Week 9 – Test Flight Problem Set

1. Say whether the following is true or false and support your answer by a proof.
   1. (∃m ∈N)(∃n ∈N)(3m + 5n = 12)
   2. **The statement is false.**
   3. Proof by contradiction.
      1. Imagine that there are n and m in the naturals such as 3m + 5n = 12
      2. Since m, n ≥ 1 (by the definition of natural numbers in this course), each of the terms (3m, 5n) must be smaller than 12: 3m < 12 ∧ 5n < 12
      3. Using some algebra: m < 4 ∧ n < 2
      4. Therefore, the only possible value for n is 1
      5. Replacing in the formula: 3m + 5 = 12
      6. Subtracting 5 on each side: 3m = 7
      7. 7 is not divisible by 7 (since it’s of form 3\*2 + 1), therefore m cannot be a natural number. That is a contradiction
      8. Therefore, the assumption is not valid: there aren’t natural numbers m, n that satisfy 3m + 5n = 12
2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder)
   1. **The statement is true**
   2. Proof: rewriting the statement using quantifiers:
      1. (∀n∈Z)(5|(n+(n+1)+(n+2)+(n+3)+(n+4))
      2. (∀n∈Z)(5|(5n+10)), by algebra
      3. (∀n∈Z)(5|(5(n+2)), by algebra
      4. That is true iff for any given n, (∃q∈Z)(5q=(5(n+2))
      5. That is true, for q = (n+2) - QED
3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number n2 + n + 1 is odd.
   1. **The statement is true**
   2. Proof: by cases
      1. An integer n can be written in the form 2x (even number) or 2x+1 (odd number), where x is an integer
      2. Case 1: n=2x:
         1. n2 + n + 1
         2. = (2x)2 + 2x + 1
         3. = 4x2 + 2x + 1
         4. = 2(2x2 + x) + 1, which is an odd number
      3. Case 2: n=2x+1
         1. n2 + n + 1
         2. = (2x+1)2 + (2x+1) + 1
         3. = 4x2 + 4x + 1 + 2x + 1 + 1
         4. = 4x2 + 6x + 2 + 1
         5. = 2(2x2 + 3x + 1) + 1, which is an odd number
      4. Since the statement is true for all cases of integer numbers, then the statement is true – QED
4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.
   1. Proof:
      1. An odd natural number p can be written in the form p=2m+1, where m is an integer.
      2. There are two cases for m: m is even or m is odd; let’s look at each one
      3. If m is even
         1. m=2n, where n is an integer
         2. Replacing in the equation: p = 2(2n)+1 = 4n+1
      4. If m is odd
         1. m=2n+1, where n is an integer
         2. Replacing in the equation: p = 2(2n + 1) + 1 = 4n+3
      5. That covers all cases, therefore, the proof is complete.
5. Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.
   1. Proof by cases: n can be written in the form n=3m, n=3m+1 or n=3m+2, where m is an integer (by the division theorem)
   2. Case 1: n = 3m: that’s trivial, n is divisible by 3 since 3\*m = n
   3. Case 2: n = 3m+1
      1. n + 2 = 3m + 1 + 2 = 3m + 3 = 3(m + 1)
      2. n + 2 is divisible by 3, since 3\*(m + 1) = n + 2
   4. Case 3: n = 3m+2
      1. n + 4 = 3m + 2 + 4 = 3m + 6 = 3(m + 2)
      2. n + 4 is divisible by 3, since 3\*(m + 2) = n + 4
   5. Having all cases covered, the proof is complete.
6. A classic unsolved problem in number theory asks if there are inﬁnitely many pairs of ‘twin primes’, pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.
   1. Rewriting using quantifiers: ¬(∃n>3)(isPrime(n) ∧ isPrime(n+2) ∧ isPrime(n+4))
   2. Proof:
      1. Moving negation to inside: (∀n>3)¬(isPrime(n) ∧ isPrime(n+2) ∧ isPrime(n+4))
      2. Rewriting: (∀n>3)(¬isPrime(n) ∨ ¬isPrime(n+2) ∨ ¬isPrime(n+4))
      3. Let n be given; Since n > 3, then n is of one of the following the forms (by the division theorem), where m is a natural number greater than 1: 3m+1, 3m+2 or 3m+3
         1. In the first case, 3m+1+2 = 3m+3 = 3(m+1), which is not prime (divisible by 3 and m+1), therefore the disjunction (¬isPrime(n) ∨ True ∨ ¬isPrime(n+4)) is true
         2. In the second case, 3m+2+4 = 3m+6 = 3(m+2), which is not prime (divisible by 3 and m+2), therefore the disjunction (¬isPrime(n) ∨ ¬isPrime(n+2) ∨ True) is true
         3. In the third case, 3m+3 = 3(m+1), which is not prime (divisible by 3 and m+1), therefore the disjunction (True ∨ ¬isPrime(n+2) ∨ ¬isPrime(n+4)) is true
      4. Having all cases covered, the proof is complete.
7. Prove that for any natural number n, 2 + 22 + 23 + ... + 2n = 2n+1 −2
   1. Proof by induction
   2. Base case: n = 1: 2 = 21+1 – 2 = 4 – 2 = 2 🡪 the identity holds for n = 1
   3. Inductive case: assuming that 2 + 22 + 23 + ... + 2n = 2n+1 −2
      1. Adding 2n+1 on both sides: 2 + 22 + 23 + ... + 2n + 2n+1 = 2n+1 −2 + 2n+1
      2. Algebra manipulation: 2 + 22 + 23 + ... + 2n + 2n+1 = 2\*2n+1 −2
      3. Algebra manipulation: 2 + 22 + 23 + ... + 2n + 2n+1 = 2n+2 −2
      4. Which is the identity for n+1, so it proves that if the identity holds for n, it holds for n+1
   4. By the principle of induction, the proof is complete
8. Prove (from the deﬁnition of a limit of a sequence) that if the sequence {an}[n=1..∞] tends to limit L as n →∞, then for any ﬁxed number M > 0, the sequence {Man}[n=1..∞] tends to the limit ML.
   1. Theorem: (∀M>0)((an → L as n → ∞) ⇒ (Man → ML as n → ∞))
   2. Proof:
      1. Let M>0 be given. Need to prove that the consequent is true if the antecedent is true.
      2. Let ε>0 be given; need to find N such as (∀m≥N)(|Mam-ML|<ε)
      3. By algebra, since M > 0, we need to find N such as (∀m≥N)(|am-L|<(ε/M))
      4. From the antecedent (the original sequence tends to limit L as n→∞), there is a n’ such that (∀m≥n’)(|am-L|<(ε/M))
      5. That gives us the N = n’, and since the value of ε is arbitrary, by the definition of limits this proves that the limit of the sequence {Man}[n=1..∞] is ML if the limit of the sequence {an}[n=1..∞] is L. That concludes the proof.
9. Given an inﬁnite collection An, n = 1, 2, ... of intervals of the real line, their intersection is deﬁned to be ⋂[n=1..∞](An) = {x|(∀n)(x ∈ An)}. Give an example of a family of intervals An, n = 1,2,..., such that An+1 ⊂ An for all n and ⋂[n=1..∞](An) = ∅. Prove that your example has the stated property.
   1. Let An=(0,1/n). Examples:
      1. A1 = (0, 1)
      2. A2 = (0, 1/2)
      3. A3 = (0, 1/3)
   2. Theorem 1: the containment property holds - An+1 ⊂ An for all n
   3. Proof 1: An = (0,1/n) = {x|0<x<1/n}; An+1 = (0,1/(n+1)) = {x|0<x<1/(n+1)}
      1. ∀x(x∈An+1 ⇒ x∈An)
      2. Rewriting, ∀x∈ℝ,∀x∈ℕ(0 < x < 1/(n+1) ⇒ 0 < x < 1/n), which is trivially true
      3. Therefore, An+1 ⊆ An.
      4. Since An+1 ≠ An (as the latter contains (1/n + 1/(n+1))/2 and the former does not), An+1 ⊂ An. The proof 1 is complete.
   4. Theorem 2: lim n →∞ An = ∅
   5. Proof 2: An = {x|0<x<(1/n)};
      1. lim n→∞ 1/n = 0, since for any given ε > 0, for all n>(1/ε), 1/n < ε
      2. Hence, lim n →∞ An = {x|0 < x < 0}, which is empty as there are no values that are both smaller than and greater than 0. The proof is complete.
   6. Theorem 3: ⋂[n=1..∞](An) = ∅
   7. Proof 3: by theorem 2, lim n →∞ An = ∅
      1. For all sets B, ∅∩B = ∅
      2. Therefore, then ⋂[n=1..∞](An) = ∅. The final proof is complete.
10. Give an example of a family of intervals An, n = 1,2,..., such that An+1 ⊂ An for all n and ⋂[n=1..∞](An) consists of a single real number. Prove that your example has the stated property.
    1. Let An=[0,1/n]. Examples:
       1. A1 = [0, 1]
       2. A2 = [0, 1/2]
       3. A3 = [0, 1/3]
    2. Theorem 1: the containment property holds - An+1 ⊂ An for all n
    3. Proof 1: An = [0,1/n] = {x|0≤x≤1/n}; An+1 = (0,1/(n+1)) = {x|0≤x≤1/(n+1)}
       1. ∀x(x∈An+1 ⇒ x∈An)
       2. Rewriting, ∀x∈ℝ,∀x∈ℕ(0 ≤ x ≤ 1/(n+1) ⇒ 0 ≤ x ≤ 1/n), which is trivially true
       3. Therefore, An+1 ⊆ An.
       4. Since An+1 ≠ An (as the latter contains (1/n + 1/(n+1))/2 and the former does not), An+1 ⊂ An. The proof 1 is complete.
    4. Theorem 2: lim n →∞ An = {0}
    5. Proof 2: An = {x|0≤x≤(1/n)};
       1. lim n→∞ 1/n = 0, since for any given ε > 0, for all n>(1/ε), 1/n < ε
       2. Hence, lim n →∞ An = {x|0 ≤ x ≤ 0}, which is the unary set with the only element that is both less than or equal and greater than or equal to 0, {0}. The proof is complete.
    6. Theorem 3: ⋂[n=1..∞](An) = {0}
    7. Proof 3: by theorem 2, lim n →∞ An = {0}
       1. The number 0 is a member of all sets
       2. Therefore, then ⋂[n=1..∞](An) = {0}. The final proof is complete.

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