

**Problem 13.4.1**

Rewrite Stirling's series to give  $\Gamma(z+1)$  instead of  $\ln \Gamma(z+1)$

$$\text{ANS. } \Gamma(z+1) = \sqrt{2\pi} z^{z+1/2} e^{-z} \left( 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51,840z^3} + \dots \right)$$

**Solution** Consider the Stirling's formula:

$$\ln \Gamma(z+1) = \frac{1}{2} \ln 2\pi + \left( z + \frac{1}{2} \right) \ln z - z + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}}$$

Where  $B_{2n}$  are the Bernoulli's numbers. Use the first few Bernoulli's numbers and rewrite the above Stirling's formula as equivalent to

$$\ln \Gamma(z+1) \sim \frac{1}{2} \ln(2\pi) + \left( z + \frac{1}{2} \right) \ln z - z + \frac{1}{12z} - \frac{1}{360z^2} + \frac{1}{1260z^3} - \dots$$

The Stirling's formula can be rewritten using Gamma function as follows. Let us take exponential form and collect similar terms to get equivalent form as follows.

$$\Gamma(z+1) \sim \sqrt{2\pi} + z^{(z+\frac{1}{2})} e^{-z} \left( 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \dots \right)$$

Hence, the required result is

$$\Gamma(z+1) \sim \sqrt{2\pi} + z^{(z+\frac{1}{2})} e^{-z} \left( 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \dots \right)$$

**Problem 13.4.5**

Test the convergence

$$\sum_{p=0}^{\infty} \left[ \frac{\Gamma(p+\frac{1}{2})}{p!} \right]^2 \frac{2p+1}{2p+2} = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

**Solution** Consider the series obtained in the magnetic field created by and enclosed by a current loop:

$$\sum_{p=0}^{\infty} \frac{\Gamma(p+\frac{1}{2})}{p!} \left( \frac{2p+1}{2p+2} \right) = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

Now, we will test the convergence of the series using Stirling asymptotic formula given by

$$\Gamma(z+1) \sim \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}$$

$$\begin{aligned} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} &\sim \sqrt{e} \frac{\left( \frac{p+\frac{1}{2}}{p+1} \right)^{p+\frac{1}{2}}}{\Gamma(p+1)} \\ &= \frac{\text{constant}}{\Gamma(p+1)} \end{aligned}$$

Hence, the series converges.

**Problem 13.4.6**

Show that

$$\lim_{x \rightarrow \infty} x^{b-a} \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1$$

**Solution** For a large  $n$ , the Stirling asymptotic formula can be taken to the  $n$  arbitrary closed to infinite. Then the expression has asymptotic limit.

$$\begin{aligned} & \ln \left[ (x^{b-a}) \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \right] \\ &= (b-a) \ln x \left( \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \right) \\ &= (b-a) \ln(x) + \ln \left( \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \right) \\ &= (b-a) \ln(x) + \ln \Gamma(x+a+1) - \ln \Gamma(x+b+1) \end{aligned}$$

Now we use

$$\ln \Gamma(z+1) = \left( z + \frac{1}{z} \right) \ln z - z$$

Now,  $(b-a) \ln(x) + \ln \Gamma(x+a+1) - \ln \Gamma(x+b+1)$  it reduces to

$$\begin{aligned} & (b-a) \ln(x) + \ln \Gamma(x+a+1) - \ln \Gamma(x+b+1) \\ &= -(x+a) - \left( x+b + \frac{1}{2} \right) \ln(x+b) + (x+b) \\ &= (b-a) \ln(x) + (a-b) \ln(x) \end{aligned}$$

Rewrite the  $\ln(x+a)$  as follows.

$$\begin{aligned} \ln(a+x) &= \ln x \left( 1 + \frac{a}{x} \right) \\ &= \ln x + \ln \left( 1 + \frac{a}{x} \right) \\ &= \ln x + \frac{a}{x} + \dots \end{aligned}$$

Now rewrite the  $\ln(x+b)$

$$\begin{aligned} \ln(b+x) &= \ln x \left( 1 + \frac{b}{x} \right) \\ &= \ln x + \ln \left( 1 + \frac{b}{x} \right) \\ &= \ln x + \frac{b}{x} + \dots \end{aligned}$$

For large  $x$ , make all the terms to exponential form. So, that  $\exp(0) = 1$ . Hence, the limit tends to 1.

$$\lim_{x \rightarrow \infty} x^{b-a} \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1$$

**Problem 13.4.7**

Show that

$$\lim_{n \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{1/2} = \pi^{-1/2}$$

**Solution** Write the limit expression in factorial notations. Then it is easy to apply the Stirling formula

$$\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \lim_{x \rightarrow \infty} \frac{(2n)!}{2^{2n}(n!)^2} n^{\frac{1}{2}}$$

Take logarithm for the limit

$$\ln \left( \lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} \right) = \ln \left( \lim_{x \rightarrow \infty} \frac{(2n)!}{2^{2n}(n!)^2} n^{\frac{1}{2}} \right)$$

Consider the right hand side of the above equation and solve.

$$\begin{aligned} & \ln \lim_{n \rightarrow \infty} \frac{(2n)! n^{\frac{1}{2}}}{2^{2n}(n!)^2} \\ &= \lim_{n \rightarrow \infty} \ln(2n)! + \frac{1}{2} \ln n - 2n \ln 2 - 2 \ln(n!) \\ &= \frac{\ln(2\pi)}{2} + \left( 2n + \frac{1}{2} \right) \ln(2n) - 2n + \frac{\ln n}{2} \\ &\approx -2n \ln 2 - \ln(2\pi) - 2 \left( n + \frac{1}{2} \right) \ln n + 2n + \dots \\ &\sim -\frac{1}{2} \ln \pi \\ &= \ln \pi^{-\frac{1}{2}} \end{aligned}$$

Substitute the value of right hand side limit

$$\begin{aligned} \ln \left( \lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} \right) &= \ln \pi^{-\frac{1}{2}} \\ \lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} &= \pi^{-\frac{1}{2}} \end{aligned}$$

Hence, the limit tends to

$$\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \pi^{-\frac{1}{2}}$$