This problem set focuses primarily on material covered in Week 6 (Lecture 8), so I recommend you to watch the lecture and attempt Assignment 8 before submitting your answers.

1. Is the following proof valid or not?

   Theorem: For any natural number n, $2^n > 2n$.

   Proof: By induction. The case $n = 1$ is obviously true, so assume the inequality holds for n.

   That is, assume $2^n > 2n$. Then

   $2^{n+1} = 2 \cdot 2^n > 2 \cdot 2n$ (by the induction hypothesis) $= 4n = 2n + 2n \geq 2n + 2$ (since $n \geq 1$) $= 2(n+1)$

   This establishes the inequality for $n + 1$. Hence, by induction, the inequality holds for all n.

2. Is the following proof valid or not?

   Theorem: If a nonempty finite set X has n elements, then X has exactly 2^n distinct subsets.

   Proof: By induction on n.

   The case $n = 1$ is true, since if X is a set with exactly one element, say $X = \{a\}$, then X has the two subsets $\emptyset$ and X itself.

   Assume the theorem is true for n. Let X be a set of $n + 1$ elements. Let $a \in X$ and let $Y = X - \{a\}$ (i.e., obtain Y by removing a from X). Then Y is a set with n elements. By the induction hypothesis, Y has 2^n subsets. List them as $Y_1, \ldots, Y_{2^n}$. Then all the subsets of X are $Y_1, \ldots, Y_{2^n}, Y_1 \cup \{a\}, \ldots, Y_{2^n} \cup \{a\}$ (i.e., the subsets of Y together with the subsets of Y with a added to each one). There are $2 \cdot 2^n = 2^{n+1}$ sets in this list. This establishes the theorem for $n+1$. Hence, by induction, it is true for all n.

3. True or false? If p is a prime number, then $\sqrt{p}$ is irrational. (Before entering your answer, you should either construct a proof of truth or find a counter-example, so you are sure. After you have completed the problem set, you should write up your proof or counter-example and share it with your study group for feedback. You can assume that if p is prime, then whenever p divides a product ab, p divides at least one of a, b.)

4. Evaluate this purported proof, and grade it according to the course rubric.

   Theorem For any natural number n,

   $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

   Proof: By induction.

   For $n = 1$, the left-hand side is $\dfrac{1}{1.2} = \dfrac{1}{2}$ and the right-hand side is $\dfrac{1}{2}$, so the identity is valid for $n = 1$.

   Assume the identity holds for n. Then:

   $$\begin{aligned} \sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}
\end{aligned}
$$

This is the identity for $n+1$. Hence, by induction, the theorem is proved.

5. This theorem is obviously false. Enter the line number of the (incorrect) statement where the proof logically breaks down.

Theorem All Americans are the same age.

Proof:

1. Let $S(n)$ be the statement: In any group of n Americans, everyone in that group has the same age.
2. We prove $S(n)$ by induction on n.
3. Since everyone in a group of one American has the same age, $S(1)$ is true.
4. Assume $S(n)$ is true for some n.
5. We prove $S(n+1)$.
6. Let G be an arbitrary group of $n+1$ Americans.
7. We show that everyone in G has the same age.
8. We do this by showing that any two members of G have the same age.
9. Let $a, b \in G$.
10. Let G_a be the result of removing a from G.
11. Since G_a has n members, b (which is in G_a) has the same age as any other person in G_a.
12. Similarly, if G_b is G with b removed, then a has the same age as any other person in G_b.
13. Now let c be any person in G other than a and b.
14. Then $c \in G_a$ and c $\in G_b$.
15. So, a and b both have the same age as c.
16. Hence a and b have the same age.
17. This proves $S(n+1)$.
18. Hence, by induction, $S(n)$ is true for all n.
19. This implies that all Americans have the same age.