This problem set focuses on material covered in Week 8 (Lecture 10), so I recommend you to watch the lecture and attempt Assignment 10 (both parts) before submitting your answers.

1. Say which of the following are true. (Leave the box empty to indicate that it's false.)
   - A set A of reals can have at most one least upper bound.
   - If a set A of reals has a lower bound, it has infinitely many lower bounds.
   - If a set A of reals has both a lower bound and an upper bound, then it is finite.
   - 0 is the least upper bound of the set of negative integers, considered as a subset of the reals.

2. Which of the following say that b is the greatest lower bound of a set A of reals? (Leave the box empty to indicate that it does not say that.)
   - $b \leq a$ for all $a \in A$ and if $c \leq a$ for all $a \in A$, then $b \geq c$.
   - $b \leq a$ for all $a \in A$ and if $c \leq a$ for all $a \in A$, then $b > c$.
   - $b < a$ for all $a \in A$ and if $c < a$ for all $a \in A$, then $b \geq c$.
   - $b < a$ for all $a \in A$ and if $c \leq a$ for all $a \in A$, then $b \geq c$.
   - $b \leq a$ for all $a \in A$ and if $\epsilon > 0$ there is an $a \in A$ such that $a < b + \epsilon$.

3. The *Sandwich Theorem* (also sometimes called the *Squeeze Theorem*) says that if $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are sequences such that, from some point n_0 onwards,

$$a_n \leq b_n \leq c_n,$$

and if

$$\lim_{n\to\infty} a_n = L \ , \ \lim_{n\to\infty} c_n = L,$$

then $\{b_n\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n\to\infty} b_n = L.$$

Taking the Sandwich Theorem to be correct (which it is), grade the following proof using the course rubric.

Theorem $\lim_{n\to\infty} \frac{\sin^2 n}{3^n} = 0$

Proof: For any n,

$$0 \leq \frac{\sin^2 n}{3^n} \leq \frac{1}{3^n}$$

Clearly, $\lim_{n\to\infty} \frac{1}{3^n} = 0$. Hence, by the Sandwich Theorem,

$$\lim_{n\to\infty} \frac{\sin^2 n}{3^n} = 0$$

as required.

4. Is the following proof of the Sandwich Theorem correct? Grade it according to the course rubric.

   Theorem (Sandwich Theorem) Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are sequences such that, from some point n_0 onwards,
   $$a_n \leq b_n \leq c_n.$$
   Suppose further that
   $$\lim_{n\to\infty} a_n = L \ , \ \lim_{n\to\infty} c_n = L.$$
   Then $\{b_n\}_{n=1}^{\infty}$ is convergent and
   $$\lim_{n\to\infty} b_n = L.$$

   Proof: Since $\lim_{n\to\infty} a_n = L$, we can find an integer n_1 such that
   $$n \geq n_1 \Rightarrow |a_n - L| < \epsilon$$
   Since $\lim_{n\to\infty} c_n = L$, we can find an integer n_2 such that
   $$n \geq n_2 \Rightarrow |c_n - L| < \epsilon$$
   Let $M = \max\{n_0, n_1, n_2\}$. Then
   $$\begin{aligned} n \geq M \quad & \Rightarrow \quad (-\epsilon < a_n - L < \epsilon) \ \wedge \ (-\epsilon < c_n - L < \epsilon) \\ & \Rightarrow \quad -\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon \ \ (\text{using } a_n \leq b_n \leq c_n) \\ & \Rightarrow \quad -\epsilon < b_n - L < \epsilon \\ & \Rightarrow \quad |b_n - L| < \epsilon \end{aligned}$$
   By the definition of a limit, this proves that $\{b_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} b_n = L$, as required.

5. Evaluate this purported proof, and grade it according to the course rubric.

   Theorem $\lim_{n\to\infty} \dfrac{n+1}{2n+1} = \dfrac{1}{2}$.

   Proof: Let $\epsilon > 0$ be given. Choose N large enough so that $N \geq \dfrac{1}{2\epsilon}$.

   Then, for $n \geq N$,
   $$\begin{aligned} \left|\frac{n+1}{2n+1} - \frac{1}{2}\right| \quad & = \quad \left|\frac{2(n+1) - (2n+1)}{2(2n+1)}\right| \\ & = \quad \left|\frac{1}{2(2n+1)}\right| \\ & = \quad \frac{1}{2(2n+1)} \\ & < \quad \frac{1}{2n+1} \\ & < \quad \frac{1}{2n} \leq \frac{1}{2N} \leq \epsilon \end{aligned}$$
   By the definition of a limit, this proves the theorem.