Week 6 – Assignment 8

1. Prove or disprove the claim that there are integers m, n such that m2 +mn+n2 is a perfect square.
   1. True: m = 0, n = 2
2. Prove or disprove the claim that for any positive integer m there is a positive integer n such that mn + 1 is a perfect square.
   1. True: for an arbitrary m > 0, take n = m + 2; mn+1 = m(m+2)+1 = m2+2m+1=(m+1)2
3. Prove that there is a quadratic f(n) = n2 + bn + c, with positive integers coeﬃcients b, c, such that f(n) is composite (i.e., not prime) for all positive integers n, or else prove that the statement is false.
   1. True: Take f(n) = n2 + 2n + 1 = (n + 1)2, which is not prime.
4. Prove that if every even natural number greater than 2 is a sum of two primes (the Goldbach Conjecture), then every odd natural number greater than 5 is a sum of three primes.
   1. Assume (∀n>2)(∃p, q)(p, q are primes and p + q = n)
   2. Prove that (∀m>5)(∃p, q, r)(p, q, r are primes and p + q + r = m)
   3. (Let m = n + 3, r = 3, since 3 is prime), p + q + 3 = n + 3
   4. (Simplifying) p + q = n
   5. (Since n = n – 3, n > 2, so based on the assumption we can find p, q which are primes
   6. QED
5. Use the method of induction to prove that the sum of the ﬁrst n odd numbers is equal to n2.
   1. Base case: 1 = 12 = 1, ok
   2. Inductive step:
      1. Assume that SUM[i=1..n](n\*2-1)=n2
      2. (Adding (n+1)\*2-1 to both sides) SUM[i=1..n+1](n\*2-1)=n2+2(n+1)-1
      3. (Rewriting) SUM[i=1..n+1](n\*2-1)=n2+2n+2-1
      4. (Rewriting) SUM[i=1..n+1](n\*2-1)=n2+2n+1
      5. (Rewriting) SUM[i=1..n+1](n\*2-1)=(n+1)2
      6. QED
6. The notation SUM[i=1..n]ai is a common abbreviation for the sum a1 + a2 + a3 + ... + an. For instance, SUM[r=1..n](r2) denotes the sum 12 + 22 + 32 + ... + n2. Prove by induction that: ∀n∈N: SUM[r=1..n](r2) = 1/6 \* n(n+1)(2n+1)
   1. Base case: 1 = 1/6 \* 1(1+1)(2+1) = 1/6 \* 1\*2\*3 = 1, ok
   2. Inductive step:
      1. Assume that SUM[r=1..n](r2) = 1/6 \* n(n+1)(2n+1)
      2. (Adding (n+1)2 to both sides) SUM[r=1..(n+1)](r2) = 1/6 \* n(n+1)(2n+1) + (n+1)2
      3. (rewriting right side) 1/6\*(n2+n)(2n+1) + n2+2n+1
      4. (rewriting right side) (2n3+3n2+n)/6 + n2+2n+1
      5. (rewriting right side) (2n3+3n2+n+6n2+12n+6)/6
      6. (rewriting right side) (2n3+9n2+13n+6)/6
      7. (rewriting right side) (n+1)(2n2+7n+6)/6
      8. (rewriting right side) (n+1)(n+2)(2n+3)/6
      9. (rewriting right side) (n+1)((n+1)+1)(2(n+1)+1)/6
      10. QED

OPTIONAL

1. In the lecture we used induction to prove the general theorem 1 + 2 + ... + n = ½\*n(n+1). There is an alternative proof that does not use induction, famous because Gauss used the key idea to solve a “busywork” arithmetic problem his teacher gave to the class when he was a small child at school. The teacher asked the class to calculate the sum of the ﬁrst hundred natural numbers. Gauss noted that if 1 + 2 + ... + 100 = N, then you can reverse the order of the addition and the answer will be the same: 100 + 99 + ... + 1 = N. So by adding those two equations, you get 101 + 101 + ... + 101 = 2N, and since there are 100 terms in the sum, this can be written as 100·101 = 2N, and hence N = ½(100·101) = 5,050. Generalize Gauss’s idea to prove the theorem without recourse to the method of induction.
   1. 1+2+…+n = N
   2. n+(n-1)+…+1=N
   3. (n+1)+(n+1)+…+(n+1)=2N
   4. n(n+1)=2N
   5. N=½\*n(n+1)
   6. QED
2. Prove that for any ﬁnite collection of points in the plane, not all collinear, there is a triangle having three of the points as its vertices, which contains none of the other points in its interior.
   1. ???
3. Prove the following by induction:
   1. 4n −1 is divisible by 3
      1. Base: 0: 40-1=0, divisible by 3
      2. Base: 1: 41-1=3, divisible by 3
      3. Step: assuming that 4n-1 is divisible by 3: there is a number p such as 3\*p=4n-1,
      4. 4(n+1)-1 = 4\*4n-1 = 3\*4n+4n-1 = 3\*4n+3\*p = 3\*(4n+p), which is divisible by 3 QED
   2. (n + 1)! > 2n+3 for all n ≥ 5
      1. Base: (5+1)!=6!=720>256=28=25+3 – OK
      2. Step:
         1. ((n+1)+1)! =
         2. (n+2)! =
         3. (n+2)\*(n+1)! >
         4. (since n > 5) 2\*(n+1)! >
         5. (assumption) 2\*2n+3 =
         6. 2(n+1)+3
         7. QED
   3. ∀n ∈N : SUM[r=1..n](r.r!) = (n + 1)! −1
      1. Base: 1.1! = 1 = 2-1 = 2!-1 = (1 + 1)!-1
      2. Step: assuming SUM[r=1..n](r.r!) = (n + 1)! −1
         1. SUM[r=1..n+1](r.r!) =
         2. SUM[r=1..n](r.r!) + (n + 1)(n + 1)! = (hypothesis)
         3. (n + 1)! −1 + (n + 1)(n + 1)! =
         4. (n + 1)! −1 + n(n + 1)! + (n + 1)! =
         5. (n + 1)!(n + 2) – 1 =
         6. (n + 2)! – 1 =
         7. ((n + 1) + 1)! – 1
         8. QED

Week 6 – Problem Set 6

1. Is the following proof valid or not?
   1. Theorem: For any natural number n, 2n > 2n.
   2. Proof: By induction. The case n = 1 is obviously true, so assume the inequality holds for n.
   3. That is, assume 2n > 2n. Then
   4. 2n+1 = 2·2n > 2·2n (by the induction hypothesis) = 4n = 2n+2n ≥ 2n+2 (since n ≥ 1) = 2(n+1) This establishes the inequality for n + 1. Hence, by induction, the inequality holds for all n.
   5. **Not valid. The base case is invalid.**
2. Is the following proof valid or not?
   1. Theorem: If a nonempty ﬁnite set X has n elements, then X has exactly 2n distinct subsets.
   2. Proof: By induction on n.
   3. The case n = 1 is true, since if X is a set with exactly one element, say X = {a}, then X has the two subsets ∅ and X itself.
   4. Assume the theorem is true for n. Let X be a set of n + 1 elements. Let a ∈ X and let Y = X −{a} (i.e., obtain Y by removing a from X). Then Y is a set with n elements. By the induction hypothesis, Y has 2n subsets. List them as Y1,...,Y2n. Then all the subsets of X are Y1,...,Y2n,Y1 ∪{a},...,Y2n ∪{a} (i.e., the subsets of Y together with the subsets of Y with a added to each one). There are 2·2n = 2n+1 sets in this list. This establishes the theorem for n+1. Hence, by induction, it is true for all n.
   5. **Valid proof**
3. True or false? If p is a prime number, then √p is irrational. (Before entering your answer, you should either construct a proof of truth or ﬁnd a counter-example, so you are sure. After you have completed the problem set, you should write up your proof or counter-example and share it with your study group for feedback. You can assume that if p is prime, then whenever p divides a product ab, p divides at least one of a, b.)
   1. True
4. Evaluate this purported proof, and grade it according to the course rubric.
   1. Theorem For any natural number n,
   2. SUM[k=1..n] 1 / (k(k + 1)) = n / (n + 1)
   3. Proof: By induction.
   4. For n = 1, the left-hand side is 1 / 1.2 = ½, and the right-hand side is ½, so the identity is valid for n = 1.
   5. Assume the identity holds for n. Then:
   6. SUM[k=1..n+1](1/(k(k+1))) = SUM[k=1..n](1/(k(k+1))) + 1/((n+1)(n+2)) =
   7. n/(n+1) + 1/((n+1)(n+2)) =
   8. [n(n+2) + 1]/[(n+1)(n+2)] =
   9. [n2+2n+1]/[(n+1)(n+2)] =
   10. [(n+1)2]/[(n+1)(n+2)] =
   11. (n+1)/(n+2)
   12. This is the identity for n + 1. Hence, by induction, the theorem is proved
   13. **Evaluation:**
       1. **Logical correctness: 4 4 (cannot use division for integers)**
          1. **Used sqrt for negative numbers**
       2. **Clarity: 4 4**
       3. **Opening: 4 4**
       4. **Stating the conclusion: 4 4**
       5. **Reasons: 2 0**
       6. **Overall: 4 3**
       7. **=> Total: 22 19**
5. This theorem is obviously false. Enter the line number of the (incorrect) statement where the proof logically breaks down.
   1. Theorem All Americans are the same age.
   2. Proof:
      1. Let S(n) be the statement: In any group of n Americans, everyone in that group has the same age.
      2. We prove S(n) by induction on n.
      3. Since everyone in a group of one American has the same age, S(1) is true.
      4. Assume S(n) is true for some n.
      5. We prove S(n + 1).
      6. Let G be an arbitrary group of n + 1 Americans.
      7. We show that everyone in G has the same age.
      8. We do this by showing that any two members of G have the same age.
      9. Let a, b ∈ G.
      10. Let Ga be the result of removing a from G
      11. Since Ga has n members, b (which is in Ga) has the same age as any other person in Ga.
      12. Similarly, if Gb is G with b removed, then a has the same age as any other person in Gb.
      13. **Now let c be any person in G other than a and b.**
          1. **This is incorrect, since there’s no assumption that G has more than 2 members**
      14. Then c ∈ Ga and c ∈ Gb.
      15. So, a and b both have the same age as c.
      16. Hence a and b have the same age.
      17. This proves S(n + 1).
      18. Hence, by induction, S(n) is true for all n.
      19. This implies that all Americans have the same age.

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