

Problem 13.4.1

Rewrite Stirling's series to give $\Gamma(z+1)$ instead of $\ln \Gamma(z+1)$

$$\text{ANS. } \Gamma(z+1) = \sqrt{2\pi} z^{z+1/2} e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51,840z^3} + \dots \right)$$

Solution Consider the Stirling's formula:

$$\ln \Gamma(z+1) = \frac{1}{2} \ln 2\pi + \left(z + \frac{1}{2} \right) \ln z - z + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}}$$

Where B_{2n} are the Bernoulli's numbers. Use the first few Bernoulli's numbers and rewrite the above Stirling's formula as equivalent to

$$\ln \Gamma(z+1) \sim \frac{1}{2} \ln(2\pi) + \left(z + \frac{1}{2} \right) \ln z - z + \frac{1}{12z} - \frac{1}{360z^2} + \frac{1}{1260z^3} - \dots$$

The Stirling's formula can be rewritten using Gamma function as follows. Let us take exponential form and collect similar terms to get equivalent form as follows.

$$\Gamma(z+1) \sim \sqrt{2\pi} + z^{(z+\frac{1}{2})} e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \dots \right)$$

Hence, the required result is

$$\Gamma(z+1) \sim \sqrt{2\pi} + z^{(z+\frac{1}{2})} e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \dots \right)$$

Problem 13.4.5

Test the convergence

$$\sum_{p=0}^{\infty} \left[\frac{\Gamma(p+\frac{1}{2})}{p!} \right]^2 \frac{2p+1}{2p+2} = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

Solution Consider the series obtained in the magnetic field created by and enclosed by a current loop:

$$\sum_{p=0}^{\infty} \frac{\Gamma(p+\frac{1}{2})}{p!} \left(\frac{2p+1}{2p+2} \right) = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}$$

Now, we will test the convergence of the series using Stirling asymptotic formula given by

$$\Gamma(z+1) \sim \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}$$

$$\begin{aligned} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} &\sim \sqrt{e} \frac{\left(\frac{p+\frac{1}{2}}{p+1} \right)^{p+\frac{1}{2}}}{\Gamma(p+1)} \\ &= \frac{\text{constant}}{\Gamma(p+1)} \end{aligned}$$

Hence, the series converges.

Problem 13.4.6

Show that

$$\lim_{x \rightarrow \infty} x^{b-a} \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1$$

Solution For a large n , the Stirling asymptotic formula can be taken to the n arbitrary closed to infinite. Then the expression has asymptotic limit.

$$\begin{aligned}
 & \ln \left[(x^{b-a}) \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \right] \\
 &= (b-a) \ln x + \ln \left(\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \right) \\
 &= (b-a) \ln(x) + \ln \left(\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \right) \\
 &= (b-a) \ln(x) + \ln \Gamma(x+a+1) - \ln \Gamma(x+b+1)
 \end{aligned}$$

Now we use

$$\ln \Gamma(z+1) = \left(z + \frac{1}{z} \right) \ln z - z$$

Now, $(b-a) \ln(x) + \ln \Gamma(x+a+1) - \ln \Gamma(x+b+1)$ it reduces to

$$\begin{aligned}
 & (b-a) \ln(x) + \ln \Gamma(x+a+1) - \ln \Gamma(x+b+1) \\
 &= -(x+a) - \left(x+b + \frac{1}{2} \right) \ln(x+b) + (x+b) \\
 &= (b-a) \ln(x) + (a-b) \ln(x)
 \end{aligned}$$

Rewrite the $\ln(x+a)$ as follows.

$$\begin{aligned}
 \ln(a+x) &= \ln x \left(1 + \frac{a}{x} \right) \\
 &= \ln x + \ln \left(1 + \frac{a}{x} \right) \\
 &= \ln x + \frac{a}{x} + \dots
 \end{aligned}$$

Now rewrite the $\ln(x+b)$

$$\begin{aligned}
 \ln(b+x) &= \ln x \left(1 + \frac{b}{x} \right) \\
 &= \ln x + \ln \left(1 + \frac{b}{x} \right) \\
 &= \ln x + \frac{b}{x} + \dots
 \end{aligned}$$

For large x , make all the terms to exponential form. So, that $\exp(0) = 1$. Hence, the limit tends to 1.

$$\lim_{x \rightarrow \infty} x^{b-a} \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1$$

Problem 13.4.7

Show that

$$\lim_{n \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{1/2} = \pi^{-1/2}$$

Solution Write the limit expression in factorial notations. Then it is easy to apply the Stirling formula

$$\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \lim_{x \rightarrow \infty} \frac{(2n)!}{2^{2n}(n!)^2} n^{\frac{1}{2}}$$

Take logarithm for the limit

$$\ln \left(\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} \right) = \ln \left(\lim_{x \rightarrow \infty} \frac{(2n)!}{2^{2n}(n!)^2} n^{\frac{1}{2}} \right)$$

Consider the right hand side of the above equation and solve.

$$\ln \lim_{n \rightarrow \infty} \frac{(2n)! n^{\frac{1}{2}}}{2^{2n}(n!)^2}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \ln(2n)! + \frac{1}{2} \ln n - 2n \ln 2 - 2 \ln(n!) \\
&\quad \frac{\ln(2\pi)}{2} + \left(2n + \frac{1}{2}\right) \ln(2n) - 2n + \frac{\ln n}{2} \\
&\approx -2n \ln 2 - \ln(2\pi) - 2 \left(n + \frac{1}{2}\right) \ln n + 2n + \dots \\
&\quad \sim -\frac{1}{2} \ln \pi \\
&= \ln \pi^{-\frac{1}{2}}
\end{aligned}$$

Substitute the value of right hand side limit

$$\begin{aligned}
&\ln \left(\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} \right) = \ln \pi^{-\frac{1}{2}} \\
&\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \pi^{-\frac{1}{2}}
\end{aligned}$$

Hence, the limit tends to

$$\lim_{x \rightarrow \infty} \frac{(2n-1)!!}{(2n)!!} n^{\frac{1}{2}} = \pi^{-\frac{1}{2}}$$