Solved Problems in Mathematical Methods for Physicist

Edited By

Carlos Faz

Disclaimer
This document shows the solution of some problems of chapters 2, 3 and 13 of the book Mathematical Methods for Physicist 7$^{\text{th}}$ edition by Arfken, Weber and Harris. This document was typeset with the help of LaTeX.

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Mathematical Preliminaries

Chapter 1.1: Infinite Series

Problem 1.1.1

(*a*) Prove that if $\lim_{n\to\infty} n^p u_n = A < \infty, p > 1$, the series $\sum_{n=1}^{\infty} u_n$ converges.

(*b*) Prove that if $\lim_{n\to\infty} n u_n = A > 0$, the series diverges. (The test fails for $A = 0$.) These two tests, known as limit tests, are often convenient for establishing the convergence of a series. They may be treated as comparison tests, comparing with

$$\sum_n n^{-q}, \quad 1 \leq q < p$$

Solution For (*a*) Let

$$\lim_{n\to\infty} n^p u_n = A < \infty, \quad p > 1$$

Define

$$u_n = \frac{1}{n^p}, \quad p > 1$$

be a series. Let

$$\lim_{m\to\infty} n^p u_n = A < \infty, \quad p > 1$$

then $\sum_{n=1}^{\infty} u_n$ is convergent by limit comparision test as

$$\lim_{n\to\infty} \frac{u_n}{u_n} = \lim_{n\to\infty} \frac{u_n}{1/n^p}$$

$$= \lim_{n\to\infty} n^p u_n = A < \infty$$

If $A \neq 0$ both, the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_n$ behave alike as

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent series. If $A \neq 0$ then by limit comparision test, if $\sum_{n=1}^{\infty} u_n$ is convergent then

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 1$$

is convergent by $p-$test. Therefore $\sum_{n=1}^{\infty} u_n$ is convergent series.

Solution For (*b*) let $\lim_{n\to\infty} n u_n = A > 0$ and define

$$u_n = \frac{1}{n}$$

then $\sum u_n$ is divergent series by limit comparision test. Because

$$\lim_{n\to\infty} \frac{u_n}{u_n} = \lim_{n\to\infty} \frac{u_n}{\frac{1}{n}} = \lim_{n\to\infty} n u_n = A \neq 0$$

then by limit comparision test, if $A \neq 0$ and finite then $\sum_{n=1}^{\infty} u_n, \sum_{n=1}^{\infty} u_n$ both behave alike since

$$\sum_{n=1}^{\infty} u_n = \sum_{m=1}^{\infty} \frac{1}{n}$$

is a divergent series by $p-$test, $\sum_{n=1}^{\infty} u_n$ is divergent series. If $A \neq 0$ and infinite, then $\sum_{n=1}^{\infty} u_n$ is divergent and $\sum_{n=1}^{\infty} u_n$ is also divergent.

Problem 1.1.2

If $\lim_{n\to\infty} \frac{b_n}{a_n} = K$, a constant with $0 < K < \infty$, show that $\Sigma_n b_n$ converges or diverges with Σa_n

Hint. If Σa_n converges, rescale b_n to $b'_n = \frac{b_n}{2K}$. If $\Sigma_n a_n$ diverges, rescale to $b''_n = \frac{2b_n}{K}$

Solution The objective is to prove that the given series converges or diverges with $\sum_{n=1}^{\infty} a_n$ It is given that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = K, \quad 0 < K < \infty$$

By the definition of limit of a sequence, for any given $\varepsilon > 0$ there exists a positive integer N such that for all

$$n \geq N, \left|\frac{b_n}{a_n} - K\right| < \varepsilon$$

This implies,

$$-\varepsilon < \frac{b_n}{a_n} - K < \varepsilon$$

for all values of $n \geq N$. From this the inequality implies,

$$K - \varepsilon < \frac{b_n}{a_n} < K + \varepsilon$$

for all $n \geq N$. From the above inequality,

$$(K - \varepsilon)a_n < b_n < (K + \varepsilon)a_n$$

for all $n \geq N$. So, $(K - \varepsilon)a_n < b_n$ and $b_n < (K + \varepsilon)a_n$ for all $n \geq N$. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then,

$$\sum_{n=1}^{\infty}(K + \varepsilon)a_n$$

also converges. By the equation above, $b_n < (K + \varepsilon)a_n$ for all $n \geq N$ By comparison test the series $\sum_{n=1}^{\infty} b_n$ also converges. Suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Then

$$\sum_{n=1}^{\infty}(K - \varepsilon)a_n$$

also diverges. By the above equation

$$(K - \varepsilon)a_n < b_n$$

for all $n \geq N$. By the condition $(K - \varepsilon)a_n < b_n$ for all $n \geq N$. Hence, by comparison test the series $\sum_{n=1}^{\infty} b_n$ also diverges.

Problem 1.1.3

(*a*) Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges.

(*b*) By direct addition

$$\sum_{n=2}^{100{,}000} \left[n(\ln n)^2\right]^{-1} = 2.02288$$

Use Eq. (1.9) to make a fivesignificant-figure estimate of the sum of this series.

$$\int_{N+1}^{\infty} f(x)dx \leq \sum_{n=N+1}^{\infty} a_n \leq \int_{N+1}^{\infty} f(x)dx + a_{N+1} \qquad (1.9)$$

Solution For (*a*) we check the convergence it is required to use the integral test. Put

$$f(x) = \frac{1}{x(\ln x)^2} \Rightarrow f(n) = \frac{1}{n(\ln n)^2}$$

Then, $f(x)$ is a continuous function and monotonically decreasing. To apply the test, the integral

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x(\ln x)^2}dx$$

should be evaluated. Let $\ln x = t \Rightarrow \frac{1}{x}dx = dt$

$$\begin{aligned}
\int f(x)dx &= \int \frac{1}{t^2}dt \\
&= \frac{-1}{t} \\
&= \frac{-1}{\ln x}
\end{aligned}$$

$$\begin{aligned}
\int_2^{\infty} f(x)dx &= \left[\frac{-1}{\ln x}\right]_2^{\infty} \\
&= 0 - \left(\frac{-1}{\ln 2}\right) \\
&= \frac{1}{\ln 2}
\end{aligned}$$

Therefore the integral of the function is a finite number. By integral test, $\sum_{n=2}^{\infty} f(n)$ is convergent if the integral is a finite number. since, the integral is a finite number,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by integral test.

Solution For (*b*) by addition, it is given that

$$\sum_{n=2}^{1{,}00{,}000} \left[n(\ln n)^2\right]^{-1} = 2 \cdot 02288$$

It is required to find the sum of the series,

$$\sum_{n=2}^{\infty} \left[n(\ln n)^2\right]^{-1}$$

But,

$$\sum_{n=2}^{\infty}\left[n(\ln n)^2\right]^{-1}=\sum_{n=2}^{N}\left[n(\ln n)^2\right]^{-1}+\sum_{n=N+1}^{\infty}\left[n(\ln n)^2\right]^{-1}$$

By the inequality,

$$\int_{N+1}^{\infty}f(x)dx\le\sum_{n=N+1}^{\infty}a_n\le\int_{N+1}^{\infty}f(x)dx+a_{N+1}$$

where

$$a_n=f(n)=\frac{1}{n(\ln n)^2}$$

$$\int_{N+1}^{\infty}\frac{1}{x(\ln x)^2}dx\le\sum_{n=N+1}^{\infty}\frac{1}{n(\ln n)^2}\le\int_{N+1}^{\infty}\frac{1}{x(\ln x)^2}dx+a_{N+1}$$

Put, the value $N=1,00,000$ in the inequality.

$$\int_{1,00,001}^{\infty}\frac{1}{x(\ln x)^2}dx\le\sum_{n=1,00,001}^{\infty}\frac{1}{n(\ln n)^2}\le\int_{1,00,001}^{\infty}\frac{1}{x(\ln x)^2}dx+\frac{1}{1,00,001\times(\ln 1,00,001)}$$

$$\left[\frac{-1}{\ln x}\right]_{1,00,001}^{\infty}\le\sum_{n=1,00,001}^{\infty}\frac{1}{n(\ln n)^2}\le\left[\frac{-1}{\ln x}\right]_{1,00,001}^{\infty}+\frac{1}{1,00,001\times(\ln 1,00,001)^2}$$

$$\frac{1}{\ln 1,00,001}\le\sum_{n=1,00,001}^{\infty}\frac{1}{n(\ln n)^2}\le\frac{1}{\ln 1,00,001}+0\cdot 0868500$$

$$0.0868588\le\sum_{n=1,00,001}^{\infty}\frac{1}{n(\ln n)^2}\le 0\cdot 0868588+0\cdot 0868500$$

$$0.0868588\le\sum_{n=1,00,001}^{\infty}\frac{1}{n(\ln n)^2}\le 0\cdot 0868588754$$

From the above the value of the sum is

$$\sum_{n=1,00,001}^{\infty}\frac{1}{n(\ln n)^2}=0\cdot 08686$$

Now, the required value of the sum of the series is

$$\begin{aligned}\sum_{n=2}^{\infty}\left[n(\ln n)^2\right]^{-1}&=\sum_{n=2}^{1,00,000}\left[n(\ln n)^2\right]^{-1}+\sum_{n=1,00,001}^{\infty}\left[n(\ln n)^2\right]^{-1}\\&=2\cdot 02288+0\cdot 08686\\&=2\cdot 10974\end{aligned}$$

Hence, the required sum is

$$\sum_{n=2}^{\infty}\left[n(\ln n)^2\right]^{-1}=2\cdot 10974$$

Problem 1.1.4

Gauss' test is often given in the form of a test of the ratio

$$\frac{u_n}{u_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0}$$

For what values of the parameters a_1 and b_1 is there convergence? divergence?
ANS: Convergent for $a_1 - b_1 > 1$, divergent for $a_1 - b_1 \leq 1$

Solution Using the division algorithm

$$\begin{aligned}
\frac{u_n}{u_{n+1}} &= 1 + \frac{(a_1 - b_1)\, n + (a_0 - b_0)}{n^2 + b_1 n + b_0} \\
&= 1 + \frac{(a_1 - b_1)\, n}{n^2 + b_1 n + b_0} + \frac{(a_0 - b_0)}{n^2 + b_1 n + b_0} \\
&= 1 + \frac{(a_1 - b_1)\, n}{n^2 \left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right)} + \frac{(a_0 - b_0)}{n^2 \left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right)} \\
&= 1 + \frac{(a_1 - b_1)}{n \left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right)} + \frac{(a_0 - b_0)}{n^2 \left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right)}
\end{aligned}$$

Compare the expression with

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2}$$

Then,

$$h = \frac{a_1 - b_1}{\left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right)}$$

$$B(n) = \frac{a_0 - b_0}{\left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right)}$$

For larger values of n,

$$\left(1 + \frac{b_1}{n} + \frac{b_0}{n^2}\right) \to 1$$

Hence, for larger values of n, $B(n)$ is bounded. By Gauss's test, the series converges if $h > 1$ and diverges if $h \leq 1$. Therefore the series converges if $a_1 - b_1 > 0$ and diverges if $a_1 - b_1 \leq 1$

Problem 1.1.5

Test for convergence

$$(a)\ \sum_{n=2}^{\infty}(\ln n)^{-1} \qquad (d)\ \sum_{n=1}^{\infty}[n(n+1)]^{-1/2}$$

$$(b)\ \sum_{n=1}^{\infty}\frac{n!}{10^n} \qquad (e)\ \sum_{n=0}^{\infty}\frac{1}{2n+1}$$

$$(c)\ \sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}$$

Solution For (a) As in all these convergence tests, it is good to first have a general idea of whether we expect this to converge or not, and then find an appropriate test to confirm our hunch. For this one, we can imagine that $\ln n$ grows very slowly, so that its inverse goes to zero very slowly - too slowly, in fact, to converge. To prove this, we can perform a simple comparison test. since $\ln n < n$ for $n \geq 2$, we see that

$$a_n = (\ln n)^{-1} > n^{-1}$$

since the harmonic series diverges, and each term is larger than the corresponding harmonic series term, this series must diverge.

Solution For (b) the factorial in the numerator will start to dominate over the power in the denominator. So we expect this to diverge. As a proof, we can perform a simple ratio test.

$$a_n = \frac{n!}{10^n} \Rightarrow \frac{a_n}{a_{n+1}} = \frac{10}{n+1}$$

Taking the limit, we obtain

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = 0$$

hence the series diverges by the ratio test.

Solution For (c) We first note that this series behaves like $1/4n^2$ for large n. As a result, we expect it to converge. To see this, we may consider a simple comparison test

$$a_n = \frac{1}{2n(2n+1)} < \frac{1}{2n\cdot 2n} = \frac{1}{4}\left(\frac{1}{n^2}\right)$$

since the series $\zeta(2) = \sum_{n=1}^{\infty}\left(1/n^2\right)$ converges, this series converges as well.

Solution For (d) we expect to diverges

$$a_n = \frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)(n+1)}} = \frac{1}{n+1}$$

Because the harmonic series diverges

Solution For (e) since this behaves as $1/2n$ for large n, the series ought to diverge. We may either compare this with the harmonic series or perform an integral test. Consider the integral test

$$\int_0^{\infty}\frac{dx}{2x+1} = \frac{1}{2}\ln(2x+1)\Big|_0^{\infty} = \infty$$

Thus the series diverges

Problem 1.1.6

Test for convergence

Solution

$$(a)\ \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$(b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

$$(c)\ \sum_{n=1}^{\infty}\frac{1}{n2^n}$$

$$(d)\ \sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$$

$$(e)\ \sum_{n=1}^{\infty}\frac{1}{n\cdot n^{1/n}}$$

Solution For (a) let the series be $\sum_{n=1}^{\infty} u_n$ where

$$u_n = \frac{1}{n(n+1)}$$

If we put,

$$f(x) = \frac{1}{x(x+1)}$$

then the function is a decreasing function. This is because the derivative of the function is

$$f'(x) = \frac{-2x-1}{(x^2+x)^2} < 0$$

Also, if

$$f(n) = \frac{1}{n(n+1)}$$

then the series is a series of positive terms. we apply integral test, it is required to find the integral of the function from 1 to infinity.

$$\begin{aligned}
\int_1^{\infty} f(x)dx &= \int_1^{\infty}\frac{1}{x(x+1)}dx \\
&= \int_1^{\infty}\left[\frac{1}{x}-\frac{1}{x+1}\right]dx \\
&= [\ln x - \ln(x+1)]_1^{\infty} \\
&= \left[\ln\left(\frac{x}{x+1}\right)\right]_1^{\infty}
\end{aligned}$$

But the value of

$$\lim_{x\to\infty}\ln\left(\frac{x}{x+1}\right) = \ln 1 = 0$$

and

$$\lim_{x\to 1}\ln\left(\frac{x}{x+1}\right) = \ln\frac{1}{2}$$

Substituting the upper and lower limits,

$$\int_1^{\infty} f(x)dx = 0 - \ln\left(\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right)$$

This is a finite number. So, by integral test, the given series converges. Therefore, the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

converges.

$\boxed{\textbf{Solution}}$ For (b) Let the series be $\sum_{n=2}^{\infty} u_n$ where

$$u_n = \frac{1}{n \ln n}$$

The series contains all positive terms. If we put,

$$f(x) = \frac{1}{x \ln x}$$

then the function is a decreasing function. This is because the derivative of the function is

$$f'(x) = \frac{-(1 + \ln x)}{(x \ln x)^2} < 0$$

Also, if

$$f(n) = \frac{1}{n \ln n}$$

then the series is a series of positive terms. To apply Integral test, it is required to find the integral of the function from 2 to infinity.

$$\begin{aligned}
\int_2^{\infty} f(x)dx &= \int_2^{\infty} \frac{1}{x \ln x} dx \\
&= \int_2^{\infty} \frac{1}{x} \times \frac{1}{\ln x} dx \\
&= [\ln(\ln x)]_2^{\infty} \\
&= \ln(\ln \infty) - \ln(\ln 2)
\end{aligned}$$

But the value of $\ln(\ln \infty)$ is infinite. So, the integral diverges So, by integral test, the given series diverges. Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

$\boxed{\textbf{Solution}}$ For (c) let the series be $\sum_{n=1}^{\infty} u_n$ where

$$u_n = \frac{1}{n2^n}$$

The series contains all positive terms. It is required to apply the ratio test. For this evaluate

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

The value of

$$u_{n+1} = \frac{1}{(n+1)2^{n+1}}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{u_{n+1}}{u_n} &= \lim_{n\to\infty} \frac{1}{(n+1)2^{n+1}} \times \frac{n2^n}{1} \\
&= \lim_{n\to\infty} \frac{n}{(n+1)2} \\
&= \frac{1}{2} < 1
\end{aligned}$$

By ratio test, the given series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.

$\boxed{\textbf{Solution}}$ For (d) let the series be $\sum_{n=1}^{\infty} u_n$ where $u_n = \ln\left(1 + \frac{1}{n}\right)$. Le the sum of first n terms be S_n

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} u_k \\
&= u_1 + u_2 + \cdots s + u_n \\
&= \ln 2 + \ln \frac{3}{2} + \ln \frac{4}{3} + \cdots + \ln \frac{n+1}{n} \\
&= \ln\left(2 \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{n+1}{n}\right) \\
&= \ln(n+1)
\end{aligned}$$

From the above equation, it is clear that $S_n = \ln(n+1)$ So, by the definition of the sum of the series,

$$\begin{aligned}\sum_{n=1}^{\infty} u_n &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \ln(n+1) \\ &= \infty\end{aligned}$$

So, the given series $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ diverges.

Solution For (e) it is required to use a test called limit comparison test. If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} \neq 0$$

then both the series converge or diverge together. Here, $u_n = \dfrac{1}{nn^{1/n}}$. Consider the series, $v_n = \dfrac{1}{n}$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{nnn}}{\frac{1}{n}} = 1$$

By limit comparison test, $\sum_{n=1}^{\infty} u_n$ converges if $\sum_{n=1}^{\infty} v_n$ converges and $\sum_{n=1}^{\infty} u_n$ diverges if $\sum_{n=1}^{\infty} v_n$ diverges. But, the series

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. So, by the test

$$\sum_{n=1}^{\infty} u_n$$

also diverges. Hence, the series

$$\sum_{n=1}^{\infty} \frac{1}{nn^{\frac{1}{n}}}$$

is divergent

Problem 1.1.7

For what values of p and q will $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}$ converge?

ANS. Convergent for $\begin{cases} p > 1, & \text{all } q, \\ p = 1, & q > 1, \end{cases}$ divergent for $\begin{cases} p < 1, & \text{all } q \\ p = 1, & q \le 1 \end{cases}$

$\boxed{\textbf{Solution}}$ since the $\ln n$ term is not as dominant as the power term n^p, we may have some idea that the series ought to converge or diverge as the $1/n^p$ series. To make this more precise, we can use Raabe's test

$$\begin{aligned}
a_n = \frac{1}{n^p(\ln n)^q} \Rightarrow \frac{a_n}{a_{n+1}} &= \frac{(n+1)^p(\ln(n+1))^q}{n^p(\ln n)^q} \\
&= \left(1 + \frac{1}{n}\right)^p \left(1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^q \\
&= \left(1 + \frac{1}{n}\right)^p \left(1 + \frac{1}{n \ln n} + \cdots\right)^q \\
&= \left(1 + \frac{p}{n} + \cdots\right)\left(1 + \frac{q}{n \ln n} + \cdots\right) \\
&= \left(1 + \frac{p}{n} + \frac{q}{n \ln n} + \cdots\right)
\end{aligned}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty}\left(p + \frac{q}{\ln n} + \cdots\right) = p$$

This gives convergence for $p > 1$ and divergence for $p < 1$. For $p = 1$, Raabe's test is ambiguous. However, in this case we can perform an integral test. since

$$p = 1 \quad \Rightarrow \quad a_n = \frac{1}{n(\ln n)^q}$$

we evaluate

$$\int_2^\infty \frac{dx}{x(\ln x)^q} = \int_{\ln 2}^\infty \frac{du}{u^q}$$

where we have used the substitution $u = \ln x$. This converges for $q > 1$ and diverges otherwise. Hence the final result is

$$\begin{array}{ll}
p > 1, \text{any } q & \text{converge} \\
p = 1, q > 1 & \text{converge} \\
p = 1, q \le 1 & \text{diverge} \\
p < 1, \text{any } q & \text{diverge}
\end{array}$$

Problem 1.1.8

Given $\sum_{n=1}^{1,000} n^{-1} = 7.485470\ldots$ set upper and lower bounds on the Euler-Mascheroni constant.
ANS: $0.5767 < \gamma < 0.5778$

Solution No solution yet.

Problem 1.1.9

(From Olbers' paradox.) Assume a static universe in which the stars are uniformly distributed. Divide all space into shells of constant thickness; the stars in any one shell by themselves subtend a solid angle of ω_0. Allowing for the blocking out of distant stars by nearer stars, show that the total net solid angle subtended by all stars, shells extending to infinity, is exactly 4π. [Therefore the night sky should be ablaze with light. For more details, see E. Harrison, Darkness at Night: A Riddle of the Universe. Cambridge, MA: Harvard University Press (1987).]

Solution No solution yet

Problem 1.1.10

Test for convergence

$$\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2=\frac{1}{4}+\frac{9}{64}+\frac{25}{256}+\cdots$$

Solution Let the series be $\sum_{n=1}^{\infty} u_n$ where $u_n=\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdot\cdots(2n)}\right]^2$. Take $a_n=n$, then $\sum_{n=1}^{\infty} a_n^{-1}$ diverges Consider the following ratio.

$$\frac{u_n}{u_{n+1}}=\frac{\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2}{\left[\frac{1\cdot 3\cdot 5\cdots(2(n+1)-1)}{2\cdot 4\cdot 6\cdots(2(n+1))}\right]^2}$$

$$\frac{u_n}{u_{n+1}}=\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2\times\left[\frac{2\cdot 4\cdot 6\cdot\cdots(2n+2)}{1\cdot 3\cdot 5\cdots(2n+1)}\right]^2$$

$$\frac{u_n}{u_{n+1}}=\left[\frac{(2n+2)}{(2n+1)}\right]^2$$

Now, consider the expression

$$\lim_{n\to\infty}\left[a_n\frac{u_n}{u_{n+1}}-a_{n+1}\right]=\lim_{n\to\infty}\left[n\left[\frac{(2n+2)}{(2n+1)}\right]^2-(n+1)\right]$$

$$\lim_{n\to\infty}\left[a_n\frac{u_n}{u_{n+1}}-a_{n+1}\right]=\lim_{n\to\infty}\left[\frac{n\left(4n^2+4+8n\right)-(n+1)\left(4n^2+1+4n\right)}{4n^2+1+4n}\right]$$

$$\lim_{n\to\infty}\left[a_n\frac{u_n}{u_{n+1}}-a_{n+1}\right]=\lim_{n\to\infty}\left[\frac{4n^3+4n+8n^2-4n^3-n-4n^2-4n^2-1-4n}{4n^2+1+4n}\right]$$

$$\lim_{n\to\infty}\left[a_n\frac{u_n}{u_{n+1}}-a_{n+1}\right]=\lim_{n\to\infty}\left[\frac{-n-1}{4n^2+1+4n}\right]$$

But the degree of the polynomial in the denominator is greater than the degree of the polynomial in the numerator.

$$\lim_{n\to\infty}\left[\frac{-n-1}{4n^2+1+4n}\right]=0$$

This implies that

$$\lim_{n\to\infty}\left[a_n\frac{u_n}{u_{n+1}}-a_{n+1}\right]=0$$

By Kummer's theorem, the given series

$$\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2$$

diverges.

Problem 1.1.11

Determine whether each of these series is convergent, and if so, whether it is absolutely convergent:

$$(a)\ \frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \frac{\ln 6}{6} - \cdots$$

$$(b)\ \frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$$

$$(c)\ 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} \cdots + \frac{1}{15} - \frac{1}{16} \cdots - \frac{1}{21} + \cdots$$

Solution No solution yet.

Problem 1.1.12

Catalan's constant $\beta(2)$ is defined by

$$\beta(2) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} \cdots$$

Calculate $\beta(2)$ to six-digit accuracy.
Hint. The rate of convergence is enhanced by pairing the terms,

$$(4k-1)^{-2} - (4k+1)^{-2} = \frac{16k}{(16k^2-1)^2}$$

If you have carried enough digits in your summation, $\sum_{1 \le k \le N} 16k/(16k^2-1)^2$, additional significant figures may be obtained by setting upper and lower bounds on the tail of the series, $\sum_{k=N+1}^{\infty}$. These bounds may be set by comparison with integrals, as in the Maclaurin integral test.
ANS. $\beta(2) = 0.915965594177 \cdots$

Solution No solution yet.

Problem 1.1.13

Show how to combine $\zeta(2) = \sum_{n=1}^{\infty} n^{-2}$ with α_1 and α_2 to obtain a series converging as n^{-4}
Note. $\zeta(2)$ has the known value $\pi^2/6$. See Eq. (12.66).

Solution No solution yet.

Problem 1.1.14

Give a method of computing

$$\lambda(3) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}$$

that converges at least as fast as n^{-8} and obtain a result good to six decimal places.
ANS. $\lambda(3) = 1.051800$

Solution No solution yet.

Problem 1.1.15

Show that (a) $\sum_{n=2}^{\infty}[\zeta(n)-1]=1$, (b) $\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]=\frac{1}{2}$, where $\zeta(n)$ is the Riemann zeta function.

Solution No solution yet.

Problem 1.1.16

The convergence improvement of 1.1.11 may be carried out more expediently (in this special case) by putting α_2, from Eq. (1.26), into a more symmetric form: Replacing n by $n-1$, we have

$$\alpha_2' = \sum_{n=2}^{\infty} \frac{1}{(n-1)n(n+1)} = \frac{1}{4}$$

(a) Combine $\zeta(3)$ and α_2' to obtain convergence as n^{-5}

(b) Let α_4' be α_4 with $n \longrightarrow n-2$. Combine $\zeta(3)$, α_2', and α_4' to obtain convergence as n^{-7}

(c) If $\zeta(3)$ is to be calculated to six-decimal place accuracy (error 5×10^{-7}), how many terms are required for $\zeta(3)$ alone? combined as in part (a)? combined as in part (b)?

Note. The error may be estimated using the corresponding integral.
ANS.

$$(a) \quad \zeta(3) = \frac{5}{4} - \sum_{n=2}^{\infty} \frac{1}{n^3\left(n^2-1\right)}$$

Solution No solution yet.

Chapter 2
Determinants and Matrices

Chapter 2.2: Matrices

Problem 2.2.1

Show that matrix multiplication is associative, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

Solution The product **BC** is defined because the column of **B** and rows of **C** are same. Suppose $\mathbf{D} = \mathbf{BC}$ Then element of **D** is of the form

$$d_{ik} = \sum_j b_{ij} c_{jk}$$

Now the product **AD** is defined because the column of **A** and rows of **D** are same. Then element of **E** is of the form

$$e_{lk} = \sum_k a_{li} \left(\sum_j b_{ij} c_{jk} \right)$$

Therefore, the matrix $\mathbf{E} = \mathbf{A}(\mathbf{BC})$ have the elements e_{lk}. The product **AB** is defined because the column of **A** and rows of **B** are same. Let, $\mathbf{D} = \mathbf{AB}$. Then element of **D** is of the form

$$d_{lj} = \sum_i a_{li} b_{ij}$$

Now the product **DC** is defined because the column of **D** and rows of **C** are same. Let, $\mathbf{E} = \mathbf{DC}$. Then element of **D** is of the form

$$e_{lk} = \sum_j \left(\sum_i a_{li} b_{ij} \right) c_{jk}$$

Therefore, the matrix $\mathbf{E} = (\mathbf{AB})\mathbf{C}$ have the elements e_{lk} Therefore,

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

Hence, matrix multiplication is associative.

Problem 2.2.2

Show that

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$$

if and only if **A** and **B** commute

$$[\mathbf{A}, \mathbf{B}] = 0$$

Solution

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}(\mathbf{A} + \mathbf{B}) - \mathbf{B}(\mathbf{A} + \mathbf{B})$$
$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} - \mathbf{BA} - \mathbf{B}^2$$
$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2 + (\mathbf{AB} - \mathbf{BA})$$

Because **A** and **B** conmute, the term $(\mathbf{AB} - \mathbf{BA})$ equals to zero, hence is proved.

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$$

Problem 2.2.3

(a) Complex numbers, $a + ib$, with a and b real, may be represented by (or are isomorphic with) 2×2 matrices:

$$a + ib \longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Show that this matrix representation is valid for

(i) addition

(ii) multiplication

(b) Find the matrix corresponding to $(a + ib)^{-1}$.

Solution (a) Let us start with addition. For complex numbers, we have (straightforwardly)

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

whereas, if we used matrices we would get

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} (a+c) & (b+d) \\ -(b+d) & (a+c) \end{bmatrix}$$

which shows that the sum of matrices yields the proper representation of the complex number $(a + c) + i(b + d)$. We now handle multiplication in the same manner. First, we have

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

while matrix multiplication gives

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} (ac-bd) & (ad+bc) \\ -(ad+bc) & (ac-bd) \end{bmatrix}$$

which is again the correct result.

Solution For (b) Find the matrix oorresponding to $(a + ib)^{-1}$ We can find the matrix in two ways. We first do standand complex arithmetic

$$(a + ib)^{-1} = \frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{1}{a^2 + b^2}(a - ib)$$

This corresponds to the 2×2 matrix

$$(a + ib)^{-1} \longleftrightarrow \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Alternatively, we first convert to a matrix representation, and then find the inverse matrix

$$(a + ib)^{-1} \leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Either way, we obtain the same result.

Problem 2.2.4

If $\mathbf{A}$ is an $n \times n$ matrix, show that

$$\det(-\mathbf{A}) = (-1)^n \det \mathbf{A}.$$

$\boxed{\textbf{Solution}}$ So from the above identity we can write

$$-A = (-I)A$$

$$\det(-A) = \det(-IA)$$

We know $\det(AB) = \det(A)\det(B)$ From this

$$\det(-IA) = \det(-I)\det(\mathrm{A})$$

$$\det(-I)\det(\mathrm{A}) = (-1)^n \det(A)$$

Problem 2.2.5

(a) The matrix equation $\mathbf{A}^2 = 0$ does not imply $\mathbf{A} = 0$. Show that the most general 2×2 matrix whose square is zero may be written as

$$\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

where a and b are real or complex numbers.

(b) If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, in general

$$\det \mathbf{C} \neq \det \mathbf{A} + \det \mathbf{B}.$$

Construct a specific numerical example to illustrate this inequality.

$\boxed{\textbf{Solution}}$ For (a) first we check the condition

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} \times \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{pmatrix} = 0$$

Therefore, the 2×2 matrix square is zero

$\boxed{\textbf{Solution}}$ For (b) we know $\mathrm{C} = \mathrm{A} + \mathrm{B}$, let us consider following matrices to show that

$$\det C \neq \det A + \det B$$

Now, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{aligned} \det A &= 1 - 0 = 1 \\ \det B &= 1 - 0 = 1 \\ \det C &= 4 - 0 = 4 \end{aligned}$$

From this

$$\begin{aligned} &\det C \neq \det A + \det B \\ &4 \neq 2 \end{aligned}$$

Therefore, the following matrix satisfies the condition

Problem 2.2.6

Given

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & i \\ -i & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

show that

$$\mathbf{K}^n = \mathbf{KKK} \cdots (n \text{ factors}) = 1$$

(with the proper choice of $n, n \neq 0$).

Solution We calculate for different n

$$\mathbf{K}^2 = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 1 \\ i & 0 & 0 \end{bmatrix} \quad \mathbf{K}^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \mathbf{K}^4 = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{K}^5 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & -1 \\ -i & 0 & 0 \end{bmatrix} \quad \mathbf{K}^6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With this, the answer is $n = 6$

Problem 2.2.7

Verify the Jacobi identity,

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{C}]] - [\mathbf{C}, [\mathbf{A}, \mathbf{B}]]$$

Solution

$$[\mathbf{A}[\mathbf{B}, \mathbf{C}]] = [\mathbf{A}, \mathbf{BC} - \mathbf{CB}] = \mathbf{A}(\mathbf{BC} - \mathbf{CB}) - (\mathbf{BC} - \mathbf{CB})\mathbf{A}$$

$$[\mathbf{A}[\mathbf{B}, \mathbf{C}]] = \mathbf{A}(\mathbf{BC}) - \mathbf{A}(\mathbf{CB}) - (\mathbf{BC})\mathbf{A} + (\mathbf{CB})\mathbf{A}$$

$$[\mathbf{B}[\mathbf{C}, \mathbf{A}]] = [\mathbf{B}, \mathbf{CA} - \mathbf{AC}] = \mathbf{B}(\mathbf{CA} - \mathbf{AC}) - (\mathbf{CA} - \mathbf{AC})\mathbf{B}$$

$$[\mathbf{B}[\mathbf{C}, \mathbf{A}]] = \mathbf{B}(\mathbf{CA}) - \mathbf{B}(\mathbf{AC}) - (\mathbf{CA})\mathbf{B} + (\mathbf{AC})\mathbf{B}$$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = [\mathbf{C}, \mathbf{AB} - \mathbf{BA}] = \mathbf{C}(\mathbf{AB} - \mathbf{BA}) - (\mathbf{AB} - \mathbf{BA})\mathbf{C}$$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = \mathbf{C}(\mathbf{AB}) - \mathbf{C}(\mathbf{BA}) - (\mathbf{AB})\mathbf{C} + (\mathbf{BA})\mathbf{C}$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}$ are obey associative law $\mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B}$, $\mathbf{C}(\mathbf{BA}) = (\mathbf{CB})\mathbf{A}$, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ and $(\mathbf{BA})\mathbf{C} = \mathbf{B}(\mathbf{AC})$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = (\mathbf{CA})\mathbf{B} - (\mathbf{CB})\mathbf{A} - \mathbf{A}(\mathbf{BC}) + \mathbf{B}(\mathbf{AC})$$

$$[\mathbf{C}[\mathbf{A}, \mathbf{B}]] = [\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}[\mathbf{C}, \mathbf{A}]] + [\mathbf{C}[\mathbf{A}, \mathbf{B}]]$$

$$= (\mathbf{A}(\mathbf{BC}) - \mathbf{A}(\mathbf{CB}) - (\mathbf{BC})\mathbf{A} + (\mathbf{CB})\mathbf{A}) + (\mathbf{B}(\mathbf{CA}) - \mathbf{B}(\mathbf{AC}) - (\mathbf{CA})\mathbf{B}$$

$$+(\mathbf{AC})\mathbf{B}) + (\mathbf{CA})\mathbf{B} - (\mathbf{CB})\mathbf{A} - \mathbf{A}(\mathbf{BC}) + \mathbf{B}(\mathbf{AC}) = 0$$

Problem 2.2.8

Show that the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy the commutation relations

$$[\mathbf{A}, \mathbf{B}] = \mathbf{C}, \quad [\mathbf{A}, \mathbf{C}] = 0, \quad \text{and} \quad [\mathbf{B}, \mathbf{C}] = 0$$

Solution We simply multiply the matrices

$$\mathbf{C} = [\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = C$$

$$[\mathbf{A}, \mathbf{C}] = A\mathbf{C} - \mathbf{CA} = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$[\mathbf{B}, \mathbf{C}] = \mathbf{BC} - \mathbf{CB} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Problem 2.2.9

Let

$$i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(a) $i^2 = j^2 = k^2 = -I$, where **I** is the unit matrix.

(b) $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$

These three matrices (i, j, and k) plus the unit matrix 1 form a basis for quaternions. An alternate basis is provided by the four 2 ×2 matrices, $i\sigma_1, i\sigma_2, -i\sigma_3$, and 1, where the σ_i are the Pauli spin matrices of Example 2.2.1.

Solution

$$i^2 = j^2 = k^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$ij = -ij = k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$jk = -kj = i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$ki = -ik = j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2.2.10

A matrix with elements $a_{ij} = 0$ for $j < i$ may be called upper right triangular. The elements in the lower left (below and to the left of the main diagonal) vanish. Show that the product of two uper right triangular matrices is an upper right triangular matrix.

Solution We build 2 matrix with terms a, b, c, x, y, z that can take any number

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x & y & z \\ 0 & u & w \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a\times x & b\times u + a\times y & b\times w + a\times z \\ 0 & d\times u & d\times w \\ 0 & 0 & 0 \end{bmatrix}$$

Hence is demostrated that the product of two upper right triangular matrices is an upper right triagular matrix.

Problem 2.2.11

The three Pauli spin matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Show that

(*a*) $(\sigma_i)^2 = \hat{1}_2$

(*b*) $\sigma_i \sigma_j = i\sigma_k, (i, j, k) = (1, 2, 3)$ or a cyclic permutation thereof,

(*c*) $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\hat{1}_2$; $\hat{1}_2$ is the 2×2 unit matrix.

Solution For $i = 1, j = 2, k = 3$

$$\sigma_i \sigma_j = \sigma_1 \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3 = i\sigma_k$$

For $i = 2, j = 3, k = 1$

$$\sigma_i \sigma_j = \sigma_2 \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\sigma_3 = i\sigma_k$$

For $i = 2, j = 3, k = 1$

$$\sigma_i \sigma_j = \sigma_3 \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_2 = i\sigma_k$$

So, we conclude that $\sigma_1 \sigma_j = i\sigma_k$

Solution (*c*) For this proof we need only to work out the commutation relation and use the proofs done in part (*a*) and (*b*)

$$\sigma_2 \sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix} = -i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -i\sigma_3$$

$$\sigma_1 \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -i \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -i\sigma_2$$

$$\sigma_3 \cdot \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -i \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} = -i\sigma_1$$

$$\left.\begin{aligned} \sigma_i \sigma_j + \sigma_j \sigma_i &= \sigma_i \sigma_j - \sigma_i \sigma_j = 0 \\ \sigma_i \sigma_j + \sigma_j \sigma_i &= 2\sigma_i^2 \end{aligned}\right\} \begin{aligned} &\text{for } j \neq i \\ &\text{for } j = i \end{aligned}$$

since $\sigma_i^2 = 1$ and using the kronecker delta we have

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} 1$$

Problem 2.2.12

One description of spin−1 particles uses the matrices

$$\mathbf{M}_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{M}_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

and

$$\mathbf{M}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) $\left[\mathbf{M}_x, \mathbf{M}_y\right] = i\mathbf{M}_z$, and so on (cyclic permutation of indices). Using the Levi-Civita symbol, we may write

$$\left[\mathbf{M}_i, \mathbf{M}_j\right] = i\sum_k \varepsilon_{ijk}\mathbf{M}_k$$

(b) $\mathbf{M}^2 \equiv \mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2 = 2\mathbf{I}_3$, where $\mathbf{I}_3$ is the 3×3 unit matrix.

(c) $\left[\mathbf{M}^2, \mathbf{M}_i\right] = 0$ $[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{L}^+$ $[\mathbf{L}^+, \mathbf{L}^-] = 2\mathbf{M}_z$ where $\mathbf{L}^+ \equiv \mathbf{M}_x + i\mathbf{M}_y$ and $\mathbf{L}^- \equiv \mathbf{M}_x - i\mathbf{M}_y$

Solution For (a)

$$\mathbf{M}_x\mathbf{M}_y = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix}$$

$$\mathbf{M}_y\mathbf{M}_x = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix}$$

$$\mathbf{M}_x\mathbf{M}_y - \mathbf{M}_y\mathbf{M}_x = \frac{1}{2}\begin{bmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{bmatrix} + \frac{1}{2}\begin{bmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{bmatrix}$$

$$\mathbf{M}_x\mathbf{M}_y - \mathbf{M}_y\mathbf{M}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{M}_x\mathbf{M}_y - \mathbf{M}_y\mathbf{M}_x = \mathbf{M}_z$$

Solution For (b)

$$\mathbf{M}_x^2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_y^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now

$$\mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2 = 2\mathbf{I}$$

Solution (c) we substitute

$$\mathbf{M}^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{M}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\left[\mathbf{M}^2, \mathbf{M}_x\right] = \mathbf{M}^2\mathbf{M}_x - \mathbf{M}_x\mathbf{M}^2$$

$$\mathbf{M}^2\mathbf{M}_x = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\mathbf{M}_x\mathbf{M}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\left[\mathbf{M}^2, \mathbf{M}_x\right] = \mathbf{M}^2\mathbf{M}_x - \mathbf{M}_x\mathbf{M}^2 = 0$$

Therefore, $\left[\mathbf{M}^2, \mathbf{M}_i\right] = 0$ Now, we substitute

$$\mathbf{M}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathbf{L}^+ = \mathbf{M}_x + i\mathbf{M}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{M}_z\mathbf{L}^+ - \mathbf{L}^+\mathbf{M}_z$$

$$\mathbf{M}_z\mathbf{L}^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{L}^+\mathbf{M}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

$$[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{M}_z\mathbf{L}^+ - \mathbf{L}^+\mathbf{M}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{L}^+$$

$$[\mathbf{M}_z, \mathbf{L}^+] = \mathbf{M}_z\mathbf{L}^+ - \mathbf{L}^+\mathbf{M}_z = \mathbf{L}^+$$

Now, substitute

$$\mathbf{L}^+ = \mathbf{M}_x + i\mathbf{M}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{L}^- = \mathbf{M}_x - i\mathbf{M}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$[\mathbf{L}^+, \mathbf{L}^-] = \mathbf{L}^+\mathbf{L}^- - \mathbf{L}^-\mathbf{L}^+$$

$$\mathbf{L}^+\mathbf{L}^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{L}^-\mathbf{L}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[\mathbf{L}^+, \mathbf{L}^-] = \mathbf{L}^+\mathbf{L}^- - \mathbf{L}^-\mathbf{L}^+ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2\mathbf{M}_z$$

Therefore, $[\mathbf{L}^+, \mathbf{L}^-] = \mathbf{L}^+\mathbf{L}^- - \mathbf{L}^-\mathbf{L}^+ = 2\mathbf{M}_z$

Problem 2.2.13

Repeat Exercise 2.2.12, using the matrices for a spin of 3/2,

$$\mathbf{M}_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \mathbf{M}_y = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and

$$\mathbf{M}_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Solution For (a) we consider that $\left[\mathbf{M}_x, \mathbf{M}_y\right]$

$$= \frac{i}{4}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} -$$

$$\frac{i}{4}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$= i\mathbf{M}_z$$

Similarly we can show that $\left[\mathbf{M}_y, \mathbf{M}_z\right] = i\mathbf{M}_x$ and $[\mathbf{M}_z, \mathbf{M}_x] = i\mathbf{M}_y$ Thus, $\left[\mathbf{M}_i, \mathbf{M}_j\right] = i\sum_k \varepsilon_{ijk}\mathbf{M}_k$ where i, j, k can take values 1,2,3 or x, y, z.

Solution For (b) we consider that

$$\mathbf{M}^2 \equiv \mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2$$

$$= \frac{1}{4}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}^2 - \frac{1}{4}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}^2 + \frac{1}{4}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}^2$$

$$= 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 21_3$$

Solution For (c) we obtain the result from the previous part

$$\begin{aligned} \left[\mathbf{M}^2, \mathbf{M}_i\right] &= [21_3, \mathbf{M}_i] \\ &= 21_3\mathbf{M}_i - 2\mathbf{M}_i 1_3 \\ &= 2\mathbf{M}_i - 2\mathbf{M}_i \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\mathbf{M}_z, \mathbf{L}^+] &= \left[\mathbf{M}_z, \mathbf{M}_x + i\mathbf{M}_y\right] \\ &= [\mathbf{M}_z, \mathbf{M}_x] - \left[i\mathbf{M}_y, \mathbf{M}_z\right] \\ &= [\mathbf{M}_z, \mathbf{M}_x] - i\left[\mathbf{M}_y, \mathbf{M}_z\right] \\ &= i\mathbf{M}_y - i\left(i\mathbf{M}_x\right) \end{aligned}$$

$$
\begin{aligned}
&= i\mathbf{M}_y + \mathbf{M}_x \\
&= \mathbf{M}_x + i\mathbf{M}_y \\
&= \mathbf{L}^+
\end{aligned}
$$

And finally

$$
\begin{aligned}
[\mathbf{L}^+, \mathbf{L}^-] &= \left[\mathbf{M}_x + i\mathbf{M}_y, \mathbf{M}_x - i\mathbf{M}_y\right] \\
&= [\mathbf{M}_x, \mathbf{M}_x] - \left[\mathbf{M}_x, i\mathbf{M}_y\right] + \left[i\mathbf{M}_y, \mathbf{M}_x\right] - \left[i\mathbf{M}_y, i\mathbf{M}_y\right], \\
&= 0 - i\left[\mathbf{M}_x, \mathbf{M}_y\right] - \left[\mathbf{M}_x, i\mathbf{M}_y\right] - 0 \\
&= -2i\,(i\mathbf{M}_z) \\
&= 2\mathbf{M}_z
\end{aligned}
$$

Problem 2.2.14

If **A** is a diagonal matrix, with all diagonal elements different, and **A** and **B** commute, show that **B** is diagonal.

Solution Given matrix **A** is diagonal matrix

$$\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3, \ldots a_n)$$

and $B = (b_{ij})$. Here **A** and **B** matrix conmute $\mathbf{AB} = \mathbf{BA}$, so

$$(a_i - a_j)b_{kl} = 0 \quad \text{for } k \neq l$$

$$b_{kl} = 0 \quad \text{for } k \neq l$$

Hence from the above statement we can say that is also a diagonal matrix

Problem 2.2.15

If **A** and **B** are diagonal, show that **A** and **B** commute.

Solution consider two $n \times n$ matrices **A** and **B**, which are diagonal.

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} b_1a_1 & 0 \\ 0 & b_2a_2 \end{bmatrix}$$

Commutative properts of addition:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}$$

$$\mathbf{B} + \mathbf{A} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 + a_1 & 0 \\ 0 & b_2 + a_2 \end{bmatrix}$$

Hence, diagonal matrices, when added commute.

Problem 2.2.16

Show that $\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{CBA})$ if any two of the three matrices commute.

Solution The trace of a matrix is the sum of its diagonal elements. Therefore, the trace of the product of three matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ is given by

$$\mathrm{Tr}(\mathbf{ABC}) = \sum_{ijk} \mathbf{A}_{ij}\mathbf{B}_{jk}\mathbf{C}_{ki}$$

By using the fact that i, j, and k are dummy summation indices with the same range, this sum can be written in the equivalent forms

$$\sum_{ijk} \mathbf{A}_{ij}\mathbf{B}_{jk}\mathbf{C}_{ki} = \sum_{ijk} \mathbf{C}_{ki}\mathbf{A}_{ij}\mathbf{B}_{jk} = \sum_{ijk} \mathbf{B}_{jk}\mathbf{C}_{ki}\mathbf{A}_{ij}$$

But the second and third of these are

$$\sum_{ijk} \mathbf{C}_{ki}\mathbf{A}_{ij}\mathbf{B}_{jk} = \mathrm{Tr}(\mathbf{CAB})$$

and

$$\sum_{ijk} \mathbf{B}_{jk}\mathbf{C}_{ki}\mathbf{A}_{ij} = \mathrm{Tr}(\mathbf{BCA})$$

respectively. Thus, we obtain the relation

$$\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{CAB}) = \mathrm{Tr}(\mathbf{BCA})$$

Problem 2.2.17

Angular momentum matrices satisfy a commutation relation

$$[\mathbf{M}_j, \mathbf{M}_k] = i\mathbf{M}_l \quad j, k, l \text{ cyclic}$$

Solution Taking the trace of both sides of the given expression, we have

$$\mathrm{Tr}\,(i\mathbf{M}_k) = \mathrm{Tr}\left(\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i\right)$$

Hence

$$i\,\mathrm{Tr}\,(\mathbf{M}_k) = \mathrm{Tr}\left(\mathbf{M}_i\mathbf{M}_j\right) - \mathrm{Tr}\left(\mathbf{M}_j\mathbf{M}_i\right)$$

since $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$, we see that $\mathrm{Tr}\,(\mathbf{M}_k) = 0$ for any k.

Problem 2.2.18

$\mathbf{A}$ and $\mathbf{B}$ anticommute: $\mathbf{AB} = -\mathbf{BA}$. Also, $\mathbf{A}^2 = 1, \mathbf{B}^2 = 1$. Show that $\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{B}) = 0$. Note. The Pauli and Dirac matrices are specific examples.

Solution Since $\mathbf{B}^2 = I$, $\mathbf{B}$ is non-singular and its inverse exists. Therefore, $\mathbf{A} = -\mathbf{B}^{-1}\mathbf{AB}$. Taking the trace, we get

$$\mathrm{Tr}(\mathbf{A}) = -\mathrm{Tr}\left(\mathbf{B}^{-1}\mathbf{AB}\right) = -\mathrm{Tr}\left(\mathbf{ABB}^{-1}\right) = -\mathrm{Tr}(\mathbf{A})$$

We see that $\mathrm{Tr}(\mathbf{A}) = 0$. Similarly, we find $\mathrm{Tr}(\mathbf{B}) = 0$

Problem 2.2.19

(a) If two nonsingular matrices anticommute, show that the trace of each one is zero. (Nonsingular means that the determinant of the matrix is nonzero.)

(b) For the conditions of part (a) to hold, $\mathbf{A}$ and $\mathbf{B}$ must be $n \times n$ matrices with n even. Show that if n is odd, a contradiction results.

Solution For (a) if the matrices are non-singular, then writing

$$\mathbf{A} = -\mathbf{B}\mathbf{A}\mathbf{B}^{-1}$$

and taking the trace, we get

$$\operatorname{Tr}\mathbf{A} = -\operatorname{Tr}\mathbf{A}$$

Hence $\operatorname{Tr}\mathbf{A} = 0$, and the procedure for $\mathbf{B}$ is analogous.

Solution For (b) now, we compute the determinant of both sides of $\mathbf{AB} = -\mathbf{BA}$: this yields det $\mathbf{A}$ det $\mathbf{B} = (-1)^N$ det $\mathbf{B}$ det $\mathbf{A}$, where N stands for size of matrices. Now since the $\mathbf{A}, \mathbf{B}$ are non-singular, both sides of the equality are non-zero and the equality is possible only for even N.

Problem 2.2.20

If $\mathbf{A}^{-1}$ has elements

$$\left(\mathbf{A}^{-1}\right)_{ij} = a_{ij}^{(-1)} = \frac{\mathbf{C}_{ji}}{|\mathbf{A}|}$$

where $\mathbf{C}_{ji}$ is the ji th cofactor of $|\mathbf{A}|$, show that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Hence $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$ (if $|\mathbf{A}| \neq 0$).

Solution We have to consider that

$$\begin{aligned}
\left(A^{-1}A\right)_{ij} &= \sum_k a_{ik}^{-1} a_{kj} \\
&= \sum_k \frac{C_{ki}}{|A|} a_{kj} \\
&= \frac{1}{|A|} \sum_k C_{ki} a_{kj} \\
&= \frac{1}{|A|} |A| \delta_{ij} \\
&= \delta_{ij}
\end{aligned}$$

Thus, $A^{-1}A = 1$. Hence, by definition of inverse of a matrix A^{-1} is the inverse of A(if $|A| \neq 0$).

Problem 2.2.21

Find the matrices $\mathbf{M}_L$ such that the product $\mathbf{M}_L\mathbf{A}$ will be $\mathbf{A}$ but with:

(a) The i th row multiplied by a constant k $\left(a_{ij} \rightarrow ka_{ij}, j = 1, 2, 3, \ldots\right)$

(b) The i th row replaced by the original i th row minus a multiple of the m^{th} row $\left(a_{ij} \rightarrow a_{ij} - Ka_{mj}, i = 1, 2, 3, \ldots\right)$

(c) The i th and m th rows interchanged $\left(a_{ij} \rightarrow a_{mj}, a_{mj} \rightarrow a_{ij}, j = 1, 2, 3, \ldots\right)$

Solution For (a) Let

$$M_L = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$M_L A = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

Solution For (b) a unit matrix except that $M_{im} = -K$.

$$M_L = \begin{bmatrix} 1 & -K \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$M_L A = \begin{bmatrix} a - Kc & b - Kd \\ c & d \end{bmatrix}$$

Solution For (c) A unit matrix except that $M_{ii} = M_{mm} = 0$ and $M_{mi} - M_{im} = 1$.

$$M_L = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M_L A = \begin{bmatrix} c & d \\ 2a & 2b \end{bmatrix}$$

Problem 2.2.22

Find the matrices $\mathbf{M}_R$ such that the product AM $_R$ will be A but with:

(a) The i th column multiplied by a constant k $(a_{ji} \to ka_{ji}, j = 1,2,3,\ldots)$

(b) The i th column replaced by the original i th column minus a multiple of the m^{th} column $(a_{ji} \to a_{ji} - ka_{jm}, j = 1,2,3,\ldots)$

(c) The i th and m th columns interchanged $(a_{ji} \to a_{jm}, a_{jm} \to a_{ji}, j = 1,2,3,\ldots)$

$\boxed{\textbf{Solution}}$ For (a), a unit matrix except that $M_{ii} = k$. Let

$$M_R = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$AM_R = \begin{bmatrix} ka & b \\ kc & d \end{bmatrix}$$

$\boxed{\textbf{Solution}}$ For (b), a unit matrix except that $M_{im} = -K$. Let

$$M_R = \begin{bmatrix} 1 & -K \\ 0 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$AM_R = \begin{bmatrix} a & b - Ka \\ c & d - Kc \end{bmatrix}$$

$\boxed{\textbf{Solution}}$ For (c), a unit matrix except that $M_{ii} = M_{mm} = 0$ and $M_{mi} - M_{im} = 1$. Example: Let

$$M_R = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$AM_R = \begin{bmatrix} 2b & a \\ 2d & c \end{bmatrix}$$

Problem 2.2.23

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution Transpose of A that are interchange rows and columns.

$$A^T = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

Now we will find the determinant of each 2×2 minor matrices we get

$$\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$$

The adjacent matrix is

$$\operatorname{Adj}(A) = \begin{pmatrix} 7 & 7 & 0 \\ 7 & 7 & 1 \\ 0 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\operatorname{Adj}(A) = \begin{pmatrix} 7 & -7 & 0 \\ -7 & 7 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{\operatorname{Adj}(A)}{\det(A)} = \frac{1}{7} \begin{pmatrix} 7 & -7 & 0 \\ -7 & 7 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Therefore, $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1} = \frac{1}{7} \begin{bmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Problem 2.2.24

Matrices are far too useful to remain the exclusive property of physicists. They may appear wherever there are linear relations. For instance, in a study of population move- ment the initial fraction of a fixed population in each of n areas (or industries or religions, etc.) is represented by an n -component column vector $\mathbf{P}$.
The movement of people from one area to another in a given time is described by an $n \times n$ (stochastic) matrix $\mathbf{T}$. Here $\mathbf{T}_{ij}$ is the fraction of the population in the j th area that moves to the i^{th} area. (Those not moving are covered by $i = j$.) With $\mathbf{P}$ describing the initial population distribution, the final population distribution is given by the matrix equation $\mathbf{TP} = \mathbf{Q}$. From its definition, $\sum_{i=1}^{n} \mathbf{P}_i = 1$

(*a*) Show that conservation of people requires that

$$\sum_{i=1}^{n} \mathbf{T}_{ij} = 1, \quad j = 1, 2, \ldots, n$$

(*b*) Prove that

$$\sum_{i=1}^{n} \mathbf{Q}_i = 1$$

continues the conservation of people.

Solution For (*a*) The equation of part states that $\mathbf{T}$ moves people from area j but does not change their total number.
Solution For (*b*) Write the component equation

$$\sum_{j} \mathbf{T}_{ij} \mathbf{P}_j = \mathbf{Q}_i$$

and sum over i. This summation replaces $\mathbf{T}_{ij}$ by unity, leaving that the sum pver $\mathbf{P}_j$ equals the sum over $\mathbf{Q}_i$, hence conserving people.

Problem 2.2.25

Given a 6×6 matrix A with elements $a_{ij} = 0.5^{|i-j|}, i, j = 0, 1, 2, \ldots, 5,$ find $\mathbf{A}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 & 0.5^5 \\ 0.5 & 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 \\ 0.5^2 & 0.5 & 1 & 0.5 & 0.5^2 & 0.5^3 \\ 0.5^3 & 0.5^2 & 0.5 & 1 & 0.5 & 0.5^2 \\ 0.5^4 & 0.5^3 & 0.5^2 & 0.5 & 1 & 0.5 \\ 0.5^5 & 0.5^4 & 0.5^3 & 0.5^2 & 0.5 & 1 \end{bmatrix}$$

Solution

$$\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 \\ -2 & 5 & -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -2 & 0 & 0 \\ 0 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix}$$

Problem 2.2.26

Show that the product of two orthogonal matrices is orthogonal.

Solution Let Q and P be orthogonal matrices. Therefore $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\mathbf{P}^T P = I$ We have that

$$(\mathbf{PQ})^T(\mathbf{PQ}) = \mathbf{Q}^T\mathbf{P}^T\mathbf{PIQ} = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

Therefore, a product of two orthogonal matrix is an orthogonal matrix.

Problem 2.2.27

If A is orthogonal, show that its determinant = ±1.

Solution We know that

$$\det\mathbf{A}^T = \det\mathbf{A}$$
$$\mathbf{A}^T\mathbf{A} = I$$
$$\det\mathbf{A}^T = \det I = 1$$
$$\det\mathbf{A}^T\mathbf{A}\ \det\mathbf{A}^T\ \det\mathbf{A}$$
$$(\det\mathbf{A})^2 = \det\mathbf{A}\ \det\mathbf{A}^T = \det\mathbf{A}^T\ \det\mathbf{A} = \det\mathbf{A}^T A = 1$$

So we must have

$$\det\mathbf{A} = \pm 1$$

Problem 2.2.28

Show that the trace of the product of a symmetric and an antisymmetric matrix is zero.

Solution If $\tilde{\mathbf{A}} = -\mathbf{A}, \tilde{\mathbf{S}} = \mathbf{S}$, then

$$\mathrm{Tr}(\widetilde{\mathbf{SA}}) = \mathrm{Tr}(\mathbf{SA}) = \mathrm{Tr}(\tilde{\mathbf{A}}\tilde{\mathbf{S}}) = -\mathrm{Tr}(\mathbf{AS})$$

Problem 2.2.29

A is 2×2 and orthogonal. Find the most general form of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution From $\tilde{\mathbf{A}} = \mathbf{A}^{-1}$ and $\det(\mathbf{A}) = 1$ we have

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

This gives $\det(\mathbf{A}) = a_{11}^2 + a_{12}^2 = 1$, hence

$$a_{11} = \cos\theta = a_{22}, \quad a_{12} = \sin\theta = -a_{21},$$

the standard 2×2 rotation matrix.

Problem 2.2.30

Show that

$$\det(\mathbf{A}^*) = (\det \mathbf{A})^* = \det\left(\mathbf{A}^\dagger\right)$$

Solution We calculate the determinant of $\mathbf{A}^*$

$$\det(\mathbf{A}^*) = \sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a_{1i_1}^* a_{2i_2}^* \cdots a_{ni_n}^* = \left(\sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}\right)$$

Because, for any **A**,

$$\det(\mathbf{A}) = \det(\tilde{\mathbf{A}}), \det(\mathbf{A}^*) = \det\left(\mathbf{A}^\dagger\right)$$

Problem 2.2.31

Three angular momentum matrices satisfy the basic commutation relation

$$[\mathbf{J}_x, \mathbf{J}_y] = i\mathbf{J}_z$$

(and cyclic permutation of indices). If two of the matrices have real elements, show that the elements of the third must be pure imaginary.

Solution We know that basic commutation relation is $\left[\mathbf{J}_i, \mathbf{J}_j\right] = i\mathbf{J}_k$, where i j and k are indices in cyclic permutation. Here it is clear that $\mathbf{J}_x, \mathbf{J}_y$ are real, so also must be their commutator. So according to commutator rule it requires that $\mathbf{J}_z$ be pure imaginary.

Problem 2.2.32

Show that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$

Solution We know that the transpose of a product is $(AB)^T = B^T A^T$. That is product is transposed by taking, in reverse order, the transpose of its factor. From this

$$(AB)^\dagger = (A^* B^*)^T = (B^*)^T (A^*)^T = B^\dagger A^\dagger$$

The complex conjugate of product is equal to conjugate of its individual factor.

Problem 2.2.33

A matrix $\mathbf{C} = \mathbf{S}^{\dagger}\mathbf{S}$. Show that the trace is positive definite unless $\mathbf{S}$ is the null matrix, in which case $\mathrm{Tr}(\mathbf{C}) = 0$.

Solution As

$$\mathbf{C}_{jk} = \sum_n S^*_{nj} S_{nk}$$

$$\mathrm{Tr}(\mathbf{C}) = \sum_{nj} \left|S_{nj}\right|^2$$

Problem 2.2.34

If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices, show that $(\mathbf{AB} + \mathbf{BA})$ and $i(\mathbf{AB} - \mathbf{BA})$ are also Hermitian.

Solution If $\mathbf{A}^{\dagger} = \mathbf{A}, \mathbf{B}^{\dagger} = B$, then

$$(\mathbf{AB} + \mathbf{BA})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger} + \mathbf{A}^{\dagger}\mathbf{B}^{\dagger} = \mathbf{AB} + \mathbf{BA}$$

$$-i\left(\mathbf{B}^{\dagger}\mathbf{A}^{\dagger} - \mathbf{A}^{\dagger}\mathbf{B}^{\dagger}\right) = i(\mathbf{AB} - \mathbf{BA})$$

Problem 2.2.35

The matrix $\mathbf{C}$ is not Hermitian. Show that then $\mathbf{C} + \mathbf{C}^{\dagger}$ and $i\left(\mathbf{C} - \mathbf{C}^{\dagger}\right)$ are Hermitian. This means that a non-Hermitian matrix may be resolved into two Hermitian parts,

$$\mathbf{C} = \frac{1}{2}\left(\mathbf{C} + \mathbf{C}^{\dagger}\right) + \frac{1}{2i} i\left(\mathbf{C} - \mathbf{C}^{\dagger}\right)$$

This decomposition of a matrix into two Hermitian matrix parts parallels the decompo- sition of a complex number z into $x + iy$, where $x = (z + z^*)/2$ and $y = (z - z^*)/2i$

Solution If $\mathbf{C}^{\dagger} \neq \mathbf{C}$, then

$$(i\mathbf{C}_{-})^{\dagger} \equiv \left(\mathbf{C}^{\dagger} - \mathbf{C}\right)^{\dagger} = \mathbf{C} - \mathbf{C}^{\dagger} = -i\mathbf{C}^{\dagger}_{-},$$

$$(\mathbf{C}_{-})^{\dagger} = \mathbf{C}_{-}$$

$$\mathbf{C}^{\dagger}_{+} = \mathbf{C}_{+} = \mathbf{C} + \mathbf{C}^{\dagger}$$

Problem 2.2.36

A and $\mathbf{B}$ are two noncommuting Hermitian matrices:

$$\mathbf{AB} - \mathbf{BA} = i\mathbf{C}$$

Prove that $\mathbf{C}$ is Hermitian.

Solution Let's consider

$$-i\mathbf{C}^{\dagger} = (\mathbf{AB} - \mathbf{BA})^{\dagger}$$

$$-i\mathbf{C}^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger} - \mathbf{A}^{\dagger}\mathbf{B}^{\dagger}$$

$$-i\mathbf{C}^{\dagger} = B\mathbf{A} - \mathbf{AB}$$

$$-i\mathbf{C}^{\dagger} = -i\mathbf{C}$$

Problem 2.2.37

Two matrices **A** and **B** are each Hermitian. Find a necessary and sufficient condition for their product **AB** to be Hermitian.

Solution

$$(\mathbf{AB})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger} = \mathbf{BA} = \mathbf{AB}$$

With this, we can say that $[\mathbf{A}, \mathbf{B}] = 0$

Problem 2.2.38

Show that the reciprocal (that is, inverse) of a unitary matrix is unitary.

Solution A matrix is said to be unitary matrix if its adjoint is equal to its inverse. Let **U** be unitary matrix. Then $\mathbf{U}^{\dagger} = \mathbf{U}^{-1}$.

$$\left(\mathbf{U}^{\dagger}\right)^{\dagger} = \mathbf{U}$$

$$\left(\mathbf{U}^{\dagger}\right)^{\dagger} = \left(\mathbf{U}^{-1}\right)^{\dagger}$$

Problem 2.2.39

Prove that the direct product of two unitary matrices is unitary.

Solution

$$(\mathbf{U}_1\mathbf{U}_2)^{\dagger} = \mathbf{U}_2^{\dagger}\mathbf{U}_1^{\dagger}$$

$$(\mathbf{U}_1\mathbf{U}_2)^{\dagger} = \mathbf{U}_2^{-1}\mathbf{U}_1^{-1}$$

$$(\mathbf{U}_1\mathbf{U}_2)^{\dagger} = (\mathbf{U}_1\mathbf{U}_2)^{-1}$$

Problem 2.2.40

If σ is the vector with the σ_i as components given in Eq. (2.61), and p is an ordinary vector, show that

$$(\sigma \cdot p)^2 = p^2 \hat{1}_2$$

where $\hat{1}_2$ is a 2×2 unit matrix.

Solution

$$(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = \left(p_x\sigma_1 + p_y\sigma_2 + p_z\sigma_3\right)^2$$

$$p_x^2\sigma_1^2 + p_y^2\sigma_2^2 + p_z^2\sigma_3^2 + p_xp_y\left(\sigma_1\sigma_2 + \sigma_2\sigma_1\right) + p_xp_z\left(\sigma_1\sigma_3 + \sigma_3\sigma_1\right)$$

$$+p_yp_z\left(\sigma_1\sigma_2 + \sigma_2\sigma_1\right) = p_x^2 + p_y^2 + p_z^2 = \mathbf{p}^2$$

Problem 2.2.41

Use the equations for the properties of direct products, Eqs. (2.57) and (2.58), to show that the four matrices $\gamma^\mu, \mu = 0, 1, 2, 3$, satisfy the conditions listed in Eqs. (2.74) and (2.75).

Solution Writing $\gamma^0 = \sigma_3 \otimes \mathbf{1}$ and $\gamma^i = \gamma \otimes \sigma_i (i = 1, 2, 3)$, where

$$\gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and noting fron Eq. (2.57) that if $\mathbf{C} = \mathbf{A} \otimes B$ and $\mathbf{C}' = \mathbf{A}' \otimes B'$ then $\mathbf{CC}' = \mathbf{AA}' \otimes BB'$

$$\left(\gamma^0\right)^2 = \sigma_3^2 \otimes \mathbf{1}_2^2 = \hat{1}_2 \otimes \mathbf{1}_2 = \hat{1}_4, \quad \left(\gamma^i\right)^2 = \gamma^2 \otimes \sigma_i^2 = \left(-\hat{1}_2\right) \otimes \mathbf{1}_2 = -\hat{1}_4$$

$$\gamma^0\gamma^i = \sigma_3\gamma \otimes \mathbf{1}_2\sigma_i = \sigma_1 \otimes \sigma_i, \quad \gamma^i\gamma^0 = \gamma\sigma_3 \otimes \sigma_i\mathbf{1}_2 = (-\sigma_1) \otimes \sigma_i$$
$$\gamma^i\gamma^j = \gamma^2 \otimes \sigma_i\sigma_j \quad \gamma^j\gamma^i = \gamma^2 \otimes \sigma_j\sigma_i$$

It is obvious from the second line of the above equation set that $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$; from the third line of the equation set we find $\gamma^i\gamma^j + \gamma^j\gamma^i$ is zero if $j \neq i$ because then $\sigma_j\sigma_i = -\sigma_i\sigma_j$

Problem 2.2.42

Show that γ^5, Eq. (2.76), anticommutes with all four γ^μ.

Solution

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \gamma^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\gamma^0\gamma^5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\gamma^5\gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\gamma^1\gamma^5 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\gamma^5\gamma^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\gamma^2\gamma^5 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$\gamma^5\gamma^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$\gamma^3\gamma^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma^5\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

This shows that γ^5 anticommutes with all four $\gamma^\mu(\mu = 0, 1, 2, 3)$

Problem 2.2.43

In this problem, the summations are over $\mu = 0, 1, 2, 3$. Define $g_{\mu v} = g^{\mu v}$ by the relations

$$g_{00} = 1; \quad g_{kk} = -1, \quad k = 1, 2, 3; \quad g_{\mu v} = 0, \quad \mu \neq v$$

and define γ_μ as $\sum g_{v\mu}\gamma^\mu$. Using these definitions, show that

(a) $\sum \gamma_\mu \gamma^\alpha \gamma^\mu = -2\gamma^\alpha$

(b) $\sum \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4g^{\alpha\beta}$

(c) $\sum \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^v \gamma^\mu = -2\gamma^v \gamma^\beta \gamma^\alpha$

Solution No solution yet.

Problem 2.2.44

If $\mathbf{M} = \frac{1}{2}\left(1+\gamma^5\right)$, where γ^5 is given in Eq. (2.76), show that

$$\mathbf{M}^2 = \mathbf{M}$$

Note that this equation is still satisfied if γ is replaced by any other Dirac matrix listed in Eq. (2.76)

$\boxed{\textbf{Solution}}$ Consider $\mathbf{M}^2 = \left[\frac{1}{2}\left(1+\gamma^5\right)\right]^2$

$$\begin{aligned}
&= \frac{1}{4}\left(\hat{1}_4 + 2\gamma^5 + \left(\gamma^5\right)^2\right) \\
&= \frac{1}{4}\left(\hat{1}_4 + 2\gamma^5 + \hat{1}_4\right) \\
&= \frac{1}{4}\left(2\hat{1}_4 + 2\gamma^5\right) \\
&= \frac{1}{2}\left(\hat{1}_4 + \gamma^5\right) \\
&= \mathbf{M}
\end{aligned}$$

Thus, $\mathbf{M}^2 = \mathbf{M}$

Problem 2.2.45

Prove that the 16 Dirac matrices form a linearly independent set.

Solution No solution yet.

Problem 2.2.46

If we assume that a given 4×4 matrix $\mathbf{A}$ (with constant elements) can be written as a linear combination of the 16 Dirac matrices (which we denote here as Γ_i)

$$\mathbf{A} = \sum_{i=1}^{16} c_i \Gamma_i$$

show that

$$c_i \sim \text{trace}\left(\mathbf{A}\Gamma_i\right)$$

The matrix $\mathbf{C} = i\gamma^2\gamma^0$ is sometimes called the charge conjugation matrix. Show that $\mathbf{C}\gamma^\mu\mathbf{C}^{-1} = -\left(\gamma^\mu\right)^T$

Solution No solution yet.

Problem 2.2.47

The matrix $\mathbf{C} = i\gamma^2\gamma^0$ is sometimes called the charge conjugation matrix. Show that $\mathbf{C}\gamma^\mu\mathbf{C}^{-1} = -\left(\gamma^\mu\right)^T$

Solution Here

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Consider $\mathbf{C}\gamma^0\mathbf{C}^{-1} = i\gamma^2\gamma^0\gamma^0\left(i\gamma^2\gamma^0\right)^{-1}$

$$\begin{aligned}
&= \gamma^2\gamma^0\gamma^0\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= -\gamma^0 \\
&= -\left(\gamma^0\right)^T \\
&\mathbf{C}\gamma^1\mathbf{C}^{-1} = i\gamma^2\gamma^0\gamma^1\left(i\gamma^2\gamma^0\right)^{-1} \\
&= \gamma^2\gamma^0\gamma^1\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= \gamma^1 \\
&= -\left(\gamma^1\right)^T
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}\gamma^2\mathbf{C}^{-1} &= i\gamma^2\gamma^0\gamma^2\left(i\gamma^2\gamma^0\right)^{-1} \\
&= \gamma^2\gamma^0\gamma^2\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= -\gamma^2 \\
&= -\left(\gamma^2\right)^T \\
\mathbf{C}\gamma^3\mathbf{C}^{-1} &= i\gamma^2\gamma^0\gamma^3\left(i\gamma^2\gamma^0\right)^{-1} \\
&= \gamma^2\gamma^0\gamma^3\left(\gamma^0\right)^{-1}\left(\gamma^2\right)^{-1} \\
&= \gamma^3 \\
&= -\left(\gamma^3\right)^T
\end{aligned}$$

Thus, $\mathbf{C}\gamma^\mu\mathbf{C}^{-1} = -\left(\gamma^\mu\right)^T$ $(\mu = 0, 1, 2, 3)$

Problem 2.2.48

(a) Show that, by substitution of the definitions of the γ^{μ} matrices from Eqs. (2.70) and (2.72), that the Dirac equation, Eq. (2.73), takes the following form when written as 2×2 blocks (with ψ_L and ψ_S column vectors of dimension 2). Here L and S stand, respectively, for "large" and "small" because of their relative size in the nonrelativistic limit):

$$\begin{bmatrix} mc^2 - E & c\left(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3\right) \\ -c\left(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3\right) & -mc^2 - E \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_S \end{bmatrix} = 0$$

(b) To reach the nonrelativistic limit, make the substitution $\mathbf{E} = mc^2 + \varepsilon$ and approximate $-2mc^2 - \varepsilon$ by $-2mc^2$. Then write the matrix equation as two simultaneous two-component equations and show that they can be rearranged to yield

$$\frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right)\psi_L = \varepsilon \psi_L$$

which is just the Schrödinger equation for a free particle.

(c) Explain why is it reasonable to call ψ_L and ψ_S "large" and "small."

Solution No solution yet.

Problem 2.2.49

Show that it is consistent with the requirements that they must satisfy to take the Dirac gamma matrices to be (in 2×2 block form)

$$\gamma^0 = \begin{bmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \quad (i = 1, 2, 3)$$

This choice for the gamma matrices is called the Weyl representation.

Solution If $\mathbf{C} = \mathbf{A} \otimes B$ and $\mathbf{C}' = \mathbf{A}' \otimes B'$ then $\mathbf{CC}' = \mathbf{AA}' \otimes BB'$ we have

$$\begin{aligned}
\left(\gamma^0\right)^2 &= \sigma_2^2 \otimes \hat{1}_2^2 \\
&= \hat{1}_2 \otimes \hat{1}_2 \quad \text{as } \sigma_i^2 = 1 \\
&= \hat{1}_4 \\
&= 1
\end{aligned}$$

as $\gamma^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\hat{1}_2$

$$\begin{aligned}
(\gamma')^2 &= \gamma^2 \otimes \sigma_i^2 \\
&= \left(-\hat{1}_2\right) \otimes \hat{1}_2 \\
&= -\hat{1}_4 \\
&= -1
\end{aligned}$$

as $\sigma_1 \gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -\sigma_3$

$$\begin{aligned}
\gamma^0 \gamma^i &= \sigma_1 \gamma \otimes \mathbf{l}_2 \sigma_i \\
&= (-\sigma_3) \otimes \sigma_i \\
\gamma^i \gamma^0 &= \gamma \sigma_1 \otimes \sigma_i \mathbf{l}_2 \\
&= \sigma_3 \otimes \sigma_i
\end{aligned}$$

as $\gamma \sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3$ Thus $\gamma^0 \gamma^i + \gamma' \gamma^0 = 0$

$$\begin{aligned}
\gamma^i \gamma^j &= \gamma^2 \otimes \sigma_i \sigma_j \\
\gamma^j \gamma^i &= \gamma^2 \otimes \sigma_j \sigma_i = \gamma^2 \otimes \left(-\sigma_i \sigma_j\right)
\end{aligned}$$

as $\sigma_i \sigma_j + \sigma_j \sigma_i = 0$. Thus, $\gamma^i \gamma^j + \gamma^j \gamma^i = 0$ if $j \neq i$

Problem 2.2.50

Show that the Dirac equation separates into independent 2×2 blocks in the Weyl representation (see Exercise 2.2 .49) in the limit that the mass m approaches zero. This observation is important in the ultra relativistic regime where the rest mass is inconsequential, or for particles of negligible mass (e.g., neutrinos).

Solution In the Weyl representation, the matrices γ^0, α_i and the wave function ψ written as 2×2 blocks take the forms

$$\gamma^0 = \begin{bmatrix} 0 & \hat{1}_2 \\ \hat{1}_2 & 0 \end{bmatrix}, \quad \alpha_i = \begin{bmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

In block form $\left[\gamma^0 mc^2 + \alpha \cdot p\right] \psi = E\psi$ becomes

$$\left[\begin{bmatrix} 0 & mc^2 \\ mc^2 & 0 \end{bmatrix} + \begin{bmatrix} -\sigma \cdot p & 0 \\ 0 & \sigma \cdot p \end{bmatrix}\right] \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = E \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

If m is zero, this matrix equation becomes two independent equations, one for ψ_1, and one for ψ_2 In this limit, one set of solutions will be with $\psi_2 = 0$ and ψ_1 a solution to $-\sigma \cdot p\psi_1 = E\psi_1$ and a second set of solutions will have $\psi_1 = 0$ and a set of ψ_2 identical to the previously found set of ψ_1 but with values of E of the opposite sign.

Problem 2.2.51

(*a*) Given $\mathbf{r}' = \mathbf{U}\mathbf{r}$, with $\mathbf{U}$ a unitary matrix and $\mathbf{r}$ a (column) vector with complex elements, show that the magnitude of $\mathbf{r}$ is invariant under this operation.

(*b*) The matrix $\mathbf{U}$ transforms any column vector $\mathbf{r}$ with complex elements into $\mathbf{r}'$, leaving the magnitude invariant: $\mathbf{r}^\dagger\mathbf{r} = \mathbf{r}'^\dagger\mathbf{r}'$. Show that $\mathbf{U}$ is unitary.

Solution For (*a*) We show that the magnitude of r is invariant i.e. $\mathbf{r}'^\dagger\mathbf{r}' = \mathbf{r}^\dagger r$. Consider $\mathbf{r}'^\dagger\mathbf{r}' = (\mathbf{U}\mathbf{r})^\dagger\mathbf{U}\mathbf{r}$

$$\begin{aligned} &= \mathbf{r}^\dagger\mathbf{U}^\dagger\mathbf{U}\mathbf{r} \\ &= \mathbf{r}^\dagger 1\mathbf{r} \\ &= \mathbf{r}^\dagger\mathbf{r} \end{aligned}$$

This shows that the magnitude of r is invariant under this operation.

Solution For (*b*) all r, $\quad \mathbf{r}'^\dagger\mathbf{r}' = \mathbf{r}^\dagger r$

$$\begin{aligned} &(\mathbf{U}\mathbf{r})^\dagger\mathbf{U}\mathbf{r} = \mathbf{r}^\dagger r \\ &\mathbf{r}^\dagger\mathbf{U}^\dagger\mathbf{U}\mathbf{r} = \mathbf{r}^\dagger 1 r \\ &\mathbf{U}^\dagger\mathbf{U} = 1 \end{aligned}$$

This shows that U is unitary.

Chapter 3
Vector Analysis

Chapter 3.2: Vectors in 3-D Space

Problem 3.2.1

If $\mathbf{P} = \hat{\mathbf{e}}_x P_x + \hat{\mathbf{e}}_y P_y$ and $\mathbf{Q} = \hat{\mathbf{e}}_x Q_x + \hat{\mathbf{e}}_y Q_y$ are any two nonparallel (Also nonantiparallectors in the xy -plane, show that $\mathbf{P} \times \mathbf{Q}$ is in the z -direction.

Solution We write the P and Q vectors as

$$\mathbf{P} = \langle P_x, P_y, 0\rangle \quad Q = \langle Q_x, Q_y, 0\rangle$$

So

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ P_x & P_y & 0 \\ Q_x & Q_y & 0 \end{vmatrix} = (P_x Q_y - Q_x P_y)\hat{\mathbf{e}}_z$$

Problem 3.2.2

Prove that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{AB})^2 - (\mathbf{A} \cdot \mathbf{B})^2$

Solution

$$(\mathbf{A} \times \mathbf{B})^2 = (|\mathbf{A}||\mathbf{B}| \sin\theta)^2$$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \times \mathbf{B}^2 \times \sin^2\theta$$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \times \mathbf{B}^2 \times \left(1 - \cos^2\theta\right)$$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \times \mathbf{B}^2 - \mathbf{A}^2 \times \mathbf{B}^2 \times \left(\cos^2\theta\right)$$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2$$

Problem 3.2.3

Using the vectors

$$\mathbf{P} = \hat{\mathbf{e}}_x \cos\theta + \hat{\mathbf{e}}_y \sin\theta$$

$$\mathbf{Q} = \hat{\mathbf{e}}_x \cos\varphi - \hat{\mathbf{e}}_y \sin\varphi$$

$$\mathbf{R} = \hat{\mathbf{e}}_x \cos\varphi + \hat{\mathbf{e}}_y \sin\varphi$$

prove the familiar trigonometric identities

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$
$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

Solution Consider $\mathbf{P} \cdot \mathbf{Q}$ as

$$\mathbf{P} \cdot \mathbf{Q} = (\hat{x}\cos\theta + \hat{y}\sin\theta) \cdot \left(\hat{x}\cos\varphi - \hat{y}\sin\varphi\right) + \hat{y}\sin\theta\hat{x}\cos\varphi - \hat{y}\sin\theta\sin\varphi$$

$$\mathbf{P} \cdot \mathbf{Q} = (1 \times \cos\theta\cos\varphi) - (0 \times \cos\theta\sin\varphi) + (0 \times \sin\theta\cos\varphi) - (1 \times \sin\theta\sin\varphi)$$

$$\mathbf{P} \cdot \mathbf{Q} = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

And by the product rule, $\mathbf{P} \cdot \mathbf{Q} = \cos(\theta + \varphi)$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

Problem 3.2.4

(*a*) Find a vector $\mathbf{A}$ that is perpendicular to

$$\mathbf{U} = 2\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z$$

$$\mathbf{V} = \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$$

(*b*) What is $\mathbf{A}$ if, in addition to this requirement, we demand that it have unit magnitude?

Solution For (*a*) we have $\mathbf{U} = 2\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z, V = \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$

$$\mathbf{U} \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{\mathbf{e}}_x(1-1) - \hat{\mathbf{e}}_y(2+1) + \hat{\mathbf{e}}_z(-2-1)$$

$$\mathbf{U} \times \mathbf{V} = -\hat{\mathbf{e}}_y(3) + \hat{\mathbf{e}}_z(-3) = -3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z$$

Solution For (*b*) We know $\mathbf{A}$ is $-3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z$, so the magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

From this

$$\mathbf{A} = \frac{-3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z}{3\sqrt{2}} = \frac{-\hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z}{\sqrt{2}}$$

Problem 3.2.5

If four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ all lie in the same plane, show that

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$$

Hint. Consider the directions of the cross-product vectors.

Solution Since all four vectors lie in the same plane, the cross product of any two of them would be orthogonal to the plane. Thus:

$$\mathbf{v}_1 = (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{v}_2 = (\mathbf{c} \times \mathbf{d})$$

By definition, it $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, so $\mathbf{v}_1 \times \mathbf{v}_2 = 0$

Problem 3.2.6

Derive the law of sines (see Fig. 3.4) :

$$\frac{\sin\alpha}{|\mathbf{A}|} = \frac{\sin\beta}{|\mathbf{B}|} = \frac{\sin\gamma}{|\mathbf{C}|}$$

Solution We have $\mathbf{A} - \mathbf{B} - \mathbf{C} = 0$ so we cross both sides by $\mathbf{A}$

$$\mathbf{A} \times \mathbf{A} - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = \mathbf{A} \times 0$$

$$0 - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = 0$$

$$-\mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = 0$$

$$-\mathbf{A} \times \mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C} \times \mathbf{A} = \mathbf{A} \times \mathbf{B}$$

$$|\mathbf{C}||\mathbf{A}| \sin\beta = |\mathbf{A}|\mathbf{B}\ | \sin\gamma$$

Again, we cross both sides of $\mathbf{A} - \mathbf{B} - \mathbf{C} = 0$ by $\mathbf{B}$

$$\mathbf{B} \times \mathbf{A} - \mathbf{B} \times \mathbf{B} - \mathbf{B} \times \mathbf{C} = \mathbf{B} \times 0$$

$$\mathbf{B} \times \mathbf{A} - \mathbf{B} \times \mathbf{C} = 0$$

$$\mathbf{B} \times \mathbf{A} = \mathbf{B} \times C$$

$$|\mathbf{B}||A| \sin\gamma = |\mathbf{B}||\mathbf{C}| \sin\alpha$$

$$|\mathbf{A}| \sin\gamma = |\mathbf{C}| \sin\alpha$$

$$\frac{\sin\gamma}{|\mathbf{C}|} = \frac{\sin\alpha}{|\mathbf{A}|}$$

Problem 3.2.7

The magnetic induction $\mathbf{B}$ is defined by the Lorentz force equation,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

Carrying out three experiments, we find that if

$$\mathbf{v} = \hat{\mathbf{e}}_x, \quad \frac{\mathbf{F}}{q} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_y$$

$$\mathbf{v} = \hat{\mathbf{e}}_y, \quad \frac{\mathbf{F}}{q} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$$

$$\mathbf{v} = \hat{\mathbf{e}}_z, \quad \frac{\mathbf{F}}{q} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$$

From the results of these three separate experiments calculate the magnetic induction $\mathbf{B}$.

$\boxed{\textbf{Solution}}$ From the first condition $\mathbf{v} = \hat{\mathbf{e}}_x, \dfrac{F}{q} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_{\dot{y}}$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 1 & 0 & 0 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = \hat{\mathbf{e}}_x(0) - \hat{\mathbf{e}}_y(\mathbf{B}_z) + \hat{\mathbf{e}}_z(\mathbf{B}_y) = -\hat{\mathbf{e}}_y(\mathbf{B}_z) + \hat{\mathbf{e}}_z(\mathbf{B}_y)$$

$$\frac{\mathbf{F}}{q} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_j,$$
$$\mathbf{v} \times \mathbf{B} = -\hat{\mathbf{e}}_y(\mathbf{B}_z) + \hat{\mathbf{e}}_z(\mathbf{B}_y)$$
$$\mathbf{B}_z = 4, \mathbf{B}_y = 2$$

Now, from the second condition $\mathbf{v} = \hat{\mathbf{e}}_y, \dfrac{F}{q} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 0 & 1 & 0 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = \hat{\mathbf{e}}_x(\mathbf{B}_z) - \hat{\mathbf{e}}_y(0) - \hat{\mathbf{e}}_z(\mathbf{B}_x) = \hat{\mathbf{e}}_x(\mathbf{B}_z) - \hat{\mathbf{e}}_z(\mathbf{B}_x)$$

$$\frac{\mathbf{F}}{q} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$$
$$\mathbf{v} \times \mathbf{B} = \hat{\mathbf{e}}_x(\mathbf{B}_z) - \hat{\mathbf{e}}_z(\mathbf{B}_x)$$
$$\mathbf{B}_z = 4, \mathbf{B}_x = 1$$

From the third condition

$$\mathbf{v} = \hat{\mathbf{e}}_z$$
$$\frac{\mathbf{F}}{q} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 0 & 0 & 1 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = \hat{\mathbf{e}}_x(-\mathbf{B}_y) - \hat{\mathbf{e}}_y(-\mathbf{B}_x) - \hat{\mathbf{e}}_z(0) = -\hat{\mathbf{e}}_x(\mathbf{B}_y) + \hat{\mathbf{e}}_y(\mathbf{B}_x)$$

$$\mathbf{v} \times \mathbf{B} = -\hat{\mathbf{e}}_x(\mathbf{B}_y) + \hat{\mathbf{e}}_y(\mathbf{B}_x)$$
$$\frac{\mathbf{F}}{q} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$$
$$\mathbf{B}_y = 2, \mathbf{B}_x = 1$$

Problem 3.2.8

You are given the three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$,

$$\mathbf{A} = \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y$$
$$\mathbf{B} = \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$$
$$\mathbf{C} = \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$$

Therefore, from above three conditions magnetic induction is given by $\mathbf{B} = \hat{x} + 2\hat{y} + 4\hat{z}$

Solution For (a), $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$. Because $\mathbf{A}$ is the plane of $\mathbf{B}$ and $\mathbf{C}$. The parallelepiped has zero height above the BC plane. So therefore volume will be zero. Therefore, the scalar triple product is zero.

Solution For (b)

$$(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \hat{\mathbf{e}}_x(-1) - \hat{\mathbf{e}}_y(-1) + \hat{\mathbf{e}}_z(-1) = -\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{vmatrix} = \hat{\mathbf{e}}_x(-1) - \hat{\mathbf{e}}_y(-1) + \hat{\mathbf{e}}_z(1+1) = -\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y + 2\hat{\mathbf{e}}_z$$

Problem 3.2.9

Prove Jacobi's identity for vector products:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

Solution From $\mathbf{BAC} - \mathbf{CAB}$ rule $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. The entire equation an written as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$
$$= [\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}]$$

since the dot product is commutative so they becomes zero. Therefore,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

Problem 3.2.10

A vector $\mathbf{A}$ is decomposed into a radial vector $\mathbf{A}_r$ and a tangential vector $\mathbf{A}_t$. If $\hat{\mathbf{r}}$ is a unit vector in the radial direction, show that (a) $\mathbf{A}_r = \hat{\mathbf{r}}(\mathbf{A}\cdot\hat{\mathbf{r}})$ and (b) $\mathbf{A}_t = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{A})$

Solution Let $\mathbf{A} = \mathbf{A}_r\hat{\mathbf{r}} + \mathbf{A}_i\hat{\theta}$

$$\mathbf{A}\cdot\hat{\mathbf{r}} = \mathbf{A}_r, \text{ as } \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$$

The left-hand side is:

$$\mathbf{A}_r = \mathbf{A}_r\hat{\mathbf{r}}$$

since $\hat{\mathbf{r}}$ is the unit vector. The right-hand side is:

$$\hat{\mathbf{r}}(\mathbf{A}\cdot\hat{\mathbf{r}}) = \hat{\mathbf{r}}(\mathbf{A}_r) = \mathbf{A}_r\hat{\mathbf{r}}$$

For (b), taking dot product of both sides of the equation $\mathbf{A}_t = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{A})$ by $\hat{\mathbf{r}}$ we get

$$\mathbf{A}_t\cdot\hat{\mathbf{r}} = [-\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{A})]\cdot\hat{\mathbf{r}}$$

The left-hand side is:

$$\mathbf{A}_t\cdot\hat{\mathbf{r}} = \mathbf{A}_t\hat{\theta}\cdot\hat{\mathbf{r}} = 0$$

$$\begin{aligned}
\cdot\hat{\mathbf{r}} &= [\hat{\mathbf{r}}\times(\mathbf{A}\times\hat{\mathbf{r}})]\cdot\hat{\mathbf{r}}\\
&= [\mathbf{A}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{A}\cdot\hat{\mathbf{r}})]\cdot\hat{\mathbf{r}}\\
&= [\mathbf{A} - \hat{\mathbf{r}}\mathbf{A}_r]\cdot\hat{\mathbf{r}}\\
&= \mathbf{A}\cdot\hat{\mathbf{r}} - \mathbf{A}_r\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\\
&= \mathbf{A}_r - \mathbf{A}_r\\
&= 0
\end{aligned}$$

Problem 3.2.11

Prove that a necessary and sufficient condition for the three (nonvanishing) vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ to be coplanar is the vanishing of the scalar triple product

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0$$

Solution It should be keep in mind that scalar triple product can also be represent as the volume of parallelepiped which is formed by three vectors. So we can say that if scalar triple product is equal to zero then vectors are coplanar as the parallelepipeds have no volume.

Problem 3.2.12

Three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are given by

$$\mathbf{A} = 3\hat{\mathbf{e}}_x - 2\hat{\mathbf{e}}_y + 2\hat{\mathbf{z}}$$
$$\mathbf{B} = 6\hat{\mathbf{e}}_x + 4\hat{\mathbf{e}}_y - 2\hat{\mathbf{z}}$$
$$\mathbf{C} = -3\hat{\mathbf{e}}_x - 2\hat{\mathbf{e}}_y - 4\hat{\mathbf{z}}$$

Compute the values of $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}), \mathbf{C} \times (\mathbf{A} \times \mathbf{B})$ and $\mathbf{B} \times (\mathbf{C} \times \mathbf{A})$

Solution First we can find $B \times C$ and then can permorm dot product

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 6 & 4 & -2 \\ -3 & -2 & -4 \end{vmatrix} = \hat{x}(-16-4) - \hat{y}(-24-6) + \hat{z}(-12+12) = -20\hat{x} + 30\hat{y}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (3\hat{x} - 2\hat{y} + 2\hat{z}) \cdot (-20\hat{x} + 30\hat{y}) = -60 - 60 = -120$$

With this, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -120$

Solution For (b) we have that the vector $\mathbf{A}$

$$\mathbf{A} = (3\hat{x} - 2\hat{y} + 2\hat{z})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & -2 & 2 \\ -20 & 30 & 0 \end{vmatrix} = \hat{x}(-60) - \hat{y}(40) + \hat{z}(50)$$

With this,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (-60)\hat{x} - (40)\hat{y} + (50)\hat{z}$$

Solution For (c)

$$(\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & -2 & 2 \\ 6 & 4 & -2 \end{vmatrix} = \hat{x}(-4) - \hat{y}(-18) + \hat{z}(24)$$

$$\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3 & -2 & -4 \\ -4 & 18 & 24 \end{vmatrix} = \hat{x}(26) - \hat{y}(-88) + \hat{z}(-62)$$

Solution For (d)

$$(\mathbf{C} \times \mathbf{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3 & -2 & -4 \\ 3 & -2 & 2 \end{vmatrix} = \hat{x}(-12) - \hat{y}(6) + \hat{z}(12)$$

$$\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 6 & 4 & -2 \\ -12 & -6 & 12 \end{vmatrix} = \hat{x}(36) - \hat{y}(48) + \hat{z}(12)$$

Problem 3.2.13

Show that
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

Solution Let $\mathbf{C}\times\mathbf{D} = m$. Now, consider the scalar triple product $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{m}$. since cross and dot product can be interchanged, we have,

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{m} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{m})$$

Resubstituting m we get

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \mathbf{A}\cdot[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})]\\ &= \mathbf{A}\cdot[(\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{D}]\\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\end{aligned}$$

Thus, $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

Problem 3.2.14

Show that $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D}$

Solution Let $\mathbf{A}\times\mathbf{B} = \mathbf{m}$

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{m}\times(\mathbf{C}\times\mathbf{D})$$
$$= (\mathbf{m}\cdot\mathbf{D})\mathbf{C} - (\mathbf{m}\cdot\mathbf{C})\mathbf{D}$$
$$= ((\mathbf{A}\times\mathbf{B})\cdot\mathbf{D})\mathbf{C} - ((\mathbf{A}\times\mathbf{B})\cdot\mathbf{C})\mathbf{D}$$
$$= (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D}))\mathbf{C} - (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}))\mathbf{D}$$

Problem 3.2.15

An electric charge q_1 moving with velocity $\mathbf{v}_1$ produces a magnetic induction $\mathbf{B}$ given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} q_1 \frac{\mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2} \qquad \text{(mks units)},$$

where $\hat{\mathbf{r}}$ is a unit vector that points from q_1 to the point at which $\mathbf{B}$ is measured (Biot and Savart law).

(*a*) Show that the magnetic force exerted by q_1 on a second charge q_2, velocity $\mathbf{v}_2$, is given by the vector triple product

$$\mathbf{F}_2 = \frac{\mu_0}{4\pi} \frac{q_1 q_2}{r^2} \mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})$$

(*b*) Write out the corresponding magnetic force $\mathbf{F}_1$ that q_2 exerts on q_1. Define your unit radial vector. How do $\mathbf{F}_1$ and $\mathbf{F}_2$ compare?

(*c*) Calculate $\mathbf{F}_1$ and $\mathbf{F}_2$ for the case of q_1 and q_2 moving along parallel trajectories side by side.

Solution For (*a*) The magnetic force $\mathbf{F}_2$ is defined by the Lorentz force equation,

$$\begin{aligned} \mathbf{F}_2 &= q_2 (\mathbf{v}_2 \times \mathbf{B}_1) \\ &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})}{r^2} \end{aligned}$$

and

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} q_1 \frac{\mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2}$$

Solution For (*b*) The magnetic force $\mathbf{F}_1$ is defined by the Lorentz force equation,

$$\begin{aligned} \mathbf{F}_1 &= q_1 (\mathbf{v}_1 \times \mathbf{B}_2) \\ &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{r}})}{r^2} \end{aligned}$$

and

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi} q_2 \frac{\mathbf{v}_2 \times (-\hat{\mathbf{r}})}{r^2}$$

From part we have,

$$\mathbf{F}_2 = \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})}{r^2}$$

since $-\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{r}}) \neq \mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}})$, $\quad \mathbf{F}_1 \neq \mathbf{F}_2$

Solution For (*c*) we have that

$$\begin{aligned} \mathbf{F}_1 &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v} \times (\mathbf{v} \times \hat{\mathbf{r}})}{r^2} \\ &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}(\mathbf{v} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\ &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{0 - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\ &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}^2 \hat{\mathbf{r}}}{r^2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{F}_2 &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v} \times (\mathbf{v} \times \hat{\mathbf{r}})}{r^2} \\ &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}(\mathbf{v} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\ &= \frac{\mu_0}{4\pi} q_1 q_2 \frac{0 - \hat{\mathbf{r}}(\mathbf{v} \cdot v)}{r^2} \\ &= -\frac{\mu_0}{4\pi} q_1 q_2 \frac{\mathbf{v}^2 \hat{\mathbf{r}}}{r^2} \end{aligned}$$

Thus, $\mathbf{F}_1 = -\mathbf{F}_2$

Chapter 3.3: Coordinate Transformation

Problem 3.3.1

A rotation $\varphi_1 + \varphi_2$ about the z -axis is carried out as two successive rotations φ_1 and φ_2, each about the z-axis. Use the matrix representation of the rotations to derive the trigonometric identities

Solution

$$\cos(\varphi_1+\varphi_2) = \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2$$
$$\sin(\varphi_1+\varphi_2) = \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2$$

$$\begin{bmatrix} \cos(\varphi_1+\varphi_2) \sin(\varphi_1+\varphi_2) \\ -\sin(\varphi_1+\varphi_2) \cos(\varphi_1+\varphi_2) \end{bmatrix} = \begin{bmatrix} \cos\varphi_2 \sin\varphi_2 \\ -\sin\varphi_2 \cos\varphi_2 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 \sin\varphi_1 \\ -\sin\varphi_1 \cos\varphi_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 \\ -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{bmatrix}$$

since we have identities $\cos(\varphi_1+\varphi_2) = \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2$

$$\sin(\varphi_1+\varphi_2) = \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2$$

Therefore the trigonometric identities follow from the rotation matrix identity.

Problem 3.3.2

A corner reflector is formed by three mutually perpendicular reflecting suffices. Show that a a ay of light incident tupon the cometor (striking all three surfaces) is reflected back along a line parallel to the line of incidence. Hint. Consider the effect of a reflection on the components of a vector describing the direction of the light ray.

Solution Here we are asked prove that the ray of light incident upon the corner reflector is reflected of back along line parallel to line of incidence. So for this align the reflecting surfaces with xy, xz, and yz planes. If an incoming ray strikes the xy plane, the z component of its direction of propagation is reversed. A strike on the xz plane reverses its y component, and a strike on yz plane reverses its x component.

Problem 3.3.3

Let x and y be column vectors. Under an orthogonal transformation S, they become $x' = Sx$ and $y' = Sy$. Show that $(x')^T y' = x^T y$, a result equivalent to the invariance of the dot product under a rotational transformation.

Solution It is given that S is orthogonal, if so its transpose is also its inverse. From this

$$(x')^T = (Sx)^T = x^T \mathbf{S}^T = x^T \mathbf{S}^{-1}$$

Then

$$(x')^T y' = x^T \mathbf{S}^{-1} S y = x^T y$$

Therefore $(x')^T y' = x^T y$

Problem 3.3.4

Given the orthogonal transformation matrix S and vectors a and **b**,

$$S = \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

(a) Calculate $\det(S)$.

(b) Verify that $\mathbf{a} \cdot \mathbf{b}$ is invariant under application of $\mathbf{S}$ to $\mathbf{a}$ and $\mathbf{b}$.

(c) Determine what happens to $\mathbf{a} \times \mathbf{b}$ under application of $\mathbf{S}$ to $\mathbf{a}$ and $\mathbf{b}$. Is this what is expected?

Solution For (a) given

$$S = \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix}$$

$$\det(S) = \det \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} = 1$$

Solution For (b) we show that $a \cdot b$ is invariant under application of $\mathbf{S}$ to a and b.

$$\begin{aligned} \mathbf{a}' &= \mathbf{S}\mathbf{a} \\ &= \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.80 \\ 0.12 \\ 1.16 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b}' &= \mathbf{S}\mathbf{b} \\ &= \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1.20 \\ 0.68 \\ -1.76 \end{bmatrix} \end{aligned}$$

$$a \cdot b = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = -1$$

$$a' \cdot b' = \begin{bmatrix} 0.80 & 0.12 & 1.16 \end{bmatrix} \begin{bmatrix} 1.20 \\ 0.68 \\ -1.76 \end{bmatrix} = -1$$

Thus, $a \cdot b$ is invariant under application of $\mathbf{S}$ to a and b.

Solution For (c) we find $\mathbf{a} \times \mathbf{b}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}(\mathbf{a} \times \mathbf{b}) &= \begin{bmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 2.8 \\ 0.4 \end{bmatrix} \end{aligned}$$

$$\mathbf{a}' \times \mathbf{b}' = \begin{vmatrix} i & j & k \\ 0.80 & 0.12 & 1.16 \\ 1.20 & 0.68 & -1.76 \end{vmatrix} = \begin{bmatrix} -1 \\ 2.8 \\ 0.4 \end{bmatrix}$$

Thus, $\mathbf{S}(\mathbf{a} \times \mathbf{b}) = \mathbf{a}' \times \mathbf{b}'$ and hence $\mathbf{a} \times \mathbf{b}$ is a vector.

Problem 3.3.5

Using **a** and **b** as defined in Exercise 3.3.5 but with

$$S = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

(a) Calculate $\det(S)$.

(b) $\mathbf{a} \times \mathbf{b}$

(c) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Solution For (a) Given that

$$S = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix}$$

Then

$$\det(S) = \det \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} = 1$$

Apply **S** to **a**, **b**, and c.

$$\begin{aligned} \mathbf{a}' &= \mathbf{Sa} \\ &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.40 \\ -0.16 \\ -0.12 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b}' &= \mathbf{Sb} \\ &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -0.80 \\ -1.68 \\ 1.24 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c}' &= \mathbf{Sc} \\ &= \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3.60 \\ -0.44 \\ 0.92 \end{bmatrix} \end{aligned}$$

Now, we determine what happen to $\mathbf{a} \times \mathbf{b}$ under application of **S** to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Solution For (b)

$$(a \times b) = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{S}(\mathbf{a}\times\mathbf{b}) = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.40 \\ 1.64 \\ 2.48 \end{bmatrix}$$

$$\mathbf{a}'\times\mathbf{b}' = \begin{vmatrix} i & j & k \\ 1.40 & -0.16 & -0.12 \\ -0.80 & -1.68 & 1.24 \end{vmatrix} = \begin{bmatrix} -0.40 \\ -1.64 \\ -2.48 \end{bmatrix}$$

Thus, $\mathbf{S}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}'\times\mathbf{b}'$

$\boxed{\textbf{Solution}}$ For (c) we determine what happen to $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ under application of $\mathbf{S}$ to $\mathbf{a},\mathbf{b},\mathbf{c}$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{bmatrix} -2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = -4+1+6 = 3$$

$$(\mathbf{a}'\times\mathbf{b}')\cdot\mathbf{c}' = \begin{bmatrix} -0.40 & -1.64 & -2.48 \end{bmatrix} \cdot \begin{bmatrix} 3.60 \\ -0.44 \\ 0.92 \end{bmatrix} = -3$$

Thus, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -(\mathbf{a}'\times\mathbf{b}')\cdot\mathbf{c}'$

$\boxed{\textbf{Solution}}$ For (d) We now determine what happen to $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ under application of $\mathbf{S}$ to $\mathbf{a},\mathbf{b},\mathbf{c}$.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$$

$$\mathbf{S}(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})) = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & 0.80 & 0.36 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix} = \begin{bmatrix} -0.40 \\ -8.84 \\ 7.12 \end{bmatrix}$$

$$\mathbf{a}'\times(\mathbf{b}'\times\mathbf{c}') = \begin{vmatrix} 1.40 & -0.16 & -0.12 \\ -0.80 & -1.68 & 1.24 \\ 3.60 & -0.44 & 0.92 \end{vmatrix} = \begin{bmatrix} -0.40 \\ -8.84 \\ 7.12 \end{bmatrix}$$

Thus, $\quad \mathbf{S}(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})) = \mathbf{a}'\times(\mathbf{b}'\times\mathbf{c}')$

Chapter 3.4: Rotations in R^3

Problem 3.4.1

Another set of Euler rotations in common use is

(a) a rotation about the x_3 -axis through an angle φ, counterclockwise,

(b) a rotation about the x_1' -axis through an angle θ, counterclockwise,

(c) a rotation about the x_3'' -axis through an angle ψ, counterclockwise.

If

$$\begin{aligned} \alpha &= \varphi - \pi/2 \\ \beta &= \theta \\ \gamma &= \psi + \pi/2 \end{aligned}$$

or

$$\begin{aligned} \varphi &= \alpha + \pi/2 \\ \theta &= \beta \\ \psi &= \gamma - \pi/2 \end{aligned}$$

show that the final systems are identical.

Solution The Euler rotations given in the text is:

1. a rotation about the x_3- axis through an angle α, counterclockwise
2. a rotation about the x_2' - axis through an angle β, counterclockwise
3. a rotation about the x_3'' -axis through an angle γ, counterclockwise.

The Euler rotation defined here differ from those in the text in that the inclination of the polar axis is about that $x_1'-$axis rather than the $x_2'-$ axis. Therefore, to achieve the same polar orientation, we must place the $x_1'-$axis where the $x_2'-$axis was using the text rotation. This requires an additional first rotation of $\frac{\pi}{2}$. After inclining the polar axis, the rotational position is now $\frac{\pi}{2}$ greater than form the text rotation, so the third Euler angle must be $\frac{\pi}{2}$ less than its original value.

Problem 3.4.2

Suppose the Earth is moved (rotated) so that the north pole goes to 30° north, 20° west (original latitude and longitude system) and the 10° west meridian points due south (also in the original system). (a) What are the Euler angles describing this rotation? (b) Find the corresponding direction cosines.

Solution No solution yet.

Problem 3.4.3

Verify that the Euler angle rotation matrix, Eq. (3.37), is invariant under the transformation

$$\alpha \to \alpha + \pi, \quad \beta \to -\beta, \quad \gamma \to \gamma - \pi$$

Solution The Euler rotation matrix $\mathbf{S}(\alpha, \beta, \gamma)$ is :

$$\mathbf{S}(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

Using the transformation $\alpha \to \alpha + \pi, \beta \to -\beta, \gamma \to \gamma - \pi$ we get,

$$\mathbf{S}(\alpha+\pi,-\beta,\gamma-\pi) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

as $\cos\alpha \to -\cos\alpha, \sin\alpha \to -\sin\alpha; \cos\beta \to \cos\beta, \sin\beta \to -\sin\beta; \sin\gamma \to -\sin\gamma$, $\cos\gamma \to -\cos\gamma$
Thus, $\mathbf{S}(\alpha,\beta,\gamma) = \mathbf{S}(\alpha+\pi,-\beta,\gamma-\pi)$ Hence, $\mathbf{S}(\alpha,\beta,\gamma)$ is invariant under the transformation
$\alpha \to \alpha + \pi, \beta \to -\beta, \gamma \to \gamma - \pi$

Problem 3.4.4

Show that the Euler angle rotation matrix $\mathbf{S}(\alpha, \beta, \gamma)$ satisfies the following relations:

(a) $\mathbf{S}^{-1}(\alpha, \beta, \gamma) = \tilde{\mathbf{S}}(\alpha, \beta, \gamma)$

(b) $\mathbf{S}^{-1}(\alpha, \beta, \gamma) = \mathbf{S}(-\gamma, -\beta, -\alpha)$

Solution For (a) The three Euler rotations $S_1(\alpha), S_2(\beta), S_3(\gamma)$ are an orthogonal matrix. So, $\mathbf{S}(\alpha, \beta, \gamma) = S_3(\gamma)S_2(\beta)S_1(\alpha)$ must also be orthogonal. Therefore $\mathbf{S}^{-1}(\alpha, \beta, \gamma) = \tilde{\mathbf{S}}(\alpha, \beta, \gamma)$, by the definition of an orthogonal matrix.

Solution For (b) we have

$$\mathbf{S}(\alpha, \beta, \gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

$$\mathbf{S}(-\gamma, -\beta, -\alpha) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & \sin\beta\cos\alpha \\ \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}^{-1}(\alpha, \beta, \gamma) &= \tilde{\mathbf{S}}(\alpha, \beta, \gamma) \\ &= \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & \sin\beta\cos\alpha \\ \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{bmatrix} \end{aligned}$$

Thus, $\mathbf{S}^{-1}(\alpha, \beta, \gamma) = \mathbf{S}(-\gamma, -\beta, -\alpha)$

Problem 3.4.5

The coordinate system (x, y, z) is rotated through an angle Φ counterclockwise about an axis defined by the unit vector $\hat{\mathbf{n}}$ into system (x', y', z'). In terms of the new coordinates the radius vector becomes

$$\mathbf{r}' = \mathbf{r}\cos\Phi + \mathbf{r}\times\mathbf{n}\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)$$

(a) Derive this expression from geometric considerations.

(b) Show that it reduces as expected for $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$. The answer, in matrix form, appears in Eq. (3.35)

(c) Verify that $r'^2 = r^2$.

Solution For (a) the projection of r on the rotation axis is not changed by the rotation; it is $(\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. The portion of r perpendicular to the rotation axis can be written $r - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. Upon rotation through an angle Φ, this vector perpendicular to the rotation axis will consist of a vector in its original direction $(r - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}})\cos\Phi$ plus a vector perpendicular both to it and to $\hat{\mathbf{n}}$ given by $(r - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}})\sin\Phi\times\hat{\mathbf{n}}$; this reduces to $\mathbf{r}\times\hat{\mathbf{n}}\sin\Phi$ Adding these contributions, we get the required result.

Solution For (b) if $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$, the formula $\mathbf{r}' = \mathbf{r}\cos\Phi + \mathbf{r}\times n\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)$ becomes

$$\begin{aligned}
\mathbf{r}' &= \left(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z\right)\cos\Phi + \left(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y\right)\sin\Phi + \hat{\mathbf{e}}_z\left(z\hat{\mathbf{e}}_z\right)(1-\cos\Phi)\\
&= \left(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z\right)\cos\Phi + \left(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y\right)\sin\Phi + z(1-\cos\Phi)\hat{\mathbf{e}}_z\\
&= x\cos\Phi\hat{\mathbf{e}}_x + y\cos\Phi\hat{\mathbf{e}}_y + z\cos\Phi\hat{\mathbf{e}}_z + y\sin\Phi\hat{\mathbf{e}}_x - x\sin\Phi\hat{\mathbf{e}}_y + z(1-\cos\Phi)\hat{\mathbf{e}}_z
\end{aligned}$$

as $r = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$, $\quad \mathbf{r}\times n = \mathbf{r}\times\hat{\mathbf{e}}_z = y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y$ and Simplifying, this reduces to

$$\mathbf{r}' = (x\cos\Phi + y\sin\Phi)\hat{\mathbf{e}}_x + (y\cos\Phi - x\sin\Phi)\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$$

This corresponds to the rotational transformation whose matrix form is

$$S_1(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1 \end{bmatrix}$$

Solution For (c) we expand r'^2, recognizing that the second term of

$$\mathbf{r}' = \mathbf{r}\cos\Phi + \mathbf{r}\times n\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)$$

$$\begin{aligned}
r'^2 &= \mathbf{r}'\cdot\mathbf{r}'\\
&= (\mathbf{r}\cos\Phi + \mathbf{r}\times\hat{\mathbf{n}}\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi))\cdot(\mathbf{r}\cos\Phi + \mathbf{r}\times\hat{\mathbf{n}}\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi))\\
&= r^2\cos^2\Phi + (\mathbf{r}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin\Phi\cos\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r})\sin\Phi\cos\Phi\\
&+ (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{r})\sin\Phi(1-\cos\Phi) + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi\\
&+ (\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{r})\sin\Phi(1-\cos\Phi) + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2\\
r'^2 &= r^2\cos^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 + 2(\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi
\end{aligned}$$

as $(\mathbf{r}\cdot\mathbf{r}\times\hat{\mathbf{n}}) = (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}) = (\mathbf{r}\times\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{r}) = 0$

$$\begin{aligned}
r'^2 &= r^2\cos^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 + 2(\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi\\
&= r^2 + (\hat{\mathbf{n}}\cdot\mathbf{r})^2\left(-\sin^2\Phi + 1 + \cos^2\Phi - 2\cos^2\Phi\right)\\
&= r^2
\end{aligned}$$

Chapter 5
Vector Spaces

Chapter 5.1: Vectors in function spaces

Problem 5.1.1

A function $f(x)$ is expanded in a series of orthonormal functions

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

Show that the series expansion is unique for a given set of $\varphi_n(x)$. The functions $\varphi_n(x)$ are being taken here as the basis vectors in an infinite-dimensional Hilbert space.

Solution Consider the Orthonormal function:

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

The objective is to show that the series expansion is unique for $\phi_n(x)$. Here, the functions $\phi_n(x)$ are as the basis vectors in an infinite-dimensional Hilbert space. If the functions ϕ_i are orthogonal and

$$f = \sum_{i=1}^{n} a_i \phi_i,$$

then the scalar

$$a_i = \frac{\langle \phi_i | f \rangle}{\langle \phi_i | \phi_i \rangle}$$

Using the Orthogonality definition, the value of $\langle \phi_n | f \rangle$ is,

$$\begin{aligned} \langle \phi_n | f \rangle &= a_n \\ &= \int_a^b w(x) f(x) \phi_n(x) dx \end{aligned}$$

This is derived from the function f. Assume that $\langle \phi_n | f \rangle = a'_n$

$$\begin{aligned} \langle \phi_n | f \rangle &= a'_n \\ &= \int_a^b w'(x) f(x) \phi_n(x) dx \end{aligned}$$

Then, $a_n = a_n{}'$ since $w(x) = w'(x)$ Therefore, $\langle \phi_n | f \rangle = a_n$ is unique.

Problem 5.1.2

A function $f(x)$ is represented by a finite set of basis functions $\varphi_i(x)$

$$f(x) = \sum_{i=1}^{N} c_i \varphi_i(x)$$

Show that the components c_i are unique, that no different set c'_i exists. Note. Your basis functions are automatically linearly independent. They are not necessarily orthogonal.

Solution Consider the function:

$$f(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

The objective is to show that the components c_i are unique. The function can be written as,

$$\begin{aligned} f(x) &= \sum_i c_i \phi_i(x) \\ &= \sum_j c'_j \phi_j(x) \end{aligned}$$

Then,

$$\begin{aligned} \sum_i \left(c_i - c'_i\right) \phi_i &= \sum_i c_i \phi_i - \sum_i c'_i \phi_i \\ &= \sum_i c_i \phi_i - \sum_i c_i \phi_i \\ &= 0 \end{aligned}$$

Assume $c_m - c'_m \neq 0$ Then,

$$\phi_m = \frac{-1}{c_m - c_m} \sum_{b=m} \left(c_i - c'_i\right) \phi_i$$

It confirms that, ϕ_m is not linearly independent of the ϕ_i, which is a contradiction to our assumption. So, $c_m - c'_m = 0$ Therefore, the scalars c_i are unique.

Problem 5.1.3

A function $f(x)$ is approximated by a power series $\sum_{i=0}^{n-1} c_i x^i$ over the interval [0,1] Show that minimizing the mean square error leads to a set of linear equations

$$\mathrm{A}\mathrm{c} = \mathbf{b}$$

where

$$A_{ij} = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1}, \quad i,j = 0,1,2,\ldots,n-1$$

and

$$b_i = \int_0^1 x^i f(x) dx, \quad i = 0,1,2,\ldots,n-1$$

Note. The A_{ij} are the elements of the Hilbert matrix of order n. The determinant of this Hilbert matrix is a rapidly decreasing function of n. For $n = 5$, $\det A = 3.7 \times 10^{-12}$ and the set of equations $\mathrm{A}c = \mathbf{b}$ is becoming ill-conditioned and unstable.

$\boxed{\textbf{Solution}}$ For

$$f(x) = \sum_{i=0}^{n-1} c_i x^i$$

we have

$$\begin{aligned}
b_j &= \int_0^1 x^j f(x) dx, \quad j = 0,1,2,\ldots,n-1 \\
&= \sum_l c_i \int_0^1 x^{i+j} dx \\
&= \sum_{i=0}^{n-1} \frac{c_i}{i+j+1} \\
&= A_{ji} c_i
\end{aligned}$$

This result also minimizing the mean square error

$$\int_0^1 \left[f(x) - \sum_{i=0}^{n-1} c_i x^i \right]^2 dx$$

upon varying the c_i

Problem 5.1.4

In place of the expansion of a function $F(x)$ given by

$$F(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

with

$$a_n = \int_a^b F(x)\varphi_n(x)w(x)dx$$

take the finite series approximation

$$F(x) \approx \sum_{n=0}^{m} c_n \varphi_n(x)$$

Show that the mean square error

$$\int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \varphi_n(x)\right]^2 w(x)dx$$

is minimized by taking $c_n = a_n$
Note. The values of the coefficients are independent of the number of terms in the finite series. This independence is a consequence of orthogonality and would not hold for a least-squares fit using powers of x.

Solution Consider the function

$$F(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

Here,

$$a_n = \int_a^b F(x)\phi_n(x)w(x)dx$$

and

$$F(x) \approx \sum_{n=0}^{m} c_n \phi_n(x)$$

The objective is to show the mean square error is minimized when $c_n = a_n$. For

$$F(x) = \sum_{n=0}^{m} a_n \phi_n(x),$$

we have

$$\begin{aligned}
c_j &= \int_0^1 x^j f(x)dx, j = 0, 1, 2, \ldots, m \\
&= \sum_i a_i \int_0^1 x^{j+j} dx \\
&= \sum_{i=0}^{m} \frac{a_i}{i+j+1} \\
&= A_{ji} a_i
\end{aligned}$$

Note that A_{ij} 's represents the elements of the Hilbert matrix of order n. The determinant of this Hilbert matrix is a decreasing function of n. Write the function as

$$F(x) = \sum_{n=0}^{m} c_n \phi_n(x)$$

$$F(x) - \sum_{n=0}^{m} c_n \phi_n(x) = 0$$

$$\int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \phi_n(x) \right]^2 w(x)dx = 0$$

$$\frac{\partial}{\partial c_l} \int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \phi_n(x) \right]^2 w(x)dx = 0$$

Remember that

$$c_n = \int_a^b F(x)\phi_n(x)w(x)dx$$

This result is also minimizing the mean square error

$$\int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \phi_n(x) \right]^2 w(x)dx$$

is minimized when $c_n = a_n$

Problem 5.1.5

The functions $\cos nx$ $(n = 0, 1, 2, \ldots)$ and $\sin nx$ $(n = 1, 2, \ldots)$ have (together) been shown to form a complete set on the interval $-\pi < x < \pi$. since this determination is obtained subject to convergence in the mean, there is the possibility of deviation at isolated points, thereby permitting the description of functions with isolated discontinuities.

$$f(x) = \begin{cases} \frac{h}{2}, & 0 < x < \pi \\ -\frac{h}{2}, & -\pi < x < 0 \end{cases} = \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$$

a) Show that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

For a finite upper limit this would be Bessel's inequality. For the upper limit ∞, this is Parseval's identity.

b) Verify that

$$\frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

by evaluating the series. Hint. The series can be expressed in terms of the Riemann zeta function $\zeta(2) = \pi^2/6$

Solution The objective is to show that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

First, we start saying that the integral $\int_{-\pi}^{\pi} [f(x)]^2 dx$ can be evaluated as

$$\begin{aligned}
\int_{-\pi}^{\pi} [f(x)]^2 dx &= \int_{-\pi}^{\pi} f(x) \cdot f(x) dx \\
&= \int_{-\pi}^{\pi} f(x) dx \cdot \int_{-\pi}^{\pi} f(x) dx \\
&= \int_{-\pi}^{\pi} \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} dx \int_{-\pi}^{\pi} \frac{2h}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1} \\
&= \left(\frac{4h^2}{\pi^2}\right) \sum_{m,n=0}^{\infty} \frac{1}{(2n+1)(2m+1)} \times \int_{-\pi}^{\pi} \sin[(2n+1)x] \sin[(2m+1)x] dx \\
&= \frac{4h^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \int_{-\pi}^{\pi} \sin^2[(2n+1)x] dx \\
&= \frac{4h^2}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) \int_{-\pi}^{\pi} \left(\frac{1 - \cos(2(2n+1)x)}{2}\right) dx \\
&= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8}\right) \left(\frac{x - \frac{\sin(2(2n+1)x)}{2(2n+1)}}{2}\right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8}\right) \left(\frac{\pi - \frac{\sin(2(2n+1)\pi)}{2(2n+1)}}{2} - \left(\frac{(-\pi) - \frac{\sin(2(2n+1)(-\pi))}{2(2n+1)}}{2}\right)\right) \\
&= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8}\right) \left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8}\right) (\pi) \\
&= \frac{h^2 \pi}{2}
\end{aligned}$$

Therefore,

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{\pi}{2}h^2$$

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{4h^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_{-\pi}^{\pi}\sin^2[(2n+1)x]dx$$

$$= \frac{4h^2}{\pi^2}\sum_{n=0}^{\infty}(2n+1)^{-2}(\pi)$$

$$= \frac{4h^2}{\pi}\sum_{n=0}^{\infty}(2n+1)^{-2}$$

Hence,

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{\pi}{2}h^2 = \frac{4h^2}{\pi}\sum_{n=0}^{\infty}(2n+1)^{-2}$$

For (b)

$$\begin{aligned}
\text{RHS} &= \frac{4h^2}{\pi}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\right)\\
&= \frac{4h^2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)\\
&= \frac{4h^2}{\pi}\left(\frac{\pi^2}{8}\right)\\
&= \frac{\pi h^2}{2}
\end{aligned}$$

Problem 5.1.6

Derive the Schwarz inequality from the identity

$$\left[\int_a^b f(x)g(x)dx\right]^2 = \int_a^b [f(x)]^2 dx \int_a^b [[g(x)]^2 dx \\ -\frac{1}{2}\int_a^b dx \int_a^b dy[f(x)g(y) - f(y)g(x)]^2$$

Solution The double integral can be written as,

$$\left[\int_a^b f(x)g(x)dx\right]^2 = \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx \\ -\frac{1}{2}\int_a^b dx \int_a^b dy[f(x)g(y) - f(y)g(x)]^2$$

$$\begin{aligned} |\langle f|g\rangle|^2 &= \langle f\rangle^2\langle g\rangle^2 - \frac{1}{2}\int_a^b \int_a^b [f(x)g(y) - f(y)g(x)]^2 \\ &\leq \langle f\rangle^2\langle g\rangle^2 \\ |\langle f|g\rangle|^2 &\leq \langle f|f\rangle\langle g|g\rangle \end{aligned}$$

since the double integral is non-negative, so $\langle f|g\rangle|^2 \geq 0$. Hence, the result of Schwarz inequality is derived.

Problem 5.1.7

Starting from

$$I = \left\langle f - \sum_i a_i\varphi_i | f - \sum_j a_j\varphi_j \right\rangle \geq 0$$

derive Bessel's inequality,

$$\langle f|f\rangle \geq \sum_n |a_n|^2$$

Solution The functions ϕ_j are assumed to be orthonormal. Expand the value of I, we have

$$\begin{aligned} I &= \left\langle f - \sum_i a_i\phi_i | f - \sum_j a_j\phi_j \right\rangle \\ &= \langle f|f\rangle - \sum_i a_i * \langle \phi_i|f\rangle - \sum_i a_i * \langle f|\phi_i\rangle + \sum_i a_i * a_j \langle \phi_i|\phi_j\rangle \\ &\geq 0 \end{aligned}$$

Hence, the result of Bessel's inequality is derived.

Problem 5.1.8

Expand the function $\sin \pi x$ in a series of functions φ_i that are orthogonal (but not normalized) on the range $0 \leq x \leq 1$ when the scalar product has definition

$$\langle f|g\rangle = \int_0^1 f^*(x)g(x)dx$$

Keep the first four terms of the expansion. The first four φ_i are:

$$\varphi_0 = 1, \quad \varphi_1 = 2x-1, \quad \varphi_2 = 6x^2-6x+1, \quad \varphi_3 = 20x^3-30x^2+12x-1$$

Note. The integrals that are needed are the subject of Example 1.10 .5 .

Solution Consider the function: $\sin(\pi x)$ Expand the function $\sin(\pi x)$ in a series of functions ϕ_i which are orthogonal. Write the function $\sin(\pi x)$ in a series of functions ϕ_i as,

$$\sin(\pi x) = \sum_i \frac{\left\langle \phi_i | \sin \pi x \right\rangle}{\left\langle \phi_i, \phi_i \right\rangle} \phi_i(x)$$

Here, $\phi_0 = 1, \phi_1 = 2x-1, \phi_2 = 6x^2-6x+1, \phi_3 = 20x^3-30x^2+12x-1$ The integrals are calculated as,

$$\begin{aligned}\left\langle \phi_0 | \phi_0 \right\rangle &= \int_0^1 dx \\ &= (x)\Big|_0^1 \\ &= 1\end{aligned}$$

$$\left\langle \phi_1 | \phi_1 \right\rangle = \int_0^1 (2x-1)^2 dx$$

$$\left\langle \phi_1 | \phi_1 \right\rangle = \int_0^1 \left(4x^2-4x+1\right) dx$$

$$\left\langle \phi_1 | \phi_1 \right\rangle = \left(\frac{4x^3}{3} - 2x^2 + x\right)\Bigg|_0^1$$

$$\left\langle \phi_1 | \phi_1 \right\rangle = \left(\frac{4}{3} - 2 + 1\right)$$

$$\left\langle \phi_1 | \phi_1 \right\rangle = \frac{1}{3}$$

$$\begin{aligned}\left\langle \phi_2 | \phi_2 \right\rangle &= \int_0^1 \left(6x^2-6x+1\right)^2 dx \\ &= \int_0^1 \left(36x^4-72x^3+48x^2-12x+1\right) dx \\ &= \left(\frac{36x^5}{5} - 18x^4 + 16x^3 - 6x^2 + x\right)\Bigg|_0^1 \\ &= \frac{36}{5} - 18 + 16 - 6 + 1 \\ &= \frac{1}{5}\end{aligned}$$

$$
\begin{aligned}
\langle \phi_3 | \phi_3 \rangle &= \int_0^1 \left(20x^3 - 30x^2 + 12x - 1\right)^2 dx \\
&= \int_0^1 \left(400x^6 - 1200x^5 + 1380x^4 - 760x^3 + 204x^2 - 24x + 1\right) dx \\
&= \left(\frac{400x^7}{7} - 200x^6 + 276x^5 - 190x^4 + 68x^3 - 12x^2 + x \right) \Big|_0^1 \\
&= \frac{400}{7} - 200 + 276 - 190 + 68 - 12 + 1 \\
&= \frac{1}{7}
\end{aligned}
$$

$$
\begin{aligned}
\langle \phi_0 | f \rangle &= \int_0^1 \sin \pi x dx \\
&= \left(\frac{-\cos \pi x}{\pi} \right) \Big|_0^1 \\
&= -\left(\frac{\cos \pi(1)}{\pi} - \frac{\cos \pi(0)}{\pi} \right) \\
&= -\left(\frac{-1}{\pi} - \frac{1}{\pi} \right) \\
&= \frac{2}{\pi}
\end{aligned}
$$

The value of $\langle \phi_1 | f \rangle$ is,

$$
\langle \phi | f \rangle = \int_0^1 (2x - 1) \sin(\pi x) dx
$$

$$
= \left(\frac{2 \sin(\pi x) + (\pi - 2\pi x) \cos(\pi x)}{\pi^2} \right) \Big|_0^1
$$

Now,

$$
\begin{aligned}
\int_0^1 (2x - 1) \sin(\pi x) dx &= \frac{2 \sin(\pi x) + (\pi - 2\pi x) \cos(\pi x)}{\pi^2} \\
&= \frac{2 \sin(\pi \cdot 1) + (\pi - 2\pi \cdot 1) \cos(\pi \cdot 1)}{\pi^2} - \frac{2 \sin(\pi \cdot 0) + (\pi - 2\pi \cdot 0) \cos(\pi \cdot 0)}{\pi^2} \\
&= \frac{2(0) + (-\pi) \cdot 1}{\pi^2} - \left(\frac{2(0) + (-\pi)1}{\pi^2} \right) \\
&= 0
\end{aligned}
$$

Using the same analysis, we get that

$$
\langle \varphi_2 | f \rangle = \frac{2}{\pi} - \frac{24}{\pi^3}
$$

$$
\langle \varphi_3 | f \rangle = 0
$$

$$
\sin \pi x = \frac{2/\pi}{1} \varphi_0 + \frac{2/\pi - 24/\pi^3}{1/5} \varphi_2 + \cdots
$$

$$
\sin(\pi x) = 0.6366 - 0.6871 \left(6x^2 - 6x + 1\right) + \cdots
$$

Problem 5.1.9

Expand the function e^{-x} in Laguerre polynomials $L_n(x)$, which are orthonormal on the range $0 \le x < \infty$ with scalar product

$$\langle f|g\rangle = \int_0^\infty f^*(x)g(x)e^{-x}dx$$

Keep the first four terms of the expansion. The first four $L_n(x)$ are

$$L_0 = 1, \quad L_1 = 1 - x, \quad L_2 = \frac{2 - 4x + x^2}{2}, \quad L_3 = \frac{6 - 18x + 9x^2 - x^3}{6}$$

$\boxed{\textbf{Solution}}$ The value of a_0 is

$$\begin{aligned} a_0 &= \int_0^\infty L_0(x)e^{-2x}dx \\ &= \int_0^\infty e^{-2x}dx \\ &= \left(\frac{e^{-2x}}{-2}\right)\Bigg|_0^\infty \\ &= \frac{-1}{2}\left(e^{-2(\alpha)} - e^0\right) \\ &= \frac{-1}{2}(0-1) \\ &= \frac{1}{2} \end{aligned}$$

The value of a_1 is

$$\begin{aligned} a_1 &= \int_0^\infty L_1(x)e^{-2x}dx \\ &= \int_0^\infty (1-x)e^{-2x}dx \\ &= \left(\frac{1}{4}e^{-2x}(2x-1)\right)\Bigg|_0^\infty \\ &= \frac{1}{4}\left(e^{-2(\infty)}(2(\infty)-1) - e^0(2(0)-1)\right) \\ &= \frac{1}{4}(0+1) \\ &= \frac{1}{4} \end{aligned}$$

The value of a_2 is

$$\begin{aligned} a_2 &= \int_0^\infty L_2(x)e^{-2x}dx \\ &= \int_0^\infty \left(\frac{2-4x+x^2}{2}\right)e^{-2x}dx \\ &= \left(\frac{-1}{8}e^{-2x}\left(1-6x+2x^2\right)\right)\Bigg|_0^\infty \end{aligned}$$

The value of a_3 is,

$$\begin{aligned} a_3 &= \int_0^\infty L_3(x)e^{-2x}dx \\ &= \int_0^\infty \left(\frac{6-18x+9x^2-x^3}{6}\right)e^{-2x}dx \\ &= \left(\frac{1}{48}e^{-2x}\left(4x^3-30x^2+42x-3\right)\right)\Bigg|_0^\infty \end{aligned}$$

$$= \frac{3}{48}$$
$$= \frac{1}{16}$$

Thus, the expansion of e^{-x} is

$$e^{-x} = a_0 L_0(x) + a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x) + \cdots$$
$$= \frac{1}{2}(1) + \frac{1}{4}(1-x) + \frac{1}{8}\left(\frac{2-4x+x^2}{2}\right) + \frac{1}{16}\left(\frac{6-18x+9x^2-x^3}{6}\right) + \cdots$$

Problem 5.1.10

The explicit form of a function f is not known, but the coefficients a_n of its expansion in the orthonormal set φ_n are available. Assuming that the φ_n and the members of another orthonormal set, χ_n, are available, use Dirac notation to obtain a formula for the coefficients for the expansion of f in the χ_n set.

Solution The coefficients of f in the ϕ basis are $a_i = \langle \phi_i | f \rangle$, so the above equation is equivalent to,

$$f = \sum_j b_j \chi_j$$

Here, $b_j = \sum_j \langle \chi_j | \phi_i \rangle a_i$

Problem 5.1.11

Using conventional vector notation, evaluate $\sum_j |\hat{\mathbf{e}}_j\rangle \langle \hat{\mathbf{e}}_j | \mathbf{a} \rangle$, where a is an arbitrary vector in the space spanned by the $\hat{\mathbf{e}}_j$

Solution We assume the unit vectors are orthogonal. Then,

$$\sum_j |\hat{\mathbf{e}}_j\rangle \langle \hat{\mathbf{e}}_j | \mathbf{a} \rangle = \sum_j (\hat{\mathbf{e}}_j \cdot \mathbf{a})\, \hat{\mathbf{e}}_j$$

This expression is a component decomposition of a.

Problem 5.1.12

Letting a $= a_1\hat{e}_1 + a_2\hat{e}_2$ and $\mathbf{b} = b_1\hat{e}_1 + b_2\hat{e}_2$ be vectors in $\mathbb{R}^2$, for what values of k, if any, is

$$\langle \mathbf{a}|\mathbf{b}\rangle = a_1b_1 - a_1b_2 - a_2b_1 + ka_2b_2$$

a valid definition of a scalar product?

Solution Consider the two vectors:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$$

and

$$b = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$$

The objective is to for what values of k the scalar product

$$\langle \mathbf{a}|\mathbf{b}\rangle = a_1b_1 - a_1b_2 - a_2b_1 + ka_2b_2$$

is valid. The scalar product $\langle \mathbf{a}|\mathbf{a}\rangle$ must be positive for every non-zero vector in the space. If we write $\langle \mathbf{a}|\mathbf{a}\rangle$ in the form,

$$\begin{aligned}
\langle \mathbf{a}|\mathbf{a}\rangle &= a_1a_1 - a_1a_2 - a_2a_1 + ka_2a_2 \\
&= a_1^2 - 2a_1a_2 + ka_2^2 \\
&= \left(a_1^2 - 2a_1a_2 + a_2^2\right) - a_2^2 + ka_2^2 \\
&= (a_1 - a_2)^2 - a_2^2 + ka_2^2 \\
&= (a_1 - a_2)^2 + (k-1)a_2^2
\end{aligned}$$

This condition is violated for some non-zero vector a unless $k > 1$. Therefore, the scalar product is valid when $k > 1$.

Chapter 5.2: Gram-Schmidt Orthogonalization

Problem 5.2.1

Following the Gram-Schmidt procedure, construct a set of polynomials $P_n^*(x)$ orthogonal (unit weighting factor) over the range [0,1] from the set $\left[1, x, x^2, \ldots\right]$. Scale so that $P_n^*(1) = 1$

$$ANS.\quad \begin{aligned} P_n^*(x) &= 1 \\ P_1^*(x) &= 2x - 1 \\ P_2^*(x) &= 6x^2 - 6x + 1 \\ P_3^*(x) &= 20x^3 - 30x^2 + 12x - 1 \end{aligned}$$

These are the first four shifted Legendre polynomials.

Solution Choose $p_0(x) = 1$

$$\begin{aligned} p_1(x) &= x - \frac{\langle x, p_0(x)\rangle}{\langle p_0(x), p_0(x)\rangle} p_0(x) \\ &= x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} \cdot 1 \\ &= x - \frac{\int_0^1 x dx}{\int_0^1 dx} \\ &= x - \frac{\left(\frac{x^2}{2}\right)_0^1}{(x)_0^1} \\ &= x - \frac{1}{2} \end{aligned}$$

The polynomial $p_2(x)$ can be calculated as,

$$\begin{aligned} p_2(x) &= x^2 - \frac{\langle x^2, p_0(x)\rangle}{\langle p_0(x), p_0(x)\rangle} \cdot p_0(x) - \frac{\langle x^2, p_1(x)\rangle}{\langle p_1(x), p_1(x)\rangle} \cdot p_1(x) \\ &= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} \cdot 1 - \frac{\langle x^2, x - \frac{1}{2}\rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2}\rangle} \cdot \left(x - \frac{1}{2}\right) \\ &= x^2 - \frac{\int_0^1 x^2 dx}{\int_0^1 dx} - \frac{\int_0^1 x^2 \left(x - \frac{1}{2}\right) dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx} \cdot \left(x - \frac{1}{2}\right) \\ &= x^2 - \frac{1}{3} - \frac{\left(\frac{1}{12}\right)}{\left(\frac{1}{12}\right)} \left(x - \frac{1}{2}\right) \end{aligned}$$

Using

$$\int_0^1 x^2 \left(x - \frac{1}{2}\right) dx = \frac{1}{12}, \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}$$

$$\begin{aligned} p_2(x) &= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

The polynomial $p_3(x)$ can be calculated as,

$$\begin{aligned} p_3(x) &= x^3 - \frac{\langle x^3, p_0(x)\rangle}{\langle p_0(x), p_0(x)\rangle} \cdot p_0(x) - \frac{\langle x^3, p_1(x)\rangle}{\langle p_1(x), p_1(x)\rangle} \cdot p_1(x) - \frac{\langle x^3, p_2(x)\rangle}{\langle p_2(x), p_2(x))} \cdot p_2(x) \\ &= x^3 - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle} \cdot 1 - \frac{\langle x^3, x - \frac{1}{2}\rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2}\rangle} \cdot \left(x - \frac{1}{2}\right) - \frac{\langle x^3, x^2 - x + \frac{1}{6}\rangle}{\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\rangle} \cdot \left(x^2 - x + \frac{1}{6}\right) \\ &= x^3 - \frac{\int_0^1 x^3 dx}{\int_0^1 dx} - \frac{\int_0^1 x^3 \left(x - \frac{1}{2}\right) dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx} \cdot \left(x - \frac{1}{2}\right) - \frac{\int_0^1 x^3 \left(x^2 - x + \frac{1}{6}\right) dx}{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx} \cdot \left(x^2 - x + \frac{1}{6}\right) \end{aligned}$$

Calculating

$$\int_0^1 x^3 \left(x^2 - x + \frac{1}{6}\right) dx = \frac{1}{120}$$

$$\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \frac{1}{180}$$

$$\begin{aligned} p_3(x) &= x^3 - \frac{1}{4} - \frac{\left(\frac{3}{40}\right)}{\left(\frac{1}{12}\right)} \left(x - \frac{1}{2}\right) - \frac{\frac{1}{120}}{\frac{1}{180}} \left(x^2 - x + \frac{1}{6}\right) \\ &= x^3 - \frac{1}{4} - \frac{9}{10} \left(x - \frac{1}{2}\right) - \frac{3}{2} \left(x^2 - x + \frac{1}{6}\right) \\ &= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20} \end{aligned}$$

Using $p_0^*(x) = c_0 p_0(x) = 1$, $p_1 * (x) = c_1 p_1(x)$, $p_2 * (x) = c_2 p_2(x)$, and $p_3^*(x) = c_3 p_3(x)$ Here, $c_0 = 1$, $c_1 = 2$, $c_2 = 6$, $c_3 = 20$. Then, the first four shifted Legendre polynomials are as shown below:

$$\begin{aligned} p_0^*(x) &= c_0 p_0(x) \\ &= 1(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} p_1^*(x) &= c_1 p_1(x) \\ &= 2\left(x - \frac{1}{2}\right) \\ &= 2\left(\frac{2x - 1}{2}\right) \\ &= 2x - 1 \end{aligned}$$

$$\begin{aligned} p_2^*(x) &= c_2 p_2(x) \\ &= 6\left(x^2 - x + \frac{1}{6}\right) \\ &= 6\left(\frac{6x^2 - 6x + 1}{6}\right) \\ &= 6x^2 - 6x + 1 \end{aligned}$$

And

$$\begin{aligned} p_3^*(x) &= c_3 p_3(x) \\ &= 20\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right) \\ &= 20\left(\frac{20x^3 - 30x^2 + 12x - 1}{20}\right) \\ &= 20x^3 - 30x^2 + 12x - 1 \end{aligned}$$

Problem 5.2.2

Apply the Gram-Schmidt procedure to form the first three Laguerre polynomials:

$$u_n(x) = x^n, \quad n = 0, 1, 2, \ldots, \quad 0 \le x < \infty, \quad w(x) = e^{-x}$$

The conventional normalization is

$$\int_0^\infty L_m(x) L_n(x) e^{-x} dx = \delta_{mn}$$

ANS. $L_0 = 1, \quad L_1 = (1 - x), \quad L_2 = \dfrac{2 - 4x + x^2}{2}$

$\boxed{\textbf{Solution}}$ The Laguerre polynomials are orthogonal on the interval $(0, \infty)$ with respect to the gamma distribution $w(x) = e^{-x} x^a$ Then,

$$L_n^{(a)}(x) = \frac{1}{n!} \frac{1}{w(x)} D^n \left[w(x) x^n \right]$$

From the hypotheses, we have $w(x) = e^{-x}$ and $\alpha = 1$ So, the first three Laguerre polynomials are calculated as shown below:

$$\begin{aligned} L_0(x) &= \frac{1}{(0)!} \frac{1}{e^{-x}} \frac{d^0}{dx^0} \left[e^{-x} x^0 \right] \\ &= \frac{1}{1} \left(\frac{1}{e^{-x}} \right) \\ &= e^x \\ L_0(0) &= e^0 \\ &= 1 \end{aligned}$$

The value of $L_1(x)$ is,

$$\begin{aligned} L_1(x) &= \frac{1}{(1)!} \frac{1}{e^{-x}} \frac{d}{dx} \left[e^{-x} x^1 \right] \\ &= e^x \left(e^{-x} - x e^{-x} \right) \\ &= e^x \left(e^{-x} \right) (1 - x) \\ &= 1 - x \end{aligned}$$

The value of $L_2(x)$ is,

$$\begin{aligned} L_2(x) &= \frac{1}{(2)!} \frac{1}{e^{-x}} \frac{d^2}{dx^2} \left[e^{-x} x^2 \right] \\ &= \frac{e^x}{2} D \left(2x e^{-x} - x^2 e^{-x} \right) \\ &= \frac{e^x}{2} \left(2e^{-x} - 4x e^{-x} + x^2 e^{-x} \right) \\ &= \frac{1}{2} (x^2 - 4x + 2) \end{aligned}$$

Problem 5.2.3

You are given

(a) a set of functions $u_n(x) = x^n, n = 0, 1, 2, \ldots,$

(b) an interval $(0, \infty)$

(c) a weighting function $w(x) = xe^{-x}$. Use the Gram-Schmidt procedure to construct the first three orthonormal functions from the set $u_n(x)$ for this interval and this weighting function.

ANS. $\varphi_0(x) = 1, \quad \varphi_1(x) = (x-2)/\sqrt{2}, \quad \varphi_2(x) = \left(x^2 - 6x + 6\right)/2\sqrt{3}$

Solution The objective is to construct the first three orthonormal functions from the set $u_n(x)$ by using the Gram-Schmidt orthogonalization. Here, $u_n(x) = x^n$, $n = 0, 1, 2, \ldots$, $0 < x < \infty$, $w(x) = xe^{-x}$. The orthonormal function ϕ_n is obtained from χ_n as shown

$$\begin{aligned}
\psi_n &= \chi_n - \sum_{\mu=0}^{n-1} \left\langle \phi_\mu \mid \chi_n \right\rangle \phi_u \\
\phi_n &= \frac{\psi_n}{\left\langle \psi_n \mid \psi_n \right\rangle^{\frac{1}{2}}}
\end{aligned}$$

Assume $\chi_n(x) = u_n(x) = x^n$, $n = 0, 1, 2, \ldots$ So, the value of $\psi_0(x) = u_0(x)$ is,

$$\begin{aligned}
\psi_0(x) &= \chi_0(x) \\
&= x^0 \\
&= 1
\end{aligned}$$

The value of ϕ_0 is,

$$\begin{aligned}
\phi_0 &= \frac{\psi_0(x)}{\left\|\psi_0\right\|} \\
&= \frac{1}{\langle 1 \mid 1 \rangle^{\frac{1}{2}}} \\
&= \frac{1}{\left|\int_0^\infty xe^{-x} dx\right|^{\frac{1}{2}}} \\
&= \frac{1}{(1)^{\frac{1}{2}}} \\
&= 1
\end{aligned}$$

The value of $\psi_1(x)$ is,

$$\begin{aligned}
\psi_1(x) &= \chi_1(x) - \left\langle \phi_0 \mid \chi_1 \right\rangle \phi_0(x) \\
&= x - \langle 1 \mid x \rangle \cdot 1 \\
&= x - \int_0^\infty x\left(xe^{-x}\right) dx \\
&= x - \int_0^\infty x^2 e^{-x} dx \\
&= x - 2
\end{aligned}$$

The value of $\phi_1(x)$ is,

$$\begin{aligned}
\phi_1 &= \frac{\psi_1(x)}{\|\psi_1\|} \\
&= \frac{x-2}{\langle x-2 \mid x-2\rangle^{\frac{1}{2}}} \\
&= \frac{x-2}{\left|\int_0^\infty (x-2)^2 e^{-x} dx\right|^{\frac{1}{2}}} \\
&= \frac{x-2}{\sqrt{2}}
\end{aligned}$$

The value of $\psi_2(x)$ is,

$$\begin{aligned}
\psi_2(x) &= \chi_2(x) - \left\langle \phi_0 \mid \chi_2 \right\rangle \phi_0(x) - \left\langle \phi \mid \chi_2 \right\rangle \phi_1(x) \\
&= x^2 - \left\langle 1 \mid x^2 \right\rangle \cdot 1 - \left\langle \frac{x-2}{\sqrt{2}} \mid x^2 \right\rangle \frac{x-2}{\sqrt{2}} \\
&= x^2 - \int_0^\infty x^2 \left(xe^{-x}\right) dx - \left(\int_0^\infty x^2 \left(\frac{x-2}{\sqrt{2}}\right) \left(xe^{-x}\right) dx \right) \frac{x-2}{\sqrt{2}} \\
&= x^2 - \int_0^\infty x^3 e^{-x} dx - \left(\frac{1}{\sqrt{2}} \int_0^\infty x^3 (x-2) e^{-x} dx \right) \frac{x-2}{\sqrt{2}}
\end{aligned}$$

Calculating

$$\int_0^\infty x^3 e^{-x} dx = 6, \int_0^\infty x^3 (x-2) e^{-x} dx = 12$$

$$\begin{aligned}
\psi_2(x) &= x^2 - 6 - \frac{1}{\sqrt{2}}(12)\frac{x-2}{\sqrt{2}} \\
&= x^2 - 6 - 6(x-2) \\
&= x^2 - 6 - 6x + 12 \\
&= x^2 - 6x + 6
\end{aligned}$$

So, the value of $\phi_2(x)$ is,

$$\begin{aligned}
\phi_2(x) &= \frac{\psi_2(x)}{\|\psi_2\|} \\
&= \frac{x^2-6x+6}{\langle x^2-6x+6 \mid x^2-6x+6\rangle^{\frac{1}{2}}} \\
&= \frac{x^2-6x+6}{\left|\int_0^\infty \left(x^2-6x+6\right)^2 xe^{-x} dx\right|^{\frac{1}{2}}} \\
&= \frac{x^2-6x+6}{(12)^{\frac{1}{2}}} \\
&= \frac{x^2-6x+6}{(4\times 3)^{\frac{1}{2}}} \\
&= \frac{x^2-6x+6}{2\sqrt{3}}
\end{aligned}$$

Therefore, $\phi_0(x) = 1$, $\phi_1(x) = \dfrac{x-2}{\sqrt{2}}$, and $\phi_3(x) = \dfrac{x^2-6x+6}{2\sqrt{3}}$

Problem 5.2.4

Using the Gram-Schmidt orthogonalization procedure, construct the lowest three Hermite polynomials:

$$u_n(x) = x^n, \quad n = 0, 1, 2, \ldots, \quad -\infty < x < \infty, \quad w(x) = e^{-x^2}$$

For this set of polynomials the usual normalization is

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) w(x) dx = \delta_{mn} 2^m m! \pi^{1/2}$$

$$ANS. \quad H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 - 2$$

Solution The Hermite polynomials are orthogonal on the interval $(-\infty, \infty)$ with respect to the normal distribution $w(x) = e^{-x^2}$. Then,

$$H_n(x) = \frac{(-1)^n}{w(x)} D^n w(x), \quad n = 0, 1, 2, \ldots$$

Plug $w(x) = e^{-x^2}$, $n = 0$ in $H_n(x) = \dfrac{(-1)^n}{w(x)} D^n w(x)$

$$\begin{aligned} H_0(x) &= \frac{(-1)^0}{e^{-x^2}} \frac{d^0}{dx^0} \left(e^{-x^2}\right) \\ &= e^{x^2} \\ H_0(0) &= e^0 \\ &= 1 \end{aligned}$$

Plug $w(x) = e^{-x^2}$, $n = 1, 2$ in $H_n(x) = \dfrac{(-1)^n}{w(x)} \dfrac{d^n}{dx^n} w(x)$

$$\begin{aligned} H_1(x) &= \frac{(-1)^1}{e^{-x^2}} \frac{d}{dx} \left(e^{-x^2}\right) \\ &= -e^{x^2} \left(e^{-x^2}\right) \frac{d}{dx} \left(-x^2\right) \\ &= -1(-2x) \\ &= 2x \end{aligned}$$

and

$$\begin{aligned} H_2(x) &= \frac{(-1)^2}{e^{-x^2}} \frac{d^2}{dx^2} \left(e^{-x^2}\right) \\ &= e^{x^2} D \left(-2x e^{-x^2}\right) \\ &= e^{x^2} \left(-2e^{-x^2} + 4x^2 e^{-x^2}\right) \\ &= e^{x^2} \left(e^{-x^2}\right) \left(-2 + 4x^2\right) \qquad = 4x^2 - 2 \end{aligned}$$

Therefore

$$H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 - 2$$

Problem 5.2.5

Use the Gram-Schmidt orthogonalization scheme to construct the first three Chebyshev polynomials (type I):

$$u_n(x) = x^n, \quad n = 0, 1, 2, \ldots, \quad -1 \le x \le 1, \quad w(x) = \left(1 - x^2\right)^{-1/2}$$

Take the normalization

$$\int_{-1}^{1} T_m(x) T_n(x) w(x) dx = \delta_{mn} \begin{cases} \pi, & m = n = 0 \\ \frac{\pi}{2}, & m = n \ge 1 \end{cases}$$

Hint. The needed integrals are given in Exercise 13.3 .2

$$ANS. \quad T_0 = 1, \quad T_1 = x, \quad T_2 = 2x^2 - 1, \quad \left(T_3 = 4x^3 - 3x\right)$$

Solution Here, $u_n(x) = x^n$, $n = 0, 1, 2, \ldots$, $-1 \le x \le 1$, $w(x) = \left(1 - x^2\right)^{\frac{-1}{2}}$. The normalization integral is,

$$\int_{-1}^{1} T_m(x) T_n(x) w(x) dx = \delta_{mn} \begin{cases} \pi, m = n = 0 \\ \frac{\pi}{2}, m = n \ge 1 \end{cases}.$$

The value of $\langle x_0 \mid x_0 \rangle$ is,

$$\begin{aligned}
\left\langle x^0 \mid x^0 \right\rangle &= \int_{-1}^{1} \left(1 - x^2\right)^{\frac{-1}{2}} dx \\
&= 2 \int_0^1 \frac{1}{\sqrt{1 - x^2}} dx \\
&= 2 \left(\sin^{-1}(x)\right)_0^1 \\
&= 2 \left(\sin^{-1}(1) - \sin^{-1}(0)\right) \\
&= 2 \left(\sin^{-1}\left(\sin \frac{\pi}{2}\right) - \sin^{-1}(\sin 0)\right) \\
&= 2 \left(\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

The value of $\left\langle x^1 \mid x^1 \right\rangle$ is calculated as shown below:

$$\begin{aligned}
\left\langle x^1 \mid x^1 \right\rangle &= \left\langle x^0 \mid x^2 \right\rangle \\
&= \int_{-1}^{1} x^2 \left(1 - x^2\right)^{\frac{-1}{2}} dx \\
&= \int_{-1}^{1} \frac{x^2}{\sqrt{1 - x^2}} dx \\
&= 2 \int_0^1 \frac{x^2}{\sqrt{1 - x^2}} dx
\end{aligned}$$

Using

$$\int \frac{x^2}{\sqrt{1 - x^2}} dx = \frac{-1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1}(x)$$

$$\begin{aligned}
\left\langle x^1 \mid x^1 \right\rangle &= 2\left(\frac{-1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right)_0^1 \\
&= \int_{-1}^{1} x^2\left(1-x^2\right)^{\frac{-1}{2}}\,dx \\
&= 2\left[\left(\frac{-1}{2}(1)\sqrt{1-(1)^2}+\frac{1}{2}\sin^{-1}(1)\right)-\left(\frac{-1}{2}(0)\sqrt{1-(0)^2}+\frac{1}{2}\sin^{-1}(0)\right)\right] \\
&= 2\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right) \\
&= 2\left(\frac{1}{2}\left(\frac{\pi}{2}\right)\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

The value of $\left\langle x^2 \mid x^2 \right\rangle$ is calculated as shown below:

$$\begin{aligned}
\left\langle x^2 \mid x^2 \right\rangle &= \int_{-1}^{1} x^4\left(1-x^2\right)^{\frac{-1}{2}}\,dx \\
&= \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}dx \\
&= 2\int_{0}^{1} \frac{x^4}{\sqrt{1-x^2}}dx \\
&= 2\left(\frac{-1}{4}x^3\sqrt{1-x^2}-\frac{3}{8}x\sqrt{1-x^2}+\frac{3}{8}\sin^{-1}(x)\right)_0^1 \\
&= 2\left[\left(\frac{-1}{4}(1)^3\sqrt{1-(1)^2}-\frac{3}{8}(1)\sqrt{1-(1)^2}+\frac{3}{8}\sin^{-1}(1)\right)-\left(\frac{-1}{4}(0)^3\sqrt{1-(0)^2}-\frac{3}{8}(0)\sqrt{1-(0)^2}+\frac{3}{8}\sin^{-1}(0)\right)\right] \\
&= 2\left(\frac{3}{8}\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right) \\
&= 2\left(\frac{3}{8}\left(\frac{\pi}{2}\right)\right) \\
&= \frac{3\pi}{8}
\end{aligned}$$

The value of $\left\langle x^0 \mid x^1 \right\rangle$ and $\left\langle x^2 \mid x^1 \right\rangle$ are calculated as shown below:

$$\begin{aligned}
\left\langle x^0 \mid x' \right\rangle &= \left\langle x^2 \mid x^1 \right\rangle \\
&= \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}dx \\
&= \left(-\sqrt{1-x^2}\right)_{-1}^{1} \\
&= 0
\end{aligned}$$

The polynomial T_0 is of the form c_0x^0, with c_0 satisfying

$$\begin{aligned}
\left\langle c_0x^0 \mid c_1x^0 \right\rangle &= |c_0|^2\left\langle x^0 \mid x^0 \right\rangle \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx \\
&= 2\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}dx \\
&= 2\left(\sin^{-1}x\right)_0^1 \\
&= 2\left(\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right) \\
&= 2\left(\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

So, $c_0 = 1$ and $T_0 = 1$. By symmetry, the polynomial T_1 is a linear combination of x^0 and x^1, this is an odd function that depends only on x^1. So, this is in the form $c_1 x$. It is orthogonal to T_0 and c_1 and satisfies,

$$\left\langle c_1 x^1 \mid c_1 x^1 \right\rangle = |c_1|^2 \left\langle x^1 \mid x^1 \right\rangle$$

Using $c_1 = 1$

$$\begin{aligned}
\left\langle c_1 x^1 \mid c_1 x^1 \right\rangle &= (1)^2 \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx \\
&= 2\left(\frac{-1}{2} x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right)_0^1 \\
&= 2\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right) \\
&= 2\left(\frac{1}{2}\left(\frac{\pi}{2}\right)\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

since $\langle x \mid x \rangle = \frac{\pi}{2}$, so we have $c_1 = 1$ and $T_1 = c_1 x = x$. The constant c_2 is determined from the normalization condition:

$$\begin{aligned}
\langle T_2 \mid T_2 \rangle &= |c_2|^2 \left\langle x^2 - \frac{1}{2} \mid x^2 - \frac{1}{2} \right\rangle \\
&= |c_2|^2 \int_{-1}^{1} \frac{\left(x^2 - \frac{1}{2}\right)^2}{\sqrt{1-x^2}} dx \\
&= 2\,|c_2|^2 \int_{0}^{1} \frac{\left(x^2 - \frac{1}{2}\right)^2}{\sqrt{1-x^2}} dx \\
&= 2\,|c_2|^2 \left(\frac{1}{8}\sin^{-1}(x) - \frac{1}{4}x^3\sqrt{1-x^2} + \frac{1}{8}x\sqrt{1-x^2}\right)_0^1 \\
&= 2|c_2|^2 \left[\left(\frac{1}{8}\sin^{-1}(1) - \frac{1}{4}(1)^3\sqrt{1-(1)^2} + \frac{1}{8}(1)\sqrt{1-(1)^2}\right) - \left(\frac{1}{8}\sin^{-1}(0) - \frac{1}{4}(0)^3\sqrt{1-(0)^2} + \frac{1}{8}(0)\sqrt{1-(0)^2}\right)\right] \\
&= 2\,|c_2|^2 \left(\frac{\pi}{16}\right) \\
&= |c_2|^2 \left(\frac{\pi}{8}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

From the above equation, the value of c_2 is $c_2 = 2$. Since T_2 is an even function, its general form is,

$$\begin{aligned}
T_2 &= c_2 \left[x^2 - \frac{\left\langle T_0 \mid x^2 \right\rangle}{\langle T_0 \mid T_0 \rangle} T_0\right] \\
&= 2\left[x^2 - \frac{\frac{\pi}{2}}{\pi} \cdot 1\right] \\
&= 2\left(x^2 - \frac{1}{2}\right) \\
&= 2x^2 - 1
\end{aligned}$$

Therefore, the first three Chebyshev polynomials are, $T_0 = 1$, $T_1 = x$ and $T_2 = 2x^2 - 1$

Chapter 13
Gamma Function

Chapter 13.1: Definitions and properties of Gamma Function

Problem 13.1.1

Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral, Eq. (13.5),

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re e(z) > 0, \quad \text{Eq.}(13.5)$$

Solution Consider the Euler integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Put, $z = z + 1$

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty e^{-t} t^{z+1-1} dt \\
&= \int_0^\infty e^{-t} t^{z} dt \\
&= t^z \int_0^\infty e^{-t} dt - \int_0^\infty \frac{dt^z}{dx} \int e^{-t} dt \\
&= -t^z e^{-t}\Big|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} dt \\
&= z\Gamma(z)
\end{aligned}$$

Problem 13.1.2

In a power-series solution for the Legendre functions of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

in which s is a positive integer.

(a) Rewrite this expression in terms of factorials.

(b) Rewrite this expression using Pochhammer symbols; see Eq. (1.72).

$$(a)_0 = 1, \quad (a)_1 = a, (a)_{n+1} = a(a+1)\cdots(a+n), (n \geq 1), \quad \text{Eq.}(1.72)$$

Solution For (a) Notice that

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2.4.6.8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$\begin{aligned}
&= \frac{[n!\ (n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)]}{n!\ s!\ 2^s\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)} \\
&= \frac{(n+2s)!(2n+1)!}{n!\ s!\ 2^s\cdot[(2n+1)!(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)} \\
&= \frac{(n+2s)!(2n+1)![(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{n!\ s!\ 2^s\cdot[(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]} \\
&= \frac{(n+2s)!(2n+1)!2^s[(n+1)(n+2)(n+3)\cdots(n+s)]}{n!\ s!\ 2^s\cdot[(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]} \\
&= \frac{(n+2s)!(2n+1)![n!\ (n+1)(n+2)(n+3)\cdots(n+s)]}{n!\ s!\ n!\ [(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]} \\
&= \frac{(n+2s)!(2n+1)!(n+s)!}{n!\ n!\ s!\ (2n+2s+1)!}
\end{aligned}$$

$\boxed{\textbf{Solution}}$ For (b) we notice that

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot4\cdot6\cdot8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$\begin{aligned}
&= \frac{(n+1)(n+2)(n+3)\cdots[(n+1)+(2s-2)][(n+1)+(2s-1)]}{(2^s[1\cdot2\cdot3\cdot\cdots\cdot(s-1)s])\cdot[(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)]}\\
&= \frac{(n+1)_{(2s-1)+1}\cdot[(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{(2^s[1\cdot2\cdot3\cdot\cdots\cdot\{1+(s-2)\}\{1+(s-1)\}])\cdot[(2n+2)(2n+3)(2n+4)(2n+5)\cdots(2n+2s)(2n+2s+1)]}\\
&= \frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots(n+s)]\cdot2^s}{2^s(1)_{(s-1)+1}\cdot[(2n+2)(2n+3)(2n+4)\cdots\{(2n+2)+(2s-1)\}]}\\
&= \frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots\{(n+1)+(s-1)\}]}{(1)_s\cdot(2n+2)_{(2s-1)+1}}\\
&= \frac{(n+1)_{2s}\cdot(n+1)_{(s-1)+1}}{(1)_s\cdot(2n+2)_{2s}}\\
&= \frac{(n+1)_{2s}\cdot(n+1)_s}{(1)_s\cdot(2n+2)_{2s}}
\end{aligned}$$

Problem 13.1.3

Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2\int_0^\infty e^{-t^2}t^{2z-1}dt, \quad \text{Re}(z) > 0$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re e(z) > 0$$

$\boxed{\textbf{Solution}}$ Changing variables $t = u^2$ and $dt = 2udu$ we have

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-u^2}u^{2z-2}udu\\
&= \int_0^\infty e^{-u^2}u^{2z-1}du\\
&= \int_0^\infty e^{-t^2}t^{2z-1}dt
\end{aligned}$$

as $t \longrightarrow 0$ to $\infty u \longrightarrow 0$ to 1 the equation takes the form of

$$\begin{aligned}
\Gamma(z) &= \int_0^1 e^{-\ln\frac{1}{u}}\left(\ln\frac{1}{u}\right)^{z-1} udu\\
&= \int_0^1 u\left(\ln\frac{1}{u}\right)^{z-1} udu\\
&= \int_0^1 \left(\ln\frac{1}{u}\right)^{z-1} du\\
&= \int_0^1 \left(\ln\frac{1}{t}\right)^{z-1} dt
\end{aligned}$$

Problem 13.1.4

In a Maxwellian distribution the fraction of particles of mass m with speed between v and $v + dv$ is

$$\frac{dN}{N} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^2 dv$$

where N is the total number of particles, k is Boltzmann's constant, and T is the absolute temperature. The average or expectation value of v^n is defined as $\langle v^n \rangle = N^{-1} \int v^n dN$. Show that

$$\langle v^n \rangle = \left(\frac{2kT}{m}\right)^{n/2} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

This is an extension of Example 13.1.1, in which the distribution was in kinetic energy $E = mv^2/2$, with $dE = mvdv$

Solution

$$\begin{aligned}
\langle v^n \rangle &= N^{-1} \int v^n dN \\
&= \int v^n \frac{dN}{N} \\
&= \int_0^\infty v^n \cdot 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{\frac{m^2}{2kT}} v^2 dv \\
&= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty v^n e^{\frac{m^2}{2kT}} v^{n+1} v dv
\end{aligned}$$

Let $\frac{mv^2}{2kT} = u^2$. Then $v = \left(\frac{2kT}{m}\right)^{\frac{1}{2}} u$ and $vdv = \frac{2kT}{m} udu$. As $v \to 0,\ u \to 0$ and as $v \to \infty,\ u \to \infty$. Then the above integral becomes

$$\begin{aligned}
\langle v^n \rangle &= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty e^{-u^2} u^{n+1} \left(\frac{2kT}{m}\right)^{\frac{n+1}{2}} \cdot \frac{2kT}{m} udu \\
&= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-u^2} u^{n+2} du
\end{aligned}$$

Let $u^2 = t\cdot$ Then $2udu = dt$ As $u \to 0,\ t \to 0$ and as $u \to \infty,\ t \to \infty$. As $u \to 0,\ t \to 0$ and as $u \to \infty,\ t \to \infty$.

$$\begin{aligned}
\langle v^n \rangle &= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+1}{2}} \frac{dt}{2} \\
&= 2\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+3}{2}} dt \\
&= 2\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+3}{2}} dt \\
&= \frac{2\pi}{\pi\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}-\frac{3}{2}} \Gamma\left(\frac{n+3}{2}\right) \\
&= \frac{2}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \Gamma\left(\frac{n+3}{2}\right) \\
&= \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}
\end{aligned}$$

since $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. Hence

$$\langle v^n \rangle = \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

Problem 13.1.5

By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

Solution Put $x = e^t$. Then $t = \ln x$ and $dx = e'dt$. As $x \to 0,\ t \to \infty$ and as $x \to 1, t \to 0$.

$$-\int_0^1 x^k \ln x dx$$

$$= -\int_\infty^0 e^{kt} t e' dt$$

$$= \int_0^\infty e^{(k+1)t} t dt$$

Now put $-(k+1)t = z$. Then

$$dt = -\frac{dz}{(k+1)}$$

As $t \to 0,\ z \to 0$ and as $t \to \infty,\ z \to 0$. Then

$$-\int_0^1 x^k \ln x dx$$

$$= \int_0^\infty e^{(k+1)t} t dt$$

$$= \int_0^\infty e^{-z} \left(\frac{z}{-(k+1)}\right)\left(\frac{dz}{-(k+1)}\right)$$

$$= \frac{1}{(k+1)^2} \int_0^\infty z e^{-z} dz$$

$$= \frac{1}{(k+1)^2} \int_0^\infty z^{2-1} e^{-z} dz$$

$$= \frac{1}{(k+1)^2} \Gamma(2)$$

$$= \frac{1}{(k+1)^2} \cdot 1!$$

$$= \frac{1}{(k+1)^2}$$

Hence

$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

Problem 13.1.6

Show that
$$\int_0^\infty e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$$

Solution Consider $x^4 = t$ and put $4x^3 dx = dt$ as $t \to 0$ to ∞, $x \to 0$ to ∞ and using
$$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z)$$
and
$$z\Gamma(z) = \Gamma(z+1)$$
the integral takes the form of
$$\begin{aligned} \frac{1}{4}\int_0^\infty e^{-t} t^{-3/4} dt &= \frac{1}{4}\int_0^\infty e^{-t} t^{1/4-1} dt \\ &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \\ &= \Gamma\left(\frac{5}{4}\right) \end{aligned}$$

Problem 13.1.7

Show that
$$\lim_{x\to 0} \frac{\Gamma(ax)}{\Gamma(x)} = \frac{1}{a}$$

Solution
$$\begin{aligned} \lim_{x\to 0} \frac{\Gamma(ax)}{\Gamma(x)} &= \lim_{x\to 0} \frac{\left(\frac{ax\Gamma(ax)}{ax}\right)}{\left(\frac{x\Gamma(x)}{x}\right)} \\ &= \lim_{x\to 0} \left(\frac{\Gamma(ax+1)}{\Gamma(x+1)} \cdot \frac{x}{ax}\right) \\ &= \frac{1}{a} \lim_{x\to 0} \frac{\Gamma(ax+1)}{\Gamma(x+1)} \\ &= \frac{1}{a} \frac{\Gamma(1)}{\Gamma(1)} \\ &= \frac{1}{a} \end{aligned}$$

Problem 13.1.8

Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

Solution Recall that

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} \cdot n^2,$$

where $z \neq 0, -1, -2, -3, \cdots$. The denominator shows that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \cdots$

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1} dt \\
&= \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-t)^n}{n!} dt + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{n+z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \left[\frac{t^{n+z}}{n+z}\right]_0^1 + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \left[\frac{1}{n+z} - 0\right] + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)} + \int_1^\infty e^{-t} t^{z-1} dt
\end{aligned}$$

The series

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)}$$

shows that the first order poles at all negative integers $z = -n$ has respective residues

$$\frac{(-1)^n}{n!}$$

Problem 13.1.10

Show that, for integer s

(a)

$$\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx = \frac{s!}{2a^{s+1}}$$

(b)

$$\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx = \frac{\Gamma\left(s+\frac{1}{2}\right)}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

$\boxed{\textbf{Solution}}$ For (a) Put $ax^2 = z\cdot$ Then $2axdx = dz$. This implies

$$dx = \frac{dz}{2\sqrt{az}}$$

As $x \longrightarrow 0,\ z \longrightarrow 0$ and as $x \longrightarrow \infty,\ z \longrightarrow \infty$. The given integral is

$$\begin{aligned}
\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx &= \int_0^\infty \left(\sqrt{\frac{z}{a}}\right)^{2s+1} e^{-z}\frac{dz}{2\sqrt{az}} \\
&= \frac{1}{2\sqrt{a}}\int_0^\infty \left(\frac{z}{a}\right)^{\frac{2s+1}{2}} e^{-z}z^{-\frac{1}{2}}dz \\
&= \frac{1}{2a^{\frac{1}{2}}}\cdot\frac{1}{a^{\frac{2s+1}{2}}}\int_0^\infty e^{-z}z^{\frac{2s+1}{2}-\frac{1}{2}}dz \\
&= \frac{1}{2a^{s+1}}\int_0^\infty e^{-z}z^s dz \\
&= \frac{1}{2a^{s+1}}\int_0^\infty e^{-z}z^{(s+1)-1}dz \\
&= \frac{1}{2a^{s+1}}\Gamma(s+1)
\end{aligned}$$

since s is an integer, therefore $\Gamma(s+1) = s!$. Hence

$$\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx = \frac{s!}{2a^{s+1}}$$

$\boxed{\textbf{Solution}}$ For (b) Put $ax^2 = z\cdot$ Then $2axdx = dz$. This implies

$$dx = \frac{dz}{2\sqrt{az}}$$

As $x \longrightarrow 0,\ z \longrightarrow 0$ and as $x \longrightarrow \infty,\ z \longrightarrow \infty$. The given integral is

$$\begin{aligned}
\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx &= \int_0^\infty \left(\sqrt{\frac{z}{a}}\right)^{2s} e^{-z}\frac{dz}{2\sqrt{az}} \\
&= \frac{1}{2\sqrt{a}}\int_0^\infty \left(\frac{z}{a}\right)^{s} e^{-z}z^{-\frac{1}{2}}dz \\
&= \frac{1}{2a^{\frac{1}{2}}}\cdot\frac{1}{a^s}\int_0^\infty e^{-z}z^{s-\frac{1}{2}}dz \\
&= \frac{1}{2a^{s+\frac{1}{2}}}\int_0^\infty e^{-z}z^{\left(s+\frac{3}{2}\right)-1}dz \\
&= \frac{1}{2a^{s+\frac{1}{2}}}\Gamma\left(s+\frac{3}{2}\right)
\end{aligned}$$

since

$$\Gamma\left(s+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^s}\cdot(2s-1)!!$$

$$= \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Thus

$$\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx = \frac{\Gamma\left(s+\frac{1}{2}\right)}{2a^{s+\frac{1}{2}}} = \frac{(2s-1)!!}{2a^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Problem 13.1.11

Express the coefficient of the n th term of the expansion of $(1+x)^{1/2}$ in powers of x

(a) in terms of factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS.\ a_n = (-1)^{n+1}\frac{(2n-3)!}{2^{2n-2}n!\,(n-2)!} = (-1)^{n+1}\frac{(2n-3)!!}{(2n)!!}, \quad n = 2, 3, \cdots$$

Solution For (a) the n th term of the expansion of $(1+x)^{1/2}$ in powers of x is:

$$\begin{aligned}
a_n &= \binom{1/2}{1} \\
&= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} \\
&= \frac{(-1)^{n-1}}{n!\,2^n}[1.3.5\cdots(2n-3)] \\
&= \frac{(-1)^{n+1}}{n!\,2^n}\left[\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdots(2n-4)\cdot(2n-3)}{2\cdot4\cdot6\cdot\cdots(2n-4)}\right] \\
&= \frac{(-1)^n}{n!\,2^n}\cdot\frac{(2n-3)!}{(n-2)!2^{n-2}} \\
&= (-1)^{n+1}\cdot\frac{(2n-3)!}{2^{2n-2}\cdot n!\,(n-2)!}
\end{aligned}$$

Therefore,

$$a_n = (-1)^{n+1}\cdot\frac{(2n-3)!}{2^{2n-2}n!\,(n-2)!}, \quad n = 1, 2, 3, \cdots$$

Solution For (b) the n th term expansion of $(1+x)^{1/2}$

$$\begin{aligned}
a_n &= \binom{-1/2}{1} \\
&= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} \\
&= \frac{(-1)^{n-1}}{n!\,2^n}[1\cdot3\cdot5\cdot\cdots(2n-3)] \\
&= (-1)^{n+1}\cdot\left[\frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdot\cdots2n}\right] \\
&= (-1)^{n+1}\cdot\frac{(2n-3)!!}{(2n)!!}
\end{aligned}$$

Therefore

$$a_n = (-1)^{n+1}\cdot\frac{(2n-3)!!}{(2n)!!}, \quad \text{for } n = 1, 2, 3, \cdots$$

Problem 13.1.12

Express the coefficient of the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x

(a) in terms of the factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. \quad a_n = (-1)^n \frac{(2n)!}{2^{2n}(n!\,)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad n = 1,2,3\cdots$$

Solution For (a) the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x is:

$$\begin{aligned}
a_n &= \binom{-1/2}{n-1} \\
&= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} \\
&= \frac{(-1)^n}{n!\;2^n}[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)] \\
&= \frac{(-1)^n}{n!\;2^n}\left[\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(2n-1)\cdot 2n}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right] \\
&= \frac{(-1)^n}{n!\;2^n}\cdot\frac{(2n)!}{n!\;2^n} \\
&= (-1)^n\cdot\frac{(2n)!}{2^{2n}\cdot(n!\,)^2}
\end{aligned}$$

Therefore,

$$a_n = (-1)^n\cdot\frac{(2n)!}{2^{2n}\cdot(n!\,)^2}, \qquad \text{for } n = 1,2,3,\cdots$$

Solution For (b) the n th term expansion of $(1+x)^{-1/2}$ in powers of x in terms of the double factorial (!!) functions.

$$\begin{aligned}
a_n &= \binom{-1/2}{n-1} \\
&= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} \\
&= \frac{(-1)^n}{n!\;2^n}[1\cdot 3\cdot 5\cdots(2n-1)] \\
&= (-1)^n\cdot\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right] \\
&= (-1)^n\cdot\frac{(2n-1)!!}{(2n)!!}
\end{aligned}$$

Therefore

$$a_n = (-1)^n\cdot\frac{(2n-1)!!}{(2n)!!}, \qquad \text{for } n = 1,2,3,\cdots$$

Problem 13.1.14

(*a*) Show that $\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$, where n is an integer.

(*b*) Express $\Gamma\left(\frac{1}{2}+n\right)$ and $\Gamma\left(\frac{1}{2}-n\right)$ separately in terms of $\pi^{1/2}$ and a double factorial function.

$$ANS. \quad \Gamma\left(\frac{1}{2}+n\right)=\frac{(2n-1)!!}{2^n}\pi^{1/2}$$

Solution For (*a*) recall that

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin \pi z}$$

Putting $z=\frac{1}{2}+n$ in the above relation, it becomes

$$\begin{aligned}\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(1-\frac{1}{2}-n\right)&=\frac{\pi}{\sin\left[\pi\left(\frac{1}{2}+n\right)\right]}\\&=\frac{\pi}{\cos(n\pi)}\\&=\frac{\pi}{(-1)^n}\end{aligned}$$

since $\cos(n\pi)=(-1)^n$ and

$$=(-1)^n\pi$$

Therefore

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$$

where n is an integer.

Solution For (*b*) recall the Legendre's duplication formula,

$$\Gamma(1+z)\Gamma\left(z+\frac{1}{2}\right)=2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

Putting $z=n$ in the above relation, it becomes

$$\begin{aligned}\Gamma(1+n)\Gamma\left(n+\frac{1}{2}\right)&=2^{-2n}\sqrt{\pi}\Gamma(2n+1)\\&=\frac{2^{-2n}\sqrt{\pi}\Gamma(2n+1)}{\Gamma(1+n)}\\&=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{(2n)!}{n!}\\&=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot 2n)}{(1\cdot 2\cdot 3\cdots n)}\\&=\frac{\sqrt{\pi}}{2^{n}}\cdot\frac{(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot 2n)}{(2\cdot 4\cdot 6\cdot\cdots\cdot 2n)}\\&=\frac{\sqrt{\pi}}{2^{n}}\cdot[1\cdot 3\cdot 5\cdots(2n-1)]\\&=\frac{\sqrt{\pi}}{2^{n}}\cdot(2n-1)!!\cdots\end{aligned}$$

From part (*a*)

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$$

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}-n\right) &= \frac{(-1)^n\pi}{\Gamma\left(\frac{1}{2}+n\right)} \\
&= \frac{(-1)^n\pi}{\left(\frac{\sqrt{\pi}}{2^n}\cdot(2n-1)!!\right)} \\
&= \frac{(-1)^n\cdot 2^n\sqrt{\pi}}{(2n-1)!!} \\
&= \frac{\sqrt{\pi}}{2^n}\cdot(2n-1)!! \quad \text{and} \quad \Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n\cdot 2^n\sqrt{\pi}}{(2n-1)!!}
\end{aligned}
$$

Problem 13.1.6

Prove that
$$|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)| \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-1/2}$$

Solution Recall
$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$
Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes
$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{\alpha+i\beta}{n}}$$
and
$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}$$
Multiplying these equations it becomes
$$\begin{aligned}
\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)} &= (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(\alpha-i\beta)} \\
&\quad \times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{\alpha+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right] \\
\frac{1}{|\Gamma(\alpha+i\beta)|^2} &= \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2\alpha}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right] \\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{2\alpha}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{2\alpha}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{\alpha}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right] \\
&= \left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]
\end{aligned}$$
Hence
$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$
$$\frac{1}{|\Gamma(\alpha+i\beta)|} = \frac{1}{|\Gamma(\alpha)|}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]^{\frac{1}{2}}$$

$$|\Gamma(\alpha + i\beta)| = |\Gamma(\alpha)| \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(\alpha + n)^2}\right]^{-\frac{1}{2}}$$

Problem 13.1.17

Show that for n, a positive integer,

$$|\Gamma(n+ib+1)| = \left(\frac{\pi b}{\sinh \pi b}\right)^{1/2} \prod_{s=1}^{n} \left(s^2+b^2\right)^{1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)} \prod_{n=1}^{\infty} \left(1+\frac{\alpha+i\beta}{n}\right) e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)} \prod_{n=1}^{\infty} \left(1+\frac{\alpha-i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+i\beta)} \cdot \frac{1}{\Gamma(\alpha-i\beta)} &= (\alpha+i\beta)e^{\gamma(\alpha+i\beta)} \cdot (\alpha-i\beta)e^{\gamma(\alpha-i\beta)} \\
&\quad \times \prod_{n=1}^{\infty} \left[\left(1+\frac{\alpha+i\beta}{n}\right) e^{\frac{\alpha+i\beta}{n}} \cdot \left(1+\frac{\alpha-i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}\right] \\
\frac{1}{|\Gamma(\alpha+i\beta)|^2} &= \left(\alpha^2+\beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2\alpha}{n}} \left[\left(1+\frac{\alpha+i\beta}{n}\right) \cdot \left(1+\frac{\alpha-i\beta}{n}\right)\right] \\
&= \left(\alpha^2+\beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right) \cdot \left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2} \cdot \left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\alpha^2+\beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{2\alpha}{n}} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right) \cdot \left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2} \cdot \left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{\alpha}{n}} \cdot \left(1+\frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right] \\
&= \left(1+\frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1+\frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1+\frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2}\right]
\end{aligned}$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha = 1$ and $\beta = b$ in the above identity. Then it becomes

$$
\begin{aligned}
\frac{1}{|\Gamma(1+ib)|^2} &= \frac{1}{\Gamma(1)^2}\prod_{n=0}^{\infty}\left[1+\frac{b^2}{(n+1)^2}\right] \\
&= \prod_{n=0}^{\infty}\left[1+\frac{b^2}{(n+1)^2}\right], \quad \text{as} \quad \Gamma(1)=1 \\
&= \prod_{n=0}^{\infty}\left[1-\frac{(ib\pi)^2}{(n+1)^2\pi^2}\right] \\
&= \prod_{n=1}^{\infty}\left[1-\frac{(ib\pi)^2}{n^2\pi^2}\right] \\
&= \frac{1}{(ib\pi)}\left\{(ib\pi)\prod_{n=1}^{\infty}\left[1-\frac{(ib\pi)^2}{n^2\pi^2}\right]\right\} \\
&= \frac{1}{ib\pi}\cdot\sin(ib\pi)
\end{aligned}
$$

Using the identy

$$
\sin z = z\prod_{n=1}^{\infty}\left[1-\frac{z^2}{n^2\pi^2}\right] \quad \text{for} \quad z = ib\pi
$$

$$
\begin{aligned}
&= \frac{1}{ib\pi}\cdot i\sinh(b\pi) \\
&= \frac{\sinh(b\pi)}{b\pi} \\
\frac{1}{|\Gamma(1+ib)|^2} &= \frac{\sinh(b\pi)}{b\pi} \\
|\Gamma(1+ib)|^2 &= \frac{b\pi}{\sinh(b\pi)}.
\end{aligned}
$$

since n is an integer, therefore

$$
\begin{aligned}
\Gamma(n+ib+1) &= \Gamma(\{1+ib+(n-1)\}+1) \\
&= \{1+ib+(n-1)\}\Gamma(\{1+ib+(n-1)\}) \\
&\quad (1+ib)(2+ib)(3+ib)\cdots(n+ib)\Gamma(1+ib) \\
\Gamma(n+ib+1) &= (1+ib)(2+ib)(3+ib)\cdots(n+ib)\Gamma(1+ib) \\
\Gamma(n-ib+1) &= (1-ib)(2-ib)(3-ib)\cdots(n-ib)\Gamma(1-ib)
\end{aligned}
$$

$$
\begin{aligned}
|\Gamma(n+ib+1)|^2 &= \Gamma(n+ib+1)\Gamma(n-ib+1) \\
&= (1+ib)(2+ib)(3+ib)\cdots(n+ib)\Gamma(1+ib)\times(1-ib)(2-ib)(3-ib)\cdots(n-ib)\Gamma(1-ib) \\
&= \{(1+ib)(1-ib)\}\{(2+ib)(2-ib)\}\{(3+ib)(3-ib)\}\cdots\{(n+ib)(n-ib)\}\Gamma(1+ib)\Gamma(1-ib) \\
&= \left(1^2+b^2\right)\left(2^2+b^2\right)\left(3^2+b^2\right)\cdots\left(n^2+b^2\right)|\Gamma(1+ib)|^2 \\
&= \prod_{s=1}^{n}\left(s^2+b^2\right)\times\frac{b\pi}{\sinh(b\pi)}
\end{aligned}
$$

Hence

$$
|\Gamma(n+ib+1)|^2 = \prod_{s=1}^{n}\left(s^2+b^2\right)\times\frac{b\pi}{\sinh(b\pi)}
$$

This gives

$$
|\Gamma(n+ib+1)| = \left(\frac{b\pi}{\sinh(b\pi)}\right)^{\frac{1}{2}}\prod_{s=1}^{n}\left(s^2+b^2\right)^{\frac{1}{2}}
$$

Problem 13.1.18

Show that for all real values of x and y, $|\Gamma(x)| \geq |\Gamma(x+iy)|$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)} &= (\alpha+i\beta)e^{\gamma(a+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(a-i\beta)}\\
&\times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{a+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right]\\
\frac{1}{|\Gamma(\alpha+i\beta)|^2} &= \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2a}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right]\\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]\\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]\\
&= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{a}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]\\
&= \left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right]\\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right]\\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]\\
&= \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]
\end{aligned}$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha=x$ and $\beta=y$ in the above identity. Then it becomes

$$\frac{1}{|\Gamma(x+iy)|^2} = \frac{1}{\Gamma(x)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+x)^2}\right]$$

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right|^2 = \prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+x)^2}\right]$$

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right|^2 \geq 1, \quad \text{since} \quad 1+\frac{\beta^2}{(n+x)^2} \geq 1$$

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right| \geq 1$$

$$|\Gamma(x)| \geq |\Gamma(x+iy)|$$

Hence is proved

Problem 13.1.19

Show that
$$\left|\Gamma(\frac{1}{2}+iy)\right|^2 = \frac{\pi}{\cosh \pi y}$$

Solution Recall
$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$
Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes
$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{\alpha+i\beta}{n}}$$
and
$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}$$
Multiplying these equations it becomes
$$\begin{aligned}
\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)} &= (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(\alpha-i\beta)} \\
&\quad\times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{\alpha+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right] \\
\frac{1}{|\Gamma(\alpha+i\beta)|^2} &= \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2\alpha}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right] \\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{a}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right] \\
&= \left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]
\end{aligned}$$
Hence
$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$
Now put $\alpha=\frac{1}{2}$ and $\beta=y$ in the above identity. Then it becomes
$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\Gamma\left(\frac{1}{2}\right)^2}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

Recall

$$\cos z = \prod_{n=1}^{\infty}\left[1-\frac{z^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right]$$

and putting $z = i\pi y$ it becomes

$$\cos(i\pi y) = \prod_{n=1}^{\infty}\left[1-\frac{i^2\pi^2y^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right]$$

$$\cosh(\pi y) = \prod_{n=1}^{\infty}\left[1+\frac{y^2}{\left(n-\frac{1}{2}\right)^2}\right]$$

$$\cosh(\pi y) = \prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+1-\frac{1}{2}\right)^2}\right]$$

$$\cosh(\pi y) = \prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\cosh(\pi y)$$

Problem 13.1.20

The probability density associated with the normal distribution of statistics is given by

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

with $(-\infty, \infty)$ for the range of x. Show that (a)

(a) $\langle x \rangle$, the mean value of x, is equal to μ

(b) the standard deviation $\left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2}$ is given by σ.

Solution For (a) For the mean

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} dx
\end{aligned}$$

Put $x - \mu = y$. Then $dx = dy$. As $x \longrightarrow 0,\ y \longrightarrow 0$ and $x \longrightarrow \infty,\ y \longrightarrow \infty$.

$$\begin{aligned}
\langle x \rangle &= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\mu + y) e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\mu + y) e^{\frac{y^2}{2\sigma^2}} dy \\
&= \frac{\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy
\end{aligned}$$

Since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{2\sigma^2}} dy = 2\int_{0}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Since $y e^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^{\infty} y e^{\frac{y^2}{2\sigma^2}} dy = 0$$

Therefore, the integral becomes

$$\langle x \rangle = \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{0}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y} dz$, that is,

$$dy = \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$$

As $y \longrightarrow 0,\ z \longrightarrow 0$ and $y \longrightarrow \infty,\ z \longrightarrow \infty$. Therefore

$$
\begin{aligned}
\langle x \rangle &= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy \\
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz \\
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz \\
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right) \\
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \sqrt{\pi} \\
&= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \sqrt{\pi} \\
&= \mu
\end{aligned}
$$

$\boxed{\textbf{Solution}}$ For (b) we start saying

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= \int_0^\infty x^2 f(x) dx \\
&= \int_{-\infty}^\infty x^2 \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{\frac{(x-\mu)^2}{2\sigma^2}} dx
\end{aligned}
$$

Put $x - \mu = y$. Then $dx = dy$. As $x \longrightarrow 0,\ y \longrightarrow 0$ and $x \longrightarrow \infty,\ y \longrightarrow \infty$.

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty (\mu + y)^2 e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty \left(\mu^2 + 2\mu y + y^2\right) e^{\frac{y^2}{2\sigma^2}} dy \\
&= \frac{\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy + \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy
\end{aligned}
$$

Since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$
\int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy = 2 \int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy
$$

Since $y e^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$
\int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy = 0
$$

Since $y e^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$
\int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy = 2 \int_0^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy
$$

Therefore the above integral becomes

$$
\left\langle x^2 \right\rangle = \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{\frac{y^2}{2\sigma^2}} dy
$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y}dz$, that is, $dy = \frac{\sigma}{\sqrt{2}}z^{-\frac{1}{2}}dz$ As $y \to 0,\ z \to 0$ and $y \to \infty,\ z \to \infty$. Therefore

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{\frac{y^2}{2\sigma^2}} dy \\
&= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{\frac{1}{2}} dz + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty 2\sigma^2 z e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz \\
&= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz + \frac{2\sqrt{2}\sigma^3}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz \\
&= \frac{\mu^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{3}{2}-1} dz \\
&= \frac{\mu^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{\mu^2}{\sqrt{\pi}} \cdot \sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\pi} \\
&= \mu^2 + \sigma^2
\end{aligned}
$$

So the standard deviation

$$\left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{\frac{1}{2}} = \left(\mu^2 + \sigma^2 - \mu^2\right)^{\frac{1}{2}}$$

$$\left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{\frac{1}{2}} = \sqrt{\sigma^2}$$

$$\left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{\frac{1}{2}} = \sigma$$

Problem 13.1.21

For the gamma distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(a) $\langle x \rangle$, the mean value of x, is equal to $\alpha\beta$

(b) σ^2, its variance, defined as $\left\langle x^2 \right\rangle - \langle x \rangle^2$, has the value $\alpha\beta^2$

Solution For (a) the mean

$$\begin{aligned} \langle x \rangle &= \int_0^\infty x f(x) dx \\ &= \int_0^\infty x \cdot \frac{1}{\beta^a \Gamma(\alpha)} x^{a-1} e^{-\frac{x}{\beta}} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^a e^{-\frac{x}{\beta}} dx \end{aligned}$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \to 0,\ z \to 0$ and $x \to \infty,\ z \to \infty$.

$$\begin{aligned} \langle x \rangle &= \frac{1}{\Gamma(\alpha)} \int_0^\infty z^a e^{-z} \beta dz \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{(a+1)-1} e^{-z} dz \\ &= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) \\ &= \frac{\beta}{\Gamma(\alpha)} \cdot \alpha \Gamma(\alpha) \\ &= \alpha\beta \end{aligned}$$

Solution For (b)

$$\begin{aligned} \left\langle x^2 \right\rangle &= \int_0^\infty x^2 f(x) dx \\ &= \int_0^\infty x^2 \cdot \frac{1}{\beta^a \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha+1} e^{-\frac{x}{\beta}} dx \end{aligned}$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \to 0,\ z \to 0$ and $x \to \infty,\ z \to \infty$

$$\begin{aligned} \left\langle x^2 \right\rangle &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{a+1} e^{-z} \beta dz \\ &= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+2)-1} e^{-z} dz \\ &= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) \\ &= \frac{\beta^2}{\Gamma(\alpha)} \cdot (\alpha+1)\alpha\Gamma(\alpha) \\ &= \alpha(\alpha+1)\beta^2 \\ &= \alpha^2\beta^2 + \alpha\beta^2 \end{aligned}$$

Hence variance,

$$\sigma^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$
$$= \alpha\beta^2$$

Chapter 13.4: Stirling's series

Problem 13.4.1

Rewrite Stirling's series to give $\Gamma(z+1)$ instead of $\ln\Gamma(z+1)$

$$\text{ANS.} \quad \Gamma(z+1) = \sqrt{2\pi} z^{z+1/2} e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51,840z^3} + \cdots\right)$$

Solution Consider the Stirling's formula:

$$\ln\Gamma(z+1) = \frac{1}{2}\ln 2\pi + \left(z + \frac{1}{2}\right)\ln z - z + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}}$$

Where B_{2n} are the Bernoulli's numbers. Use the first few Bernoulli's numbers and rewrite the above Stirling's formula as equivalent to

$$\ln\Gamma(z+1) \sim \frac{1}{2}\ln(2\pi) + \left(z + \frac{1}{2}\right)\ln z - z + \frac{1}{12z} - \frac{1}{360z^2} + \frac{1}{1260z^3} - \ldots$$

The Stirling's formula can be rewritten using Gamma function as follows. Let us take exponential form and collect similar terms to get equivalent form as follows.

$$\Gamma(z+1) \sim \sqrt{2\pi} + z^{\left(z+\frac{1}{2}\right)} e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \ldots\right)$$

Hence, the required result is

$$\Gamma(z+1) \sim \sqrt{2\pi} + z^{\left(z+\frac{1}{2}\right)} e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \ldots\right)$$

Problem 13.4.5

Test the convergence

$$\sum_{p=0}^{\infty} \left[\frac{\Gamma\left(p+\frac{1}{2}\right)}{p!}\right]^2 \frac{2p+1}{2p+2} = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

Solution Consider the series obtained in the magnetic field created by and enclosed by a current loop:

$$\sum_{p=0}^{\infty} \frac{\Gamma\left(p+\frac{1}{2}\right)}{p!} \left(\frac{2p+1}{2p+2}\right) = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

Now, we will test the convergence of the series using Stirling asymptotic formula given by

$$\Gamma(z+1) \sim \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}$$

$$\begin{aligned} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} &\sim \sqrt{e}\, \frac{\left(\frac{p+\frac{1}{2}}{p+1}\right)^{p+\frac{1}{2}}}{\Gamma(p+1)} \\ &= \frac{\text{constant}}{\Gamma(p+1)} \end{aligned}$$

Hence, the series converges.

Problem 13.4.6

Show that

$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1$$

$\boxed{\textbf{Solution}}$ For a large n, the Stirling asymptotic formula can be taken to the n arbitrary closed to infinite Then the expression has asymptotic limit.

$$\begin{aligned}
&\ln\left[\left(x^{b-a}\right)\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right]\\
&= (b-a)\ln x\left(\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right)\\
&= (b-a)\ln(x)+\ln\left(\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right)\\
&= (b-a)\ln(x)+\ln\Gamma(x+a+1)-\ln\Gamma(x+b+1)
\end{aligned}$$

Now we use

$$\ln\Gamma(z+1) = \left(z+\frac{1}{z}\right)\ln z - z$$

Now, $(b-a)\ln(x)+\ln\Gamma(x+a+1)-\ln\Gamma(x+b+1)$ it reduces to

$$\begin{aligned}
&(b-a)\ln(x)+\ln\Gamma(x+a+1)-\ln\Gamma(x+b+1)\\
&-(x+a)-\left(x+b+\frac{1}{2}\right)\ln(x+b)+(x+b)\\
&= (b-a)\ln(x)+(a-b)\ln(x)
\end{aligned}$$

Rewrite the $\ln(x+a)$ as follows.

$$\begin{aligned}
\ln(a+x) &= \ln x\left(1+\frac{a}{x}\right)\\
&= \ln x+\ln\left(1+\frac{a}{x}\right)\\
&= \ln x+\frac{a}{x}+\dots
\end{aligned}$$

Now rewrite the $\ln(x+b)$

$$\begin{aligned}
\ln(b+x) &= \ln x\left(1+\frac{b}{x}\right)\\
&= \ln x+\ln\left(1+\frac{b}{x}\right)\\
&= \ln x+\frac{b}{x}+\dots
\end{aligned}$$

For large x, make all the terms to exponential form. So, that $\exp(0)=1$. Hence, the limit tends to 1.

$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1$$

Problem 13.4.7

Show that
$$\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{1/2}=\pi^{-1/2}$$

Solution Write the limit expression in factorial notations. Then it is easy to apply the Stirling formula
$$\lim_{x\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{\frac{1}{2}}=\lim_{x\to\infty}\frac{(2n)!}{2^{2n}(n!)^2}n^{\frac{1}{2}}$$

Take logarithm for the limit
$$\ln\left(\lim_{x\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{\frac{1}{2}}\right)=\ln\left(\lim_{x\to\infty}\frac{(2n)!}{2^{2n}(n!)^2}n^{\frac{1}{2}}\right)$$

Consider the right hand side of the above equation and solve.
$$\begin{aligned}
&\ln\lim_{n\to\infty}\frac{(2n)!n^{\frac{1}{2}}}{2^{2n}(n!)^2}\\
&=\lim_{n\to\infty}\ln(2n)!+\frac{1}{2}\ln n-2n\ln 2-2\ln(n!)\\
&\frac{\ln(2\pi)}{2}+\left(2n+\frac{1}{2}\right)\ln(2n)-2n+\frac{\ln n}{2}\\
&\approx -2n\ln 2-\ln(2\pi)-2\left(n+\frac{1}{2}\right)\ln n+2n+\ldots\\
&\sim -\frac{1}{2}\ln\pi\\
&=\ln\pi^{-\frac{1}{2}}
\end{aligned}$$

Substitute the value of right hand side limit
$$\ln\left(\lim_{x\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{\frac{1}{2}}\right)=\ln\pi^{-\frac{1}{2}}$$
$$\lim_{x\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{\frac{1}{2}}=\pi^{-\frac{1}{2}}$$

Hence, the limit tends to
$$\lim_{x\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{\frac{1}{2}}=\pi^{-\frac{1}{2}}$$

Chapter 14.3: Neumann Function

Problem 14.3.1

Prove that the Neumann functions Y_n (with n an integer) satisfy the recurrence relations

$$\begin{aligned} Y_{n-1}(x) + Y_{n+1}(x) &= \tfrac{2n}{x} Y_n(x) \\ Y_{n-1}(x) - Y_{n+1}(x) &= 2Y_n'(x) \end{aligned}$$

Hint. These relations may be proved by differentiating the recurrence relations for J_v or by using the limit form of Y_v but not dividing everything by zero.

Solution As

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

with n as an integer, and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} + \lim_{v \to n} \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi}$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} - \lim_{v \to n} \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi}$$

As $Y_n(x) = \lim_{v \to n} Y_v(x)$ exists and is not identically zero, we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} + \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi} \right)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos(v-1)\pi J_{v-1}(x) - J_{-v+1}(x)}{\sin(v-1)\pi} - \frac{\cos(v+1)\pi J_{v+1}(x) - J_{-v-1}(x)}{\sin(v+1)\pi} \right)$$

As $\cos(v-1)\pi = \cos(\pi - v\pi) = -\cos v\pi$, $\cos(v+1)\pi = -\cos v\pi$ $\sin(v-1)\pi = -\sin(\pi - v\pi) = \sin v\pi$ and $\sin(v+1)\pi = -\sin v\pi$ we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{-\cos v\pi J_{v-1}(x) - J_{-v+1}(x)}{-\sin v\pi} + \frac{-\cos v\pi J_{v+1}(x) - J_{-v-1}(x)}{-\sin v\pi} \right)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \left(\frac{-\cos v\pi J_{v-1}(x) - J_{-v+1}(x)}{-\sin v\pi} - \frac{-\cos v\pi J_{v+1}(x) - J_{-v-1}(x)}{-\sin v\pi} \right)$$

Thus,

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} + \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi} \right)$$

and

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to \pi} \left(\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} - \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi} \right)$$

Also,

$$\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} + \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

can be written as

$$\frac{\cos v\pi J_{v-1}(x) + \cos v\pi J_{v+1}(x) + J_{-v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

and hence we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos v\pi \left(J_{v-1}(x) + J_{v+1}(x)\right) + \left(J_{-v+1}(x) + J_{-v-1}(x)\right)}{\sin v\pi} \right)$$

Similarly,

$$\frac{\cos v\pi J_{v-1}(x) + J_{-v+1}(x)}{\sin v\pi} - \frac{\cos v\pi J_{v+1}(x) + J_{-v-1}(x)}{\sin v\pi}$$

can be written as
$$\frac{\cos v\pi J_{v-1}(x) - \cos v\pi J_{v+1}(x) + J_{-v+1}(x) - J_{-v-1}(x)}{\sin v\pi}$$
and hence we get that
$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\longrightarrow n}\left(\frac{\cos v\pi\left(J_{v-1}(x) - J_{v+1}(x)\right) - \left(J_{-v-1}(x) - J_{-v+1}(x)\right)}{\sin v\pi}\right)$$
We now have to proove that
$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$$
$$J_{v-1}(x) - J_{v+1}(x) = 2J'_v(x)$$
$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(x^v \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$
which implies that
$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v}}{s!\,\Gamma(v+s+1)2^{2s+v}}\right)$$
$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v}}{s!\,\Gamma(v+s+1)2^{2s+v}}\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(2s+2v)(x)^{2s+2v-1}}{s!\,\Gamma(v+s+1)2^{2s+v}} = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}$$
As
$$J_{v-1}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}$$
and
$$\frac{d}{dx}\left(x^v J_v(x)\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}$$
we get that
$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}\right) = x^v J_{v-1}(x)$$
Similarly,
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(x^{-v} \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$
which implies that
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s}}{s!\,\Gamma(v+s+1)2^{2s+v}}\right)$$
Also
$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{s!\,\Gamma(v+s+1)2^{2s+\gamma}}\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(2s)(x)^{2s-1}}{s!\,\Gamma(v+s+1)2^{2s+v}} = \sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}$$
$$\sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\,\Gamma(v+s)2^{2s+v-1}} = -\sum_{s=1}^{\infty}\frac{(-1)^{s-1}x^{2(s-1)+1}}{(s-1)!\,\Gamma(v+s-1+1)2^{2(s-1)+*+1}} = -\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!\,\Gamma(v+k+1)2^{2k+w+1}}$$
As
$$J_{v+1}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma+1}}{s!\,\Gamma(v+s+1)2^{2s+v+1}}$$
and
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+1}}{s!\,\Gamma(v+s)2^{2s+v+1}}$$
we get that
$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -x^{-v}\left(\sum_{x=0}^{\infty}\frac{(-1)^s(x)^{2s+v+1}}{s!\,\Gamma(v+s)2^{2s+v+1}}\right) = -x^{-v}J_{v+1}(x)$$

Then

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v \frac{d}{dx}\left(J_v(x)\right) + \frac{d}{dx}\left(x^v\right) J_v(x) = x^v J_v'(x) + v x^{v-1} J_v(x)$$

as

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v J_v'(x) + v x^{v-1} J_v(x)$$

and

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v J_{v-1}(x)$$

we get that

$$x^v J_v'(x) + v x^{v-1} J_v(x) = x^v J_{v-1}(x)$$

and

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

Also

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = x^{-v} \frac{d}{dx}\left(J_v(x)\right) + \frac{d}{dx}\left(x^{-v}\right) J_v(x) = x^{-v} J_v'(x) - v x^{-v-1} J_v(x)$$

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = x^{-v} J_v'(x) - v x^{-\gamma-1} J_v(x)$$

and

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -x^{-v} J_{v+1}(x)$$

we get that

$$x^{-v} J_v'(x) - v x^{-\gamma-1} J_v(x) = -x^{-v} J_{v+1}(x)$$

and

$$J_v'(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x)$$

As

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

and

$$J_v'(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x),$$

we get that

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

and

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

As

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(J_{v-1}(x) + J_{v+1}(x)\right) + \left(J_{-v+1}(x) + J_{-v-1}(x)\right)}{\sin v\pi}\right)$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{x\to n}\left(\frac{\cos v\pi\left(\frac{2v}{x} J_v(x)\right) + \left(\frac{2(-v)}{x} J_{-v}(x)\right)}{\sin v\pi}\right)$$

As

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(J_{v-1}(x) - J_{v+1}(x)\right) - \left(J_{-v-1}(x) - J_{-v+1}(x)\right)}{\sin v\pi}\right)$$

we get that

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v\to n}\left(\frac{\cos v\pi\left(2J_v'(x)\right) - \left(2J_{-v}'(x)\right)}{\sin v\pi}\right)$$

Also

$$\frac{\cos v\pi\left(\frac{2v}{x} J_v(x)\right) + \left(\frac{2(-v)}{x} J_{-v}(x)\right)}{\sin v\pi} = \frac{2v}{x}\left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}\right)$$

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \frac{2v}{x} \left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} \right)$$

and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

we get that

$$Y_{n-1}(x) + Y_{n+1}(x) = \lim_{v \to n} \frac{2v}{x} Y_v(x) = \frac{2n}{x} Y_n(x)$$

Also,

$$Y_{n-1}(x) - Y_{n+1}(x) = \lim_{v \to n} \left(\frac{\cos v\pi \left(2J'_v(x)\right) - \left(2J'_{-v}(x)\right)}{\sin v\pi} \right) = 2 \lim_{v \to n} \left(\frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} \right)$$

Now,

$$\frac{d}{dx}\left(Y_v(x)\right) = \frac{d}{dx}\left(\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} \right) = \frac{\cos v\pi \frac{d}{dx}\left(J_v(x)\right) - \frac{d}{dx}\left(J_{-v}(x)\right)}{\sin v\pi}$$

Hence, we get

$$Y'_v(x) = \frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi}$$

As

$$Y_{n-1}(x) - Y_{n+1}(x) = 2 \lim_{v \to n} \left(\frac{\cos v\pi J'_v(x) - J'_{-v}(x)}{\sin v\pi} \right)$$

and

$$Y'_n(x) = \frac{\cos v\pi J'_n(x) - J'_{-n}(x)}{\sin v\pi}$$

we get

$$Y_{n-1}(x) - Y_{n+1}(x) = 2 \lim_{v \to n} Y'_v(x) = 2Y'_n(x)$$

Therefore, the recurrence relations are

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y'_n(x)$$

are true when n is an integer

Problem 14.3.2

Show that for integer n
$$Y_{-n}(x) = (-1)^n Y_n(x)$$

Solution We know that for an integer n,
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
Clearly, the statement is true for $n = 0$. Clearly, for any integer n,
$$Y_{-n}(x) = (-1)^n Y_n(x)$$
can be rewritten as
$$(-1)^{-n} Y_{-n}(x) = Y_n(x)$$
which implies that
$$Y_{-(-n)}(x) = (-1)^{-n} Y_{-n}(x)$$
This implies that if the statement is true for any positive integer n, then it is true for any integer n. Assume that the statement
$$Y_{-n}(x) = (-1)^n Y_n(x)$$
is true for any non-negative integer $n \leq k$ where k is any arbitrary non-negative integer. Now we have to prove that $Y_{-k-1}(x) = (-1)^{k+1} Y_{k+1}(x)$ i.e. the statement is true for $n = k + 1$. Also, by substituting $n = 0$ in
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
and hence
$$Y_{-1}(x) = -Y_1(x)$$
As $Y_{-1}(x) = -Y_1(x)$, we get that the statement is true for $n = 1$. As the statement is true for $n = 0$ and $n = 1$, we can assume that $k \geq 1$ As the statement $Y_{-n}(x) = (-1)^n Y_n(x)$ is true for any non-negative integer $n \leq k$, we get that
$$Y_{-k}(x) = (-1)^k Y_k(x)$$
and
$$Y_{-k+1}(x) = (-1)^{k-1} Y_{k-1}(x)$$
As
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$
for any integer n, we get that
$$Y_{-k-1}(x) + Y_{-k+1}(x) = \frac{2(-k)}{x} Y_{-k}(x)$$
and hence
$$Y_{-k-1}(x) = -\frac{2k}{x} Y_{-k}(x) - Y_{-k+1}(x)$$
As
$$Y_{-k-1}(x) = -\frac{2k}{x} Y_{-k}(x) - Y_{-k+1}(x), \quad Y_{-k}(x) = (-1)^k Y_k(x)$$
and
$$Y_{-k+1}(x) = (-1)^{k-1} Y_{k-1}(x)$$
we get that
$$Y_{-k-1}(x) = -\frac{2k}{x} \left((-1)^k Y_k(x) \right) - (-1)^{k-1} Y_{k-1}(x)$$
Also
$$Y_{-k-1}(x) = -\frac{2k}{x} \left((-1)^k Y_k(x) \right) - (-1)^{k-1} Y_{k-1}(x) = (-1)^{k+1} \left(\frac{2k}{x} Y_k(x) - Y_{k-1}(x) \right)$$
As
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

for any integer n, we get that

$$Y_{k-1}(x) + Y_{k+1}(x) = \frac{2k}{x} Y_k(x)$$

and hence

$$Y_{k+1}(x) = \frac{2k}{x} Y_k(x) - Y_{k-1}(x)$$

As

$$Y_{k+1}(x) = \frac{2k}{x} Y_k(x) - Y_{k-1}(x)$$

and

$$Y_{-k-1}(x) = (-1)^{k+1} \left(\frac{2k}{x} Y_k(x) - Y_{k-1}(x) \right)$$

we get that

$$Y_{-k-1}(x) = (-1)^{k+1} Y_{k+1}(x)$$

Therefore, by Mathematical induction, we get that the statement

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

for any non−negative integer n.

Problem 14.4.3

Show that
$$Y_0'(x) = -Y_1(x)$$

Solution As
$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x}Y_n(x)$$
for any integer n, we get that the statement is true for $n = 0$ which implies that
$$Y_{-1}(x) + Y_1(x) = \frac{2(0)}{x}Y_0(x) = 0$$
and hence $Y_{-1}(x) = -Y_1(x)$. As
$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x)$$
for any integer n, we get that the statement is true for $n = 0$ which implies that
$$Y_{-1}(x) - Y_1(x) = 2Y_0'(x)$$
As
$$Y_{-1}(x) = -Y_1(x)$$
and
$$Y_{-1}(x) - Y_1(x) = 2Y_0'(x)$$
we get that
$$2Y_0'(x) = Y_{-1}(x) - Y_1(x) = -Y_1(x) - Y_1(x) = -2Y_1(x)$$
and hence
$$Y_0'(x) = -Y_1(x)$$
Therefore, the statement
$$Y_0'(x) = -Y_1(x)$$
is true.

Problem 14.3.4

If X and Z are any two solutions of Bessel's equation, show that

$$X_\nu(x)Z_v'(x) - X_v'(x)Z_v(x) = \frac{A_v}{x}$$

in which A_v may depend on v but is independent of x. This is a special case of Exercise 7.6.11

Solution We know that for a linear second order homogeneous ODE of form

$$y'' + P(x)y' + Q(x)y = 0$$

and two solutions y_1, y_2 of this ODE, we have that the Wronskian W of y_1 and y_2 satisfies the equation

$$W(x) = W(a)\exp\left[-\int_a^x P(t)dt\right]$$

Thus, the Wronskian W of $X_v(x)$ and $Z_v(x)$ is satisfies the equation

$$W(x) = W(a)\exp\left[-\int_a^x P(t)dt\right]$$

As any Bessel's equation is of the form

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0,$$

we get that $P(x) = \dfrac{1}{x}$ and hence

$$\int_a^x P(t)dt = \int_a^x \frac{1}{t}dt = \ln x - \ln a = \ln\frac{x}{a}$$

Thus,

$$W(x) = W(a)\exp\left[-\int_a^x P(t)dt\right] = W(a)\exp\left[-\ln\frac{x}{a}\right] = W(a)\exp\left[\ln\frac{a}{x}\right] = W(a)\frac{a}{x}$$

Clearly,

$$W(a)a = \left(X_v(a)Z_v'(a) - X_v'(a)Z_v(a)\right)a$$

which implies that $W(a)a$ is a constant independent of x but it may depend on v. Thus, by taking

$$W(a)a = A_v,$$

we get that

$$W(x) = \frac{A_v}{x}$$

where A_v may depend on v but is independent of x. As the Wronskian W of $X_v(x)$ and $Z_v(x)$ is equal to

$$X_v(x)Z_v'(x) - X_v'(x)Z_v(x)$$

and $W(x) = \dfrac{A_v}{x}$, we get that

$$X_v(x)Z_v'(x) - X_v'(x)Z_v(x) = \frac{A_v}{x}$$

where A_v may depend on v but is independent of x. Therefore, the statement

$$X_v(x)Z_v'(x) - X_v'(x)Z_v(x) = \frac{A_v}{x}$$

in which A_v, may depend on v but is independent of x is true when X and Z are any two solutions of Bessel's equation.

Problem 14.3.5

Verify the Wronskian formulas

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = \frac{2\sin v\pi}{\pi x}$$

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = \frac{2}{\pi x}$$

Solution As

$$Y_v'(x) = \frac{\cos v\pi J_v'(x) - J_{-v}'(x)}{\sin v\pi}$$

and

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin vx}$$

we get that

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = J_v(x)\frac{\cos v\pi J_v'(x) - J_{-v}'(x)}{\sin v\pi} - J_v'(x)\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin vx}$$

Also

$$J_v(x)\frac{\cos v\pi J_v'(x) - J_{-v}'(x)}{\sin v\pi} - J_v'(x)\frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin vx}$$

is equal to

$$\frac{-J_v(x)J_{-v}'(x) + J_v'(x)J_{-v}(x)}{\sin vx}$$

which implies that

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = \frac{-J_v(x)J_{-v}'(x) + J_v'(x)J_{-v}(x)}{\sin vx}$$

Clearly,

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(x^v\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\;\Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$

which implies that

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v}}{s!\;\Gamma(v+s+1)2^{2s+v}}\right)$$

Also

$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v}}{s!\;\Gamma(v+s+1)2^{2s+v}}\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(2s+2v)(x)^{2s+2v-1}}{s!\;\Gamma(v+s+1)2^{2s+v}} = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v-1}}{s!\;\Gamma(v+s)2^{2s+v-1}}$$

As

$$J_{v-1}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma-1}}{s!\;\Gamma(v+s)2^{2s+v-1}}$$

and

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+2v-1}}{s!\;\Gamma(v+s)2^{2s+v-1}},$$

we get that

$$\frac{d}{dx}\left(x^v J_v(x)\right) = x^v\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+\gamma-1}}{s!\;\Gamma(v+s)2^{2s+v-1}}\right) = x^v J_{v-1}(x)$$

Similarly,

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(x^{-v}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\;\Gamma(v+s+1)}\left(\frac{x}{2}\right)^{2s+v}\right)$$

which implies that

$$\frac{d}{dx}\left(x^{-v} J_v(x)\right) = \frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s}}{s!\;\Gamma(v+s+1)2^{2s+v}}\right)$$

Also

$$\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{s!\,\Gamma(v+s+1)2^{2s+v}}\right)=\sum_{s=0}^{\infty}\frac{(-1)^s(2s)(x)^{2s-1}}{s!\,\Gamma(v+s+1)2^{2s+v}}=\sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}$$

Also

$$\sum_{s=1}^{\infty}\frac{(-1)^s(s)x^{2s-1}}{s!\,\Gamma(v+s)2^{2s+v-1}}=-\sum_{s=1}^{\infty}\frac{(-1)^{s-1}x^{2(s-1)+1}}{(s-1)!\,\Gamma(v+s-1+1)2^{2(s-1)+v+1}}=-\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!\,\Gamma(v+k+1)2^{2k+v+1}}$$

As

$$J_{v+1}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+v+1}}{s!\,\Gamma(v+s+1)2^{2s+v+1}}$$

and

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=-\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+1}}{s!\,\Gamma(v+s)2^{2s+v+1}},$$

we get that

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=-x^{-v}\left(\sum_{s=0}^{\infty}\frac{(-1)^s(x)^{2s+v+1}}{s!\,\Gamma(v+s)2^{2s+v+1}}\right)=-x^{-v}J_{v+1}(x)$$

Also

$$\frac{d}{dx}\left(x^vJ_v(x)\right)=x^{\nu}\frac{d}{dx}\left(J_v(x)\right)+\frac{d}{dx}\left(x^v\right)J_v(x)=x^vJ_v'(x)+vx^{v-1}J_v(x)$$

As

$$\frac{d}{dx}\left(x^vJ_v(x)\right)=x^vJ_v'(x)+vx^{v-1}J_v(x)$$

and

$$\frac{d}{dx}\left(x^vJ_v(x)\right)=x^vJ_{v-1}(x),$$

we get that

$$x^vJ_v'(x)+vx^{v-1}J_v(x)=x^vJ_{v-1}(x)$$

and hence

$$J_v'(x)+\frac{v}{x}J_v(x)=J_{v-1}(x)$$

Also

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=x^{-v}\frac{d}{dx}\left(J_v(x)\right)+\frac{d}{dx}\left(x^{-v}\right)J_v(x)=x^{-v}J_v'(x)-vx^{-v-1}J_v(x)$$

As

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=x^{-v}J_v'(x)-vx^{-v-1}J_v(x)$$

and

$$\frac{d}{dx}\left(x^{-v}J_v(x)\right)=-x^{-v}J_{v+1}(x),$$

we get that

$$x^{-v}J_v'(x)-vx^{-y-1}J_v(x)=-x^{-v}J_{v+1}(x)$$

and hence

$$J_v'(x)-\frac{v}{x}J_v(x)=-J_{v+1}(x)$$

As

$$J_v'(x)-\frac{v}{x}J_v(x)=-J_{v+1}(x),$$

we get that

$$J_{-v}'(x)-\frac{-v}{x}J_{-v}(x)=-J_{-v+1}(x)$$

by substituting $-v$ in place of v, which implies that

$$J_{-v+1}(x)=-J_{-v}'(x)-\frac{v}{x}J_{-v}(x)$$

As

$$J_{-v+1}(x)=-J_{-v}'(x)-\frac{v}{x}J_{-v}(x)$$

and

$$J'_v(x) + \frac{v}{x} J_v(x) = J_{v-1}(x),$$

we get that

$$J_v(x) J_{-v+1}(x) + J_{-v}(x) J_{v-1}(x)$$

can be written as

$$J_v(x) \left(-J'_{-v}(x) - \frac{v}{x} J_{-v}(x) \right) + J_{-v}(x) \left(J'_v(x) + \frac{v}{x} J_v(x) \right)$$

Also

$$J_v(x) \left(-J'_{-v}(x) - \frac{v}{x} J_{-v}(x) \right) + J_{-v}(x) \left(J'_v(x) + \frac{v}{x} J_v(x) \right)$$

is equal to

$$-J_v(x) J'_{-v}(x) - \frac{v}{x} J_v(x) J_{-v}(x) + J_{-v}(x) J'_v(x) + \frac{v}{x} J_{-v}(x) J_v(x)$$

which is nothing but

$$J_{-v}(x) J'_v(x) - J_v(x) J'_{-v}(x)$$

Thus,

$$J_v(x) J_{-v+1}(x) + J_{-v}(x) J_{v-1}(x) = J_{-v}(x) J'_v(x) - J_v(x) J'_{-v}(x)$$

Therefore, we got that

$$J_v(x) J_{-v+1}(x) + J_{-v}(x) J_{v-1}(x) = J_{-v}(x) J'_v(x) - J_v(x) J'_{-v}(x)$$

and

$$J_v(x) Y'_v(x) - J'_v(x) Y_v(x) = \frac{-J_v(x) J'_{-v}(x) + J'_v(x) J_{-v}(x)}{\sin vx}$$

As $J_v(x)$ and $J_{-v}(x)$ are solutions to the same Bessel's equation, we get that

$$J_{-v}(x) J'_v(x) - J_v(x) J'_{-v}(x) = \frac{A_v}{x}$$

where A, may depend on v but is independent of x. As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\, \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{2s+v},$$

we get that

$$J'_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s+v)}{s!\, \Gamma(v+s+1) 2} \left(\frac{x}{2} \right)^{2s+v-1}$$

Similarly,

$$J_{-v}(x) = \sum_{x=0}^{\infty} \frac{(-1)^s}{s!\, \Gamma(-v+s+1)} \left(\frac{x}{2} \right)^{2s-v}$$

and hence

$$J'_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s-v)}{s!\, \Gamma(-v+s+1) 2} \left(\frac{x}{2} \right)^{2s-v-1}$$

As

$$\frac{1}{\Gamma(v+1)} \left(\frac{x}{2} \right)^{v}$$

is the leading power of

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\, \Gamma(v+s+1)} \left(\frac{x}{2} \right)^{2s+v}$$

and

$$\frac{(-v)}{\Gamma(-v+1) 2} \left(\frac{x}{2} \right)^{-v-1}$$

is the leading power of

$$J'_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (2s-v)}{s!\, \Gamma(-v+s+1) 2} \left(\frac{x}{2} \right)^{2s-\gamma-1},$$

we get that

$$\frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^{v}\frac{(-v)}{\Gamma(-v+1)2}\left(\frac{x}{2}\right)^{-v-1}$$

is the leading power of $J_v(x)J'_{-v}(x)$ Similarly,

$$\frac{1}{\Gamma(-v+1)}\left(\frac{x}{2}\right)^{-v}$$

is the leading power of

$$J_{-v}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(-v+s+1)}\left(\frac{x}{2}\right)^{2s-v}$$

and

$$\frac{v}{\Gamma(v+1)2}\left(\frac{x}{2}\right)^{v-1}$$

is the leading power of

$$J'_v(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(2s+v)}{s!\,\Gamma(v+s+1)2}\left(\frac{x}{2}\right)^{2s+\gamma-1},$$

we get that

$$\frac{1}{\Gamma(-v+1)}\left(\frac{x}{2}\right)^{-v}\frac{v}{\Gamma(v+1)2}\left(\frac{x}{2}\right)^{v-1}$$

is the leading power of $J_{-v}(x)J'_v(x)$. Also

$$\frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^{v}\frac{(-v)}{\Gamma(-v+1)2}\left(\frac{x}{2}\right)^{-\gamma-1}=\frac{-v}{\Gamma(v+1)\Gamma(-v+1)x}$$

and

$$\frac{1}{\Gamma(-v+1)}\left(\frac{x}{2}\right)^{-v}\frac{v}{\Gamma(v+1)2}\left(\frac{x}{2}\right)^{v-1}=\frac{v}{\Gamma(v+1)\Gamma(-v+1)x}$$

Thus, we get that

$$\frac{-v}{\Gamma(v+1)\Gamma(-v+1)x}$$

is the leading power of

$$J_v(x)J'_{-v}(x)$$

and

$$\frac{v}{\Gamma(v+1)\Gamma(-v+1)x}$$

is the leading power of

$$J_{-v}(x)J'_v(x)$$

which implies that the coefficient of x^{-1} in

$$J_{-v}(x)J'_v(x)-J_v(x)J'_{-v}(x)$$

is equal to

$$\frac{2v}{\Gamma(v+1)\Gamma(1-v)x}$$

From reflection formula, we get that

$$\Gamma(v)\Gamma(1-v)=\frac{\pi}{\sin v\pi}$$

As

$$\Gamma(v+1)=v\Gamma(v)$$

and

$$\Gamma(v)\Gamma(1-v)=\frac{\pi}{\sin v\pi},$$

we get that

$$\frac{v}{\Gamma(v+1)\Gamma(1-v)}=\frac{\sin v\pi}{\pi}$$

Therefore, coefficient of x^{-1} in $J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x)$ is equal to $\dfrac{2\sin v\pi}{\pi x}$. As

$$J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x) = \frac{A_v}{x}$$

where A_v, may depend on v but is independent of x and the coefficient of x^{-1} in

$$J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x)$$

is equal to

$$\frac{2\sin v\pi}{\pi x},$$

we get that

$$A_v = \frac{2\sin v\pi}{\pi}$$

and all coefficients of x (except x^{-1}) are zero. Therefore,

$$J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x) = \frac{2\sin v\pi}{\pi x}$$

As

$$J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x) = \frac{2\sin v\pi}{\pi x}$$

and

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x),$$

we get that

$$J_v(x)J_{-v+1}(x) + J_{-v}(x)J_{v-1}(x) = \frac{2\sin v\pi}{\pi x}$$

As

$$J_{-v}(x)J_v'(x) - J_v(x)J_{-v}'(x) = \frac{2\sin v\pi}{\pi x}$$

and

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = \frac{-J_v(x)J_{-v}'(x) + J_v'(x)J_{-v}(x)}{\sin vx},$$

we get that

$$J_v(x)Y_v'(x) - J_v'(x)Y_v(x) = \frac{2}{\pi x}$$

and hence the given statements are true.

Problem 14.3.6

As an alternative to letting x approach zero in the evaluation of the Wronskian constant, we may invoke the uniqueness of power-series expansions. The coefficient of x^{-1} in the series expansion of

$$u_v(x)v_v'(x) - u_v'(x)v_v(x)$$

is then A_v Show by series expansion that the coefficients of x^0 and x^1 of

$$J_v(x)J_{-v}'(x) - J_v'(x)J_{-v}(x)$$

are each zero.

Solution To prove that the coefficients of x^0 and x^1 in

$$J_v(x)J_{-v}'(x) - J_v'(x)J_{-v}(x)$$

are both zero by using power series expansions. As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(v+s+1)} \left(\frac{x}{2}\right)^{2s+v},$$

we get that

$$J_v'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s+v)}{s!\ \Gamma(v+s+1)2} \left(\frac{x}{2}\right)^{2s+v-1}$$

Similarly,

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\ \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

and hence

$$J_{-v}'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(2s-v)}{s!\ \Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2s-v-1}$$

Also

$$J_v(x)J_{-v}'(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k+v} \frac{(-1)^s(2s-v)}{s!\ \Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2s-v-1}$$

and

$$J_v'(x)J_{-v}(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(2k+v)}{k!\ \Gamma(v+k+1)2} \left(\frac{x}{2}\right)^{2k+v-1} \frac{(-1)^s}{s!\ \Gamma(-v+s+1)} \left(\frac{x}{2}\right)^{2s-v}$$

As

$$J_v(x)J_{-v}'(x) = \sum_{x=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2s-v)}{k!\ s!\ \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that the coefficient of x^0 in

$$J_v(x)J_{-v}'(x)$$

is equal to zero (as $2k+2s-1$ is odd for any $s, k \in \mathbb{Z}$, we get that $2k+2s-1$ is never 0 and hence there is no constant term in $J_v(x)J_{-v}'(x)$) Similarly, as

$$J_v'(x)J_{-v}(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2k+v)}{k!\ s!\ \Gamma(v+k+1)\Gamma(-v+s+1)2} \left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that coefficient of x^0 in

$$J_v'(x)J_{-v}(x)$$

is equal to zero (as $2k+2s-1$ is odd for any $s, k \in \mathbb{Z}$, we get that $2k+2s-1$ is never 0 and hence there is no constant term in $J_v'(x)J_{-v}(x)$) As there are constant terms in both $J_v'(x)J_{-v}(x)$ and $J_v(x)J_{-v}'(x)$, we get that the coefficient of x^0 in

$$J_v(x)J_{-v}'(x) - J_v'(x)J_{-v}(x)$$

is zero. As

$$J_v(x)J'_{-v}(x) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+s}(2s-v)}{k!\ s!\ \Gamma(v+k+1)\Gamma(-v+s+1)2}\left(\frac{x}{2}\right)^{2k+2s-1}$$

we get that the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{(-1)^{1+0}(2(0)-v)}{1!\ 0!\ \Gamma(v+(1)+1)\Gamma(-v+(0)+1)2}\left(\frac{1}{2}\right) + \frac{(-1)^{0+1}(2(1)-v)}{0!\ 1!\ \Gamma(v+(0)+1)\Gamma(-v+(1)+1)2}\left(\frac{1}{2}\right)$$

when $s, k \in \mathbb{Z}$, we get that $2k+2s-1=1$ if and only if $k+s=1$ which implies that either $k=0, s=1$ or $k=1, s=0$. Also

$$\frac{(-1)^{1+0}(2(1)+v)}{1!\ 0!\ \Gamma(v+(1)+1)\Gamma(-v+(0)+1)2}\left(\frac{1}{2}\right) + \frac{(-1)^{0+1}(2(0)+v)}{0!\ 1!\ \Gamma(v+(0)+1)\Gamma(-v+(1)+1)2}\left(\frac{1}{2}\right)$$

is equal to

$$\frac{-v-2}{4\Gamma(v+2)\Gamma(1-v)} + \frac{-v}{4\Gamma(v+1)\Gamma(2-v)}$$

Thus, the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{-v-2}{4\Gamma(v+2)\Gamma(1-v)} + \frac{-v}{4\Gamma(v+1)\Gamma(2-v)}$$

As the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{-v-2}{4\Gamma(v+2)\Gamma(1-v)} + \frac{-v}{4\Gamma(v+1)\Gamma(2-v)}$$

and the coefficient of x^1 in $J_v(x)J'_{-v}(x)$ is equal to

$$\frac{v}{4\Gamma(v+2)\Gamma(1-v)} + \frac{v-2}{4\Gamma(v+1)\Gamma(2-v)}$$

we get that the coefficient of x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ is equal to

$$\frac{v}{4\Gamma(v+2)\Gamma(1-v)} + \frac{v-2}{4\Gamma(v+1)\Gamma(2-v)} - \left(\frac{-v-2}{4\Gamma(v+2)\Gamma(1-v)} + \frac{-v}{4\Gamma(v+1)\Gamma(2-v)}\right)$$

which is equal to

$$\frac{2v+2}{4\Gamma(v+2)\Gamma(1-v)} + \frac{2v-2}{4\Gamma(v+1)\Gamma(2-v)}$$

Also

$$\frac{2v+2}{4\Gamma(v+2)\Gamma(1-v)} = \frac{2v+2}{4(v+1)\Gamma(v+1)\Gamma(1-v)} = \frac{1}{2\Gamma(v+1)\Gamma(1-v)}$$

and

$$\frac{2v-2}{4\Gamma(v+1)\Gamma(2-v)} = \frac{2v-2}{4\Gamma(v+1)(1-v)\Gamma(1-v)} = \frac{-1}{2\Gamma(v+1)\Gamma(1-v)}$$

Thus,

$$\frac{2v+2}{4\Gamma(v+2)\Gamma(1-v)} + \frac{2v-2}{4\Gamma(v+1)\Gamma(2-v)} = \frac{1}{\Gamma(v+1)\Gamma(1-v)2} + \frac{-1}{2\Gamma(v+1)\Gamma(1-v)} = 0.$$

Thus, the coefficient of x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ is equal to zero. Therefore, the coefficients of x^0 and x^1 in $J_v(x)J'_{-v}(x) - J'_v(x)J_{-v}(x)$ are both zero.

Problem 14.3.8

Verify the expansion formula for $Y_n(x)$ given in Eq. (14.61).

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n}$$
$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} [\psi(k+1) + \psi(n+k+1)] \left(\frac{x}{2}\right)^{2k+n} \quad (14.61)$$

Hint. Start from Eq. (14.60)

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v}{dv} - (-1)^n \frac{dJ_{-v}}{dv} \right]_{v=n} \quad (14.60)$$

and perform the indicated differentiations on the powerseries expansions of J_v and J_{-v}. The digamma functions ψ arise from the differentiation of the gamma function. You will need the identity (not derived in this book) $\lim_{z \to -n} \psi(z)/\Gamma(z) = (-1)^{n-1} n!$, where n is a positive integer.

Solution To prove that

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} - A$$

where

$$A = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} [\psi(k+1) + \psi(n+k+1)] \left(\frac{x}{2}\right)^{2k+n}$$

We know that

$$Y_n(x) = \frac{1}{\pi} \left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv} \right]_{v=n}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s}$$

we get that

$$\frac{dJ_v(x)}{dv} = \frac{d}{dv} \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s} \right)$$

and hence

$$\frac{dJ_v(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \frac{d}{dv} \left(\frac{x}{2}\right)^{v+2s} + \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{d}{dv} \frac{(-1)^s}{s!\,\Gamma(v+s+1)}$$

Also

$$\frac{d}{dv} \left(\frac{x}{2}\right)^{v+2s} = \left(\frac{x}{2}\right)^{v+2s} \ln\left(\frac{x}{2}\right)$$

and

$$\frac{d}{dv} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} = -\frac{(-1)^s}{s!\,(\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

Thus,

$$\frac{dJ_v(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s} \ln\left(\frac{x}{2}\right) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s}{s!\,(\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s},$$

we get that

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s} \ln\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2}\right) \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s} = \ln\left(\frac{x}{2}\right) J_v(x)$$

Therefore,

$$\frac{dJ_v(x)}{dv} = \ln\left(\frac{x}{2}\right) J_v(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s}{s!\,(\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv}$$

As

$$\psi(z+1) = \frac{d\ln\Gamma(z+1)}{dv} = \frac{1}{\Gamma(z+1)} \frac{d\Gamma(z+1)}{dv}$$

where ψ is the digamma function, we get that

$$\frac{(-1)^s}{s!\,(\Gamma(v+s+1))^2} \frac{d\Gamma(v+s+1)}{dv} = \frac{(-1)^s \psi(v+s+1)}{s!\,\Gamma(v+s+1)}$$

Thus,

$$\frac{dJ_v(x)}{dv} = \ln\left(\frac{x}{2}\right) J_v(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{v+2s} \frac{(-1)^s \psi(v+s+1)}{s!\,\Gamma(v+s+1)}$$

Therefore,

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{n+2s} \frac{(-1)^s \psi(n+s+1)}{s!\,\Gamma(n+s+1)}$$

which implies that

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^s \psi(n+s+1)}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

As

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(v+s+1)} \left(\frac{x}{2}\right)^{v+2s},$$

we get that

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v}$$

Also

$$\frac{dJ_{-v}(x)}{dv} = \frac{d}{dv}\left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v}\right)$$

and hence we get that

$$\frac{dJ_{-v}(x)}{dv} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \frac{d}{dv} \left(\frac{x}{2}\right)^{2s-v} + \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{d}{dv} \frac{(-1)^s}{s!\,\Gamma(s-v+1)}$$

Also

$$\frac{d}{dv}\left(\frac{x}{2}\right)^{2s-v} = \frac{d(2s-v)}{dv} \frac{d}{d(2s-v)} \left(\frac{x}{2}\right)^{2s-v} = -\left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right)$$

and

$$\frac{d}{dv} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} = -\frac{(-1)^s}{s!\,(\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

Thus,

$$\frac{dJ_{-v}(x)}{dv} = -\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s}{s!\,(\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

As

$$J_{-v}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v},$$

we get that

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} \ln\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2}\right) \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s-v+1)} \left(\frac{x}{2}\right)^{2s-v} = \ln\left(\frac{x}{2}\right) J_{-v}(x)$$

Therefore,

$$\frac{dJ_{-v}(x)}{dv} = -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s}{s!\,(\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv}$$

As

$$\psi(z+1) = \frac{d\ln\Gamma(z+1)}{dv} = \frac{1}{\Gamma(z+1)} \frac{d\Gamma(z+1)}{dv},$$

we get that

$$\frac{(-1)^s}{s!\,(\Gamma(s-v+1))^2} \frac{d\Gamma(s-v+1)}{dv} = \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)}$$

and hence

$$\frac{dJ_{-v}(x)}{dv} = -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-v} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)}$$

Thus,

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)}$$

As

$$\lim_{v\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)} = \frac{(-1)^s}{s!} \lim_{v\to n} \frac{\psi(s-v+1)}{\Gamma(s-v+1)},$$

we get that

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^x \psi(s-v+1)}{s!\,\Gamma(s-v+1)} - \sum_{s=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)}$$

(by dividing the summation into $s < n$ and $s \geq n$ parts). Also

$$\sum_{s=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n}}{(k+n)!} \frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)!\,k!}$$

and

$$\sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)} = \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} \lim_{k\to s-n} \frac{\psi(k+1)}{\Gamma(k+1)}$$

As

$$\lim_{z\to -n} \frac{\psi(z+1)}{\Gamma(z+1)} = (-1)^{n-1} n!\,,$$

we get that

$$\sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} \lim_{k\to s-n} \frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1} (s-n-1)!$$

As

$$\sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} \lim_{k\to s-n} \frac{\psi(k+1)}{\Gamma(k+1)} = \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \frac{(-1)^s}{s!} (-1)^{s-n-1} (s-n-1)!$$

$$\sum_{s=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^{k+n} \psi(k+1)}{(k+n)!\,k!}$$

and

$$\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n} = -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \sum_{s=0}^{n-1} \left(\frac{x}{2}\right)^{2s-n} \lim_{x\to n} \frac{(-1)^s \psi(s-v+1)}{s!\,\Gamma(s-v+1)} - \sum_{x=n}^{\infty} \left(\frac{x}{2}\right)^{2s-n} \lim_{v\to n} \frac{(-1)^x \psi(s-v+1)}{s!\,\Gamma(s-v+1)}$$

we get that $\left(\dfrac{dJ_{-v}(x)}{dv}\right)_{v=n}$ is equal to

$$-\ln\left(\frac{x}{2}\right)J_{-n}(x)-\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}(-1)^{s-n-1}(s-n-1)!\;-\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^{k+n}\psi(k+1)}{(k+n)!\;k!}$$

As $J_{-n}(x)=(-1)^nJ_n(x)$

$$-\ln\left(\frac{x}{2}\right)J_{-n}(x)-\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(-1)^s}{s!}(-1)^{s-n-1}(s-n-1)!\;-\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^{k+n}\psi(k+1)}{(k+n)!\;k!}$$

can be written as

$$-(-1)^n\ln\left(\frac{x}{2}\right)J_n(x)-\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(s-n-1)!}{s!}(-1)^{n+1}-(-1)^{n+1}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^k\psi(k+1)}{(k+n)!\;k!}$$

$$(-1)^{s-n-1}(-1)^s=(-1)^{s-n-1+s-n-1}(-1)^{n+1}=(-1)^{n+1}(-1)^{2(s-n-1)}=(-1)^{n+1}$$

Thus

$$(-1)^{n+1}\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n}=\ln\left(\frac{x}{2}\right)J_n(x)-\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(s-n-1)!}{s!}-\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^k\psi(k+1)}{(k+n)!\;k!}$$

As

$$\left(\frac{dJ_v(x)}{dv}\right)_{v=n}=\ln\left(\frac{x}{2}\right)J_n(x)-\sum_{s=0}^{\infty}\frac{(-1)^s\psi(n+s+1)}{s!\;(n+s)!}\left(\frac{x}{2}\right)^{n+2}$$

$$(-1)^{n+1}\left(\frac{dJ_{-v}(x)}{dv}\right)_{v=n}=\ln\left(\frac{x}{2}\right)J_n(x)-\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(s-n-1)!}{s!}-\sum_{s=0}^{\infty}\frac{(-1)^s\psi(s+1)}{(s+n)!\;s!}\left(\frac{x}{2}\right)^{2s+n}$$

and

$$Y_n(x)=\frac{1}{\pi}\left[\frac{dJ_v(x)}{dv}-(-1)^n\frac{dJ_{-v}(x)}{dv}\right]_{v=n}$$

we get

$$\frac{2}{\pi}\ln\left(\frac{x}{2}\right)J_n(x)-\frac{1}{\pi}\sum_{s=0}^{n-1}\left(\frac{x}{2}\right)^{2s-n}\frac{(s-n-1)!}{s!}-\sum_{s=0}^{\infty}\frac{(-1)^s(\psi(s+1)+\psi(n+s+1))}{(s+n)!\;s!}\left(\frac{x}{2}\right)^{2s+n}$$

Therefore,

$$Y_n(x)=\frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right)-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}-A$$

where

$$A=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\;(n+k)!}[\psi(k+1)+\psi(n+k+1)]\left(\frac{x}{2}\right)^{2k+n}$$

Problem 14.3.9

If Bessel's ODE (with solution J_v) is differentiated with respect to v, one obtains

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right) + \left(x^2 - v^2\right)\frac{\partial J_v}{\partial v} = 2vJ_v$$

Use the above equation to show that $Y_n(x)$ is a solution to Bessel's ODE.

Solution We know that

$$Y_n(x) = \frac{1}{\pi}\left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv}\right]_{v=n}$$

As

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right) + \left(x^2 - v^2\right)\frac{\partial J_v}{\partial v} = 2vJ_v,$$

we get that

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial(-v)}\right) + x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial(-v)}\right) + \left(x^2 - (-v)^2\right)\frac{\partial J_{-v}}{\partial(-v)} = 2(-v)J_{-v}$$

As $\frac{\partial J_{-v}}{\partial(-v)} = -\frac{\partial J_{-v}}{\partial v}$, we get that

$$-x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right) - x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right) - \left(x^2 - v^2\right)\frac{\partial J_{-v}}{\partial v} = -2vJ_{-v}$$

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right) + \left(x^2 - v^2\right)\frac{\partial J_{-v}}{\partial v} = 2vJ_{-v}$$

As

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right) + \left(x^2 - v^2\right)\frac{\partial J_v}{\partial v} = 2vJ_v$$

and

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right) + \left(x^2 - v^2\right)\frac{\partial J_{-v}}{\partial v} = 2vJ_{-v},$$

we get that $2vJ_v - 2v(-1)^n J_{-v}$ is equal to

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v} - (-1)^n\frac{\partial J_{-v}}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v} - (-1)^n\frac{\partial J_{-v}}{\partial v}\right) + \left(x^2 - v^2\right)\left(\frac{\partial J_v}{\partial v} - (-1)^n\frac{\partial J_{-v}}{\partial v}\right)$$

Thus, $2nJ_n - 2n(-1)^n J_{-n}$ is equal to

$$\left[x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v} - (-1)^n\frac{\partial J_{-v}}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v} - (-1)^n\frac{\partial J_{-v}}{\partial v}\right) + \left(x^2 - v^2\right)\left(\frac{\partial J_v}{\partial v} - (-1)^n\frac{\partial J_{-v}}{\partial v}\right)\right]_{v=n}$$

As

$$Y_n(x) = \frac{1}{\pi}\left[\frac{dJ_v(x)}{dv} - (-1)^n \frac{dJ_{-v}(x)}{dv}\right]_{v=n},$$

the above equation implies that

$$x^2\frac{d^2Y_n(x)}{dx^2} + x\frac{dY_n(x)}{dx} + \left(x^2 - n^2\right)Y_n(x) = \frac{1}{\pi}\left(2nJ_n - 2n(-1)^n J_{-n}\right)$$

As $J_{-n} = (-1)^n J_n$, we get that $2nJ_n - 2n(-1)^n J_{-n} = 0$. As $2nJ_n - 2n(-1)^n J_{-n} = 0$ and

$$x^2\frac{d^2Y_n(x)}{dx^2} + x\frac{dY_n(x)}{dx} + \left(x^2 - n^2\right)Y_n(x) = \frac{1}{\pi}\left(2nJ_n - 2n(-1)^n J_{-n}\right),$$

we get that

$$x^2\frac{d^2Y_n(x)}{dx^2} + x\frac{dY_n(x)}{dx} + \left(x^2 - n^2\right)Y_n(x) = 0$$

Therefore, $Y_n(x)$ is a solution to the Bessel's ODE.