Title

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Problem 13.1.1

Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral, Eq. (13.5),

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$$

Solution Consider the Euler integral

$$\Gamma z = \int_0^\infty e^{-t}t^{z-1}dt$$

Put, $z = z+1$

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t}t^{z+1-1}dt \\ &= \int_0^\infty e^{-t}t^{z}dt \\ &= t^z\int_0^\infty e^{-t}dt - \int_0^\infty \frac{dt^z}{dx}\int e^{-t}dt \\ =& -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}dt \\ =& z\Gamma(z)\end{aligned}$$

Problem 13.1.2

In a power-series solution for the Legendre functions of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

in which s is a positive integer.

(a) Rewrite this expression in terms of factorials.

(b) Rewrite this expression using Pochhammer symbols; see Eq. (1.72).

Solution For (a) Notice that

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2.4.6.8\cdots\cdot(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$= \frac{[n!(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)]}{n!s!2^s\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$= \frac{(n+2s)!(2n+1)!}{n!s!2^s\cdot[(2n+1)!(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$= \frac{(n+2s)!(2n+1)![(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{n!s!2^s\cdot[(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]}$$

$$= \frac{(n+2s)!(2n+1)!2^s[(n+1)(n+2)(n+3)\cdots(n+s)]}{n!s!2^s\cdot[(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]}$$

$$= \frac{(n+2s)!(2n+1)![n!(n+1)(n+2)(n+3)\cdots(n+s)]}{n!s!n![(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]}$$

$$= \frac{(n+2s)!(2n+1)!(n+s)!}{n!n!s!(2n+2s+1)!}$$

Solution For (b) we notice that

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdot\cdots\cdot(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$= \frac{(n+1)(n+2)(n+3)\cdots[(n+1)+(2s-2)][(n+1)+(2s-1)]}{(2^s[1\cdot 2\cdot 3\cdot\cdots\cdot(s-1)s])\cdot[(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)]}$$

$$= \frac{(n+1)_{(2s-1)+1}\cdot[(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{(2^s[1\cdot 2\cdot 3\cdot\cdots\cdot\{1+(s-2)\}\{1+(s-1)\}\})\cdot[(2n+2)(2n+3)(2n+4)(2n+5)\cdots(2n+2s)(2n+2s+1)]}$$

$$= \frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots(n+s)]\cdot 2^s}{2^s(1)_{(s-1)+1}\cdot[(2n+2)(2n+3)(2n+4)\cdots\{(2n+2)+(2s-1)\}]}$$

$$= \frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots\{(n+1)+(s-1)\}]}{(1)_s\cdot(2n+2)_{(2s-1)+1}}$$

$$= \frac{(n+1)_{2s}\cdot(n+1)_{(s-1)+1}}{(1)_s\cdot(2n+2)_{2s}}$$

$$= \frac{(n+1)_{2s}\cdot(n+1)_s}{(1)_s\cdot(2n+2)_{2s}}$$

Problem 13.1.3

Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2\int_0^\infty e^{-t^2}t^{2z-1}dt, \quad \mathrm{Re}(z) > 0$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re e(z) > 0$$

Solution Changing variables $t = u^2$ and $dt = 2udu$ we have

$$\begin{aligned}\Gamma z &= \int_0^\infty e^{-u^2}u^{2z-2}udu \\ &= \int_0^\infty e^{-u^2}u^{2z-1}du \\ &= \int_0^\infty e^{-t^2}t^{2z-1}dt\end{aligned}$$

as $t \to 0$ to $\infty u \to 0$ to 1 the equation takes the form of

$$\begin{aligned}\Gamma z &= \int_0^1 e^{-\ln\frac{1}{u}}\left(\ln\frac{1}{u}\right)^{z-1} udu \\ &= \int_0^1 u\left(\ln\frac{1}{u}\right)^{z-1} udu \\ &= \int_0^1 \left(\ln\frac{1}{u}\right)^{z-1} du \\ &= \int_0^1 \left(\ln\frac{1}{t}\right)^{z-1} dt\end{aligned}$$

Problem 13.1.4

In a Maxwellian distribution the fraction of particles of mass m with speed between v and $v+dv$ is

$$\frac{dN}{N} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{mv^2}{2kT}\right)v^2dv$$

where N is the total number of particles, k is Boltzmann's constant, and T is the absolute temperature. The average or expectation value of v^n is defined as $\langle v^n\rangle = N^{-1}\int v^n dN$. Show that

$$\langle v^n\rangle = \left(\frac{2kT}{m}\right)^{n/2}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

This is an extension of Example 13.1.1, in which the distribution was in kinetic energy $E = mv^2/2$, with $dE = mvdv$

Solution

$$\begin{aligned}
\langle v^n\rangle &= N^{-1}\int v^n dN \\
&= \int v^n \frac{dN}{N} \\
&= \int_0^\infty v^n\cdot 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{\frac{m^2}{2kT}}v^2dv \\
&= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^\infty v^n e^{\frac{m^2}{2kT}}v^{n+1}vdv
\end{aligned}$$

Let $\frac{mv^2}{2kT} = u^2$. Then $v = \left(\frac{2kT}{m}\right)^{\frac{1}{2}}u$ and $vdv = \frac{2kT}{m}udu$. As $v\to 0, u\to 0$ and as $v\to\infty, u\to\infty$. Then the above integral becomes

$$\begin{aligned}
\langle v^n\rangle &= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^\infty e^{-u^2}u^{n+1}\left(\frac{2kT}{m}\right)^{\frac{n+1}{2}}\cdot\frac{2kT}{m}udu \\
&= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot\left(\frac{2kT}{m}\right)^{\frac{n+3}{2}}\int_0^\infty e^{-u^2}u^{n+2}du
\end{aligned}$$

Let $u^2 = t\cdot$ Then $2udu = dt$ As $u\to 0, t\to 0$ and as $u\to\infty, t\to\infty$. As $u\to 0, t\to 0$ and as $u\to\infty, t\to\infty$.

$$\begin{aligned}
\langle v^n\rangle &= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot\left(\frac{2kT}{m}\right)^{\frac{n+3}{2}}\int_0^\infty e^{-t}t^{\frac{n+1}{2}}\frac{dt}{2} \\
&= 2\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot\left(\frac{2kT}{m}\right)^{\frac{n+3}{2}}\int_0^\infty e^{-t}t^{\frac{n+3}{2}}dt \\
&= \frac{2\pi}{\pi\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{\frac{n+3}{2}-\frac{3}{2}}\Gamma\left(\frac{n+3}{2}\right) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{\frac{n}{2}}\Gamma\left(\frac{n+3}{2}\right) \\
&= \left(\frac{2kT}{m}\right)^{\frac{n}{2}}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}
\end{aligned}$$

since $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. Hence

$$\langle v^n\rangle = \left(\frac{2kT}{m}\right)^{\frac{n}{2}}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

Problem 13.1.5

By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k\ln x dx = \frac{1}{(k+1)^2},\quad k>-1$$

$\boxed{\textbf{Solution}}$ Put $x = e^t$. Then $t = \ln x$ and $dx = e'dt$. As $x \to 0, t \to \infty$ and as $x \to 1, t \to 0$.

$$-\int_0^1 x^k \ln x dx$$

$$= -\int_\infty^0 e^{kt} te' dt$$

$$= \int_0^\infty e^{(k+1)t} t dt$$

Now put $-(k+1)t = z$. Then

$$dt = -\frac{dz}{(k+1)}$$

As $t \to 0, z \to 0$ and as $t \to \infty, z \to 0$. Then

$$-\int_0^1 x^k \ln x dx$$

$$= \int_0^\infty e^{(k+1)t} t dt$$

$$= \int_0^\infty e^{-z}\left(\frac{z}{-(k+1)}\right)\left(\frac{dz}{-(k+1)}\right)$$

$$= \frac{1}{(k+1)^2}\int_0^\infty ze^{-z} dz$$

$$= \frac{1}{(k+1)^2}\int_0^\infty z^{2-1}e^{-z} dz$$

$$= \frac{1}{(k+1)^2}\Gamma(2)$$

$$= \frac{1}{(k+1)^2} \cdot 1!$$

$$= \frac{1}{(k+1)^2}$$

Hence

$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

Problem 13.1.6

Show that

$$\int_0^\infty e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$$

Solution Consider $x^4 = t$ and put $4x^3 dx = dt$ as $t \to 0$ to $\infty x \to 0$ to ∞ and using

$$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma z$$

and

$$z\Gamma z = \Gamma(z+1)$$

the integral takes the form of

$$\frac{1}{4}\int_0^\infty e^{-t} t^{-3/4} dt = \frac{1}{4}\int_0^\infty e^{-t} t^{1/4-1} dt$$

$$= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$= \Gamma\left(\frac{5}{4}\right)$$

Problem 13.1.7

Show that
$$\lim_{x\to 0}\frac{\Gamma(ax)}{\Gamma(x)}=\frac{1}{a}$$

Solution

$$\begin{aligned}
&=\lim_{x\to 0}\frac{\left(\frac{ax\Gamma(ax)}{ax}\right)}{\left(\frac{x\Gamma(x)}{x}\right)}\\
&=\lim_{x\to 0}\left(\frac{\Gamma(ax+1)}{\Gamma(x+1)}\cdot\frac{x}{ax}\right)\\
&=\frac{1}{a}\lim_{x\to 0}\frac{\Gamma(ax+1)}{\Gamma(x+1)}\\
&=\frac{1}{a}\frac{\Gamma(1)}{\Gamma(1)}\\
&=\frac{1}{a}
\end{aligned}$$

Problem 13.1.8

Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

Solution Recall that
$$\Gamma(z)=\lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots\cdot n}{z(z+1)(z+2)\cdots(z+n)}\cdot n^2,$$
where $z\neq 0,-1,-2,-3,\cdots$. The denominator shows that $\Gamma(z)$ has simple poles at $z=0,-1,-2,-3,\cdots$

$$\begin{aligned}
\Gamma(z)&=\int_0^\infty e^{-t}t^{z-1}dt\\
&=\int_0^1 e^{-t}t^{z-1}dt+\int_1^\infty e^{-t}t^{z-1}dt\\
&=\int_0^1 t^{z-1}\sum_{n=0}^\infty\frac{(-t)^n}{n!}dt+\int_1^\infty e^{-t}t^{z-1}dt\\
&=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\int_0^1 t^{n+z-1}dt+\int_1^\infty e^{-t}t^{z-1}dt\\
&=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\cdot\left[\frac{t^{n+z}}{n+z}\right]_0^1+\int_1^\infty e^{-t}t^{z-1}dt\\
&=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\cdot\left[\frac{1}{n+z}-0\right]+\int_1^\infty e^{-t}t^{z-1}dt\\
&=\sum_{n=0}^\infty\frac{(-1)^n}{n!(n+z)}+\int_1^\infty e^{-t}t^{z-1}dt
\end{aligned}$$

The series
$$\sum_{n=0}^\infty\frac{(-1)^n}{n!(n+z)}$$
shows that the first order poles at all negative integers $z=-n$ has respective residues
$$\frac{(-1)^n}{n!}$$

Problem 13.1.10

Show that, for integer s

(a)

$$\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx = \frac{s!}{2a^{s+1}}$$

(b)

$$\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx = \frac{\Gamma\left(s+\frac{1}{2}\right)}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Solution For (a) Put $ax^2 = z\cdot$ Then $2axdx = dz$. This implies

$$dx = \frac{dz}{2\sqrt{az}}$$

As $x \to 0, z \to 0$ and as $x \to \infty, z \to \infty$. The given integral is

$$\begin{aligned}
&\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx \\
&= \int_0^\infty \left(\sqrt{\frac{z}{a}}\right)^{2s+1} e^{-z}\frac{dz}{2\sqrt{az}} \\
&= \frac{1}{2\sqrt{a}}\int_0^\infty \left(\frac{z}{a}\right)^{\frac{2s+1}{2}} e^{-z}z^{-\frac{1}{2}}dz \\
&= \frac{1}{2a^{\frac{1}{2}}}\cdot\frac{1}{a^{\frac{2s+1}{2}}}\int_0^\infty e^{-z}z^{\frac{2s+1}{2}-\frac{1}{2}}dz \\
&= \frac{1}{2a^{s+1}}\int_0^\infty e^{-z}z^s dz \\
&= \frac{1}{2a^{s+1}}\int_0^\infty e^{-z}z^{(s+1)-1}dz \\
&= \frac{1}{2a^{s+1}}\Gamma(s+1)
\end{aligned}$$

since s is an integer, therefore $\Gamma(s+1) = s!$. Hence

$$\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx = \frac{s!}{2a^{s+1}}$$

Solution For (b) Put $ax^2 = z\cdot$ Then $2axdx = dz$. This implies

$$dx = \frac{dz}{2\sqrt{az}}$$

As $x \to 0, z \to 0$ and as $x \to \infty, z \to \infty$. The given integral is

$$\begin{aligned}
&\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx \\
&= \int_0^\infty \left(\sqrt{\frac{z}{a}}\right)^{2s} e^{-z}\frac{dz}{2\sqrt{az}} \\
&= \frac{1}{2\sqrt{a}}\int_0^\infty \left(\frac{z}{a}\right)^{s} e^{-z}z^{-\frac{1}{2}}dz \\
&= \frac{1}{2a^{\frac{1}{2}}}\cdot\frac{1}{a^s}\int_0^\infty e^{-z}z^{s-\frac{1}{2}}dz \\
&= \frac{1}{2a^{s+\frac{1}{2}}}\int_0^\infty e^{-z}z^{\left(s+\frac{3}{2}\right)-1}dz \\
&= \frac{1}{2a^{s+\frac{1}{2}}}\Gamma\left(s+\frac{3}{2}\right)
\end{aligned}$$

since

$$\Gamma\left(s+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^s}\cdot(2s-1)!!$$

$$=\frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Thus

$$\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx=\frac{\Gamma\left(s+\frac{1}{2}\right)}{2a^{s+\frac{1}{2}}}=\frac{(2s-1)!!}{2a^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Problem 13.1.11

Express the coefficient of the n th term of the expansion of $(1+x)^{1/2}$ in powers of x

(a) in terms of factorials of integers,

(b) in terms of the double factorial $(!!)$ functions.

$$ANS.\ a_n=(-1)^{n+1}\frac{(2n-3)!}{2^{2n-2}n!(n-2)!}=(-1)^{n+1}\frac{(2n-3)!!}{(2n)!!},\quad n=2,3,\ldots$$

Solution For (a) the n th term of the expansion of $(1+x)^{1/2}$ in powers of x is:

$$a_n=\binom{\frac{1}{2}}{n-1}$$

$$=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!}$$

$$=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!}$$

$$=\frac{(-1)^{n-1}}{n!2^n}[1.3.5\ldots\cdot(2n-3)]$$

$$=\frac{(-1)^{n+1}}{n!2^n}\left[\frac{1.2.3.4.5.6\cdots\cdot(2n-4)\cdot(2n-3)}{2.4.6.\cdots.(2n-4)}\right]$$

$$=\frac{(-1)^n}{n!2^n}\cdot\frac{(2n-3)!}{(n-2)!2^{n-2}}$$

$$=(-1)^{n+1}\cdot\frac{(2n-3)!}{2^{2n-2}\cdot n!(n-2)!}$$

Therefore,

$$a_n=(-1)^{n+1}\cdot\frac{(2n-3)!}{2^{2n-2}n!(n-2)!},\quad n=1,2,3,\cdots$$

Solution For (b) the n th term expansion of $(1+x)^{1/2}$

$$a_n=\binom{-\frac{1}{2}}{n-1}$$

$$=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!}$$

$$=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!}$$

$$=\frac{(-1)^{n-1}}{n!2^n}[1.3.5\cdot\cdots\cdot(2n-3)]$$

$$=(-1)^{n+1}\cdot\left[\frac{1.3.5\cdots\cdot(2n-3)}{2.4.6\cdots\cdot.2n}\right]$$

$$=(-1)^{n+1}\cdot\frac{(2n-3)!!}{(2n)!!}$$

Therefore

$$a_n = (-1)^{n+1} \cdot \frac{(2n-3)!!}{(2n)!!}, \qquad \text{for } n = 1, 2, 3, \cdots$$

Problem 13.1.12

Express the coefficient of the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x

(a) in terms of the factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. \quad a_n = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad n = 1, 2, 3 \ldots$$

Solution For (a) the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x is:

$$a_n = \binom{-\frac{1}{2}}{n-1}$$

$$= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}$$

$$= \frac{(-1)^n}{n!2^n}[1.3.5.\cdots.(2n-1)]$$

$$= \frac{(-1)^n}{n!2^n}\left[\frac{1.2.3.4.5.6.\cdots.(2n-1)\cdot 2n}{2.4.6.\cdots.2n}\right]$$

$$= \frac{(-1)^n}{n!2^n} \cdot \frac{(2n)!}{n!2^n}$$

$$= (-1)^n \cdot \frac{(2n)!}{2^{2n}\cdot(n!)^2}$$

Therefore,

$$a_n = (-1)^n \cdot \frac{(2n)!}{2^{2n}\cdot(n!)^2}, \qquad \text{for } n = 1, 2, 3, \cdots$$

Solution For (b) the n th term expansion of $(1+x)^{-1/2}$ in powers of x in terms of the double factorial (!!) functions.

$$a_n = \binom{-\frac{1}{2}}{n-1}$$

$$= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}$$

$$= \frac{(-1)^n}{n!2^n}[1.3.5\ldots.(2n-1)]$$

$$= (-1)^n \cdot \left[\frac{1.3.5\ldots.(2n-1)}{2.4.6.\cdots.2n}\right]$$

$$= (-1)^n \cdot \frac{(2n-1)!!}{(2n)!!}$$

Therefore

$$a_n = (-1)^n \cdot \frac{(2n-1)!!}{(2n)!!}, \qquad \text{for } n = 1, 2, 3, \cdots$$

Problem 13.1.14

(*a*) Show that $\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$, where n is an integer.

(*b*) Express $\Gamma\left(\frac{1}{2}+n\right)$ and $\Gamma\left(\frac{1}{2}-n\right)$ separately in terms of $\pi^{1/2}$ and a double factorial function.

$$ANS. \quad \Gamma\left(\frac{1}{2}+n\right)=\frac{(2n-1)!!}{2^n}\pi^{1/2}$$

Solution For (*a*) recall that

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

Putting $z=\frac{1}{2}+n$ in the above relation, it becomes

$$\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(1-\frac{1}{2}-n\right)=\frac{\pi}{\sin\left[\pi\left(\frac{1}{2}+n\right)\right]}$$

$$=\frac{\pi}{\cos(n\pi)}$$

$$=\frac{\pi}{(-1)^n}$$

since $\cos(n\pi)=(-1)^n$ and

$$=(-1)^n\pi$$

Therefore

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$$

where n is an integer.

Solution For (*b*) recall the Legendre's duplication formula,

$$\Gamma(1+z)\Gamma\left(z+\frac{1}{2}\right)=2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

Putting $z=n$ in the above relation, it becomes

$$\Gamma(1+n)\Gamma\left(n+\frac{1}{2}\right)=2^{-2n}\sqrt{\pi}\Gamma(2n+1)$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{2^{-2n}\sqrt{\pi}\Gamma(2n+1)}{\Gamma(1+n)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{(2n)!}{n!}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{(1.2.3.4.5\ldots..2n)}{(1.2.3\ldots n)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{n}}\cdot\frac{(1.2.3.4.5\ldots..2n)}{(2.4.6\ldots..2n)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{n}}\cdot[1.3.5\ldots\ldots(2n-1)]$$

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{\sqrt{\pi}}{2^{n}}\cdot(2n-1)!!\cdots$$

From part (*a*)

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$$

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n\pi}{\Gamma\left(\frac{1}{2}+n\right)}$$

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n\pi}{\left(\frac{\sqrt{\pi}}{2^n}\cdot(2n-1)!!\right)}$$

$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n \cdot 2^n\sqrt{\pi}}{(2n-1)!!}$$

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{\sqrt{\pi}}{2^n}\cdot(2n-1)!! \text{ and } \Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n \cdot 2^n\sqrt{\pi}}{(2n-1)!!}$$

Problem 13.1.6

Prove that

$$|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)|\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha+i\beta)e^{\gamma(a+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(a-i\beta)}$$

$$\times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{a+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2a}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right]$$

$$= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{\frac{2a}{n}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{a}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]$$

$$= \left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|} = \frac{1}{|\Gamma(\alpha)|}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]^{\frac{1}{2}}$$

$$|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)|\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-\frac{1}{2}}$$

Problem 13.1.17

Show that for n, a positive integer,

$$|\Gamma(n+ib+1)| = \left(\frac{\pi b}{\sinh \pi b}\right)^{1/2}\prod_{s=1}^{n}\left(s^2+b^2\right)^{1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{a-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha+i\beta)e^{\gamma(a+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(a-i\beta)}$$

$$\times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{a+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2a}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right]$$

$$= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{\frac{2a}{n}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{a}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]$$

$$= \left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]$$

Hence

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha = 1$ and $\beta = b$ in the above identity. Then it becomes

$$\frac{1}{|\Gamma(1 + ib)|^2} = \frac{1}{\Gamma(1)^2} \prod_{n=0}^{\infty} \left[1 + \frac{b^2}{(n+1)^2}\right]$$

$$= \prod_{n=0}^{\infty} \left[1 + \frac{b^2}{(n+1)^2}\right], \quad \text{as} \quad \Gamma(1) = 1$$

$$= \prod_{n=0}^{\infty} \left[1 - \frac{(ib\pi)^2}{(n+1)^2\pi^2}\right]$$

$$= \prod_{n=1}^{\infty} \left[1 - \frac{(ib\pi)^2}{n^2\pi^2}\right]$$

$$= \frac{1}{(ib\pi)} \left\{(ib\pi) \prod_{n=1}^{\infty} \left[1 - \frac{(ib\pi)^2}{n^2\pi^2}\right]\right\}$$

$$= \frac{1}{ib\pi} \cdot \sin(ib\pi)$$

Using the identy

$$\sin z = z \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{n^2\pi^2}\right] \quad \text{for} z = ib\pi$$

$$= \frac{1}{ib\pi} \cdot i \sinh(b\pi)$$

$$= \frac{\sinh(b\pi)}{b\pi}$$

$$\frac{1}{|\Gamma(1 + ib)|^2} = \frac{\sinh(b\pi)}{b\pi}$$

$$|\Gamma(1 + ib)|^2 = \frac{b\pi}{\sinh(b\pi)}.$$

since n is an integer, therefore

$$\Gamma(n + ib + 1) = \Gamma(\{1 + ib + (n-1)\} + 1)$$

$$= \{1 + ib + (n-1)\}\Gamma(\{1 + ib + (n-1)\})$$

$$(1 + ib)(2 + ib)(3 + ib) \cdots (n + ib)\Gamma(1 + ib)$$

$$\Gamma(n + ib + 1) = (1 + ib)(2 + ib)(3 + ib) \cdots (n + ib)\Gamma(1 + ib)$$

$$\Gamma(n - ib + 1) = (1 - ib)(2 - ib)(3 - ib) \cdots (n - ib)\Gamma(1 - ib)$$

$$|\Gamma(n + ib + 1)|^2$$

$$= \Gamma(n + ib + 1)\Gamma(n - ib + 1)$$

$$= (1 + ib)(2 + ib)(3 + ib) \cdots (n + ib)\Gamma(1 + ib) \times (1 - ib)(2 - ib)(3 - ib) \cdots (n - ib)\Gamma(1 - ib)$$

$$= \{(1 + ib)(1 - ib)\}\{(2 + ib)(2 - ib)\}\{(3 + ib)(3 - ib)\} \cdots \{(n + ib)(n - ib)\}\Gamma(1 + ib)\Gamma(1 - ib)$$

$$= \left(1^2 + b^2\right)\left(2^2 + b^2\right)\left(3^2 + b^2\right) \cdots \left(n^2 + b^2\right)|\Gamma(1 + ib)|^2$$

$$= \prod_{s=1}^{n} \left(s^2 + b^2\right) \times \frac{b\pi}{\sinh(b\pi)}$$

Hence

$$|\Gamma(n + ib + 1)|^2 = \prod_{s=1}^{n} \left(s^2 + b^2\right) \times \frac{b\pi}{\sinh(b\pi)}$$

This gives

$$|\Gamma(n + ib + 1)| = \left(\frac{b\pi}{\sinh(b\pi)}\right)^{\frac{1}{2}} \prod_{s=1}^{n} \left(s^2 + b^2\right)^{\frac{1}{2}}$$

Problem 13.1.18

Show that for all real values of x and y, $|\Gamma(x)| \geq |\Gamma(x + iy)|$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} = (\alpha + i\beta)e^{\gamma(\alpha+i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha + i\beta}{n}\right) e^{-\frac{a+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha - i\beta)} = (\alpha - i\beta)e^{\gamma(\alpha-i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha - i\beta}{n}\right) e^{\frac{a-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} \cdot \frac{1}{\Gamma(\alpha - i\beta)} = (\alpha + i\beta)e^{\gamma(a+i\beta)} \cdot (\alpha - i\beta)e^{\gamma(a-i\beta)}$$

$$\times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha + i\beta}{n}\right) e^{\frac{a+i\beta}{n}} \cdot \left(1 + \frac{\alpha - i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \left(\alpha^2 + \beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2a}{n}} \left[\left(1 + \frac{\alpha + i\beta}{n}\right) \cdot \left(1 + \frac{\alpha - i\beta}{n}\right)\right]$$

$$= \left(\alpha^2 + \beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{\frac{2a}{n} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2 + \beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)}{\left(1 + \frac{\alpha}{n}\right)^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2 + \beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)}{\left(1 + \frac{\alpha}{n}\right)^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\frac{\alpha^2 + \beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{a}{n}} \cdot \left(1 + \frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{\left(1 + \frac{2\alpha}{n} + \frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]$$

$$= \left(1 + \frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{\left(1 + 2\alpha n + \alpha^2 + \beta^2\right)}{(n + \alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n + \alpha)^2 + \beta^2}{(n + \alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha = x$ and $\beta = y$ in the above identity. Then it becomes

$$\frac{1}{|\Gamma(x+iy)|^2} = \frac{1}{\Gamma(x)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+x)^2}\right]$$

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right|^2 = \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+x)^2}\right]$$

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right|^2 \geq 1, \quad \text{since} \quad 1 + \frac{\beta^2}{(n+x)^2} \geq 1$$

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right| \geq 1$$

$$|\Gamma(x)| \geq |\Gamma(x+iy)|$$

Hence is proved

Problem 13.1.19

Show that

$$\left|\Gamma(\frac{1}{2} + iy)\right|^2 = \frac{\pi}{\cosh \pi y}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha+i\beta}{n}\right) e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha-i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} \cdot \frac{1}{\Gamma(\alpha-i\beta)} = (\alpha+i\beta)e^{\gamma(a+i\beta)} \cdot (\alpha-i\beta)e^{\gamma(a-i\beta)}$$

$$\times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha+i\beta}{n}\right) e^{\frac{a+i\beta}{n}} \cdot \left(1 + \frac{\alpha-i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \left(\alpha^2+\beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2a}{n}} \left[\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)\right]$$

$$= \left(\alpha^2+\beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{\frac{2a}{n} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2} \cdot \left(1+\frac{\alpha}{n}\right)^2 \right]$$

$$= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{a}{n}} \cdot \left(1+\frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}} \right]$$

$$= \left(1+\frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2} \right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1+\frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2} \right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1+\frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2} \right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2} \right]$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2} \right]$$

Now put $\alpha = \frac{1}{2}$ and $\beta = y$ in the above identity. Then it becomes

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\Gamma\left(\frac{1}{2}\right)^2} \prod_{n=0}^{\infty} \left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2} \right]$$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi} \prod_{n=0}^{\infty} \left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2} \right]$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi} \prod_{n=0}^{\infty} \left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2} \right]$$

Recall

$$\cos z = \prod_{n=1}^{\infty} \left[1-\frac{z^2}{\left(n-\frac{1}{2}\right)^2 \pi^2} \right]$$

and putting $z = i\pi y$ it becomes

$$\cos(i\pi y) = \prod_{n=1}^{\infty} \left[1-\frac{i^2\pi^2 y^2}{\left(n-\frac{1}{2}\right)^2 \pi^2} \right]$$

$$\cosh(\pi y) = \prod_{n=1}^{\infty} \left[1+\frac{y^2}{\left(n-\frac{1}{2}\right)^2} \right]$$

$$\cosh(\pi y) = \prod_{n=0}^{\infty} \left[1+\frac{y^2}{\left(n+1-\frac{1}{2}\right)^2} \right]$$

$$\cosh(\pi y) = \prod_{n=0}^{\infty} \left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2} \right]$$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\cosh(\pi y)$$

Problem 13.1.20

The probability density associated with the normal distribution of statistics is given by

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

with $(-\infty, \infty)$ for the range of x. Show that (a)

(a) $\langle x \rangle$, the mean value of x, is equal to μ

(b) the standard deviation $\left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2}$ is given by σ.

Solution For (a) For the mean

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $x - \mu = y$. Then $dx = dy$. As $x \to 0, y \to 0$ and $x \to \infty, y \to \infty$.

$$\langle x \rangle = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\mu + y) e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\mu + y) e^{\frac{y^2}{2\sigma^2}} dy$$

$$= \frac{\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy$$

since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{2\sigma^2}} dy = 2 \int_{0}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

and since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^{\infty} y e^{\frac{y^2}{2\sigma^2}} dy = 0$$

Therefore, the integral becomes

$$\langle x \rangle = \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{0}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y} dz$, that is, $dy = \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$ As $y \to 0, z \to 0$ and $y \to \infty, z \to \infty$. Therefore

$$\langle x \rangle = \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{0}^{\infty} e^{\frac{y^2}{2\sigma^2}} dy$$

$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{0}^{\infty} e^{-z} \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$$

$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_{0}^{\infty} e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}}\sqrt{\pi}$$
$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}}\sqrt{\pi}$$
$$= \mu$$

$\boxed{\textbf{Solution}}$ For (b) we start saying

$$\langle x^2 \rangle = \int_0^\infty x^2 f(x)dx$$
$$= \int_{-\infty}^\infty x^2 \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $x - \mu = y$. Then $dx = dy$. As $x \to 0, y \to 0$ and $x \to \infty, y \to \infty$.

$$\langle x^2 \rangle = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty (\mu + y)^2 e^{-\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty \left(\mu^2 + 2\mu y + y^2\right) e^{\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy + \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy$$

since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy = 2\int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy$$

since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy = 0$$

since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy = 2\int_0^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy$$

Therefore the above integral becomes

$$\langle x^2 \rangle = \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y}dz$, that is, $dy = \frac{\sigma}{\sqrt{2}}z^{-\frac{1}{2}}dz$ As $y \to 0, z \to 0$ and $y \to \infty, z \to \infty$. Therefore

$$\langle x^2 \rangle = \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{\frac{1}{2}} dz + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty 2\sigma^2 z e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$$
$$= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz + \frac{2\sqrt{2}\sigma^3}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz$$
$$= \frac{\mu^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{3}{2}-1} dz$$

$$= \frac{\mu^2}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) + \frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\mu^2}{\sqrt{\pi}} \cdot \sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\pi}$$

$$= \mu^2 + \sigma^2$$

So the standard deviation

$$\left(\left\langle x^2\right\rangle - \langle x\rangle^2\right)^{\frac{1}{2}} = \left(\mu^2 + \sigma^2 - \mu^2\right)^{\frac{1}{2}}$$

$$\left(\left\langle x^2\right\rangle - \langle x\rangle^2\right)^{\frac{1}{2}} = \sqrt{\sigma^2}$$

$$\left(\left\langle x^2\right\rangle - \langle x\rangle^2\right)^{\frac{1}{2}} = \sigma$$

Problem 13.1.21

For the gamma distribution

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(a) $\langle x\rangle$, the mean value of x, is equal to $\alpha\beta$

(b) σ^2, its variance, defined as $\left\langle x^2\right\rangle - \langle x\rangle^2$, has the value $\alpha\beta^2$

Solution For (a) the mean

$$\langle x\rangle = \int_0^\infty x f(x) dx$$

$$= \int_0^\infty x \cdot \frac{1}{\beta^a \Gamma(\alpha)} x^{a-1} e^{-\frac{x}{\beta}} dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^a e^{-\frac{x}{\beta}} dx$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \to 0, z \to 0$ and $x \to \infty, z \to \infty$.

$$\langle x\rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty z^a e^{-z} \beta dz$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{(a+1)-1} e^{-z} dz$$

$$= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1)$$

$$= \frac{\beta}{\Gamma(\alpha)} \cdot \alpha\Gamma(\alpha)$$

$$= \alpha\beta$$

Solution For (b)

$$\left\langle x^2\right\rangle = \int_0^\infty x^2 f(x) dx$$

$$= \int_0^\infty x^2 \cdot \frac{1}{\beta^a \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha+1} e^{-\frac{x}{\beta}} dx$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \to 0, z \to 0$ and $x \to \infty, z \to \infty$

$$\left\langle x^2\right\rangle = \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{a+1} e^{-z} \beta dz$$

$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+2)-1} e^{-z} dz$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha + 2)$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \cdot (\alpha + 1)\alpha\Gamma(\alpha)$$

$$= \alpha(\alpha + 1)\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

Hence variance, $\sigma^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$