1. MLE of the Laplace Distribution

   Let X have a Laplace distribution with density

   $$p(x; \mu, b) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$$

   Suppose that n samples $x_1, \ldots, x_n$ are drawn independently according to $p(x; \mu, b)$.

   (a) Find the maximum likelihood estimate of μ.

   (b) Find the maximum likelihood estimate of b.

   (c) Assume that μ is given. Show that b_{MLE} is an unbiased estimator (to show that the estimator is unbiased, show that $\mathrm{E}\left[b_{\text{MLE}} - b\right] = 0$).

2. Transforming a Standard Normal Multivariate Gaussian

   We are given a 2 dimensional multivariate Gaussian random variable Z, with mean 0 and covariance I. We want to transform this into something cooler. Find the covariance matrix of a multivariate Gaussian such that the axes x_1 and x_2 of the isocontours of the density are elliptically shaped with major/minor axis lengths in a 4:3 ratio, and the axes are rotated 45 degrees counterclockwise.

3. Multivariate Gaussian

   a) True or False

      (i) If X_1 and X_2 are both normally distributed and independent, then (X_1, X_2) must have multivariate normal distribution.

      (ii) If (X_1, X_2) has multivariate normal distribution, then X_1 and X_2 are independent.

   b) Affine transformation

      $X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}^{\mathrm{T}}$ is a n-dimensional random vector which has multivariate normal distribution. If $X \sim \mathcal{N}(\mu, \Sigma)$ and $Y = BX + c$ is an affine transformation of X, where c is a constant $m \times 1$ vector and B is a constant $m \times n$ matrix, what is the expectation and variance of Y?

4. [Extra for Experts] Linear Algebra

   a) Let A be a square matrix. Show that we can write A as the sum of a symmetric matrix A_+ and an antisymmetric matrix A_-:

   $$A = A_+ + A_-$$

   where $A_+ = A_+^{\mathrm{T}}$ and $A_- = -A_-^{\mathrm{T}}$.

   b) Show that if A_- is antisymmetric, then $x^{\mathrm{T}} A_- x = 0$ for all nonzero x.

   c) Show that the inverse of a positive definite matrix is positive definite.

   d) Any multivariate Gaussian distribution can be defined by two parameters, μ and Σ. It is common to assume that Σ is a positive definite matrix. Explain how we can find a Gaussian distribution corresponding to any square matrix Λ, which satisfies only $z^{\mathrm{T}} \Lambda z > 0 \ \forall z \neq 0$, but is not necessarily symmetric, and therefore not PD. Hint: make use of parts a), b), and c).