Week 6 – Assignment 8

1. Prove or disprove the claim that there are integers m, n such that m2 +mn+n2 is a perfect square.
   1. True: m = 0, n = 2
2. Prove or disprove the claim that for any positive integer m there is a positive integer n such that mn + 1 is a perfect square.
   1. True: for an arbitrary m > 0, take n = m + 2; mn+1 = m(m+2)+1 = m2+2m+1=(m+1)2
3. Prove that there is a quadratic f(n) = n2 + bn + c, with positive integers coeﬃcients b, c, such that f(n) is composite (i.e., not prime) for all positive integers n, or else prove that the statement is false.
   1. True: Take f(n) = n2 + 2n + 1 = (n + 1)2, which is not prime.
4. Prove that if every even natural number greater than 2 is a sum of two primes (the Goldbach Conjecture), then every odd natural number greater than 5 is a sum of three primes.
   1. Assume (∀n>2)(∃p, q)(p, q are primes and p + q = n)
   2. Prove that (∀m>5)(∃p, q, r)(p, q, r are primes and p + q + r = m)
   3. (Let m = n + 3, r = 3, since 3 is prime), p + q + 3 = n + 3
   4. (Simplifying) p + q = n
   5. (Since n = n – 3, n > 2, so based on the assumption we can find p, q which are primes
   6. QED
5. Use the method of induction to prove that the sum of the ﬁrst n odd numbers is equal to n2.
   1. Base case: 1 = 12 = 1, ok
   2. Inductive step:
      1. Assume that SUM[i=1..n](n\*2-1)=n2
      2. (Adding (n+1)\*2-1 to both sides) SUM[i=1..n+1](n\*2-1)=n2+2(n+1)-1
      3. (Rewriting) SUM[i=1..n+1](n\*2-1)=n2+2n+2-1
      4. (Rewriting) SUM[i=1..n+1](n\*2-1)=n2+2n+1
      5. (Rewriting) SUM[i=1..n+1](n\*2-1)=(n+1)2
      6. QED
6. The notation SUM[i=1..n]ai is a common abbreviation for the sum a1 + a2 + a3 + ... + an. For instance, SUM[r=1..n](r2) denotes the sum 12 + 22 + 32 + ... + n2. Prove by induction that: ∀n∈N: SUM[r=1..n](r2) = 1/6 \* n(n+1)(2n+1)
   1. Base case: 1 = 1/6 \* 1(1+1)(2+1) = 1/6 \* 1\*2\*3 = 1, ok
   2. Inductive step:
      1. Assume that SUM[r=1..n](r2) = 1/6 \* n(n+1)(2n+1)
      2. (Adding (n+1)2 to both sides) SUM[r=1..(n+1)](r2) = 1/6 \* n(n+1)(2n+1) + (n+1)2
      3. (rewriting right side) 1/6\*(n2+n)(2n+1) + n2+2n+1
      4. (rewriting right side) (2n3+3n2+n)/6 + n2+2n+1
      5. (rewriting right side) (2n3+3n2+n+6n2+12n+6)/6
      6. (rewriting right side) (2n3+9n2+13n+6)/6
      7. (rewriting right side) (n+1)(2n2+7n+6)/6
      8. (rewriting right side) (n+1)(n+2)(2n+3)/6
      9. (rewriting right side) (n+1)((n+1)+1)(2(n+1)+1)/6
      10. QED

OPTIONAL

1. In the lecture we used induction to prove the general theorem 1 + 2 + ... + n = ½\*n(n+1). There is an alternative proof that does not use induction, famous because Gauss used the key idea to solve a “busywork” arithmetic problem his teacher gave to the class when he was a small child at school. The teacher asked the class to calculate the sum of the ﬁrst hundred natural numbers. Gauss noted that if 1 + 2 + ... + 100 = N, then you can reverse the order of the addition and the answer will be the same: 100 + 99 + ... + 1 = N. So by adding those two equations, you get 101 + 101 + ... + 101 = 2N, and since there are 100 terms in the sum, this can be written as 100·101 = 2N, and hence N = ½(100·101) = 5,050. Generalize Gauss’s idea to prove the theorem without recourse to the method of induction.
   1. 1+2+…+n = N
   2. n+(n-1)+…+1=N
   3. (n+1)+(n+1)+…+(n+1)=2N
   4. n(n+1)=2N
   5. N=½\*n(n+1)
   6. QED
2. Prove that for any ﬁnite collection of points in the plane, not all collinear, there is a triangle having three of the points as its vertices, which contains none of the other points in its interior.
   1. ???
3. Prove the following by induction:
   1. 4n −1 is divisible by 3
      1. Base: 0: 40-1=0, divisible by 3
      2. Base: 1: 41-1=3, divisible by 3
      3. Step: assuming that 4n-1 is divisible by 3: there is a number p such as 3\*p=4n-1,
      4. 4(n+1)-1 = 4\*4n-1 = 3\*4n+4n-1 = 3\*4n+3\*p = 3\*(4n+p), which is divisible by 3 QED
   2. (n + 1)! > 2n+3 for all n ≥ 5
      1. Base: (5+1)!=6!=720>256=28=25+3 – OK
      2. Step:
         1. ((n+1)+1)! =
         2. (n+2)! =
         3. (n+2)\*(n+1)! >
         4. (since n > 5) 2\*(n+1)! >
         5. (assumption) 2\*2n+3 =
         6. 2(n+1)+3
         7. QED
   3. ∀n ∈N : SUM[r=1..n](r.r!) = (n + 1)! −1
      1. Base: 1.1! = 1 = 2-1 = 2!-1 = (1 + 1)!-1
      2. Step: assuming SUM[r=1..n](r.r!) = (n + 1)! −1
         1. SUM[r=1..n+1](r.r!) =
         2. SUM[r=1..n](r.r!) + (n + 1)(n + 1)! = (hypothesis)
         3. (n + 1)! −1 + (n + 1)(n + 1)! =
         4. (n + 1)! −1 + n(n + 1)! + (n + 1)! =
         5. (n + 1)!(n + 2) – 1 =
         6. (n + 2)! – 1 =
         7. ((n + 1) + 1)! – 1
         8. QED

Week 6 – Problem Set 6

1. PPPPPPPPPPPPPPPPPP
2. Let m,n denote any two natural numbers. Is the following a valid proof that mn is odd iﬀ m and n are odd?
   1. If m,n are odd there are integers p,q such that m = 2p+1,n = 2q+1. Then mn = (2p+1)(2q+1) = 2(2pq + p + q) + 1, so mn is odd. That completes the proof.
   2. **INVALID**
      1. **Needs to prove the other side (mn is odd => m, n are odd)**
3. Take the sentence: You can fool some of the people some of the time, but you cannot fool all of the people all the time. Let x be a variable for a person, t a variable for a period of time, and let F(x,t) mean you can fool x at time t. Which of the following mathematical formulas is equivalent to the given statement?
   1. **∃x∃tF(x,t)∧∃x∃t¬F(x,t)**
   2. ∃x∃tF(x,t)∧¬∀x∃tF(x,t)
   3. ∃x∃tF(x,t)∧¬∃x∃tF(x,t)
   4. None of the above.
4. True or false? For any two statements φ and ψ, either φ ⇒ ψ or its converse is true (or both)
   1. **True.**
      1. (φ ⇒ ψ) ∨ (ψ ⇒ φ) = ¬φ ∨ ψ ∨ ¬ψ ∨ φ = True
5. Are the following two statements equivalent? ¬(φ ⇒ ψ) and φ∧(¬ψ)
   1. **Yes**
      1. ¬(φ ⇒ ψ) = ¬(¬φ ∨ ψ) = ¬¬φ ∧ ¬ψ = φ ∧ ¬ψ
6. Are the following two statements equivalent? (φ∨ψ) ⇒ θ and (φ ⇒ θ)∧(ψ ⇒ θ)
   1. **Yes.**
      1. (φ∨ψ) ⇒ θ
         1. (Implication equivalence) ¬(φ∨ψ) ∨ θ
         2. (NOR) (¬φ∧¬ψ) ∨ θ
         3. (Distributive) (¬φ∨ θ) ∧ (¬ψ ∨ θ)
         4. (Implication equivalence) (φ ⇒ θ)∧(ψ ⇒ θ)
      2. Truth table

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| φ | ψ | θ | φ∨ψ | (φ∨ψ) ⇒ θ | φ ⇒ θ | ψ ⇒ θ | (φ ⇒ θ)∧(ψ ⇒ θ) |
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | F |
| T | F | T | T | T | T | T | T |
| T | F | F | T | F | F | T | F |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | T | F | F |
| F | F | T | F | T | T | T | T |
| F | F | F | F | T | T | T | T |

1. True or false? There are inﬁnitely many natural numbers n for which √n is rational. (Before entering your answer, you should construct a proof of the statement or its negation, so you are sure.)
   1. **True**
      1. Assume there are a finite number of natural numbers n such as √n is rational, 1, 4, …, p, where p is the highest of these numbers
      2. Let q=(p+1)2
      3. √q is rational (p+1), and larger than p
      4. Therefore, p is not the highest of the numbers with such property, which is a contradiction
      5. Hence, there are infinitely many natural numbers
2. This argument is a proof that 1 = 2. Obviously it is incorrect. Identify exactly what the error is, and grade the purported proof according to the course rubric. (Remember, this is not a regular mathematics course of the kind you are probably familiar with. We are working on various elements of mathematical thinking, mathematical exposition, and the communication of mathematics. The rubric is designed to focus attention on all of those factors.) Your “Overall valuation” ﬁgure is the grade you would assign a student if s/he submitted this proof in a ﬁrst-year college mathematics course
   1. Argument to show that 1 = 2.
   2. We start with the identity
      1. 1−3 = 4−6
   3. Adding 9/4 to both sides to complete the squares, we get 1−3 + 9/4 = 4−6 + 9/4
   4. This factors as (1− 3/2)2 = (2− 3/2)2
   5. Taking the square root of both sides, 1− 3/2 = 2− 3/2
   6. Hence 1 = 2
   7. Evaluation:
      1. Logical correctness: 1 0 (cannot use division for integers)
         1. Used sqrt for negative numbers
      2. Clarity: 4 4
      3. Opening: 4 4
      4. Stating the conclusion: 4 4
      5. Reasons: 3 4
      6. Overall: 0 0
      7. => Total: 16 19

∀ ∃ φ ψ ¬ ∧ ∨ℕℚ∈∉ℝℤ≠⇐⇒⇔∃∄∀∴∧∨≤≥⊗