

Problem 14.3.1

Prove that the Neumann functions Y_n (with n an integer) satisfy the recurrence relations

$$\begin{aligned} Y_{n-1}(x) + Y_{n+1}(x) &= \frac{2n}{x} Y_n(x) \\ Y_{n-1}(x) - Y_{n+1}(x) &= 2Y'_n(x) \end{aligned}$$

Hint. These relations may be proved by differentiating the recurrence relations for J_v or by using the limit form of Y_v but not dividing everything by zero.

Solution As

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

with n as an integer, and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} + \lim_{v \rightarrow n} \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi}$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} - \lim_{v \rightarrow n} \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi}$$

As $Y_n(x) = \lim_{v \rightarrow n} Y_v(x)$ exists and is not identically zero, we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} + \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi} \right)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} - \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi} \right)$$

As $\cos(v-1)\pi = \cos(\pi - v\pi) = -\cos v\pi$, $\cos(v+1)\pi = -\cos v\pi$, $\sin(v-1)\pi = -\sin(\pi - v\pi) = \sin v\pi$ and $\sin(v+1)\pi = -\sin v\pi$ we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{-\cos v\pi J_{v-1}(x) - J_{-v+1}(x)}{-\sin v\pi} + \frac{-\cos v\pi J_{v+1}(x) - J_{-v-1}(x)}{-\sin v\pi} \right)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{-\cos v\pi J_{v-1}(x) - J_{-v+1}(x)}{-\sin v\pi} - \frac{-\cos v\pi J_{v+1}(x) - J_{-v-1}(x)}{-\sin v\pi} \right)$$

Thus,

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} + \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi} \right)$$

and

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} - \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi} \right)$$

Also,

$$\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} + \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

can be written as

$$\frac{\cos v\pi J_{v-1}(x) + \cos v\pi J_{v+1}(x) + J_{-v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

and hence we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi (J_{v-1}(x) + J_{v+1}(x)) + (J_{-v+1}(x) + J_{-v-1}(x))}{\sin v\pi} \right)$$

Similarly,

$$\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} - \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

can be written as

$$\frac{\cos v\pi J_{v-1}(x) - \cos v\pi J_{v+1}(x) + J_{-v+1}(x) - J_{-v-1}(x)}{\sin v\pi}$$

and hence we get that

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi (J_{v-1}(x) - J_{v+1}(x)) - (J_{-v-1}(x) - J_{-v+1}(x))}{\sin v\pi} \right)$$

We now have to prove that

$$\begin{aligned} J_{v-1}(x) + J_{v+1}(x) &= \frac{2v}{x} J_v(x) \\ J_{v-1}(x) - J_{v+1}(x) &= 2J'_v(x) \\ \frac{d}{dx} (x^v J_v(x)) &= \frac{d}{dx} \left(x^v \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v} \right) \end{aligned}$$

which implies that

$$\begin{aligned} \frac{d}{dx} (x^v J_v(x)) &= \frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v}}{s! \Gamma(v+s+1) 2^{2s+v}} \right) \\ \frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v}}{s! \Gamma(v+s+1) 2^{2s+v}} \right) &= \sum_{s=0}^{\infty} \frac{(-1)^s (2s+2v) (x)^{2s+2v-1}}{s! \Gamma(v+s+1) 2^{2s+v}} = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v-1}}{s! \Gamma(v+s) 2^{2s+v-1}} \end{aligned}$$

As

$$J_{v-1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma-1}}{s! \Gamma(v+s) 2^{2s+v-1}}$$

and

$$\frac{d}{dx} (x^v J_v(x)) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v-1}}{s! \Gamma(v+s) 2^{2s+v-1}}$$

we get that

$$\frac{d}{dx} (x^v J_v(x)) = x^v \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma-1}}{s! \Gamma(v+s) 2^{2s+v-1}} \right) = x^v J_{v-1}(x)$$

Similarly,

$$\frac{d}{dx} (x^{-v} J_v(x)) = \frac{d}{dx} \left(x^{-v} \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v} \right)$$

which implies that

$$\frac{d}{dx} (x^{-v} J_v(x)) = \frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s}}{s! \Gamma(v+s+1) 2^{2s+v}} \right)$$

Also

$$\begin{aligned} \frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s! \Gamma(v+s+1) 2^{2s+v}} \right) &= \sum_{s=0}^{\infty} \frac{(-1)^s (2s) (x)^{2s-1}}{s! \Gamma(v+s+1) 2^{2s+v}} = \sum_{s=1}^{\infty} \frac{(-1)^s (s) x^{2s-1}}{s! \Gamma(v+s) 2^{2s+v-1}} \\ \sum_{s=1}^{\infty} \frac{(-1)^s (s) x^{2s-1}}{s! \Gamma(v+s) 2^{2s+v-1}} &= - \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x^{2(s-1)+1}}{(s-1)! \Gamma(v+s-1+1) 2^{2(s-1)+*+1}} = - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k! \Gamma(v+k+1) 2^{2k+w+1}} \end{aligned}$$

As

$$J_{v+1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma+1}}{s! \Gamma(v+s+1) 2^{2s+v+1}}$$

and

$$\frac{d}{dx} (x^{-v} J_v(x)) = - \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+1}}{s! \Gamma(v+s) 2^{2s+v+1}}$$

we get that

$$\frac{d}{dx} (x^{-v} J_v(x)) = -x^{-v} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+v+1}}{s! \Gamma(v+s) 2^{2s+v+1}} \right) = -x^{-v} J_{v+1}(x)$$

Then

$$\frac{d}{dx} (x^v J_v(x)) = x^v \frac{d}{dx} (J_v(x)) + \frac{d}{dx} (x^v) J_v(x) = x^v J'_v(x) + v x^{v-1} J_v(x)$$

as

$$\frac{d}{dx} (x^v J_v(x)) = x^v J'_v(x) + vx^{v-1} J_v(x)$$

and

$$\frac{d}{dx} (x^v J_v(x)) = x^v J_{v-1}(x)$$

we get that

$$x^v J'_v(x) + vx^{v-1} J_v(x) = x^v J_{v-1}(x)$$

and

$$J'_v(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

Also

$$\frac{d}{dx} (x^{-v} J_v(x)) = x^{-v} \frac{d}{dx} (J_v(x)) + \frac{d}{dx} (x^{-v}) J_v(x) = x^{-v} J'_v(x) - vx^{-v-1} J_v(x)$$

$$\frac{d}{dx} (x^{-v} J_v(x)) = x^{-v} J'_v(x) - vx^{-v-1} J_v(x)$$

and

$$\frac{d}{dx} (x^{-v} J_v(x)) = -x^{-v} J_{v+1}(x)$$

we get that

$$x^{-v} J'_v(x) - vx^{-v-1} J_v(x) = -x^{-v} J_{v+1}(x)$$

and

$$J'_v(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x)$$

As

$$J'_v(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

and

$$J'_v(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x),$$

we get that

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

and

$$J_{v-1}(x) - J_{v+1}(x) = 2J'_v(x)$$

As

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi (J_{v-1}(x) + J_{v+1}(x)) + (J_{-v+1}(x) + J_{-v-1}(x))}{\sin v\pi} \right)$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{x \rightarrow n} \left(\frac{\cos v\pi \left(\frac{2v}{x} J_v(x) \right) + \left(\frac{2(-v)}{x} J_{-v}(x) \right)}{\sin v\pi} \right)$$

As

$$J_{v-1}(x) - J_{v+1}(x) = 2J'_v(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi (J_{v-1}(x) - J_{v+1}(x)) - (J_{-v-1}(x) - J_{-v+1}(x))}{\sin v\pi} \right)$$

we get that

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi (2J'_v(x)) - (2J'_{-v}(x))}{\sin v\pi} \right)$$

Also

$$\frac{\cos v\pi \left(\frac{2v}{x} J_v(x) \right) + \left(\frac{2(-v)}{x} J_{-v}(x) \right)}{\sin v\pi} = \frac{2v}{x} \left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} \right)$$

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \frac{2v}{x} \left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} \right)$$

and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \rightarrow n} \frac{2v}{x} Y_v(x) = \frac{2n}{x} Y_n(x)$$

Also,

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \rightarrow n} \left(\frac{\cos v\pi (2J'_v(x)) - (2J'_{-v}(x))}{\sin v\pi} \right) = 2 \lim_{v \rightarrow n} \left(\frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} \right)$$

Now,

$$\frac{d}{dx} (Y_v(x)) = \frac{d}{dx} \left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} \right) = \frac{\cos v\pi \frac{d}{dx} (J_v(x)) - \frac{d}{dx} (J_{-v}(x))}{\sin v\pi}$$

Hence, we get

$$Y'_v(x) = \frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi}$$

As

$$Y_{n-1}(x) - Y_{n+1}(x) = 2 \lim_{v \rightarrow n} \left(\frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} \right)$$

and

$$Y'_n(x) = \frac{\cos v\pi J'_n(x) - J'_{-n}(x)}{\sin v\pi}$$

we get

$$Y_{n-1}(x) - Y_{n+1}(x) = 2 \lim_{v \rightarrow n} Y'_v(x) = 2Y'_n(x)$$

Therefore, the recurrence relations are

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y'_n(x)$$

are true when n is an integer

Problem 14.3.2

Show that for integer n

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

Solution We know that for an integer n ,

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

Clearly, the statement is true for $n = 0$. Clearly, for any integer n ,

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

can be rewritten as

$$(-1)^{-n} Y_{-n}(x) = Y_n(x)$$

which implies that

$$Y_{-(-n)}(x) = (-1)^{-n} Y_{-n}(x)$$

This implies that if the statement is true for any positive integer n , then it is true for any integer n . Assume that the statement

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

is true for any non-negative integer $n \leq k$ where k is any arbitrary non-negative integer. Now we have to prove that $Y_{-k-1}(x) = (-1)^{k+1} Y_{k+1}(x)$ i.e. the statement is true for $n = k + 1$. Also, by substituting $n = 0$ in

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

and hence

$$Y_{-1}(x) = -Y_1(x)$$

As $Y_{-1}(x) = -Y_1(x)$, we get that the statement is true for $n = 1$. As the statement is true for $n = 0$ and $n = 1$, we can assume that $k \geq 1$. As the statement $Y_{-n}(x) = (-1)^n Y_n(x)$ is true for any non-negative integer $n \leq k$, we get that

$$Y_{-k}(x) = (-1)^k Y_k(x)$$

and

$$Y_{-k+1}(x) = (-1)^{k-1} Y_{k-1}(x)$$

As

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

for any integer n , we get that

$$Y_{-k-1}(x) + Y_{-k+1}(x) = \frac{2(-k)}{x} Y_{-k}(x)$$

and hence

$$Y_{-k-1}(x) = -\frac{2k}{x} Y_{-k}(x) - Y_{-k+1}(x)$$

As

$$Y_{-k-1}(x) = -\frac{2k}{x} Y_{-k}(x) - Y_{-k+1}(x), \quad Y_{-k}(x) = (-1)^k Y_k(x)$$

and

$$Y_{-k+1}(x) = (-1)^{k-1} Y_{k-1}(x)$$

we get that

$$Y_{-k-1}(x) = -\frac{2k}{x} ((-1)^k Y_k(x)) - (-1)^{k-1} Y_{k-1}(x)$$

Also

$$Y_{-k-1}(x) = -\frac{2k}{x} ((-1)^k Y_k(x)) - (-1)^{k-1} Y_{k-1}(x) = (-1)^{k+1} \left(\frac{2k}{x} Y_k(x) - Y_{k-1}(x) \right)$$

As

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

for any integer n , we get that

$$Y_{k-1}(x) + Y_{k+1}(x) = \frac{2k}{x} Y_k(x)$$

and hence

$$Y_{k+1}(x) = \frac{2k}{x} Y_k(x) - Y_{k-1}(x)$$

As

$$Y_{k+1}(x) = \frac{2k}{x} Y_k(x) - Y_{k-1}(x)$$

and

$$Y_{-k-1}(x) = (-1)^{k+1} \left(\frac{2k}{x} Y_k(x) - Y_{k-1}(x) \right)$$

we get that

$$Y_{-k-1}(x) = (-1)^{k+1} Y_{k+1}(x)$$

Therefore, by Mathematical induction, we get that the statement

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

for any non-negative integer n .

Problem 14.4.3

Show that

$$Y_0'(x) = -Y_1(x)$$

Solution As

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

for any integer n , we get that the statement is true for $n = 0$ which implies that

$$Y_{-1}(x) + Y_1(x) = \frac{2(0)}{x} Y_0(x) = 0$$

and hence $Y_{-1}(x) = -Y_1(x)$. As

$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x)$$

for any integer n , we get that the statement is true for $n = 0$ which implies that

$$Y_{-1}(x) - Y_1(x) = 2Y_0'(x)$$

As

$$Y_{-1}(x) = -Y_1(x)$$

and

$$Y_{-1}(x) - Y_1(x) = 2Y_0'(x)$$

we get that

$$2Y_0'(x) = Y_{-1}(x) - Y_1(x) = -Y_1(x) - Y_1(x) = -2Y_1(x)$$

and hence

$$Y_0'(x) = -Y_1(x)$$

Therefore, the statement

$$Y_0'(x) = -Y_1(x)$$

is true.

Problem 14.3.4

If X and Z are any two solutions of Bessel's equation, show that

$$X_\nu(x)Z'_\nu(x) - X'_\nu(x)Z_\nu(x) = \frac{A_\nu}{x}$$

in which A_ν may depend on ν but is independent of x . This is a special case of Exercise 7.6.11

Solution We know that for a linear second order homogeneous ODE of form

$$y'' + P(x)y' + Q(x)y = 0$$

and two solutions y_1, y_2 of this ODE, we have that the Wronskian W of y_1 and y_2 satisfies the equation

$$W(x) = W(a) \exp \left[- \int_a^x P(t) dt \right]$$

Thus, the Wronskian W of $X_\nu(x)$ and $Z_\nu(x)$ satisfies the equation

$$W(x) = W(a) \exp \left[- \int_a^x P(t) dt \right]$$

As any Bessel's equation is of the form

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0,$$

we get that $P(x) = \frac{1}{x}$ and hence

$$\int_a^x P(t) dt = \int_a^x \frac{1}{t} dt = \ln x - \ln a = \ln \frac{x}{a}$$

Thus,

$$W(x) = W(a) \exp \left[- \int_a^x P(t) dt \right] = W(a) \exp \left[- \ln \frac{x}{a} \right] = W(a) \exp \left[\ln \frac{a}{x} \right] = W(a) \frac{a}{x}$$

Clearly,

$$W(a)a = (X_\nu(a)Z'_\nu(a) - X'_\nu(a)Z_\nu(a))a$$

which implies that $W(a)a$ is a constant independent of x but it may depend on ν . Thus, by taking

$$W(a)a = A_\nu,$$

we get that

$$W(x) = \frac{A_\nu}{x}$$

where A_ν may depend on ν but is independent of x . As the Wronskian W of $X_\nu(x)$ and $Z_\nu(x)$ is equal to

$$X_\nu(x)Z'_\nu(x) - X'_\nu(x)Z_\nu(x)$$

and $W(x) = \frac{A_\nu}{x}$, we get that

$$X_\nu(x)Z'_\nu(x) - X'_\nu(x)Z_\nu(x) = \frac{A_\nu}{x}$$

where A_ν may depend on ν but is independent of x . Therefore, the statement

$$X_\nu(x)Z'_\nu(x) - X'_\nu(x)Z_\nu(x) = \frac{A_\nu}{x}$$

in which A_ν , may depend on ν but is independent of x is true when X and Z are any two solutions of Bessel's equation.

Problem 14.3.5

Verify the Wronskian formulas

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = \frac{2 \sin v\pi}{\pi x}$$

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{2}{\pi x}$$

Solution As

$$Y'_v(x) = \frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi}$$

and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = J_v(x) \frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} - J'_v(x) \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

Also

$$J_v(x) \frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} - J'_v(x) \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

is equal to

$$\frac{-J_v(x)J'_{-v}(x) + J'_v(x)J_{-v}(x)}{\sin v\pi}$$

which implies that

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{-J_v(x)J'_{-v}(x) + J'_v(x)J_{-v}(x)}{\sin v\pi}$$

Clearly,

$$\frac{d}{dx} (x^v J_v(x)) = \frac{d}{dx} \left(x^v \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v} \right)$$

which implies that

$$\frac{d}{dx} (x^v J_v(x)) = \frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v}}{s! \Gamma(v+s+1) 2^{2s+v}} \right)$$

Also

$$\frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v}}{s! \Gamma(v+s+1) 2^{2s+v}} \right) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s+2v)(x)^{2s+2v-1}}{s! \Gamma(v+s+1) 2^{2s+v}} = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v-1}}{s! \Gamma(v+s+1) 2^{2s+v-1}}$$

As

$$J_{v-1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma-1}}{s! \Gamma(v+s) 2^{2s+v-1}}$$

and

$$\frac{d}{dx} (x^v J_v(x)) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+2v-1}}{s! \Gamma(v+s) 2^{2s+v-1}},$$

we get that

$$\frac{d}{dx} (x^v J_v(x)) = x^v \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+\gamma-1}}{s! \Gamma(v+s) 2^{2s+v-1}} \right) = x^v J_{v-1}(x)$$

Similarly,

$$\frac{d}{dx} (x^{-v} J_v(x)) = \frac{d}{dx} \left(x^{-v} \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v} \right)$$

which implies that

$$\frac{d}{dx} (x^{-v} J_v(x)) = \frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s}}{s! \Gamma(v+s+1) 2^{2s+v}} \right)$$

Also

$$\frac{d}{dx} \left(\sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s! \Gamma(v+s+1) 2^{2s+v}} \right) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s)(x)^{2s-1}}{s! \Gamma(v+s+1) 2^{2s+v}} = \sum_{s=1}^{\infty} \frac{(-1)^s (s) x^{2s-1}}{s! \Gamma(v+s) 2^{2s+v-1}}$$

Also

$$\sum_{s=1}^{\infty} \frac{(-1)^s (s) x^{2s-1}}{s! \Gamma(v+s) 2^{2s+v-1}} = - \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x^{2(s-1)+1}}{(s-1)! \Gamma(v+s-1+1) 2^{2(s-1)+v+1}} = - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k! \Gamma(v+k+1) 2^{2k+v+1}}$$

As

$$J_{v+1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+v+1}}{s! \Gamma(v+s+1) 2^{2s+v+1}}$$

and

$$\frac{d}{dx} (x^{-v} J_v(x)) = - \sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+1}}{s! \Gamma(v+s) 2^{2s+v+1}},$$

we get that

$$\frac{d}{dx} (x^{-v} J_v(x)) = -x^{-v} \left(\sum_{s=0}^{\infty} \frac{(-1)^s (x)^{2s+v+1}}{s! \Gamma(v+s) 2^{2s+v+1}} \right) = -x^{-v} J_{v+1}(x)$$

Also

$$\frac{d}{dx} (x^v J_v(x)) = x^v \frac{d}{dx} (J_v(x)) + \frac{d}{dx} (x^v) J_v(x) = x^v J'_v(x) + v x^{v-1} J_v(x)$$

As

$$\frac{d}{dx} (x^v J_v(x)) = x^v J'_v(x) + v x^{v-1} J_v(x)$$

and

$$\frac{d}{dx} (x^v J_v(x)) = x^v J_{v-1}(x),$$

we get that

$$x^v J'_v(x) + v x^{v-1} J_v(x) = x^v J_{v-1}(x)$$

and hence

$$J'_v(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

Also

$$\frac{d}{dx} (x^{-v} J_v(x)) = x^{-v} \frac{d}{dx} (J_v(x)) + \frac{d}{dx} (x^{-v}) J_v(x) = x^{-v} J'_v(x) - v x^{-v-1} J_v(x)$$

As

$$\frac{d}{dx} (x^{-v} J_v(x)) = x^{-v} J'_v(x) - v x^{-v-1} J_v(x)$$

and

$$\frac{d}{dx} (x^{-v} J_v(x)) = -x^{-v} J_{v+1}(x),$$

we get that

$$x^{-v} J'_v(x) - v x^{-v-1} J_v(x) = -x^{-v} J_{v+1}(x)$$

and hence

$$J'_v(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x)$$

As

$$J'_v(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x),$$

we get that

$$J'_{-v}(x) - \frac{-v}{x} J_{-v}(x) = -J_{-v+1}(x)$$

by substituting $-v$ in place of v , which implies that

$$J_{-v+1}(x) = -J'_{-v}(x) - \frac{v}{x} J_{-v}(x)$$

As

$$J_{-v+1}(x) = -J'_{-v}(x) - \frac{v}{x} J_{-v}(x)$$

and

$$J'_v(x) + \frac{v}{x} J_v(x) = J_{v-1}(x),$$

we get that

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x)$$

can be written as

$$J_v(x) \left(-J'_{-v}(x) - \frac{v}{x} J_{-v}(x) \right) + J_{-v}(x) \left(J'_v(x) + \frac{v}{x} J_v(x) \right)$$

Also

$$J_v(x) \left(-J'_{-v}(x) - \frac{v}{x} J_{-v}(x) \right) + J_{-v}(x) \left(J'_v(x) + \frac{v}{x} J_v(x) \right)$$

is equal to

$$-J_v(x)J'_{-v}(x) - \frac{v}{x} J_v(x)J_{-v}(x) + J_{-v}(x)J'_v(x) + \frac{v}{x} J_{-v}(x)J_v(x)$$

which is nothing but

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$$

Thus,

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$$

Therefore, we got that

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$$

and

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{-J_v(x)J'_{-v}(x) + J'_v(x)J_{-v}(x)}{\sin vx}$$

As $J_v(x)$ and $J_{-v}(x)$ are solutions to the same Bessel's equation, we get that

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{A_v}{x}$$

where A , may depend on v but is independent of x . As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{2s+v},$$

we get that

$$J'_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s+v)}{s! \Gamma(v+s+1) 2} \left(\frac{x}{2} \right)^{2s+v-1}$$

Similarly,

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(-v+s+1)} \left(\frac{x}{2} \right)^{2s-v}$$

and hence

$$J'_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s-v)}{s! \Gamma(-v+s+1) 2} \left(\frac{x}{2} \right)^{2s-v-1}$$

As

$$\frac{1}{\Gamma(v+1)} \left(\frac{x}{2} \right)^v$$

is the leading power of

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{2s+v}$$

and

$$\frac{(-v)}{\Gamma(-v+1) 2} \left(\frac{x}{2} \right)^{-v-1}$$

is the leading power of

$$J'_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s-v)}{s! \Gamma(-v+s+1) 2} \left(\frac{x}{2} \right)^{2s-v-1},$$

we get that

$$\frac{1}{\Gamma(v+1)} \left(\frac{x}{2}\right)^v \frac{(-v)}{\Gamma(-v+1)2} \left(\frac{x}{2}\right)^{-v-1}$$

is the leading power of $J_v(x)J'_{-v}(x)$ Similarly,

$$\frac{1}{\Gamma(-v+1)} \left(\frac{x}{2}\right)^{-v}$$

is the leading power of

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

and

$$\frac{v}{\Gamma(v+1)2} \left(\frac{x}{2}\right)^{v-1}$$

is the leading power of

$$J'_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s+v)}{s! \Gamma(v+s+1)2} \left(\frac{x}{2}\right)^{2s+\gamma-1},$$

we get that

$$\frac{1}{\Gamma(-v+1)} \left(\frac{x}{2}\right)^{-v} \frac{v}{\Gamma(v+1)2} \left(\frac{x}{2}\right)^{v-1}$$

is the leading power of $J_{-v}(x)J'_v(x)$. Also

$$\frac{1}{\Gamma(v+1)} \left(\frac{x}{2}\right)^v \frac{(-v)}{\Gamma(-v+1)2} \left(\frac{x}{2}\right)^{-\gamma-1} = \frac{-v}{\Gamma(v+1)\Gamma(-v+1)x}$$

and

$$\frac{1}{\Gamma(-v+1)} \left(\frac{x}{2}\right)^{-v} \frac{v}{\Gamma(v+1)2} \left(\frac{x}{2}\right)^{v-1} = \frac{v}{\Gamma(v+1)\Gamma(-v+1)x}$$

Thus, we get that

$$\frac{-v}{\Gamma(v+1)\Gamma(-v+1)x}$$

is the leading power of

$$J_v(x)J'_{-v}(x)$$

and

$$\frac{v}{\Gamma(v+1)\Gamma(-v+1)x}$$

is the leading power of

$$J_{-v}(x)J'_v(x)$$

which implies that the coefficient of x^{-1} in

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$$

is equal to

$$\frac{2v}{\Gamma(v+1)\Gamma(1-v)x}$$

From reflection formula, we get that

$$\Gamma(v)\Gamma(1-v) = \frac{\pi}{\sin v\pi}$$

As

$$\Gamma(v+1) = v\Gamma(v)$$

and

$$\Gamma(v)\Gamma(1-v) = \frac{\pi}{\sin v\pi},$$

we get that

$$\frac{v}{\Gamma(v+1)\Gamma(1-v)} = \frac{\sin v\pi}{\pi}$$

Therefore, coefficient of x^{-1} in $J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$ is equal to $\frac{2 \sin v\pi}{\pi x}$. As

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{A_v}{x}$$

where A_v , may depend on v but is independent of x and the coefficient of x^{-1} in

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x)$$

is equal to

$$\frac{2 \sin v\pi}{\pi x},$$

we get that

$$A_v = \frac{2 \sin v\pi}{\pi}$$

and all coefficients of x (except x^{-1}) are zero. Therefore,

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{2 \sin v\pi}{\pi x}$$

As

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{2 \sin v\pi}{\pi x}$$

and

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x),$$

we get that

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = \frac{2 \sin v\pi}{\pi x}$$

As

$$J_{-v}(x)J'_v(x) - J_v(x)J'_{-v}(x) = \frac{2 \sin v\pi}{\pi x}$$

and

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{-J_v(x)J'_{-v}(x) + J'_v(x)J_{-v}(x)}{\sin vx},$$

we get that

$$J_v(x)Y'_v(x) - J'_v(x)Y_v(x) = \frac{2}{\pi x}$$

and hence the given statements are true.

Problem 14.3.6

As an alternative to letting x approach zero in the evaluation of the Wronskian constant, we may invoke the uniqueness of power-series expansions. The coefficient of x^{-1} in the series expansion of

$$u_v(x)v'_v(x) - u'_v(x)v_v(x)$$

is then A_v . Show by series expansion that the coefficients of x^0 and x^1 of

$$J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$$

are each zero.

Solution To prove that the coefficients of x^0 and x^1 in

$$J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$$

are both zero by using power series expansions. As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v},$$

we get that

$$J'_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s+v)}{s! \Gamma(v+s+1)2} \left(\frac{x}{2}\right)^{2s+v-1}$$

Similarly,

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

and hence

$$J'_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s-v)}{s! \Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2s-v-1}$$

Also

$$J_v(x)J'_{-v}(x) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k+v} \frac{(-1)^s(2s-v)}{s! \Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2s-v-1}$$

and

$$J'_v(x)J_{-v}(x) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k(2k+v)}{k! \Gamma(v+k+1)2} \left(\frac{x}{2}\right)^{2k+v-1} \frac{(-1)^s}{s! \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

As

$$J_v(x)J'_{-v}(x) = \sum_{x=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2s-v)}{k! s! \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that the coefficient of x^0 in

$$J_v(x)J'_{-v}(x)$$

is equal to zero (as $2k+2s-1$ is odd for any $s, k \in \mathbb{Z}$, we get that $2k+2s-1$ is never 0 and hence there is no constant term in $J_v(x)J'_{-v}(x)$). Similarly, as

$$J'_v(x)J_{-v}(x) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2k+v)}{k! s! \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that coefficient of x^0 in

$$J'_v(x)J_{-v}(x)$$

is equal to zero (as $2k+2s-1$ is odd for any $s, k \in \mathbb{Z}$, we get that $2k+2s-1$ is never 0 and hence there is no constant term in $J'_v(x)J_{-v}(x)$). As there are constant terms in both $J'_v(x)J_{-v}(x)$ and $J_v(x)J'_{-v}(x)$, we get that the coefficient of x^0 in

$$J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$$

is zero. As

$$J_v(x)J'_{-v}(x) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2s-v)}{k! s! \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{(-1)^{1+0}(2(0)-v)}{1! 0! \Gamma(v+(1)+1)\Gamma(-v+(0)+1)2} \left(\frac{1}{2}\right) + \frac{(-1)^{0+1}(2(1)-v)}{0! 1! \Gamma(v+(0)+1)\Gamma(-v+(1)+1)2} \left(\frac{1}{2}\right)$$

when $s, k \in \mathbb{Z}$, we get that $2k+2s-1=1$ if and only if $k+s=1$ which implies that either $k=0, s=1$ or $k=1, s=0$. Also

$$\frac{(-1)^{1+0}(2(1)+v)}{1! 0! \Gamma(v+(1)+1)\Gamma(-v+(0)+1)2} \left(\frac{1}{2}\right) + \frac{(-1)^{0+1}(2(0)+v)}{0! 1! \Gamma(v+(0)+1)\Gamma(-v+(1)+1)2} \left(\frac{1}{2}\right)$$

is equal to

$$\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4}$$

Thus, the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4}$$

As the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4}$$

and the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{v}{\Gamma(v+2)\Gamma(1-v)4} + \frac{v-2}{\Gamma(v+1)\Gamma(2-v)4}$$

we get that the coefficient of x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ is equal to

$$\frac{v}{\Gamma(v+2)\Gamma(1-v)4} + \frac{v-2}{\Gamma(v+1)\Gamma(2-v)4} - \left(\frac{-v-2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{-v}{\Gamma(v+1)\Gamma(2-v)4} \right)$$

which is equal to

$$\frac{2v+2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{2v-2}{\Gamma(v+1)\Gamma(2-v)4}$$

Also

$$\frac{2v+2}{\Gamma(v+2)\Gamma(1-v)4} = \frac{2v+2}{(v+1)\Gamma(v+1)\Gamma(1-v)4} = \frac{1}{\Gamma(v+1)\Gamma(1-v)2}$$

and

$$\frac{2v-2}{\Gamma(v+1)\Gamma(2-v)4} = \frac{2v-2}{\Gamma(v+1)(1-v)\Gamma(1-v)4} = \frac{-1}{\Gamma(v+1)\Gamma(1-v)2}$$

Thus,

$$\frac{2v+2}{\Gamma(v+2)\Gamma(1-v)4} + \frac{2v-2}{\Gamma(v+1)\Gamma(2-v)4} = \frac{1}{\Gamma(v+1)\Gamma(1-v)2} + \frac{-1}{\Gamma(v+1)\Gamma(1-v)2} = 0.$$

Thus, the coefficient of x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ is equal to zero. Therefore, the coefficients of x^0 and x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ are both zero.

Problem 14.3.8

Verify the expansion formula for $Y_n(x)$ given in Eq. (14.61).

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln \left(\frac{x}{2} \right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} [\psi(k+1) + \psi(n+k+1)] \left(\frac{x}{2} \right)^{2k+n} \quad (14.61)$$

Hint. Start from Eq. (14.60)

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v}{dv} - (-1)^n \frac{dJ_{-v}}{dv} \right]_{v=n} \quad (14.60)$$

and perform the indicated differentiations on the powerseries expansions of J_v and J_{-v} . The digamma functions ψ arise from the differentiation of the gamma function. You will need the identity (not derived in this book) $\lim_{z \rightarrow -n} \psi(z)/\Gamma(z) = (-1)^{n-1}n!$, where n is a positive integer.

Solution To prove that

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln \left(\frac{x}{2} \right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} - A$$

where

$$A = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} [\psi(k+1) + \psi(n+k+1)] \left(\frac{x}{2} \right)^{2k+n}$$

We know that

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv} \right]_{v=n}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{v+2s}$$

we get that

$$\frac{dJ_v(x)}{dv} = \frac{d}{dv} \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{v+2s} \right)$$

and hence

$$\frac{dJ_v(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \frac{d}{dv} \left(\frac{x}{2} \right)^{v+2s} + \sum_{s=0}^{\infty} \left(\frac{x}{2} \right)^{v+2s} \frac{d}{dv} \frac{(-1)^s}{s! \Gamma(v+s+1)}$$

Also

$$\frac{d}{dv} \left(\frac{x}{2} \right)^{v+2s} = \left(\frac{x}{2} \right)^{v+2s} \ln \left(\frac{x}{2} \right)$$

and

$$\frac{d}{dv} \frac{(-1)^s}{s! \Gamma(v+s+1)} = - \frac{(-1)^s}{s! (\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

Thus,

$$\frac{dJ_v(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{v+2s} \ln \left(\frac{x}{2} \right) - \sum_{s=0}^{\infty} \left(\frac{x}{2} \right)^{v+2s} \frac{(-1)^s}{s! (\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{v+2s},$$

we get that

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{v+2s} \ln \left(\frac{x}{2} \right) = \ln \left(\frac{x}{2} \right) \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{v+2s} = \ln \left(\frac{x}{2} \right) J_v(x)$$

Therefore,

$$\frac{dJ_v(x)}{dv} = \ln\left(\frac{x}{2}\right) J_v(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s}{s! (\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

As

$$\psi(z+1) = \frac{d \ln \Gamma(z+1)}{dz} = \frac{1}{\Gamma(z+1)} \frac{d\Gamma(z+1)}{dz}$$

where ψ is the digamma function, we get that

$$\frac{(-1)^s}{s! (\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv} = \frac{(-1)^s \psi(v+s+1)}{s! \Gamma(v+s+1)}$$

Thus,

$$\frac{dJ_v(x)}{dv} = \ln\left(\frac{x}{2}\right) J_v(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s \psi(v+s+1)}{s! \Gamma(v+s+1)}$$

Therefore,

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{n+2s} \frac{(-1)^s \psi(n+s+1)}{s! \Gamma(n+s+1)}$$

which implies that

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^s \psi(n+s+1)}{s! (n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s},$$

we get that

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v}$$

Also

$$\frac{dJ_{-v}(x)}{dv} = \frac{d}{dv} \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \right)$$

and hence we get that

$$\frac{dJ_{-v}(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \frac{d}{dv} \left(\frac{x}{2}\right)^{2s-v} + \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{d}{dv} \frac{(-1)^s}{s! \Gamma(s-v+1)}$$

Also

$$\frac{d}{dv} \left(\frac{x}{2}\right)^{2s-v} = \frac{d(2s-v)}{dv} \frac{d}{d(2s-v)} \left(\frac{x}{2}\right)^{2s-v} = -\left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right)$$

and

$$\frac{d}{dv} \frac{(-1)^s}{s! \Gamma(s-v+1)} = -\frac{(-1)^s}{s! (\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

Thus,

$$\frac{dJ_{-v}(x)}{dv} = -\sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s}{s! (\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

As

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v},$$

we get that

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2}\right) \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} = \ln\left(\frac{x}{2}\right) J_{-v}(x)$$

Therefore,

$$\frac{dJ_{-v}(x)}{dv} = -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s}{s! (\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

As

$$\psi(z+1) = \frac{d \ln \Gamma(z+1)}{dz} = \frac{1}{\Gamma(z+1)} \frac{d\Gamma(z+1)}{dz},$$

we get that

$$\frac{(-1)^s}{s! (\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv} = \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)}$$

and hence

$$\frac{dJ_{-v}(x)}{dv} = -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)}$$

Thus,

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)}$$

As

$$\lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)} = \frac{(-1)^s}{s!} \lim_{v \rightarrow n} \frac{\psi(s-v+1)}{\Gamma(s-v+1)},$$

we get that

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)} - \sum_{s=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)}$$

(by dividing the summation into $s < n$ and $s \geq n$ parts). Also

$$\sum_{s=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)! \Gamma(k+1)} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)! k!}$$

and

$$\sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)} = \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} \lim_{k \rightarrow s-n} \frac{\psi(k+1)}{\Gamma(k+1)}$$

As

$$\lim_{z \rightarrow -n} \frac{\psi(z+1)}{\Gamma(z+1)} = (-1)^{n-1} n!,$$

we get that

$$\sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} \lim_{k \rightarrow s-n} \frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1} (s-n-1)!$$

As

$$\sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} \lim_{k \rightarrow s-n} \frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1} (s-n-1)!$$

$$\sum_{s=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)! k!}$$

and

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \lim_{x \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)} - \sum_{x=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v \rightarrow n} \frac{(-1)^s \psi(s-v+1)}{s! \Gamma(s-v+1)}$$

we get that $\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n}$ is equal to

$$-\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1} (s-n-1)! - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)! k!}$$

As $J_{-n}(x) = (-1)^n J_n(x)$

$$-\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1} (s-n-1)! - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)! k!}$$

can be written as

$$-(-1)^n \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} (-1)^{n+1} - (-1)^{n+1} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k \psi(k+1)}{(k+n)! k!}$$

$$(-1)^{s-n-1} (-1)^s = (-1)^{s-n-1+s-n-1} (-1)^{n+1} = (-1)^{n+1} (-1)^{2(s-n-1)} = (-1)^{n+1}$$

Thus

$$(-1)^{n+1} \left(\frac{dJ_{-v}(x)}{dv} \right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k \psi(k+1)}{(k+n)! k!}$$

As

$$\left(\frac{dJ_v(x)}{dv} \right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^s \psi(n+s+1)}{s! (n+s)!} \left(\frac{x}{2}\right)^{n+2}$$

$$(-1)^{n+1} \left(\frac{dJ_{-v}(x)}{dv} \right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} - \sum_{s=0}^{\infty} \frac{(-1)^s \psi(s+1)}{(s+n)! s!} \left(\frac{x}{2}\right)^{2s+n}$$

and

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv} \right]_{v=n}$$

we get

$$\frac{2}{\pi} \ln\left(\frac{x}{2}\right) J_n(x) - \frac{1}{\pi} \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(s-n-1)!}{s!} - \sum_{s=0}^{\infty} \frac{(-1)^s (\psi(s+1) + \psi(n+s+1))}{(s+n)! s!} \left(\frac{x}{2}\right)^{2s+n}$$

Therefore,

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} - A$$

where

$$A = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} [\psi(k+1) + \psi(n+k+1)] \left(\frac{x}{2}\right)^{2k+n}$$

Problem 14.3.9

If Bessel's ODE (with solution J_v) is differentiated with respect to v , one obtains

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_v}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_v}{\partial v} \right) + (x^2 - v^2) \frac{\partial J_v}{\partial v} = 2vJ_v$$

Use the above equation to show that $Y_n(x)$ is a solution to Bessel's ODE.

Solution We know that

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv} \right]_{v=n}$$

As

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_v}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_v}{\partial v} \right) + (x^2 - v^2) \frac{\partial J_v}{\partial v} = 2vJ_v,$$

we get that

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_{-v}}{\partial (-v)} \right) + x \frac{d}{dx} \left(\frac{\partial J_{-v}}{\partial (-v)} \right) + (x^2 - (-v)^2) \frac{\partial J_{-v}}{\partial (-v)} = 2(-v)J_{-v}$$

As $\frac{\partial J_{-v}}{\partial (-v)} = -\frac{\partial J_{-v}}{\partial v}$, we get that

$$-x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_{-v}}{\partial v} \right) - x \frac{d}{dx} \left(\frac{\partial J_{-v}}{\partial v} \right) - (x^2 - v^2) \frac{\partial J_{-v}}{\partial v} = -2vJ_{-v}$$

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_{-v}}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_{-v}}{\partial v} \right) + (x^2 - v^2) \frac{\partial J_{-v}}{\partial v} = 2vJ_{-v}$$

As

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_v}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_v}{\partial v} \right) + (x^2 - v^2) \frac{\partial J_v}{\partial v} = 2vJ_v$$

and

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_{-v}}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_{-v}}{\partial v} \right) + (x^2 - v^2) \frac{\partial J_{-v}}{\partial v} = 2vJ_{-v},$$

we get that $2vJ_v - 2v(-1)^n J_{-v}$ is equal to

$$x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v} \right) + (x^2 - v^2) \left(\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v} \right)$$

Thus, $2nJ_n - 2n(-1)^n J_{-n}$ is equal to

$$\left[x^2 \frac{d^2}{dx^2} \left(\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v} \right) + x \frac{d}{dx} \left(\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v} \right) + (x^2 - v^2) \left(\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v} \right) \right]_{v=n}$$

As

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv} \right]_{v=n},$$

the above equation implies that

$$x^2 \frac{d^2 Y_n(x)}{dx^2} + x \frac{dY_n(x)}{dx} + (x^2 - n^2) Y_n(x) = \frac{1}{\pi} (2nJ_n - 2n(-1)^n J_{-n})$$

As $J_{-n} = (-1)^n J_n$, we get that $2nJ_n - 2n(-1)^n J_{-n} = 0$. As $2nJ_n - 2n(-1)^n J_{-n} = 0$ and

$$x^2 \frac{d^2 Y_n(x)}{dx^2} + x \frac{dY_n(x)}{dx} + (x^2 - n^2) Y_n(x) = \frac{1}{\pi} (2nJ_n - 2n(-1)^n J_{-n}),$$

we get that

$$x^2 \frac{d^2 Y_n(x)}{dx^2} + x \frac{dY_n(x)}{dx} + (x^2 - n^2) Y_n(x) = 0$$

Therefore, $Y_n(x)$ is a solution to the Bessel's ODE.