Proposta de resolução - Prova Modelo 4

1. $P(A) = P(C)$

   $P(A\cap B) + 5P(C) = 2P(B) \Leftrightarrow P(A\cap B) = 2P(B) - 5P(C)$

   $P(A\cup B) = 3P(C) \Leftrightarrow$

   $P(A) + P(B) - P(A\cap B) = 3P(C) \Leftrightarrow$

   $P(C) + P(B) - 2P(B) + 5P(C) = 3P(C) \Leftrightarrow$

   $P(B) = 3P(C)$

   $P(A\cap B) = 2P(B) - 5P(C) = P(C)$

   $P\left(A|B\right) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(C)}{3P(C)} = \dfrac{1}{3}$

   Opção C

2. $(-z)^n + (\overline{z})^n = \left[\rho e^{(\theta+\pi)i}\right]^n + \left[\rho e^{(-\theta)i}\right]^n = \rho^n\left[e^{(n\theta+n\pi)i} + e^{(-n\theta)i}\right] =$

   $\rho^n\left[\cos(n\theta+n\pi) + i\sin(n\theta+n\pi) + \cos(-n\theta) + i\sin(-n\theta)\right] =$

   $\rho^n\left[\cos(n\theta+n\pi) + i\sin(n\theta+n\pi) + \cos(n\theta) - i\sin(n\theta)\right]$

   - Se n é par, então $\cos(n\theta+n\pi) = \cos(n\theta)$ e $\sin(n\theta+n\pi) = \sin(n\theta)$. Assim, $(-z)^n + (\overline{z})^n = \rho^n\left[\cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)\right] =$ $2\rho^n\cos(n\theta)$, que é um número real.
   - Se n é ímpar, então $\cos(n\theta+n\pi) = -\cos(n\theta)$ e $\sin(n\theta+n\pi) = -\sin(n\theta)$. Assim, $(-z)^n + (\overline{z})^n = \rho^n\left[-\cos(n\theta) - i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)\right] =$ $-2i\rho^n\sin(n\theta)$, que é um imaginário puro.

3. Opção A

4.

   4.1.
   $f'(x) = 1 - \ln x + x\left(0 - \dfrac{1}{x}\right) = 1 - \ln x - 1 = -\ln x$

   $f'(x) = 0 \Leftrightarrow -\ln x = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$

x	0		1	$+\infty$
f'	n.d.	$+$	0	$-$
f	n.d.	$\uparrow$	Máx.	$\downarrow$

   f é estritamente crescente em $]0,\ 1]$

   f é estritamente decrescente em $[1,\ +\infty[$

   $f(1)$ é máximo relativo.

   4.2. g é contínua em $x = 0$ se e só se $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^-} g(x) = g(0)$

   $g(0) = 0$

   $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} x(1-\ln x) = \lim\limits_{y\to+\infty} \dfrac{1}{y}\left(1 - \ln\left(\dfrac{1}{y}\right)\right) \lim\limits_{y\to+\infty}\left(\dfrac{1}{y} + \dfrac{\ln y}{y}\right) = \lim\limits_{y\to+\infty}\dfrac{1}{y} + \lim\limits_{y\to+\infty}\dfrac{\ln y}{y} = 0 + 0 = 0$

   $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \dfrac{1-\sin\left(\dfrac{\pi}{2}+x\right)}{e^x - 1} = \dfrac{\lim\limits_{x\to 0^-}\dfrac{1-\cos x}{x}}{\lim\limits_{x\to 0^-}\dfrac{e^x-1}{x}} = \lim\limits_{x\to 0^-}\dfrac{1-\cos^2 x}{x(1+\cos x)} =$

$$\lim_{x\to 0^-} \frac{\sin^2 x}{x\left(1+\cos x\right)} = \lim_{x\to 0^-} \frac{\sin x}{x} \times \frac{\sin x}{1+\cos x} = 1 \times \frac{0}{1+1} = 0$$

Como $\lim\limits_{x\to 0^+} g\left(x\right) = \lim\limits_{x\to 0^-} g\left(x\right) = g\left(0\right)$ então g é contínua em $x=0$.

5. $\overline{AB} = \dfrac{6}{3} = 2$

   $\overline{OA}^2 + \overline{OB}^2 = \overline{AB}^2 \Leftrightarrow 2\overline{OA}^2 = 4 \Rightarrow \overline{OA} = \sqrt{2}$

   $A\left(\sqrt{2},\ 0,\ 0\right)$

   $B\left(0,\ \sqrt{2},\ 0\right)$

   $C\left(0,\ 0,\ \sqrt{2}\right)$

   Seja $\overrightarrow{n} = (a,\ b,\ c) \perp ABC$

   $$\begin{cases} \overrightarrow{n}\cdot\overrightarrow{AB} = 0 \\ \\ \overrightarrow{n}\cdot\overrightarrow{AC} = 0 \end{cases} \Leftrightarrow \begin{cases} (a,\ b,\ c)\cdot\left(-\sqrt{2},\ \sqrt{2}, 0\right) = 0 \\ \\ (a,\ b,\ c)\cdot\left(-\sqrt{2},\ 0,\ \sqrt{2}\right) = 0 \end{cases} \Leftrightarrow \begin{cases} -\sqrt{2}a + \sqrt{2}b = 0 \\ \\ -\sqrt{2}a + \sqrt{2}c = 0 \end{cases} \Leftrightarrow \begin{cases} b = a \\ \\ c = a \end{cases}$$

   $\overrightarrow{n} = \left(a,\ a,\ a\right),\ a \in \mathbb{R}\setminus\{0\}$

   Se $a=1$, tem-se $\overrightarrow{n} = \left(1,\ 1,\ 1\right)$

   Opção C

6. $h'\left(x\right) = 1 + \cos x e^{\sin x}$

   $m_r = 1 \Leftrightarrow h'\left(a\right) = 1 \Leftrightarrow 1 + \cos a e^{\sin a} = 1 \Leftrightarrow \cos a e^{\sin a} = 0 \Leftrightarrow \cos a = 0 \vee \underbrace{e^{\sin a} = 0}_{eq.imp.} \Leftrightarrow$

   $a = \dfrac{\pi}{2} + k\pi,\ k\in\mathbb{Z}$

   A única solução em $[0,\ \pi]$ é $a = \dfrac{\pi}{2}$

   $h\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2} + e^{\sin\frac{\pi}{2}} = \dfrac{\pi}{2} + e$

   $y - \dfrac{\pi}{2} - e = 1\left(x - \dfrac{\pi}{2}\right) \Leftrightarrow y = x + e$

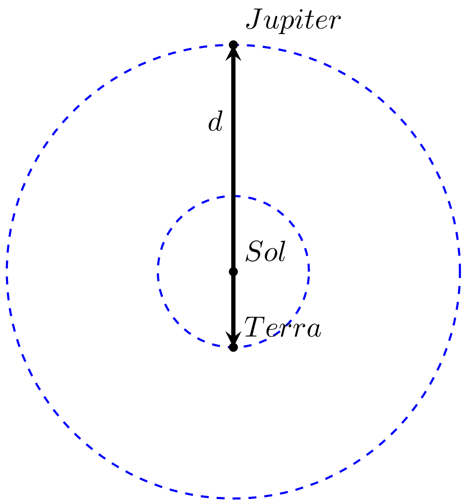
7. $u_3 = \ln\left(u_2\right) + \ln\left(u_1\right) = \ln 3 + \ln 2 = \ln 6$

   $u_4 = \ln\left(u_3\right) + \ln\left(u_2\right) = \ln\left(\ln 6\right) + \ln 3 = \ln\left(3\ln 6\right) = \ln\left(\ln 6^3\right) = \ln\left[\ln\left(216\right)\right]$

   Opção D

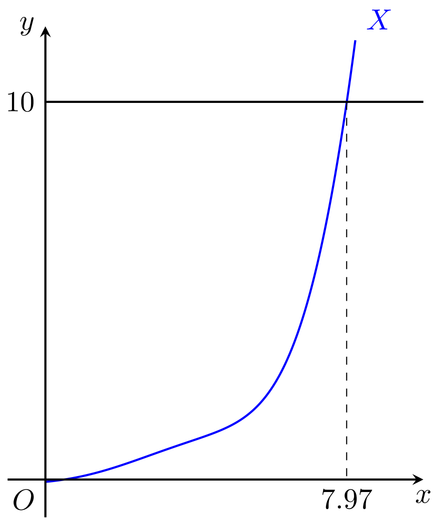
8.

8.1.



$d = \big(X(7) + X(4)\big) \times 150 =$
$150e^{0.422\times 7}\left[0.36\sin(9\times 6.5) + 0.08\sin(49.5\times 6.5)\right]$
$+150e^{0.422\times 4}\left[0.36\sin(9\times 3.5) + 0.08\sin(49.5\times 3.5)\right] \approx 901$ milhões de quilómetros.

8.2. 1.5 milhões de quilómetros $=$ 10 u.a
$X(n) = 10$



O asteróide está mais próximo da órbita de Saturno.

9. $\lim\limits_{x\to+\infty}\left[2x + \frac{1}{e^x} + 3h(x) - \frac{\ln x}{x}\right] = 3 \Leftrightarrow \lim\limits_{x\to+\infty}\big(3h(x) + 2x\big) + \lim\limits_{x\to+\infty}\frac{1}{e^x} - \lim\limits_{x\to+\infty}\frac{\ln x}{x} = 3 \Leftrightarrow$
$3\lim\limits_{x\to+\infty}\left(h(x) + \frac{2}{3}x\right) + \frac{1}{e^{+\infty}} - 0 = 3 \Leftrightarrow \lim\limits_{x\to+\infty}\left[h(x) - \left(-\frac{2}{3}x + 1\right)\right] = 0$

$r\colon\ y = -\frac{2}{3}x + 1$

Opção D

10. $f(x) > g(x) \Leftrightarrow \log_2(x+2) > \log_{\sqrt{2}} x \Leftrightarrow \log_2(x+2) > \dfrac{\log_2 x}{\log_2 \sqrt{2}} \Leftrightarrow \log_2(x+2) > 2\log_2 x \Leftrightarrow$
$\log_2(x+2) > \log_2\left(x^2\right) \Leftrightarrow x+2 > x^2 \Leftrightarrow -x^2+x+2>0$

C.A.

$-x^2+x+2=0 \Leftrightarrow x=-1 \vee x=2$

$D=\mathbb{R}^+$

$S=]0,\ 2[$

Opção A

11.

11.1. ${}^5C_3 \times 2! = 20$

11.2.
$$\frac{2}{9}=\frac{{}^5C_2 +^4 C_2 +^3 C_2 +^n C_2}{{}^{12+n}C_2} \Leftrightarrow \frac{2}{9}=\frac{10+6+3+\frac{n(n+1+)}{2}}{\frac{(12+n)(11+n)}{2}} \Leftrightarrow$$

$$\frac{2}{9}=\frac{38+n^2-n}{n^2+23n+132} \Leftrightarrow 7n^2-55n+78=0 \Rightarrow n=6$$

6 fidget spinners são cinzentos

$$p=\frac{{}^{18}C_2 -^{15} C_2}{{}^{18}C_2}=\frac{16}{51}$$

12.
$$m_t=\frac{1}{2}$$

$$t: y=\frac{1}{2}x-1$$

$$g(4)=1$$

$$\left(f' \times g\right)(4)=2\times 1=2$$

$$\lim_{x\to 4}\frac{\left(f'\times g\right)(x)-2}{x-4}=\left(f'\times g\right)'(4)=f''(4)\times g(4)+f'(4)\times g'(4)=1\times 1+2\times\frac{1}{2}=2$$

Opção B

13.

13.1. $0^2+0^2-6\times 0-2a\times 0=16-a^2 \Leftrightarrow a^2=16 \Rightarrow a=4$
$x^2+y^2-6x-8y=0 \Leftrightarrow x^2-6x+9+y^2-8y+16=25 \Leftrightarrow (x-3)^2+(y-4)^2=25$
$C(3,\ 4)$ e $r=5$

13.2. $A_{\text{Setor circular}}=\dfrac{\alpha}{2}r^2 \Leftrightarrow \dfrac{25\pi}{6}=\dfrac{\alpha}{2}\times 5^2 \Leftrightarrow \alpha=\dfrac{\pi}{3} \Leftrightarrow A\hat{C}B=\dfrac{\pi}{3}$
$\overrightarrow{DA}\cdot\overrightarrow{DB}=\overrightarrow{DA}\cdot\left(\overrightarrow{DC}+\overrightarrow{CB}\right)=\overrightarrow{DA}\cdot\overrightarrow{DC}+\overrightarrow{DA}\cdot\overrightarrow{CB}=10\times 5\times\cos 0+10\times 5\times\cos\dfrac{\pi}{3}=$
$50+50\times\dfrac{1}{2}=75$

13.3. $x^2 + 1^2 - 6x - 8 = 0 \Leftrightarrow x^2 - 6x - 7 = 0 \Leftrightarrow x = \dfrac{6 \pm \sqrt{36 - 4 \times 1 \times (-7)}}{2} \Leftrightarrow$

$x = 7 \vee x = -1 \Rightarrow x = 7$

$A\,(7,\ 1)$

$\overrightarrow{CA} = (7,\ 1) - (3,\ 4) = (4,\ \ -3)$

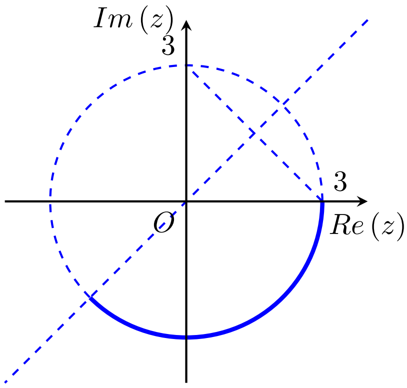
$m_{CA} = -\dfrac{3}{4} \Rightarrow m_t = \dfrac{4}{3}$, pois $t \perp CA$

$\overrightarrow{t}\,(3,\ 4)$ é um vetor diretor da reta t.

t: $(x,\ y) = (7,\ 1) + k\,(3,\ 4)\,,\ \ k \in \mathbb{R}$

14.

$$|z - 3i| \geq |z - 3| \wedge \operatorname{Im}(z) \leq 0 \wedge |z| = 3$$



Opção B