This problem set focuses on material covered in Week 7 (Lecture 9), so I recommend you to watch the lecture and attempt Assignment 9 before submitting your answers.

1. Say which of the following statements are true. (Leave the box blank to indicate that it is false.)

   - $20|300$
   - $17|35$
   - $5|0$
   - $0|5$
   - $21|(-21)$

2. Say whether the following proof is valid or not.

   Theorem. The square of any odd number is 1 more than a multiple of 8. (For example, $3^2 = 9 = 8 + 1, 5^2 = 25 = 3 \cdot 8 + 1$.)

   Proof: By the Division Theorem, any number can be expressed in one of the forms $4q,\ 4q+1,\ 4q+2,\ 4q+3$. So any odd number has one of the forms $4q+1, 4q+3$. Squaring each of these gives

   $$\begin{aligned} (4q+1)^2 &= 16q^2 + 8q + 1 &= 8(2q^2+q)+1 \\ (4q+3)^2 &= 16q^2 + 24q + 9 &= 8(2q^2+3q+1)+1 \end{aligned}$$

   In both cases the result is one more than a multiple of 8. This proves the theorem.

3. Say whether the following verification of the method of induction is valid or not?

   Proof: We have to prove that if

   - $A(1)$
   - $(\forall n)[A(n) \Rightarrow A(n+1)]$

   then $(\forall n)A(n)$.

   We argue by contradiction. Suppose the conclusion is false. Then there will be a natural number n such that $\neg A(n)$. Let m be the least such number. By the first condition, $m > 1$, so $m = n+1$ for some n. Since $n < m$, $A(n)$. Then by the second condition, $A(n+1)$, i.e., $A(m)$. This is a contradiction, and that proves the result.

4. Evaluate this purported proof, and grade it according to the course rubric.

   The *Fibonacci sequence* is defined iteratively by setting $F_1 = F_2 = 1$ and thereafter letting $F_{n+2} = F_n + F_{n+1}$. This fascinating sequence has been known for at least 1500 years. It has several connections to the natural world, some of which are described in the second lecture of Devlin's mathematics survey course on iTunes University, listed as recommended supplementary viewing to this course. It also has a number of pleasing mathematical connections. Here is one:

Theorem For any natural number n,

$$\sum_{k=1}^{n} {F_k}^2 = F_n F_{n+1}$$

Proof: By induction.

For $n = 1$, the left-hand side is ${F_1}^2 = 1^2 = 1$ and the right-hand side is $F_1 F_2 = 1.1 = 1$, so the identity is valid for $n = 1$.

Assume the identity holds for n. Then:

$$\begin{aligned}
\sum_{k=1}^{n+1} {F_k}^2 &= \sum_{k=1}^{n} {F_k}^2 \ + \ {F_{n+1}}^2 \\
&= F_n F_{n+1} \ + \ {F_{n+1}}^2, \ \text{ by the induction hypothesis} \\
&= F_{n+1}(F_n + F_{n+1}), \ \text{ by algebra} \\
&= F_{n+1} F_{n+2}, \ \text{ by the definition of } F_{n+2}
\end{aligned}$$

which is the identity for $n + 1$. The proof is complete.

5. Evaluate this purported proof, and grade it according to the course rubric.

   The *Fibonacci sequence* is defined iteratively by setting $F_1 = F_2 = 1$ and thereafter letting $F_{n+2} = F_n + F_{n+1}$.

   Theorem For any natural number n,

   $$\sum_{k=1}^{n} F_k = F_{n+2}$$

   Proof: By induction.

   For $n = 1$, the left-hand side is $F_1 = 1$ and the right-hand side is $F_2 = 1$, so the identity is valid for $n = 1$.

   Assume the identity holds for n. Then:

   $$\begin{aligned}
   \sum_{k=1}^{n+1} F_k &= \sum_{k=1}^{n} F_k \ + \ F_{n+1} \\
   &= F_{n+2} \ + \ F_{n+1}, \ \text{ by the induction hypothesis} \\
   &= F_{n+3}, \ \text{ by the definition of } F_{n+3}
   \end{aligned}$$

   which is the identity for $n + 1$. The proof is complete.

6. Evaluate this purported proof, and grade it according to the course rubric.

   The *Fibonacci sequence* is defined iteratively by setting $F_1 = F_2 = 1$ and thereafter letting $F_{n+2} = F_n + F_{n+1}$.

   Theorem For any natural number n,

$$F_n \geq \left(\frac{3}{2}\right)^{n-2}$$

   Proof: Proof: We have $F_1 = 1 \geq \frac{2}{3} = \left(\frac{3}{2}\right)^{-1}$ and $F_2 = 1 = \left(\frac{3}{2}\right)^{0}$, so the inequality is valid for $n = 1, 2$.

   Now assume the inequality holds for n, where $n \geq 2$. Then:

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&\geq \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-3} \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{3}{2}+1\right), \text{ by algebra} \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{5}{2}\right) \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{10}{4}\right) \\
&\geq \left(\frac{3}{2}\right)^{n-3}\left(\frac{9}{4}\right) \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{3}{2}\right)^{2} \\
&= \left(\frac{3}{2}\right)^{n-1}
\end{aligned}$$

   which establishes the inequality for $n + 1$.