Title

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Problem 3.3.1

A rotation $\varphi_1+\varphi_2$ about the z -axis is carried out as two successive rotations φ_1 and φ_2, each about the z-axis. Use the matrix representation of the rotations to derive the trigonometric identities

Solution

$$\cos(\varphi_1+\varphi_2) = \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2$$
$$\sin(\varphi_1+\varphi_2) = \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2$$

$$\begin{bmatrix} \cos(\varphi_1+\varphi_2) \sin(\varphi_1+\varphi_2) \\ -\sin(\varphi_1+\varphi_2)\cos(\varphi_1+\varphi_2) \end{bmatrix} = \begin{bmatrix} \cos\varphi_2 \sin\varphi_2 \\ -\sin\varphi_2 \cos\varphi_2 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 \sin\varphi_1 \\ -\sin\varphi_1 \cos\varphi_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 \\ -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{bmatrix}$$

Problem 3.3.2

A corner reflector is formed by three mutually perpendicular reflecting suffices. Show that a a ay of light incident tupon the cometor (striking all three surfaces) is reflected back along a line parallel to the line of incidence. Hint. Consider the effect of a reflection on the components of a vector describing the direction of the light ray.

Solution Here we are asked prove that the ray of light incident upon the corner reflector is reflected of back along line parallel to line of incidence. So for this align the reflecting surfaces with xy, xz, and yz planes. If an incoming ray strikes the xy plane, the z component of its direction of propagation is reversed. A strike on the xz plane reverses its y component, and a strike on yz plane reverses its x component.

Problem 3.3.3

Let x and y be column vectors. Under an orthogonal transformation S, they become $x' = Sx$ and $y' = Sy$. Show that $(x')^T y' = x^T y$, a result equivalent to the invariance of the dot product under a rotational transformation.

Solution It is given that S is orthogonal, if so its transpose is also its inverse. From this

$$(x')^T = (Sx)^T = x^T\mathbf{S}^T = x^T\mathbf{S}^{-1}$$

Then

$$(x')^T y' = x^T\mathbf{S}^{-1}Sy = x^T y$$

Therefore $(x')^T y' = x^T y$

Problem 3.3.4

Given the orthogonal transformation matrix S and vectors a and $\mathbf{b}$,

$$S = \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

(a) Calculate $\det(S)$.

(b) Verify that $\mathbf{a}\cdot\mathbf{b}$ is invariant under application of $\mathbf{S}$ to $\mathbf{a}$ and $\mathbf{b}$.

(c) Determine what happens to $\mathbf{a}\times\mathbf{b}$ under application of $\mathbf{S}$ to $\mathbf{a}$ and $\mathbf{b}$. Is this what is expected?

$\boxed{\textbf{Solution}}$ For (a) given

$$S = \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix}$$

$$\det(S) = \det \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} = 1$$

$\boxed{\textbf{Solution}}$ For (b) we show that $a \cdot b$ is invariant under application of $\mathbf{S}$ to a and b.

$$\begin{aligned} \mathbf{a}' &= \mathbf{Sa} \\ &= \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.80 \\ 0.12 \\ 1.16 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b}' &= \mathbf{Sb} \\ &= \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1.20 \\ 0.68 \\ -1.76 \end{bmatrix} \end{aligned}$$

$$a \cdot b = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = -1$$

$$a' \cdot b' = \begin{bmatrix} 0.80 & 0.12 & 1.16 \end{bmatrix} \begin{bmatrix} 1.20 \\ 0.68 \\ -1.76 \end{bmatrix} = -1$$

Thus, $a \cdot b$ is invariant under application of $\mathbf{S}$ to a and b.

$\boxed{\textbf{Solution}}$ For (c) we find $\mathbf{a} \times \mathbf{b}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}(\mathbf{a} \times \mathbf{b}) &= \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 2.8 \\ 0.4 \end{bmatrix} \end{aligned}$$

$$\mathbf{a}' \times \mathbf{b}' = \begin{vmatrix} i & j & k \\ 0.80 & 0.12 & 1.16 \\ 1.20 & 0.68 & -1.76 \end{vmatrix} = \begin{bmatrix} -1 \\ 2.8 \\ 0.4 \end{bmatrix}$$

Thus, $\mathbf{S}(\mathbf{a} \times \mathbf{b}) = \mathbf{a}' \times \mathbf{b}'$ and hence $\mathbf{a} \times \mathbf{b}$ is a vector.

Problem 3.3.5

Using $\mathbf{a}$ and $\mathbf{b}$ as defined in Exercise 3.3.5 but with

$$S = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

(a) Calculate $\det(S)$.

(b) $\mathbf{a} \times \mathbf{b}$

(c) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Solution For (a) Given that

$$S = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix}$$

Then

$$\det(S) = \det \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} = 1$$

Apply $\mathbf{S}$ to $\mathbf{a}, \mathbf{b}$, and c.

$$\begin{aligned} \mathbf{a}' &= \mathbf{S}\mathbf{a} \\ &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.40 \\ -0.16 \\ -0.12 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b}' &= \mathbf{S}\mathbf{b} \\ &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -0.80 \\ -1.68 \\ 1.24 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c}' &= \mathbf{S}\mathbf{c} \\ &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3.60 \\ -0.44 \\ 0.92 \end{bmatrix} \end{aligned}$$

Now, we determine what happen to $\mathbf{a} \times \mathbf{b}$ under application of $\mathbf{S}$ to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Solution For (b)

$$(a \times b) = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}(\mathbf{a} \times \mathbf{b}) &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0.40 \\ 1.64 \\ 2.48 \end{bmatrix} \end{aligned}$$

$$\mathbf{a}' \times \mathbf{b}' = \begin{vmatrix} i & j & k \\ 1.40 & -0.16 & -0.12 \\ -0.80 & -1.68 & 1.24 \end{vmatrix} = \begin{bmatrix} -0.40 \\ -1.64 \\ -2.48 \end{bmatrix}$$

Thus, $\mathbf{S}(\mathbf{a} \times \mathbf{b}) = \mathbf{a}' \times \mathbf{b}'$

$\boxed{\textbf{Solution}}$ For (c) we determine what happen to $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ under application of $\mathbf{S}$ to $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{bmatrix} -2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = -4 + 1 + 6 = 3$$

$$(\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}' = \begin{bmatrix} -0.40 & -1.64 & -2.48 \end{bmatrix} \cdot \begin{bmatrix} 3.60 \\ -0.44 \\ 0.92 \end{bmatrix} = -3$$

Thus, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}'$

$\boxed{\textbf{Solution}}$ For (d) We now determine what happen to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ under application of $\mathbf{S}$ to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$$

$$\mathbf{S}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix} = \begin{bmatrix} -0.40 \\ -8.84 \\ 7.12 \end{bmatrix}$$

$$\mathbf{a}' \times (\mathbf{b}' \times \mathbf{c}') = \begin{vmatrix} 1.40 & -0.16 & -0.12 \\ -0.80 & -1.68 & 1.24 \\ 3.60 & -0.44 & 0.92 \end{vmatrix} = \begin{bmatrix} -0.40 \\ -8.84 \\ 7.12 \end{bmatrix}$$

Thus, $\quad \mathbf{S}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \mathbf{a}' \times (\mathbf{b}' \times \mathbf{c}')$