Problem 5.1.1

A function $f(x)$ is expanded in a series of orthonormal functions

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

Show that the series expansion is unique for a given set of $\varphi_n(x)$. The functions $\varphi_n(x)$ are being taken here as the basis vectors in an infinite-dimensional Hilbert space.

Solution Consider the Orthonormal function:

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

The objective is to show that the series expansion is unique for $\phi_n(x)$. Here, the functions $\phi_n(x)$ are as the basis vectors in an infinite-dimensional Hilbert space. If the functions ϕ_i are orthogonal and

$$f = \sum_{i=1}^{n} a_i \phi_i,$$

then the scalar

$$a_i = \frac{\langle \phi_i | f \rangle}{\langle \phi_i | \phi_i \rangle}$$

Using the Orthogonality definition, the value of $\langle \phi_n | f \rangle$ is,

$$\begin{aligned} \langle \phi_n | f \rangle &= a_n \\ &= \int_a^b w(x) f(x) \phi_n(x) dx \end{aligned}$$

This is derived from the function f. Assume that $\langle \phi_n | f \rangle = a'_n$

$$\begin{aligned} \langle \phi_n | f \rangle &= a'_n \\ &= \int_a^b w'(x) f(x) \phi_n(x) dx \end{aligned}$$

Then, $a_n = a_n{}'$ since $w(x) = w'(x)$ Therefore, $\langle \phi_n | f \rangle = a_n$ is unique.

Problem 5.1.2

A function $f(x)$ is represented by a finite set of basis functions $\varphi_i(x)$

$$f(x) = \sum_{i=1}^{N} c_i \varphi_i(x)$$

Show that the components c_i are unique, that no different set c_i' exists. Note. Your basis functions are automatically linearly independent. They are not necessarily orthogonal.

Solution Consider the function:

$$f(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

The objective is to show that the components c_i are unique. The function can be written as,

$$\begin{aligned} f(x) &= \sum_i c_i \phi_i(x) \\ &= \sum_j c_j' \phi_j(x) \end{aligned}$$

Then,

$$\begin{aligned} \sum_i (c_i - c_i')\, \phi_i &= \sum_i c_i \phi_i - \sum_i c_i' \phi_i \\ &= \sum_i c_i \phi_i - \sum_i c_i \phi_i \\ &= 0 \end{aligned}$$

Assume $c_m - c_m' \neq 0$ Then,

$$\phi_m = \frac{-1}{c_m - c_m} \sum_{b=m} (c_i - c_i')\, \phi_i$$

It confirms that, ϕ_m is not linearly independent of the ϕ_i, which is a contradiction to our assumption. So, $c_m - c_m' = 0$ Therefore, the scalars c_i are unique.

Problem 5.1.3

A function $f(x)$ is approximated by a power series $\sum_{i=0}^{n-1} c_i x^i$ over the interval [0,1] Show that minimizing the mean square error leads to a set of linear equations

$$\mathrm{Ac} = \mathbf{b}$$

where

$$A_{ij} = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1}, \quad i, j = 0, 1, 2, \ldots, n-1$$

and

$$b_i = \int_0^1 x^i f(x) dx, \quad i = 0, 1, 2, \ldots, n-1$$

Note. The A_{ij} are the elements of the Hilbert matrix of order n. The determinant of this Hilbert matrix is a rapidly decreasing function of n. For $n = 5$, $\det A = 3.7 \times 10^{-12}$ and the set of equations $\mathrm{A}c = \mathbf{b}$ is becoming ill-conditioned and unstable.

Solution For

$$f(x) = \sum_{i=0}^{n-1} c_i x^i$$

we have

$$\begin{aligned} b_j &= \int_0^1 x^j f(x) dx, \quad j = 0, 1, 2, \ldots, n-1 \\ &= \sum_l c_i \int_0^1 x^{i+j} dx \\ &= \sum_{i=0}^{n-1} \frac{c_i}{i+j+1} \\ &= A_{ji} c_i \end{aligned}$$

This result also minimizing the mean square error

$$\int_0^1 \left[f(x) - \sum_{i=0}^{n-1} c_i x^i \right]^2 dx$$

upon varying the c_i

Problem 5.1.4

In place of the expansion of a function $F(x)$ given by

$$F(x)=\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

with

$$a_n=\int_a^b F(x)\varphi_n(x)w(x)dx$$

take the finite series approximation

$$F(x)\approx\sum_{n=0}^{m}c_n\varphi_n(x)$$

Show that the mean square error

$$\int_a^b\left[F(x)-\sum_{n=0}^{m}c_n\varphi_n(x)\right]^2 w(x)dx$$

is minimized by taking $c_n=a_n$
Note. The values of the coefficients are independent of the number of terms in the finite series. This independence is a consequence of orthogonality and would not hold for a least-squares fit using powers of x.

Solution Consider the function

$$F(x)=\sum_{n=0}^{\infty}a_n\phi_n(x)$$

Here,

$$a_n=\int_a^b F(x)\phi_n(x)w(x)dx$$

and

$$F(x)\approx\sum_{n=0}^{m}c_n\phi_n(x)$$

The objective is to show the mean square error is minimized when $c_n=a_n$. For

$$F(x)=\sum_{n=0}^{m}a_n\phi_n(x),$$

we have

$$\begin{aligned}c_j&=\int_0^1 x^j f(x)dx, j=0,1,2,\dots,m\\&=\sum_i a_i\int_0^1 x^{j+j}dx\\&=\sum_{i=0}^{m}\frac{a_i}{i+j+1}\\&=A_{ji}a_i\end{aligned}$$

Note that A_{ij} 's represents the elements of the Hilbert matrix of order n. The determinant of this Hilbert matrix is a decreasing function of n. Write the function as

$$F(x)=\sum_{n=0}^{m}c_n\phi_n(x)$$

$$F(x)-\sum_{n=0}^{m}c_n\phi_n(x)=0$$

$$\int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \phi_n(x)\right]^2 w(x)dx = 0$$

$$\frac{\partial}{\partial c_l} \int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \phi_n(x)\right]^2 w(x)dx = 0$$

Remember that

$$c_n = \int_a^b F(x)\phi_n(x)w(x)dx$$

This result is also minimizing the mean square error

$$\int_a^b \left[F(x) - \sum_{n=0}^{m} c_n \phi_n(x)\right]^2 w(x)dx$$

is minimized when $c_n = a_n$

Problem 5.1.5

The functions $\cos nx$ $(n = 0, 1, 2, \ldots)$ and $\sin nx$ $(n = 1, 2, \ldots)$ have (together) been shown to form a complete set on the interval $-\pi < x < \pi$. since this determination is obtained subject to convergence in the mean, there is the possibility of deviation at isolated points, thereby permitting the description of functions with isolated discontinuities.

$$f(x) = \left\{ \begin{array}{ll} \dfrac{h}{2}, & 0 < x < \pi \\ -\dfrac{h}{2}, & -\pi < x < 0 \end{array} \right\} = \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$$

a) Show that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

For a finite upper limit this would be Bessel's inequality. For the upper limit ∞, this is Parseval's identity.

b) Verify that

$$\frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

by evaluating the series. Hint. The series can be expressed in terms of the Riemann zeta function $\zeta(2) = \pi^2/6$

Solution The objective is to show that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

First, we start saying that the integral $\int_{-\pi}^{\pi} [f(x)]^2 dx$ can be evaluated as

$$\begin{aligned}
\int_{-\pi}^{\pi} [f(x)]^2 dx &= \int_{-\pi}^{\pi} f(x) \cdot f(x) dx \\
&= \int_{-\pi}^{\pi} f(x) dx \cdot \int_{-\pi}^{\pi} f(x) dx \\
&= \int_{-\pi}^{\pi} \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} dx \int_{-\pi}^{\pi} \frac{2h}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1} \\
&= \left(\frac{4h^2}{\pi^2} \right) \sum_{m,n=0}^{\infty} \frac{1}{(2n+1)(2m+1)} \times \int_{-\pi}^{\pi} \sin[(2n+1)x] \sin[(2m+1)x] dx
\end{aligned}$$

$$= \frac{4h^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \int_{-\pi}^{\pi} \sin^2[(2n+1)x] dx$$

$$= \frac{4h^2}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right) \int_{-\pi}^{\pi} \left(\frac{1 - \cos(2(2n+1)x)}{2} \right) dx$$

$$= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8} \right) \left(\frac{x - \frac{\sin(2(2n+1)x)}{2(2n+1)}}{2} \right) \Bigg|_{-\pi}^{\pi}$$

$$= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8} \right) \left(\frac{\pi - \frac{\sin(2(2n+1)\pi)}{2(2n+1)}}{2} - \left(\frac{(-\pi) - \frac{\sin(2(2n+1)(-\pi))}{2(2n+1)}}{2} \right) \right)$$

$$= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8} \right) \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$= \frac{4h^2}{\pi^2} \left(\frac{\pi^2}{8} \right) (\pi)$$

$$= \frac{h^2\pi}{2}$$

Therefore,

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{\pi}{2} h^2$$

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{4h^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \int_{-\pi}^{\pi} \sin^2[(2n+1)x] dx$$

$$= \frac{4h^2}{\pi^2} \sum_{n=0}^{\infty} (2n+1)^{-2} (\pi)$$

$$= \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

Hence,

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{\pi}{2} h^2 = \frac{4h^2}{\pi} \sum_{n=0}^{\infty} (2n+1)^{-2}$$

For (b)

$$\begin{aligned}
\text{RHS} &= \frac{4h^2}{\pi} \left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \right) \\
&= \frac{4h^2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right) \\
&= \frac{4h^2}{\pi} \left(\frac{\pi^2}{8} \right) \\
&= \frac{\pi h^2}{2}
\end{aligned}$$

Problem 5.1.6

Derive the Schwarz inequality from the identity

$$\left[\int_a^b f(x)g(x)dx\right]^2 = \int_a^b [f(x)]^2 dx \int_a^b [[g(x)]^2 dx \\ - \frac{1}{2}\int_a^b dx \int_a^b dy[f(x)g(y) - f(y)g(x)]^2$$

Solution The double integral can be written as,

$$\left[\int_a^b f(x)g(x)dx\right]^2 = \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx \\ - \frac{1}{2}\int_a^b dx \int_a^b dy[f(x)g(y) - f(y)g(x)]^2$$

$$\begin{aligned} |\langle f|g\rangle|^2 &= \langle f\rangle^2\langle g\rangle^2 - \frac{1}{2}\int_a^b\int_a^b [f(x)g(y) - f(y)g(x)]^2 \\ &\leq \langle f\rangle^2\langle g\rangle^2 \\ |\langle f|g\rangle|^2 &\leq \langle f|f\rangle\langle g|g\rangle \end{aligned}$$

since the double integral is non-negative, so $\langle f|g\rangle|^2 \geq 0$. Hence, the result of Schwarz inequality is derived.

Problem 5.1.7

Starting from

$$I = \left\langle f - \sum_i a_i\varphi_i | f - \sum_j a_j\varphi_j \right\rangle \geq 0$$

derive Bessel's inequality,

$$\langle f|f\rangle \geq \sum_n |a_n|^2$$

Solution The functions ϕ_j are assumed to be orthonormal. Expand the value of I, we have

$$\begin{aligned} I &= \left\langle f - \sum_i a_i\phi_i | f - \sum_j a_j\phi_j \right\rangle \\ &= \langle f|f\rangle - \sum_i a_i * \langle \phi_i|f\rangle - \sum_i a_i * \langle f|\phi_i\rangle + \sum_i a_i * a_j \langle \phi_i|\phi_j\rangle \\ &\geq 0 \end{aligned}$$

Hence, the result of Bessel's inequality is derived.

Problem 5.1.8

Expand the function $\sin \pi x$ in a series of functions φ_i that are orthogonal (but not normalized) on the range $0 \leq x \leq 1$ when the scalar product has definition

$$\langle f|g\rangle = \int_0^1 f^*(x)g(x)dx$$

Keep the first four terms of the expansion. The first four φ_i are:

$$\varphi_0 = 1, \quad \varphi_1 = 2x-1, \quad \varphi_2 = 6x^2-6x+1, \quad \varphi_3 = 20x^3-30x^2+12x-1$$

Note. The integrals that are needed are the subject of Example 1.10 .5 .

$\boxed{\textbf{Solution}}$ Consider the function: $\sin(\pi x)$ Expand the function $\sin(\pi x)$ in a series of functions ϕ_i which are orthogonal. Write the function $\sin(\pi x)$ in a series of functions ϕ_i as,

$$\sin(\pi x) = \sum_i \frac{\langle \phi_i|\sin \pi x\rangle}{\langle \phi_i, \phi_i\rangle}\phi_i(x)$$

Here, $\phi_0 = 1, \phi_1 = 2x-1, \phi_2 = 6x^2-6x+1, \phi_3 = 20x^3-30x^2+12x-1$ The integrals are calculated as,

$$\begin{aligned}\langle \phi_0|\phi_0\rangle &= \int_0^1 dx \\ &= (x)\Big|_0^1 \\ &= 1\end{aligned}$$

$$\langle \phi_1|\phi_1\rangle = \int_0^1 (2x-1)^2 dx$$

$$\langle \phi_1|\phi_1\rangle = \int_0^1 \left(4x^2-4x+1\right) dx$$

$$\langle \phi_1|\phi_1\rangle = \left(\frac{4x^3}{3} - 2x^2 + x\right)\Bigg|_0^1$$

$$\langle \phi_1|\phi_1\rangle = \left(\frac{4}{3} - 2 + 1\right)$$

$$\langle \phi_1|\phi_1\rangle = \frac{1}{3}$$

$$\begin{aligned}\langle \phi_2|\phi_2\rangle &= \int_0^1 \left(6x^2-6x+1\right)^2 dx \\ &= \int_0^1 \left(36x^4-72x^3+48x^2-12x+1\right) dx \\ &= \left(\frac{36x^5}{5} - 18x^4 + 16x^3 - 6x^2 + x\right)\Bigg|_0^1 \\ &= \frac{36}{5} - 18 + 16 - 6 + 1 \\ &= \frac{1}{5}\end{aligned}$$

$$
\begin{aligned}
\langle \phi_3|\phi_3\rangle &= \int_0^1 \left(20x^3-30x^2+12x-1\right)^2 dx \\
&= \int_0^1 \left(400x^6-1200x^5+1380x^4-760x^3+204x^2-24x+1\right) dx \\
&= \left(\frac{400x^7}{7}-200x^6+276x^5-190x^4+68x^3-12x^2+x\right)\Bigg|_0^1 \\
&= \frac{400}{7}-200+276-190+68-12+1 \\
&= \frac{1}{7}
\end{aligned}
$$

$$
\begin{aligned}
\langle \phi_0|f\rangle &= \int_0^1 \sin \pi x dx \\
&= \left(\frac{-\cos \pi x}{\pi}\right)\Bigg|_0^1 \\
&= -\left(\frac{\cos \pi(1)}{\pi}-\frac{\cos \pi(0)}{\pi}\right) \\
&= -\left(\frac{-1}{\pi}-\frac{1}{\pi}\right) \\
&= \frac{2}{\pi}
\end{aligned}
$$

The value of $\langle \phi_1|f\rangle$ is,

$$
\langle \phi|f\rangle = \int_0^1 (2x-1)\sin(\pi x)dx
$$

$$
= \left(\frac{2\sin(\pi x)+(\pi-2\pi x)\cos(\pi x)}{\pi^2}\right)\Bigg|_0^1
$$

Using $\int_0^1 (2x-1)\sin(\pi x)dx = \frac{2\sin(\pi x)+(\pi-2\pi x)\cos(\pi x)}{\pi^2}$

$$
= \frac{2\sin(\pi\cdot 1)+(\pi-2\pi\cdot 1)\cos(\pi\cdot 1)}{\pi^2} -
$$

$$
= \frac{2\sin(\pi\cdot 0)+(\pi-2\pi\cdot 0)\cos(\pi\cdot 0)}{\pi^2}
$$

$$
= \frac{2(0)+(-\pi)\cdot 1}{\pi^2} - \left(\frac{2(0)+(-\pi)1}{\pi^2}\right)
$$

$$
= 0
$$

$$
\langle \varphi_2|f\rangle = \frac{2}{\pi}-\frac{24}{\pi^3}
$$

$$
\langle \varphi_3|f\rangle = 0
$$

$$
\sin \pi x = \frac{2/\pi}{1}\varphi_0 + \frac{2/\pi-24/\pi^3}{1/5}\varphi_2 + \cdots
$$

$$
\sin(\pi x) = 0.6366 - 0.6871\left(6x^2-6x+1\right) + \cdots
$$

Problem 5.1.9

Expand the function e^{-x} in Laguerre polynomials $L_n(x)$, which are orthonormal on the range $0 \leq x < \infty$ with scalar product

$$\langle f|g\rangle = \int_0^\infty f^*(x)g(x)e^{-x}dx$$

Keep the first four terms of the expansion. The first four $L_n(x)$ are

$$L_0 = 1, \quad L_1 = 1 - x, \quad L_2 = \frac{2 - 4x + x^2}{2}, \quad L_3 = \frac{6 - 18x + 9x^2 - x^3}{6}$$

Solution The value of a_0 is

$$\begin{aligned} a_0 &= \int_0^\infty L_0(x)e^{-2x}dx \\ &= \int_0^\infty e^{-2x}dx \\ &= \left(\frac{e^{-2x}}{-2}\right)\Big|_0^\infty \\ &= \frac{-1}{2}\left(e^{-2(\alpha)} - e^0\right) \\ &= \frac{-1}{2}(0-1) \\ &= \frac{1}{2} \end{aligned}$$

The value of a_1 is

$$\begin{aligned} a_1 &= \int_0^\infty L_1(x)e^{-2x}dx \\ &= \int_0^\infty (1-x)e^{-2x}dx \\ &= \left(\frac{1}{4}e^{-2x}(2x-1)\right)\Big|_0^\infty \\ &= \frac{1}{4}\left(e^{-2(\infty)}(2(\infty)-1) - e^0(2(0)-1)\right) \\ &= \frac{1}{4}(0+1) \\ &= \frac{1}{4} \end{aligned}$$

The value of a_2 is

$$\begin{aligned} a_2 &= \int_0^\infty L_2(x)e^{-2x}dx \\ &= \int_0^\infty \left(\frac{2-4x+x^2}{2}\right)e^{-2x}dx \\ &= \left(\frac{-1}{8}e^{-2x}\left(1-6x+2x^2\right)\right)\Big|_0^\infty \end{aligned}$$

The value of a_3 is,

$$\begin{aligned} a_3 &= \int_0^\infty L_3(x)e^{-2x}dx \\ &= \int_0^\infty \left(\frac{6-18x+9x^2-x^3}{6}\right)e^{-2x}dx \\ &= \left(\frac{1}{48}e^{-2x}\left(4x^3-30x^2+42x-3\right)\right)\Big|_0^\infty \\ &= \frac{3}{48} \end{aligned}$$

$$= \frac{1}{16}$$

Thus, the expansion of e^{-x} is

$$e^{-x} = a_0 L_0(x) + a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x) + \cdots$$

$$= \frac{1}{2}(1) + \frac{1}{4}(1-x) + \frac{1}{8}\left(\frac{2-4x+x^2}{2}\right) + \frac{1}{16}\left(\frac{6-18x+9x^2-x^3}{6}\right) + \cdots$$

Problem 5.1.10

The explicit form of a function f is not known, but the coefficients a_n of its expansion in the orthonormal set φ_n are available. Assuming that the φ_n and the members of another orthonormal set, χ_n, are available, use Dirac notation to obtain a formula for the coefficients for the expansion of f in the χ_n set.

Solution The coefficients of f in the ϕ basis are $a_i = \langle \phi_i | f \rangle$, so the above equation is equivalent to,

$$f = \sum_j b_j \chi_j$$

Here, $b_j = \sum_j \langle \chi_j | \phi_i \rangle \, a_i$

Problem 5.1.11

Using conventional vector notation, evaluate $\sum_j |\hat{\mathbf{e}}_j\rangle \langle \hat{\mathbf{e}}_j | \mathbf{a} \rangle$, where a is an arbitrary vector in the space spanned by the $\hat{\mathbf{e}}_j$

Solution We assume the unit vectors are orthogonal. Then,

$$\sum_j |\hat{\mathbf{e}}_j\rangle \langle \hat{\mathbf{e}}_j | \mathbf{a} \rangle = \sum_j (\hat{\mathbf{e}}_j \cdot \mathbf{a})\, \hat{\mathbf{e}}_j$$

This expression is a component decomposition of a.

Problem 5.1.12

Letting a $= a_1\hat{\text{e}}_1 + a_2\hat{\text{e}}_2$ and $\mathbf{b} = b_1\hat{\text{e}}_1 + b_2\hat{\text{e}}_2$ be vectors in $\mathbb{R}^2$, for what values of k, if any, is

$$\langle \mathbf{a} | \mathbf{b} \rangle = a_1 b_1 - a_1 b_2 - a_2 b_1 + k a_2 b_2$$

a valid definition of a scalar product?

Solution Consider the two vectors:

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2$$

and

$$b = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2$$

The objective is to for what values of k the scalar product

$$\langle \mathbf{a} | \mathbf{b} \rangle = a_1 b_1 - a_1 b_2 - a_2 b_1 + k a_2 b_2$$

is valid. The scalar product $\langle \mathbf{a} | \mathbf{a} \rangle$ must be positive for every non-zero vector in the space. If we write $\langle \mathbf{a} | \mathbf{a} \rangle$ in the form,

$$\begin{aligned}
\langle \mathbf{a} | \mathbf{a} \rangle &= a_1 a_1 - a_1 a_2 - a_2 a_1 + k a_2 a_2 \\
&= a_1^2 - 2a_1 a_2 + k a_2^2 \\
&= \left(a_1^2 - 2a_1 a_2 + a_2^2\right) - a_2^2 + k a_2^2 \\
&= (a_1 - a_2)^2 - a_2^2 + k a_2^2 \\
&= (a_1 - a_2)^2 + (k-1) a_2^2
\end{aligned}$$

This condition is violated for some non-zero vector a unless $k > 1$. Therefore, the scalar product is valid when $k > 1$.