Proposta de resolução

1.

$$\overrightarrow{AM}\cdot\overrightarrow{AC}=\left(\overrightarrow{AD}+\overrightarrow{DM}\right)\cdot\left(\overrightarrow{AD}+\overrightarrow{DC}\right)=\overrightarrow{AD}\cdot\overrightarrow{AD}+\overrightarrow{AD}\cdot\overrightarrow{DC}+\overrightarrow{DM}\cdot\overrightarrow{AD}+\overrightarrow{DM}\cdot\overrightarrow{DC}=$$

$$=\overline{AD}^2+\underbrace{0}_{\overrightarrow{AD}\perp\overrightarrow{DC}}+\underbrace{0}_{\overrightarrow{DM}\perp\overrightarrow{AC}}+\frac{1}{2}\overrightarrow{DC}\cdot\overrightarrow{DC}=\overline{AD}^2+\overline{DC}^2-\frac{1}{2}\overline{DC}^2=\underbrace{\overline{AC}^2}_{\text{T. Pitágoras}}-\frac{1}{2}\overline{AB}^2$$

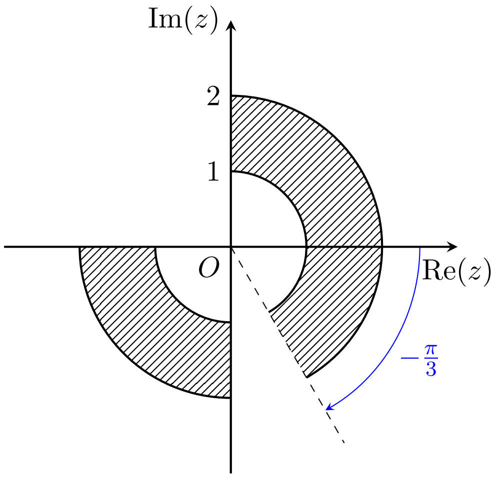
Opção **(D)**

2.

2.1.

$$1\leq|z|\leq 2\ \wedge\ \left[\operatorname{Im}(z)\times\operatorname{Re}(z)\geq 0\ \vee\ \left|\operatorname{Arg}(z)\right|<\frac{\pi}{3}\right]\Leftrightarrow$$

$$1\leq|z|\leq 2\wedge\left[\left(\operatorname{Im}(z)\geq 0\ \wedge\ \operatorname{Re}(z)\geq 0\right)\ \vee\ \left(\operatorname{Im}(z)\leq 0\ \wedge\ \operatorname{Re}(z)\leq 0\right)\ \vee\ -\frac{\pi}{3}<\operatorname{Arg}(z)<\frac{\pi}{3}\right]$$



$$A=\frac{\pi+\frac{\pi}{3}}{2}\times 2^2-\frac{\pi+\frac{\pi}{3}}{2}\times 1^2=\frac{8\pi}{3}-\frac{2\pi}{3}=2\pi$$

2.2.

$$355=4\times 88+3\Rightarrow i^{355}=i^3=-i$$

$$\left|\frac{1}{2}-\frac{\sqrt{3}}{2}i\right|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\operatorname{Arg}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-\arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=-\arctan\sqrt{3}=-\frac{\pi}{3}$$

$$\frac{\left[(w_1)^7-i^{355}\right]\times e^{i\frac{\pi}{3}}}{\left(\overline{w_2}\right)^3}=\frac{\left[\left(e^{i\frac{\pi}{14}}\right)^7+i\right]\times e^{i\frac{\pi}{3}}}{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3}=\frac{\left(e^{i\frac{\pi}{2}+i}\right)\times e^{i\frac{\pi}{3}}}{\left(e^{i\left(-\frac{\pi}{3}\right)}\right)^3}=\frac{2ie^{i\frac{\pi}{3}}}{e^{i(-\pi)}}=\frac{2e^{i\frac{\pi}{2}}\times e^{i\frac{\pi}{3}}}{e^{i(-\pi)}}=$$

$$= 2e^{i\left(\frac{\pi}{2}+\frac{\pi}{3}+\pi\right)} = 2e^{i\left(\frac{11\pi}{6}\right)} = 2e^{i\left(-\frac{\pi}{6}\right)}$$

Como o módulo do complexo é 2 e o seu argumento principal é $-\frac{\pi}{6}$ (note-se que $-\frac{\pi}{3} < -\frac{\pi}{6} < \frac{\pi}{3}$), então a imagem geométrica do complexo pertence à fronteira da região.

3.

$$a = 4b \Leftrightarrow a = 2^{2b} \Leftrightarrow \log_2 a = 2b$$

$$\log_8\left(32a^3\right) - \log_{32}\left(16a^2\right) = \frac{\log_2\left(32a^3\right)}{\log_2 8} - \frac{\log_2\left(16a^2\right)}{\log_2 32} = \frac{1}{3}\left(\log_2 32 + 3\log_2 a\right) - \frac{1}{5}\left(\log_2 16 + 2\log_2 a\right) =$$

$$= \frac{1}{3}\left(5 + 3\times 2b\right) - \frac{1}{5}\left(4 + 2\times 2b\right) = \frac{5}{3} + 2b - \frac{4}{5} - \frac{4}{5}b = \frac{13}{15} + \frac{6}{5}b$$

Opção **(C)**

4.

$$P\left(\overline{A}\cup\overline{B}\right) = \frac{7}{9} \Leftrightarrow P\left(\overline{A\cap B}\right) = \frac{7}{9} \Leftrightarrow 1 - P\left(A\cap B\right) = \frac{7}{9} \Leftrightarrow P\left(A\cap B\right) = \frac{2}{9}$$

$$P\left(A|B\right) = \frac{1}{3} \Leftrightarrow \frac{P\left(A\cap B\right)}{P\left(B\right)} = \frac{1}{3} \Leftrightarrow \frac{\frac{2}{9}}{P\left(B\right)} = \frac{1}{3} \Leftrightarrow P\left(B\right) = \frac{2}{3} \Leftrightarrow P\left(\overline{B}\right) = \frac{1}{3}$$

$$\therefore P\left(\text{“Sair uma bola com número 1”}\right) = P\left(\overline{B}\right) = \frac{1}{3}$$

5. Se a reta de equação $y = 3$ é uma reta tangente ao gráfico de f no ponto P, então $f'\left(a\right) = 0$.

$$\lim_{x\to a}\frac{f'\left(x\right)}{x-a} = 2 \Leftrightarrow \underbrace{\lim_{x\to a}\frac{f'\left(x\right)-f'\left(a\right)}{x-a}}_{\text{definição de derivada}} = 2 \Leftrightarrow f''\left(a\right) = 2$$

Como $f''\left(a\right) > 0$ e $f'\left(a\right) = 0$, então $f\left(a\right)$ é mínimo relativo de f.

Opção **(C)**

6.

$$a_1 = \frac{1\times f\left(1\right)}{2} = \frac{1\times 2^2}{2} = 2$$

$$a_2 = \frac{1\times f\left(2\right)}{2} = \frac{1\times 2}{2} = 1$$

$$a_3 = \frac{1\times f\left(3\right)}{2} = \frac{1\times 2^0}{2} = \frac{1}{2}$$

...

$\left(a_n\right)$ é uma progressão geométrica de razão $\frac{1}{2}$.

Note-se que:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1\times f\left(n+1\right)}{2}}{\frac{1\times f\left(n\right)}{2}} = \frac{2^{3-n-1}}{2^{3-n}} = 2^{3-n-1-3+n} = \frac{1}{2} = r$$

A soma das áreas de todos os triângulos é a soma de todos os termos de uma progressão geométrica (de razão r, com $|r| < 1$), que é dada por:

$$S = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$$

7.

$$^{10}C_p\left(\frac{3}{\sqrt{x}}\right)^{10-p}\times\left(-\frac{x^2}{2}\right)^p={}^{10}C_p\frac{3^{10-p}\times(-1)^p}{2^p}\times\frac{x^{2p}}{x^{5-\frac{p}{2}}}={}^{10}C_p\frac{3^{10-p}\times(-1)^p}{2^p}\times x^{\frac{5}{2}p-5}$$

$$\frac{5}{2}p-5=10\Leftrightarrow p=6$$

$$\therefore {}^{10}C_6\frac{3^4\times(-1)^6}{2^6}=\frac{8505}{32}$$

Opção **(A)**

8.

$$f\left(-\frac{1}{2}\right)=3-\log_2\left(1-\frac{1}{2}\right)=3-\log_2\frac{1}{2}=3+1=4$$

$$P_1\left(-\frac{1}{2},4\right)$$

$$f(0)=3-\log_2(1+0)=3$$

$$P_2(0,3)$$

$$f(1)=3-\log_2(1+1)=3-1=2$$

$$P_3(1,2)$$

$$f(7)=3-\log_2(1+7)=3-3=0$$

$$P_4(7,0)$$

$$f(15)=3-\log_2(1+15)=3-4=-1$$

$$P_5(15,-1)$$

Para o segmento de reta intersetar a reta de equação $y=1$ um dos extremos tem que ter ordenada maior que 1 e o outro extremo tem que ter ordenada menor que 1. Assim,

$$p=\frac{3\times 2}{^5C_2}=0.6$$

Opção **(C)**

9.

9.1.

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\left[3-3(x+1)e^{2x}\right]=\lim_{x\to-\infty}3-3\lim_{x\to-\infty}(x+1)e^{2x}=$$

$$\underbrace{=}_{y=-2x}3-3\lim_{y\to+\infty}\left(-\frac{1}{2}y+1\right)e^{-y}=3-3\lim_{y\to+\infty}\frac{-\frac{1}{2}y+1}{e^y}=3-3\times\frac{\lim\limits_{y\to+\infty}\left(-\frac{1}{2}+\frac{1}{y}\right)}{\underbrace{\lim\limits_{y\to+\infty}\frac{e^y}{y}}_{\text{limite notável}}}=$$

$$=3-3\times\frac{-\frac{1}{2}}{+\infty}=3-3\times 0=3$$

A reta de equação $y=3$ é assíntota horizontal ao gráfico da função f quando $x\to-\infty$.

$$m=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x\ln x}{x}=\lim_{x\to+\infty}\ln x=\ln(+\infty)=+\infty$$

O gráfico de f não admite assíntotas não verticais quando $x\to+\infty$.

9.2.

$$A\left(-\frac{3}{2},f\left(-\frac{3}{2}\right)\right)\Leftrightarrow A\left(-\frac{3}{2},3+\frac{3}{2e^3}\right)\Leftrightarrow A\left(-\frac{3}{2},\frac{6e^3+3}{2e^3}\right)$$

$$B\left(b,f(b)\right)\Leftrightarrow B(b,b\ln b)$$

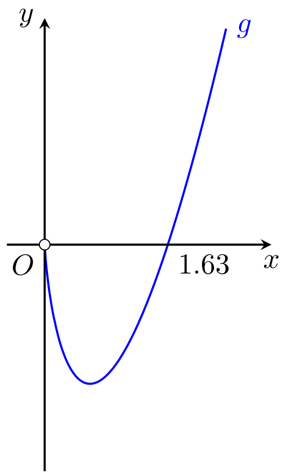
$$\overrightarrow{OA} = A - O = \left(-\frac{3}{2}, \frac{6e^3+3}{2e^3}\right)$$

$$\overrightarrow{OB} = B - O = (b, b\ln b)$$

AOB é obtuso se e só se:

$$\overrightarrow{OA} \cdot \overrightarrow{OB} < 0 \Leftrightarrow \left(-\frac{3}{2}, \frac{6e^3+3}{2e^3}\right) \cdot (b, b\ln b) < 0 \Leftrightarrow -\frac{3}{2}b + \frac{6e^3+3}{2e^3} \times b\ln b < 0$$

Seja g a função definida por $g(x) = -\frac{3}{2}x + \frac{6e^3+3}{2e^3}x\ln x$



$$\therefore AOB \text{ é obtuso se } b \in \left]0, 1.63\right[$$

10.

$$A_{[OPQ]} = \frac{\overline{OP} \times \overline{RQ}}{2} = \frac{1 \times (-\tan\theta)}{2} = -\frac{\tan\theta}{2}$$

Opção **(D)**

11.

$${}^5C_2 \times 3! - 4 \times 3! = 36$$

- ${}^5C_2$ é o número de formas de escolher as duas posições para os livros iguais
- $3!$ é o número de trocas entre os livros diferentes
- 4 é o número de formas de colocar os livros iguais lado a lado

12.

12.1. $MV \perp ABC$, portanto vamos determinar um vetor normal ao plano ABC, que será um vetor diretor da reta MV.

$$\overline{AB} = \sqrt{0^2 + 4^2 + 3^2} = 5 = \overline{AD}$$

$A(0,0,3)$

$$\overrightarrow{AD} = (5,0,0)$$

$B(0,4,0)$

$$\overrightarrow{AB} = B - A = (0,4,-3)$$

Seja $\overrightarrow{n} = (a,b,c)$ um vetor normal ao plano ABC. Logo,

$$\begin{cases} \overrightarrow{n} \cdot \overrightarrow{AD} = 0 \\ \\ \overrightarrow{n} \cdot \overrightarrow{AB} = 0 \end{cases} \Leftrightarrow \begin{cases} 5a = 0 \\ \\ 4b - 3c = 0 \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ \\ b = \dfrac{3}{4}c \end{cases}$$

Fazendo $c = 4$, por exemplo, obtém-se $\overrightarrow{n} = (0,3,4)$

$$\overrightarrow{n} \parallel MV$$

M é o ponto médio de $[AC]$

$$C = (5,4,0)$$

$$M\left(\frac{0+5}{2}, \frac{0+4}{2}, \frac{3+0}{2}\right) \Leftrightarrow M\left(\frac{5}{2}, 2, \frac{3}{2}\right) \in MV$$

$$\therefore MV : (x,y,z) = \left(\frac{5}{2}, 2, \frac{3}{2}\right) + k\,(0,3,4)\,,\ k \in \mathbb{R}$$

12.2.

$$V = 25 \Leftrightarrow 25 = \frac{1}{3}A_b \times h \Leftrightarrow 25 = \frac{1}{3} \times \frac{5^2}{2} \times \overline{MV} \Leftrightarrow \overline{MV} = 6$$

$$V = M + \overrightarrow{MV}$$

Vamos determinar $\overrightarrow{MV}$:

$$\overrightarrow{MV} \parallel \overrightarrow{n} = (0,3,4)$$

$$\|\overrightarrow{n}\| = \sqrt{0^2 + 3^2 + 4^2} = 5$$

Assim,

$$\overrightarrow{MV} = \frac{6}{5}\overrightarrow{n} \vee \overrightarrow{MV} = -\frac{6}{5}\overrightarrow{n} \Leftrightarrow \overrightarrow{MV} = \left(0, \frac{18}{5}, \frac{24}{5}\right) \vee \underbrace{\overrightarrow{MV} = \left(0, -\frac{18}{5}, -\frac{24}{5}\right)}_{\text{tem sentido contrário}} \Rightarrow \overrightarrow{MV} = \left(0, \frac{18}{5}, \frac{24}{5}\right)$$

$$\therefore V = M + \overrightarrow{MV} = \left(\frac{5}{2}, 2, \frac{3}{2}\right) + \left(0, \frac{18}{5}, \frac{24}{5}\right) = \left(\frac{5}{2}, \frac{28}{5}, \frac{63}{10}\right)$$

13.

$$m_r = \frac{1-0}{0+2} = \frac{1}{2}$$

$$r : y = \frac{1}{2}x + 1$$

$$f'(0) = m_r = \frac{1}{2}$$

$$f(0) = 1$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = m_r = \frac{1}{2}$$

$$\lim_{x \to +\infty} \left[x\left(f\left(x^{-1}\right) - 1\right) + \frac{x}{2f(x)}\right] = \lim_{x \to +\infty} \frac{f\left(\frac{1}{x}\right) - 1}{\frac{1}{x}} + \frac{1}{2}\lim_{x \to +\infty} \frac{x}{f(x)} =$$

$$\underbrace{=}_{y=\frac{1}{x}} \underbrace{\lim_{y \to 0^+} \frac{f(y) - f(0)}{y - 0}}_{\text{definição de derivada}} + \frac{1}{2} \times \frac{1}{\underbrace{\lim_{x \to +\infty} \frac{f(x)}{x}}_{\text{declive a.o.}}} = f'(0) + \frac{1}{2} \times \frac{1}{\frac{1}{2}} = \frac{1}{2} + 1 = \frac{3}{2}$$

Opção **(C)**

14.

14.1.

$$g'(x) = \Big[4\big(\sin(2x) + x\big)\Big]' = 4\big(\sin(2x) + x\big)' = 4\big(2\cos(2x) + 1\big)$$

$$g\left(\frac{\pi}{12}\right) = 4\left[\sin\left(2 \times \frac{\pi}{12}\right) + \frac{\pi}{12}\right] = 4\left(1 + \frac{\pi}{12}\right) = 2 + \frac{\pi}{3} = \frac{6+\pi}{3}$$

$$A\left(\frac{\pi}{12}, \frac{6+\pi}{3}\right)$$

$$g'(x) = 0 \Leftrightarrow 4\big(2\cos(2x) + 1\big) = 0 \Leftrightarrow \cos(2x) = -\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow 2x = \frac{2\pi}{3} + 2k\pi \ \vee \ 2x = -\frac{2\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\pi}{3} + k\pi \ \vee \ x = -\frac{\pi}{3} + k\pi,\ k \in \mathbb{Z}$$

As únicas soluções em $]0, \pi[$ são: $x = \dfrac{\pi}{3} \ \vee \ x = \dfrac{2\pi}{3}$

x	0		$\frac{\pi}{3}$		$\frac{2\pi}{3}$		π
$g'(x)$		$+$	0	$-$	0	$+$	
g		↗	Max.	↘	Min.	↗	

$$g\left(\frac{\pi}{3}\right) = 4\left[\sin\left(\frac{2\pi}{3}\right) + \frac{\pi}{3}\right] = 4\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) = \frac{6\sqrt{3} + 4\pi}{3}$$

$$B\left(\frac{\pi}{3}, \frac{6\sqrt{3} + 4\pi}{3}\right)$$

$$D\left(\frac{2\pi}{3}, \frac{6\sqrt{3} + 4\pi}{3}\right)$$

$$\overline{BD} = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

$$h = y_B - y_A = \frac{6\sqrt{3} + 4\pi}{3} - \frac{6+\pi}{3} = 2\sqrt{3} - 2 + \pi = \pi + 2\left(\sqrt{3} - 1\right)$$

$$\therefore A_{[ABD]} = \frac{\overline{BD} \times h}{2} = \frac{\frac{\pi}{3} \times \left(\pi + 2\left(\sqrt{3} - 1\right)\right)}{2} = \frac{\pi}{6}\left[\pi + 2\left(\sqrt{3} - 1\right)\right]$$

14.2.

$$g''(x) = \Big[4\big(2\cos(2x) + 1\big)\Big]' = 4\big(2\cos(2x) + 1\big)' = 4\big(-4\sin(2x) + 0\big) = -16\sin(2x)$$

$$g''(x) = 0 \Leftrightarrow -16\sin(2x) = 0 \Leftrightarrow \sin(2x) = 0 \Leftrightarrow 2x = k\pi,\ k \in \mathbb{Z} \Leftrightarrow x = \frac{k\pi}{2},\ k \in \mathbb{Z}$$

A única solução em $]0, \pi[$ é $x = \dfrac{\pi}{2}$.

x	0		$\frac{\pi}{2}$		π
$g''(x)$		$-$	0	$+$	
g'		$\searrow$	Min.	$\nearrow$	

$$g\left(\frac{\pi}{2}\right) = 4\left[\sin\left(2 \times \frac{\pi}{2}\right) + \frac{\pi}{2}\right] = 2\pi$$

$$P\left(\frac{\pi}{2}, 2\pi\right)$$

$$m_r = g'\left(\frac{\pi}{2}\right) = 4\left(2\cos\left(2 \times \frac{\pi}{2}\right) + 1\right) = -4$$

$$\therefore y - 2\pi = -1\left(x - \frac{\pi}{2}\right) \Leftrightarrow y - 2\pi = -4x + 2\pi \Leftrightarrow 4x + y = 4\pi$$