Problem 3.4.1

Another set of Euler rotations in common use is
(a) a rotation about the x_3 -axis through an angle φ, counterclockwise,

(b) a rotation about the x_1' -axis through an angle θ, counterclockwise,

(c) a rotation about the x_3'' -axis through an angle ψ, counterclockwise.

If

$$\begin{aligned} \alpha &= \varphi - \pi/2 \\ \beta &= \theta \\ \gamma &= \psi + \pi/2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \varphi &= \alpha + \pi/2 \\ \theta &= \beta \\ \psi &= \gamma - \pi/2 \end{aligned}$$

show that the final systems are identical.

Solution The Euler rotations given in the text is:

1. a rotation about the x_3- axis through an angle α, counterclockwise
2. a rotation about the x_2' - axis through an angle β, counterclockwise
3. a rotation about the x_3'' -axis through an angle γ, counterclockwise.

The Euler rotation defined here differ from those in the text in that the inclination of the polar axis is about that $x_1'-$axis rather than the $x_2'-$ axis. Therefore, to achieve the same polar orientation, we must place the $x_1'-$axis where the $x_2'-$axis was using the text rotation. This requires an additional first rotation of $\frac{\pi}{2}$. After inclining the polar axis, the rotational position is now $\frac{\pi}{2}$ greater than form the text rotation, so the third Euler angle must be $\frac{\pi}{2}$ less than its original value.

Problem 3.4.2

Suppose the Earth is moved (rotated) so that the north pole goes to 30° north, 20° west (original latitude and longitude system) and the 10° west meridian points due south (also in the original system). (a) What are the Euler angles describing this rotation? (b) Find the corresponding direction cosines.

Solution No solution yet.

Problem 3.4.3

Verify that the Euler angle rotation matrix, Eq. (3.37), is invariant under the transformation

$$\alpha \to \alpha + \pi, \quad \beta \to -\beta, \quad \gamma \to \gamma - \pi$$

Solution The Euler rotation matrix $\mathbf{S}(\alpha, \beta, \gamma)$ is :

$$\mathbf{S}(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

Using the transformation $\alpha \to \alpha + \pi, \beta \to -\beta, \gamma \to \gamma - \pi$ we get,

$$\mathbf{S}(\alpha+\pi,-\beta,\gamma-\pi) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

as $\cos\alpha \to -\cos\alpha, \sin\alpha \to -\sin\alpha; \cos\beta \to \cos\beta, \sin\beta \to -\sin\beta; \sin\gamma \to -\sin\gamma$, $\cos\gamma \to -\cos\gamma$ Thus, $\mathbf{S}(\alpha,\beta,\gamma) = \mathbf{S}(\alpha+\pi,-\beta,\gamma-\pi)$ Hence, $\mathbf{S}(\alpha,\beta,\gamma)$ is invariant under the transformation
$\alpha \to \alpha + \pi, \beta \to -\beta, \gamma \to \gamma - \pi$

Problem 3.4.4

Show that the Euler angle rotation matrix $\mathbf{S}(\alpha,\beta,\gamma)$ satisfies the following relations:

(a) $\mathbf{S}^{-1}(\alpha,\beta,\gamma) = \tilde{\mathbf{S}}(\alpha,\beta,\gamma)$

(b) $\mathbf{S}^{-1}(\alpha,\beta,\gamma) = \mathbf{S}(-\gamma,-\beta,-\alpha)$

$\boxed{\textbf{Solution}}$ For (a) The three Euler rotations $\boldsymbol{S}_1(\alpha), \boldsymbol{S}_2(\beta), \boldsymbol{S}_3(\gamma)$ are an orthogonal matrix. So, $\mathbf{S}(\alpha,\beta,\gamma) = \boldsymbol{S}_3(\gamma)\boldsymbol{S}_2(\beta)\boldsymbol{S}_1(\alpha)$ must also be orthogonal. Therefore $\mathbf{S}^{-1}(\alpha,\beta,\gamma) = \tilde{\mathbf{S}}(\alpha,\beta,\gamma)$, by the definition of an orthogonal matrix.

$\boxed{\textbf{Solution}}$ For (b) we have

$$\mathbf{S}(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

$$\mathbf{S}(-\gamma,-\beta,-\alpha) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & \sin\beta\cos\alpha \\ \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}^{-1}(\alpha,\beta,\gamma) &= \tilde{\mathbf{S}}(\alpha,\beta,\gamma) \\ &= \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & \sin\beta\cos\alpha \\ \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{bmatrix} \end{aligned}$$

Thus, $\mathbf{S}^{-1}(\alpha,\beta,\gamma) = \mathbf{S}(-\gamma,-\beta,-\alpha)$

Problem 3.4.5

The coordinate system (x, y, z) is rotated through an angle Φ counterclockwise about an axis defined by the unit vector $\hat{\mathbf{n}}$ into system (x', y', z'). In terms of the new coordinates the radius vector becomes

$$\mathbf{r}' = \mathbf{r}\cos\Phi + \mathbf{r}\times\mathbf{n}\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)$$

(a) Derive this expression from geometric considerations.

(b) Show that it reduces as expected for $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$. The answer, in matrix form, appears in Eq. (3.35)

(c) Verify that $r'^2 = r^2$.

$\boxed{\textbf{Solution}}$ For (a) the projection of r on the rotation axis is not changed by the rotation; it is $(\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. The portion of r perpendicular to the rotation axis can be written $r - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. Upon rotation through an angle Φ, this vector perpendicular to the rotation axis will consist of a vector in its original direction $(r - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}})\cos\Phi$ plus a vector perpendicular both to it and to $\hat{\mathbf{n}}$ given by $(r - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}})\sin\Phi\times\hat{\mathbf{n}}$; this reduces to $\mathbf{r}\times\hat{\mathbf{n}}\sin\Phi$ Adding these contributions, we get the required result.

$\boxed{\textbf{Solution}}$ For (b) if $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$, the formula $\mathbf{r}' = \mathbf{r}\cos\Phi + \mathbf{r}\times n\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)$ becomes

$$\begin{aligned} \mathbf{r}' &= (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z)\cos\Phi + (y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\sin\Phi + \hat{\mathbf{e}}_z\,(z\hat{\mathbf{e}}_z)\,(1-\cos\Phi) \\ &= (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z)\cos\Phi + (y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\sin\Phi + z(1-\cos\Phi)\hat{\mathbf{e}}_z \\ &= x\cos\Phi\hat{\mathbf{e}}_x + y\cos\Phi\hat{\mathbf{e}}_y + z\cos\Phi\hat{\mathbf{e}}_z + y\sin\Phi\hat{\mathbf{e}}_x - x\sin\Phi\hat{\mathbf{e}}_y + z(1-\cos\Phi)\hat{\mathbf{e}}_z \end{aligned}$$

as $r = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$, $\quad \mathbf{r}\times n = \mathbf{r}\times\hat{\mathbf{e}}_z = y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y$ and Simplifying, this reduces to

$$\mathbf{r}' = (x\cos\Phi + y\sin\Phi)\hat{\mathbf{e}}_x + (y\cos\Phi - x\sin\Phi)\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$$

This corresponds to the rotational transformation whose matrix form is

$$\boldsymbol{S}_1(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\textbf{Solution}}$ For (c) we expand r'^2, recognizing that the second term of

$$\mathbf{r}' = \mathbf{r}\cos\Phi + \mathbf{r}\times n\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)$$

$$\begin{aligned} r'^2 &= \mathbf{r}'\cdot\mathbf{r}' \\ &= (\mathbf{r}\cos\Phi + \mathbf{r}\times\hat{\mathbf{n}}\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi))\cdot(\mathbf{r}\cos\Phi + \mathbf{r}\times\hat{\mathbf{n}}\sin\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi)) \\ &= r^2\cos^2\Phi + (\mathbf{r}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin\Phi\cos\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r})\sin\Phi\cos\Phi \\ &+ (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{r})\sin\Phi(1-\cos\Phi) + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi \\ &+ (\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{r})\sin\Phi(1-\cos\Phi) + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 \end{aligned}$$

$$r'^2 = r^2\cos^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 + 2(\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi$$

as $(\mathbf{r}\cdot\mathbf{r}\times\hat{\mathbf{n}}) = (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}) = (\mathbf{r}\times\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{r}) = 0$

$$\begin{aligned}
r'^2 &= r^2\cos^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}}\cdot\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 + 2(\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)\cos\Phi \\
&= r^2 + (\hat{\mathbf{n}}\cdot\mathbf{r})^2\left(-\sin^2\Phi + 1 + \cos^2\Phi - 2\cos^2\Phi\right) \\
&= r^2
\end{aligned}$$