Problem 13.4.1

Rewrite Stirling's series to give $\Gamma(z+1)$ instead of $\ln\Gamma(z+1)$

$$\text{ANS.}\quad \Gamma(z+1)=\sqrt{2\pi}z^{z+1/2}e^{-z}\left(1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51,840z^3}+\cdots\right)$$

Solution Consider the Stirling's formula:

$$\ln\Gamma(z+1)=\frac{1}{2}\ln 2\pi+\left(z+\frac{1}{2}\right)\ln z-z+\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)z^{2n-1}}$$

Where B_{2n} are the Bernoulli's numbers. Use the first few Bernoulli's numbers and rewrite the above Stirling's formula as equivalent to

$$\ln\Gamma(z+1)\sim\frac{1}{2}\ln(2\pi)+\left(z+\frac{1}{2}\right)\ln z-z+\frac{1}{12z}-\frac{1}{360z^2}+\frac{1}{1260z^3}-\ldots$$

The Stirling's formula can be rewritten using Gamma function as follows. Let us take exponential form and collect similar terms to get equivalent form as follows.

$$\Gamma(z+1)\sim\sqrt{2\pi}+z^{\left(z+\frac{1}{2}\right)}e^{-z}\left(1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}+\ldots\right)$$

Hence, the required result is

$$\Gamma(z+1)\sim\sqrt{2\pi}+z^{\left(z+\frac{1}{2}\right)}e^{-z}\left(1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}+\ldots\right)$$

Problem 13.4.5

Test the convergence

$$\sum_{p=0}^{\infty}\left[\frac{\Gamma\left(p+\frac{1}{2}\right)}{p!}\right]^2\frac{2p+1}{2p+2}=\pi\sum_{p=0}^{\infty}\frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

Solution Consider the series obtained in the magnetic field created by and enclosed by a current loop:

$$\sum_{p=0}^{\infty}\frac{\Gamma\left(p+\frac{1}{2}\right)}{p!}\left(\frac{2p+1}{2p+2}\right)=\pi\sum_{p=0}^{\infty}\frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

Now, we will test the convergence of the series using Stirling asymptotic formula given by

$$\Gamma(z+1)\sim\sqrt{2\pi}z^{z+\frac{1}{2}}e^{-z}$$

$$\begin{aligned}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}&\sim\sqrt{e}\frac{\left(\frac{p+\frac{1}{2}}{p+1}\right)^{p+\frac{1}{2}}}{\Gamma(p+1)}\\&=\frac{\text{constant}}{\Gamma(p+1)}\end{aligned}$$

Hence, the series converges.

Problem 13.4.6

Show that
$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}=1$$

$\boxed{\textbf{Solution}}$ For a large n, the Stirling asymptotic formula can be taken to the n arbitrary closed to infinite Then the expression has asymptotic limit.

$$\ln\left[\left(x^{b-a}\right)\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right]$$

$$=(b-a)\ln x\left(\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right)$$

$$=(b-a)\ln(x)+\ln\left(\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right)$$

$$=(b-a)\ln(x)+\ln\Gamma(x+a+1)-\ln\Gamma(x+b+1)$$

Now we use
$$\ln\Gamma(z+1)=\left(z+\frac{1}{z}\right)\ln z-z$$

Now, $(b-a)\ln(x)+\ln\Gamma(x+a+1)-\ln\Gamma(x+b+1)$ it reduces to

$$(b-a)\ln(x)+\ln\Gamma(x+a+1)-\ln\Gamma(x+b+1)$$

$$-(x+a)-\left(x+b+\frac{1}{2}\right)\ln(x+b)+(x+b)$$

$$=(b-a)\ln(x)+(a-b)\ln(x)$$

Rewrite the $\ln(x+a)$ as follows.
$$\begin{aligned}\ln(a+x)&=\ln x\left(1+\frac{a}{x}\right)\\&=\ln x+\ln\left(1+\frac{a}{x}\right)\\&=\ln x+\frac{a}{x}+\dots\end{aligned}$$

Now rewrite the $\ln(x+b)$
$$\begin{aligned}\ln(b+x)&=\ln x\left(1+\frac{b}{x}\right)\\&=\ln x+\ln\left(1+\frac{b}{x}\right)\\&=\ln x+\frac{b}{x}+\dots\end{aligned}$$

For large x, make all the terms to exponential form. So, that $\exp(0)=1$. Hence, the limit tends to 1.

$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}=1$$

Problem 13.4.7

Show that

$$\lim_{n\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{1/2} = \pi^{-1/2}$$

$\boxed{\textbf{Solution}}$ Write the limit expression in factorial notations. Then it is easy to apply the Stirling formula

$$\lim_{x\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \lim_{x\to\infty} \frac{(2n)!}{2^{2n}(n!)^2} n^{\frac{1}{2}}$$

Take logarithm for the limit

$$\ln\left(\lim_{x\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}}\right) = \ln\left(\lim_{x\to\infty} \frac{(2n)!}{2^{2n}(n!)^2} n^{\frac{1}{2}}\right)$$

Consider the right hand side of the above equation and solve.

$$\ln \lim_{n\to\infty} \frac{(2n)! n^{\frac{1}{2}}}{2^{2n}(n!)^2}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \ln(2n)! + \frac{1}{2}\ln n - 2n\ln 2 - 2\ln(n!) \\
&\frac{\ln(2\pi)}{2} + \left(2n + \frac{1}{2}\right)\ln(2n) - 2n + \frac{\ln n}{2} \\
&\approx -2n\ln 2 - \ln(2\pi) - 2\left(n + \frac{1}{2}\right)\ln n + 2n + \dots \\
&\sim -\frac{1}{2}\ln \pi \\
&= \ln \pi^{-\frac{1}{2}}
\end{aligned}$$

Substitute the value of right hand side limit

$$\ln\left(\lim_{x\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}}\right) = \ln \pi^{-\frac{1}{2}}$$

$$\lim_{x\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \pi^{-\frac{1}{2}}$$

Hence, the limit tends to

$$\lim_{x\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \pi^{-\frac{1}{2}}$$