Problem 14.3.1

Prove that the Neumann functions Y_n (with n an integer) satisfy the recurrence relations

$$\begin{aligned} Y_{n-1}(x) + Y_{n+1}(x) &= \tfrac{2n}{x} Y_n(x) \\ Y_{n-1}(x) - Y_{n+1}(x) &= 2Y_n'(x) \end{aligned}$$

Hint. These relations may be proved by differentiating the recurrence relations for J_v or by using the limit form of Y_v but not dividing everything by zero.

$\boxed{\textbf{Solution}}$ As

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

with n as an integer, and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} + \lim_{v \to n} \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi}$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} - \lim_{v \to n} \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi}$$

As $Y_n(x) = \lim_{v \to n} Y_v(x)$ exists and is not identically zero, we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} + \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi} \right)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} - \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi} \right)$$

As $\cos(v-1)\pi = \cos(\pi - v\pi) = -\cos v\pi$, $\cos(v+1)\pi = -\cos v\pi$ $\sin(v-1)\pi = -\sin(\pi - v\pi) = \sin v\pi$ and $\sin(v+1)\pi = -\sin v\pi$ we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{-\cos v\pi J_{v-1}(x) - J_{-v+1}(x)}{-\sin v\pi} + \frac{-\cos v\pi J_{v+1}(x) - J_{-v-1}(x)}{-\sin v\pi} \right)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \left(\frac{-\cos v\pi J_{v-1}(x) - J_{-v+1}(x)}{-\sin v\pi} - \frac{-\cos v\pi J_{v+1}(x) - J_{-v-1}(x)}{-\sin v\pi} \right)$$

Thus,

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} + \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi} \right)$$

and

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to \pi} \left(\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} - \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi} \right)$$

Also,

$$\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} + \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

can be written as

$$\frac{\cos v\pi J_{v-1}(x) + \cos v\pi J_{v+1}(x) + J_{-v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

and hence we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos v\pi \left(J_{v-1}(x) + J_{v+1}(x)\right) + \left(J_{-v+1}(x) + J_{-v-1}(x)\right)}{\sin v\pi} \right)$$

Similarly,

$$\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} - \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

can be written as
$$\frac{\cos v\pi J_{v-1}(x) - \cos v\pi J_{v+1}(x) + J_{-v+1}(x) - J_{-v-1}(x)}{\sin v\pi}$$
and hence we get that
$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(J_{v-1}(x) - J_{v+1}(x)\right) - \left(J_{-v-1}(x) - J_{-v+1}(x)\right)}{\sin v\pi}\right)$$
We now have to proove that
$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$$
$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$
$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(x^v\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$
which implies that
$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v}}{s!\ \Gamma(v+s+1)2^{2s+v}}\right)$$
$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v}}{s!\ \Gamma(v+s+1)2^{2s+v}}\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(2s+2v)(x)^{2s+2v-1}}{s!\ \Gamma(v+s+1)2^{2s+v}} = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}$$
As
$$J_{v-1}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}$$
and
$$\frac{d}{dx}\left(x^v J_v(x)\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}$$
we get that
$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}\right) = x^v J_{v-1}(x)$$
Similarly,
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(x^{-v}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$
which implies that
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s}}{s!\ \Gamma(v+s+1)2^{2s+v}}\right)$$
Also
$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{s!\ \Gamma(v+s+1)2^{2s+\gamma}}\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(2s)(x)^{2s-1}}{s!\ \Gamma(v+s+1)2^{2s+v}} = \sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}$$
$$\sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\ \Gamma(v+s)2^{2s+v-1}} = -\sum_{s=1}^{\infty}\frac{(-1)^{s-1}x^{2(s-1)+1}}{(s-1)!\ \Gamma(v+s-1+1)2^{2(s-1)+*+1}} = -\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!\ \Gamma(v+k+1)2^{2k+w+1}}$$
As
$$J_{v+1}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma+1}}{s!\ \Gamma(v+s+1)2^{2s+v+1}}$$
and
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+1}}{s!\ \Gamma(v+s)2^{2s+v+1}}$$
we get that
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -x^{-v}\left(\sum_{x=0}^{\infty}\frac{(-1)^s(x)^{2s+v+1}}{s!\ \Gamma(v+s)2^{2s+v+1}}\right) = -x^{-v}J_{v+1}(x)$$
Then
$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v\frac{d}{dx}\left(J_v(x)\right) + \frac{d}{dx}\left(x^v\right)J_v(x) = x^v J_v'(x) + vx^{v-1}J_v(x)$$

as

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v J_v'(x) + v x^{v-1} J_v(x)$$

and

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v J_{v-1}(x)$$

we get that

$$x^v J_v'(x) + v x^{v-1} J_v(x) = x^v J_{v-1}(x)$$

and

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

Also

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = x^{-v}\frac{d}{dx}\left(J_v(x)\right) + \frac{d}{dx}\left(x^{-v}\right) J_v(x) = x^{-v} J_v'(x) - v x^{-v-1} J_v(x)$$

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = x^{-v} J_v'(x) - v x^{-\gamma-1} J_v(x)$$

and

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -x^{-v} J_{v+1}(x)$$

we get that

$$x^{-v} J_v'(x) - v x^{-\gamma-1} J_v(x) = -x^{-v} J_{v+1}(x)$$

and

$$J_v'(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x)$$

As

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

and

$$J_v'(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x),$$

we get that

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

and

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

As

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(J_{v-1}(x) + J_{v+1}(x)\right) + \left(J_{-v+1}(x) + J_{-v-1}(x)\right)}{\sin v\pi}\right)$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{x\to n}\left(\frac{\cos v\pi\left(\frac{2v}{x} J_v(x)\right) + \left(\frac{2(-v)}{x} J_{-v}(x)\right)}{\sin v\pi}\right)$$

As

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(J_{v-1}(x) - J_{v+1}(x)\right) - \left(J_{-v-1}(x) - J_{-v+1}(x)\right)}{\sin v\pi}\right)$$

we get that

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(2J_v'(x)\right) - \left(2J_{-v}'(x)\right)}{\sin v\pi}\right)$$

Also

$$\frac{\cos v\pi\left(\frac{2v}{x} J_v(x)\right) + \left(\frac{2(-v)}{x} J_{-v}(x)\right)}{\sin v\pi} = \frac{2v}{x}\left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}\right)$$

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v\to n}\frac{2v}{x}\left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}\right)$$

and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v\to n} \frac{2v}{x} Y_v(x) = \frac{2n}{x} Y_n(x)$$

Also,

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\to n} \left(\frac{\cos v\pi \left(2J'_v(x)\right) - \left(2J'_{-v}(x)\right)}{\sin v\pi} \right) = 2\lim_{v\to n} \left(\frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} \right)$$

Now,

$$\frac{d}{dx}\left(Y_v(x)\right) = \frac{d}{dx}\left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} \right) = \frac{\cos v\pi \frac{d}{dx}\left(J_v(x)\right) - \frac{d}{dx}\left(J_{-v}(x)\right)}{\sin v\pi}$$

Hence, we get

$$Y'_v(x) = \frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi}$$

As

$$Y_{n-1}(x) - Y_{n+1}(x) = 2\lim_{v\to n} \left(\frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} \right)$$

and

$$Y'_n(x) = \frac{\cos v\pi J'_n(x) - J'_{-n}(x)}{\sin v\pi}$$

we get

$$Y_{n-1}(x) - Y_{n+1}(x) = 2\lim_{v\to n} Y'_v(x) = 2Y'_n(x)$$

Therefore, the recurrence relations are

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y'_n(x)$$

are true when n is an integer

Problem 14.3.2

Show that for integer n
$$Y_{-n}(x) = (-1)^n Y_n(x)$$

Solution We know that for an integer n,
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
Clearly, the statement is true for $n = 0$. Clearly, for any integer n,
$$Y_{-n}(x) = (-1)^n Y_n(x)$$
can be rewritten as
$$(-1)^{-n} Y_{-n}(x) = Y_n(x)$$
which implies that
$$Y_{-(-n)}(x) = (-1)^{-n} Y_{-n}(x)$$
This implies that if the statement is true for any positive integer n, then it is true for any integer n. Assume that the statement
$$Y_{-n}(x) = (-1)^n Y_n(x)$$
is true for any non-negative integer $n \leq k$ where k is any arbitrary non-negative integer. Now we have to prove that $Y_{-k-1}(x) = (-1)^{k+1} Y_{k+1}(x)$ i.e. the statement is true for $n = k+1$. Also, by substituting $n = 0$ in
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
and hence
$$Y_{-1}(x) = -Y_1(x)$$
As $Y_{-1}(x) = -Y_1(x)$, we get that the statement is true for $n = 1$. As the statement is true for $n = 0$ and $n = 1$, we can assume that $k \geq 1$ As the statement $Y_{-n}(x) = (-1)^n Y_n(x)$ is true for any non-negative integer $n \leq k$, we get that
$$Y_{-k}(x) = (-1)^k Y_k(x)$$
and
$$Y_{-k+1}(x) = (-1)^{k-1} Y_{k-1}(x)$$
As
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
for any integer n, we get that
$$Y_{-k-1}(x) + Y_{-k+1}(x) = \frac{2(-k)}{x} Y_{-k}(x)$$
and hence
$$Y_{-k-1}(x) = -\frac{2k}{x} Y_{-k}(x) - Y_{-k+1}(x)$$
As
$$Y_{-k-1}(x) = -\frac{2k}{x} Y_{-k}(x) - Y_{-k+1}(x), \quad Y_{-k}(x) = (-1)^k Y_k(x)$$
and
$$Y_{-k+1}(x) = (-1)^{k-1} Y_{k-1}(x)$$
we get that
$$Y_{-k-1}(x) = -\frac{2k}{x} \left((-1)^k Y_k(x)\right) - (-1)^{k-1} Y_{k-1}(x)$$
Also
$$Y_{-k-1}(x) = -\frac{2k}{x} \left((-1)^k Y_k(x)\right) - (-1)^{k-1} Y_{k-1}(x) = (-1)^{k+1} \left(\frac{2k}{x} Y_k(x) - Y_{k-1}(x)\right)$$
As
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

for any integer n, we get that

$$Y_{k-1}(x) + Y_{k+1}(x) = \frac{2k}{x} Y_k(x)$$

and hence

$$Y_{k+1}(x) = \frac{2k}{x} Y_k(x) - Y_{k-1}(x)$$

As

$$Y_{k+1}(x) = \frac{2k}{x} Y_k(x) - Y_{k-1}(x)$$

and

$$Y_{-k-1}(x) = (-1)^{k+1} \left(\frac{2k}{x} Y_k(x) - Y_{k-1}(x) \right)$$

we get that

$$Y_{-k-1}(x) = (-1)^{k+1} Y_{k+1}(x)$$

Therefore, by Mathematical induction, we get that the statement

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

for any non−negative integer n.

Problem 14.4.3

Show that
$$Y_0'(x) = -Y_1(x)$$

$\boxed{\textbf{Solution}}$ As
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
for any integer n, we get that the statement is true for $n = 0$ which implies that
$$Y_{-1}(x) + Y_1(x) = \frac{2(0)}{x} Y_0(x) = 0$$
and hence $Y_{-1}(x) = -Y_1(x)$. As
$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x)$$
for any integer n, we get that the statement is true for $n = 0$ which implies that
$$Y_{-1}(x) - Y_1(x) = 2Y_0'(x)$$
As
$$Y_{-1}(x) = -Y_1(x)$$
and
$$Y_{-1}(x) - Y_1(x) = 2Y_0'(x)$$
we get that
$$2Y_0'(x) = Y_{-1}(x) - Y_1(x) = -Y_1(x) - Y_1(x) = -2Y_1(x)$$
and hence
$$Y_0'(x) = -Y_1(x)$$
Therefore, the statement
$$Y_0'(x) = -Y_1(x)$$
is true.

Problem 14.3.4

If X and Z are any two solutions of Bessel's equation, show that

$$X_\nu(x)Z_v'(x) - X_v'(x)Z_v(x) = \frac{A_v}{x}$$

in which A_v may depend on v but is independent of x. This is a special case of Exercise 7.6.11

Solution We know that for a linear second order homogeneous ODE of form

$$y'' + P(x)y' + Q(x)y = 0$$

and two solutions y_1, y_2 of this ODE, we have that the Wronskian W of y_1 and y_2 satisfies the equation

$$W(x) = W(a)\exp\left[-\int_a^x P(t)dt\right]$$

Thus, the Wronskian W of $X_v(x)$ and $Z_v(x)$ is satisfies the equation

$$W(x) = W(a)\exp\left[-\int_a^x P(t)dt\right]$$

As any Bessel's equation is of the form

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0,$$

we get that $P(x) = \dfrac{1}{x}$ and hence

$$\int_a^x P(t)dt = \int_a^x \frac{1}{t}dt = \ln x - \ln a = \ln\frac{x}{a}$$

Thus,

$$W(x) = W(a)\exp\left[-\int_a^x P(t)dt\right] = W(a)\exp\left[-\ln\frac{x}{a}\right] = W(a)\exp\left[\ln\frac{a}{x}\right] = W(a)\frac{a}{x}$$

Clearly,

$$W(a)a = \left(X_v(a)Z_v'(a) - X_v'(a)Z_v(a)\right)a$$

which implies that $W(a)a$ is a constant independent of x but it may depend on v. Thus, by taking

$$W(a)a = A_v,$$

we get that

$$W(x) = \frac{A_v}{x}$$

where A_v may depend on v but is independent of x. As the Wronskian W of $X_v(x)$ and $Z_v(x)$ is equal to

$$X_v(x)Z_v'(x) - X_v'(x)Z_v(x)$$

and $W(x) = \dfrac{A_v}{x}$, we get that

$$X_v(x)Z_v'(x) - X_v'(x)Z_v(x) = \frac{A_v}{x}$$

where A_v may depend on v but is independent of x. Therefore, the statement

$$X_v(x)Z_v'(x) - X_v'(x)Z_v(x) = \frac{A_v}{x}$$

in which A_v, may depend on v but is independent of x is true when X and Z are any two solutions of Bessel's equation.

Problem 14.3.5

Verify the Wronskian formulas

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = \frac{2\sin v\pi}{\pi x}$$
$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = \frac{2}{\pi x}$$

Solution As

$$Y_v'(x) = \frac{\cos v\pi J_v'(x) - J_{-v}'(x)}{\sin v\pi}$$

and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin vx}$$

we get that

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = J_v(x)\frac{\cos v\pi J_v'(x) - J_{-v}'(x)}{\sin v\pi} - J_v'(x)\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin vx}$$

Also

$$J_v(x)\frac{\cos v\pi J_v'(x) - J_{-v}'(x)}{\sin v\pi} - J_v'(x)\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin vx}$$

is equal to

$$\frac{-J_v(x)J_{-v}'(x) + J_v'(x)J_{-v}(x)}{\sin vx}$$

which implies that

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = \frac{-J_v(x)J_{-v}'(x) + J_v'(x)J_{-v}(x)}{\sin vx}$$

Clearly,

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(x^v \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$

which implies that

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v}}{s!\ \Gamma(v+s+1)2^{2s+v}}\right)$$

Also

$$\frac{d}{dx}\left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v}}{s!\ \Gamma(v+s+1)2^{2s+v}}\right) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s+2v)(x)^{2s+2v-1}}{s!\ \Gamma(v+s+1)2^{2s+v}} = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}$$

As

$$J_{v-1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}$$

and

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v-1}}{s!\ \Gamma(v+s)2^{2s+v-1}},$$

we get that

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v\left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma-1}}{s!\ \Gamma(v+s)2^{2s+v-1}}\right) = x^v J_{v-1}(x)$$

Similarly,

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(x^{-v} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$

which implies that

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s}}{s!\ \Gamma(v+s+1)2^{2s+v}}\right)$$

Also

$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{s!\;\Gamma(v+s+1)2^{2s+v}}\right)=\sum_{s=0}^{\infty}\frac{(-1)^s(2s)(x)^{2s-1}}{s!\;\Gamma(v+s+1)2^{2s+v}}=\sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\;\Gamma(v+s)2^{2s+v-1}}$$

Also

$$\sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\;\Gamma(v+s)2^{2s+v-1}}=-\sum_{s=1}^{\infty}\frac{(-1)^{s-1}x^{2(s-1)+1}}{(s-1)!\;\Gamma(v+s-1+1)2^{2(s-1)+v+1}}=-\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!\;\Gamma(v+k+1)2^{2k+v+1}}$$

As

$$J_{v+1}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+v+1}}{s!\;\Gamma(v+s+1)2^{2s+v+1}}$$

and

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=-\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+1}}{s!\;\Gamma(v+s)2^{2s+v+1}},$$

we get that

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=-x^{-v}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+v+1}}{s!\;\Gamma(v+s)2^{2s+v+1}}\right)=-x^{-v}J_{v+1}(x)$$

Also

$$\frac{d}{dx}\left(x^vJ_v(x)\right)=x^{\nu}\frac{d}{dx}\left(J_v(x)\right)+\frac{d}{dx}\left(x^v\right)J_v(x)=x^vJ_v'(x)+vx^{v-1}J_v(x)$$

As

$$\frac{d}{dx}\left(x^vJ_v(x)\right)=x^vJ_v'(x)+vx^{v-1}J_v(x)$$

and

$$\frac{d}{dx}\left(x^vJ_v(x)\right)=x^vJ_{v-1}(x),$$

we get that

$$x^vJ_v'(x)+vx^{v-1}J_v(x)=x^vJ_{v-1}(x)$$

and hence

$$J_v'(x)+\frac{v}{x}J_v(x)=J_{v-1}(x)$$

Also

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=x^{-v}\frac{d}{dx}\left(J_v(x)\right)+\frac{d}{dx}\left(x^{-v}\right)J_v(x)=x^{-v}J_v'(x)-vx^{-v-1}J_v(x)$$

As

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=x^{-v}J_v'(x)-vx^{-v-1}J_v(x)$$

and

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=-x^{-v}J_{v+1}(x),$$

we get that

$$x^{-v}J_v'(x)-vx^{-y-1}J_v(x)=-x^{-v}J_{v+1}(x)$$

and hence

$$J_v'(x)-\frac{v}{x}J_v(x)=-J_{v+1}(x)$$

As

$$J_v'(x)-\frac{v}{x}J_v(x)=-J_{v+1}(x),$$

we get that

$$J_{-v}'(x)-\frac{-v}{x}J_{-v}(x)=-J_{-v+1}(x)$$

by substituting $-v$ in place of v, which implies that

$$J_{-v+1}(x)=-J_{-v}'(x)-\frac{v}{x}J_{-v}(x)$$

As

$$J_{-v+1}(x)=-J_{-v}'(x)-\frac{v}{x}J_{-v}(x)$$

and

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x),$$

we get that

$$J_v(x) J_{-v+1}(x) + J_{-v}(x) J_{v-1}(x)$$

can be written as

$$J_v(x) \left(-J_{-v}'(x) - \frac{v}{x} J_{-v}(x) \right) + J_{-v}(x) \left(J_v'(x) + \frac{v}{x} J_v(x) \right)$$

Also

$$J_v(x) \left(-J_{-v}'(x) - \frac{v}{x} J_{-v}(x) \right) + J_{-v}(x) \left(J_v'(x) + \frac{v}{x} J_v(x) \right)$$

is equal to

$$-J_v(x) J_{-v}'(x) - \frac{v}{x} J_v(x) J_{-v}(x) + J_{-v}(x) J_v'(x) + \frac{v}{x} J_{-v}(x) J_v(x)$$

which is nothing but

$$J_{-v}(x) J_v'(x) - J_v(x) J_{-v}'(x)$$

Thus,

$$J_v(x) J_{-v+1}(x) + J_{-v}(x) J_{v-1}(x) = J_{-v}(x) J_v'(x) - J_v(x) J_{-v}'(x)$$

Therefore, we got that

$$J_v(x) J_{-v+1}(x) + J_{-v}(x) J_{v-1}(x) = J_{-v}(x) J_v'(x) - J_v(x) J_{-v}'(x)$$

and

$$J_v(x) Y_v'(x) - J_v'(x) Y_v(x) = \frac{-J_v(x) J_{-v}'(x) + J_v'(x) J_{-v}(x)}{\sin vx}$$

As $J_v(x)$ and $J_{-v}(x)$ are solutions to the same Bessel's equation, we get that

$$J_{-v}(x) J_v'(x) - J_v(x) J_{-v}'(x) = \frac{A_v}{x}$$

where A, may depend on v but is independent of x. As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{2s+v},$$

we get that

$$J_v'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s+v)}{s!\ \Gamma(v+s+1) 2} \left(\frac{x}{2} \right)^{2s+v-1}$$

Similarly,

$$J_{-v}(x) = \sum_{x=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(-v+s+1)} \left(\frac{x}{2} \right)^{2s-v}$$

and hence

$$J_{-v}'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s-v)}{s!\ \Gamma(-v+s+1) 2} \left(\frac{x}{2} \right)^{2s-v-1}$$

As

$$\frac{1}{\Gamma(v+1)} \left(\frac{x}{2} \right)^{v}$$

is the leading power of

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{2s+v}$$

and

$$\frac{(-v)}{\Gamma(-v+1) 2} \left(\frac{x}{2} \right)^{-v-1}$$

is the leading power of

$$J_{-v}'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s-v)}{s!\ \Gamma(-v+s+1) 2} \left(\frac{x}{2} \right)^{2s-\gamma-1},$$

we get that

$$\frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^{v}\frac{(-v)}{\Gamma(-v+1)2}\left(\frac{x}{2}\right)^{-v-1}$$

is the leading power of $J_v(x)J'_{-v}(x)$ Similarly,

$$\frac{1}{\Gamma(-v+1)}\left(\frac{x}{2}\right)^{-v}$$

is the leading power of

$$J_{-v}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\;\Gamma(-v+s+1)}\left(\frac{x}{2}\right)^{2s-v}$$

and

$$\frac{v}{\Gamma(v+1)2}\left(\frac{x}{2}\right)^{v-1}$$

is the leading power of

$$J'_v(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(2s+v)}{s!\;\Gamma(v+s+1)2}\left(\frac{x}{2}\right)^{2s+\gamma-1},$$

we get that

$$\frac{1}{\Gamma(-v+1)}\left(\frac{x}{2}\right)^{-v}\frac{v}{\Gamma(v+1)2}\left(\frac{x}{2}\right)^{v-1}$$

is the leading power of $J_{-v}(x)J'_v(x)$. Also

$$\frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^{v}\frac{(-v)}{\Gamma(-v+1)2}\left(\frac{x}{2}\right)^{-\gamma-1}=\frac{-v}{\Gamma(v+1)\Gamma(-v+1)x}$$

and

$$\frac{1}{\Gamma(-v+1)}\left(\frac{x}{2}\right)^{-v}\frac{v}{\Gamma(v+1)2}\left(\frac{x}{2}\right)^{v-1}=\frac{v}{\Gamma(v+1)\Gamma(-v+1)x}$$

Thus, we get that

$$\frac{-v}{\Gamma(v+1)\Gamma(-v+1)x}$$

is the leading power of

$$J_v(x)J'_{-v}(x)$$

and

$$\frac{v}{\Gamma(v+1)\Gamma(-v+1)x}$$

is the leading power of

$$J_{-v}(x)J'_v(x)$$

which implies that the coefficient of x^{-1} in

$$J_{-v}(x)J'_v(x)-J_v(x)J'_{-v}(x)$$

is equal to

$$\frac{2v}{\Gamma(v+1)\Gamma(1-v)x}$$

From reflection formula, we get that

$$\Gamma(v)\Gamma(1-v)=\frac{\pi}{\sin v\pi}$$

As

$$\Gamma(v+1)=v\Gamma(v)$$

and

$$\Gamma(v)\Gamma(1-v)=\frac{\pi}{\sin v\pi},$$

we get that

$$\frac{v}{\Gamma(v+1)\Gamma(1-v)}=\frac{\sin v\pi}{\pi}$$

Therefore, coefficient of x^{-1} in $J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$ is equal to $\dfrac{2\sin v\pi}{\pi x}$. As

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{A_v}{x}$$

where A_v, may depend on v but is independent of x and the coefficient of x^{-1} in

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$$

is equal to

$$\frac{2\sin v\pi}{\pi x},$$

we get that

$$A_v = \frac{2\sin v\pi}{\pi}$$

and all coefficients of x (except x^{-1}) are zero. Therefore,

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{2\sin v\pi}{\pi x}$$

As

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{2\sin v\pi}{\pi x}$$

and

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x),$$

we get that

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = \frac{2\sin v\pi}{\pi x}$$

As

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{2\sin v\pi}{\pi x}$$

and

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{-J_v(x)J'_{-v}(x) + J'_v(x)J_{-v}(x)}{\sin vx},$$

we get that

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{2}{\pi x}$$

and hence the given statements are true.

Problem 14.3.6

As an alternative to letting x approach zero in the evaluation of the Wronskian constant, we may invoke the uniqueness of power-series expansions. The coefficient of x^{-1} in the series expansion of

$$u_v(x)v_v'(x) - u_v'(x)v_v(x)$$

is then A_v Show by series expansion that the coefficients of x^0 and x^1 of

$$J_v(x)J_{-v}'(x) - J_v'(x)J_{-v}(x)$$

are each zero.

Solution To prove that the coefficients of x^0 and x^1 in

$$J_v(x)J_{-v}'(x) - J_v'(x)J_{-v}(x)$$

are both zero by using power series expansions. As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v},$$

we get that

$$J_v'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s+v)}{s!\ \Gamma(v+s+1)2} \left(\frac{x}{2}\right)^{2s+v-1}$$

Similarly,

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

and hence

$$J_{-v}'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s-v)}{s!\ \Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2s-v-1}$$

Also

$$J_v(x)J_{-v}'(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k+v} \frac{(-1)^s(2s-v)}{s!\ \Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2s-v-1}$$

and

$$J_v'(x)J_{-v}(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(2k+v)}{k!\ \Gamma(v+k+1)2} \left(\frac{x}{2}\right)^{2k+v-1} \frac{(-1)^s}{s!\ \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

As

$$J_v(x)J_{-v}'(x) = \sum_{x=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2s-v)}{k!\ s!\ \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that the coefficient of x^0 in

$$J_v(x)J_{-v}'(x)$$

is equal to zero (as $2k+2s-1$ is odd for any $s,k \in \mathbb{Z}$, we get that $2k+2s-1$ is never 0 and hence there is no constant term in $J_v(x)J_{-v}'(x)$) Similarly, as

$$J_v'(x)J_{-v}(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2k+v)}{k!\ s!\ \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that coefficient of x^0 in

$$J_v'(x)J_{-v}(x)$$

is equal to zero (as $2k+2s-1$ is odd for any $s,k \in \mathbb{Z}$, we get that $2k+2s-1$ is never 0 and hence there is no constant term in $J_v'(x)J_{-v}(x)$) As there are constant terms in both $J_v'(x)J_{-v}(x)$ and $J_v(x)J_{-v}'(x)$, we get that the coefficient of x^0 in

$$J_v(x)J_{-v}'(x) - J_v'(x)J_{-v}(x)$$

is zero. As

$$J_v(x)J'_{-v}(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2s-v)}{k!\ s!\ \Gamma(v+k+1)\Gamma(-v+s+1)2}\left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{(-1)^{1+0}(2(0)-v)}{1!\ 0!\ \Gamma(v+(1)+1)\Gamma(-v+(0)+1)2}\left(\frac{1}{2}\right) + \frac{(-1)^{0+1}(2(1)-v)}{0!\ 1!\ \Gamma(v+(0)+1)\Gamma(-v+(1)+1)2}\left(\frac{1}{2}\right)$$

when $s, k \in \mathbb{Z}$, we get that $2k+2s-1 = 1$ if and only if $k+s=1$ which implies that either $k=0, s=1$ or $k=1, s=0$. Also

$$\frac{(-1)^{1+0}(2(1)+v)}{1!\ 0!\ \Gamma(v+(1)+1)\Gamma(-v+(0)+1)2}\left(\frac{1}{2}\right) + \frac{(-1)^{0+1}(2(0)+v)}{0!\ 1!\ \Gamma(v+(0)+1)\Gamma(-v+(1)+1)2}\left(\frac{1}{2}\right)$$

is equal to

$$\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)^4}$$

Thus, the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)^4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4}$$

As the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)^4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4}$$

and the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{v}{\Gamma(v+2)\Gamma(1-v)4} + \frac{v-2}{\Gamma(v+1)\Gamma(2-v)4}$$

we get that the coefficient of x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ is equal to

$$\frac{v}{\Gamma(v+2)\Gamma(1-v)4} + \frac{v-2}{\Gamma(v+1)\Gamma(2-v)4} - \left(\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)^4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4}\right)$$

which is equal to

$$\frac{2v+2}{\Gamma(v+2)\Gamma(1-v)^4} + \frac{2v-2}{\Gamma(v+1)\Gamma(2-v)^4}$$

Also

$$\frac{2v+2}{\Gamma(v+2)\Gamma(1-v)4} = \frac{2v+2}{(v+1)\Gamma(v+1)\Gamma(1-v)^4} = \frac{1}{\Gamma(v+1)\Gamma(1-v)^2}$$

and

$$\frac{2v-2}{\Gamma(v+1)\Gamma(2-v)4} = \frac{2v-2}{\Gamma(v+1)(1-v)\Gamma(1-v)^4} = \frac{-1}{\Gamma(v+1)\Gamma(1-v)^2}$$

Thus,

$$\frac{2v+2}{\Gamma(v+2)\Gamma(1-v)^4} + \frac{2v-2}{\Gamma(v+1)\Gamma(2-v)4} = \frac{1}{\Gamma(v+1)\Gamma(1-v)2} + \frac{-1}{\Gamma(v+1)\Gamma(1-v)^2} = 0.$$

Thus, the coefficient of x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ is equal to zero. Therefore, the coefficients of x^0 and x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ are both zero.

Problem 14.3.8

Verify the expansion formula for $Y_n(x)$ given in Eq. (14.61).

$$Y_n(x) = \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ (n+k)!}[\psi(k+1)+\psi(n+k+1)]\left(\frac{x}{2}\right)^{2k+n} \tag{14.61}$$

Hint. Start from Eq. (14.60)

$$Y_n(x) = \frac{1}{\pi}\left[\frac{dJ_v}{dv} - (-1)^n\frac{dJ_{-v}}{dv}\right]_{v=n} \tag{14.60}$$

and perform the indicated differentiations on the powerseries expansions of J_v and J_{-v}. The digamma functions ψ arise from the differentiation of the gamma function. You will need the identity (not derived in this book) $\lim_{z\to -n}\psi(z)/\Gamma(z) = (-1)^{n-1}n!$, where n is a positive integer.

Solution To prove that

$$Y_n(x) = \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - A$$

where

$$A = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ (n+k)!}[\psi(k+1)+\psi(n+k+1)]\left(\frac{x}{2}\right)^{2k+n}$$

We know that

$$Y_n(x) = \frac{1}{\pi}\left[\frac{dJ_v(x)}{dv} - (-1)^n\frac{dJ_{-v}(x)}{dv}\right]_{v=n}$$

As

$$J_v(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{v+2s}$$

we get that

$$\frac{dJ_v(x)}{dv} = \frac{d}{dv}\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{v+2s}\right)$$

and hence

$$\frac{dJ_v(x)}{dv} = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\frac{d}{dv}\left(\frac{x}{2}\right)^{v+2s} + \sum_{s=0}^{\infty}\left(\frac{x}{2}\right)^{v+2s}\frac{d}{dv}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}$$

Also

$$\frac{d}{dv}\left(\frac{x}{2}\right)^{v+2s} = \left(\frac{x}{2}\right)^{v+2s}\ln\left(\frac{x}{2}\right)$$

and

$$\frac{d}{dv}\frac{(-1)^s}{s!\ \Gamma(v+s+1)} = -\frac{(-1)^s}{s!\ (\Gamma(v+s+1))^2}\frac{d\Gamma(v+s+1)}{dv}$$

Thus,

$$\frac{dJ_v(x)}{dv} = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{v+2s}\ln\left(\frac{x}{2}\right) - \sum_{s=0}^{\infty}\left(\frac{x}{2}\right)^{v+2s}\frac{(-1)^s}{s!\ (\Gamma(v+s+1))^2}\frac{d\Gamma(v+s+1)}{dv}$$

As

$$J_v(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{v+2s},$$

we get that

$$\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{v+2s}\ln\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2}\right)\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(v+s+1)}\left(\frac{x}{2}\right)^{v+2s} = \ln\left(\frac{x}{2}\right)J_v(x)$$

Therefore,

$$\frac{dJ_v(x)}{dv} = \ln\left(\frac{x}{2}\right) J_v(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s}{s!\,(\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

As

$$\psi(z+1) = \frac{d \ln \Gamma(z+1)}{dv} = \frac{1}{\Gamma(z+1)} \frac{d\Gamma(z+1)}{dv}$$

where ψ is the digamma function, we get that

$$\frac{(-1)^s}{s!\,(\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv} = \frac{(-1)^s \psi(v+s+1)}{s!\,\Gamma(v+s+1)}$$

Thus,

$$\frac{dJ_v(x)}{dv} = \ln\left(\frac{x}{2}\right) J_v(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s \psi(v+s+1)}{s!\,\Gamma(v+s+1)}$$

Therefore,

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{n+2s} \frac{(-1)^s \psi(n+s+1)}{s!\,\Gamma(n+s+1)}$$

which implies that

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^s \psi(n+s+1)}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s},$$

we get that

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v}$$

Also

$$\frac{dJ_{-v}(x)}{dv} = \frac{d}{dv}\left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v}\right)$$

and hence we get that

$$\frac{dJ_{-v}(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \frac{d}{dv} \left(\frac{x}{2}\right)^{2s-v} + \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{d}{dv} \frac{(-1)^s}{s!\,\Gamma(s-v+1)}$$

Also

$$\frac{d}{dv}\left(\frac{x}{2}\right)^{2s-v} = \frac{d(2s-v)}{dv} \frac{d}{d(2s-v)} \left(\frac{x}{2}\right)^{2s-v} = -\left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right)$$

and

$$\frac{d}{dv} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} = -\frac{(-1)^s}{s!\,(\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

Thus,

$$\frac{dJ_{-v}(x)}{dv} = -\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s}{s!\,(\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

As

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v},$$

we get that

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2}\right) \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} = \ln\left(\frac{x}{2}\right) J_{-v}(x)$$

Therefore,

$$\frac{dJ_{-v}(x)}{dv} = -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{s=0}^{\infty}\left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s}{s!\ (\Gamma(s-v+1))^2}\frac{d\Gamma(s-v+1)}{dv}$$

As

$$\psi(z+1) = \frac{d\ln\Gamma(z+1)}{dv} = \frac{1}{\Gamma(z+1)}\frac{d\Gamma(z+1)}{dv},$$

we get that

$$\frac{(-1)^s}{s!\ (\Gamma(s-v+1))^2}\frac{d\Gamma(s-v+1)}{dv} = \frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)}$$

and hence

$$\frac{dJ_{-v}(x)}{dv} = -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{s=0}^{\infty}\left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)}$$

Thus,

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{\infty}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)}$$

As

$$\lim_{v\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)} = \frac{(-1)^s}{s!}\lim_{v\to n}\frac{\psi(s-v+1)}{\Gamma(s-v+1)},$$

we get that

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^x\psi(s-v+1)}{s!\ \Gamma(s-v+1)} - \sum_{s=n}^{\infty}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)}$$

(by dividing the summation into $s < n$ and $s \geq n$ parts). Also

$$\sum_{s=n}^{\infty}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)} = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^{k+n}}{(k+n)!}\frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^{k+n}\psi(k+1)}{(k+n)!\ k!}$$

and

$$\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)} = \sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}\lim_{k\to s-n}\frac{\psi(k+1)}{\Gamma(k+1)}$$

As

$$\lim_{z\to -n}\frac{\psi(z+1)}{\Gamma(z+1)} = (-1)^{n-1}n!\ ,$$

we get that

$$\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}\lim_{k\to s-n}\frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}(-1)^{s-n-1}(s-n-1)!$$

As

$$\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}\lim_{k\to s-n}\frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}(-1)^{s-n-1}(s-n-1)!$$

$$\sum_{s=n}^{\infty}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)} = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^{k+n}\psi(k+1)}{(k+n)!\ k!}$$

and

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n} \lim_{x\to n}\frac{(-1)^s\psi(s-v+1)}{s!\ \Gamma(s-v+1)} - \sum_{x=n}^{\infty}\left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n}\frac{(-1)^x\psi(s-v+1)}{s!\ \Gamma(s-v+1)}$$

we get that $\left(\dfrac{dJ_{-v}(x)}{dv}\right)_{v=n}$ is equal to

$$-\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}(-1)^{s-n-1}(s-n-1)!\ - \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^{k+n}\psi(k+1)}{(k+n)!\ k!}$$

As $J_{-n}(x) = (-1)^n J_n(x)$

$$-\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1}(s-n-1)! \; - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n}\psi(k+1)}{(k+n)!\; k!}$$

can be written as

$$-(-1)^n \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} (-1)^{n+1} - (-1)^{n+1} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k\psi(k+1)}{(k+n)!\; k!}$$

$$(-1)^{s-n-1}(-1)^s = (-1)^{s-n-1+s-n-1}(-1)^{n+1} = (-1)^{n+1}(-1)^{2(s-n-1)} = (-1)^{n+1}$$

Thus

$$(-1)^{n+1} \left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k\psi(k+1)}{(k+n)!\; k!}$$

As

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^s\psi(n+s+1)}{s!\;(n+s)!} \left(\frac{x}{2}\right)^{n+2}$$

$$(-1)^{n+1} \left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} - \sum_{s=0}^{\infty} \frac{(-1)^s\psi(s+1)}{(s+n)!\; s!} \left(\frac{x}{2}\right)^{2s+n}$$

and

$$Y_n(x) = \frac{1}{\pi}\left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv}\right]_{v=n}$$

we get

$$\frac{2}{\pi} \ln\left(\frac{x}{2}\right) J_n(x) - \frac{1}{\pi} \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} - \sum_{s=0}^{\infty} \frac{(-1)^s(\psi(s+1)+\psi(n+s+1))}{(s+n)!\; s!} \left(\frac{x}{2}\right)^{2s+n}$$

Therefore,

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} - A$$

where

$$A = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\;(n+k)!} [\psi(k+1) + \psi(n+k+1)] \left(\frac{x}{2}\right)^{2k+n}$$

Problem 14.3.9

If Bessel's ODE (with solution J_v) is differentiated with respect to v, one obtains

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right)+\left(x^2-v^2\right)\frac{\partial J_v}{\partial v}=2vJ_v$$

Use the above equation to show that $Y_n(x)$ is a solution to Bessel's ODE.

Solution We know that

$$Y_n(x)=\frac{1}{\pi}\left[\frac{dJ_v(x)}{dv}-(-1)^n\frac{dJ_{-v}(x)}{dv}\right]_{v=n}$$

As

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right)+\left(x^2-v^2\right)\frac{\partial J_v}{\partial v}=2vJ_v,$$

we get that

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial(-v)}\right)+x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial(-v)}\right)+\left(x^2-(-v)^2\right)\frac{\partial J_{-v}}{\partial(-v)}=2(-v)J_{-v}$$

As $\frac{\partial J_{-v}}{\partial(-v)}=-\frac{\partial J_{-v}}{\partial v}$, we get that

$$-x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right)-x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right)-\left(x^2-v^2\right)\frac{\partial J_{-v}}{\partial v}=-2vJ_{-v}$$

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right)+\left(x^2-v^2\right)\frac{\partial J_{-v}}{\partial v}=2vJ_{-v}$$

As

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right)+\left(x^2-v^2\right)\frac{\partial J_v}{\partial v}=2vJ_v$$

and

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right)+\left(x^2-v^2\right)\frac{\partial J_{-v}}{\partial v}=2vJ_{-v},$$

we get that $2vJ_v-2v(-1)^nJ_{-v}$ is equal to

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}-(-1)^n\frac{\partial J_{-v}}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}-(-1)^n\frac{\partial J_{-v}}{\partial v}\right)+\left(x^2-v^2\right)\left(\frac{\partial J_v}{\partial v}-(-1)^n\frac{\partial J_{-v}}{\partial v}\right)$$

Thus, $2nJ_n-2n(-1)^nJ_{-n}$ is equal to

$$\left[x^2\frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}-(-1)^n\frac{\partial J_{-v}}{\partial v}\right)+x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}-(-1)^n\frac{\partial J_{-v}}{\partial v}\right)+\left(x^2-v^2\right)\left(\frac{\partial J_v}{\partial v}-(-1)^n\frac{\partial J_{-v}}{\partial v}\right)\right]_{v=n}$$

As

$$Y_n(x)=\frac{1}{\pi}\left[\frac{dJ_v(x)}{dv}-(-1)^n\frac{dJ_{-v}(x)}{dv}\right]_{v=n},$$

the above equation implies that

$$x^2\frac{d^2Y_n(x)}{dx^2}+x\frac{dY_n(x)}{dx}+\left(x^2-n^2\right)Y_n(x)=\frac{1}{\pi}\left(2nJ_n-2n(-1)^nJ_{-n}\right)$$

As $J_{-n}=(-1)^nJ_n$, we get that $2nJ_n-2n(-1)^nJ_{-n}=0$. As $2nJ_n-2n(-1)^nJ_{-n}=0$ and

$$x^2\frac{d^2Y_n(x)}{dx^2}+x\frac{dY_n(x)}{dx}+\left(x^2-n^2\right)Y_n(x)=\frac{1}{\pi}\left(2nJ_n-2n(-1)^nJ_{-n}\right),$$

we get that

$$x^2\frac{d^2Y_n(x)}{dx^2}+x\frac{dY_n(x)}{dx}+\left(x^2-n^2\right)Y_n(x)=0$$

Therefore, $Y_n(x)$ is a solution to the Bessel's ODE.