Week 9 – Evaluation exercises

Grade the answers according to the course rubric. Enter your grade (which should be a whole number between 0 and 24, inclusive) in the box below.

1. True or false? (∃m∈N)(∃n∈N)(3m+5n=12)
   1. It’s true. Let m=4,n=0. Then 3m+5n=12.
   2. **Evaluation:**
      1. **Logical correctness: 0 – by the definition in the course, 0 is not natural**
      2. **Clarity: 2 – ok**
      3. **Opening: 0 – there’s no opening**
      4. **Stating the conclusion: 2 – it’s stated, but not the existence**
      5. **Reasons: 4 – if 0 were a natural number, it would have been good**
      6. **Overall: 0**
      7. **=> Total: 8**
         1. **Rubric: 18**
2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).
   1. True. 1 + 2 + 3 + 4 + 5 = 15, which is divisible by 5.
   2. **Evaluation:**
      1. **Logical correctness: 0 – it conflates ∀ and ∃**
      2. **Clarity: 2 – ok**
      3. **Opening: 0 – there’s no opening**
      4. **Stating the conclusion: 4 – it’s stated**
      5. **Reasons: 2 – at least the arithmetic is correct**
      6. **Overall: 0**
      7. **=> Total: 8**
         1. **Rubric: 0**
3. Say whether the following is true or false and support your answer by a proof: For any integer *n*, the number *n*2+*n*+1 is odd.
   1. We prove it by induction.
   2. For n=1, n2+n+1=1+1+1=3, which is odd.
   3. Suppose n2+n+1 is odd. Then
   4. (n+1)2+(n+1)+1=n2+2n+1+n+1+1=n2+3n+2+1=(n+1)(n+2)+1
   5. But one of (n+1), (n+2) must be even, so (n+1)(n+2) is even. Hence (n+1)2+(n+1)+1 is odd. This proves the result by induction.
   6. **Evaluation:**
      1. **Logical correctness: 0 – the statement is about integers, proof only for positives**
      2. **Clarity: 4 – ok**
      3. **Opening: 4**
      4. **Stating the conclusion: 4 – it’s stated**
      5. **Reasons: 0 – Only “half” of a proof, didn’t use induction hypothesis**
      6. **Overall: 2**
      7. **=> Total: 14**
         1. **Rubric: 11**
4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.
   1. We prove it by induction. For n=1,4n+1=5, which is odd
   2. If it’s true for n, then 4(n+1)+1=4n+4+1=4n+5 and 4(n+1)+3=4n+4+3=4n+7, which are both odd. This proves the result by induction
   3. **Evaluation:**
      1. **Logical correctness: 0 – missing cases for 1 or 3; it’s a proof that the numbers are odd, not that it contains all natural numbers**
      2. **Clarity: 3 – ok**
      3. **Opening: 4**
      4. **Stating the conclusion: 4 – it’s stated**
      5. **Reasons: 2 – Didn’t use induction hypothesis, didn’t prove what was asked**
      6. **Overall: 1**
      7. **=> Total: 14**
         1. **Rubric: 0**
5. Prove that for any integer n, at least one of the integers n, n+2,n+4 is divisible by 3.
   1. Given m, by the Division Theorem, m=4n+q, where 0≤q<4. If we divide n by 3, either it divides evenly or it leaves a remainder of 1 or 2. So 3 has to divide one of n,n+2,n+4.
   2. **Evaluation:**
      1. **Logical correctness: 0 – has nothing to do with the problem**
      2. **Clarity: 1**
      3. **Opening: 3**
      4. **Stating the conclusion: 4 – it’s stated**
      5. **Reasons: 0**
      6. **Overall: 1**
      7. **=> Total: 9**
         1. **Rubric: 0**
6. A classic unsolved problem in number theory asks if there are infinitely many pairs of ‘twin primes’, pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.
   1. Suppose p,q is a pair of twin primes, where p>5. We show that it is impossible to extend p,q to be a prime triple Let N=p.q+1. Then, either N is prime or else there is a prime r such that r|N. It follows that there is no prime that can be added to give a prime triple.
   2. **Evaluation:**
      1. **Logical correctness: 0 – the reasoning has nothing to do with the problem**
      2. **Clarity: 2**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 0 – opening doesn’t lead to conclusion at all**
      6. **Overall: 0**
      7. **=> Total: 10**
         1. **Rubric: 4**
7. Prove that for any natural number n: 2 + 22 + 23 + . . . + 2n = 2n+1 – 2
   1. For n=1, the identity reduces to 2=22−2, which is true.
   2. Assume it hold for n. Then, adding 2n+1 to both sides of the identity,
   3. 2+22+23+...+2n+2n+1=2n+1−2+2n+1=2.2n+1−2=2n+2−2
   4. This is the identity at n+1. That completes the proof.
   5. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 2**
      4. **Stating the conclusion: 1 – missing “by the principle of induction”**
      5. **Reasons: 0 – opening doesn’t lead to conclusion at all**
      6. **Overall: 2**
      7. **=> Total: 13**
         1. **Rubric 12**
8. Prove (from the definition of a limit of a sequence) that if the sequence {an}[n=1..∞] tends to limit L as n→∞, then for any fixed number M>0, the sequence {Man}∞n=1 tends to the limit ML.
   1. By the assumption, we can find an N such that
   2. n≥N→|an=L|<ϵ/M
   3. Then,
   4. n≥N⇒|Man−ML|=M.|an−L|<M.ϵ/M=ϵ
   5. which shows that {Man}[n=1..∞] tends to the limit ML
   6. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 3 – missing “by the principle of induction”**
      5. **Reasons: 4 – opening doesn’t lead to conclusion at all**
      6. **Overall: 4**
      7. **=> Total: 23**
         1. **Rubric 15**
9. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An=∅. Prove that your example has the stated property.
   1. Let An=(1/n+1,1/n).
   2. For any x>0, we can find an m such that 1/m<x, and then x∉(1/m+1,1/m).
   3. Hence ⋂[n=1..∞]An=∅
   4. **Evaluation:**
      1. **Logical correctness: 3**
      2. **Clarity: 3**
      3. **Opening: 4**
      4. **Stating the conclusion: 3 – didn’t state the containment property**
      5. **Reasons: 2 – didn’t prove the containment property**
      6. **Overall: 3**
      7. **=> Total: 18**
         1. **Rubric 8**
10. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An consists of a single real number. Prove that your example has the stated property.
    1. Let An=(−1/n, +1/n).
    2. For any n,0∈An, so 0∈⋂[n=1..∞]An.
    3. On the other hand, if x≠0, then there is an m such that 1/m<|x|, and for that m, x∉Am, so x∉⋂[n=1..∞]An.
    4. Hence ⋂[n=1..∞]An={0}.
    5. **Evaluation:**
       1. **Logical correctness: 3 – missed containment property**
       2. **Clarity: 3**
       3. **Opening: 4**
       4. **Stating the conclusion: 3 – missing containment property**
       5. **Reasons: 2 – didn’t prove the containment property**
       6. **Overall: 3**
       7. **=> Total: 18**
          1. **Rubric 24**

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