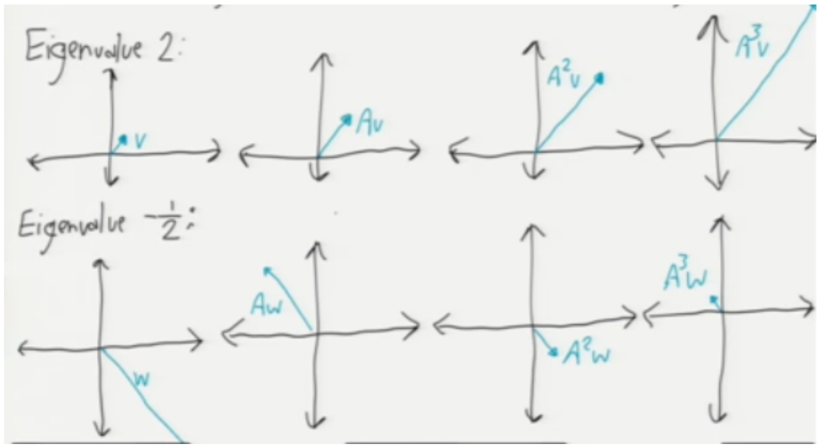
02/17/2016

Eigenvectors

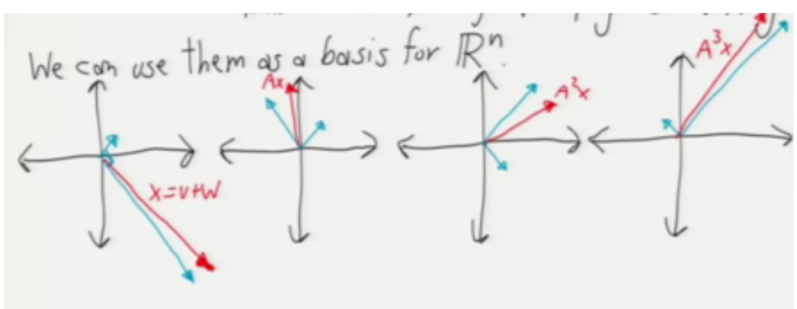
- Given a matrix A, if $Av = \lambda v$ for some vector $v \neq 0$, scalar λ, then v is an eigenvector of A and λ is the associated eigenvalue of A.



- Theorem: if v is an eigenvector of A with eigenvalue λ, then v is an eigenvector of A^k with eigenvalue λ^k.
- Proof: $A^2v = A(\lambda v) = \lambda^2 v$ etc.
- Theorem: moreover, if A is invertible, then v is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.
- Proof: $A^{-1}v = \frac{1}{\lambda}A^{-1}Av = \frac{1}{\lambda}v$.
- Spectral Theorem: Every symmetric nxn matrix has n eigenvectors that are mutually orthogonal,

$$v_i^T v_j = 0, \forall i \neq j$$

- We can use them as a basis for $\mathbb{R}^n$.

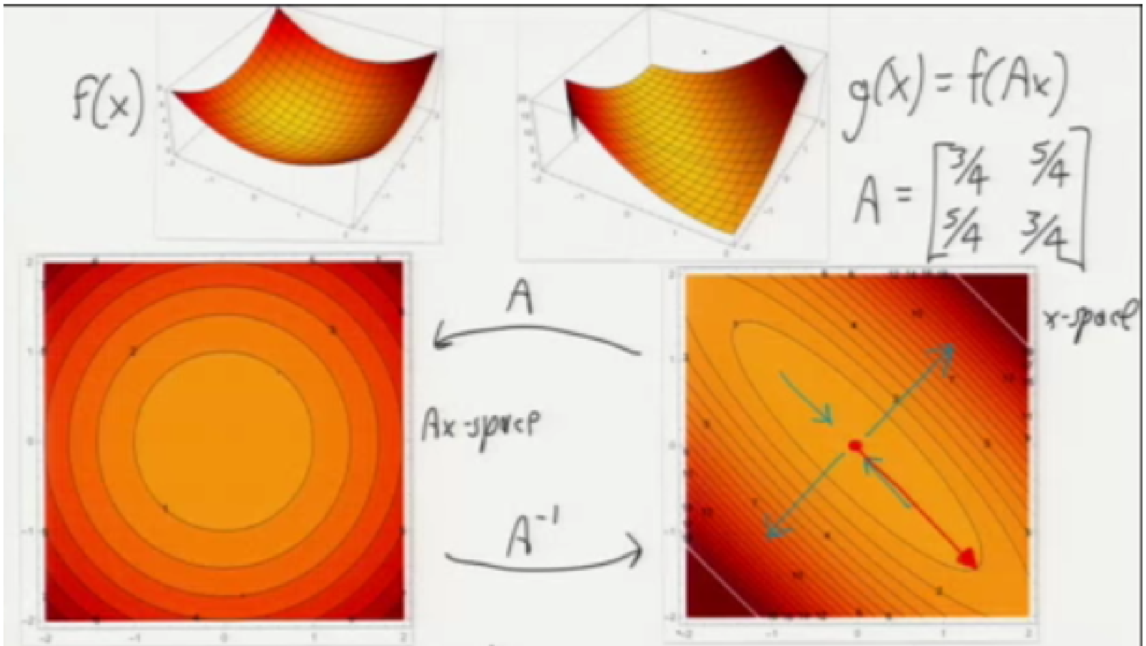


- Write x as a linear combination of eigenvectors:

$$x = \alpha v + \beta w$$
$$A^k x = \alpha \lambda_v^k v + \beta \lambda_w^k w$$

- Ellipsoids

$$f(x) = x^T x \Leftarrow \text{quadratic; isotropic; isosurfaces are spheres.}$$
$$g(x) = f(Ax) \Leftarrow A \text{symmetric.}$$
$$= x^T A^2 x \Leftarrow \text{quadratic form of the matrix } A^2 \text{ anisotropic; isosurfaces are ellipsoids}$$



  - $g(x) = 1$ is an ellipsoid with axes $v_1, v_2, \ldots, v_n$ and radii $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ (eigevalues of A) because if v_i has length $\frac{1}{\lambda_i}$ (red arrow), $g(v_i) = f(\lambda_i v_i) = 1 \Rightarrow v_i$ lies on the ellipsoid.

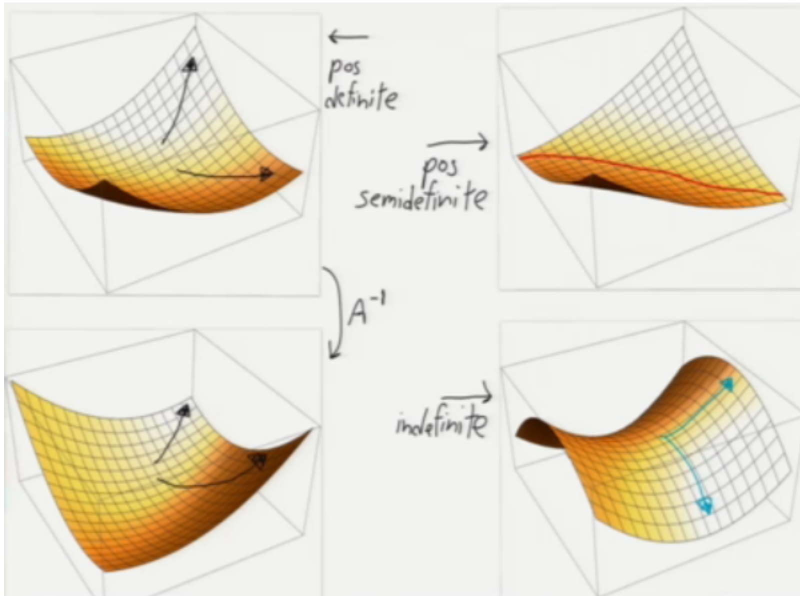
- Bigger eigenvalue $\Leftrightarrow$ steeper slope $\Leftrightarrow$ shorter ellipsoid radius.

- Alternative interpretation:

  - Ellipsoids are spheres in a distance metric A^2.
  - Call $M = A^2$ a metric tensor because the distance between samples x and z in stretched space is $d(x, z) = |Ax - Az| = \sqrt{(x - z)^T M (x - z)}$.
  - Ellipsoids are "spheres" in this metric: $\{x : d(x, \text{center}) = \text{isovalue}\}$.

- A square matrix B is,

  - positive definite if $w^T B w > 0$ for all $w \neq 0 \Leftrightarrow$ all positive eigenvalues.
  - positive semidefinite if $w^T B w \geq 0$ for all $w \neq 0 \Leftrightarrow$ all non-eigenvalues.
  - indefinite if at least one positive eigenvalue and one negative eigenvalue.
  - invertible if no zero eigenvalue.



- Metric tensor must be symmetric positive definite (SPD).
- Special case: M and A are diagonal matrices $\Leftrightarrow$ eigenvectors are coordinate axes $\Leftrightarrow$ ellipsoids are axis-aligned.

Building a Quadratic with specified eigenvectors and eigenvalues

- Choose n mutually orthogonal unit n-vectors $v_1, \ldots, v_n$ Let, $V = [v_1, v_2, \ldots, v_n]$
- Observe: $V^T V = I \Rightarrow V^T = V^{-1} \Rightarrow V V^T = I$.
- V is orthogonal matrix: acts like rotation (or reflection).
- Choose some inverse radii λ_i:
- Let,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

- Theorem: $A = V \Lambda V^T = \sum_{i=1}^{n} \lambda_i v_i v_i^T$ has chosen eigenvectors and eigenvalues.
- Proof: $AV = V\Lambda \Leftarrow$ definition of eigenvectors! (in matrix form).
- This is a matrix factorization called the eigen-decomposition
- Λ is the diagonalized version of A.
- V^T rotates the ellipsoid to be axis-aligned.
- This is also a recipe for building quadratics with axes v_i, radii $\frac{1}{\lambda_i}$.
- Given SPD metric tensor M, we can find symmetric square root $A = M^{\frac{1}{2}}$:
  - compute eigenvectors and eigenvalues of M
  - take square roots of $M's$ eigenvalues
  - reassemble matrix A.