

Title

Carlos Faz

Problem 13.1.1

Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral, Eq. (13.5),

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Solution Consider the Euler integral

$$\Gamma z = \int_0^\infty e^{-t} t^{z-1} dt$$

Put, $z = z + 1$

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t} t^{z+1-1} dt \\ &= \int_0^\infty e^{-t} t^z dt \\ &= t^z \int_0^\infty e^{-t} dt - \int_0^\infty \frac{dt}{dx} \int e^{-t} dt \\ &= -t^z e^{-t} \Big|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} dt \\ &= z\Gamma(z)\end{aligned}$$

Problem 13.1.2

In a power-series solution for the Legendre functions of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3) \cdots (n+2s-1)(n+2s)}{2 \cdot 4 \cdot 6 \cdots (2s-2)(2s) \cdot (2n+3)(2n+5)(2n+7) \cdots (2n+2s+1)}$$

in which s is a positive integer.

- (a) Rewrite this expression in terms of factorials.
- (b) Rewrite this expression using Pochhammer symbols; see Eq. (1.72).

Solution For (a) Notice that

$$\begin{aligned}&\frac{(n+1)(n+2)(n+3) \cdots (n+2s-1)(n+2s)}{2 \cdot 4 \cdot 6 \cdots (2s-2)(2s) \cdot (2n+3)(2n+5)(2n+7) \cdots (2n+2s+1)} \\ &= \frac{[n!(n+1)(n+2)(n+3) \cdots (n+2s-1)(n+2s)]}{n!s!2^s \cdot (2n+3)(2n+5)(2n+7) \cdots (2n+2s+1)} \\ &= \frac{(n+2s)!(2n+1)!}{n!s!2^s \cdot [(2n+1)!(2n+3)(2n+5)(2n+7) \cdots (2n+2s+1)]} \\ &= \frac{(n+2s)!(2n+1)![(2n+2)(2n+4)(2n+6) \cdots (2n+2s)]}{n!s!2^s \cdot [(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7) \cdots (2n+2s)(2n+2s+1)]} \\ &= \frac{(n+2s)!(2n+1)!2^s[(n+1)(n+2)(n+3) \cdots (n+s)]}{n!s!2^s \cdot [(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7) \cdots (2n+2s)(2n+2s+1)]} \\ &= \frac{(n+2s)!(2n+1)!n!(n+1)(n+2)(n+3) \cdots (n+s)}{n!s!n![(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7) \cdots (2n+2s)(2n+2s+1)]}\end{aligned}$$

$$= \frac{(n+2s)!(2n+1)!(n+s)!}{n!n!s!(2n+2s+1)!}$$

Solution For (b) we notice that

$$\begin{aligned} & \frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)} \\ &= \frac{(n+1)(n+2)(n+3)\cdots[(n+1)+(2s-2)][(n+1)+(2s-1)]}{(2^s[1\cdot 2\cdot 3\cdots(s-1)s])\cdot[(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)]} \\ &= \frac{(n+1)_{(2s-1)+1}\cdot[(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{(2^s[1\cdot 2\cdot 3\cdots\{1+(s-2)\}\{1+(s-1)\}]\cdot[(2n+2)(2n+3)(2n+4)(2n+5)\cdots(2n+2s)(2n+2s+1)])} \\ &= \frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots(n+s)]\cdot 2^s}{2^s(1)_{(s-1)+1}\cdot[(2n+2)(2n+3)(2n+4)\cdots\{(2n+2)+(2s-1)\}]} \\ &= \frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots\{(n+1)+(s-1)\}]}{(1)_s\cdot(2n+2)_{(2s-1)+1}} \\ &= \frac{(n+1)_{2s}\cdot(n+1)_{(s-1)+1}}{(1)_s\cdot(2n+2)_{2s}} \\ &= \frac{(n+1)_{2s}\cdot(n+1)_s}{(1)_s\cdot(2n+2)_{2s}} \end{aligned}$$

Problem 13.1.3

Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2 \int_0^\infty e^{-t^2} t^{2z-1} dt, \quad \operatorname{Re}(z) > 0$$

$$\Gamma(z) = \int_0^1 \left[\ln \left(\frac{1}{t} \right) \right]^{z-1} dt, \quad \operatorname{Re}(z) > 0$$

Solution Changing variables $t = u^2$ and $dt = 2u du$ we have

$$\begin{aligned} \Gamma z &= \int_0^\infty e^{-u^2} u^{2z-2} u du \\ &= \int_0^\infty e^{-u^2} u^{2z-1} du \\ &= \int_0^\infty e^{-t^2} t^{2z-1} dt \end{aligned}$$

as $t \rightarrow 0$ to ∞ $u \rightarrow 0$ to 1 the equation takes the form of

$$\begin{aligned} \Gamma z &= \int_0^1 e^{-\ln \frac{1}{u}} \left(\ln \frac{1}{u} \right)^{z-1} u du \\ &= \int_0^1 u \left(\ln \frac{1}{u} \right)^{z-1} u du \\ &= \int_0^1 \left(\ln \frac{1}{u} \right)^{z-1} du \\ &= \int_0^1 \left(\ln \frac{1}{t} \right)^{z-1} dt \end{aligned}$$

Problem 13.1.4

In a Maxwellian distribution the fraction of particles of mass m with speed between v and $v + dv$ is

$$\frac{dN}{N} = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left(-\frac{mv^2}{2kT} \right) v^2 dv$$

where N is the total number of particles, k is Boltzmann's constant, and T is the absolute temperature. The average or expectation value of v^n is defined as $\langle v^n \rangle = N^{-1} \int v^n dN$. Show that

$$\langle v^n \rangle = \left(\frac{2kT}{m} \right)^{n/2} \frac{\Gamma \left(\frac{n+3}{2} \right)}{\Gamma \left(\frac{3}{2} \right)}$$

This is an extension of Example 13.1.1, in which the distribution was in kinetic energy $E = mv^2/2$, with $dE = mv dv$

Solution

$$\begin{aligned} \langle v^n \rangle &= N^{-1} \int v^n dN \\ &= \int v^n \frac{dN}{N} \\ &= \int_0^\infty v^n \cdot 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} v^2 dv \\ &= 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^\infty v^n e^{-\frac{mv^2}{2kT}} v^{n+1} dv \end{aligned}$$

Let $\frac{mv^2}{2kT} = u^2$. Then $v = \left(\frac{2kT}{m} \right)^{1/2} u$ and $v dv = \frac{2kT}{m} u du$. As $v \rightarrow 0, u \rightarrow 0$ and as $v \rightarrow \infty, u \rightarrow \infty$. Then the above integral becomes

$$\begin{aligned} \langle v^n \rangle &= 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^\infty e^{-u^2} u^{n+1} \left(\frac{2kT}{m} \right)^{\frac{n+1}{2}} \cdot \frac{2kT}{m} u du \\ &= 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \cdot \left(\frac{2kT}{m} \right)^{\frac{n+3}{2}} \int_0^\infty e^{-u^2} u^{n+2} du \end{aligned}$$

Let $u^2 = t$. Then $2u du = dt$. As $u \rightarrow 0, t \rightarrow 0$ and as $u \rightarrow \infty, t \rightarrow \infty$. As $u \rightarrow 0, t \rightarrow 0$ and as $u \rightarrow \infty, t \rightarrow \infty$.

$$\begin{aligned} \langle v^n \rangle &= 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \cdot \left(\frac{2kT}{m} \right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+1}{2}} \frac{dt}{2} \\ &= 2\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \cdot \left(\frac{2kT}{m} \right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+3}{2}} dt \\ &= \frac{2\pi}{\pi\sqrt{\pi}} \left(\frac{2kT}{m} \right)^{\frac{n+3}{2} - \frac{3}{2}} \Gamma \left(\frac{n+3}{2} \right) \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{2kT}{m} \right)^{\frac{n}{2}} \Gamma \left(\frac{n+3}{2} \right) \\ &= \left(\frac{2kT}{m} \right)^{\frac{n}{2}} \frac{\Gamma \left(\frac{n+3}{2} \right)}{\Gamma \left(\frac{3}{2} \right)} \end{aligned}$$

since $\Gamma \left(\frac{3}{2} \right) = \frac{\sqrt{\pi}}{2}$. Hence

$$\langle v^n \rangle = \left(\frac{2kT}{m} \right)^{\frac{n}{2}} \frac{\Gamma \left(\frac{n+3}{2} \right)}{\Gamma \left(\frac{3}{2} \right)}$$

Problem 13.1.5

By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

Solution Put $x = e^t$. Then $t = \ln x$ and $dx = e^t dt$. As $x \rightarrow 0, t \rightarrow -\infty$ and as $x \rightarrow 1, t \rightarrow 0$.

$$\begin{aligned} & - \int_0^1 x^k \ln x dx \\ &= - \int_{-\infty}^0 e^{kt} t e^t dt \\ &= \int_0^{\infty} e^{(k+1)t} t dt \end{aligned}$$

Now put $-(k+1)t = z$. Then

$$dt = -\frac{dz}{(k+1)}$$

As $t \rightarrow 0, z \rightarrow 0$ and as $t \rightarrow \infty, z \rightarrow -\infty$. Then

$$\begin{aligned} & - \int_0^1 x^k \ln x dx \\ &= \int_0^{\infty} e^{(k+1)t} t dt \\ &= \int_0^{\infty} e^{-z} \left(\frac{z}{-(k+1)} \right) \left(-\frac{dz}{(k+1)} \right) \\ &= \frac{1}{(k+1)^2} \int_0^{\infty} z e^{-z} dz \\ &= \frac{1}{(k+1)^2} \int_0^{\infty} z^{2-1} e^{-z} dz \\ &= \frac{1}{(k+1)^2} \Gamma(2) \\ &= \frac{1}{(k+1)^2} \cdot 1! \\ &= \frac{1}{(k+1)^2} \end{aligned}$$

Hence

$$- \int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

Problem 13.1.6

Show that

$$\int_0^{\infty} e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$$

Solution Consider $x^4 = t$ and put $4x^3 dx = dt$ as $t \rightarrow 0$ to $\infty, x \rightarrow 0$ to ∞ and using

$$\int_0^{\infty} e^{-t} t^{z-1} dt = \Gamma z$$

and

$$z\Gamma z = \Gamma(z+1)$$

the integral takes the form of

$$\begin{aligned} \frac{1}{4} \int_0^{\infty} e^{-t} t^{-3/4} dt &= \frac{1}{4} \int_0^{\infty} e^{-t} t^{1/4-1} dt \\ &= \frac{1}{4} \Gamma\left(\frac{1}{4}\right) \\ &= \Gamma\left(\frac{5}{4}\right) \end{aligned}$$

Problem 13.1.7

Show that

$$\lim_{x \rightarrow 0} \frac{\Gamma(ax)}{\Gamma(x)} = \frac{1}{a}$$

Solution

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\left(\frac{ax\Gamma(ax)}{ax} \right)}{\left(\frac{x\Gamma(x)}{x} \right)} \\ &= \lim_{x \rightarrow 0} \left(\frac{\Gamma(ax+1)}{\Gamma(x+1)} \cdot \frac{x}{ax} \right) \\ &= \frac{1}{a} \lim_{x \rightarrow 0} \frac{\Gamma(ax+1)}{\Gamma(x+1)} \\ &= \frac{1}{a} \frac{\Gamma(1)}{\Gamma(1)} \\ &= \frac{1}{a} \end{aligned}$$

Problem 13.1.8

Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

Solution Recall that

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} \cdot n^z,$$

where $z \neq 0, -1, -2, -3, \dots$. The denominator shows that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \dots$

$$\begin{aligned} \Gamma(z) &= \int_0^\infty e^{-t} t^{z-1} dt \\ &= \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-t)^n}{n!} dt + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{n+z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \left[\frac{t^{n+z}}{n+z} \right]_0^1 + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \left[\frac{1}{n+z} - 0 \right] + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+z)} + \int_1^\infty e^{-t} t^{z-1} dt \end{aligned}$$

The series

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+z)}$$

shows that the first order poles at all negative integers $z = -n$ has respective residues

$$\frac{(-1)^n}{n!}$$

Problem 13.1.10

Show that, for integer s

(a)

$$\int_0^\infty x^{2s+1} \exp(-ax^2) dx = \frac{s!}{2a^{s+1}}$$

(b)

$$\int_0^\infty x^{2s} \exp(-ax^2) dx = \frac{\Gamma\left(s + \frac{1}{2}\right)}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s} \sqrt{\frac{\pi}{a}}$$

Solution For (a) Put $ax^2 = z$. Then $2axdx = dz$. This implies

$$dx = \frac{dz}{2\sqrt{az}}$$

As $x \rightarrow 0, z \rightarrow 0$ and as $x \rightarrow \infty, z \rightarrow \infty$. The given integral is

$$\begin{aligned} & \int_0^\infty x^{2s+1} \exp(-ax^2) dx \\ &= \int_0^\infty \left(\sqrt{\frac{z}{a}}\right)^{2s+1} e^{-z} \frac{dz}{2\sqrt{az}} \\ &= \frac{1}{2\sqrt{a}} \int_0^\infty \left(\frac{z}{a}\right)^{\frac{2s+1}{2}} e^{-z} z^{-\frac{1}{2}} dz \\ &= \frac{1}{2a^{\frac{1}{2}}} \cdot \frac{1}{a^{\frac{2s+1}{2}}} \int_0^\infty e^{-z} z^{\frac{2s+1}{2}-\frac{1}{2}} dz \\ &= \frac{1}{2a^{s+1}} \int_0^\infty e^{-z} z^s dz \\ &= \frac{1}{2a^{s+1}} \int_0^\infty e^{-z} z^{(s+1)-1} dz \\ &= \frac{1}{2a^{s+1}} \Gamma(s+1) \end{aligned}$$

since s is an integer, therefore $\Gamma(s+1) = s!$. Hence

$$\int_0^\infty x^{2s+1} \exp(-ax^2) dx = \frac{s!}{2a^{s+1}}$$

Solution For (b) Put $ax^2 = z$. Then $2axdx = dz$. This implies

$$dx = \frac{dz}{2\sqrt{az}}$$

As $x \rightarrow 0, z \rightarrow 0$ and as $x \rightarrow \infty, z \rightarrow \infty$. The given integral is

$$\begin{aligned} & \int_0^\infty x^{2s} \exp(-ax^2) dx \\ &= \int_0^\infty \left(\sqrt{\frac{z}{a}}\right)^{2s} e^{-z} \frac{dz}{2\sqrt{az}} \\ &= \frac{1}{2\sqrt{a}} \int_0^\infty \left(\frac{z}{a}\right)^s e^{-z} z^{-\frac{1}{2}} dz \\ &= \frac{1}{2a^{\frac{1}{2}}} \cdot \frac{1}{a^s} \int_0^\infty e^{-z} z^{s-\frac{1}{2}} dz \\ &= \frac{1}{2a^{s+\frac{1}{2}}} \int_0^\infty e^{-z} z^{(s+\frac{3}{2})-1} dz \\ &= \frac{1}{2a^{s+\frac{1}{2}}} \Gamma\left(s + \frac{3}{2}\right) \end{aligned}$$

since

$$\begin{aligned}\Gamma\left(s + \frac{1}{2}\right) &= \frac{\sqrt{\pi}}{2^s} \cdot (2s-1)!! \\ &= \frac{(2s-1)!!}{2^{s+1}a^s} \sqrt{\frac{\pi}{a}}\end{aligned}$$

Thus

$$\int_0^\infty x^{2s} \exp(-ax^2) dx = \frac{\Gamma\left(s + \frac{1}{2}\right)}{2a^{s+\frac{1}{2}}} = \frac{(2s-1)!!}{2a^{s+1}a^s} \sqrt{\frac{\pi}{a}}$$

Problem 13.1.11

Express the coefficient of the n th term of the expansion of $(1+x)^{1/2}$ in powers of x

(a) in terms of factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. a_n = (-1)^{n+1} \frac{(2n-3)!}{2^{2n-2}n!(n-2)!} = (-1)^{n+1} \frac{(2n-3)!!}{(2n)!!}, \quad n = 2, 3, \dots$$

Solution For (a) the n th term of the expansion of $(1+x)^{1/2}$ in powers of x is:

$$\begin{aligned}a_n &= \binom{\frac{1}{2}}{n-1} \\ &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-(n-1))}{n!} \\ &= \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-3}{2})}{n!} \\ &= \frac{(-1)^{n-1}}{n!2^n} [1.3.5\cdots(2n-3)] \\ &= \frac{(-1)^{n+1}}{n!2^n} \left[\frac{1.2.3.4.5.6\cdots(2n-4) \cdot (2n-3)}{2.4.6\cdots(2n-4)} \right] \\ &= \frac{(-1)^n}{n!2^n} \cdot \frac{(2n-3)!}{(n-2)!2^{n-2}} \\ &= (-1)^{n+1} \cdot \frac{(2n-3)!}{2^{2n-2} \cdot n!(n-2)!}\end{aligned}$$

Therefore,

$$a_n = (-1)^{n+1} \cdot \frac{(2n-3)!}{2^{2n-2}n!(n-2)!}, \quad n = 1, 2, 3, \dots$$

Solution For (b) the n th term expansion of $(1+x)^{1/2}$

$$\begin{aligned}a_n &= \binom{-\frac{1}{2}}{n-1} \\ &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-(n-1))}{n!} \\ &= \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-3}{2})}{n!} \\ &= \frac{(-1)^{n-1}}{n!2^n} [1.3.5\cdots(2n-3)] \\ &= (-1)^{n+1} \cdot \left[\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots 2n} \right] \\ &= (-1)^{n+1} \cdot \frac{(2n-3)!!}{(2n)!!}\end{aligned}$$

Therefore

$$a_n = (-1)^{n+1} \cdot \frac{(2n-3)!!}{(2n)!!}, \quad \text{for } n = 1, 2, 3, \dots$$

Problem 13.1.12

Express the coefficient of the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x

(a) in terms of the factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. \quad a_n = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad n = 1, 2, 3, \dots$$

Solution For (a) the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x is:

$$\begin{aligned} a_n &= \binom{-\frac{1}{2}}{n-1} \\ &= \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right) \left(-\frac{1}{2} - 2\right) \left(-\frac{1}{2} - 3\right) \cdots \left(-\frac{1}{2} - (n-1)\right)}{n!} \\ &= \frac{\left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \cdots \left(-\frac{2n-1}{2}\right)}{n!} \\ &= \frac{(-1)^n}{n!2^n} [1.3.5 \cdots (2n-1)] \\ &= \frac{(-1)^n}{n!2^n} \left[\frac{1.2.3.4.5.6 \cdots (2n-1) \cdot 2n}{2.4.6 \cdots 2n} \right] \\ &= \frac{(-1)^n}{n!2^n} \cdot \frac{(2n)!}{n!2^n} \\ &= (-1)^n \cdot \frac{(2n)!}{2^{2n} \cdot (n!)^2} \end{aligned}$$

Therefore,

$$a_n = (-1)^n \cdot \frac{(2n)!}{2^{2n} \cdot (n!)^2}, \quad \text{for } n = 1, 2, 3, \dots$$

Solution For (b) the n th term expansion of $(1+x)^{-1/2}$ in powers of x in terms of the double factorial (!!) functions.

$$\begin{aligned} a_n &= \binom{-\frac{1}{2}}{n-1} \\ &= \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right) \left(-\frac{1}{2} - 2\right) \left(-\frac{1}{2} - 3\right) \cdots \left(-\frac{1}{2} - (n-1)\right)}{n!} \\ &= \frac{\left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \cdots \left(-\frac{2n-1}{2}\right)}{n!} \\ &= \frac{(-1)^n}{n!2^n} [1.3.5 \cdots (2n-1)] \\ &= (-1)^n \cdot \left[\frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n} \right] \\ &= (-1)^n \cdot \frac{(2n-1)!!}{(2n)!!} \end{aligned}$$

Therefore

$$a_n = (-1)^n \cdot \frac{(2n-1)!!}{(2n)!!}, \quad \text{for } n = 1, 2, 3, \dots$$

Problem 13.1.14

(a) Show that $\Gamma\left(\frac{1}{2} - n\right) \Gamma\left(\frac{1}{2} + n\right) = (-1)^n \pi$, where n is an integer.

(b) Express $\Gamma\left(\frac{1}{2} + n\right)$ and $\Gamma\left(\frac{1}{2} - n\right)$ separately in terms of $\pi^{1/2}$ and a double factorial function.

$$ANS. \quad \Gamma\left(\frac{1}{2} + n\right) = \frac{(2n-1)!!}{2^n} \pi^{1/2}$$

Solution For (a) recall that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Putting $z = \frac{1}{2} + n$ in the above relation, it becomes

$$\begin{aligned} \Gamma\left(\frac{1}{2} + n\right) \Gamma\left(1 - \frac{1}{2} - n\right) &= \frac{\pi}{\sin\left[\pi\left(\frac{1}{2} + n\right)\right]} \\ &= \frac{\pi}{\cos(n\pi)} \\ &= \frac{\pi}{(-1)^n} \end{aligned}$$

since $\cos(n\pi) = (-1)^n$ and

$$= (-1)^n \pi$$

Therefore

$$\Gamma\left(\frac{1}{2} - n\right) \Gamma\left(\frac{1}{2} + n\right) = (-1)^n \pi$$

where n is an integer.

Solution For (b) recall the Legendre's duplication formula,

$$\Gamma(1+z)\Gamma\left(z + \frac{1}{2}\right) = 2^{-2z} \sqrt{\pi} \Gamma(2z+1)$$

Putting $z = n$ in the above relation, it becomes

$$\Gamma(1+n)\Gamma\left(n + \frac{1}{2}\right) = 2^{-2n} \sqrt{\pi} \Gamma(2n+1)$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{2^{-2n} \sqrt{\pi} \Gamma(2n+1)}{\Gamma(1+n)}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}} \cdot \frac{(2n)!}{n!}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}} \cdot \frac{(1.2.3.4.5 \dots 2n)}{(1.2.3 \dots n)}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} \cdot \frac{(1.2.3.4.5 \dots 2n)}{(2.4.6 \dots 2n)}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} \cdot [1.3.5 \dots (2n-1)]$$

$$\Gamma\left(\frac{1}{2} + n\right) = \frac{\sqrt{\pi}}{2^n} \cdot (2n-1)!! \dots$$

From part (a)

$$\Gamma\left(\frac{1}{2} - n\right) \Gamma\left(\frac{1}{2} + n\right) = (-1)^n \pi$$

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n \pi}{\Gamma\left(\frac{1}{2} + n\right)}$$

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n \pi}{\left(\frac{\sqrt{\pi}}{2^n} \cdot (2n-1)!!\right)}$$

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n \cdot 2^n \sqrt{\pi}}{(2n-1)!!}$$

$$\Gamma\left(\frac{1}{2} + n\right) = \frac{\sqrt{\pi}}{2^n} \cdot (2n-1)!! \text{ and } \Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n \cdot 2^n \sqrt{\pi}}{(2n-1)!!}$$

Problem 13.1.6

Prove that

$$|\Gamma(\alpha + i\beta)| = |\Gamma(\alpha)| \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(\alpha + n)^2}\right]^{-1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} = (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha + i\beta}{n}\right) e^{-\frac{\alpha + i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha - i\beta)} = (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha - i\beta}{n}\right) e^{-\frac{\alpha - i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned} \frac{1}{\Gamma(\alpha + i\beta)} \cdot \frac{1}{\Gamma(\alpha - i\beta)} &= (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \cdot (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \\ &\quad \times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha + i\beta}{n}\right) e^{-\frac{\alpha + i\beta}{n}} \cdot \left(1 + \frac{\alpha - i\beta}{n}\right) e^{-\frac{\alpha - i\beta}{n}} \right] \\ \frac{1}{|\Gamma(\alpha + i\beta)|^2} &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2\alpha}{n}} \left[\left(1 + \frac{\alpha + i\beta}{n}\right) \cdot \left(1 + \frac{\alpha - i\beta}{n}\right) \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{\frac{2\alpha}{n} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2}} \cdot \left(1 + \frac{\alpha}{n}\right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2 \right] \\ &= \left(\frac{\alpha^2 + \beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{\alpha}{n}} \cdot \left(1 + \frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{(1 + \frac{2\alpha}{n} + \frac{\alpha^2 + \beta^2}{n^2})}{\frac{(n + \alpha)^2}{n^2}}\right] \\ &= \left(1 + \frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{(1 + 2\alpha n + \alpha^2 + \beta^2)}{(n + \alpha)^2}\right] \\ &= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n + \alpha)^2 + \beta^2}{(n + \alpha)^2}\right] \\ &= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right] \\ &= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right] \end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{|\Gamma(\alpha + i\beta)|^2} &= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2} \right] \\ \frac{1}{|\Gamma(\alpha + i\beta)|} &= \frac{1}{|\Gamma(\alpha)|} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2} \right]^{\frac{1}{2}} \\ |\Gamma(\alpha + i\beta)| &= |\Gamma(\alpha)| \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(\alpha + n)^2} \right]^{-\frac{1}{2}}\end{aligned}$$

Problem 13.1.17

Show that for n , a positive integer,

$$|\Gamma(n + ib + 1)| = \left(\frac{\pi b}{\sinh \pi b} \right)^{1/2} \prod_{s=1}^n (s^2 + b^2)^{1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} = (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha + i\beta}{n} \right) e^{-\frac{\alpha + i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha - i\beta)} = (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha - i\beta}{n} \right) e^{-\frac{\alpha - i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned}\frac{1}{\Gamma(\alpha + i\beta)} \cdot \frac{1}{\Gamma(\alpha - i\beta)} &= (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \cdot (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \\ &\quad \times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha + i\beta}{n} \right) e^{-\frac{\alpha + i\beta}{n}} \cdot \left(1 + \frac{\alpha - i\beta}{n} \right) e^{-\frac{\alpha - i\beta}{n}} \right] \\ \frac{1}{|\Gamma(\alpha + i\beta)|^2} &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2\alpha}{n}} \left[\left(1 + \frac{\alpha + i\beta}{n} \right) \cdot \left(1 + \frac{\alpha - i\beta}{n} \right) \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{\frac{2\alpha}{n} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2}} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= \left(\frac{\alpha^2 + \beta^2}{\alpha^2} \right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{\alpha}{n}} \cdot \left(1 + \frac{\alpha}{n} \right) \right] \right)^2 \prod_{n=1}^{\infty} \left[\frac{(1 + \frac{2\alpha}{n} + \frac{\alpha^2 + \beta^2}{n^2})}{\frac{(n + \alpha)^2}{n^2}} \right] \\ &= \left(1 + \frac{\beta^2}{\alpha^2} \right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{(1 + 2\alpha n + \alpha^2 + \beta^2)}{(n + \alpha)^2} \right] \\ &= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2} \right) \prod_{n=1}^{\infty} \left[\frac{(n + \alpha)^2 + \beta^2}{(n + \alpha)^2} \right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]
\end{aligned}$$

Hence

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha = 1$ and $\beta = b$ in the above identity. Then it becomes

$$\begin{aligned}
\frac{1}{|\Gamma(1 + ib)|^2} &= \frac{1}{\Gamma(1)^2} \prod_{n=0}^{\infty} \left[1 + \frac{b^2}{(n+1)^2}\right] \\
&= \prod_{n=0}^{\infty} \left[1 + \frac{b^2}{(n+1)^2}\right], \quad \text{as } \Gamma(1) = 1 \\
&= \prod_{n=0}^{\infty} \left[1 - \frac{(ib\pi)^2}{(n+1)^2\pi^2}\right] \\
&= \prod_{n=1}^{\infty} \left[1 - \frac{(ib\pi)^2}{n^2\pi^2}\right] \\
&= \frac{1}{(ib\pi)} \left\{ (ib\pi) \prod_{n=1}^{\infty} \left[1 - \frac{(ib\pi)^2}{n^2\pi^2}\right] \right\} \\
&= \frac{1}{ib\pi} \cdot \sin(ib\pi)
\end{aligned}$$

Using the identity

$$\begin{aligned}
\sin z &= z \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{n^2\pi^2}\right] \quad \text{for } z = ib\pi \\
&= \frac{1}{ib\pi} \cdot i \sinh(b\pi) \\
&= \frac{\sinh(b\pi)}{b\pi} \\
\frac{1}{|\Gamma(1 + ib)|^2} &= \frac{\sinh(b\pi)}{b\pi} \\
|\Gamma(1 + ib)|^2 &= \frac{b\pi}{\sinh(b\pi)}.
\end{aligned}$$

since n is an integer, therefore

$$\begin{aligned}
\Gamma(n + ib + 1) &= \Gamma(\{1 + ib + (n-1)\} + 1) \\
&= \{1 + ib + (n-1)\} \Gamma(\{1 + ib + (n-1)\}) \\
&= (1 + ib)(2 + ib)(3 + ib) \cdots (n + ib) \Gamma(1 + ib) \\
\Gamma(n + ib + 1) &= (1 + ib)(2 + ib)(3 + ib) \cdots (n + ib) \Gamma(1 + ib) \\
\Gamma(n - ib + 1) &= (1 - ib)(2 - ib)(3 - ib) \cdots (n - ib) \Gamma(1 - ib) \\
|\Gamma(n + ib + 1)|^2 &= \Gamma(n + ib + 1) \Gamma(n - ib + 1) \\
&= (1 + ib)(2 + ib)(3 + ib) \cdots (n + ib) \Gamma(1 + ib) \times (1 - ib)(2 - ib)(3 - ib) \cdots (n - ib) \Gamma(1 - ib) \\
&= \{(1 + ib)(1 - ib)\} \{(2 + ib)(2 - ib)\} \{(3 + ib)(3 - ib)\} \cdots \{(n + ib)(n - ib)\} \Gamma(1 + ib) \Gamma(1 - ib) \\
&= (1^2 + b^2) (2^2 + b^2) (3^2 + b^2) \cdots (n^2 + b^2) |\Gamma(1 + ib)|^2
\end{aligned}$$

$$= \prod_{s=1}^n (s^2 + b^2) \times \frac{b\pi}{\sinh(b\pi)}$$

Hence

$$|\Gamma(n + ib + 1)|^2 = \prod_{s=1}^n (s^2 + b^2) \times \frac{b\pi}{\sinh(b\pi)}$$

This gives

$$|\Gamma(n + ib + 1)| = \left(\frac{b\pi}{\sinh(b\pi)} \right)^{\frac{1}{2}} \prod_{s=1}^n (s^2 + b^2)^{\frac{1}{2}}$$

Problem 13.1.18

Show that for all real values of x and y , $|\Gamma(x)| \geq |\Gamma(x + iy)|$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} = (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha + i\beta}{n} \right) e^{-\frac{\alpha + i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha - i\beta)} = (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha - i\beta}{n} \right) e^{-\frac{\alpha - i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned} \frac{1}{\Gamma(\alpha + i\beta)} \cdot \frac{1}{\Gamma(\alpha - i\beta)} &= (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \cdot (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \\ &\quad \times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha + i\beta}{n} \right) e^{\frac{\alpha + i\beta}{n}} \cdot \left(1 + \frac{\alpha - i\beta}{n} \right) e^{\frac{\alpha - i\beta}{n}} \right] \\ \frac{1}{|\Gamma(\alpha + i\beta)|^2} &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2\alpha}{n}} \left[\left(1 + \frac{\alpha + i\beta}{n} \right) \cdot \left(1 + \frac{\alpha - i\beta}{n} \right) \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{\frac{2\alpha}{n} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2}} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= \left(\frac{\alpha^2 + \beta^2}{\alpha^2} \right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{\alpha}{n}} \cdot \left(1 + \frac{\alpha}{n} \right) \right] \right)^2 \prod_{n=1}^{\infty} \left[\frac{(1 + \frac{2\alpha}{n} + \frac{\alpha^2 + \beta^2}{n^2})}{\frac{(n + \alpha)^2}{n^2}} \right] \\ &= \left(1 + \frac{\beta^2}{\alpha^2} \right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{(1 + 2\alpha n + \alpha^2 + \beta^2)}{(n + \alpha)^2} \right] \\ &= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2} \right) \prod_{n=1}^{\infty} \left[\frac{(n + \alpha)^2 + \beta^2}{(n + \alpha)^2} \right] \\ &= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2} \right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2} \right] \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2} \right]$$

Hence

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2} \right]$$

Now put $\alpha = x$ and $\beta = y$ in the above identity. Then it becomes

$$\begin{aligned} \frac{1}{|\Gamma(x + iy)|^2} &= \frac{1}{\Gamma(x)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+x)^2} \right] \\ \left| \frac{\Gamma(x)}{\Gamma(x + iy)} \right|^2 &= \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n+x)^2} \right] \\ \left| \frac{\Gamma(x)}{\Gamma(x + iy)} \right|^2 &\geq 1, \quad \text{since} \quad 1 + \frac{\beta^2}{(n+x)^2} \geq 1 \\ \left| \frac{\Gamma(x)}{\Gamma(x + iy)} \right| &\geq 1 \\ |\Gamma(x)| &\geq |\Gamma(x + iy)| \end{aligned}$$

Hence is proved

Problem 13.1.19

Show that

$$\left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2 = \frac{\pi}{\cosh \pi y}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} = (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha + i\beta}{n} \right) e^{-\frac{\alpha + i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha - i\beta)} = (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha - i\beta}{n} \right) e^{-\frac{\alpha - i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned} \frac{1}{\Gamma(\alpha + i\beta)} \cdot \frac{1}{\Gamma(\alpha - i\beta)} &= (\alpha + i\beta) e^{\gamma(\alpha + i\beta)} \cdot (\alpha - i\beta) e^{\gamma(\alpha - i\beta)} \\ &\quad \times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha + i\beta}{n} \right) e^{\frac{\alpha + i\beta}{n}} \cdot \left(1 + \frac{\alpha - i\beta}{n} \right) e^{\frac{\alpha - i\beta}{n}} \right] \\ \frac{1}{|\Gamma(\alpha + i\beta)|^2} &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2\alpha}{n}} \left[\left(1 + \frac{\alpha + i\beta}{n} \right) \cdot \left(1 + \frac{\alpha - i\beta}{n} \right) \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{\frac{2\alpha}{n} \cdot \frac{(1 + \frac{\alpha + i\beta}{n}) \cdot (1 + \frac{\alpha - i\beta}{n})}{(1 + \frac{\alpha}{n})^2}} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \\ &= (\alpha^2 + \beta^2) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{\left(1 + \frac{\alpha + i\beta}{n} \right) \cdot \left(1 + \frac{\alpha - i\beta}{n} \right)}{\left(1 + \frac{\alpha}{n} \right)^2} \cdot \left(1 + \frac{\alpha}{n} \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
&= (\alpha^2 + \beta^2) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)}{\left(1 + \frac{\alpha}{n}\right)^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2 \right] \\
&= \left(\frac{\alpha^2 + \beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma \alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{a}{n}} \cdot \left(1 + \frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{\left(1 + \frac{2\alpha}{n} + \frac{\alpha^2 + \beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right] \\
&= \left(1 + \frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{(1 + 2\alpha n + \alpha^2 + \beta^2)}{(n + \alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n + \alpha)^2 + \beta^2}{(n + \alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right]
\end{aligned}$$

Hence

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right]$$

Now put $\alpha = \frac{1}{2}$ and $\beta = y$ in the above identity. Then it becomes

$$\begin{aligned}
\frac{1}{|\Gamma(\frac{1}{2} + iy)|^2} &= \frac{1}{\Gamma(\frac{1}{2})^2} \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(n + \frac{1}{2})^2}\right] \\
\frac{1}{|\Gamma(\frac{1}{2} + iy)|^2} &= \frac{1}{\pi} \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(n + \frac{1}{2})^2}\right]
\end{aligned}$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\frac{1}{|\Gamma(\frac{1}{2} + iy)|^2} = \frac{1}{\pi} \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(n + \frac{1}{2})^2}\right]$$

Recall

$$\cos z = \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{(n - \frac{1}{2})^2 \pi^2}\right]$$

and putting $z = i\pi y$ it becomes

$$\begin{aligned}
\cos(i\pi y) &= \prod_{n=1}^{\infty} \left[1 - \frac{i^2 \pi^2 y^2}{(n - \frac{1}{2})^2 \pi^2}\right] \\
\cosh(\pi y) &= \prod_{n=1}^{\infty} \left[1 + \frac{y^2}{(n - \frac{1}{2})^2}\right] \\
\cosh(\pi y) &= \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(n + 1 - \frac{1}{2})^2}\right] \\
\cosh(\pi y) &= \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(n + \frac{1}{2})^2}\right] \\
\frac{1}{|\Gamma(\frac{1}{2} + iy)|^2} &= \frac{1}{\pi} \cosh(\pi y)
\end{aligned}$$

Problem 13.1.20

The probability density associated with the normal distribution of statistics is given by

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right]$$

with $(-\infty, \infty)$ for the range of x . Show that (a)

(a) $\langle x \rangle$, the mean value of x , is equal to μ

(b) the standard deviation $(\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ is given by σ .

Solution For (a) For the mean

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma(2\pi)^{1/2}} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right] dx \\ &= \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\infty} x e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx \end{aligned}$$

Put $x - \mu = y$. Then $dx = dy$. As $x \rightarrow 0, y \rightarrow 0$ and $x \rightarrow \infty, y \rightarrow \infty$.

$$\begin{aligned} \langle x \rangle &= \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\infty} x e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\infty} (\mu + y) e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\infty} (\mu + y) e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \frac{\mu}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy \end{aligned}$$

since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy = 2 \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

and since $y e^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = 0$$

Therefore, the integral becomes

$$\langle x \rangle = \frac{2\mu}{\sigma(2\pi)^{1/2}} \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2y dy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y} dz$, that is, $dy = \frac{\sigma}{\sqrt{2}} z^{-1/2} dz$. As $y \rightarrow 0, z \rightarrow 0$ and $y \rightarrow \infty, z \rightarrow \infty$. Therefore

$$\begin{aligned} \langle x \rangle &= \frac{2\mu}{\sigma(2\pi)^{1/2}} \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \frac{2\mu}{\sigma(2\pi)^{1/2}} \int_0^{\infty} e^{-z} \frac{\sigma}{\sqrt{2}} z^{-1/2} dz \\ &= \frac{2\mu}{\sigma(2\pi)^{1/2}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz \\ &= \frac{2\mu}{\sigma(2\pi)^{1/2}} \cdot \frac{\sigma}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \sqrt{\pi} \\
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \sqrt{\pi} \\
&= \mu
\end{aligned}$$

Solution For (b) we start saying

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^\infty x^2 f(x) dx \\
&= \int_{-\infty}^\infty x^2 \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp \left[-\frac{(x-\mu)^2}{2\sigma^2} \right] dx \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx
\end{aligned}$$

Put $x - \mu = y$. Then $dx = dy$. As $x \rightarrow 0, y \rightarrow 0$ and $x \rightarrow \infty, y \rightarrow \infty$.

$$\begin{aligned}
\langle x^2 \rangle &= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty (\mu + y)^2 e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty (\mu^2 + 2\mu y + y^2) e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy + \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy
\end{aligned}$$

since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy = 2 \int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy$$

since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^\infty ye^{-\frac{y^2}{2\sigma^2}} dy = 0$$

since $y^2 e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy = 2 \int_0^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy$$

Therefore the above integral becomes

$$\langle x^2 \rangle = \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y} dz$, that is, $dy = \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$. As $y \rightarrow 0, z \rightarrow 0$ and $y \rightarrow \infty, z \rightarrow \infty$. Therefore

$$\begin{aligned}
\langle x^2 \rangle &= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{\frac{1}{2}} dz + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty 2\sigma^2 z e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz \\
&= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz + \frac{2\sqrt{2}\sigma^3}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz \\
&= \frac{\mu^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{3}{2}-1} dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mu^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{\mu^2}{\sqrt{\pi}} \cdot \sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \sqrt{\pi} \\
&= \mu^2 + \sigma^2
\end{aligned}$$

So the standard deviation

$$\begin{aligned}
(\langle x^2 \rangle - \langle x \rangle^2)^{\frac{1}{2}} &= (\mu^2 + \sigma^2 - \mu^2)^{\frac{1}{2}} \\
(\langle x^2 \rangle - \langle x \rangle^2)^{\frac{1}{2}} &= \sqrt{\sigma^2} \\
(\langle x^2 \rangle - \langle x \rangle^2)^{\frac{1}{2}} &= \sigma
\end{aligned}$$

Problem 13.1.21

For the gamma distribution

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(a) $\langle x \rangle$, the mean value of x , is equal to $\alpha\beta$

(b) σ^2 , its variance, defined as $\langle x^2 \rangle - \langle x \rangle^2$, has the value $\alpha\beta^2$

Solution For (a) the mean

$$\begin{aligned}
\langle x \rangle &= \int_0^\infty x f(x) dx \\
&= \int_0^\infty x \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^\alpha e^{-\frac{x}{\beta}} dx
\end{aligned}$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \rightarrow 0, z \rightarrow 0$ and $x \rightarrow \infty, z \rightarrow \infty$.

$$\begin{aligned}
\langle x \rangle &= \frac{1}{\Gamma(\alpha)} \int_0^\infty z^\alpha e^{-z} \beta dz \\
&= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+1)-1} e^{-z} dz \\
&= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) \\
&= \frac{\beta}{\Gamma(\alpha)} \cdot \alpha \Gamma(\alpha) \\
&= \alpha\beta
\end{aligned}$$

Solution For (b)

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^\infty x^2 f(x) dx \\
&= \int_0^\infty x^2 \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha+1} e^{-\frac{x}{\beta}} dx
\end{aligned}$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \rightarrow 0, z \rightarrow 0$ and $x \rightarrow \infty, z \rightarrow \infty$

$$\langle x^2 \rangle = \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{\alpha+1} e^{-z} \beta dz$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+2)-1} e^{-z} dz$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha + 2)$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \cdot (\alpha + 1) \alpha \Gamma(\alpha)$$

$$= \alpha(\alpha + 1) \beta^2$$

$$= \alpha^2 \beta^2 + \alpha \beta^2$$

Hence variance, $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$= \alpha^2 \beta^2 + \alpha \beta^2 - \alpha^2 \beta^2$$

$$= \alpha \beta^2$$