Week 7 – Assignment 9

1. Express as concisely and accurately as you can the relationship between b|a and a/b.
   1. b|a ⇔ a/b∈ℤ
2. Determine whether each of the following is true or false and prove your answer. (You saw these questions in the in-lecture quiz, so th ﬁrst part is a repeat, except that now you should know the right answers.) The focus of this assignment is to prove each of your answers.
   1. 0|7
      1. False. Divisibility (b|a) is not defined for b=0
   2. 9|0
      1. True: ∃q(9\*q = 0): q = 0
   3. 0|0
      1. False: Divisibility (b|a) is not defined for b=0
   4. 1|1
      1. True: ∃q(1\*q = 1): q = 1
   5. 7|44
      1. False
      2. Expressing: ¬∃q∈Z(7\*q=44)
      3. Rewriting: ∀q∈Z(7\*q≠44)
      4. Proof by contradiction
         1. Assume there’s a q∈Z such as 7\*q=44
         2. Since 7\*7=49, and 44 < 49, q < 7
         3. Since 7\*6=42, and 44 > 42, q > 6
         4. (q < 7) ∧ (q > 6) is false (contradiction)
         5. Therefore, there is no such q that satisfies the condition
   6. 7|(−42)
      1. True: ∃q∈Z(7\*q = -42): q = -6
   7. (−7)|49
      1. True: ∃q∈Z(-7\*q = 49): q = -7
   8. (−7)|(−56)
      1. True: ∃q∈Z(-7\*q = -56): q = 8
   9. (∀n ∈Z)[1|n]
      1. True: (∀n∈Z)(∃q∈Z)(1\*q=n): q = n
   10. (∀n ∈N)[n|0]
       1. True: (∀n∈ℕ)(∃q∈Z)(n\*q=0): q = 0 (assuming 0 does not belong to ℕ)
   11. (∀n ∈Z)[n|0]
       1. False: Counter-example: n = 0 (divisibility not defined)
3. Prove all the parts of the theorem in the lecture, giving the basic properties of divisibility. Namely, show that for any integers a,b,c,d, with a ≠ 0
   1. a|0, a|a
      1. a|0 ⇔ ∃q(aq = 0) 🡪 q = 0
      2. a|a ⇔ ∃q(aq = a) 🡪 q = a
   2. a|1 if and only if a = ±1
      1. <= side (a = ±1 ⇒ a|1):
         1. If a=1, 1|1, since ∃q(1q = 1) 🡪 q = 1
         2. If a=-1, -1|1, since ∃q(-1q = 1) 🡪 q = -1
      2. => side (a|1 ⇒ a = ±1):
         1. ∃q(aq = 1)
         2. Abs value: |aq| = |1|
         3. Expanding: |a|.|q|=1
         4. Only possible values are |a| = |q| = 1
         5. Therefore, a=±1
         6. Other alternative:
            1. From (f), |a|≤1
            2. From the definition of divisibility, a ≠ 0
            3. Therefore, a=±1
   3. if a|b and c|d, then ac|bd (for c ≠ 0)
      1. By definition of divisibility: ∃p,q(ap=b) ∧ (cq=d)
      2. So, ap.cq = bd
      3. Rewriting: ac (pq) = bd
      4. By the definition, ac|bd
   4. if a|b and b|c, then a|c (for b ≠ 0)
      1. (Definition): ∃p,q(ap=b) ∧ (bq=c)
      2. Rewriting: apq = c
      3. Grouping: a(pq) = c
   5. [a|b and b|a] if and only if a = ±b
      1. <= side (a = ±b ⇒ a|b):
         1. Take q=±1, existence is proven
      2. => side (a|b and b|a) ⇒ a = ±b
         1. ∃p,q(ap=b) ∧ (bq=a)
         2. Rewriting: a=bq=apq
         3. Dividing by a: 1 = pq
         4. Either p=q=1 or p=q=-1
         5. Hence, a = ±b
         6. Alternative:
            1. From f, a|b => |a|≤|b|
            2. From f, b|a => |b|≤|a|
            3. |a|≤|b| and |b|≤|a|
            4. Hence, a = ±b
   6. if a|b and b ≠ 0, then |a|≤|b|
      1. Since a|b, ∃q(b=d.a)
      2. |b|=|d|.|a|
      3. Since b≠0, |d| ≥ 1
      4. Hence |a|≤|b|
   7. if a|b and a|c, then a|(bx + cy) for any integers x,y.
      1. ∃p,q(a.p = b ∧ a.q = c)
      2. (bx + cy) = (apx + aqy) = a(px + qy)
      3. a|(bx + cy) ⇔ ∃r[ar = (bx + cy)]
      4. By 2, r = (px + qy).

OPTIONAL PROBLEMS

1. It is a standard result about primes (known as Euclid’s Lemma) that if p is prime, then whenever p divides a product ab, p divides at least one of a,b. Prove the converse, that any natural number having this property (for any pair a,b) must be prime.
   1. Not true. Take n=6, a=12, b=24. N divides ab, n divides a, n divides b, but n is not prime.
2. Try to prove Euclid’s Lemma. If you do not succeed, you can ﬁnd proofs in most textbooks on elementary number theory, and on the Web. Find a proof and make sure you understand it. If you ﬁnd a proof on the Web, you will need to check that it is correct. There are false mathematical proofs all over the Internet. Though false proofs on Wikipedia usually get corrected fairly quickly, they also occasionally become corrupted when a well-intentioned contributor makes an attempted simpliﬁcation that renders the proof incorrect. Learning how to make good use of Web resources is an important part of being a good mathematical thinker.

Week 7 – Problem Set 7

1. Say which of the following statements are true. (Leave the box blank to indicate that it is false.)
   1. 20|300
      1. **True: 15.20 = 300**
   2. 17|35
      1. False: 35/17 is not integer
   3. 5|0
      1. **True: 5.0 = 0**
   4. 0|5
      1. False: a|b is not defined for a=0
   5. 21|(−21)
      1. **True: 21.-1=-21**
2. Say whether the following proof is valid or not.
   1. Theorem. The square of any odd number is 1 more than a multiple of 8. (For example, 32 = 9 = 8 + 1, 52 = 25 = 3·8 + 1.)
   2. *Proof*: By the Division Theorem, any number can be expressed in one of the forms 4q, 4q+1, 4q+ 2, 4q + 3. So any odd number has one of the forms 4q + 1,4q + 3. Squaring each of these gives
   3. (4q + 1)2 = 16q2 + 8q + 1 = 8(2q2 + q) + 1
   4. (4q + 3)2 = 16q2 + 24q + 9 = 8(2q2 + 3q + 1) + 1
   5. In both cases the result is one more than a multiple of 8. This proves the theorem.
   6. **Valid proof**
3. Say whether the following veriﬁcation of the method of induction is valid or not?
   1. *Proof*: We have to prove that if
      1. A(1)
      2. (∀n)[A(n) ⇒ A(n + 1)]
   2. then (∀n)A(n).
   3. We argue by contradiction. Suppose the conclusion is false. Then there will be a natural number n such that ¬A(n). Let m be the least such number. By the ﬁrst condition, m > 1, so m = n + 1 for some n. Since n < m, A(n). Then by the second condition, A(n + 1), i.e., A(m). This is a contradiction, and that proves the result.
   4. **Valid proof**
4. Evaluate this purported proof, and grade it according to the course rubric.
   1. The *Fibonacci sequence* is deﬁned iteratively by setting F1 = F2 = 1 and thereafter letting Fn+2 = Fn + Fn+1. This fascinating sequence has been known for at least 1500 years. It has several connections to the natural world, some of which are described in the second lecture of Devlin’s mathematics survey course on iTunes University, listed as recommended supplementary viewing to this course. It also has a number of pleasing mathematical connections. Here is one:
   2. Theorem For any natural number n
      1. SUM[k=1..n]Fk2=FnFn+1
   3. Proof: By induction.
   4. For n = 1, the left-hand side is F12 = 12 = 1 and the right-hand side is F1F2 = 1.1 = 1, so the identity is valid for n = 1.
   5. Assume the identity holds for n. Then:
   6. SUM[k=1..n+1](Fk2) = SUM[k=1..n](Fk2) + Fn+12
      1. = FnFn+1 + Fn+12, by the induction hypothesis
      2. = Fn+1(Fn + Fn+1), by algebra
      3. = Fn+1Fn+2, by the deﬁnition of Fn+2
   7. which is the identity for n + 1. The proof is complete.
   8. **Evaluation:**
      1. **Logical correctness: 4 4**
      2. **Clarity: 2 4**
      3. **Opening: 4 4**
      4. **Stating the conclusion: 3 4**
      5. **Reasons: 4 2 (should explain first part, then explain that “The proof is complete *by the principle of induction*).**
      6. **Overall: 4 4**
      7. **=> Total: 21 22**
5. Evaluate this purported proof, and grade it according to the course rubric.
   1. The *Fibonacci sequence* is deﬁned iteratively by setting F1 = F2 = 1 and thereafter letting Fn+2 = Fn + Fn+1.
   2. Theorem For any natural number n
      1. SUM[k=1..n]Fk=Fn+2
   3. Proof: By induction.
   4. For n = 1, the left-hand side is F1 = 1 and the right-hand side is F2 = 1, so the identity is valid for n = 1.
      1. **Incorrect, the RHS is F3=2, so the identity is not valid for n=1**
   5. Assume the identity holds for n. Then:
   6. SUM[k=1..n+1](Fk) = SUM[k=1..n](Fk) + Fn+1
      1. = Fn+2 + Fn+1, by the induction hypothesis
      2. = Fn+3, by the deﬁnition of Fn+3
   7. which is the identity for n + 1. The proof is complete.
   8. **Evaluation:**
      1. **Logical correctness: 0 0**
      2. **Clarity: 4 4**
      3. **Opening: 4 4**
      4. **Stating the conclusion: 4 4**
      5. **Reasons: 2 2 (similar as previous)**
      6. **Overall: 0 2**
      7. **=> Total: 14 16**
6. Evaluate this purported proof, and grade it according to the course rubric.
   1. The *Fibonacci sequence* is deﬁned iteratively by setting F1 = F2 = 1 and thereafter letting Fn+2 = Fn + Fn+1.
   2. Theorem For any natural number n
      1. Fn ≥ (3/2)n-2
   3. Proof: We have F1 = 1 ≥ 2/3 = (3/2)-1 and F2 = 1 = (3/2)0, so the inequality is valid for n = 1, 2.
   4. Now assume the inequality holds for n, where n ≥ 2. Then:
   5. Fn+1 = Fn + Fn-1
      1. ≥ (3/2)n-2 + (3/2)n-3
      2. = (3/2)n-3 (3/2 + 1), by algebra
      3. = (3/2)n-3 (5/2)
      4. = (3/2)n-3 (10/4)
      5. ≥ (3/2)n-3 (9/4)
      6. = (3/2)n-3 (3/2)2
      7. = (3/2)n-1
   6. Which establishes the inequality for n + 1
   7. **Evaluation:**
      1. **Logical correctness: 3 4**
      2. **Clarity: 2 4**
      3. **Opening: 4 0 (there wasn’t any)**
      4. **Stating the conclusion: 3 0 (there wasn’t; missing something like “hence, by induction, the theorem is proved”)**
      5. **Reasons: 2 0**
      6. **Overall: 4 2**
      7. **=> Total: 18 10**

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