Week 5 – Assignment 7

1. Prove or disprove the statement “All birds can ﬂy.”
   1. FALSE. Counter-example: chicken.
2. Prove or disprove the claim (∀x,y ∈R)[(x−y)2 > 0].
   1. FALSE. Counter-example, x = y = 0
3. Prove that between any two unequal rationals there is a third rational.
   1. (∀x,y ∈ ℚ)(x ≠ y => (∃z ∈ ℚ)(x < z < y)) (assuming that x < y)
   2. Assume x ≠ y
   3. Since x,y are rationals, can be rewritten as x=p/m, y=q/n, where p, q, r, s are integers
   4. Rewriting, x=2pn/2mn, y=2qm/2mn
   5. Since x ≠ y, pn ≠ qm. The number Z = pn+qm/2mn is between the two numbers
   6. Therefore: there is a number between x and y.
4. Explain why proving φ ⇒ ψ and ψ ⇒ φ establishes the truth of φ ⇔ ψ.
   1. Because that’s the exact definition of φ ⇔ ψ
5. Explain why proving φ ⇒ ψ and (¬φ) ⇒ (¬ψ) establishes the truth of φ ⇔ ψ.
   1. Because (¬φ) ⇒ (¬ψ) is the same as ψ ⇒ φ, and from the previous question, that, alongside with φ ⇒ ψ, establishes the truth of φ ⇔ ψ.
6. Prove that if five investors split a payout of $2M, at least one investor receives at least $400,000.
   1. Assume that all investors receive less than $400k
   2. Then the sum of all 5 investors is less than 5\*$400k
   3. The sum of all 5 investors is less than $2M – that is false, they received $2M
   4. Therefore, the assumption is incorrect – at least one investor received at least $400K
7. Prove that √3 is irrational.
   1. Assume √3 is rational.
   2. (by definition), there are 2 numbers p, q with no common factors such as √3 = p/q
   3. (squaring both sides) 3 = p2/q2
   4. (rearranging) p2 = 3q2
   5. (by sub-proof) p is divisible by 3
      1. Sub-claim: (∀x ∈ ℕ)(x2 divisible by 3 ⇒ x divisible by 3)
      2. (counter-positive): x not divisible by 3 ⇒ x2 not divisible by 3
      3. (rewriting) x = 3p+1 or x = 3p+2
      4. (squaring) x2 = 9p2 + 6p + 1 or x2 = 9p2 + 12p + 4
      5. (rewriting) x2 = 3(3p2 + 2p) + 1 or x2 = 3(3p2 + 4p + 1) + 1
      6. Hence, x2 is not divisible by 3 (since there is a reminder of 1)
      7. QED
   6. (rewriting p = 3r) (3r)2 = 3q2
   7. (expanding) 9r2=3q2
   8. (simplifying) 3r2=q2
   9. (by sub-proof) q is divisible by 3
   10. That is false, contradicting (b)
   11. Since the negation of the claim is false, the claim is true
8. .
   1. If the Yuan rises, the Dollar falls.
   2. If −y < −x then x < y.
   3. If two triangles have the same area then they are congruent.
   4. Whenever b2 ≥ 4ac PPPPP The quadratic equation ax2 + bx + c = 0 has a solution whenever. (Where a,b,c,x denote real numbers and a 6= 0.) (e) Let ABCD be a quadrilateral. If the opposite sides of ABCD are pairwise equal, then the opposite angles are pairwise equal. (f) Let ABCD be a quadrilateral. If all four sides of ABCD are equal, then all four angles are equal. (g) If n is not divisible by 3 then n2 + 5 is divisible by 3. (For n a natural number.)
   5. 🡪 direction
      1. Assume ¬[∃xA(x)]
      2. (rewrite in English) It is not the case that there is an x such that A(x) is true
      3. (therefore) For all values of x, it is the case that A(x) is false
      4. (rewrite) For all values of x, it is not the case that A(x) is true
      5. (rewrite) ∀x[¬A(x)]
   6. 🡨 direction
      1. Assume ∀x[¬A(x)]
      2. (rewrite in English) For all values of x, it is not the case that A(x) is true
      3. (rewrite) For all values of x, it is the case that A(x) is false
      4. (same as) There are no values x such as A(x) is true
      5. (rewrite) It is not the case that there is a value x such that A(x) is true
      6. (rewrite) ¬[∃xA(x)]
9. Prove that “There is an even prime bigger than 2” is false
   1. (rewrite) There are no even prime numbers bigger than 2
   2. (Equivalent to) All prime numbers bigger than 2 are not even
   3. Let p be a prime bigger than 2; if p is even, then p can be written in the form p=2\*p’
   4. (therefore) p cannot be prime (since there is a number p’ greater than 1 and smaller than p that divides it)
10. .
    1. ∀p(IsStudent(p) => LikesPizza(p))
    2. ∃p(IsMyFriend(p) ∧ ¬HasCar(p))
    3. ∃a(IsElephant(a) ∧ ¬LikesMuffin(a))
    4. ∀x(IsTriangle(x) => IsIsoceles(x))
    5. ∃p(IsStudentInThisClass(p) ∧ IsNotHereToday(p))
    6. ∀p∃q(Loves(p, q))
    7. ∀p∃q(¬Loves(p, q))
    8. ∀p1((Man(p1) ∧ Comes(p1)) => ∀p2(Woman(p2) => Leaves(p2)))
    9. ∀p(Tall(p) OR Short(p))
    10. ∀p(Tall(p)) OR ∀p(Short(p))
    11. ∃s(Precious(s) AND ¬Beautiful(s))
    12. ∀p(¬Loves(p, me))
    13. ∃s(American(s) ∧ Poisonous(s))
    14. ∃s(Snake(s) ∧ American(s) ∧ Poisonous(s))
11. .
    1. ∃p(IsStudent(p) ∧ ¬LikesPizza(p))
       1. There is a student that does not like pizza.
    2. ∀p(IsMyFriend(p) => HasCar(p))
       1. All my friends have cars.
    3. ∀a(IsElephant(a) => LikesMuffin(a))
       1. All elephants like muffins
    4. ∃x(IsTriangle(x) ∧ ¬IsIsoceles(x))
       1. Some triangles are not isosceles.
    5. ∀p(IsStudentInThisClass(p) => IsHereToday(p))
       1. All students in this class are here today
    6. ∃p∀q(¬Loves(p, q))
       1. Someone doesn’t love anybody
    7. ∃p∀q(Loves(p, q))
       1. Someone loves everyone
    8. ∃p1(Man(p1) ∧ Comes(p1) ∧ ∃p2(Woman(p2) ∧ => ∀p2(Woman(p2) => ¬Leaves(p2)))
       1. For some men that come, not all women will leave
    9. ∃p(¬Tall(p) AND ¬Short(p))
       1. Someone is both not tall and not short
    10. ∃p(¬Tall(p)) AND ∃p(¬Short(p))
        1. Someone is not tall and someone is not short
    11. ∀s(Precious(s) => Beautiful(s))
        1. All precious stones are beautiful
    12. ∃p(Loves(p, me))
        1. Someone loves me
    13. ∀s(American(s) => ¬Poisonous(s))
        1. No American snakes are poisonous
    14. ∀s((Snake(s) ∧ American(s)) => ¬Poisonous(s))
        1. No American snakes are poisonous
12. .
    1. FALSE (x = 2/3, not natural)
    2. FALSE (x = SQRT(2), not rational)
    3. TRUE
    4. TRUE
    5. FALSE (y = 1, it’s not the case for all z)
    6. FALSE
    7. FALSE (e.g., x = -1)
    8. TRUE (by default)
13. .
    1. ∀x(2x + 3 ≠ 5x + 1)
    2. ∀x(x2 ≠ 2)
    3. ∃x∀y(y ≠ x2)
    4. ∃x∀y(y ≠ x2)
    5. ∃x∀y∃z(xy ≠ xz)
    6. ∃x∀y∃z(xy ≠ xz)
    7. ∃x(x < 0 ∧ ∀y(y2 ≠ x))
    8. ∃x(x < 0 ∧ ∀y(y2 ≠ x))
14. .
    1. (∃x ∊ ℕ)(∀y ∊ ℕ)(x + y ≠ 1)
    2. (∃x > 0)(∀y < 0)(x + y ≠ 0)
    3. ∀x(∃ε > 0)(x ≤ -ε OR x ≥ ε)
    4. (∃x ∊ ℕ)(∃y ∊ ℕ)(∀z ∊ ℕ) (x + y ≠ z2)
15. Negation of [((∃p)(∀t)(CanFoolAt(p,t))) AND  
     ((∃t)(∀p)(CanFoolAt(p,t))) AND ¬((∀t)(∀p)(CanFoolAt(p,t)))]
    1. [((∀p)(∃t)(¬CanFoolAt(p,t))) OR   
        ((∀t)(∃p)(¬CanFoolAt(p,t))) OR ((∀t)(∀p)(CanFoolAt(p,t)))]
    2. For every person, there’s a time you can’t fool them OR at any time there is someone you can’t fool OR you can always fool all the people all the time
16. Negation of (∀ε > 0)(∃δ > 0)(∀x)[|x-a| < δ => |f(x)-f(a)| < ε]
    1. (∃ε > 0)(∀δ > 0)(∃x)[|x-a| < δ AND |f(x)-f(a)| ≥ ε]

Week 4 – Problem Set 4

1. Which of the following is equivalent to ¬∀x[P(x) ⇒ (Q(x)∨R(x))]? (Only one is.)
   1. ∃x[P(x)∨¬Q(x)∨¬R(x)]
   2. ∃x[¬P(x)∧Q(x)∧R(x)]
   3. **∃x[P(x)∧¬Q(x)∧¬R(x)]**
   4. ∃x[P(x)∧(¬Q(x)∨¬R(x))]
   5. ∃x[P(x)∨(¬Q(x)∧¬R(x))]
      1. Demonstration: ¬∀x[P(x) ⇒ (Q(x)∨R(x))]
      2. ∃x[¬(P(x) ⇒ (Q(x)∨R(x)))]
      3. ∃x[¬(¬P(x) ∨ Q(x) ∨ R(x))]
      4. ∃x[(P(x) ∧ ¬Q(x) ∧ ¬R(x))]
2. Let p, q be variables denoting tennis players, let t be a variable denoting games of tennis, and let W(p,q,t) mean that p plays against q in game t and wins. Which of the following claims about tennis players mean the same as the symbolic formula ∀p∃q∃tW(p,q,t)? Select all that have that meaning.
   1. **Everyone wins a game**
   2. Everyone loses a game (e.g., AxB, AxC, BxC, CxB)
   3. For every player there is another player they beat all the time (e.g., AxB, BxA, AxC, CxA, BxC, CxB)
   4. There is a player who loses every game (same as c)
   5. There is a player who wins every game (same as c)
3. Let p,q be variables denoting the tennis players in a club, let t be a variable denoting the club’s games of tennis, and let W(p,q,t) mean that p plays against q in game t and wins. Assuming that there are at least two tennis players and games between them do take place, which (if any) of the following symbolic formula cannot possibly be true? Select all you think cannot possibly be true
   1. ∀p∃q∃tW(p,q,t)
   2. **∀p∀q∃tW(p,q,t)** (not conditioned that p != q)
   3. ∀q∃p∃tW(p,q,t)
4. Which of the following means “Everybody loves a lover”, where L(x,y) means (person) x loves (person) y and a lover is deﬁned to be someone in a mutual loving relationship? [If English is not your native language, you might want to discuss this sentence with a native English speaker before you answer. It’s an idiomatic expression.]
   1. **∀x∀y[∃z(L(x,z)∧L(z,x)) ⇒ L(y,x)]**
   2. ∀x∀y[∀z(L(x,z)∨L(z,x)) ⇒ L(y,x)]
   3. ∀x[∃z(L(x,z)∧L(z,x))∧∀yL(y,x)]
5. Which of the following statements about the order relation on the real line is/are false?TBC
   1. ∀x∀y∀z[(x ≤ y)∧(y ≤ z) ⇒ (x ≤ z)] TRUE (definition of total order)
   2. ∀x∀y[(x ≤ y)∧(y ≤ x) ⇒ (x = y)] TRUE
   3. ∀x∃y[(x ≤ y)∧(y ≤ x)] TRUE (x == y)
   4. **∃x∀y[(y < x)∨(x < y)]** FALSE (y == x)
6. A student produced this purported proof while trying to understand Euclid’s proof of the inﬁnitude of the primes. Grade it according to the course rubric.
   1. If an integer N is divisible by a prime P, then N+1 is not divisible by P.
   2. Proof: Suppose N is divisible by P. Then there is an integer Q such that N=PQ
   3. So N+1=PQ+1. Then
   4. (N+1)/P = Q + 1/P
   5. But Q+1/P is not an integer. So N+1 is not divisible by P.
   6. Logical correctness: 4 3 (cannot use division for integers)
   7. Clarity: 4 4
   8. Opening: 4 4
   9. Stating the conclusion: 4 4
   10. Reasons: 4 4
   11. Overall: 4 0
   12. => Total: 24 19

∀ ∃ φ ψ ¬ ∧ ∨ℕℚ∈∉ℝℤ≠⇐⇒⇔∃∄∀∴∧∨≤≥⊗