Show that matrix multiplication is associative, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

Solution The product $\mathbf{BC}$ is defined because the column of $\mathbf{B}$ and rows of $\mathbf{C}$ are same. Suppose $\mathbf{D} = \mathbf{BC}$ Then element of $\mathbf{D}$ is of the form

$$d_{ik} = \sum_{j} b_{ij} c_{jk}$$

Now the product $\mathbf{AD}$ is defined because the column of $\mathbf{A}$ and rows of $\mathbf{D}$ are same. Then element of $\mathbf{E}$ is of the form

$$e_{lk} = \sum_{k} a_{li} \left(\sum_{j} b_{ij} c_{jk} \right)$$

Therefore, the matrix $\mathbf{E} = \mathbf{A}(\mathbf{BC})$ have the elements e_{lk}. The product $\mathbf{AB}$ is defined because the column of $\mathbf{A}$ and rows of $\mathbf{B}$ are same. Let, $\mathbf{D} = \mathbf{AB}$. Then element of $\mathbf{D}$ is of the form

$$d_{lj} = \sum_{i} a_{li} b_{ij}$$

Now the product $\mathbf{DC}$ is defined because the column of $\mathbf{D}$ and rows of $\mathbf{C}$ are same. Let, $\mathbf{E} = \mathbf{DC}$. Then element of $\mathbf{D}$ is of the form

$$e_{lk} = \sum_{j} \left(\sum_{i} a_{li} b_{ij} \right) c_{jk}$$

Therefore, the matrix $\mathbf{E} = (\mathbf{AB})\mathbf{C}$ have the elements e_{lk} Therefore,

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

Hence, matrix multiplication is associative.

Show that
$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$$

if and only if $\mathbf{A}$ and $\mathbf{B}$ commute
$$[\mathbf{A}, \mathbf{B}] = 0$$

Solution

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}(\mathbf{A} + \mathbf{B}) - \mathbf{B}(\mathbf{A} + \mathbf{B})$$

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 + \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} - \mathbf{B}^2$$

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2 + (\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})$$

Because $\mathbf{A}$ and $\mathbf{B}$ conmute, the term $(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})$ equals to zero, hence is proved.

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$$

Problem 2.2.3

(a) Complex numbers, $a + ib$, with a and b real, may be represented by (or are isomorphic with) 2×2 matrices:
$$a + ib \longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Show that this matrix representation is valid for

(i) addition

(ii) multiplication

(b) Find the matrix corresponding to $(a + ib)^{-1}$.

Solution (a) Let us start with addition. For complex numbers, we have (straightforwardly)
$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

whereas, if we used matrices we would get
$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} (a + c) & (b + d) \\ -(b + d) & (a + c) \end{bmatrix}$$

which shows that the sum of matrices yields the proper representation of the complex number $(a + c) + i(b + d)$. We now handle multiplication in the same manner. First, we have
$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

while matrix multiplication gives
$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} (ac - bd) & (ad + bc) \\ -(ad + bc) & (ac - bd) \end{bmatrix}$$

which is again the correct result.

Solution For (b) Find the matrix oorresponding to $(a + ib)^{-1}$ We can find the matrix in two ways. We first do standand complex arithmetic
$$(a + ib)^{-1} = \frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{1}{a^2 + b^2}(a - ib)$$

This corresponds to the 2×2 matrix
$$(a + ib)^{-1} \longleftrightarrow \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Alternatively, we first convert to a matrix representation, and then find the inverse matrix
$$(a + ib)^{-1} \leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Either way, we obtain the same result.

Problem 2.2.4

If $\mathbf{A}$ is an $n \times n$ matrix, show that

$$\det(-\mathbf{A}) = (-1)^n \det \mathbf{A}.$$

Solution So from the above identity we can write

$$-A = (-I)A$$

$$\det(-A) = \det(-IA)$$

We know $\det(AB) = \det(A)\det(B)$ From this

$$\det(-IA) = \det(-I)\det(\mathrm{A})$$

$$\det(-I)\det(\mathrm{A}) = (-1)^n \det(A)$$

Problem 2.2.5

(a) The matrix equation $\mathbf{A}^2 = 0$ does not imply $\mathbf{A} = 0$. Show that the most general 2×2 matrix whose square is zero may be written as

$$\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

where a and b are real or complex numbers.

(b) If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, in general

$$\det \mathbf{C} \neq \det \mathbf{A} + \det \mathbf{B}.$$

Construct a specific numerical example to illustrate this inequality.

Solution For (a) first we check the condition

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} \times \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{pmatrix} = 0$$

Therefore, the 2×2 matrix square is zero

Solution For (b) we know $C = A + B$, let us consider following matrices to show that

$$\det C \neq \det A + \det B$$

Now, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\det A = 1 - 0 = 1$$
$$\det B = 1 - 0 = 1$$
$$\det C = 4 - 0 = 4$$

From this

$$\det C \neq \det A + \det B$$
$$4 \neq 2$$

Therefore, the following matrix satisfies the condition

Problem 2.2.6

Given
$$\mathbf{K} = \begin{bmatrix} 0 & 0 & i \\ -i & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

show that
$$\mathbf{K}^n = \mathbf{K}\mathbf{K}\mathbf{K} \;\cdots\; (n \text{ factors}) = 1$$

(with the proper choice of n, $n \neq 0$).

Solution We calculate for different n

$$\mathbf{K}^2 = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 1 \\ i & 0 & 0 \end{bmatrix} \quad \mathbf{K}^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \mathbf{K}^4 = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{K}^5 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & -1 \\ -i & 0 & 0 \end{bmatrix} \quad \mathbf{K}^6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With this, the answer is $n = 6$

Verify the Jacobi identity,

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{C}]] - [\mathbf{C}, [\mathbf{A}, \mathbf{B}]]$$

Solution

$$[\mathbf{A}[\mathbf{B}, \mathbf{C}]] = [\mathbf{A}, \mathbf{BC} - \mathbf{CB}] = \mathbf{A}(\mathbf{BC} - \mathbf{CB}) - (\mathbf{BC} - \mathbf{CB})\mathbf{A}$$

$$[\mathbf{A}[\mathbf{B}, \mathbf{C}]] = \mathbf{A}(\mathbf{BC}) - \mathbf{A}(\mathbf{CB}) - (\mathbf{BC})\mathbf{A} + (\mathbf{CB})\mathbf{A}$$

$$[\mathbf{B}[\mathbf{C}, \mathbf{A}]] = [\mathbf{B}, \mathbf{CA} - \mathbf{AC}] = \mathbf{B}(\mathbf{CA} - \mathbf{AC}) - (\mathbf{CA} - \mathbf{AC})\mathbf{B}$$

$$[\mathbf{B}[\mathbf{C}, \mathbf{A}]] = \mathbf{B}(\mathbf{CA}) - \mathbf{B}(\mathbf{AC}) - (\mathbf{CA})\mathbf{B} + (\mathbf{AC})\mathbf{B}$$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = [\mathbf{C}, \mathbf{AB} - \mathbf{BA}] = \mathbf{C}(\mathbf{AB} - \mathbf{BA}) - (\mathbf{AB} - \mathbf{BA})\mathbf{C}$$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = \mathbf{C}(\mathbf{AB}) - \mathbf{C}(\mathbf{BA}) - (\mathbf{AB})\mathbf{C} + (\mathbf{BA})\mathbf{C}$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}$ are obey associative law $\mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B}$, $\mathbf{C}(\mathbf{BA}) = (\mathbf{CB})\mathbf{A}$, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ and $(\mathbf{BA})\mathbf{C} = \mathbf{B}(\mathbf{AC})$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = (\mathbf{CA})\mathbf{B} - (\mathbf{CB})\mathbf{A} - \mathbf{A}(\mathbf{BC}) + \mathbf{B}(\mathbf{AC})$$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = [\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}[\mathbf{C}, \mathbf{A}]] + [\mathbf{C}[\mathbf{A}, \mathbf{B}]]$$

$$= (\mathbf{A}(\mathbf{BC}) - \mathbf{A}(\mathbf{CB}) - (\mathbf{BC})\mathbf{A} + (\mathbf{CB})\mathbf{A}) + (\mathbf{B}(\mathbf{CA}) - \mathbf{B}(\mathbf{AC}) - (\mathbf{CA})\mathbf{B}$$

$$+ (\mathbf{AC})\mathbf{B}) + (\mathbf{CA})\mathbf{B} - (\mathbf{CB})\mathbf{A} - \mathbf{A}(\mathbf{BC}) + \mathbf{B}(\mathbf{AC}) = 0$$

Problem 2.2.8

Show that the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy the commutation relations

$$[\mathbf{A}, \mathbf{B}] = \mathbf{C}, \quad [\mathbf{A}, \mathbf{C}] = 0, \quad \text{and} \quad [\mathbf{B}, \mathbf{C}] = 0$$

Solution We simply multiply the matrices

$$\mathbf{C} = [\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = C$$

$$[\mathbf{A}, \mathbf{C}] = A\mathbf{C} - \mathbf{C}A = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$[\mathbf{B}, \mathbf{C}] = \mathbf{BC} - \mathbf{CB} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Problem 2.2.9

Let

$$i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(a) $i^2 = j^2 = k^2 = -I$, where $\mathbf{I}$ is the unit matrix.

(b) $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$

These three matrices (i, j, and k) plus the unit matrix 1 form a basis for quaternions. An alternate basis is provided by the four 2 ×2 matrices, $i\sigma_1, i\sigma_2, -i\sigma_3$, and 1, where the σ_i are the Pauli spin matrices of Example 2.2.1.

Solution

$$i^2 = j^2 = k^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$ij = -ij = k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$jk = -kj = i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$ki = -ik = j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2.2.10

A matrix with elements $a_{ij} = 0$ for $j < i$ may be called upper right triangular. The elements in the lower left (below and to the left of the main diagonal) vanish. Show that the product of two uper right triangular matrices is an upper right triangular matrix.

Solution We build 2 matrix with terms a, b, c, x, y, z that can take any number

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x & y & z \\ 0 & u & w \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a \times x & b \times u + a \times y & b \times w + a \times z \\ 0 & d \times u & d \times w \\ 0 & 0 & 0 \end{bmatrix}$$

Hence is demostrated that the product of two upper right triangular matrices is an upper right triagular matrix.

Problem 2.2.11

The three Pauli spin matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Show that

(a) $(\sigma_i)^2 = \hat{1}_2$

(b) $\sigma_i \sigma_j = i\sigma_k, (i, j, k) = (1, 2, 3)$ or a cyclic permutation thereof,

(c) $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\hat{1}_2$; $\quad \hat{1}_2$ is the 2×2 unit matrix.

Solution For $i = 1, j = 2, k = 3$

$$\sigma_i \sigma_j = \sigma_1 \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3 = i\sigma_k$$

For $i = 2, j = 3, k = 1$

$$\sigma_i \sigma_j = \sigma_2 \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\sigma_3 = i\sigma_k$$

For $i = 2, j = 3, k = 1$

$$\sigma_i \sigma_j = \sigma_3 \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_2 = i\sigma_k$$

So, we conclude that $\sigma_1 \sigma_j = i\sigma_k$

Solution (c) For this proof we need only to work out the commutation relation and use the proofs done in part (a) and (b)

$$\sigma_2 \sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix} = -i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -i\sigma_3$$

$$\sigma_1 \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -i \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -i\sigma_2$$

$$\sigma_3 \cdot \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -i \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} = -i\sigma_1$$

$$\left. \begin{aligned} \sigma_i \sigma_j + \sigma_j \sigma_i = \sigma_i \sigma_j - \sigma_i \sigma_j = 0 \\ \sigma_i \sigma_j + \sigma_j \sigma_i = 2\sigma_i^2 \end{aligned} \right\} \quad \begin{aligned} \text{for } j \neq i \\ \text{for } j = i \end{aligned}$$

since $\sigma_i^2 = 1$ and using the kronecker delta we have

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} 1$$

Problem 2.2.12

One description of spin-1 particles uses the matrices

$$\mathbf{M}_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{M}_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

and

$$\mathbf{M}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) $[\mathbf{M}_x, \mathbf{M}_y] = i\mathbf{M}_z$, and so on (cyclic permutation of indices). Using the Levi-Civita symbol, we may write

$$[\mathbf{M}_i, \mathbf{M}_j] = i\sum_k \varepsilon_{ijk}\mathbf{M}_k$$

(b) $\mathbf{M}^2 \equiv \mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2 = 2\mathbf{I}_3$, where $\mathbf{I}_3$ is the 3×3 unit matrix.

(c) $\left[\mathbf{M}^2, \mathbf{M}_i\right] = 0$ $[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{L}^+$ $[\mathbf{L}^+, \mathbf{L}^-] = 2\mathbf{M}_z$ where $\mathbf{L}^+ \equiv \mathbf{M}_x + i\mathbf{M}_y$ and $\mathbf{L}^- \equiv \mathbf{M}_x - i\mathbf{M}_y$

Solution For (a)

$$\mathbf{M}_x\mathbf{M}_y = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix}$$

$$\mathbf{M}_y\mathbf{M}_x = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix}$$

$$\mathbf{M}_x\mathbf{M}_y - \mathbf{M}_y\mathbf{M}_x = \frac{1}{2}\begin{bmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{bmatrix} + \frac{1}{2}\begin{bmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{bmatrix}$$

$$\mathbf{M}_x\mathbf{M}_y - \mathbf{M}_y\mathbf{M}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{M}_x\mathbf{M}_y - \mathbf{M}_y\mathbf{M}_x = \mathbf{M}_z$$

Solution For (b)

$$\mathbf{M}_x^2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_y^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now

$$\mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2 = 2\mathbf{I}$$

Solution (c) we substitute

$$\mathbf{M}^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{M}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\left[\mathbf{M}^2, \mathbf{M}_x\right] = \mathbf{M}^2\mathbf{M}_x - \mathbf{M}_x\mathbf{M}^2$$

$$\mathbf{M}^2\mathbf{M}_x = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\mathbf{M}_x\mathbf{M}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$[\mathbf{M}^2, \mathbf{M}_x] = \mathbf{M}^2\mathbf{M}_x - \mathbf{M}_x\mathbf{M}^2 = 0$$

Therefore, $[\mathbf{M}^2, \mathbf{M}_i] = 0$ Now, we substitute

$$\mathbf{M}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathbf{L}^+ = \mathbf{M}_x + i\mathbf{M}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{M}_z\mathbf{L}^+ - \mathbf{L}^+\mathbf{M}_z$$

$$\mathbf{M}_z\mathbf{L}^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{L}^+\mathbf{M}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

$$[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{M}_z\mathbf{L}^+ - \mathbf{L}^+\mathbf{M}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{L}^+$$

$$[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{M}_z\mathbf{L}^+ - \mathbf{L}^+\mathbf{M}_z = \mathbf{L}^+$$

Now, substitute

$$\mathbf{L}^+ = \mathbf{M}_x + i\mathbf{M}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{L}^- = \mathbf{M}_x - i\mathbf{M}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$[\mathbf{L}^+, \mathbf{L}^-] = \mathbf{L}^+\mathbf{L}^- - \mathbf{L}^-\mathbf{L}^+$$

$$\mathbf{L}^+\mathbf{L}^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{L}^-\mathbf{L}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[\mathbf{L}^+, \mathbf{L}^-] = \mathbf{L}^+\mathbf{L}^- - \mathbf{L}^-\mathbf{L}^+ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2\mathbf{M}_z$$

Therefore, $[\mathbf{L}^+, \mathbf{L}^-] = \mathbf{L}^+\mathbf{L}^- - \mathbf{L}^-\mathbf{L}^+ = 2\mathbf{M}_z$

Problem 2.2.13

Repeat Exercise 2.2.12, using the matrices for a spin of 3/2,

$$\mathbf{M}_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \mathbf{M}_y = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and

$$\mathbf{M}_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Solution For (a) we consider that $[\mathbf{M}_x, \mathbf{M}_y]$

$$= \frac{i}{4}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} -$$

$$\frac{i}{4}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$= i\mathbf{M}_z$$

Similarly we can show that $[\mathbf{M}_y, \mathbf{M}_z] = i\mathbf{M}_x$ and $[\mathbf{M}_z, \mathbf{M}_x] = i\mathbf{M}_y$ Thus, $[\mathbf{M}_i, \mathbf{M}_j] = i\sum_k \varepsilon_{ijk}\mathbf{M}_k$ where i, j, k can take values 1,2,3 or x, y, z.

Solution For (b) we consider that

$$\mathbf{M}^2 \equiv \mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2$$

$$= \frac{1}{4}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}^2 - \frac{1}{4}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}^2 + \frac{1}{4}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}^2$$

$$= 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 2\mathbf{1}_3$$

Solution For (c) we obtain the result from the previous part

$$[\mathbf{M}^2, \mathbf{M}_i] = [2\mathbf{1}_3, \mathbf{M}_i]$$
$$= 2\mathbf{1}_3\mathbf{M}_i - 2\mathbf{M}_i\mathbf{1}_3$$
$$= 2\mathbf{M}_i - 2\mathbf{M}_i$$
$$= 0$$

$$[\mathbf{M}_z, \mathbf{L}^+] = [\mathbf{M}_z, \mathbf{M}_x + i\mathbf{M}_y]$$
$$= [\mathbf{M}_z, \mathbf{M}_x] - [i\mathbf{M}_y, \mathbf{M}_z]$$
$$= [\mathbf{M}_z, \mathbf{M}_x] - i[\mathbf{M}_y, \mathbf{M}_z]$$
$$= i\mathbf{M}_y - i(i\mathbf{M}_x)$$
$$= i\mathbf{M}_y + \mathbf{M}_x$$
$$= \mathbf{M}_x + i\mathbf{M}_y$$
$$= \mathbf{L}^+$$

And finally

$$
\begin{aligned}
\left[\mathbf{L}^{+}, \mathbf{L}^{-}\right] &= \left[\mathbf{M}_x + i\mathbf{M}_y, \mathbf{M}_x - i\mathbf{M}_y\right] \\
&= \left[\mathbf{M}_x, \mathbf{M}_x\right] - \left[\mathbf{M}_x, i\mathbf{M}_y\right] + \left[i\mathbf{M}_y, \mathbf{M}_x\right] - \left[i\mathbf{M}_y, i\mathbf{M}_y\right], \\
&= 0 - i\left[\mathbf{M}_x, \mathbf{M}_y\right] - \left[\mathbf{M}_x, i\mathbf{M}_y\right] - 0 \\
&= -2i\left(i\mathbf{M}_z\right) \\
&= 2\mathbf{M}_z
\end{aligned}
$$

Problem 2.2.14

If **A** is a diagonal matrix, with all diagonal elements different, and **A** and **B** commute, show that **B** is diagonal.

Solution Given matrix **A** is diagonal matrix

$$\mathbf{A} = \text{diag}(a_1, a_2, a_3, ...a_n)$$

and $B = (b_{ij})$. Here **A** and **B** matrix conmute $\mathbf{AB} = \mathbf{BA}$, so

$$(a_i - a_j)b_{kl} = 0 \quad \text{for } k \neq l$$

$$b_{kl} = 0 \quad \text{for } k \neq l$$

Hence from the above statement we can say that is also a diagonal matrix

If $\mathbf{A}$ and $\mathbf{B}$ are diagonal, show that $\mathbf{A}$ and $\mathbf{B}$ commute.

Solution consider two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, which are diagonal.

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 a_1 & 0 \\ 0 & b_2 a_2 \end{bmatrix}$$

Commutative properts of addition:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}$$

$$\mathbf{B} + \mathbf{A} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 + a_1 & 0 \\ 0 & b_2 + a_2 \end{bmatrix}$$

Hence, diagonal matrices, when added commute.

Problem 2.2.16

Show that $\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{CBA})$ if any two of the three matrices commute.

Solution The trace of a matrix is the sum of its diagonal elements. Therefore, the trace of the product of three matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ is given by

$$\mathrm{Tr}(\mathbf{ABC}) = \sum_{ijk} \mathbf{A}_{ij}\mathbf{B}_{jk}\mathbf{C}_{ki}$$

By using the fact that i, j, and k are dummy summation indices with the same range, this sum can be written in the equivalent forms

$$\sum_{ijk} \mathbf{A}_{ij}\mathbf{B}_{jk}\mathbf{C}_{ki} = \sum_{ijk} \mathbf{C}_{ki}\mathbf{A}_{ij}\mathbf{B}_{jk} = \sum_{ijk} \mathbf{B}_{jk}\mathbf{C}_{ki}\mathbf{A}_{ij}$$

But the second and third of these are

$$\sum_{ijk} \mathbf{C}_{ki}\mathbf{A}_{ij}\mathbf{B}_{jk} = \mathrm{Tr}(\mathbf{CAB})$$

and

$$\sum_{ijk} \mathbf{B}_{jk}\mathbf{C}_{ki}\mathbf{A}_{ij} = \mathrm{Tr}(\mathbf{BCA})$$

respectively. Thus, we obtain the relation

$$\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{CAB}) = \mathrm{Tr}(\mathbf{BCA})$$

Problem 2.2.17

Angular momentum matrices satisfy a commutation relation

$$[\mathbf{M}_j, \mathbf{M}_k] = i\mathbf{M}_l \quad j, k, l \text{ cyclic}$$

Solution Taking the trace of both sides of the given expression, we have

$$\mathrm{Tr}\,(i\mathbf{M}_k) = \mathrm{Tr}\,(\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i)$$

Hence

$$i\,\mathrm{Tr}\,(\mathbf{M}_k) = \mathrm{Tr}\,(\mathbf{M}_i\mathbf{M}_j) - \mathrm{Tr}\,(\mathbf{M}_j\mathbf{M}_i)$$

since $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$, we see that $\mathrm{Tr}\,(\mathbf{M}_k) = 0$ for any k.

Problem 2.2.18

A and **B** anticommute: $\mathbf{AB} = -\mathbf{BA}$. Also, $\mathbf{A}^2 = 1, \mathbf{B}^2 = 1$. Show that $\text{Tr}(\mathbf{A}) = \text{Tr}(\mathbf{B}) = 0$. Note. The Pauli and Dirac matrices are specific examples.

Solution Since $\mathbf{B}^2 = I$, $\mathbf{B}$ is non-singular and its inverse exists. Therefore, $\mathbf{A} = -\mathbf{B}^{-1}\mathbf{AB}$. Taking the trace, we get

$$\text{Tr}(\mathbf{A}) = -\text{Tr}\left(\mathbf{B}^{-1}\mathbf{AB}\right) = -\text{Tr}\left(\mathbf{ABB}^{-1}\right) = -\text{Tr}(\mathbf{A})$$

We see that $\text{Tr}(\mathbf{A}) = 0$. Similarly, we find $\text{Tr}(\mathbf{B}) = 0$

Problem 2.2.19

(a) If two nonsingular matrices anticommute, show that the trace of each one is zero. (Nonsingular means that the determinant of the matrix is nonzero.)

(b) For the conditions of part (a) to hold, $\mathbf{A}$ and $\mathbf{B}$ must be $n \times n$ matrices with n even. Show that if n is odd, a contradiction results.

$\boxed{\textbf{Solution}}$ For (a) if the matrices are non-singular, then writing $\mathbf{A} = -\mathbf{BAB}^{-1}$ and taking the trace, we get $\operatorname{Tr} \mathbf{A} = -\operatorname{Tr} \mathbf{A}$. Hence $\operatorname{Tr} \mathbf{A} = 0$, and the procedure for $\mathbf{B}$ is analogous.

$\boxed{\textbf{Solution}}$ For (b) now, we compute the determinant of both sides of $\mathbf{AB} = -\mathbf{BA}$: this yields det $\mathbf{A}$ det $\mathbf{B} = (-1)^N$ det $\mathbf{B}$ det $\mathbf{A}$, where N stands for size of matrices. Now since the $\mathbf{A}, \mathbf{B}$ are non-singular, both sides of the equality are non-zero and the equality is possible only for even N.

Problem 2.2.20

If $\mathbf{A}^{-1}$ has elements

$$\left(\mathbf{A}^{-1}\right)_{ij} = a_{ij}^{(-1)} = \frac{\mathbf{C}_{ji}}{|\mathbf{A}|}$$

where $\mathbf{C}_{ji}$ is the ji th cofactor of $|\mathbf{A}|$, show that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Hence $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$ (if $|\mathbf{A}| \neq 0$).

Solution We have to consider that

$$\left(A^{-1}A\right)_{ij} = \sum_k a_{ik}^{-1} a_{kj}$$

$$= \sum_k \frac{C_{ki}}{|A|} a_{kj}$$

$$= \frac{1}{|A|} \sum_k C_{ki} a_{kj}$$

$$= \frac{1}{|A|} |A| \delta_{ij}$$

$$= \delta_{ij}$$

Thus, $A^{-1}A = 1$. Hence, by definition of inverse of a matrix A^{-1} is the inverse of A(if $|A| \neq 0$).

Find the matrices $\mathbf{M}_L$ such that the product $\mathbf{M}_L \mathbf{A}$ will be $\mathbf{A}$ but with:

(a) The i th row multiplied by a constant k $(a_{ij} \rightarrow ka_{ij}, j = 1, 2, 3, \ldots)$

(b) The i th row replaced by the original i th row minus a multiple of the mth row $(a_{ij} \rightarrow a_{ij} - Ka_{mj}, i = 1, 2, 3, \ldots)$

(c) The i th and m th rows interchanged $(a_{ij} \rightarrow a_{mj}, a_{mj} \rightarrow a_{ij}, j = 1, 2, 3, \ldots)$

Solution For (a) Let

$$M_L = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$M_L A = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

Solution For (b) a unit matrix except that $M_{im} = -K$.

$$M_L = \begin{bmatrix} 1 & -K \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$M_L A = \begin{bmatrix} a - Kc & b - Kd \\ c & d \end{bmatrix}$$

Solution For (c) A unit matrix except that $M_{ii} = M_{mm} = 0$ and $M_{mi} - M_{im} = 1$.

$$M_L = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M_L A = \begin{bmatrix} c & d \\ 2a & 2b \end{bmatrix}$$

Find the matrices $\mathbf{M}_R$ such that the product AM $_R$ will be A but with:

(a) The i th column multiplied by a constant k $(a_{ji} \rightarrow ka_{ji}, j = 1, 2, 3, \ldots)$

(b) The i th column replaced by the original i th column minus a multiple of the mth column $(a_{ji} \rightarrow a_{ji} - ka_{jm}, j = 1, 2, 3, \ldots)$

(c) The i th and m th columns interchanged $(a_{ji} \rightarrow a_{jm}, a_{jm} \rightarrow a_{ji}, j = 1, 2, 3, \ldots)$

Solution For (a), a unit matrix except that $M_{ii} = k$. Let

$$M_R = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$AM_R = \begin{bmatrix} ka & b \\ kc & d \end{bmatrix}$$

Solution For (b), a unit matrix except that $M_{im} = -K$. Let

$$M_R = \begin{bmatrix} 1 & -K \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$AM_R = \begin{bmatrix} a & b - Ka \\ c & d - Kc \end{bmatrix}$$

Solution For (c), a unit matrix except that $M_{ii} = M_{mm} = 0$ and $M_{mi} - M_{im} = 1$. Example: Let

$$M_R = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$AM_R = \begin{bmatrix} 2b & a \\ 2d & c \end{bmatrix}$$

Problem 2.2.23

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution Calculating $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1} = \frac{1}{7} \begin{bmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Problem 2.2.24

Matrices are far too useful to remain the exclusive property of physicists. They may appear wherever there are linear relations. For instance, in a study of population move- ment the initial fraction of a fixed population in each of n areas (or industries or religions, etc.) is represented by an n-component column vector $\mathbf{P}$.

The movement of people from one area to another in a given time is described by an $n \times n$ (stochastic) matrix $\mathbf{T}$. Here $\mathbf{T}_{ij}$ is the fraction of the population in the j th area that moves to the ith area. (Those not moving are covered by $i = j$.) With $\mathbf{P}$ describing the initial population distribution, the final population distribution is given by the matrix equation $\mathbf{TP} = \mathbf{Q}$. From its definition, $\sum_{i=1}^{n} \mathbf{P}_i = 1$

(a) Show that conservation of people requires that

$$\sum_{i=1}^{n} \mathbf{T}_{ij} = 1, \quad j = 1, 2, \ldots, n$$

(b) Prove that

$$\sum_{i=1}^{n} \mathbf{Q}_i = 1$$

continues the conservation of people.

Solution For (a) The equation of part states that $\mathbf{T}$ moves people from area j but does not change their total number.

Solution For (b) Write the component equation $\sum_j \mathbf{T}_{ij}\mathbf{P}_j = \mathbf{Q}_i$ and sum over i. This summation replaces $\mathbf{T}_{ij}$ by unity, leaving that the sum pver $\mathbf{P}_j$ equals the sum over $\mathbf{Q}_i$, hence conserving people.

Problem 2.2.25

Given a 6×6 matrix A with elements $a_{ij} = 0.5^{|i-j|}, i, j = 0, 1, 2, \ldots, 5$, find $\mathbf{A}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 & 0.5^5 \\ 0.5 & 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 \\ 0.5^2 & 0.5 & 1 & 0.5 & 0.5^2 & 0.5^3 \\ 0.5^3 & 0.5^2 & 0.5 & 1 & 0.5 & 0.5^2 \\ 0.5^4 & 0.5^3 & 0.5^2 & 0.5 & 1 & 0.5 \\ 0.5^5 & 0.5^4 & 0.5^3 & 0.5^2 & 0.5 & 1 \end{bmatrix}$$

Solution

$$\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 \\ -2 & 5 & -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -2 & 0 & 0 \\ 0 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix}$$

> **Problem 2.2.26**
>
> Show that the product of two orthogonal matrices is orthogonal.

Solution Let Q and P be orthogonal matrices. Therefore $\mathbf{Q}^T Q = I$ and $\mathbf{P}^T P = I$. We have that

$$(PQ)^T (PQ) = \mathbf{Q}^T \mathbf{P}^T P I Q = \mathbf{Q}^T Q = I$$

Therefore, a product of two orthogonal matrix is an orthogonal matrix.

Problem 2.2.27

If A is orthogonal, show that its determinant $= \pm 1$.

Solution We know that

$$\det \mathbf{A}^T = \det \mathbf{A}$$

$$\mathbf{A}^T \mathbf{A} = I$$

$$\det \mathbf{A}^T = \det I = 1$$

$$\det \mathbf{A}^T \mathbf{A} \det \mathbf{A}^T \det \mathbf{A}$$

$$(\det \mathbf{A})^2 = \det \mathbf{A} \det \mathbf{A}^T = \det \mathbf{A}^T \det \mathbf{A} = \det \mathbf{A}^T A = 1$$

So we must have

$$\det \mathbf{A} = \pm 1$$

> **Problem 2.2.28**
>
> Show that the trace of the product of a symmetric and an antisymmetric matrix is zero.

Solution If $\tilde{\mathbf{A}} = -\mathbf{A}, \tilde{\mathbf{S}} = \mathbf{S}$, then

$$\mathrm{Tr}(\widetilde{\mathbf{S}\mathbf{A}}) = \mathrm{Tr}(\mathbf{S}\mathbf{A}) = \mathrm{Tr}(\tilde{\mathbf{A}}\tilde{\mathbf{S}}) = -\mathrm{Tr}(\mathbf{A}\mathbf{S})$$

Problem 2.2.29

A is 2×2 and orthogonal. Find the most general form of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution From $\tilde{\mathbf{A}} = \mathbf{A}^{-1}$ and $\det(\mathbf{A}) = 1$ we have

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

This gives $\det(\mathbf{A}) = a_{11}^2 + a_{12}^2 = 1$, hence

$$a_{11} = \cos\theta = a_{22}, \quad a_{12} = \sin\theta = -a_{21},$$

the standard 2×2 rotation matrix.

Problem 2.2.30

Show that
$$\det\left(\mathbf{A}^*\right) = (\det \mathbf{A})^* = \det\left(\mathbf{A}^\dagger\right)$$

Solution We calculate the determinant of $\mathbf{A}^*$

$$\det\left(\mathbf{A}^*\right) = \sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a^*_{1 i_1} a^*_{2 i_2} \cdots a^*_{n i_n} = \left(\sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a_{1 i_1} a_{2 i_2} \cdots a_{n i_n}\right)$$

Because, for any $\mathbf{A}$,

$$\det(\mathbf{A}) = \det(\tilde{\mathbf{A}}), \det\left(\mathbf{A}^*\right) = \det\left(\mathbf{A}^\dagger\right)$$

Problem 2.2.31

Three angular momentum matrices satisfy the basic commutation relation

$$[\mathbf{J}_x, \mathbf{J}_y] = i\mathbf{J}_z$$

(and cyclic permutation of indices). If two of the matrices have real elements, show that the elements of the third must be pure imaginary.

Solution We know that basic commutation relation is $[\mathbf{J}_i, \mathbf{J}_j] = i\mathbf{J}_k$, where i j and k are indices in cyclic permutation. Here it is clear that $\mathbf{J}_x, \mathbf{J}_y$ are real, so also must be their commutator. So according to commutator rule it requires that $\mathbf{J}_z$ be pure imaginary.

Problem 2.2.32

Show that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$

Solution We know that the transpose of a product is $(AB)^T = B^T A^T$. That is product is transposed by taking, in reverse order, the transpose of its factor. From this

$$(AB)^\dagger = (A^* B^*)^T = (B^*)^T (A^*)^T = B^\dagger A^\dagger$$

The complex conjugate of product is equal to conjugate of its individual factor.

Problem 2.2.33

A matrix $\mathbf{C} = \mathbf{S}^\dagger \mathbf{S}$. Show that the trace is positive definite unless $\mathbf{S}$ is the null matrix, in which case $\mathrm{Tr}(\mathbf{C}) = 0$.

Solution As

$$\mathbf{C}_{jk} = \sum_n \boldsymbol{S}_{nj}^* \boldsymbol{S}_{nk}$$

$$\mathrm{Tr}(\mathbf{C}) = \sum_{nj} |\boldsymbol{S}_{nj}|^2$$

Problem 2.2.34

If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices, show that $(\mathbf{AB} + \mathbf{BA})$ and $i(\mathbf{AB} - \mathbf{BA})$ are also Hermitian.

$\boxed{\textbf{Solution}}$ If $\mathbf{A}^\dagger = \mathbf{A}, \mathbf{B}^\dagger = B$, then

$$(\mathbf{AB} + \mathbf{BA})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger + \mathbf{A}^\dagger\mathbf{B}^\dagger = \mathbf{AB} + \mathbf{BA}$$

$$-i\left(\mathbf{B}^\dagger\mathbf{A}^\dagger - \mathbf{A}^\dagger\mathbf{B}^\dagger\right) = i(\mathbf{AB} - \mathbf{BA})$$

Problem 2.2.35

The matrix $\mathbf{C}$ is not Hermitian. Show that then $\mathbf{C} + \mathbf{C}^\dagger$ and $i\left(\mathbf{C} - \mathbf{C}^\dagger\right)$ are Hermitian. This means that a non-Hermitian matrix may be resolved into two Hermitian parts,

$$\mathbf{C} = \frac{1}{2}\left(\mathbf{C} + \mathbf{C}^\dagger\right) + \frac{1}{2i}i\left(\mathbf{C} - \mathbf{C}^\dagger\right)$$

This decomposition of a matrix into two Hermitian matrix parts parallels the decompo- sition of a complex number z into $x + iy$, where $x = \left(z + z^*\right)/2$ and $y = \left(z - z^*\right)/2i$

$\boxed{\text{Solution}}$ If $\mathbf{C}^\dagger \neq \mathbf{C}$, then

$$\left(i\mathbf{C}_-\right)^\dagger \equiv \left(\mathbf{C}^\dagger - \mathbf{C}\right)^\dagger = \mathbf{C} - \mathbf{C}^\dagger = -i\mathbf{C}_-^\dagger,$$

$$\left(\mathbf{C}_-\right)^\dagger = \mathbf{C}_-$$

$$\mathbf{C}_+^\dagger = \mathbf{C}_+ = \mathbf{C} + \mathbf{C}^\dagger$$

Problem 2.2.36

A and **B** are two noncommuting Hermitian matrices:

$$\mathbf{AB} - \mathbf{BA} = i\mathbf{C}$$

Prove that **C** is Hermitian.

Solution Let's consider

$$-i\mathbf{C}^\dagger = (\mathbf{AB} - \mathbf{BA})^\dagger$$
$$-i\mathbf{C}^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger - \mathbf{A}^\dagger\mathbf{B}^\dagger$$
$$-i\mathbf{C}^\dagger = B\mathbf{A} - \mathbf{AB}$$
$$-i\mathbf{C}^\dagger = -i\mathbf{C}$$

Problem 2.2.37

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are each Hermitian. Find a necessary and sufficient condition for their product $\mathbf{AB}$ to be Hermitian.

Solution

$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger = \mathbf{BA} = \mathbf{AB}$$

With this, we can say that $[\mathbf{A}, \mathbf{B}] = 0$

Problem 2.2.38

Show that the reciprocal (that is, inverse) of a unitary matrix is unitary.

Solution A matrix is said to be unitary matrix if its adjoint is equal to its inverse. Let $\mathbf{U}$ be unitary matrix. Then $\mathbf{U}^\dagger = \mathbf{U}^{-1}$.

$$\left(\mathbf{U}^\dagger\right)^\dagger = \mathbf{U}$$

$$\left(\mathbf{U}^\dagger\right)^\dagger = \left(\mathbf{U}^{-1}\right)^\dagger$$

Problem 2.2.39

Prove that the direct product of two unitary matrices is unitary.

Solution

$$(\mathbf{U}_1\mathbf{U}_2)^\dagger = \mathbf{U}_2^\dagger \mathbf{U}_1^\dagger$$

$$(\mathbf{U}_1\mathbf{U}_2)^\dagger = \mathbf{U}_2^{-1} \mathbf{U}_1^{-1}$$

$$(\mathbf{U}_1\mathbf{U}_2)^\dagger = (\mathbf{U}_1\mathbf{U}_2)^{-1}$$

Problem 2.2.40

If σ is the vector with the σ_i as components given in Eq. (2.61), and p is an ordinary vector, show that

$$(\sigma \cdot p)^2 = p^2 \hat{1}_2$$

where $\hat{1}_2$ is a 2×2 unit matrix.

Solution

$$(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = (p_x \sigma_1 + p_y \sigma_2 + p_z \sigma_3)^2$$

$$p_x^2 \sigma_1^2 + p_y^2 \sigma_2^2 + p_z^2 \sigma_3^2 + p_x p_y (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) + p_x p_z (\sigma_1 \sigma_3 + \sigma_3 \sigma_1)$$

$$+ p_y p_z (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) = p_x^2 + p_y^2 + p_z^2 = \mathbf{p}^2$$

Problem 2.2.41

Use the equations for the properties of direct products, Eqs. (2.57) and (2.58), to show that the four matrices $\gamma^\mu, \mu = 0, 1, 2, 3$, satisfy the conditions listed in Eqs. (2.74) and (2.75).

Solution Writing $\gamma^0 = \sigma_3 \otimes \mathbf{1}$ and $\gamma^i = \gamma \otimes \sigma_i (i = 1, 2, 3)$, where

$$\gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and noting fron Eq. (2.57) that if $\mathbf{C} = \mathbf{A} \otimes B$ and $\mathbf{C}' = \mathbf{A}' \otimes B'$ then $\mathbf{CC}' = \mathbf{AA}' \otimes BB'$

$$\left(\gamma^0\right)^2 = \sigma_3^2 \otimes \mathbf{1}_2^2 = \hat{1}_2 \otimes \mathbf{1}_2 = \hat{1}_4, \quad \left(\gamma^i\right)^2 = \gamma^2 \otimes \sigma_i^2 = \left(-\hat{1}_2\right) \otimes \mathbf{1}_2 = -\hat{1}_4$$

$$\gamma^0 \gamma^i = \sigma_3 \gamma \otimes \mathbf{1}_2 \sigma_i = \sigma_1 \otimes \sigma_i, \quad \gamma^i \gamma^0 = \gamma \sigma_3 \otimes \sigma_i \mathbf{1}_2 = \left(-\sigma_1\right) \otimes \sigma_i$$

$$\gamma^i \gamma^j = \gamma^2 \otimes \sigma_i \sigma_j \quad \gamma^j \gamma^i = \gamma^2 \otimes \sigma_j \sigma_i$$

It is obvious from the second line of the above equation set that $\gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$; from the third line of the equation set we find $\gamma^i \gamma^j + \gamma^j \gamma^i$ is zero if $j \neq i$ because then $\sigma_j \sigma_i = -\sigma_i \sigma_j$

Show that γ^5, Eq. (2.76), anticommutes with all four γ^μ.

Solution

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \gamma^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\gamma^0\gamma^5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\gamma^5\gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\gamma^1\gamma^5 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\gamma^5\gamma^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\gamma^2\gamma^5 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$\gamma^5\gamma^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$\gamma^3\gamma^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma^5\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

This shows that γ^5 anticommutes with all four $\gamma^\mu (\mu = 0, 1, 2, 3)$

Problem 2.2.43

In this problem, the summations are over $\mu = 0, 1, 2, 3$. Define $g_{\mu v} = g^{\mu v}$ by the relations

$$g_{00} = 1; \quad g_{kk} = -1, \quad k = 1, 2, 3; \quad g_{\mu v} = 0, \quad \mu \neq v$$

and define γ_μ as $\sum g_{v\mu} \gamma^\mu$. Using these definitions, show that

(a) $\sum \gamma_\mu \gamma^\alpha \gamma^\mu = -2\gamma^\alpha$

(b) $\sum \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4g^{\alpha\beta}$

(c) $\sum \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^v \gamma^\mu = -2\gamma^v \gamma^\beta \gamma^\alpha$

Solution No solution yet.

Problem 2.2.44

If $\mathbf{M} = \frac{1}{2}\left(1 + \gamma^5\right)$, where γ^5 is given in Eq. (2.76), show that

$$\mathbf{M}^2 = \mathbf{M}$$

Note that this equation is still satisfied if γ is replaced by any other Dirac matrix listed in Eq. (2.76)

Solution Consider $\mathbf{M}^2 = \left[\frac{1}{2}\left(1 + \gamma^5\right)\right]^2$

$$
\begin{aligned}
&= \frac{1}{4}\left(\hat{1}_4 + 2\gamma^5 + \left(\gamma^5\right)^2\right) \\
&= \frac{1}{4}\left(\hat{1}_4 + 2\gamma^5 + \hat{1}_4\right) \\
&= \frac{1}{4}\left(2\hat{1}_4 + 2\gamma^5\right) \\
&= \frac{1}{2}\left(\hat{1}_4 + \gamma^5\right) \\
&= \mathbf{M}
\end{aligned}
$$

Thus, $\mathbf{M}^2 = \mathbf{M}$

Problem 2.2.45

Prove that the 16 Dirac matrices form a linearly independent set.

Solution No solution yet.

Problem 2.2.46

If we assume that a given 4×4 matrix $\mathbf{A}$ (with constant elements) can be written as a linear combination of the 16 Dirac matrices (which we denote here as Γ_i)

$$\mathbf{A} = \sum_{i=1}^{16} c_i \Gamma_i$$

show that

$$c_i \sim \operatorname{trace}(\mathbf{A}\Gamma_i)$$

The matrix $\mathbf{C} = i\gamma^2\gamma^0$ is sometimes called the charge conjugation matrix. Show that $\mathbf{C}\boldsymbol{\gamma}^\mu \mathbf{C}^{-1} = -\left(\boldsymbol{\gamma}^\mu\right)^T$

Solution No solution yet.

Problem 2.2.47

The matrix $\mathbf{C} = i\gamma^2\gamma^0$ is sometimes called the charge conjugation matrix. Show that $\mathbf{C}\gamma^\mu\mathbf{C}^{-1} = -\left(\gamma^\mu\right)^T$

Solution Here

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Consider $\mathbf{C}\gamma^0\mathbf{C}^{-1} = i\gamma^2\gamma^0\gamma^0\left(i\gamma^2\gamma^0\right)^{-1}$

$$\begin{aligned}
&= \gamma^2\gamma^0\gamma^0\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= -\gamma^0 \\
&= -\left(\gamma^0\right)^T \\
\mathbf{C}\gamma^1\mathbf{C}^{-1} &= i\gamma^2\gamma^0\gamma^1\left(i\gamma^2\gamma^0\right)^{-1} \\
&= \gamma^2\gamma^0\gamma^1\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= \gamma^1 \\
&= -\left(\gamma^1\right)^T
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}\gamma^2\mathbf{C}^{-1} &= i\gamma^2\gamma^0\gamma^2\left(i\gamma^2\gamma^0\right)^{-1} \\
&= \gamma^2\gamma^0\gamma^2\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= -\gamma^2 \\
&= -\left(\gamma^2\right)^T \\
\mathbf{C}\gamma^3\mathbf{C}^{-1} &= i\gamma^2\gamma^0\gamma^3\left(i\gamma^2\gamma^0\right)^{-1} \\
&= \gamma^2\gamma^0\gamma^3\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= \gamma^3 \\
&= -\left(\gamma^3\right)^T
\end{aligned}$$

Thus, $\mathbf{C}\gamma^\mu\mathbf{C}^{-1} = -\left(\gamma^\mu\right)^T \ (\mu = 0, 1, 2, 3)$

Problem 2.2.48

(a) Show that, by substitution of the definitions of the γ^μ matrices from Eqs. (2.70) and (2.72), that the Dirac equation, Eq. (2.73), takes the following form when written as 2×2 blocks (with ψ_L and ψ_S column vectors of dimension 2). Here L and S stand, respectively, for "large" and "small" because of their relative size in the nonrelativistic limit):

$$\begin{bmatrix} mc^2 - E & c\left(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3\right) \\ -c\left(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3\right) & -mc^2 - E \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_S \end{bmatrix} = 0$$

(b) To reach the nonrelativistic limit, make the substitution $\mathbf{E} = mc^2 + \varepsilon$ and approximate $-2mc^2 - \varepsilon$ by $-2mc^2$. Then write the matrix equation as two simultaneous two-component equations and show that they can be rearranged to yield

$$\frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right)\psi_L = \varepsilon\psi_L$$

which is just the Schrödinger equation for a free particle.

(c) Explain why is it reasonable to call ψ_L and ψ_S "large" and "small."

Solution No solution yet.

Problem 2.2.49

Show that it is consistent with the requirements that they must satisfy to take the Dirac gamma matrices to be (in 2×2 block form)

$$\gamma^0 = \begin{bmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \quad (i = 1, 2, 3)$$

This choice for the gamma matrices is called the Weyl representation.

Solution If $\mathbf{C} = \mathbf{A} \otimes B$ and $\mathbf{C}' = \mathbf{A}' \otimes B'$ then $\mathbf{CC}' = \mathbf{AA}' \otimes BB'$ we have

$$\begin{aligned}
\left(\gamma^0\right)^2 &= \sigma_2^2 \otimes \hat{1}_2^2 \\
&= \hat{1}_2 \otimes \hat{1}_2 \quad \text{as } \sigma_i^2 = 1 \\
&= \hat{1}_4 \\
&= 1
\end{aligned}$$

as $\gamma^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\hat{1}_2$

$$\begin{aligned}
\left(\gamma'\right)^2 &= \gamma^2 \otimes \sigma_i^2 \\
&= \left(-\hat{1}_2\right) \otimes \hat{1}_2 \\
&= -\hat{1}_4 \\
&= -1
\end{aligned}$$

as $\sigma_1\gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -\sigma_3$

$$\begin{aligned}
\gamma^0\gamma^i &= \sigma_1\gamma \otimes \mathbf{1}_2\sigma_i \\
&= \left(-\sigma_3\right) \otimes \sigma_i \\
\gamma^i\gamma^0 &= \gamma\sigma_1 \otimes \sigma_i\mathbf{1}_2 \\
&= \sigma_3 \otimes \sigma_i
\end{aligned}$$

as $\gamma\sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3$ Thus $\gamma^0\gamma^i + \gamma'\gamma^0 = 0$

$$\begin{aligned}
\gamma^i\gamma^j &= \gamma^2 \otimes \sigma_i\sigma_j \\
\gamma^j\gamma^i &= \gamma^2 \otimes \sigma_j\sigma_i = \gamma^2 \otimes \left(-\sigma_i\sigma_j\right)
\end{aligned}$$

as $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$. Thus, $\gamma^i\gamma^j + \gamma^j\gamma^i = 0$ if $j \neq i$

Problem 2.2.50

Show that the Dirac equation separates into independent 2×2 blocks in the Weyl representation (see Exercise 2.2 .49) in the limit that the mass m approaches zero. This observation is important in the ultra relativistic regime where the rest mass is inconsequential, or for particles of negligible mass (e.g., neutrinos).

Solution In the Weyl representation, the matrices γ^0, α_i and the wave function ψ written as 2×2 blocks take the forms

$$\gamma^0 = \begin{bmatrix} 0 & \hat{1}_2 \\ \hat{1}_2 & 0 \end{bmatrix}, \quad \alpha_i = \begin{bmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

In block form $\left[\gamma^0 mc^2 + \alpha \cdot p\right]\psi = E\psi$ becomes

$$\left[\begin{bmatrix} 0 & mc^2 \\ mc^2 & 0 \end{bmatrix} + \begin{bmatrix} -\sigma \cdot p & 0 \\ 0 & \sigma \cdot p \end{bmatrix}\right]\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = E \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

If m is zero, this matrix equation becomes two independent equations, one for ψ_1, and one for ψ_2 In this limit, one set of solutions will be with $\psi_2 = 0$ and ψ_1 a solution to $-\sigma \cdot p\psi_1 = E\psi_1$ and a second set of solutions will have $\psi_1 = 0$ and a set of ψ_2 identical to the previously found set of ψ_1 but with values of E of the opposite sign.

Problem 2.2.51

(a) Given $\mathbf{r}' = \mathbf{U}\mathbf{r}$, with $\mathbf{U}$ a unitary matrix and $\mathbf{r}$ a (column) vector with complex elements, show that the magnitude of $\mathbf{r}$ is invariant under this operation.

(b) The matrix $\mathbf{U}$ transforms any column vector $\mathbf{r}$ with complex elements into $\mathbf{r}'$, leaving the magnitude invariant: $\mathbf{r}^\dagger \mathbf{r} = \mathbf{r}'^\dagger \mathbf{r}'$. Show that $\mathbf{U}$ is unitary.

$\boxed{\textbf{Solution}}$ For (a) We show that the magnitude of r is invariant i.e. $\mathbf{r}'^\dagger \mathbf{r}' = \mathbf{r}^\dagger r$. Consider
$\mathbf{r}'^\dagger \mathbf{r}' = (\mathbf{U}\mathbf{r})^\dagger \mathbf{U}\mathbf{r}$

$$= \mathbf{r}^\dagger \mathbf{U}^\dagger \mathbf{U}\mathbf{r}$$
$$= \mathbf{r}^\dagger 1 \mathbf{r}$$
$$= \mathbf{r}^\dagger \mathbf{r}$$

This shows that the magnitude of r is invariant under this operation.

$\boxed{\textbf{Solution}}$ For (b) all r, $\quad \mathbf{r}'^\dagger \mathbf{r}' = \mathbf{r}^\dagger r$

$$(\mathbf{U}\mathbf{r})^\dagger \mathbf{U}\mathbf{r} = \mathbf{r}^\dagger r$$
$$\mathbf{r}^\dagger \mathbf{U}^\dagger \mathbf{U}\mathbf{r} = \mathbf{r}^\dagger 1 r$$
$$\mathbf{U}^\dagger \mathbf{U} = 1$$

This shows that U is unitary.