \quad (3.95c)$$

$$= -i\Omega(\overline{\psi}, \chi) \quad (3.95d)$$

$$= -i\Omega(\overline{\psi}, \chi) \quad (3.95e)$$

$$= -i\Omega(\overline{K_2\psi + \overline{K}_2\psi}, \chi) \quad (3.95f)$$

$$= -i\Omega(\overline{K_2\psi}, \chi) \quad (3.95g)$$

$$= \langle K_2\psi, \chi \rangle_{\mathcal{H}_2} \quad (3.95h)$$

$$= \langle \overline{D}\psi, \chi \rangle_{\mathcal{H}_2} \quad (3.95i)$$

$$= -\langle \psi, \overline{D}^\dagger \chi \rangle_{\mathcal{H}_1}, \quad (3.95j)$$

hence,

$$B = -\overline{D}^\dagger. \quad (3.96)$$

¹⁶ *why?*

The properties [Equations \(3.87\) and \(3.91\)](#) implies¹⁷ that A^{-1} and C^{-1} exist as bounded operators. An unitary map S that satisfies [Equation \(3.85\)](#) and such that the operators A , B , C and D satisfies the properties [Equations \(3.87\), \(3.90\) to \(3.92\), \(3.94\) and \(3.96\)](#) is called a Bogoliubov transformation.

Explicit expression

Suppose that this transformation exists and we shall try to find an expression for it. We want to find $\Psi_0 \in \mathcal{F}_2$ such that

$$\Psi_0 = S|0\rangle_1. \quad (3.97)$$

The most general expression for Ψ_0 is

$$\Psi_0 = c(1, \psi^{a_1}, \psi^{a_1 a_2}, \dots). \quad (3.98)$$

Take $\xi \in \mathcal{H}_2$ and define $\chi = C^{-1}\xi$. Applying the unitary mp in this state, we have

$$Sa_1(\bar{\chi}) S^{-1}\Psi_0 = \left[a_2(\overline{C\chi}) - a_2^\dagger(\overline{D\chi}) \right] \Psi_0 \quad (3.99a)$$

$$Sa_1(\bar{\chi}) |0\rangle_1 = \left[a_2(\bar{\xi}) - a_2^\dagger(\overline{DC^{-1}\xi}) \right] \Psi_0 \quad (3.99b)$$

$$0 = \left[a_2(\bar{\xi}) - a_2^\dagger(E\bar{\xi}) \right] \Psi_0, \quad (3.99c)$$

i.e.,

$$a_2(\bar{\xi}) \Psi_0 = a_2^\dagger(E\bar{\xi}), \quad (3.100)$$

where $E : \overline{\mathcal{H}}_2 \rightarrow \mathcal{H}_2$ is defined as

$$E \equiv \overline{DC}^{-1}. \quad (3.101)$$

From property expressed in Equation (3.92), we have

$$C^\dagger \overline{DC}^{-1} = D^\dagger \implies \overline{DC}^{-1} = (C^\dagger)^{-1} D^\dagger \implies E = (DC^{-1}) = \overline{E}^\dagger, \quad (3.102)$$

i.e., E is symmetric. From the definitions of creation and annihilation operators, we have

$$\begin{cases} \bar{\xi}_{a_1} \psi^{a_1} = 0 \\ \sqrt{2} \bar{\xi}_{a_1} \psi^{a_1 a_2} = (E\bar{\xi})^{a_2} \\ \sqrt{3} \bar{\xi}_{a_1} \psi^{a_1 a_2 a_3} = \sqrt{2} (E\bar{\xi})^{(a_2} \psi^{a_3)} \\ \sqrt{4} \bar{\xi}_{a_1} \psi^{a_1 a_2 a_3 a_4} = \sqrt{3} (E\bar{\xi})^{(a_2} \psi^{a_3 a_4)} \\ \vdots \end{cases} \quad (3.103)$$

Since these relations are true for any ξ , then $\psi^a = 0$. From that, follows the any odd rank tensor must vanish as well. Abusing notation considering that $E \in \overline{\mathcal{H}}_2 \otimes \overline{\mathcal{H}}_2$, we have

$$\psi^{a_1 a_2} = \frac{1}{\sqrt{2}} E^{a_1 a_2}. \quad (3.104)$$

Proposition 3.6. The mapped ground state is given by

$$\Psi_0 = c \left(1, 0, \sqrt{\frac{1}{2}} E^{a_1 a_2}, 0, \dots, \sqrt{\frac{(n-1)!!}{n!!}} E^{(a_1 a_2} \dots E^{a_{n-1} a_n)}, \dots \right), \quad (3.105)$$

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where n is an even number.

Proof. show it! □

Notice that we can always choose c such that $\|\Psi_0\| = 1$. Our assumption to hold, i.e., $\Psi_0 = S|0\rangle_1$ exists, if, and only if,

$$\mathrm{Tr} \left(E^\dagger E \right) < \infty. \quad (3.106)$$

Since C and C^{-1} are bounded¹⁸, it is equivalent to

$$\mathrm{Tr} \left(D^\dagger D \right) < \infty, \quad (3.107)$$

that can also be expressed as

$$\mathrm{Tr} \left(B^\dagger B \right) < \infty. \quad (3.108)$$

Now, we can consider the action on a 1-particle state considering the adjoint of [Equation \(3.85\)](#),

$$S a_1^\dagger(\chi) S^{-1} = a_2^\dagger(C\chi) - a_2(D\chi) \quad (3.109a)$$

$$S a_1^\dagger(\chi) |0\rangle_1 = \left[a_2^\dagger(C\chi) - a_2(D\chi) \right] \Psi_0. \quad (3.109b)$$

Henceforth, by induction we extend the action of S to any element of \mathcal{F}_1 .



CHAPTER 4

UNRUH EFFECT

We shall explore our construction of quantum field theory in stationary spacetimes returning to Minkowski spacetime. There we find a striking feature that explicates the fact that particle concept is observer dependent, known as Unruh effect. Moreover, we generalize this effect to general spacetimes that possess a bifurcate Killing horizon.

4.1 Unruh effect in Minkowski

Setting up the background

First, in Minkowski spacetime, let ξ^a be the Killing field associated with time translation attributed by a family of inertial observers. We denote the vacuum of this construction as Minkowski vacuum, $|0\rangle_M$.

Furthermore, there other isometries available, for example the one parameter group of Lorentz's boost, with Killing field

$$\chi^a = a [x (\partial_t)^a + t (\partial_x)^a], \quad (4.1)$$

where a is an arbitrary constant and x and t are global coordinates. We can divide the spacetime in four regions whose boundaries are the surfaces where χ^a is null¹⁹.

¹⁸ check all that follows!

Take a orbit of χ^a such that $\chi^a \chi_a = -1$, then χ^a can be interpreted as the 4-velocity of a family of observers. Therefore, the 4-acceleration is

$$\alpha^a = \chi^b \nabla_b \chi^a \quad (4.2a)$$

$$= a^2 \left(x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right) [x (\partial_t)^a + t (\partial_x)^a] \quad (4.2b)$$

$$= a^2 [t (\partial_t)^a + x (\partial_x)^a]. \quad (4.2c)$$

Since

$$\chi^a \chi_a = a^2 (-x^2 + t^2) = -1, \quad (4.3)$$

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then,

$$\alpha^a \alpha_a = a^4 (-t^2 + x^2) = a^2. \quad (4.4)$$

Thus, we interpret this family of observers as being uniformly accelerated that move along the orbits of χ^a . Let Σ be a Cauchy surface, we denote by Σ_I and Σ_{II} its restriction to the regions I and II respectively, hence each region is globally hyperbolic. Thereby, we construct our theory picking the one-particle state to be $\mathcal{H}_2 = \mathcal{H}_I \oplus \mathcal{H}_{II}$, in which \mathcal{H}_I is obtained by the usual construction in region I with the notion arising from χ^a and, similarly, \mathcal{H}_{II} is constructed with the aid of $-\chi^a$ (that is future directed) in region II.

Now, we want to represent Minkowski vacuum in the Fock space

$$\mathcal{F}_S(\mathcal{H}_2) \cong \mathcal{F}_S(\mathcal{H}_I) \otimes \mathcal{F}_S(\mathcal{H}_{II}). \quad (4.5)$$

The vacuum notion generated by this construction is called Rindler vacuum, $|0\rangle_R$. Since accelerated observers, for example, in region I does not have access to region II, we specifically want to express $|0\rangle_M$ in $\mathcal{F}_S(\mathcal{H}_I)$. To do it, we shall find an S matrix, $S : \mathcal{F}_S(\mathcal{H}_I) \rightarrow \mathcal{F}_S(\mathcal{H}_2)$, apply it on $|0\rangle_M$ and trace out the states of \mathcal{H}_2 .

Before we perform these calculations, the following results simplify the calculations. We define the retarded and advanced null coordinates, U and V , as

$$U = t - x \quad V = t + x. \quad (4.6)$$

We claim that, in the horizon \mathfrak{h}_A (\mathfrak{h}_B), solution is of positive frequency with respect to the retarded (advanced) time V (U) if, and only if, it is of positive frequency with respect to the inertial time t . Indeed, take $\varphi \in \mathcal{H}_1$, i.e., of positive frequency, then

$$\varphi = \int \frac{d^3k}{(2\pi)^3} \tilde{\varphi}(k) \exp \left[i \left(-\omega_k t + k_x x + \vec{x}_\perp \cdot \vec{k}_\perp \right) \right] \quad (4.7a)$$

$$= \int \frac{d^3k}{(2\pi)^3} \tilde{\varphi}(k) \exp \left[i \left(-\omega_k \frac{U+V}{2} + k_x \frac{V-U}{2} + \vec{x}_\perp \cdot \vec{k}_\perp \right) \right], \quad (4.7b)$$

where $\vec{x}_\perp \cdot \vec{k}_\perp = k_y y + k_z z$. On \mathfrak{h}_A , $U = 0$, then

$$\varphi = \int \frac{d^3k}{(2\pi)^3} \tilde{\varphi}(k) \exp \left[i \left(-\frac{\omega_k - k_x}{2} V + \vec{x}_\perp \cdot \vec{k}_\perp \right) \right], \quad (4.8)$$

since $\omega_k = \sqrt{k_x^2 + k_\perp^2 + m^2} > k_x$, then a positive frequency solution with respect to t is also of positive frequency with respect to V on \mathfrak{h}_A . Similarly,

in \mathfrak{h}_B , $V = 0$, then

$$\varphi = \int \frac{d^3k}{(2\pi)^3} \tilde{\varphi}(k) \exp \left[i \left(-\frac{\omega_k + k_x}{2} U + \vec{x}_\perp \cdot \vec{k}_\perp \right) \right], \quad (4.9)$$

hence a positive frequency solution with respect to t is also of positive frequency with respect to U on \mathfrak{h}_B .

Moreover, any solution in Minkowski spacetime is uniquely determined by its value on $\mathfrak{h}_A \cup \mathfrak{h}_B$. In other words, the dependence domain of this union satisfies $D(\mathfrak{h}_A \cup \mathfrak{h}_B) = \mathcal{M}$.

Finally, consider the retarded Killing time v such that $\chi^a \nabla_a v = 1$. In terms of the inertial retarded and accelerated times, we have

$$\chi^a = a [x (\partial_t) + t (\partial_x)^a] \quad (4.10a)$$

$$= \frac{a}{2} [(V - U) (\partial_U + \partial_V)^a + (U + V) (\partial_V - \partial_U)^a] \quad (4.10b)$$

$$= a [-U (\partial_U)^a + V (\partial_V)^a]. \quad (4.10c)$$

Thus, in \mathfrak{h}_A , we have

$$1 = \chi^a \nabla_a v \quad (4.11a)$$

$$= a \left(-U \frac{\partial v}{\partial U} + V \frac{\partial v}{\partial V} \right) \quad (4.11b)$$

$$= aV \frac{\partial v}{\partial V}, \quad (4.11c)$$

hence,

$$v = \frac{1}{a} \ln |V|. \quad (4.12)$$

Similarly, for the advanced Killing time u in \mathfrak{h}_B , we have

$$1 = \chi^a \nabla_a u \quad (4.13a)$$

$$= -aU \frac{\partial u}{\partial U}, \quad (4.13b)$$

thus,

$$u = -\frac{1}{a} \ln |U|. \quad (4.14)$$

From these remarks, we can look at positive frequency solutions in $\mathfrak{h}_A \cup \mathfrak{h}_B$ and decompose them as function of the inertial time, we take the Fourier transform with respect to V (U) in \mathfrak{h}_A (\mathfrak{h}_B). This way, we avoid an explicit solutions of Klein-Gordon equation in different coordinate systems.

Effective calculation

Let ψ_ω^I be a solution of Klein-Gordon equation that vanishes in region II and oscillate harmonically with frequency $\omega > 0$ (with respect to the accelerated Killing time) in region I. A solution of this type has divergent norm, however it can be helped by putting it into a suitable wavepacket. We denote the restriction of ψ_ω^I to \mathfrak{h}_A as

$$f_\omega^I(V, y, z) = \begin{cases} h(y, z)e^{-i\omega V} & , \quad V > 0 \\ 0 & , \quad V < 0 \end{cases} . \quad (4.15)$$

Taking the Fourier transform with respect to V , whose associated frequency we shall denote by σ , we have

$$\hat{f}_\omega^I(\sigma, y, z) = \int_{-\infty}^{\infty} dV \psi_\omega^I(V, y, z) e^{i\sigma V} \quad (4.16a)$$

$$= \int_0^{\infty} e^{i\sigma V} e^{-i\frac{\omega}{a} \ln |V|} h(y, z) \quad (4.16b)$$

$$\equiv h(y, z) I(\sigma) \quad (4.16c)$$

Now, suppose that $\sigma > 0$ and that \log denotes the principal branch of logarithm. If we make $V = iy$, and use the integration path²⁰, we have

$$I(\sigma) = \int_0^{\infty} dV e^{i\sigma V} e^{-i\frac{\omega}{a} \ln V} \quad (4.17a)$$

$$= i \int_0^{-i\infty} dy e^{-\sigma y} e^{-i\frac{\omega}{a} \log(iy)} \quad (4.17b)$$

$$= i e^{-i\frac{\omega}{a} \log\left(e^{i\frac{\pi}{2}}\right)} \int_0^{-i\infty} dy e^{-\sigma y} e^{-i\frac{\omega}{a} \log y} \quad (4.17c)$$

$$= i e^{\frac{\pi\omega}{2a}} \int_0^{\infty} dy e^{-\sigma y} e^{-i\frac{\omega}{a} \log y}, \quad (4.17d)$$

where in the last passage we used the residue theorem to exchange the former integral for two (omitted) integrals around paths with arbitraly small and large radius and the latter one.

Similarly, if we take $V = -iy$ ²¹,

¹⁹ *diagrama!*

²⁰ *imaginary path*

$$I(-\sigma) = \int_0^\infty dV e^{-i\sigma V} e^{-i\frac{\omega}{a} \ln V} \quad (4.18a)$$

$$= -i \int_0^{i\infty} dy e^{-\sigma y} e^{-i\frac{\omega}{a} \log(-iy)} \quad (4.18b)$$

$$= -ie^{-i\frac{\omega}{a} \log(-i)} \int_0^{i\infty} dy e^{-\sigma y} e^{-i\frac{\omega}{a} \log y} \quad (4.18c)$$

$$= -ie^{-\frac{\pi\omega}{2a}}. \quad (4.18d)$$

Therefore, we conclude that

$$\hat{f}_\omega^{\text{I}}(-\sigma, y, z) = -e^{\frac{\pi\omega}{a}} \hat{f}_\omega^{\text{I}}(\sigma, y, z), \quad \forall \sigma > 0. \quad (4.19)$$

Now, to study region II, notice that the reflection transformation $(t, x, y, z) \rightarrow (-t, -x, y, z)$ maps region I in region II. As a consequence, it maps solutions of \mathcal{H}_{I} in \mathcal{H}_{II} . Since \mathcal{H}_{II} is obtained with $-\chi^a$, the restriction to \mathfrak{h}_A of the image of ψ_ω^{I} has the form

$$\bar{f}_\omega^{\text{II}}(V, y, z) = \begin{cases} 0, & V > 0 \\ h(y, z)e^{-i\omega v}, & V < 0 \end{cases}. \quad (4.20)$$

We obtained $\bar{f}_\omega^{\text{II}}$ by a time reflection of f_ω^{I} , hence

$$\hat{\bar{f}}_\omega^{\text{II}}(\sigma, y, z) = \hat{f}_\omega^{\text{I}}(-\sigma, y, z). \quad (4.21)$$

Inspired by the last results obtained, we define

$$F_\omega \equiv f_\omega^{\text{I}} + e^{-\frac{\pi\omega}{a}} \bar{f}_\omega^{\text{II}}. \quad (4.22)$$

Notice that in \mathfrak{h}_A , for $\sigma > 0$, we have

$$\hat{F}_\omega(-\sigma, y, z) = \hat{f}_\omega^{\text{I}}(-\sigma, y, z) + e^{-\frac{\pi\omega}{a}} \hat{\bar{f}}_\omega^{\text{II}}(-\sigma, y, z) \quad (4.23a)$$

$$= -e^{-\frac{\pi\omega}{a}} \hat{f}_\omega^{\text{I}}(\sigma, y, z) + e^{-\frac{\pi\omega}{a}} \hat{f}_\omega^{\text{I}}(\sigma, y, z) \quad (4.23b)$$

$$= 0, \quad (4.23c)$$

i.e., F_ω is a solution of positive frequency with respect to the inertial time²². ²¹ *imaginary path*

Henceforth, in order to solve the divergence problem, we take an orthonormal basis $\{\psi_i^{\text{I}}\}$ of \mathcal{H}_{I} formed of thin peaks around $\omega_i > 0$. From the reflection transformation, we get an associated basis $\{\psi_i^{\text{II}}\}$ of \mathcal{H}_{II} . Therefore,

$$\Psi_i = \psi_i^{\text{I}} + e^{-\frac{\pi\omega_i}{a}} \psi_i^{\text{II}}, \quad (4.24)$$

4. Unruh effect

is a positive frequency solution with respect to the inertial time, i.e., $\Psi_i \in \mathcal{H}_1$. Similarly, we also have

$$\Psi'_i = \psi_i^{\text{II}} + e^{-\frac{\pi\omega_i}{a}} \bar{\psi}_i^{\text{I}} \in \mathcal{H}_1. \quad (4.25)$$

Now, we can calculate the S matrix. Remember that $C : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $D : \mathcal{H}_1 \rightarrow \bar{\mathcal{H}}_2$, then

$$C\Psi_i = \psi_i^{\text{I}} \quad C\Psi'_i = \psi_i^{\text{II}} \quad (4.26a)$$

$$D\Psi_i = e^{-\frac{\pi\omega_i}{a}} \bar{\psi}_i^{\text{II}} \quad D\Psi'_i = e^{-\frac{\pi\omega_i}{a}} \bar{\psi}_i^{\text{I}}. \quad (4.26b)$$

Therefore, we have

$$DC^{-1}\psi_i^{\text{I}} = e^{-\frac{\pi\omega_i}{a}} \quad \text{and} \quad DC^{-1}\bar{\psi}_i^{\text{II}} = e^{-\frac{\pi\omega_i}{a}} \bar{\psi}_i^{\text{I}}. \quad (4.27)$$

Furthermore, since $\mathcal{H}_2 = \mathcal{H}_I \oplus \mathcal{H}_{\text{II}}$, $\{\psi_i^{\text{I}}\}$ and $\{\psi_i^{\text{II}}\}$ generate \mathcal{H}_2 , thus the action of the operator $E = \overline{DC}^{-1}$ is completely defined by [Equation \(4.27\)](#). The 2-particle state correspondent to this operator is

$$E^{ab} = \sum_i 2e^{-\frac{\pi\omega_i}{a}} \left(\psi_i^{\text{I}} \right)^{(a)} \left(\psi_i^{\text{II}} \right)^{(b)}. \quad (4.28)$$

Proposition 4.1. The S matrix obtained from [Equation \(4.28\)](#) acting on Minkowski vacuum is

$$S|0\rangle_M = \prod_j \left(\sum_{n=0}^{\infty} e^{-\frac{\pi n\omega_j}{a}} |n_j\rangle_{\text{I}} \otimes |n_j\rangle_{\text{II}} \right), \quad (4.29)$$

where $|n_j\rangle_{\text{I}}$ represents a state of $\mathcal{F}_S(\mathcal{H}_I)$ with n particles in the mode ψ_i^{I} and the same holds for $|n_j\rangle_{\text{II}}$.

■ **Proof.** will show it. □

The density matrix of the state mapped by the vacuum is

$$\rho = \prod_{i,j} \left(\sum_{n,m} e^{-\frac{\pi n\omega_i}{a}} e^{-\frac{\pi m\omega_j}{a}} |n_{i\text{I}} n_{i\text{II}}\rangle \langle m_{j\text{I}} m_{j\text{II}}| \right). \quad (4.30)$$

Tracing out the region II, we get

$$\rho^{\text{I}} = \text{Tr}_{\text{II}} \rho \quad (4.31a)$$

$$= \prod_{i,j,k} \left(\sum_{n,m,\ell} e^{-\frac{\pi n\omega_i}{a}} e^{-\frac{\pi m\omega_j}{a}} \langle \ell_{k\text{II}} | n_{i\text{II}} \rangle \langle m_{i\text{II}} | \ell_{k\text{II}} \rangle |n_i\rangle_{\text{I}} \langle m_j |_{\text{I}} \right) \quad (4.31b)$$

$$= \prod_j \left(\sum_n e^{-\frac{2\pi n\omega_j}{a}} |n_j\rangle_{\text{I}} \langle n_j |_{\text{I}} \right). \quad (4.31c)$$

An accelerated observer attribute energy $n\omega_j$ to a state $|n_j\rangle_I$, then, if we compare Equation (4.31) with a thermal state

$$\rho_T = e^{-\frac{H_T}{T}}, \quad (4.32)$$

we conclude that the accelerated observers that follows the orbit of χ^a interpret the Minkowski vacuum as thermal bath of particles at a temperature

$$T = \frac{a}{2\pi} = \frac{a\hbar}{2\pi k_B c}. \quad (4.33)$$

Remark. Generalize for curved space.

4. Unruh effect

APPENDIX A

MATHEMATICAL IDENTITIES

This chapter intends to develop mathematical identities used throughout the text.

Proposition A.1. It holds

$$\lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x} f(x) dx = 0, \quad (\text{A.1})$$

i.e.,

$$\lim_{\epsilon \rightarrow 0^+} \frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x} = \delta(x). \quad (\text{A.2})$$



APPENDIX B

FOCK SPACE

The symmetric Fock space is the Hilbert space used in the construction of our quantum field theory in curved spacetimes for a scalar field. However, it must be constructed from a existent Hilbert space, that physically can have the interpretation of the space of one particle.

Let \mathcal{H} be a Hilbert space, then the Fock space associated to it, $\mathcal{F}(\mathcal{H})$, is defined as

$$\mathcal{F}(\mathcal{H}) \equiv \oplus_{n=0}^{\infty} (\otimes^n \mathcal{H}). \quad (\text{B.1})$$

Therefore, an element $\Psi \in \mathcal{F}(\mathcal{H})$ is infinite dimension vector whose each entry is a tensor product of elements of \mathcal{H} , i.e.,

$$\Psi = (\psi, \psi^a, \psi^{ab}, \dots), \quad (\text{B.2})$$

where

$$\psi^{a_1 \dots a_j} \in \otimes^j \mathcal{H}. \quad (\text{B.3})$$

The symmetric Fock space, $\mathcal{F}_S(\mathcal{H})$, is defined as

$$\mathcal{F}_S(\mathcal{H}) \equiv \oplus_{n=0}^{\infty} (\otimes_S^n \mathcal{H}), \quad (\text{B.4})$$

that is very similar to the ordinary Fock space, however, the possible entries must be symmetric, i.e.,

$$\psi^{a_1 \dots a_j} = \psi^{(a_1 \dots a_j)}. \quad (\text{B.5})$$

In order to the Fock space be a Hilbert space, we equip it with inner product that act in two elements $\Psi, \Phi \in \mathcal{F}_S(\mathcal{H})$ as

$$\langle \Psi, \Phi \rangle_{\mathcal{F}_S} \equiv \bar{\psi} \phi + \bar{\psi}_a \phi^a + \dots, \quad (\text{B.6})$$

in which the elements $\bar{\psi}_a$ are the dual vectors correspondent to the conjugated one in $\overline{\mathcal{H}}$.

Algebra of operators

Now, we define the annihilation and creation operators in the Fock space. Let $\xi^a \in \mathcal{H}$, then the annihilation operator $a(\bar{\xi}) : \mathcal{F}_S(\mathcal{H}) \rightarrow \mathcal{F}_S(\mathcal{H})$ is defined as

$$a(\bar{\xi})\Psi \equiv \left(\bar{\xi}_{a_1} \psi^{a_1}, \sqrt{2} \bar{\xi}_{a_1} \psi^{a_1 a_2}, \sqrt{3} \bar{\xi}_{a_1} \psi^{a_1 a_2 a_3}, \dots \right). \quad (\text{B.7})$$

Similarly, the creation operator $a^\dagger(\xi) : \mathcal{F}_S(\mathcal{H}) \rightarrow \mathcal{F}_S(\mathcal{H})$ is defined as

$$a^\dagger(\xi)\Psi \equiv \left(0, \psi \xi^a, \sqrt{2} \xi^{(a_1} \psi^{a_2)}, \sqrt{3} \xi^{(a_1} \psi^{a_2 a_3)}, \dots \right). \quad (\text{B.8})$$

Proposition B.1. Let $\xi, \eta \in \mathcal{H}$, then the annihilation and creation operators has commutation relation given by

$$[a(\bar{\xi}), a(\bar{\eta})] = [a^\dagger(\xi), a^\dagger(\eta)] = 0 \quad (\text{B.9})$$

and

$$[a(\bar{\xi}), a^\dagger(\eta)] = \bar{\xi}_a \eta^a \mathbb{I}. \quad (\text{B.10})$$

Proof. We shall restrict our proof to the first two non-trivial terms. Using our definitions, we have

$$a(\bar{\xi})a(\bar{\eta})\Psi = \left(\sqrt{2} \bar{\xi}_{a_2} \bar{\eta}_{a_1} \psi^{a_1 a_2}, \sqrt{6} \bar{\xi}_{a_2} \bar{\eta}_{a_1} \psi^{a_1 a_2 a_3}, \dots \right). \quad (\text{B.11})$$

Similarly, we have

$$a(\bar{\eta})a(\bar{\xi})\Psi = \left(\sqrt{2} \bar{\eta}_{a_2} \bar{\xi}_{a_1} \psi^{a_1 a_2}, \sqrt{6} \bar{\eta}_{a_2} \bar{\xi}_{a_1} \psi^{a_1 a_2 a_3}, \dots \right). \quad (\text{B.12})$$

However, from the Fock space symmetry,

$$\bar{\xi}_{a_2} \bar{\eta}_{a_1} \psi^{a_1 a_2} = \bar{\eta}_{a_2} \bar{\xi}_{a_1} \psi^{a_1 a_2} \quad \text{and} \quad \bar{\xi}_{a_2} \bar{\eta}_{a_1} \psi^{a_1 a_2 a_3} = \bar{\eta}_{a_2} \bar{\xi}_{a_1} \psi^{a_1 a_2 a_3}. \quad (\text{B.13})$$

Arguments exploring the symmetry also holds for the other components. Since it is applied in an arbitrary vector Ψ , is evident that the commutator vanishes. Similarly, from the definition of creation operator, we have

$$\begin{aligned} a^\dagger(\xi)a^\dagger(\eta)\Psi &= \left(0, 0, \sqrt{2} \psi \xi^{(a_1} \eta^{a_2)}, \sqrt{6} \xi^{(a_1} \eta^{a_2} \psi^{a_3)}, \dots \right) \\ a^\dagger(\eta)a^\dagger(\xi)\Psi &= \left(0, 0, \sqrt{2} \psi \eta^{(a_1} \xi^{a_2)}, \sqrt{6} \eta^{(a_1} \xi^{a_2} \psi^{a_3)}, \dots \right) \end{aligned} \quad (\text{B.14})$$

Again, invoking the symmetry of the Fock space, we have

$$\psi_0 \xi^{(a_1} \eta^{a_2)} = \psi_0 \eta^{(a_1} \xi^{a_2)} \quad \text{and} \quad \xi^{(a_1} \eta^{a_2} \psi_1^{a_3)} = \eta^{(a_1} \xi^{a_2} \psi_1^{a_3)}. \quad (\text{B.15})$$

With the same strategies for the other components, we find that the commutator of creation operators are null.

Finally, the elements for the last commutator are

$$a(\bar{\xi})a^\dagger(\eta)\Psi = \left(\psi\bar{\xi}_a\eta^a, 2\bar{\xi}_a\eta^{(a}\psi^{a_2)}, 3\bar{\xi}_a\eta^{(a}\psi^{a_1a_2)}, \dots\right) \quad (\text{B.16a})$$

$$a^\dagger(\eta)a(\bar{\xi})\Psi = \left(0, \bar{\xi}_a\psi^a\eta^{a_1}, 2\bar{\xi}_a\psi^{a(a_1}\eta^{a_2)}, \dots\right). \quad (\text{B.16b})$$

Notice that

$$2\bar{\xi}_a\eta^{(a}\psi^{a_1)} = \bar{\xi}_a\eta^a\psi^{a_1} + \bar{\xi}_a\psi^a\eta^{a_1} \quad (\text{B.17})$$

and

$$3\bar{\xi}_{a_1}\eta^{(a_1}\psi^{a_2a_3)} = \frac{1}{2}\bar{\xi}_a[\eta^{a_1}\psi^{a_2a_3} + \eta^{a_1}\psi^{a_3a_2} + \eta^{a_2}\psi^{a_1a_3} \quad (\text{B.18a})$$

$$+ \eta^{a_2}\psi^{a_3a_1} + \eta^{a_3}\psi^{a_1a_2} + \eta^{a_3}\psi^{a_2a_1}] \quad (\text{B.18b})$$

$$= \bar{\xi}_{a_1}\eta^{a_1}\psi^{(a_2a_3)} + \bar{\xi}_{a_1}[\psi^{a_1a_3}\eta^{a_2} + \psi^{a_1a_2}\eta^{a_3}] \quad (\text{B.18c})$$

$$= \bar{\xi}_{a_1}\eta^{a_1}\psi_2^{(a_2a_3)} + 2\bar{\xi}_{a_1}\psi_2^{(a_1a_2}\eta^{a_3)}. \quad (\text{B.18d})$$

Thus, if we procede with similar manipulations for the other components, we find that

$$[a(\bar{\xi}), a^\dagger(\eta)]\Psi = \bar{\xi}_a\eta^a\Psi, \quad (\text{B.19})$$

that proves the desired relations. \square