Title

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Problem 3.2.1

If $\mathbf{P} = \hat{\mathbf{e}}_x P_x + \hat{\mathbf{e}}_y P_y$ and $\mathbf{Q} = \hat{\mathbf{e}}_x Q_x + \hat{\mathbf{e}}_y Q_y$ are any two nonparallel (Also nonantiparallectors in the xy -plane, show that $\mathbf{P} \times \mathbf{Q}$ is in the z -direction.

Solution We write the P and Q vectors as

$$\mathbf{P} = \langle P_x, P_y, 0 \rangle \quad Q = \langle Q_x, Q_y, 0 \rangle$$

So

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ P_x & P_y & 0 \\ Q_x & Q_y & 0 \end{vmatrix} = (P_x Q_y - Q_x P_y)\hat{\mathbf{e}}_z$$

Problem 3.2.2

Prove that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{AB})^2 - (\mathbf{A} \cdot \mathbf{B})^2$

Solution

$$(\mathbf{A} \times \mathbf{B})^2 = (|\mathbf{A}||\mathbf{B}| \sin\theta)^2$$
$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \times \mathbf{B}^2 \times \sin^2\theta$$
$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \times \mathbf{B}^2 \times \left(1 - \cos^2\theta\right)$$
$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \times \mathbf{B}^2 - \mathbf{A}^2 \times \mathbf{B}^2 \times \left(\cos^2\theta\right)$$
$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2$$

Problem 3.2.3

Using the vectors

$$\mathbf{P} = \hat{\mathbf{e}}_x \cos\theta + \hat{\mathbf{e}}_y \sin\theta$$
$$\mathbf{Q} = \hat{\mathbf{e}}_x \cos\varphi - \hat{\mathbf{e}}_y \sin\varphi$$
$$\mathbf{R} = \hat{\mathbf{e}}_x \cos\varphi + \hat{\mathbf{e}}_y \sin\varphi$$

prove the familiar trigonometric identities

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$
$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

Solution Consider $\mathbf{P} \cdot \mathbf{Q}$ as

$$\mathbf{P} \cdot \mathbf{Q} = (\hat{x}\cos\theta + \hat{y}\sin\theta) \cdot (\hat{x}\cos\varphi - \hat{y}\sin\varphi) + \hat{y}\sin\theta\hat{x}\cos\varphi - \hat{y}\sin\theta\sin\varphi$$

$$\mathbf{P} \cdot \mathbf{Q} = (1 \times \cos\theta\cos\varphi) - (0 \times \cos\theta\sin\varphi) + (0 \times \sin\theta\cos\varphi) - (1 \times \sin\theta\sin\varphi)$$
$$\mathbf{P} \cdot \mathbf{Q} = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

And by the product rule, $\mathbf{P} \cdot \mathbf{Q} = \cos(\theta + \varphi)$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

Problem 3.2.4

(*a*) Find a vector **A** that is perpendicular to

$$\mathbf{U} = 2\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z$$

$$\mathbf{V} = \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$$

(*b*) What is **A** if, in addition to this requirement, we demand that it have unit magnitude?

$\boxed{\textbf{Solution}}$ For (a) we have $\mathbf{U} = 2\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z, V = \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$

$$\mathbf{U} \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{\mathbf{e}}_x(1-1) - \hat{\mathbf{e}}_y(2+1) + \hat{\mathbf{e}}_z(-2-1)$$

$$\mathbf{U} \times \mathbf{V} = -\hat{\mathbf{e}}_y(3) + \hat{\mathbf{e}}_z(-3) = -3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z$$

$\boxed{\textbf{Solution}}$ For (b) We know **A** is $-3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z$, so the magnitude of **A** is

$$|\mathbf{A}| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

From this

$$\mathbf{A} = \frac{-3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z}{3\sqrt{2}} = \frac{-\hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z}{\sqrt{2}}$$

Problem 3.2.5

If four vectors **a**, **b**, **c**, and **d** all lie in the same plane, show that

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$$

Hint. Consider the directions of the cross-product vectors.

$\boxed{\textbf{Solution}}$ Since all four vectors lie in the same plane, the cross product of any two of them would be orthogonal to the plane. Thus:

$$\mathbf{v}_1 = (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{v}_2 = (\mathbf{c} \times \mathbf{d})$$

By definition, it $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, so $\mathbf{v}_1 \times \mathbf{v}_2 = 0$

Problem 3.2.6

Derive the law of sines (see Fig. 3.4) :

$$\frac{\sin\alpha}{|\mathbf{A}|} = \frac{\sin\beta}{|\mathbf{B}|} = \frac{\sin\gamma}{|\mathbf{C}|}$$

$\boxed{\textbf{Solution}}$ We have $\mathbf{A} - \mathbf{B} - \mathbf{C} = 0$ so we cross both sides by **A**

$$\mathbf{A} \times \mathbf{A} - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = \mathbf{A} \times 0$$

$$0 - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = 0$$

$$-\mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = 0$$

$$-\mathbf{A} \times \mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C} \times \mathbf{A} = \mathbf{A} \times \mathbf{B}$$

$$|\mathbf{C}||\mathbf{A}|\sin\beta = |\mathbf{A}|\mathbf{B}\mid\sin\gamma$$

Again, we cross both sides of $\mathbf{A} - \mathbf{B} - \mathbf{C} = 0$ by **B**

$$\mathbf{B} \times \mathbf{A} - \mathbf{B} \times \mathbf{B} - \mathbf{B} \times \mathbf{C} = \mathbf{B} \times 0$$

$$\mathbf{B} \times \mathbf{A} - \mathbf{B} \times \mathbf{C} = 0$$

$$\mathbf{B} \times \mathbf{A} = \mathbf{B} \times C$$
$$|\mathbf{B}||A| \sin\gamma = |\mathbf{B}||\mathbf{C}| \sin\alpha$$
$$|\mathbf{A}| \sin\gamma = |\mathbf{C}| \sin\alpha$$
$$\frac{\sin\gamma}{|\mathbf{C}|} = \frac{\sin\alpha}{|\mathbf{A}|}$$

Problem 3.2.7

The magnetic induction $\mathbf{B}$ is defined by the Lorentz force equation,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

Carrying out three experiments, we find that if

$$\mathbf{v} = \hat{\mathbf{e}}_x, \quad \frac{\mathbf{F}}{q} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_y$$
$$\mathbf{v} = \hat{\mathbf{e}}_y, \quad \frac{\mathbf{F}}{q} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$$
$$\mathbf{v} = \hat{\mathbf{e}}_z, \quad \frac{\mathbf{F}}{q} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$$

From the results of these three separate experiments calculate the magnetic induction $\mathbf{B}$.

Solution From the first condition $\mathbf{v} = \hat{\mathbf{e}}_x, \dfrac{F}{q} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_{\dot{y}}$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 1 & 0 & 0 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = \hat{\mathbf{e}}_x(0) - \hat{\mathbf{e}}_y(\mathbf{B}_z) + \hat{\mathbf{e}}_z(\mathbf{B}_y) = -\hat{\mathbf{e}}_y(\mathbf{B}_z) + \hat{\mathbf{e}}_z(\mathbf{B}_y)$$

$$\frac{\mathbf{F}}{q} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_j,$$
$$\mathbf{v} \times \mathbf{B} = -\hat{\mathbf{e}}_y(\mathbf{B}_z) + \hat{\mathbf{e}}_z(\mathbf{B}_y)$$
$$\mathbf{B}_z = 4, \mathbf{B}_y = 2$$

Now, from the second condition $\mathbf{v} = \hat{\mathbf{e}}_y, \dfrac{F}{q} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 0 & 1 & 0 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = \hat{\mathbf{e}}_x(\mathbf{B}_z) - \hat{\mathbf{e}}_y(0) - \hat{\mathbf{e}}_z(\mathbf{B}_x) = \hat{\mathbf{e}}_x(\mathbf{B}_z) - \hat{\mathbf{e}}_z(\mathbf{B}_x)$$

$$\frac{\mathbf{F}}{q} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$$
$$\mathbf{v} \times \mathbf{B} = \hat{\mathbf{e}}_x(\mathbf{B}_z) - \hat{\mathbf{e}}_z(\mathbf{B}_x)$$
$$\mathbf{B}_z = 4, \mathbf{B}_x = 1$$

From the third condition

$$\mathbf{v} = \hat{\mathbf{e}}_z$$
$$\frac{\mathbf{F}}{q} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 0 & 0 & 1 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = \hat{\mathbf{e}}_x(-\mathbf{B}_y) - \hat{\mathbf{e}}_y(-\mathbf{B}_x) - \hat{\mathbf{e}}_z(0) = -\hat{\mathbf{e}}_x(\mathbf{B}_y) + \hat{\mathbf{e}}_y(\mathbf{B}_x)$$

$$\mathbf{v} \times \mathbf{B} = -\hat{\mathbf{e}}_x(\mathbf{B}_y) + \hat{\mathbf{e}}_y(\mathbf{B}_x)$$
$$\frac{\mathbf{F}}{q} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$$
$$\mathbf{B}_y = 2, \mathbf{B}_x = 1$$

Problem 3.2.8

You are given the three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$,

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y \\ \mathbf{B} &= \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z \\ \mathbf{C} &= \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z\end{aligned}$$

Therefore, from above three conditions magnetic induction is given by $\mathbf{B} = \hat{x} + 2\hat{y} + 4\hat{z}$

Solution For (a), $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$. Because $\mathbf{A}$ is the plane of $\mathbf{B}$ and $\mathbf{C}$. The parallelepiped has zero height above the BC plane. So therefore volume will be zero. Therefore, the scalar triple product is zero.

Solution For (b)

$$(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \hat{\mathbf{e}}_x(-1) - \hat{\mathbf{e}}_y(-1) + \hat{\mathbf{e}}_z(-1) = -\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{vmatrix} = \hat{\mathbf{e}}_x(-1) - \hat{\mathbf{e}}_y(-1) + \hat{\mathbf{e}}_z(1+1) = -\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y + 2\hat{\mathbf{e}}_z$$

Problem 3.2.9

Prove Jacobi's identity for vector products:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

Solution From $\mathbf{BAC} - \mathbf{CAB}$ rule $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. The entire equation an written as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$

$$= [\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}]$$

since the dot product is commutative so they becomes zero. Therefore,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

Problem 3.2.10

A vector $\mathbf{A}$ is decomposed into a radial vector $\mathbf{A}_r$ and a tangential vector $\mathbf{A}_t$. If $\hat{\mathbf{r}}$ is a unit vector in the radial direction, show that (a) $\mathbf{A}_r = \hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}})$ and (b) $\mathbf{A}_t = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{A})$

Solution Let $\mathbf{A} = \mathbf{A}_r\hat{\mathbf{r}} + \mathbf{A}_i\hat{\theta}$

$$\mathbf{A} \cdot \hat{\mathbf{r}} = \mathbf{A}_r, \text{ as } \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$$

The left-hand side is:

$$\mathbf{A}_r = \mathbf{A}_r\hat{\mathbf{r}}$$

since $\hat{\mathbf{r}}$ is the unit vector. The right-hand side is:

$$\hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}}) = \hat{\mathbf{r}}\,(\mathbf{A}_r) = \mathbf{A}_r\hat{\mathbf{r}}$$

For (b), taking dot product of both sides of the equation $\mathbf{A}_t = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{A})$ by $\hat{\mathbf{r}}$ we get

$$\mathbf{A}_t \cdot \hat{\mathbf{r}} = [-\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{A})] \cdot \hat{\mathbf{r}}$$

The left-hand side is:

$$\mathbf{A}_t \cdot \hat{\mathbf{r}} = \mathbf{A}_t\hat{\theta} \cdot \hat{\mathbf{r}} = 0$$

$$\begin{aligned}\cdot\hat{\mathbf{r}} &= [\hat{\mathbf{r}} \times (\mathbf{A} \times \hat{\mathbf{r}})] \cdot \hat{\mathbf{r}} \\ &= [\mathbf{A}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}})] \cdot \hat{\mathbf{r}} \\ &= [\mathbf{A} - \hat{\mathbf{r}}\mathbf{A}_r] \cdot \hat{\mathbf{r}} \\ &= \mathbf{A} \cdot \hat{\mathbf{r}} - \mathbf{A}_r\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \\ &= \mathbf{A}_r - \mathbf{A}_r \\ &= 0\end{aligned}$$

Problem 3.2.11

Prove that a necessary and sufficient condition for the three (nonvanishing) vectors **A**, **B**, and **C** to be coplanar is the vanishing of the scalar triple product

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=0$$

$\boxed{\textbf{Solution}}$ It should be keep in mind that scalar triple product can also be represent as the volume of parallelepiped which is formed by three vectors. So we can say that if scalar triple product is equal to zero then vectors are coplanar as the parallelepipeds have no volume.

Problem 3.2.12

Three vectors $\mathbf{A},\mathbf{B}$, and $\mathbf{C}$ are given by

$$\begin{aligned}\mathbf{A}&=3\hat{\mathbf{e}}_x-2\hat{\mathbf{e}}_y+2\hat{\mathbf{z}}\\ \mathbf{B}&=6\hat{\mathbf{e}}_x+4\hat{\mathbf{e}}_y-2\hat{\mathbf{z}}\\ \mathbf{C}&=-3\hat{\mathbf{e}}_x-2\hat{\mathbf{e}}_y-4\hat{\mathbf{z}}\end{aligned}$$

Compute the values of $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}),\mathbf{C}\times(\mathbf{A}\times\mathbf{B})$ and $\mathbf{B}\times(\mathbf{C}\times\mathbf{A})$

$\boxed{\textbf{Solution}}$ First we can find $B\times C$ and then can permorm dot product

$$\mathbf{B}\times\mathbf{C}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\6&4&-2\\-3&-2&-4\end{vmatrix}=\hat{x}(-16-4)-\hat{y}(-24-6)+\hat{z}(-12+12)=-20\hat{x}+30\hat{y}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(3\hat{x}-2\hat{y}+2\hat{z})\cdot(-20\hat{x}+30\hat{y})=-60-60=-120$$

With this, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=-120$

$\boxed{\textbf{Solution}}$ For (b) we have that the vector **A**

$$\mathbf{A}=(3\hat{x}-2\hat{y}+2\hat{z})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\3&-2&2\\-20&30&0\end{vmatrix}=\hat{x}(-60)-\hat{y}(40)+\hat{z}(50)$$

With this,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(-60)\hat{x}-(40)\hat{y}+(50)\hat{z}$$

$\boxed{\textbf{Solution}}$ For (c)

$$(\mathbf{A}\times\mathbf{B})=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\3&-2&2\\6&4&-2\end{vmatrix}=\hat{x}(-4)-\hat{y}(-18)+\hat{z}(24)$$

$$\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\-3&-2&-4\\-4&18&24\end{vmatrix}=\hat{x}(26)-\hat{y}(-88)+\hat{z}(-62)$$

$\boxed{\textbf{Solution}}$ For (d)

$$(\mathbf{C}\times\mathbf{A})=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\-3&-2&-4\\3&-2&2\end{vmatrix}=\hat{x}(-12)-\hat{y}(6)+\hat{z}(12)$$

$$\mathbf{B}\times(\mathbf{C}\times\mathbf{A})=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\6&4&-2\\-12&-6&12\end{vmatrix}=\hat{x}(36)-\hat{y}(48)+\hat{z}(12)$$

Problem 3.2.13

Show that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

Solution Let $\mathbf{C} \times \mathbf{D} = m$. Now, consider the scalar triple product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{m}$. since cross and dot product can be interchanged, we have,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{m} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{m})$$

Resubstituting m we get

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \\ &= \mathbf{A} \cdot [(\mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}] \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})\end{aligned}$$

Thus, $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$

Problem 3.2.14

Show that $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{D})\mathbf{C} - (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})\mathbf{D}$

Solution Let $\mathbf{A} \times \mathbf{B} = \mathbf{m}$

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= \mathbf{m} \times (\mathbf{C} \times \mathbf{D}) \\ &= (\mathbf{m} \cdot \mathbf{D})\mathbf{C} - (\mathbf{m} \cdot \mathbf{C})\mathbf{D} \\ &= ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D})\mathbf{C} - ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C})\mathbf{D} \\ &= (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D}))\mathbf{C} - (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))\mathbf{D}\end{aligned}$$

Problem 3.2.15

An electric charge q_1 moving with velocity $\mathbf{v}_1$ produces a magnetic induction $\mathbf{B}$ given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} q_1 \frac{\mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2} \qquad \text{(mks units)},$$

where $\hat{\mathbf{r}}$ is a unit vector that points from q_1 to the point at which $\mathbf{B}$ is measured (Biot and Savart law).

(*a*) Show that the magnetic force exerted by q_1 on a second charge q_2, velocity $\mathbf{v}_2$, is given by the vector triple product

$$\mathbf{F}_2 = \frac{\mu_0}{4\pi} \frac{q_1 q_2}{r^2} \mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})$$

(*b*) Write out the corresponding magnetic force $\mathbf{F}_1$ that q_2 exerts on q_1. Define your unit radial vector. How do $\mathbf{F}_1$ and $\mathbf{F}_2$ compare?

(*c*) Calculate $\mathbf{F}_1$ and $\mathbf{F}_2$ for the case of q_1 and q_2 moving along parallel trajectories side by side.

Solution For (*a*) The magnetic force $\mathbf{F}_2$ is defined by the Lorentz force equation,

$$\begin{aligned}\mathbf{F}_2 &= q_2 (\mathbf{v}_2 \times \mathbf{B}_1) \\ &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})}{r^2}\end{aligned}$$

and

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} q_1 \frac{\mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2}$$

For (*b*) The magnetic force $\mathbf{F}_1$ is defined by the Lorentz force equation,

$$\begin{aligned}\mathbf{F}_1 &= q_1 (\mathbf{v}_1 \times \mathbf{B}_2) \\ &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{r}})}{r^2}\end{aligned}$$

and

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi} q_2 \frac{\mathbf{v}_2 \times (-\hat{\mathbf{r}})}{r^2}$$

From part we have,

$$\mathbf{F}_2 = \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})}{r^2}$$

since $-\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{r}}) \neq \mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})\,, \quad \mathbf{F}_1 \neq \mathbf{F}_2$ For (c) we have that

$$\begin{aligned}
\mathbf{F}_1 &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v} \times (\mathbf{v} \times \hat{\mathbf{r}})}{r^2} \\
&= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}(\mathbf{v} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\
&= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{0 - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\
&= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}^2 \hat{\mathbf{r}}}{r^2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}_2 &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v} \times (\mathbf{v} \times \hat{\mathbf{r}})}{r^2} \\
&= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}(\mathbf{v} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\
&= \frac{\mu_0}{4\pi} q_1 q_2 \frac{0 - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\
&= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}^2 \hat{\mathbf{r}}}{r^2}
\end{aligned}$$

Thus, $\mathbf{F}_1 = -\mathbf{F}_2$