Week 5 – Assignment 7

1. Prove or disprove the statement “All birds can ﬂy.”
   1. FALSE. Counter-example: chicken.
2. Prove or disprove the claim (∀x,y ∈R)[(x−y)2 > 0].
   1. FALSE. Counter-example, x = y = 0
3. Prove that between any two unequal rationals there is a third rational.
   1. (∀x,y ∈ ℚ)(x ≠ y => (∃z ∈ ℚ)(x < z < y)) (assuming that x < y)
   2. Assume x ≠ y
   3. Since x,y are rationals, can be rewritten as x=p/m, y=q/n, where p, q, r, s are integers
   4. Rewriting, x=2pn/2mn, y=2qm/2mn
   5. Since x ≠ y, pn ≠ qm. The number Z = pn+qm/2mn is between the two numbers
   6. Therefore: there is a number between x and y.
4. Explain why proving φ ⇒ ψ and ψ ⇒ φ establishes the truth of φ ⇔ ψ.
   1. Because that’s the exact definition of φ ⇔ ψ
5. Explain why proving φ ⇒ ψ and (¬φ) ⇒ (¬ψ) establishes the truth of φ ⇔ ψ.
   1. Because (¬φ) ⇒ (¬ψ) is the same as ψ ⇒ φ, and from the previous question, that, alongside with φ ⇒ ψ, establishes the truth of φ ⇔ ψ.
6. Prove that if five investors split a payout of $2M, at least one investor receives at least $400,000.
   1. Assume that all investors receive less than $400k
   2. Then the sum of all 5 investors is less than 5\*$400k
   3. The sum of all 5 investors is less than $2M – that is false, they received $2M
   4. Therefore, the assumption is incorrect – at least one investor received at least $400K
7. Prove that √3 is irrational.
   1. Assume √3 is rational.
   2. (by definition), there are 2 numbers p, q with no common factors such as √3 = p/q
   3. (squaring both sides) 3 = p2/q2
   4. (rearranging) p2 = 3q2
   5. (by sub-proof) p is divisible by 3
      1. Sub-claim: (∀x ∈ ℕ)(x2 divisible by 3 ⇒ x divisible by 3)
      2. (counter-positive): x not divisible by 3 ⇒ x2 not divisible by 3
      3. (rewriting) x = 3p+1 or x = 3p+2
      4. (squaring) x2 = 9p2 + 6p + 1 or x2 = 9p2 + 12p + 4
      5. (rewriting) x2 = 3(3p2 + 2p) + 1 or x2 = 3(3p2 + 4p + 1) + 1
      6. Hence, x2 is not divisible by 3 (since there is a reminder of 1)
      7. QED
   6. (rewriting p = 3r) (3r)2 = 3q2
   7. (expanding) 9r2=3q2
   8. (simplifying) 3r2=q2
   9. (by sub-proof) q is divisible by 3
   10. That is false, contradicting (b)
   11. Since the negation of the claim is false, the claim is true
8. .
   1. If the Yuan rises, the Dollar falls.
   2. If −y < −x then x < y.
   3. If two triangles have the same area then they are congruent.
   4. If the quadratic equation ax2 + bx + c = 0 has a solution, then b2 ≥ 4ac.
   5. Let ABCD be a quadrilateral. If the opposite angles of ABCD are pairwise equal, then the opposite sides are pairwise equal.
   6. Let ABCD be a quadrilateral. If all four angles of ABCD are equal, then all four sides are equal.
   7. If n2 + 5 is divisible by 3 then n is not divisible by 3. (For n a natural number.)
9. .
   1. N/A
   2. Equivalent
      1. ∀x,y(x < y => -y < -x) – rule of ordering when multiplied by negative number
      2. ∀x,y(-y < -x => x < y) – rule of ordering when multiplied by negative number
   3. Statement is true, converse is not
      1. Statement: congruent are the same modulo rotation; rotation does not alter area
      2. Counter-example: triangle with sides (3,4,5) and equilateral with area = 12 (s=sqrt(6/sqrt(2)))
   4. Equivalent
      1. b2 ≥ 4ac, one solution is –b+sqrt(b2 - 4ac)/2a
      2. If has solution, it was found with the formula –b+sqrt(b2 - 4ac)/2a, which is real only if b2 - 4ac ≥ 0 (or b2 ≥ 4ac)
   5. Equivalent. Opposite sides, or opposite angles, of a quadrilateral being equal implies it’s a parallelogram. Parallelogram have equal opposite angles and sides.
   6. Statement is false, converse is false
      1. Counter-example: rectangle with sides 10, 5, 10, 5
      2. Counter-example: diamond with equal sides and angles 60, 120, 60, 120
   7. Equivalent
      1. n is of form (3n’+1 or 3n’+2)
         1. n2 = 9n’2+[12 or 6]n’ + [1 or 4]
         2. Adding 5 and rearranging, n2 = 3(3n2 + [4 or 2]n’ + [2 or 3])
         3. which is a multiple of 3 QED
      2. Proving the counter-positive
         1. n is divisible by 3
         2. n2 is divisible by 3 (multiplication)
         3. n2+5 is not divisible by 3 (as 5 is not divisible by 3) QED
10. False. Counter-example: 6. 63 = 216 which is divisible by 12. But 2 is not divisible by 12.
11. .
    1. Assume r+3 is rational
       1. There exist integers p, q such as p/q = r+3
       2. (Subtracting 3) p/q – 3 = r
       3. (Rewriting) (p – 3q) / q = r
       4. Let p’ = p – 3q, which is integer
       5. (Rewriting) r = p’ / q, which contradicts the hypothesis
       6. (Therefore) r+3 is irrational
    2. Assume 5r is rational
       1. There exist integers p, q such as p/q = 5r
       2. Rewriting p/q: 5p/5q = 5r
       3. (5p is divisible by 5, rewriting): p/5q = r, which contradicts the hypothesis
       4. Therefore, 5r is irrational
    3. Not necessarily. Counter-example: sqrt(2) and –sqrt(2) (r+s = 0)
    4. Not necessarily. Counter-example: sqrt(2) and sqrt(2) (rs = 2)
    5. Assume sqrt(r) is rational
       1. There exist integers p, q such a p/q = sqrt(r)
       2. (squaring both sides) p2/q2=r, which contradicts the hypothesis (p2 and q2 are integers)
       3. Therefore, sqrt(r) is irrational
    6. Not necessarily. Counter-example: r=2√2 and s=√2.
       1. r^s = (2√2)^√2 = 2^(√2\*√2) = 2^2 = 4
12. .
    1. Let m = 2p and n = 2q; m+n = 2p+2q = 2(p+q), which is even
    2. Let m = 2p and n = 2q; mn=2p\*2q=4\*pq, which is divisible by 4
    3. Let m = 2p+1 and n = 2q+1; m+n=2p+1+2q+1=2(p+q+1), which is even
    4. Let m=2p and n=2q+1, m+n=2p+2q+1 = 2(p+q)+1, which is odd
    5. Let m=2p and n=2q+1, mn=2p\*(2q+1)=2\*(2pq+1), which is even
13. OPTIONAL
    1. True: x=0, y=1
    2. True: y=-x
    3. False. Counter-example, a=6, b=2, c=3
    4. True; x is of form a+0i, y is of form a’+b’i (a, a’ are real, b’ ≠ 0)
       1. x+y=(a+a’)+b’i, which is irrational since b’ ≠ 0
    5. True
       1. Counter-positive: x and y are rational
       2. Therefore, there are integers p,q,m,n such as x=p/q, y=m/n
       3. x+y=p/q+m/n = (pn+qm)/qn, which is rational
    6. False. Counter-example: x=√2, y=2-√2, x+y=2

Week 5 – Problem Set 5

1. Let m,n denote any two natural numbers. Is the following a valid proof that mn is odd iﬀ m and n are odd?
   1. If m,n are odd there are integers p,q such that m = 2p+1,n = 2q+1. Then mn = (2p+1)(2q+1) = 2(2pq + p + q) + 1, so mn is odd. That completes the proof.
2. Take the sentence: You can fool some of the people some of the time, but you cannot fool all of the people all the time. Let x be a variable for a person, t a variable for a period of time, and let F(x,t) mean you can fool x at time t. Which of the following mathematical formulas is equivalent to the given statement?
   1. ∃x∃tF(x,t)∧∃x∃t¬F(x,t)
   2. ∃x∃tF(x,t)∧¬∀x∃tF(x,t)
   3. ∃x∃tF(x,t)∧¬∃x∃tF(x,t)
   4. None of the above.
3. True or false? For any two statements φ and ψ, either φ ⇒ ψ or its converse is true (or both)
4. Are the following two statements equivalent? ¬(φ ⇒ ψ) and φ∧(¬ψ)
5. Are the following two statements equivalent? (φ∨ψ) ⇒ θ and (φ ⇒ θ)∧(ψ ⇒ θ)
6. True or false? There are inﬁnitely many natural numbers n for which √n is rational. (Before entering your answer, you should construct a proof of the statement or its negation, so you are sure.)
7. This argument is a proof that 1 = 2. Obviously it is incorrect. Identify exactly what the error is, and grade the purported proof according to the course rubric. (Remember, this is not a regular mathematics course of the kind you are probably familiar with. We are working on various elements of mathematical thinking, mathematical exposition, and the communication of mathematics. The rubric is designed to focus attention on all of those factors.) Your “Overall valuation” ﬁgure is the grade you would assign a student if s/he submitted this proof in a ﬁrst-year college mathematics course
   1. Argument to show that 1 = 2.
   2. We start with the identity
      1. 1−3 = 4−6
   3. Adding 9/4 to both sides to complete the squares, we get 1−3 + 9/4 = 4−6 + 9/4
   4. This factors as (1− 3/2)2 = (2− 3/2)2
   5. Taking the square root of both sides, 1− 3/2 = 2− 3/2
   6. Hence 1 = 2
   7. Evaluation:
      1. Logical correctness: 4 3 (cannot use division for integers)
      2. Clarity: 4 4
      3. Opening: 4 4
      4. Stating the conclusion: 4 4
      5. Reasons: 4 4
      6. Overall: 4 0
      7. => Total: 24 19

∀ ∃ φ ψ ¬ ∧ ∨ℕℚ∈∉ℝℤ≠⇐⇒⇔∃∄∀∴∧∨≤≥⊗