Week 8 – Assignment 10.1

Definitions:

* Upper bound: b is an upper bound of set A if (∀a∈A)(a≤b)
* Least upper bound - lub(A): b is lub(A) if b is upper bound of A and ∀c, c is upper bound of A ⇒ b ≤ c

1. Prove that the intersection of two intervals is again an interval. Is the same true for unions?
   1. Given i1: (a1, a2) and i2: (b1, b2), assuming a1 ≤ b1, the intersection is given as:
      1. Empty, if a2 < b1.
      2. Given by (b1, min(a2, b2))
   2. Not the case: for example, if i1: (1, 2) and i2: (3, 4), their union is not an interval.
2. Taking R as the universal set, express the following as simply as possible in terms of intervals and unions of intervals. (Note that A’ denotes the complement of the set A relative to the given universal set, which in this case is R. See the module on set theory.)
   1. [1,3]’
      1. (-∞,1) ∪ (3,∞)
   2. (1,7]’
      1. (-∞,1] ∪ (7,∞)
   3. (5,8]’
      1. (-∞,5] ∪ (8,∞)
   4. (3,7)∪[6,8]
      1. (3,8]
   5. (−∞,3)’∪(6,∞)
      1. [3,∞)
   6. {π}’
      1. (-∞,π) ∪ (π,∞)
   7. (1,4]∩[4,10]
      1. {4}
   8. (1,2)∩[2,3)
      1. ∅
   9. A’, where A = (6,8)∩(7,9]
      1. (-∞,7) ∪ [8,∞)
   10. A’, where A = (−∞,5]∪(7,∞)
       1. (5,7]
3. Prove that if a set A of integers/rationals/reals has an upper bound, then it has inﬁnitely many diﬀerent upper bounds.
   1. Proof by contradiction. Assume that there is a finite number of upper bounds.
   2. Let b’ be the largest of these upper bounds: (∀a∈A)(a ≤ b’).
   3. If we take b’ + 1, it will be in the same number system as b’ (integer if integer, rational if rational, real if real)
   4. By algebra, if a ≤ b’, then a ≤ b’+1 for integers, rationals and reals.
   5. Hence, b’+1 is an upper bound of A, which is a contradiction (since b’ was assumed to be the largest of such upper bounds, and b’+1 is larger than b’)
   6. Therefore, the assumption is false: there are infinitely many different upper bounds.
4. Prove that if a set A of integers/rationals/reals has a least upper bound, then it is unique
   1. Let b1 and b2 be arbitrary lub(A).
   2. By the definition (part 1), both b1 and b2 are upper bounds of A
   3. By the definition (part 2), b1 ≤ b2 and b2 ≤ b1.
   4. The only case where this is true is where b2 = b1
   5. Therefore, the lub(A) is unique (if it exists)
5. Let A be a set of integers, rationals, or reals. Prove that b is the least upper bound of A iﬀ:
   1. (∀a ∈ A)(a ≤ b); and
   2. whenever c < b there is an a ∈ A such that a > c.
      1. First statement says that b is an upper bound of A
      2. By definition, b is lub(A) iff there is no c < b such that c is an upper bound of A
      3. If and only if: for all c < b, there is an a ∈ A such that ¬(a≤c)
      4. If and only if: for all c < b, there is an a ∈ A such that a>c, which is the second point
6. The following variant of the above characterization is often found. Show that b is the lub of A iﬀ:
   1. (∀a ∈ A)(a ≤ b) ; and
   2. (∀ε > 0)(∃a ∈ A)(a > b− ε).
   3. PPPPPPPPPPPPPPPP
7. Give an example of a set of integers that has no upper bound
   1. {x ∈ ℤ | x ≥ 2}
8. Show that any ﬁnite set of integers/rationals/reals has a least upper bound.
   1. Let A be the set of finite numbers, and let m be the largest number of the set. Let’s show that lub(A)=m:
      1. m is an upper bound: (∀a ∈ A)(a ≤ m), since m is the largest number in the set
      2. m is the least upper bound: proof by contradiction
         1. Assume that there is another number n, n < m, such as n is upper bound of A
         2. Since m ∈ A, (∃a ∈ A)¬(a ≤ n)
         3. Rewriting, ¬(∀a ∈ A)(a ≤ n)
         4. Which means that n is not an upper build of A
         5. Therefore all other upper bounds of A are ≥ m
         6. Which by definition, makes m = lub(A)
9. Intervals:
   1. What is lub (a,b)?
      1. b
   2. What is lub [a,b]?
      1. b
   3. What is max (a,b)?
      1. Undefined
   4. What is max [a,b]?
      1. b
10. Let A = {|x−y| | x,y ∈ (a,b)}. Prove that A has an upper bound. What is lub A?
    1. A = [0, (b – a))
    2. PPPPPPPPPPPPPPPP
    3. lub(A) = (b – a)
11. Deﬁne the notion of a lower bound of a set of integers/rationals/reals.
    1. Definition of lower bound: b is a lower bound of set A if (∀a∈A)(b≤a)
12. Deﬁne the notion of a greatest lower bound (glb) of a set of integers/rationals/reals by analogy with our original deﬁnition of lub.
    1. glb(A) = b iff
       1. b is a lower bound of A
       2. ∀c, c is lower bound of A ⇒ c ≤ b
13. State and prove the analog of question 5 for greatest lower bounds.
    1. PPPPP
14. State and prove the analog of question 6 for greatest lower bounds.
    1. PPPPP
15. Show that the Completeness Property for the real number system could equally well have been deﬁned by the statement, “Any nonempty set of reals that has a lower bound has a greatest lower bound.”
    1. PPPPPPPPPPPPPPP
16. The integers satisfy the Completeness Property, but for a trivial reason. What is that reason?
    1. It is not dense PPPPPPPPPPP

Week 8 – Assignment 10.2

Definitions:

* an → a as n → ∞ iff (∀ε>0)(∃n∈ℕ)(∀m≥n)(|am-a|<ε)

1. Let A = {r ∈Q| r > 0 ∧ r2 > 3}. Show that A has a lower bound in Q but no greatest lower bound in Q. Give all details of the proof along the lines of the proof given in the lecture that the rationals are not complete.
2. In addition to the completeness property, the Archimedean property is an important fundamental property of R. It says is that if x,y ∈R and x,y > 0, there is an n ∈N such that nx > y. Use the Archimedean property to show that if r,s ∈R and r < s, there is a q ∈Q such that r < q < s. (Hint: pick n ∈N, n > 1/(s−r), and ﬁnd an m ∈N such that r < (m/n) < s.)
3. Formulate both in symbols and in words what it means to say that an ↛ a as n →∞.
4. Prove that (n/(n + 1))2 → 1 as n →∞.
   1. Need to prove that (∀ε>0)(∃n∈ℕ)(∀m≥n)(|(m/(m+1))2 - 1|<ε)
   2. i.e., (∀ε>0)(∃n∈ℕ)(∀m≥n)|(m2-m2-2m-1)/(m+1)2|<ε)
   3. i.e., (∀ε>0)(∃n∈ℕ)(∀m≥n)((2m-1)/(m+1)2) < ε)
   4. Need n such that (n+1)2/(2n+1) > 1/ε
   5. Pick n so big that (n+1)2/(2n+1) > 1/ε, since squared term grows faster than linear
   6. Let ε be given, need to find n such as (∀m≥n)(1-(m/(m+1))2 < ε
   7. PPPPPPPPPPPP
5. Prove that 1/n2 → 0 as n →∞.
   1. Need to prove that (∀ε>0)(∃n∈ℕ)(∀m≥n)(|(1/m2|<ε)
   2. i.e., (∀ε>0)(∃n∈ℕ)(∀m≥n)(1/m2 < ε)
   3. Let ε be given; if ε ≥ 1, take n = 2. For all m≥n, 1/m2 < 1 ≤ ε, this case is ok
   4. If ε < 1, take n = ceiling(1/ε), which is a number > 1. Then for all m≥n:
      1. 1/m2 ≤ 1/n2 < 1/(1/ε)2 = ε2 < ε
6. Prove that 1/2n → 0 as n →∞.
   1. Need to prove that (∀ε>0)(∃n∈ℕ)(∀m≥n)(|(1/2m|<ε)
   2. i.e., (∀ε>0)(∃n∈ℕ)(∀m≥n)(1/2m < ε)
   3. Let ε be given; if ε ≥ 1, take n = 2. For all m≥n, 1/2m < 1 ≤ ε, this case is ok
   4. If ε < 1, take n = ceiling(-log2(ε)), which is a number > 1. Then for all m≥n:
      1. 1/2m ≤ 1/2n ≤ 1/2-log2(ε) = 2log2(ε) = ε
7. We say a sequence {an}[n=1..∞] tends to inﬁnity if, as n increases, an increases without bound. For instance, the sequence {n}[n=1..∞] tends to inﬁnity, as does the sequence {2n}[n=1..∞]. Formulate a precise deﬁnition of this notion, and prove that both of these examples fulﬁll the deﬁnition.
   1. PPPPPPPPPPPPPP
8. Let {an}[n=1..∞] be an increasing sequence (i.e. an < an+1 for each n). Suppose that an → a as n →∞. Prove that a = lub{an|n ∈N}.
   1. PPPPPPPPPPPP
9. Prove that if {an}[n=1..∞] is increasing and bounded above, then it tends to a limit.
   1. PPPPPPPPPPPPPP

Week 8 – Problem Set 8

1. Say which of the following are true. (Leave the box empty to indicate that it’s false.)
   1. A set A of reals can have at most one least upper bound.
      1. **True**
   2. If a set A of reals has a lower bound, it has inﬁnitely many lower bounds.
      1. **True**
   3. If a set A of reals has both a lower bound and an upper bound, then it is ﬁnite.
      1. **False (an interval has both upper/lower bounds, but is not finite)**
   4. 0 is the least upper bound of the set of negative integers, considered as a subset of the reals.
      1. **False (since it’s considered in the subset of reals; -0.5 is less than 0 and greater than all members of the set**
2. 2. Which of the following say that b is the greatest lower bound of a set A of reals? (Leave the box empty to indicate that it does not say that.)
   1. **b ≤ a for all a ∈ A and if c ≤ a for all a ∈ A, then b ≥ c.**
   2. b ≤ a for all a ∈ A and if c ≤ a for all a ∈ A, then b > c.
   3. b < a for all a ∈ A and if c < a for all a ∈ A, then b ≥ c.
   4. b < a for all a ∈ A and if c ≤ a for all a ∈ A, then b ≥ c.
   5. **b ≤ a for all a ∈ A and if ε > 0 there is an a ∈ A such that a < b + ε.**
3. The Sandwich Theorem (also sometimes called the Squeeze Theorem) says that if{an}[n=1..∞], {bn}[n=1..∞], {cn}[n=1..∞] are sequences such that, from some point n0 onwards, an ≤ bn ≤ cn, and if lim n→∞ an = L , lim n→∞ cn = L, then {bn}[i=1..∞] is convergent and lim n→∞ bn = L. Taking the Sandwich Theorem to be correct (which it is), grade the following proof using the course rubric.
   1. Theorem lim n→∞ sin2 n/3n=0
   2. Proof: For any n, 0 ≤ sin2 n/3n ≤ 1/3n
   3. Clearly, lim n→∞ 1/3n = 0.
   4. Hence, by the Sandwich Theorem,
   5. lim n→∞ sin2 n/3n = 0 as required.
   6. **Evaluation:**
      1. **Logical correctness: 4 4**
      2. **Clarity: 3 4**
      3. **Opening: 3 4**
      4. **Stating the conclusion: 4 4**
      5. **Reasons: 4 4**
      6. **Overall: 4 4**
      7. **=> Total: 22 24**
4. Is the following proof of the Sandwich Theorem correct? Grade it according to the course rubric.
   1. Theorem (Sandwich Theorem) Suppose {an}[n=1..∞], {bn}[n=1..∞], {cn}[n=1..∞] are sequences such that, from some point n0 onwards, an ≤ bn ≤ cn.
   2. Suppose further that lim n→∞ an = L , lim n→∞ cn = L.
   3. Then {bn}[i=1..∞] is convergent and lim n→∞ bn = L.
   4. Proof: Since lim n→∞ an = L, we can ﬁnd an integer n1 such that n ≥ n1 ⇒|an −L| < ε
   5. Since lim n→∞ cn = L, we can ﬁnd an integer n2 such that n ≥ n2 ⇒|cn −L| < ε
   6. Let M = max{n0,n1,n2}. Then
      1. n ≥ M ⇒
      2. (−ε < an−L < ε) ∧ (−ε < cn−L < ε) ⇒
      3. − ε < an−L ≤ bn−L ≤ cn−L < ε (using an ≤ bn ≤ cn) ⇒
      4. − ε < bn−L < ε ⇒ |bn −L| < ε
   7. By the deﬁnition of a limit, this proves that {bn}[n=1..∞] is convergent and lim n→∞ bn = L, as required.
   8. **Evaluation:**
      1. **Logical correctness: 3 4**
      2. **Clarity: 3 4**
      3. **Opening: 4 0 (missing assumption that ε>0 is given)**
      4. **Stating the conclusion: 4 4**
      5. **Reasons: 2 4**
      6. **Overall: 4 0**
      7. **=> Total: 20 16**
5. Evaluate this purported proof, and grade it according to the course rubric.
   1. Theorem lim n→∞ (n + 1) / (2n + 1)=1/2.
   2. Proof: Let ε > 0 be given. Choose N large enough so that N ≥ 1/2ε.
   3. Then, for n ≥ N,
   4. |(n + 1)/(2n + 1) – 1/2|
   5. = |(2(n + 1)−(2n + 1))/2(2n + 1)|
   6. = |1/(2(2n + 1))|
   7. = 1/(2(2n + 1))
   8. < 1/2n + 1
   9. < 1/2n ≤ 1/2N ≤ ε
   10. By the deﬁnition of a limit, this proves the theorem.
   11. **Evaluation:**
       1. **Logical correctness: 4 4**
       2. **Clarity: 3 4**
       3. **Opening: 2 4**
       4. **Stating the conclusion: 4 4**
       5. **Reasons: 4 4**
       6. **Overall: 4 4**
       7. **=> Total: 21 24**

∀ ∃ φ ψ ¬ ∧ ∨ℕℚ∈∉ℝℤ≠⇐⇒⇔∃∄∀∴∧∨≤≥ε⊗∅