1. Prove that the intersection of two intervals is again an interval. Is the same true for unions?

2. Taking $\mathcal{R}$ as the universal set, express the following as simply as possible in terms of intervals and unions of intervals. (Note that $A'$ denotes the complement of the set $A$ relative to the given universal set, which in this case is $\mathcal{R}$. See the module on set theory.)

| | |
|---|---|
| (a) $[1,3]'$ | (b) $(1,7]'$ |
| (c) $(5,8]'$ | (d) $(3,7) \cup [6,8]$ |
| (e) $(-\infty,3)' \cup (6,\infty)$ | (f) $\{\pi\}'$ |
| (g) $(1,4] \cap [4,10]$ | (h) $(1,2) \cap [2,3)$ |
| (i) $A',$ where $A = (6,8) \cap (7,9]$ | (j) $A',$ where $A = (-\infty,5] \cup (7,\infty)$ |

3. Prove that if a set $A$ of integers/rationals/reals has an upper bound, then it has infinitely many different upper bounds.

4. Prove that if a set $A$ of integers/rationals/reals has a least upper bound, then it is unique.

5. Let $A$ be a set of integers, rationals, or reals. Prove that $b$ is the least upper bound of $A$ iff:

   (a) $(\forall a \in A)(a \leq b)$; and

   (b) whenever $c < b$ there is an $a \in A$ such that $a > c$.

6. The following variant of the above characterization is often found. Show that $b$ is the lub of $A$ iff:

   (a) $(\forall a \in A)(a \leq b)$ ; and

   (b) $(\forall \epsilon > 0)(\exists a \in A)(a > b - \epsilon)$.

7. Give an example of a set of integers that has no upper bound.

8. Show that any finite set of integers/rationals/reals has a least upper bound.

9. Intervals: What is lub $(a,b)$? What is lub $[a,b]$? What is max $(a,b)$? What is max $[a,b]$?

10. Let $A = \{|x-y| \mid x,y \in (a,b)\}$. Prove that $A$ has an upper bound. What is lub $A$ ?

11. Define the notion of a *lower bound* of a set of integers/rationals/reals.

12. Define the notion of a *greatest lower bound* (glb) of a set of integers/rationals/reals by analogy with our original definition of lub.

13. State and prove the analog of question 5 for greatest lower bounds.

14. State and prove the analog of question 6 for greatest lower bounds.

15. Show that the Completeness Property for the real number system could equally well have been defined by the statement, "Any nonempty set of reals that has a lower bound has a greatest lower bound."

16. The integers satisfy the Completeness Property, but for a trivial reason. What is that reason?