Week 9 – Test Flight Problem Set

1. Say whether the following is true or false and support your answer by a proof.
   1. (∃m ∈N)(∃n ∈N)(3m + 5n = 12)
   2. **The statement is false.**
   3. Proof by contradiction.
      1. Imagine that there are n and m in the naturals such as 3m + 5n = 12
      2. Since m,n ≥ 1 (by the definition of natural numbers in this course), each of the terms (3m, 5n) must be smaller than 12: 3m < 12 ∧ 5n < 12
      3. Using some algebra: m < 4 ∧ n < 2
      4. Therefore the only possible value for n is 1
      5. Replacing in the formula: 3m + 5 = 12
      6. Subtracting 5 on each side: 3m = 7
      7. 7 is not divisible by 7 (since it’s of form 3\*2 + 1), therefore m cannot be a natural number. That is a contradiction
      8. Therefore, the assumption is not valid: there aren’t natural numbers m, n that satisfy 3m + 5n = 12
2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder)
   1. **The statement is true**
   2. Proof: rewriting the statement using quantifiers:
      1. (∀n∈Z)(5|(n+(n+1)+(n+2)+(n+3)+(n+4))
      2. (∀n∈Z)(5|(5n+10)), by algebra
      3. (∀n∈Z)(5|(5(n+2)), by algebra
      4. That is true iff for any given n, (∃q∈Z)(5q=(5(n+2))
      5. That is true, for q = (n+2) - QED
3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number n2 + n + 1 is odd.
   1. **The statement is true**
   2. Proof: by cases
      1. An integer n can be written in the form 2x (even number) or 2x+1 (odd number), where x is an integer
      2. Case 1: n=2x:
         1. n2 + n + 1
         2. = (2x)2 + 2x + 1
         3. = 4x2 + 2x + 1
         4. = 2(2x2 + x) + 1, which is an odd number
      3. Case 2: n=2x+1
         1. n2 + n + 1
         2. = (2x+1)2 + (2x+1) + 1
         3. = 4x2 + 4x + 1 + 2x + 1 + 1
         4. = 4x2 + 6x + 2 + 1
         5. = 2(2x2 + 3x + 1) + 1, which is an odd number
      4. Since the statement is true for all cases of integer numbers, then the statement is true – QED
4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.
   1. Proof:
      1. An odd natural number p can be written in the form p=2m+1, where m is an integer.
      2. There are two cases for m: m is even or m is odd; let’s look at each one
      3. If m is even
         1. m=2n, where n is an integer
         2. Replacing in the equation: p = 2(2n)+1 = 4n+1
      4. If m is odd
         1. m=2n+1, where n is an integer
         2. Replacing in the equation: p = 2(2n + 1) + 1 = 4n+3
      5. That covers all cases, therefore, the proof is complete.
5. Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.
   1. Proof by cases: n can be written in the form n=3m, n=3m+1 or n=3m+2, where m is an integer (by the division theorem)
   2. Case 1: n = 3m: that’s trivial, n is divisible by 3 since 3\*m = n
   3. Case 2: n = 3m+1
      1. n + 2 = 3m + 1 + 2 = 3m + 3 = 3(m + 1)
      2. n + 2 is divisible by 3, since 3\*(m + 1) = n + 2
   4. Case 3: n = 3m+2
      1. n + 4 = 3m + 2 + 4 = 3m + 6 = 3(m + 2)
      2. n + 4 is divisible by 3, since 3\*(m + 2) = n + 4
   5. Having all cases covered, the proof is complete.
6. A classic unsolved problem in number theory asks if there are inﬁnitely many pairs of ‘twin primes’, pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

∀ ∃ φ ψ ¬ ∧ ∨ℕℚ∈∉ℝℤ≠⇐⇒⇔∃∄∀∴∧∨≤≥ε⊗∅