Problem 13.1.1

Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral, Eq. (13.5),

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$\boxed{\textbf{Solution}}$ Consider the Euler integral

$$\Gamma z = \int_0^\infty e^{-t} t^{z-1} dt$$

Put, $z = z+1$

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty e^{-t} t^{z+1-1} dt \\
&= \int_0^\infty e^{-t} t^{z} dt \\
&= t^z \int_0^\infty e^{-t} dt - \int_0^\infty \frac{dt^z}{dx} \int e^{-t} dt \\
&= - t^z e^{-t}\Big|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} dt \\
&= z\Gamma(z)
\end{aligned}$$

Problem 13.1.2

In a power-series solution for the Legendre functions of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

in which s is a positive integer.

(a) Rewrite this expression in terms of factorials.

(b) Rewrite this expression using Pochhammer symbols; see Eq. (1.72).

Solution For (a) Notice that

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2.4.6.8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$=\frac{[n!\ (n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)]}{n!\ s!\ 2^s\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$=\frac{(n+2s)!(2n+1)!}{n!\ s!\ 2^s\cdot[(2n+1)!(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$=\frac{(n+2s)!(2n+1)![(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{n!\ s!\ 2^s\cdot[(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]}$$

$$=\frac{(n+2s)!(2n+1)!2^s[(n+1)(n+2)(n+3)\cdots(n+s)]}{n!\ s!\ 2^s\cdot[(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]}$$

$$=\frac{(n+2s)!(2n+1)![n!\ (n+1)(n+2)(n+3)\cdots(n+s)]}{n!\ s!\ n!\ [(2n+1)!(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)\cdots(2n+2s)(2n+2s+1)]}$$

$$=\frac{(n+2s)!(2n+1)!(n+s)!}{n!\ n!\ s!\ (2n+2s+1)!}$$

Solution For (b) we notice that

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)}$$

$$=\frac{(n+1)(n+2)(n+3)\cdots[(n+1)+(2s-2)][(n+1)+(2s-1)]}{(2^s[1\cdot 2\cdot 3\cdot\cdots\cdot(s-1)s])\cdot[(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)]}$$

$$=\frac{(n+1)_{(2s-1)+1}\cdot[(2n+2)(2n+4)(2n+6)\cdots(2n+2s)]}{(2^s[1\cdot 2\cdot 3\cdot\cdots\cdot\{1+(s-2)\}\{1+(s-1)\}])\cdot[(2n+2)(2n+3)(2n+4)(2n+5)\cdots(2n+2s)(2n+2s+1)]}$$

$$=\frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots(n+s)]\cdot 2^s}{2^s(1)_{(s-1)+1}\cdot[(2n+2)(2n+3)(2n+4)\cdots\{(2n+2)+(2s-1)\}]}$$

$$=\frac{(n+1)_{2s}\cdot[(n+1)(n+2)(n+3)\cdots\{(n+1)+(s-1)\}]}{(1)_s\cdot(2n+2)_{(2s-1)+1}}$$

$$=\frac{(n+1)_{2s}\cdot(n+1)_{(s-1)+1}}{(1)_s\cdot(2n+2)_{2s}}$$

$$=\frac{(n+1)_{2s}\cdot(n+1)_s}{(1)_s\cdot(2n+2)_{2s}}$$

Problem 13.1.3

Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt, \quad \mathrm{Re}(z) > 0$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re e(z) > 0$$

$\boxed{\textbf{Solution}}$ Changing variables $t = u^2$ and $dt = 2udu$ we have

$$\begin{aligned}
\Gamma z &= \int_0^\infty e^{-u^2} u^{2z-2} u du \\
&= \int_0^\infty e^{-u^2} u^{2z-1} du \\
&= \int_0^\infty e^{-t^2} t^{2z-1} dt
\end{aligned}$$

as $t \to 0$ to $\infty u \to 0$ to 1 the equation takes the form of

$$\begin{aligned}
\Gamma z &= \int_0^1 e^{-\ln\frac{1}{u}} \left(\ln\frac{1}{u}\right)^{z-1} udu \\
&= \int_0^1 u \left(\ln\frac{1}{u}\right)^{z-1} udu \\
&= \int_0^1 \left(\ln\frac{1}{u}\right)^{z-1} du \\
&= \int_0^1 \left(\ln\frac{1}{t}\right)^{z-1} dt
\end{aligned}$$

Problem 13.1.4

In a Maxwellian distribution the fraction of particles of mass m with speed between v and $v+dv$ is

$$\frac{dN}{N} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^2 dv$$

where N is the total number of particles, k is Boltzmann's constant, and T is the absolute temperature. The average or expectation value of v^n is defined as $\langle v^n\rangle = N^{-1}\int v^n dN$. Show that

$$\langle v^n\rangle = \left(\frac{2kT}{m}\right)^{n/2} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

This is an extension of Example 13.1.1, in which the distribution was in kinetic energy $E = mv^2/2$, with $dE = mvdv$

Solution

$$\begin{aligned}
\langle v^n\rangle &= N^{-1}\int v^n dN \\
&= \int v^n \frac{dN}{N} \\
&= \int_0^\infty v^n \cdot 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{\frac{m^2}{2kT}} v^2 dv \\
&= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty v^n e^{\frac{m^2}{2kT}} v^{n+1} v dv
\end{aligned}$$

Let $\frac{mv^2}{2kT} = u^2$. Then $v = \left(\frac{2kT}{m}\right)^{\frac{1}{2}} u$ and $vdv = \frac{2kT}{m} udu$. As $v \to 0,\ u \to 0$ and as $v \to \infty,\ u \to \infty$. Then the above integral becomes

$$\begin{aligned}
\langle v^n\rangle &= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty e^{-u^2} u^{n+1} \left(\frac{2kT}{m}\right)^{\frac{n+1}{2}} \cdot \frac{2kT}{m} udu \\
&= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-u^2} u^{n+2} du
\end{aligned}$$

Let $u^2 = t\cdot$ Then $2udu = dt$ As $u \to 0,\ t \to 0$ and as $u \to \infty,\ t \to \infty$. As $u \to 0,\ t \to 0$ and as $u \to \infty,\ t \to \infty$.

$$\begin{aligned}
\langle v^n\rangle &= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+1}{2}} \frac{dt}{2} \\
&= 2\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-t} t^{\frac{n+3}{2}} dt \\
&= \frac{2\pi}{\pi\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}-\frac{3}{2}} \Gamma\left(\frac{n+3}{2}\right) \\
&= \frac{2}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \Gamma\left(\frac{n+3}{2}\right) \\
&= \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}
\end{aligned}$$

since $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. Hence

$$\langle v^n\rangle = \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

Problem 13.1.5

By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

$\boxed{\textbf{Solution}}$ Put $x = e^t$. Then $t = \ln x$ and $dx = e^\prime dt$. As $x \to 0,\ t \to \infty$ and as $x \to 1, t \to 0$.

$$\begin{aligned}
&-\int_0^1 x^k \ln x dx \\
&= -\int_\infty^0 e^{kt} t e^\prime dt \\
&= \int_0^\infty e^{(k+1)t} t dt
\end{aligned}$$

Now put $-(k+1)t = z$. Then

$$dt = -\frac{dz}{(k+1)}$$

As $t \to 0,\ z \to 0$ and as $t \to \infty,\ z \to 0$. Then

$$\begin{aligned}
&-\int_0^1 x^k \ln x dx \\
&= \int_0^\infty e^{(k+1)t} t dt \\
&= \int_0^\infty e^{-z} \left(\frac{z}{-(k+1)}\right)\left(\frac{dz}{-(k+1)}\right) \\
&= \frac{1}{(k+1)^2} \int_0^\infty z e^{-z} dz \\
&= \frac{1}{(k+1)^2} \int_0^\infty z^{2-1} e^{-z} dz \\
&= \frac{1}{(k+1)^2} \Gamma(2) \\
&= \frac{1}{(k+1)^2} \cdot 1! \\
&= \frac{1}{(k+1)^2}
\end{aligned}$$

Hence

$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2}, \quad k > -1$$

Problem 13.1.6

Show that

$$\int_0^\infty e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$$

$\boxed{\textbf{Solution}}$ Consider $x^4 = t$ and put $4x^3 dx = dt$ as $t \to 0$ to $\infty x \to 0$ to ∞ and using

$$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma z$$

and

$$z\Gamma z = \Gamma(z+1)$$

the integral takes the form of

$$\begin{aligned}\frac{1}{4}\int_0^\infty e^{-t} t^{-3/4} dt &= \frac{1}{4}\int_0^\infty e^{-t} t^{1/4-1} dt \\ &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \\ &= \Gamma\left(\frac{5}{4}\right)\end{aligned}$$

Problem 13.1.7

Show that

$$\lim_{x\to 0} \frac{\Gamma(ax)}{\Gamma(x)} = \frac{1}{a}$$

Solution

$$= \lim_{x\to 0} \frac{\left(\frac{ax\Gamma(ax)}{ax}\right)}{\left(\frac{x\Gamma(x)}{x}\right)}$$

$$= \lim_{x\to 0} \left(\frac{\Gamma(ax+1)}{\Gamma(x+1)} \cdot \frac{x}{ax}\right)$$

$$= \frac{1}{a} \lim_{x\to 0} \frac{\Gamma(ax+1)}{\Gamma(x+1)}$$

$$= \frac{1}{a} \frac{\Gamma(1)}{\Gamma(1)}$$

$$= \frac{1}{a}$$

Problem 13.1.8

Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

$\boxed{\textbf{Solution}}$ Recall that

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} \cdot n^2,$$

where $z \neq 0, -1, -2, -3, \cdots$. The denominator shows that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \cdots$

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1} dt \\
&= \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-t)^n}{n!} dt + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{n+z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \left[\frac{t^{n+z}}{n+z}\right]_0^1 + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \left[\frac{1}{n+z} - 0\right] + \int_1^\infty e^{-t} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)} + \int_1^\infty e^{-t} t^{z-1} dt
\end{aligned}$$

The series

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)}$$

shows that the first order poles at all negative integers $z = -n$ has respective residues

$$\frac{(-1)^n}{n!}$$

Problem 13.1.10

Show that, for integer s

(a)

$$\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx=\frac{s!}{2a^{s+1}}$$

(b)

$$\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx=\frac{\Gamma\left(s+\frac{1}{2}\right)}{2a^{s+1/2}}=\frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Solution For (a) Put $ax^2=z\cdot$ Then $2axdx=dz$. This implies

$$dx=\frac{dz}{2\sqrt{az}}$$

As $x\to 0,\ z\to 0$ and as $x\to\infty,\ z\to\infty$. The given integral is

$$\begin{aligned}
&\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx\\
&=\int_0^\infty\left(\sqrt{\frac{z}{a}}\right)^{2s+1}e^{-z}\frac{dz}{2\sqrt{az}}\\
&=\frac{1}{2\sqrt{a}}\int_0^\infty\left(\frac{z}{a}\right)^{\frac{2s+1}{2}}e^{-z}z^{-\frac{1}{2}}dz\\
&=\frac{1}{2a^{\frac{1}{2}}}\cdot\frac{1}{a^{\frac{2s+1}{2}}}\int_0^\infty e^{-z}z^{\frac{2s+1}{2}-\frac{1}{2}}dz\\
&=\frac{1}{2a^{s+1}}\int_0^\infty e^{-z}z^{s}dz\\
&=\frac{1}{2a^{s+1}}\int_0^\infty e^{-z}z^{(s+1)-1}dz\\
&=\frac{1}{2a^{s+1}}\Gamma(s+1)
\end{aligned}$$

since s is an integer, therefore $\Gamma(s+1)=s!$. Hence

$$\int_0^\infty x^{2s+1}\exp\left(-ax^2\right)dx=\frac{s!}{2a^{s+1}}$$

Solution For (b) Put $ax^2=z\cdot$ Then $2axdx=dz$. This implies

$$dx=\frac{dz}{2\sqrt{az}}$$

As $x\to 0,\ z\to 0$ and as $x\to\infty,\ z\to\infty$. The given integral is

$$\begin{aligned}
&\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx\\
&=\int_0^\infty\left(\sqrt{\frac{z}{a}}\right)^{2s}e^{-z}\frac{dz}{2\sqrt{az}}\\
&=\frac{1}{2\sqrt{a}}\int_0^\infty\left(\frac{z}{a}\right)^{s}e^{-z}z^{-\frac{1}{2}}dz\\
&=\frac{1}{2a^{\frac{1}{2}}}\cdot\frac{1}{a^s}\int_0^\infty e^{-z}z^{s-\frac{1}{2}}dz\\
&=\frac{1}{2a^{s+\frac{1}{2}}}\int_0^\infty e^{-z}z^{\left(s+\frac{3}{2}\right)-1}dz\\
&=\frac{1}{2a^{s+\frac{1}{2}}}\Gamma\left(s+\frac{3}{2}\right)
\end{aligned}$$

since

$$\Gamma\left(s+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^s}\cdot(2s-1)!!$$

$$= \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Thus

$$\int_0^\infty x^{2s}\exp\left(-ax^2\right)dx = \frac{\Gamma\left(s+\frac{1}{2}\right)}{2a^{s+\frac{1}{2}}} = \frac{(2s-1)!!}{2a^{s+1}a^s}\sqrt{\frac{\pi}{a}}$$

Problem 13.1.11

Express the coefficient of the n th term of the expansion of $(1+x)^{1/2}$ in powers of x

(a) in terms of factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS.\ a_n = (-1)^{n+1}\frac{(2n-3)!}{2^{2n-2}n!\,(n-2)!} = (-1)^{n+1}\frac{(2n-3)!!}{(2n)!!}, \quad n=2,3,\cdots$$

$\boxed{\textbf{Solution}}$ For (a) the n th term of the expansion of $(1+x)^{1/2}$ in powers of x is:

$$
\begin{aligned}
a_n &= \begin{pmatrix} \frac{1}{2} \\ n-1 \end{pmatrix} \\
&= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} \\
&= \frac{(-1)^{n-1}}{n!\ 2^n}[1.3.5\cdots(2n-3)] \\
&= \frac{(-1)^{n+1}}{n!\ 2^n}\left[\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots(2n-4)\cdot(2n-3)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n-4)}\right] \\
&= \frac{(-1)^n}{n!\ 2^n}\cdot\frac{(2n-3)!}{(n-2)!2^{n-2}} \\
&= (-1)^{n+1}\cdot\frac{(2n-3)!}{2^{2n-2}\cdot n!\ (n-2)!}
\end{aligned}
$$

Therefore,

$$a_n = (-1)^{n+1}\cdot\frac{(2n-3)!}{2^{2n-2}n!\ (n-2)!}, \quad n=1,2,3,\cdots$$

$\boxed{\textbf{Solution}}$ For (b) the n th term expansion of $(1+x)^{1/2}$

$$
\begin{aligned}
a_n &= \begin{pmatrix} -\frac{1}{2} \\ n-1 \end{pmatrix} \\
&= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} \\
&= \frac{(-1)^{n-1}}{n!\ 2^n}[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)] \\
&= (-1)^{n+1}\cdot\left[\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right] \\
&= (-1)^{n+1}\cdot\frac{(2n-3)!!}{(2n)!!}
\end{aligned}
$$

Therefore

$$a_n = (-1)^{n+1}\cdot\frac{(2n-3)!!}{(2n)!!}, \quad \text{for } n=1,2,3,\cdots$$

Problem 13.1.12

Express the coefficient of the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x

(a) in terms of the factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. \quad a_n = (-1)^n \frac{(2n)!}{2^{2n}(n!\,)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad n = 1, 2, 3\cdots$$

Solution For (a) the n th term of the expansion of $(1+x)^{-1/2}$ in powers of x is:

$$a_n = \binom{-\frac{1}{2}}{n-1}$$
$$= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!}$$
$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}$$
$$= \frac{(-1)^n}{n!\,2^n}[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]$$
$$= \frac{(-1)^n}{n!\,2^n}\left[\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(2n-1)\cdot 2n}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$
$$= \frac{(-1)^n}{n!\,2^n}\cdot\frac{(2n)!}{n!\,2^n}$$
$$= (-1)^n\cdot\frac{(2n)!}{2^{2n}\cdot(n!\,)^2}$$

Therefore,

$$a_n = (-1)^n\cdot\frac{(2n)!}{2^{2n}\cdot(n!\,)^2}, \qquad \text{for } n = 1, 2, 3, \cdots$$

Solution For (b) the n th term expansion of $(1+x)^{-1/2}$ in powers of x in terms of the double factorial (!!) functions.

$$a_n = \binom{-\frac{1}{2}}{n-1}$$
$$= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!}$$
$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}$$
$$= \frac{(-1)^n}{n!\,2^n}[1\cdot 3\cdot 5\cdots(2n-1)]$$
$$= (-1)^n\cdot\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$
$$= (-1)^n\cdot\frac{(2n-1)!!}{(2n)!!}$$

Therefore

$$a_n = (-1)^n\cdot\frac{(2n-1)!!}{(2n)!!}, \qquad \text{for } n = 1, 2, 3, \cdots$$

Problem 13.1.14

(*a*) Show that $\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$, where n is an integer.

(*b*) Express $\Gamma\left(\frac{1}{2}+n\right)$ and $\Gamma\left(\frac{1}{2}-n\right)$ separately in terms of $\pi^{1/2}$ and a double factorial function.

$$ANS. \quad \Gamma\left(\frac{1}{2}+n\right)=\frac{(2n-1)!!}{2^n}\pi^{1/2}$$

Solution For (*a*) recall that

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

Putting $z=\frac{1}{2}+n$ in the above relation, it becomes

$$\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(1-\frac{1}{2}-n\right)=\frac{\pi}{\sin\left[\pi\left(\frac{1}{2}+n\right)\right]}$$

$$=\frac{\pi}{\cos(n\pi)}$$

$$=\frac{\pi}{(-1)^n}$$

since $\cos(n\pi)=(-1)^n$ and

$$=(-1)^n\pi$$

Therefore

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$$

where n is an integer.

Solution For (*b*) recall the Legendre's duplication formula,

$$\Gamma(1+z)\Gamma\left(z+\frac{1}{2}\right)=2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

Putting $z=n$ in the above relation, it becomes

$$\Gamma(1+n)\Gamma\left(n+\frac{1}{2}\right)=2^{-2n}\sqrt{\pi}\Gamma(2n+1)$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{2^{-2n}\sqrt{\pi}\Gamma(2n+1)}{\Gamma(1+n)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{(2n)!}{n!}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{(1\cdot2\cdot3\cdot4\cdot5\cdot\cdots\cdot2n)}{(1\cdot2\cdot3\cdots n)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{n}}\cdot\frac{(1\cdot2\cdot3\cdot4\cdot5\cdot\cdots\cdot2n)}{(2\cdot4\cdot6\cdot\cdots\cdot2n)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{n}}\cdot[1\cdot3\cdot5\cdots(2n-1)]$$

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{\sqrt{\pi}}{2^{n}}\cdot(2n-1)!!\cdots$$

From part (*a*)

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=(-1)^n\pi$$

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n\pi}{\Gamma\left(\frac{1}{2}+n\right)}$$

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n\pi}{\left(\frac{\sqrt{\pi}}{2^n}\cdot(2n-1)!!\right)}$$

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n\cdot 2^n\sqrt{\pi}}{(2n-1)!!}$$

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{\sqrt{\pi}}{2^n}\cdot(2n-1)!! \text{ and } \Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n\cdot 2^n\sqrt{\pi}}{(2n-1)!!}$$

Problem 13.1.6

Prove that

$$|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)| \prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{a+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{a-i\beta}{n}}$$

Multiplying these equations it becomes

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)} &= (\alpha+i\beta)e^{\gamma(a+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(a-i\beta)} \\
&\times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{a+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right] \\
\frac{1}{|\Gamma(\alpha+i\beta)|^2} &= \left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2a}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right] \\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right] \\
&= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{a}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right] \\
&= \left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right] \\
&= \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]
\end{aligned}$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|} = \frac{1}{|\Gamma(\alpha)|}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]^{\frac{1}{2}}$$

$$|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)|\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-\frac{1}{2}}$$

Problem 13.1.17

Show that for n, a positive integer,

$$|\Gamma(n+ib+1)| = \left(\frac{\pi b}{\sinh \pi b}\right)^{1/2} \prod_{s=1}^{n} \left(s^2+b^2\right)^{1/2}$$

Solution Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$

Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} = (\alpha+i\beta)e^{\gamma(\alpha+i\beta)} \prod_{n=1}^{\infty} \left(1+\frac{\alpha+i\beta}{n}\right) e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)} = (\alpha-i\beta)e^{\gamma(\alpha-i\beta)} \prod_{n=1}^{\infty} \left(1+\frac{\alpha-i\beta}{n}\right) e^{\frac{a-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)} \cdot \frac{1}{\Gamma(\alpha-i\beta)} = (\alpha+i\beta)e^{\gamma(a+i\beta)} \cdot (\alpha-i\beta)e^{\gamma(a-i\beta)}$$

$$\times \prod_{n=1}^{\infty} \left[\left(1+\frac{\alpha+i\beta}{n}\right) e^{\frac{a+i\beta}{n}} \cdot \left(1+\frac{\alpha-i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \left(\alpha^2+\beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2a}{n}} \left[\left(1+\frac{\alpha+i\beta}{n}\right) \cdot \left(1+\frac{\alpha-i\beta}{n}\right)\right]$$

$$= \left(\alpha^2+\beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right) \cdot \left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2} \cdot \left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2+\beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1+\frac{\alpha+i\beta}{n}\right) \cdot \left(1+\frac{\alpha-i\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^2} \cdot \left(1+\frac{\alpha}{n}\right)^2\right]$$

$$= \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{a}{n}} \cdot \left(1+\frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]$$

$$= \left(1+\frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{\left(1+2\alpha n+\alpha^2+\beta^2\right)}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1+\frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1+\frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha=1$ and $\beta=b$ in the above identity. Then it becomes

$$\frac{1}{|\Gamma(1+ib)|^2} = \frac{1}{\Gamma(1)^2} \prod_{n=0}^{\infty} \left[1+\frac{b^2}{(n+1)^2}\right]$$

$$= \prod_{n=0}^{\infty}\left[1+\frac{b^2}{(n+1)^2}\right], \quad \text{as} \quad \Gamma(1)=1$$

$$= \prod_{n=0}^{\infty}\left[1-\frac{(ib\pi)^2}{(n+1)^2\pi^2}\right]$$

$$= \prod_{n=1}^{\infty}\left[1-\frac{(ib\pi)^2}{n^2\pi^2}\right]$$

$$= \frac{1}{(ib\pi)}\left\{(ib\pi)\prod_{n=1}^{\infty}\left[1-\frac{(ib\pi)^2}{n^2\pi^2}\right]\right\}$$

$$= \frac{1}{ib\pi}\cdot\sin(ib\pi)$$

Using the identy

$$\sin z = z\prod_{n=1}^{\infty}\left[1-\frac{z^2}{n^2\pi^2}\right] \quad \text{for} z = ib\pi$$

$$= \frac{1}{ib\pi}\cdot i\sinh(b\pi)$$

$$= \frac{\sinh(b\pi)}{b\pi}$$

$$\frac{1}{|\Gamma(1+ib)|^2} = \frac{\sinh(b\pi)}{b\pi}$$

$$|\Gamma(1+ib)|^2 = \frac{b\pi}{\sinh(b\pi)}.$$

since n is an integer, therefore

$$\Gamma(n+ib+1) = \Gamma(\{1+ib+(n-1)\}+1)$$

$$= \{1+ib+(n-1)\}\Gamma(\{1+ib+(n-1)\})$$

$$(1+ib)(2+ib)(3+ib)\cdots(n+ib)\Gamma(1+ib)$$

$$\Gamma(n+ib+1) = (1+ib)(2+ib)(3+ib)\cdots(n+ib)\Gamma(1+ib)$$

$$\Gamma(n-ib+1) = (1-ib)(2-ib)(3-ib)\cdots(n-ib)\Gamma(1-ib)$$

$$|\Gamma(n+ib+1)|^2$$

$$= \Gamma(n+ib+1)\Gamma(n-ib+1)$$

$$= (1+ib)(2+ib)(3+ib)\cdots(n+ib)\Gamma(1+ib)\times(1-ib)(2-ib)(3-ib)\cdots(n-ib)\Gamma(1-ib)$$

$$= \{(1+ib)(1-ib)\}\{(2+ib)(2-ib)\}\{(3+ib)(3-ib)\}\cdots\{(n+ib)(n-ib)\}\Gamma(1+ib)\Gamma(1-ib)$$

$$= \left(1^2+b^2\right)\left(2^2+b^2\right)\left(3^2+b^2\right)\cdots\left(n^2+b^2\right)|\Gamma(1+ib)|^2$$

$$= \prod_{s=1}^{n}\left(s^2+b^2\right)\times\frac{b\pi}{\sinh(b\pi)}$$

Hence

$$|\Gamma(n+ib+1)|^2 = \prod_{s=1}^{n}\left(s^2+b^2\right)\times\frac{b\pi}{\sinh(b\pi)}$$

This gives

$$|\Gamma(n+ib+1)| = \left(\frac{b\pi}{\sinh(b\pi)}\right)^{\frac{1}{2}}\prod_{s=1}^{n}\left(s^2+b^2\right)^{\frac{1}{2}}$$

Problem 13.1.18

Show that for all real values of x and y, $|\Gamma(x)| \geq |\Gamma(x+iy)|$

$\boxed{\textbf{Solution}}$ Recall

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Putting $z = \alpha + i\beta$ and $z = \alpha - i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} = (\alpha + i\beta)e^{\gamma(\alpha+i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha + i\beta}{n}\right) e^{-\frac{\alpha+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha - i\beta)} = (\alpha - i\beta)e^{\gamma(\alpha-i\beta)} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha - i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha + i\beta)} \cdot \frac{1}{\Gamma(\alpha - i\beta)} = (\alpha + i\beta)e^{\gamma(a+i\beta)} \cdot (\alpha - i\beta)e^{\gamma(a-i\beta)}$$

$$\times \prod_{n=1}^{\infty} \left[\left(1 + \frac{\alpha + i\beta}{n}\right) e^{\frac{a+i\beta}{n}} \cdot \left(1 + \frac{\alpha - i\beta}{n}\right) e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \left(\alpha^2 + \beta^2\right) e^{2\gamma\alpha} \prod_{n=1}^{\infty} e^{-\frac{2a}{n}} \left[\left(1 + \frac{\alpha + i\beta}{n}\right) \cdot \left(1 + \frac{\alpha - i\beta}{n}\right)\right]$$

$$= \left(\alpha^2 + \beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)}{\left(1 + \frac{\alpha}{n}\right)^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\alpha^2 + \beta^2\right) e^{2\gamma a} \prod_{n=1}^{\infty} \left[e^{-\frac{2a}{n}} \cdot \frac{\left(1 + \frac{\alpha+i\beta}{n}\right) \cdot \left(1 + \frac{\alpha-i\beta}{n}\right)}{\left(1 + \frac{\alpha}{n}\right)^2} \cdot \left(1 + \frac{\alpha}{n}\right)^2\right]$$

$$= \left(\frac{\alpha^2 + \beta^2}{\alpha^2}\right) \left(\alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left[e^{-\frac{a}{n}} \cdot \left(1 + \frac{\alpha}{n}\right)\right]\right)^2 \prod_{n=1}^{\infty} \left[\frac{\left(1 + \frac{2\alpha}{n} + \frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]$$

$$= \left(1 + \frac{\beta^2}{\alpha^2}\right) \frac{1}{\Gamma(\alpha)^2} \prod_{n=1}^{\infty} \left[\frac{\left(1 + 2\alpha n + \alpha^2 + \beta^2\right)}{(n + \alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[\frac{(n + \alpha)^2 + \beta^2}{(n + \alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right]$$

$$= \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right]$$

Hence

$$\frac{1}{|\Gamma(\alpha + i\beta)|^2} = \frac{1}{\Gamma(\alpha)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + \alpha)^2}\right]$$

Now put $\alpha = x$ and $\beta = y$ in the above identity. Then it becomes

$$\frac{1}{|\Gamma(x + iy)|^2} = \frac{1}{\Gamma(x)^2} \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + x)^2}\right]$$

$$\left|\frac{\Gamma(x)}{\Gamma(x + iy)}\right|^2 = \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(n + x)^2}\right]$$

$$\left|\frac{\Gamma(x)}{\Gamma(x + iy)}\right|^2 \geq 1, \quad \text{since} \quad 1 + \frac{\beta^2}{(n + x)^2} \geq 1$$

$$\left|\frac{\Gamma(x)}{\Gamma(x + iy)}\right| \geq 1$$

$$|\Gamma(x)| \geq |\Gamma(x + iy)|$$

Hence is proved

Problem 13.1.19

Show that

$$\left|\Gamma(\frac{1}{2}+iy)\right|^2=\frac{\pi}{\cosh \pi y}$$

Solution Recall

$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Putting $z=\alpha+i\beta$ and $z=\alpha-i\beta$ successively in the above relation, it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)}=(\alpha+i\beta)e^{\gamma(\alpha+i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha+i\beta}{n}\right)e^{-\frac{a+i\beta}{n}}$$

and

$$\frac{1}{\Gamma(\alpha-i\beta)}=(\alpha-i\beta)e^{\gamma(\alpha-i\beta)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{a-i\beta}{n}}$$

Multiplying these equations it becomes

$$\frac{1}{\Gamma(\alpha+i\beta)}\cdot\frac{1}{\Gamma(\alpha-i\beta)}=(\alpha+i\beta)e^{\gamma(a+i\beta)}\cdot(\alpha-i\beta)e^{\gamma(a-i\beta)}$$

$$\times\prod_{n=1}^{\infty}\left[\left(1+\frac{\alpha+i\beta}{n}\right)e^{\frac{a+i\beta}{n}}\cdot\left(1+\frac{\alpha-i\beta}{n}\right)e^{\frac{\alpha-i\beta}{n}}\right]$$

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2}=\left(\alpha^2+\beta^2\right)e^{2\gamma\alpha}\prod_{n=1}^{\infty}e^{-\frac{2a}{n}}\left[\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)\right]$$

$$=\left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{(1+\frac{\alpha}{n})^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$=\left(\alpha^2+\beta^2\right)e^{2\gamma a}\prod_{n=1}^{\infty}\left[e^{-\frac{2a}{n}}\cdot\frac{\left(1+\frac{\alpha+i\beta}{n}\right)\cdot\left(1+\frac{\alpha-i\beta}{n}\right)}{(1+\frac{\alpha}{n})^2}\cdot\left(1+\frac{\alpha}{n}\right)^2\right]$$

$$=\left(\frac{\alpha^2+\beta^2}{\alpha^2}\right)\left(\alpha e^{\gamma\alpha}\prod_{n=1}^{\infty}\left[e^{-\frac{a}{n}}\cdot\left(1+\frac{\alpha}{n}\right)\right]\right)^2\prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2+\beta^2}{n^2}\right)}{\frac{(n+\alpha)^2}{n^2}}\right]$$

$$=\left(1+\frac{\beta^2}{\alpha^2}\right)\frac{1}{\Gamma(\alpha)^2}\prod_{n=1}^{\infty}\left[\frac{(1+2\alpha n+\alpha^2+\beta^2)}{(n+\alpha)^2}\right]$$

$$=\frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[\frac{(n+\alpha)^2+\beta^2}{(n+\alpha)^2}\right]$$

$$=\frac{1}{\Gamma(\alpha)^2}\cdot\left(1+\frac{\beta^2}{\alpha^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

$$=\frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Hence

$$\frac{1}{|\Gamma(\alpha+i\beta)|^2}=\frac{1}{\Gamma(\alpha)^2}\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(n+\alpha)^2}\right]$$

Now put $\alpha=\frac{1}{2}$ and $\beta=y$ in the above identity. Then it becomes

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2}=\frac{1}{\Gamma\left(\frac{1}{2}\right)^2}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

Recall

$$\cos z = \prod_{n=1}^{\infty}\left[1-\frac{z^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right]$$

and putting $z = i\pi y$ it becomes

$$\cos(i\pi y) = \prod_{n=1}^{\infty}\left[1-\frac{i^2\pi^2y^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right]$$

$$\cosh(\pi y) = \prod_{n=1}^{\infty}\left[1+\frac{y^2}{\left(n-\frac{1}{2}\right)^2}\right]$$

$$\cosh(\pi y) = \prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+1-\frac{1}{2}\right)^2}\right]$$

$$\cosh(\pi y) = \prod_{n=0}^{\infty}\left[1+\frac{y^2}{\left(n+\frac{1}{2}\right)^2}\right]$$

$$\frac{1}{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2} = \frac{1}{\pi}\cosh(\pi y)$$

Problem 13.1.20

The probability density associated with the normal distribution of statistics is given by

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

with $(-\infty, \infty)$ for the range of x. Show that (a)

(a) $\langle x \rangle$, the mean value of x, is equal to μ

(b) the standard deviation $\left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2}$ is given by σ.

Solution For (a) For the mean

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx$$
$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $x - \mu = y$. Then $dx = dy$. As $x \to 0,\ y \to 0$ and $x \to \infty,\ y \to \infty$.

$$\langle x \rangle = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\mu + y) e^{-\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\mu + y) e^{\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy$$

Since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{2\sigma^2}} dy = 2 \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^{\infty} y e^{\frac{y^2}{2\sigma^2}} dy = 0$$

Therefore, the integral becomes

$$\langle x \rangle = \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz\cdot$ This implies $dy = \frac{\sigma^2}{y} dz$, that is,

$$dy = \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$$

As $y \to 0,\ z \to 0$ and $y \to \infty,\ z \to \infty$. Therefore

$$\langle x \rangle = \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-z} \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$$
$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}}\sqrt{\pi}$$
$$= \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}}\sqrt{\pi}$$
$$= \mu$$

$\boxed{\textbf{Solution}}$ For (b) we start saying

$$\langle x^2 \rangle = \int_0^\infty x^2 f(x)dx$$
$$= \int_{-\infty}^\infty x^2 \cdot \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $x - \mu = y$. Then $dx = dy$. As $x \to 0,\ y \to 0$ and $x \to \infty,\ y \to \infty$.

$$\langle x^2 \rangle = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty (\mu + y)^2 e^{-\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty \left(\mu^2 + 2\mu y + y^2\right) e^{\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy + \frac{2\mu}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy + \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy$$

Since $e^{-\frac{y^2}{2\sigma^2}}$ is an even function, therefore

$$\int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}} dy = 2\int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy$$

Since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^\infty y e^{-\frac{y^2}{2\sigma^2}} dy = 0$$

Since $ye^{-\frac{y^2}{2\sigma^2}}$ is an odd function, therefore

$$\int_{-\infty}^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy = 2\int_0^\infty y^2 e^{-\frac{y^2}{2\sigma^2}} dy$$

Therefore the above integral becomes

$$\langle x^2 \rangle = \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{\frac{y^2}{2\sigma^2}} dy$$

Put $\frac{y^2}{2\sigma^2} = z$, then $2ydy = 2\sigma^2 dz$. This implies $dy = \frac{\sigma^2}{y}dz$, that is, $dy = \frac{\sigma}{\sqrt{2}}z^{-\frac{1}{2}}dz$ As $y \to 0,\ z \to 0$ and $y \to \infty,\ z \to \infty$. Therefore

$$\langle x^2 \rangle = \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{\frac{y^2}{2\sigma^2}} dy + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty y^2 e^{\frac{y^2}{2\sigma^2}} dy$$
$$= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{\frac{1}{2}} dz + \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty 2\sigma^2 z e^{-z} \cdot \frac{\sigma}{\sqrt{2}} z^{-\frac{1}{2}} dz$$
$$= \frac{2\mu^2}{\sigma(2\pi)^{\frac{1}{2}}} \cdot \frac{\sigma}{\sqrt{2}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz + \frac{2\sqrt{2}\sigma^3}{\sigma(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-z} z^{\frac{1}{2}} dz$$

$$
\begin{aligned}
&= \frac{\mu^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{\frac{3}{2}-1} dz \\
&= \frac{\mu^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{\mu^2}{\sqrt{\pi}} \cdot \sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\pi} \\
&= \mu^2 + \sigma^2
\end{aligned}
$$

So the standard deviation

$$
\begin{aligned}
\left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{\frac{1}{2}} &= \left(\mu^2 + \sigma^2 - \mu^2\right)^{\frac{1}{2}} \\
\left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{\frac{1}{2}} &= \sqrt{\sigma^2} \\
\left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{\frac{1}{2}} &= \sigma
\end{aligned}
$$

Problem 13.1.21

For the gamma distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(a) $\langle x \rangle$, the mean value of x, is equal to $\alpha\beta$

(b) σ^2, its variance, defined as $\langle x^2 \rangle - \langle x \rangle^2$, has the value $\alpha\beta^2$

$\boxed{\textbf{Solution}}$ For (a) the mean

$$\langle x \rangle = \int_0^\infty x f(x) dx$$

$$= \int_0^\infty x \cdot \frac{1}{\beta^a \Gamma(\alpha)} x^{a-1} e^{-\frac{x}{\beta}} dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^a e^{-\frac{x}{\beta}} dx$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \to 0,\ z \to 0$ and $x \to \infty,\ z \to \infty$.

$$\langle x \rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty z^a e^{-z} \beta dz$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{(a+1)-1} e^{-z} dz$$

$$= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha + 1)$$

$$= \frac{\beta}{\Gamma(\alpha)} \cdot \alpha \Gamma(\alpha)$$

$$= \alpha\beta$$

$\boxed{\textbf{Solution}}$ For (b)

$$\langle x^2 \rangle = \int_0^\infty x^2 f(x) dx$$

$$= \int_0^\infty x^2 \cdot \frac{1}{\beta^a \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha+1} e^{-\frac{x}{\beta}} dx$$

Put $\frac{x}{\beta} = z$. Then $dx = \beta dz$. As $x \to 0,\ z \to 0$ and $x \to \infty,\ z \to \infty$

$$\langle x^2 \rangle = \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{a+1} e^{-z} \beta dz$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+2)-1} e^{-z} dz$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha + 2)$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \cdot (\alpha + 1)\alpha\Gamma(\alpha)$$

$$= \alpha(\alpha + 1)\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

Hence variance, $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$