Week 9 – Evaluation exercises

Grade the answers according to the course rubric. Enter your grade (which should be a whole number between 0 and 24, inclusive) in the box below.

Evaluation exercise 1

1. True or false? (∃m∈N)(∃n∈N)(3m+5n=12)
   1. It’s true. Let m=4,n=0. Then 3m+5n=12.
   2. **Evaluation:**
      1. **Logical correctness: 0 – by the definition in the course, 0 is not natural**
      2. **Clarity: 4 – ok**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4 – if 0 were a natural number, it would have been good**
      6. **Overall: 0**
      7. **=> Total: 16**
         1. **Rubric: 2, 4, 4, 4, 4, 0 = 18**
2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).
   1. True. 1 + 2 + 3 + 4 + 5 = 15, which is divisible by 5.
   2. **Evaluation:**
      1. **Logical correctness: 0 – it conflates ∀ and ∃**
      2. **Clarity: 0**
      3. **Opening: 0 – there’s no opening**
      4. **Stating the conclusion: 2 – it’s stated**
      5. **Reasons: 0 – at least the arithmetic is correct**
      6. **Overall: 0**
      7. **=> Total: 2**
         1. **Rubric: 0**
3. Say whether the following is true or false and support your answer by a proof: For any integer *n*, the number *n*2+*n*+1 is odd.
   1. We prove it by induction.
   2. For n=1, n2+n+1=1+1+1=3, which is odd.
   3. Suppose n2+n+1 is odd. Then
   4. (n+1)2+(n+1)+1=n2+2n+1+n+1+1=n2+3n+2+1=(n+1)(n+2)+1
   5. But one of (n+1), (n+2) must be even, so (n+1)(n+2) is even. Hence (n+1)2+(n+1)+1 is odd. This proves the result by induction.
   6. **Evaluation:**
      1. **Logical correctness: 0 – the statement is about integers, proof only for positives**
      2. **Clarity: 4 – ok**
      3. **Opening: 4**
      4. **Stating the conclusion: 4 – it’s stated**
      5. **Reasons: 0 – Only “half” of a proof, didn’t use induction hypothesis**
      6. **Overall: 2**
      7. **=> Total: 14**
         1. **Rubric: 1, 2, 2, 4, 2, 0 = 11**
4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.
   1. We prove it by induction. For n=1,4n+1=5, which is odd
   2. If it’s true for n, then 4(n+1)+1=4n+4+1=4n+5 and 4(n+1)+3=4n+4+3=4n+7, which are both odd. This proves the result by induction
   3. **Evaluation:**
      1. **Logical correctness: 0 – missing cases for 1 or 3; it’s a proof that the numbers are odd, not that it contains all natural numbers**
      2. **Clarity: 0 – ok**
      3. **Opening: 0**
      4. **Stating the conclusion: 2 – it’s stated**
      5. **Reasons: 0 – Didn’t use induction hypothesis, didn’t prove what was asked**
      6. **Overall: 0**
      7. **=> Total: 2**
         1. **Rubric: 0**
5. Prove that for any integer n, at least one of the integers n, n+2,n+4 is divisible by 3.
   1. Given m, by the Division Theorem, m=4n+q, where 0≤q<4. If we divide n by 3, either it divides evenly or it leaves a remainder of 1 or 2. So 3 has to divide one of n,n+2,n+4.
   2. **Evaluation:**
      1. **Logical correctness: 0 – has nothing to do with the problem**
      2. **Clarity: 0**
      3. **Opening: 2**
      4. **Stating the conclusion: 2 – it’s stated**
      5. **Reasons: 0**
      6. **Overall: 0**
      7. **=> Total: 4**
         1. **Rubric: 0**
6. A classic unsolved problem in number theory asks if there are infinitely many pairs of ‘twin primes’, pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.
   1. Suppose p,q is a pair of twin primes, where p>5. We show that it is impossible to extend p,q to be a prime triple Let N=p.q+1. Then, either N is prime or else there is a prime r such that r|N. It follows that there is no prime that can be added to give a prime triple.
   2. **Evaluation:**
      1. **Logical correctness: 0 – the reasoning has nothing to do with the problem**
      2. **Clarity: 0**
      3. **Opening: 2**
      4. **Stating the conclusion: 4**
      5. **Reasons: 0 – opening doesn’t lead to conclusion at all**
      6. **Overall: 0**
      7. **=> Total: 6**
         1. **Rubric: 0, 0, 4, 0, 0, 0 = 4**
7. Prove that for any natural number n: 2 + 22 + 23 + . . . + 2n = 2n+1 – 2
   1. For n=1, the identity reduces to 2=22−2, which is true.
   2. Assume it hold for n. Then, adding 2n+1 to both sides of the identity,
   3. 2+22+23+...+2n+2n+1=2n+1−2+2n+1=2.2n+1−2=2n+2−2
   4. This is the identity at n+1. That completes the proof.
   5. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 2**
      4. **Stating the conclusion: 1 – missing “by the principle of induction”**
      5. **Reasons: 0 – opening doesn’t lead to conclusion at all**
      6. **Overall: 2**
      7. **=> Total: 13**
         1. **Rubric 4, 4, 0 (missing stating induction), 0, 2, 2 = 12**
8. Prove (from the definition of a limit of a sequence) that if the sequence {an}[n=1..∞] tends to limit L as n→∞, then for any fixed number M>0, the sequence {Man}[n=1..∞] tends to the limit ML.
   1. By the assumption, we can find an N such that
   2. n≥N→|an=L|<ϵ/M
   3. Then,
   4. n≥N⇒|Man−ML|=M.|an−L|<M.ϵ/M=ϵ
   5. which shows that {Man}[n=1..∞] tends to the limit ML
   6. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 3**
      3. **Opening: 2**
      4. **Stating the conclusion: 2**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 19**
         1. **Rubric 4, 2, 0, 4, 2, 3 = 15**
9. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An=∅. Prove that your example has the stated property.
   1. Let An=(1/n+1,1/n).
   2. For any x>0, we can find an m such that 1/m<x, and then x∉(1/m+1,1/m).
   3. Hence ⋂[n=1..∞]An=∅
   4. **Evaluation:**
      1. **Logical correctness: 2**
      2. **Clarity: 2**
      3. **Opening: 2**
      4. **Stating the conclusion: 2 – didn’t state the containment property**
      5. **Reasons: 2 – didn’t prove the containment property**
      6. **Overall: 2**
      7. **=> Total: 12**
         1. **Rubric 0 (doesn’t satisfy containment requirement), 2, 2, 2, 2, 0 = 8**
10. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An consists of a single real number. Prove that your example has the stated property.
    1. Let An=(−1/n, +1/n).
    2. For any n,0∈An, so 0∈⋂[n=1..∞]An.
    3. On the other hand, if x≠0, then there is an m such that 1/m<|x|, and for that m, x∉Am, so x∉⋂[n=1..∞]An.
    4. Hence ⋂[n=1..∞]An={0}.
    5. **Evaluation:**
       1. **Logical correctness: 3 – missed containment property**
       2. **Clarity: 4**
       3. **Opening: 4**
       4. **Stating the conclusion: 3 – missing containment property**
       5. **Reasons: 3 – didn’t prove the containment property**
       6. **Overall: 3**
       7. **=> Total: 20**
          1. **Rubric 24**

Evaluation exercise 2

1. Say whether the following is true or false and support your answer by a proof. (∃m∈N)(∃n∈N)(3m+5n=12)
   1. It’s false. We need only look at values of m from 1 to 3 (since 3×4 = 12, which already gives the right-hand side) and values of n from 1 to 2 (since 5 × 3 = 15 ≥ 12). If you calculate 3m+5n for the six possible pairs in this range, you find that the answer is never 12. This proves the result.
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 2**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 22**
         1. **Rubric: 4, 4, 4, 4, 4, 4 = 24**
2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).
   1. False. Let n, n+1, n+2, n+3, n+4 be any five consecutive integers. Then
   2. n+(n+1)+(n+2)+(n+3)+(n+4)=5n+1+2+3+4=5n+8=5(n+1)+3
   3. which is not a multiple of 5 since in the Division Theorem it leaves a remainder of 3..
   4. **Evaluation:**
      1. **Logical correctness: 0**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 2 – it’s stated**
      5. **Reasons: 2**
      6. **Overall: 2**
      7. **=> Total: 14**
         1. **Rubric: 2, 4, 4, 4, 4, 2 = 20**
3. Say whether the following is true or false and support your answer by a proof: For any integer *n*, the number *n*2+*n*+1 is odd.
   1. For any n, n2+n+1=n(n+1)+1. But n(n+1) is always even (since one of n, n+1 is even and the other odd). Hence n(n+1) is always odd, as claimed.
   2. **Evaluation:**
      1. **Logical correctness: 3 – the statement is about integers, proof only for positives**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 0 – it’s stated incorrectly**
      5. **Reasons: 3**
      6. **Overall: 3**
      7. **=> Total: 17**
         1. **Rubric: 4, 3, 4, 3, 4, 2 = 20**
4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.
   1. This is not true. For example, if n=−1, which is an integer, then 4n+1=−3 and 4n+3=−1. But −3 and −1 are not natural numbers.
   2. **Evaluation:**
      1. **Logical correctness: 0 – missing cases for 1 or 3; it’s a proof that the numbers are odd, not that it contains all natural numbers**
      2. **Clarity: 0**
      3. **Opening: 0**
      4. **Stating the conclusion: 2**
      5. **Reasons: 0**
      6. **Overall: 0**
      7. **=> Total: 2**
         1. **Rubric: X, X, X, X, X, X = 0**
5. Prove that for any integer n, at least one of the integers n, n+2,n+4 is divisible by 3.
   1. n can be expressed in one of the forms 3q,3q+1,3q+2, for some q.
   2. In the first case, n is divisible by 3.
   3. In the second case n+2=3q+3=3(q+1), so n+2 is divisible by 3.
   4. In the third case n+4=3q+6=3(q+2), so n+4 is divisible by 3.
   5. **Evaluation:**
      1. **Logical correctness: 4 – has nothing to do with the problem**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 0**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 20**
         1. **Rubric: 4, 3, 3, 3, 0 (missing “by the division theorem”), 2 = 15**
6. A classic unsolved problem in number theory asks if there are infinitely many pairs of ‘twin primes’, pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.
   1. Let n,n+2,n+4 be any three successive natural numbers, where n>3. I show that 3 divides one of these numbers. If 3 does not divide n, then by the Division Theorem, n=3q+1 or n=3q+2, for some q. In the first case, n+2=3q+3, so 3|n, and in the second case n+4=3q+644, so again 3|n. Thus 3 must divide one of the three numbers, which means they cannot all be prime.
   2. **Evaluation:**
      1. **Logical correctness: 4 – the reasoning has nothing to do with the problem**
      2. **Clarity: 4**
      3. **Opening: 2: n, n+2, n+4 cannot be “successive natural numbers”**
      4. **Stating the conclusion: 4**
      5. **Reasons: 2 (typo on 644, stating that 3|n for all cases)**
      6. **Overall: 3**
      7. **=> Total: 19**
         1. **Rubric: 4, 4, 4, 4, 4, 0 (too many typos) = 20**
7. Prove that for any natural number n: 2 + 22 + 23 + . . . + 2n = 2n+1 – 2
   1. We prove the result by induction. For n=1, the identity reduces to 2=22−2, which is true. Assume it hold for n.
   2. Then, 2+22+23+...+2n+2n+1=2n+1−2+2n+1=2.2n+1−2=2n+2−2
   3. This is the identity at n+1. The result follows by induction.
   4. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 24**
         1. **Rubric: 4, 2, 4, 4, 2, 2 = 18**
8. Prove (from the definition of a limit of a sequence) that if the sequence {an}[n=1..∞] tends to limit L as n→∞, then for any fixed number M>0, the sequence {Man}[n=1..∞] tends to the limit ML.
   1. Pick ϵ > 0. Since {an}[n=1..∞] tends to limit L as n→∞, there is an N such that an is within a distance of ϵ/M of L whenever n>N. For any such n, Man is within a distance M(ϵ/M) = ϵ of ML. Hence {Man}[n=1..∞] tends to ML as n tend to ∞
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 24**
         1. **Rubric: 4, 3 (“pick ϵ > 0” not saying arbitrarily, definition of limits), 3, 4, 3, 3 = 20**
9. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An=∅. Prove that your example has the stated property.
   1. Take the sequence (0, 1), (0, 1/2). (0, 1/4), (0, 1/8), . . . That is, An = (0, 1/2n−1). Since {1/2n}[n=1..∞] tends to 0 as n→∞, ⋂[n=1..∞]An=∅
   2. **Evaluation:**
      1. **Logical correctness: 2 – the limit is not properly defined**
      2. **Clarity: 3**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 2 – big jump**
      6. **Overall: 3**
      7. **=> Total: 18**
         1. **Rubric: 4, 3, 4, 4, 2, 3 = 20**
10. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An consists of a single real number. Prove that your example has the stated property.
    1. Take An=[0,1/2n]. Then 0∈An for all n, so 0 ∈ ⋂[n=1..∞]An. By the same argument as in question 9 above, it follows that ⋂[n=1..∞]An = {0}
    2. **Evaluation:**
       1. **Logical correctness: 4**
       2. **Clarity: 3**
       3. **Opening: 4**
       4. **Stating the conclusion: 2 – missing containment property**
       5. **Reasons: 2 – big jump**
       6. **Overall: 2**
       7. **=> Total: 17**
          1. **Rubric: 4, 3, 4, 4, 3, 3 = 20**

Evaluation exercise 3

1. Say whether the following is true or false and support your answer by a proof. (∃m∈N)(∃n∈N)(3m+5n=12)
   1. It’s false. If n≥2, then for any m,3m+5n≥13, so we need only show that there is no m such that 3m+5=12, i.e. no m such that 3m=7. This is immediate.
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 2**
      4. **Stating the conclusion: 2**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 20**
         1. **Rubric: X, X, X, X, X, X = X**
2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).
   1. True. Let n,n+1,n+2,n+3,n+4 be any five consecutive integers. Then
   2. n+(n+1)+(n+2)+(n+3)+(n+4)=5n+1+2+3+4=5n+10=5(n+2)
   3. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 2**
      3. **Opening: 4**
      4. **Stating the conclusion: 2**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 20**
         1. **Rubric: X, X, X, X, X, X = X**
3. Say whether the following is true or false and support your answer by a proof: For any integer *n*, the number *n*2+*n*+1 is odd.
   1. True. Consider the two case n even and n odd separately. If n is even, say n=2k, then
   2. n2+n+1=4k2+2k+1=2(2k2+k)+1
   3. which is odd. If n is odd, say n=2k+1, then
   4. n2+n+1=(2k+1)2+(2k+1)+1=4k2+4k+1+2k+1+1=4k2+6k+2+1=2(2k2+3k+1)+1
   5. which is odd. In both cases, n2+n+1 is odd.
   6. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 24**
         1. **Rubric: X, X, X, X, X, X = X**
4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.
   1. Let m be a natural number. By the Division Theorem, there are unique numbers n, r such that m=4n+r, where 0≤r<4. Thus m is one of 4n,4n+1,4n+2,4n+3. Since 4n and 4n+2 are even, if m is odd, the only possibilities are 4n+1 and 4n+3.
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 24**
         1. **Rubric: X, X, X, X, X, X = X**
5. Prove that for any integer n, at least one of the integers n, n+2,n+4 is divisible by 3.
   1. By the Division Theorem, n can be expressed in one of the forms 3q,3q+1,3q+2, for some q. In the first case, n is divisible by 3. In the second case n+2=3q+3=3(q+1), so n+2 is divisible by 3. In the third case n+4=3q+6=3(q+2), so n+4 is divisible by 3.
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 2**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 22**
         1. **Rubric: X, X, X, X, X, X = X**
6. A classic unsolved problem in number theory asks if there are infinitely many pairs of ‘twin primes’, pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.
   1. Consider any three numbers of the form n, n+2, n+4, where n>3. By the answer to the previous question, one of these numbers is divisible by 3, and hence is not prime.
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 24**
         1. **Rubric: X, X, X, X, X, X = X**
7. Prove that for any natural number n: 2 + 22 + 23 + . . . + 2n = 2n+1 – 2
   1. Let S=2+22+23+...+2n. Then 2S=22+23+24+...+2n+2n+1. Subtracting the first identity from the second gives 2S−S=2n+1−2. But 2S−S=S, so this establishes the stated identity.
   2. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 4**
      6. **Overall: 4**
      7. **=> Total: 24**
         1. **Rubric: X, X, X, X, X, X = X**
8. Prove (from the definition of a limit of a sequence) that if the sequence {an}[n=1..∞] tends to limit L as n→∞, then for any fixed number M>0, the sequence {Man}[n=1..∞] tends to the limit ML.
   1. Let ϵ > 0 be given. By the assumption, we can find an N such that
   2. n≥N⇒|an−L|<ϵ/M
   3. Then,
   4. n≥N⇒|Man−ML|=M.|an−L|<M.ϵ/M=ϵ
   5. which shows that {Man}[n=1..∞] tends to the limit ML
   6. **Evaluation:**
      1. **Logical correctness: 4**
      2. **Clarity: 4**
      3. **Opening: 4**
      4. **Stating the conclusion: 2**
      5. **Reasons: 3**
      6. **Overall: 4**
      7. **=> Total: 22**
         1. **Rubric: X, X, X, X, X, X = X**
9. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An=∅. Prove that your example has the stated property.
   1. Let An=(0,1/n). Clearly, ⋂[n=1..∞]An ⊆A1=(0,1). Hence any element of the intersection must be a member of (0,1). But if x∈(0,1), we can find a natural number n such that 1/n<x. Then x∉An, so x∉⋂[n=1..∞]An. Thus ⋂[n=1..∞]An = ∅.
   2. **Evaluation:**
      1. **Logical correctness: 3 – didn’t prove the containment property for all**
      2. **Clarity: 3**
      3. **Opening: 4**
      4. **Stating the conclusion: 4**
      5. **Reasons: 3**
      6. **Overall: 3**
      7. **=> Total: 21**
         1. **Rubric: X, X, X, X, X, X = X**
10. Given a collection An, n=1,2,… of intervals of the real line, their intersection is defined to be ⋂[n=1..∞]An={x|(∀n)(x∈An)}. Give an example of a family of intervals An, n=1,2,…, such that An+1⊂An for all n and ⋂[n=1..∞]An consists of a single real number. Prove that your example has the stated property.
    1. Let An = [0, 1/n). Clearly, 0∈⋂[n=1..∞]An. But the same argument as above shows that no other number is in the intersection. Hence ⋂[n=1..∞]An = {0}.
    2. **Evaluation:**
       1. **Logical correctness: 2 (no containment proof)**
       2. **Clarity: 3**
       3. **Opening: 4**
       4. **Stating the conclusion: 4**
       5. **Reasons: 2**
       6. **Overall: 3**
       7. **=> Total: 18**
          1. **Rubric: X, X, X, X, X, X = X**

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