1. Probability Review

   There are n archers all shooting at the same target (bullseye) of radius 1. Let the score for a particular archer be defined to be the distance away from the center (the lower the score the better, and 0 is the optimal score). Each archer's score is independent of the others, and is distributed uniformly between 0 and 1. What is the expected value of the worst (highest) score?

Solution: For this problem, we compute the CDF, take the derivative to get the PDF, then calculate the expectation. We defined a random variable $Z = \max\{X_1, \ldots, X_n\}$.

- Computing the CDF:
  $F(z) = P(Z \leq z) = P(X_1 \leq z)P(X_2 \leq z)\ldots P(X_n \leq z) = \prod_{i=1}^n P(X_i \leq z)$

$$= \begin{cases} 0 & \text{if } z < 0 \\ z^n & \text{if } 0 \leq z \leq 1 \\ 1 & \text{if } z > 1 \end{cases}$$

- Computing the PDF:

$$f(z) = \frac{d}{dz}F(z) = \begin{cases} nz^{n-1} & \text{if } 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Computing $E[Z]$:

$$E[Z] = \int_{-\infty}^{\infty} zf(z)\mathrm{d}z = \int_0^1 znz^{n-1}\mathrm{d}z = n\int_0^1 z^n\mathrm{d}z = \frac{n}{n+1}$$

2. Maximum Likelihood Estimation

   Given N i.i.d. Poisson random variables, $x_1, x_2, ..., x_N$, find the maximum likelihood estimator for the parameter of the distribution, λ. Recall for a Poisson R.V., $p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$.

Solution:

$$\begin{aligned}
\mathcal{L}(x_1, x_2, ..., x_N; \lambda) &= p(x_1, x_2, ..., x_N; \lambda) \\
&= p(x_1; \lambda)p(x_2; \lambda) \cdots p(x_N; \lambda) \\
&= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \cdots \frac{e^{-\lambda}\lambda^{x_N}}{x_N!} \\
&= \frac{e^{-N\lambda}\lambda^{\sum x_i}}{\prod x_i!}
\end{aligned}$$

Taking the log yields the log-likelihood:

$$\ln \mathcal{L}(x_1, x_2, ..., x_N; \lambda) = -N\lambda + (\ln \lambda) \sum x_i - \ln \prod x_i!$$

Taking the derivative and setting it to zero:

$$\begin{aligned}
\frac{d(\ln \mathcal{L})}{d\lambda} &= -N + \frac{\sum x_i}{\lambda} = 0 \\
\hat{\lambda} &= \frac{\sum x_i}{N}
\end{aligned}$$

3. Linear Algebra

Find the eigenvalues and corresponding eigenvectors of the following matrix.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

Solution: Remember that an eigenvector is a vector v such that $Av = \lambda v$, where the constant λ is the eigenvalue corresponding to v. We manipulate the above equation to be $(A - \lambda I)v = 0$, which implies that $A - \lambda I$ is a singular matrix since it has an eigenvalue of 0.

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix}$$

We can take the determinant of the above matrix and set it to zero in order for the matrix to be singular, giving us the following characteristic polynomial:

$$-(2-\lambda)(1+\lambda) = 0$$

$$\lambda_1 = 2, \lambda_2 = -1$$

We can plug each λ back into $A - \lambda I$ and solve for the corresponding eigenvectors. Note that the eigenvectors can be scaled up or down arbitrarily by a constant factor.

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} v_1 = 0 \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} v_2 = 0 \rightarrow v_2 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

4. Projections

Given a plane $x + y + z = 4$ and point A located at $(2, 6, 8)$, find the coordinates of the closest point B on the plane to A. What is the distance between A and B?

Solution:

The plane $x + y + z = 4$ has normal vector $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, which is by definition perpendicular to the plane. The vector between points A and B will also be perpendicular to the plane. Thus, we can follow the normal vector from A back to the plane; the point of intersection will be point B.

Therefore, we define w as a constant attached to the normal vector. Note that if the normal vector had a length of 1, then w would be the distance from point A to point B.

$$\begin{aligned}
\left(\begin{pmatrix} 2 & 6 & 8 \end{pmatrix} - w\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} &= 4 \\
\begin{pmatrix} 2 - w & 6 - w & 8 - w \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} &= 4 \\
16 - 3w &= 4 \\
w &= 4
\end{aligned}$$

Then, the coordinates of B are:

$$\left(\begin{pmatrix} 2 & 6 & 8 \end{pmatrix} - 4\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} -2 & 2 & 4 \end{pmatrix}$$

We can then use the Euclidean distance formula to determine the distance between A and B:

$$\sqrt{(2+2)^2 + (6-2)^2 + (8-4)^2} = 4\sqrt{3}$$