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Two Pumping Lemmata for Petri Nets

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Abstract. A careful analysis of the coverability tree for Petri nets shows that each infinite reachability set contains infinite linear subsets. This can be formulated as a pumping lemma for markings. It can also be shown that the nonterminal Petri net languages satisfy a pumping lemma similar to that for regular languages.

1. Preliminaries. The coverability tree

$\mathbb{N}$ is the set of all non-negative integers. For a finite alphabet A, A^* is the free monoid generated by A with the empty word e. A (generalized initial) *Petri net* is given by

$$\mathcal{N} = (P, T, F, m_0) ,$$

where P and T are the finite disjoint sets of places and transitions, respectively. $F\colon (P \times T) \cup (T \times P) \to \mathbb{N}$ is the flow function and $m_0 \in \mathbb{N}^P$ is the initial marking.

For a transition $t \in T$ we define $t^-, t^+ \in \mathbb{N}^P$ by

$$t^-(p) := F(p, t) , \qquad t^+(p) := F(t, p) \qquad (p \in P) .$$

The transition t is *firable* at a marking $m \in \mathbb{N}^P$ iff $t^- \leqq m$. After its firing, the new marking is $m + \Delta t$, where $\Delta t := t^+ - t^-$. A sequence $u = t_1 \dots t_n \in T^*$ is a firing sequence iff each transition t_i $(i = 1, \dots, n)$ is firable at $m_0 = \sum_{j=1}^{i-1} \Delta t_j$, it leads to the new marking $m_0 + \Delta u$, where $\Delta u := \sum_{i=1}^{n} \Delta t_i$.

Operations and relations on vectors are understood componentwise.

For the construction of the coverability tree generalized markings $m \in (\mathbb{N} \cup \{\omega\})^P$ are needed, $m(p) = \omega$ represents the possibility of arbitrarily many tokens in the place p. As usual,

$$\omega \geqq \omega , \qquad \omega > n , \qquad \omega \pm n = \omega \qquad \text{for all} \quad n \in \mathbb{N} .$$

Different definitions of the coverability tree were given by several authors, cf. for instance [3], [5], [6], [9]. Some other authors tried to simplify the definition, but not all these simplifications are correct. We propose the following one:

The *coverability tree* $\tau_{\mathcal{N}}$ of a Petri net $\mathcal{N} = (P, T, F, m_0)$ is given by $\tau_{\mathcal{N}} = (S, E, \mu)$. $S \subset T^*$ is the set of nodes. It is to be defined together with the node labelling function $\mu\colon S \to (\mathbb{N} \cup \{\omega\})^P$. $E := \{(r, rt) \mid r \in T^* \wedge t \in T \wedge r, rt \in S\}$ is the set of directed edges. S and μ are defined recursively:

(0) $e \in S, \mu(e) := m_0$ (e is the root of the tree).

(1) If $r \in S$ and $\mu(r) \neq \mu(s)$ for all proper prefixes s of r, then $rt \in S$ for all $t \in T$ with $t^- \leqq \mu(r)$. For these rt the function μ is defined by

$$\mu(rt)\,(p) := \begin{cases} \omega\,, & \text{if } \exists s, w \in T^*\colon rt = sw \wedge w \neq e \wedge \\ & \wedge\, \mu(s) \leqq \mu(r) + \Delta t \,\wedge\, \mu(s)\,(p) < \big(\mu(r) + \Delta t\big)\,(p)\,, \\ (\mu(r) + \Delta t)\,(p)\,, & \text{otherwise}\,. \end{cases}$$

(2) No other r are in S.

Lemma 1 ([3], [5]). *For each Petri net $\mathcal{N}$ the coverability tree $\tau_{\mathcal{N}}$ is finite and can be constructed effectively.*

Proof. At first we state that each infinite sequence $m_0, m_1, m_2, \ldots$ of vectors from $(\mathbb{N} \cup \{\omega\})^P$ contains an infinite not decreasing subsequence. To show this one has to choose an infinite subsequence not decreasing in the first coordinate, from this sequence an infinite subsequence is to be chosen which is not decreasing in the second coordinate and so on.

Now let us assume that there is an infinite path $r_0, r_1, r_2, \ldots$ in $\tau_{\mathcal{N}}$, where $\mu(r_0)$, $\mu(r_1), \mu(r_2), \ldots$ is the sequence of node labels. Then there is an infinite not decreasing subsequence $\mu(r_{i_0}), \mu(r_{i_1}), \mu(r_{i_2}), \ldots$. Since the tree would end in the case $\mu(r_{i_j}) = \mu(r_{i_{j+1}})$, this sequence must be strongly increasing. By the definition of $\tau_{\mathcal{N}}$, each $\mu(r_{i_{j+1}})$ has more ω-coordinates than $\mu(r_{i_j})$ in contradiction to the finite number of coordinates. Thus, all pathes in $\tau_{\mathcal{N}}$ are finite and hence $\tau_{\mathcal{N}}$ is finite by König's lemma [7]:

Let τ be a tree where each node has only a finite number of successors. If there is no infinite path starting in the root, then τ is finite.

The construction of $\tau_{\mathcal{N}}$ starts in the root and is to be continued by constructing successors step by step. □

For a convenient work with the coverability tree some additional notions are introduced.

The set of places with ω-coordinates is given by $\Omega(r) := \{p \mid \mu(r)\,(p) = \omega\}$ for a node $r \in S$.

If r is a successor of s in the coverability tree, then we have $\Omega(s) \subseteqq \Omega(r)$ by the definition.

The *edge-labelling function* $\delta\colon E \to T$ is given by $\delta\big((r, rt)\big) := t$ for the edges $(r, rt) \in E$. With this function, the sequence of edge labels on the path from a node $s \in S$ to a node $sv \in S$ is identical with v.

If $\mu(s) = \mu(sv)$ $(v \neq e)$, then sv has no successor in $\tau_{\mathcal{N}}$. But if we defined $\mu(svt)$ in the same way, then we would have $\mu(svt) = \mu(st)$ and so on. Hence successors of sv would give no additional information (cf. Lemma 2). Thus, arriving at a node sv with $\mu(sv) = \mu(s)$, we can go back to the node s. We call the path from s to sv a *loop*, the sequence v is the *loop-sequence*, s is the *loop-start*, and sv is the *loop-end*. The so-called *loop-backpointers* [3] are additional arcs labelled by e which lead from the loop-ends to the related loop-starts (from sv to v).

It can be possible to return to a certain loop-start after performing different loops as in Fig. 1. We refer to such cases as *generalized loops*, the related *generalized loop-sequences* are shuffled from loop-sequences (for instance the generalized loop-sequence $t_1t_2t_3t_2t_4$ in the right hand picture of Fig. 1).

The next lemma shows that there is a concatenation of paths in the coverability tree (linked by loop-backpointers) for each firing sequence u such that the sequence

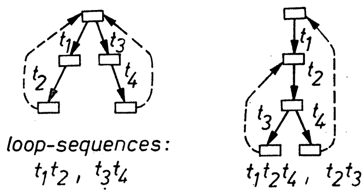

Fig. 1

of edge labels is identical with u. The node $s(u)$ at the end of these paths has a label $\mu(s(u))$ which coincides with $m_0 + \Delta u$ in its finite coordinates.

Lemma 2 ([3], [5]). *Each firing sequence u has a unique "loop-decomposition"*

$$u = s_1 v_1 s_2 v_2 \dots s_n v_n s_{n+1}$$

such that

(1) $s(u) := s_1 s_2 \dots s_{n+1}$ *is a node (but not a loop-end) of the coverability tree, and* s_1, $s_1 s_2, \dots, s_1 \dots s_n$ *are loop-starts,*
(2) $v_1, \dots, v_n$ *are generalized loop-sequences,*
(3) $\mu(s(u))\,(p) = (m_0 + \Delta u)\,(p)$ *for all* $p \in P \setminus \Omega(s(u))$.

Note that s_{n+1} may be the empty word e.

The proof is carried out by induction on u, the basis for $u = e$ is trivial.

We assume that the lemma holds for some firing sequence u. Let ut be a firing sequence too, i.e. $t^- \leqq m_0 + \Delta u$. By (3) we have $t^- \leqq \mu(s_1 \dots s_{n+1})$ and therefore $s_1 \dots s_{n+1} t$ is a successor of $s_1 \dots s_{n+1}$ in the coverability tree with

$$\begin{aligned} \mu(s_1 \dots s_{n+1} t)\,(p) &= \big(\mu(s_1 \dots s_{n+1}) + \Delta t\big)\,(p) \\ &= (m_0 + \Delta u)\,(p) + \Delta t(p) \\ &= (m_0 + \Delta ut)\,(p) \quad \text{for} \quad p \in P \setminus \Omega(s_1 \dots s_{n+1} t)\,. \end{aligned}$$

If $s_1 \dots s_{n+1} t$ is not a loop-end, we have the loop-decomposition

$$ut = s_1 v_1 s_2 v_2 \dots s_n v_n (s_{n+1} t)$$

and

$$s(ut) := s_1 s_2 \dots s_n (s_{n+1} t)\,.$$

Otherwise, if $s_1 \dots s_{n+1} t$ is a loop-end, there must be some $s_i = s_i' s_i''$ such that $s_1 \dots s_{i-1} s_i'$ is the loop-start and $s_i'' s_{i+1} \dots s_{n+1} t$ is the loop-sequence. Then we have the loop-decomposition

$$ut = s_1 v_1 s_2 v_2 \dots s_{i-1} v_{i-1} s_i' v' s_{i+1}'$$

whereby $v' = s_i'' v_i s_{i+1} \dots v_n s_{n+1} t$ is a generalized loop-sequence and $s_{i+1}' = e$, and we have

$$s(ut) := s_1 \dots s_{i-1} s_i' s_{i+1}'\,. \ \square$$

By Lemma 2 a node $s(u)$ is assigned to each firing sequence u. Now, conversely certain firing sequences u are assigned to the nodes s. Thereby arbitrary submarkings on the places $p \in \Omega(s)$ can be covered by $m_0 + \Delta u$ (a related firing sequence u can be constructed to the given submarking), while $m_0 + \Delta u$ and $\mu(s)$ agree in the finite coordinates of $\mu(s)$.

The nodes rt with new ω-coordinates (i.e. $\Omega(rt) \supsetneqq \Omega(r)$) are called *$\omega$-nodes*. There must be at least one antecedent s of rt permitting the introduction of "ω" for each new ω-coordinate. We call such a node s an *ω-source* and the related sequence w with

$rt = sw$ (cf. the definition of the coverability tree) is called an *ω-sequence*. By the definition it holds for ω-sequences w:

$$\begin{aligned} \Delta w(p) &= 0 \quad \text{for } p \in P \setminus \Omega(rt)\,, \\ \Delta w(p) &\geqq 0 \quad \text{for } p \in \Omega(rt) \setminus \Omega(r)\,, \\ \Delta w(p) &> 0 \quad \text{for at least one } p \in \Omega(rt) \setminus \Omega(r)\,. \end{aligned}$$

Now let $\{w_1, \dots, w_l\}$ be a set of ω-sequences related to the ω-node rt such that for each $p \in \Omega(rt) \setminus \Omega(r)$ there is some w_i generating the "ω" for the p-coordinate, i.e. $\Delta w_i(p) > 0$. Furthermore let α_i be a non-negative integer such that w_i is firable in the marking $(\alpha_i, \dots, \alpha_i)$.

By the construction of the coverability tree, w_i is firable with respect to the places $p \in P \setminus \Omega(r)$ in a marking that agrees with $\mu(r) + \Delta t \geqq \mu(s)$. Hence the sequence $w_1^K \dots w_l^K$ is firable for $K \in \mathbb{N}$ in a marking which covers

$$m_K^*(p) := \begin{cases} (\mu(r) + \Delta t)\,(p) & \text{for } p \in P \setminus \Omega(r)\,, \\ K \cdot \sum\limits_{i=1}^{l} \alpha_i & \text{for } p \in \Omega(r)\,. \end{cases} \qquad (*)$$

Thereby we have

$$\Delta w_1^K \dots w_l^K \begin{cases} = 0 & \text{for } p \in P \setminus \Omega(rt)\,, \\ \geqq K & \text{for } p \in \Omega(rt) \setminus \Omega(r)\,, \\ \geqq -K \cdot \sum\limits_{i=1}^{l} \alpha_i & \text{for } p \in \Omega(r)\,. \end{cases}$$

We shall refer to (*) in the proofs of Lemma 3 and Lemma 4: By the ω-sequences $w_1, \dots, w_l$ an arbitrarily large number of tokens can be given to the places $p \in \Omega(rt) \setminus \Omega(r)$.

In general the sequence s for a node $s \in S$ is not a firing sequence. But, as Lemma 3 will show, by inserting appropriate ω-sequences we can construct a firing sequence u. If we add the so-called *ω-backpointers* [3] leading from ω-nodes to the related ω-sources and labelled by e (and for a convenient work they may additionally be marked by the related new ω-coordinates), then we shall have a concatenation of pathes in $\tau_{\mathcal{N}}$ linked by ω-backpointers such that the sequence of edge labels will be identical to u.

Lemma 3 ([3], [5]). *If r is a node of the coverability tree and $m \in \mathbb{N}^{\Omega(r)}$ is a partial marking on the places $p \in \Omega(r)$, then there are*

- *a decomposition $r = r_1 r_2 \dots r_{m+1}$ (where $r_1, r_1 r_2, \dots, r_1 \dots r_m$ are the ω-nodes on the path r, some r_i may be the empty word),*
- *ω-sequences $w_1, \dots, w_m$ (where w_i is related to $r_1 \dots r_i$) and*
- *numbers $k_1, \dots, k_m \in \mathbb{N}$*

such that

$$u := r_1 w_1^{k_1} r_2 w_2^{k_2} \dots r_m w_m^{k_m} r_{m+1}$$

is a firing sequence with

$$\begin{aligned} (m_0 + \Delta u)\,(p) &= \mu(r)\,(p) \quad \textit{for } p \in P \setminus \Omega(r)\,, \\ (m_0 + \Delta u)\,(p) &\geqq m(p) \quad \textit{for } p \in \Omega(r)\,. \end{aligned}$$

Note that the sequences r_i may differ from the sequences s_i in the loop-decomposition of u by Lemma 2, and the node $s(u)$ need not be identical with r.

The proof is by induction on r, the basis is trivial. We assume that the lemma holds for some $r \in S$. Let rt be a node of the coverability tree, i.e. $t^- \leqq \mu(r)$.

If rt is an ω-node, let $w_1', \dots, w_l'$ be a set of ω-sequences related to rt with $\sum_{i=1}^{l} \Delta w_i'(p) > 0$ for all $p \in \Omega(rt) \setminus \Omega(r)$ as in (*). Again we take α_i such that w_i' is firable in $(\alpha_i, \dots, \alpha_i)$.

There is a firing sequence $u = r_1 w_1^{k_1} \dots r_{m+1}$ for the node $r = r_1 \dots r_{m+1}$ and an arbitrary number $K \in N$ such that (by induction)

$$(m_0 + \Delta u)(p) = \mu(r)(p) \qquad \text{for } p \in P \setminus \Omega(r),$$
$$(m_0 + \Delta u)(p) \geqq K \cdot \left(1 + \sum_{i=1}^{l} \alpha_i\right) + t^-(p) \qquad \text{for } p \in \Omega(r).$$

Now ut is a firing sequence since $t^- \leqq m_0 + \Delta u$, and we have

$$(m_0 + \Delta ut)(p) = (m_0 + \Delta u)(p) + \Delta t(p) = \mu(r)(p) + \Delta t(p) \qquad \text{for } p \in P \setminus \Omega(r),$$
$$(m_0 + \Delta ut)(p) \geqq K \cdot \left(1 + \sum_{i=1}^{l} \alpha_i\right) + t^+(p) \qquad \text{for } p \in \Omega(r).$$

Hence $u' := utw_1'^K \dots w_l'^K$ is a firing sequence by (*) with

$$(m_0 + \Delta u')(p) = (m_0 + \Delta ut)(p) + \Delta w_1'^K \dots w_l'^K(p)$$

for $p \in P \setminus \Omega(rt)$: $\quad = (m_0 + \Delta ut)(p) = \mu(r)(p) + \Delta t(p) = \big(\mu(rt)\big)(p)$,

for $p \in \Omega(rt) \setminus \Omega(r)$: $\quad \geqq (m_0 + \Delta ut)(p) + K \geqq K$,

for $p \in \Omega(r)$: $\quad \geqq K + K \cdot \sum_{i=1}^{l} \alpha_i + \Delta w_1'^K \dots w_l'^K(p) \geqq K$.

Since K may be an arbitrarily large number, all submarkings on $\Omega(rt)$ can be covered by $m_0 + \Delta u'$. We found $rt = r_1 \dots r_m(r_{m+1}t)\, r_{m+2} \dots r_{m+1+l}$ whereby $r_{m+2} = \dots = r_{m+1+l} := e$ and

$$u' t w_1'^K \dots w_l'^K = r_1 w_1^{k_1} \dots r_m w_m^{k_m}(r_{m+1}t)\, w_1'^K r_{m+2} \dots w_l'^K r_{m+1+l}.$$

If rt is no ω-node we only have to use a firing sequence u with $(m_0 + \Delta u)(p) \geqq K + t^-(p)$ (by induction). We get the firing sequence $ut = r_1 w_1^{k_1} \dots r_m w_m^{k_m}(r_{m+1}t)$ for $rt = r_1 \dots r_m(r_{m+1}t)$. □

As an example we construct a part of the coverability tree (Fig. 3) for the well-known net of Fig. 2 which weakly computes 2^n (on the places x_1 and x_2). The markings are given by

$$m := \big(m(x_1), m(x_2), m(y_1), m(y_2), m(y_3)\big),$$

$m_0 = (0, 1, 0, 0, 1)$ is the initial marking.

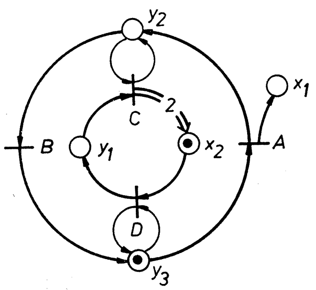

Fig. 2

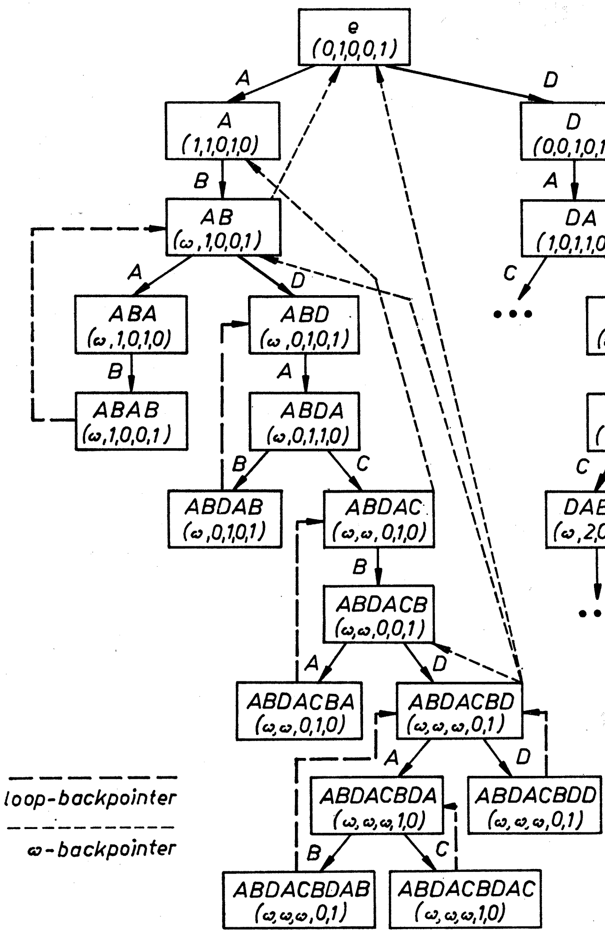

Fig. 3

An example of the loop-decomposition of a firing sequence u is given by

$$u = \underbrace{AB}_{s_1}\,(\underbrace{ABAB}_{w_1})\,\underbrace{DAC}_{s_2}\,(\underbrace{BA}_{v_2})\,\underbrace{BD}_{s_3}\,(\underbrace{ABACBD}_{v_3})\,\underbrace{A}_{s_4}$$

with $s(u) = AB\,DAC\,BD\,A$ (Lemma 2).

As an example of Lemma 3 we consider the node

$$r = \underbrace{AB}_{r_1}\,\underbrace{DAC}_{r_2}\,\underbrace{BD}_{r_3}\,\underbrace{AC}_{r_4}$$

and the submarking $m(x_1) = m(x_2) = m(y_1) = 2$ on the places of $\Omega(r) = \{x_1, x_2, y_1\}$. Then the firing sequence

$$u = \underbrace{AB}_{r_1}\,\underbrace{(AB)^{14}}_{w_1}\,\underbrace{DAC}_{r_2}\,\underbrace{(BDAC)^{7}}_{w_2}\,\underbrace{BD}_{r_3}\,\underbrace{(D)^{3}}_{w_3}\,\underbrace{AC}_{r_4}$$

can be derived by our proof but, of course, this firing sequence is not the optimal one.

Remarks. (1) Nodes with identical labels may have successors with different labels: In our example we have $\mu(ABDA) = \mu(DABA)$ but $\mu(ABDAC) \neq \mu(DABAC)$. This shows that identifying nodes with the same label is problematic in some sense. For this reason the "reachability graph" construction in [10] (as mentioned by the authors) has no unique result.

(2) It is possible to consider an automaton instead of a tree such that the firing sequences u are input sequences leading to $s(u)$ (but in general not vice versa). For instance, such an automaton can have the state set $S \setminus \{r \mid r \text{ is a loop-end}\}$ with the initial state e, the input set T and the partially defined next state function f with

$$f(r, t) = \begin{cases} rt & \text{if } rt \in S \text{ and } rt \text{ is not a loop-end}\,, \\ s & \text{if } rt \text{ is a loop-end, whereby } \mu(s) = \mu(rt)\,. \end{cases}$$

The "reachability graphs" in [10] might be interpreted as automata, too. An axiomatic approach to related automata was given in [2].

2. Pumping for markings

From Lemma 2 and Lemma 3 it follows that boundedness is a decidable property [3], [5] (a place p is bounded iff $\mu(s) \neq \omega$ for all $s \in S$). Furthermore, a set P' of places is simultaneously unbounded (i.e. each submarking on these places can be covered by a reachable marking) iff $P' \subseteq \Omega(s)$ for some $s \in S$ [3]. It was already mentioned by *Hack* [3] that the covering marking can be reached by a firing sequence of a length proportional to the submarking to be covered.

We are now interested in the study of these linearity aspects from a certain point of view which leads to the "pumping lemma" for markings. We are going to prove a lemma which can also be considered as a connection of Lemma 2 and Lemma 3. A similar lemma was proved in [1].

Lemma 4. *Let s be a node of the coverability tree. Then there are*

— *fixed ω-sequences $w_1, \ldots, w_n$ and*
— *fixed numbers $K_1, \ldots, K_n \in \mathbb{N}$*
such that each firing sequence u satisfying $s(u) = s$ has an "ω-decomposition" $u = u_1u_2 \ldots u_{n+1}$ (some u_i may be the empty word) such that the following holds:

$$u(\nu) := u_1(w_1^{K_1})^\nu\, u_2(w_2^{K_2})^\nu\, u_3 \ldots (w_n^{K_n})^\nu\, u_{n+1}$$

is a firing sequence for each $\nu \in \mathbb{N}$ and

$$\begin{aligned} &(m_0 + \Delta u(\nu))\,(p) = (m_0 + \Delta u)\,(p) && \text{for } p \in P \setminus \Omega(s)\,, \\ &(m_0 = \Delta u(\nu))\,(p) \geqq (m_0 + \Delta u)\,(p) + \nu && \text{for } p \in \Omega(s)\,. \end{aligned}$$

Proof. As we have seen, we can assign a node $s(u)$ of the coverability tree to each tiring sequence u. By Lemma 2 we decomposed $u = s_1v_1s_2v_2 \ldots s_nv_ns_{n+1}$ with respect fo generalized loop-sequences contained in u. Now we are interested in an ω-decomposition of u with respect to the insertion of additional ω-sequences.

The insertion of ω-sequences is related to certain ω-nodes. Since the loop-start and the loop-end of a loop must have identical labels, no ω-node can be on a loop except that the loop-start itself may be an ω-node. Hence if we consider the pathes linked by loop-backpointers for the firing sequence u as in Lemma 2, we can meet ω-nodes only on the path given by $s(u) = s_1 \ldots s_{n+1}$.

The way to find the ω-decomposition of u is now by splitting u each time we meet an ω-node on the paths linked by the loop-backpointers for the first time:

We consider the decomposition $s(u) = s_1 \dots s_{n+1}$ and the loop-decomposition $u = s_1 v_1 \dots s_n v_n s_{n+1}$ by Lemma 2 together with the decomposition $s(u) = r_1 \dots r_{m+1}$ by Lemma 3, which gives us information about the possibilities of inserting ω-sequences (since the nodes $r_1 \dots r_i$, $i = 1, \dots, m$, are the ω-nodes to be considered).

We construct the sequences u_i for the ω-decomposition of u from the sequences r_i taking into account the loop-starts on the r_i-path where the node $r_1 \dots r_i$ has to be excluded. Thus we decompose $r_i = r_{i,0} r_{i,1} \dots r_{i,k+1}$, such that $r_{i,k+1} \neq e$, but $r_{i,0}$ may be the empty word, such that each node $r_1 \dots r_{i-1} r_{i,0} \dots r_{i,\varkappa}$, $\varkappa = 0, \dots, k$, is identical with a loop-start $s_1 \dots s_{l+\varkappa}$ on the r_i-subpath of $s_1 \dots s_{n+1}$ and $u = s_1 v_1 \dots s_l v_l \dots s_{l+k} v_{l+k} \dots s_{n+1}$. (The node $r_1 \dots r_i$ may also be a loop-start, but then it is to be considered as $r_1 \dots r_i r_{i+1,0}$ with $r_{i+1,0} = e$ in the construction of u_{i+1}.) Now u_i is constructed from r_i by inserting the generalized loop-sequences $v_{l+\varkappa}$ after $r_{i,\varkappa}$:

$$u_i := r_{i,0} v_l r_{i,1} v_{l+1} \dots r_{i,k} v_{l+k} r_{i,k+1} .$$

An example is given in Fig. 4 where we have

a) $u = s_1 v_1 s_2 v_2 s_3 v_3 s_4 v_4 s_5$ (loop-decomposition by Lemma 2),
b) $s(u) = s_1 s_2 s_3 s_4 s_5$,
c) $s(u) = r_1 r_2 r_3 r_4 r_5 r_6$ (decomposition by Lemma 3),
d) $u = (s_1 v_1 s_2')(s_2'' v_2 s_3')(s_3'')(s_3''' v_3 s_4)(v_4 s_5')(s_5'')$ is the ω-decomposition of u

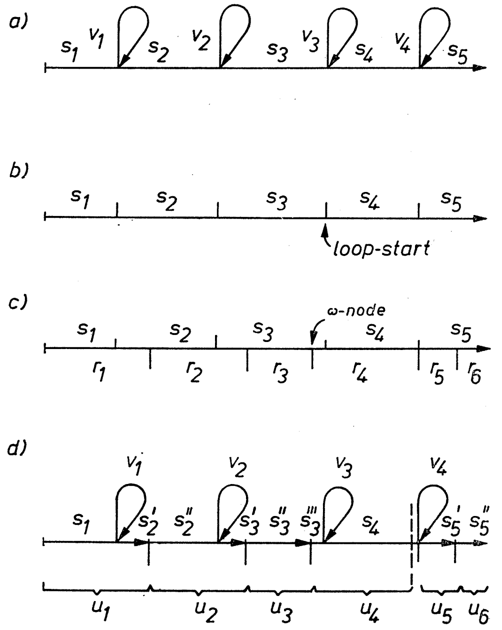

Fig. 4

Remark. To get the final form of Lemma 3 it may be necessary to insert some sequences u_j equal to the empty word (if more than one ω-sequences are to be used in some ω-node, and at the end if $s(u)$ is an ω-node, respectively).

Now the proof is similar to that of Lemma 3. It is by induction on u whereby the basis is trivial. We consider a firing sequence ut and assume that the lemma holds for u.

If $u = u_1 \dots u_{n+1}$ is the ω-decomposition of u, then the ω-decomposition of ut is given by $ut = u_1 \dots u_n(u_{n+1}t)$, where we have to add $u_{n+2} = e$ if $s(ut)$ is an ω-node. Since $m_0 + \Delta u(\nu) \geqq m_0 + \Delta u$, the sequence $u(\nu)\, t$ is a firing sequence for all $\nu \in \mathbb{N}$. By induction we have

$$\begin{aligned}(m_0 + \Delta u(\nu)\, t)\,(p) &= (m_0 + \Delta u(\nu) + \Delta t)\,(p)\\ &= (m_0 + \Delta u + \Delta t)\,(p)\\ &= (m_0 + \Delta ut)\,(p) \quad \text{for} \quad p \in P \setminus \Omega(s(u))\end{aligned}$$

and

$$(m_0 + \Delta u(\nu)\, t)\,(p) \geqq (m_0 + \Delta ut)\,(p) + \nu \qquad \text{for } p \in \Omega(s(u))\,.$$

If $s(ut)$ is no ω-node we have $\Omega(s(ut)) = \Omega(s(u))$, and the proof can be finished by $ut(\nu) := u(\nu)\, t$.

If $s(ut)$ is an ω-node, let $w_1', \dots, w_l'$ be a set of ω-sequences related to the node $s(ut)$ such that

$$\sum_{i=1}^{l} \Delta w_i'(p) > 0 \quad \text{for all} \quad p \in \Omega(s(ut)) \setminus \Omega(s(u))$$

as in Lemma 3. Again let α_i be a number such that w_i is firable in $(\alpha_i, \dots, \alpha_i)$.

We get (by induction and by Lemma 2) for

$$\nu := \left(1 + \sum_{i=1}^{l} \alpha_i\right) \cdot \nu' \quad (\nu' \in \mathbb{N})$$

and for the marking $m_{\nu'}^*$ as in (*) (see p. 352):

$$\begin{aligned}(m_0 + \Delta u(\nu)\, t)\,(p) &= (m_0 + \Delta u + \Delta t)\,(p)\\ &= (\mu(s(u)) + \Delta t)\,(p)\\ &= m_{\nu'}^*(p) \qquad \text{for } p \in P \setminus \Omega(s(u))\,,\end{aligned}$$

$$\begin{aligned}(m_0 = \Delta u(\nu)\, t)\,(p) &\geqq (m_0 + \Delta u + \Delta t)\,(p) + \left(1 + \sum_{i=1}^{l} \alpha_i\right) \cdot \nu'\\ &= (m_0 + \Delta ut)\,(p) + \nu' + m_{\nu'}^*(p) \qquad \text{for } p \in \Omega(s(u))\,.\end{aligned}$$

Now the premises of (*) are satisfied since $s(ut) = s(u)\, t$ by our construction. Hence we have found the firing sequence

$$\begin{aligned}ut(\nu') &:= u(\nu)\, tw_1' \dots w_l'\\ &= u_1(w_1^{K_1})^{\left(1+\sum_{i=1}^{l}\alpha_i\right)\cdot\nu'} u_2 \dots (w_n^{K_n})^{\left(1+\sum_{i=1}^{l}\alpha_i\right)\cdot\nu'} u_{n+1}tw_1' \dots w_l'\\ &= u_1\left(w_1^{K_1\left(1+\sum_{i=1}^{l}\alpha_i\right)}\right)^{\nu'} u_2 \dots \left(w_n^{K_n\left(1+\sum_{i=1}^{l}\alpha_i\right)}\right)^{\nu'} u_{n+1}tw_1' \dots w_l'\\ &= u_1(w_1^{K_1'})^{\nu'}\, u_2 \dots (w_n^{K_n'})^{\nu'}\, u_{n+1}'(w_1'^{K_{n+1}'})^{\nu'}\, u_{n+2}' \dots (w_l'^{K_{n+l}'})^{\nu'}\, u_{n+1+l}'\,,\end{aligned}$$

whereby

$$\begin{aligned}K_i' &:= K_i\left(1 + \sum_{i=1}^{l} \alpha_i\right) && \text{if } i = 1, \dots, n\,,\\ K_i' &:= 1 && \text{if } i = n+1, \dots, n+l\,,\\ u_{n+1}' &:= u_{n+1}t\,,\\ u_{n+2}' &= \dots = u_{n+1+l}' := e\,.\end{aligned}$$

Note that the ω-sequences and the numbers K_i' depend only on $s(ut)$.

For the marking $m_0 + \Delta ut(\nu')$ we get by (*):

$$\big(m_0 + \Delta ut(\nu')\big)\,(p) = \big(m_0 + \Delta u(\nu)\,t\big)\,(p) + \Delta w_1^{\nu'} \dots w_l^{\nu'}(p)$$

(a) for $p \in P \setminus \Omega\big(s(u)\big)$:

$$= \big(m_0 + \Delta u(\nu)\,t\big)\,(p)$$
$$= (m_0 + \Delta ut)\,(p)\,,$$

(b) for $p \in \Omega\big(s(ut)\big) \setminus \Omega\big(s(u)\big)$:

$$\geqq (m_0 + \Delta ut)\,(p) + \nu'\,,$$

(c) for $p \in \Omega\big(s(u)\big)$:

$$\geqq (m_0 + \Delta ut)\,(p) + \nu' + \nu' \cdot \sum_{i=1}^{l} \alpha_i + \Delta w_1^{\nu'} \dots w_l^{\nu'}(p)$$
$$\geqq (m_0 + \Delta ut)\,(p) + \nu'\,. \;\square$$

In our net example (see p. 354) the firing sequence

$$u = ABABABDACBABDABACBDA$$

has the ω-decomposition

$$u = \underbrace{AB}_{u_1}\,\underbrace{ABABDAC}_{u_2}\,\underbrace{BABD}_{u_3}\,\underbrace{ABACBDA}_{u_4}$$

and we derive by our proof

$$u(\nu) = AB\,[(AB)^4]^\nu\,ABABDAC\,[(BDAC)^2]^\nu\,BABD\,[D]^\nu\,ABACBDA$$

with

$$m_0 + \Delta u(\nu) = m_0 + \Delta u = (6\nu, \nu, \nu, 0, 0)\,.$$

Let $R_{\mathcal{N}} := \{m_0 + \Delta u \mid u \text{ is a firing sequence of } \mathcal{N}\}$ be the set of all reachable markings. The following theorem is the "pumping lemma" for markings: If a reachable marking m is sufficiently large on some places, then all markings $m + \nu \cdot a$, $\nu \in \mathbb{N}$, for some vector a are reachable too. Thereby the vector a can be chosen from a finite set of vectors which are all increasing the marking on those places.

Theorem 1 (Pumping for markings). *Let $P' \subseteqq P$ be a subset of places. Then there are*

— *a submarking $m' \in \mathbb{N}^{P'}$ on the places of P' and*
— *a finite set $A \subset \mathbb{N}^P$ of vectors with $a(p) \geqq 1$ for $a \in A$, $p \in P'$, such that the following holds:*

If a reachable marking m covers m' on P', then there is at least one vector $a \in A$ such that

$$m + \nu \cdot a \in R_{\mathcal{N}} \quad \text{for all} \quad \nu \in \mathbb{N}\,.$$

Proof. For a node s of the coverability tree let

$$a_s := \sum_{i=1}^{n} \Delta w_i^{K_i}\,,$$

where $w_1, \dots, w_n$ and $K_1, \dots, K_n$ are fixed ω-sequences and numbers, respectively, as in Lemma 4. Then we have

$$a_s(p) \geqq 1 \quad \text{for} \quad p \in \Omega(s)\,, \qquad a_s(p) = 0 \quad \text{for} \quad p \in P \setminus \Omega(s)\,,$$

and $A := \{a_s \mid s \in S\}$ is a finite set.

For a given subset $P' \subseteqq P$ let m' be a submarking on P' such that $m'(p) > \mu(s)\,(p)$ for all $s \in S$ with $\mu(s) \neq \omega$.

Now let the marking m be reachable by a firing sequence u, i.e. $m = m_0 + \Delta u$. If m covers m' on P', then $P' \subseteqq \Omega(s(u))$ by Lemma 2. By Lemma 4 all markings

$$m_0 + \Delta u(\nu) = m_0 + \Delta u + \nu \cdot \sum_{i=1}^{n} \Delta w_i^{K_i}$$
$$= m_0 + \Delta u + \nu \cdot a_{s(u)} = m + \nu \cdot a_{s(u)}$$

are reachable, and $a_{s(u)}(p) \geqq 1$ for all $p \in P' \subseteqq \Omega(s(u))$. □

Corollary. *If $P' \subseteqq P$ is a maximal set of simultaneously unbounded places, then there is a linear subset $M \subseteqq R_{\mathcal{N}}$ which is also simultaneously unbounded exactly in the places of P'.*

$\Big($*M is called linear if there are vectors $a_0, a_1, \dots, a_k \in \mathbb{N}^P$ such that* $M = \Big\{a_0 + \sum_{i=1}^{k} n_i a_i \mid n_i \in \mathbb{N}\Big\}.\Big)$

The reachability set $R_{\mathcal{N}}$ may also contain linear subsets which are unbounded only in smaller sets of places (for instance in the sets $\Omega(s)$). We also mention that there are infinitely many possibilities to fix the numbers $K_1, \dots, K_n$ such that Lemma 4 holds. Hence we can obtain different vectors $a_{s(u)}^{(i)}$ from different numbers $K_1^{(i)}, \dots, K_n^{(i)}$, $i \in I$, for some index set I. Now it is not difficult to show that there are also subsets $M \subseteqq R_{\mathcal{N}}$ of the form

$$M = \Big\{m + \sum_{i \in I'} \nu_i \cdot a_{s(u)}^{(i)} \,\Big|\, I' \text{ is a finite subset of } I \text{ and } \nu_i \in \mathbb{N}\Big\}$$

whereby $m = m_0 + \Delta u$ is a reachable marking which covers the submarking m' given by Theorem 1 on the places of P'.

By a result of *van Leeuwen* [8], there is a marking m for each net $\mathcal{N}$ such that $R_{\mathcal{N}} \cap \{m' \mid m' \geqq m\}$ is a semilinear set. But, if the set P of all places is not simultaneously unbounded, we can choose a marking m which is not coverable in $\mathcal{N}$ and then the proposition is trivial since we consider the empty set. This is the case for our net example (where we have no other possibility to satisfy this proposition). By our results the reachability set of this net contains linear subsets which are unbounded in the places x_1, x_2 and y_1.

3. Pumping for firing sequences

Let u be a firing sequence where the length of u is greater than the height of the coverability tree. Then u can be decomposed by $u = u'u''u'''$ such that $u'u''$ is a loop-end or an ω-node of the coverability tree and u'' is the related loop-sequence or a related ω-sequence.

Furthermore the predecessors of $u'u''$ must not be loop-ends or ω-nodes. Then $u'(u'')^{\nu+1} u'''$ is a firing sequence for all $\nu \in \mathbb{N}$. Thus the languages of firing sequences of Petri nets have to satisfy a pumping lemma similar to that one for regular languages (even though they need not be regular themselves). Moreover, such a pumping lemma holds for all nonterminal Petri net languages $L_{\mathcal{N}}$ of labelled Petri nets $\mathcal{N}$ which are defined by

$$L_{\mathcal{N}} := \{h(t_1) \dots h(t_n) \mid t_1 \dots t_n \text{ is a firing sequence of } \mathcal{N}\} ,$$

thereby $h\colon T \to \Sigma \cup \{e\}$ is a labelling function and Σ is a finite set of labels.

Theorem 2 (Pumping for nonterminal Petri net languages). *There are numbers k, l for each nonterminal Petri net language $L_{\mathcal{N}}$ such that he following holds:*

If the length of a sequence $r \in L_{\mathcal{N}}$ is greater than k, then there is a decomposition $r = xyz$ with $1 \leqq$ length of $y \leqq l$ such that $xy^{\nu+1} z \in L_{\mathcal{N}}$ for all $\nu \in \mathbb{N}$.

Proof. Let k be the height of the coverability tree and let l be the maximum length of all loop-sequences and all ω-sequences. Let $r \in L_{\mathcal{N}}$ have a length greater than k and let $u = t_1 \dots t_n$ be a firing sequence with $h(u) := h(t_1) \dots h(t_n) = r$. Then we can decompose $u = u'u''u'''$ such that

(a) u'' is a generalized loop-sequence in the loop-decomposition of u (Lemma 2) or u'' is an ω-sequence,
(b) u'' is not labelled by the empty word, i.e. $h(u'') \neq e$,
(c) if a predecessor of $s(u'u'')$ is an ω-node, then the related ω-sequences w are labelled by e, i.e. $h(w) = e$.

Let be $u'u'' = \bar{u}t$, $t \in T$. By the construction of the coverability tree and by Lemma 2 we have for all places $p \in P \setminus \Omega(s(\bar{u})) \subseteqq P \setminus \Omega(s(u'))$:

$$\begin{aligned}(m_0 + \Delta u')(p) &= \mu(s(u'))(p)\\ &\leqq \mu(s(\bar{u}) + \Delta t)(p)\\ &= (m_0 + \Delta\bar{u} + \Delta t)(p)\\ &= (m_0 + \Delta\bar{u}t)(p) = (m_0 + \Delta u'u'')(p)\,.\end{aligned}$$

Hence u'' is firable in $m_0 + \Delta u'u''$ with respect to the places $p \in P \setminus \Omega(s(\bar{u}))$ (since it is firable in $m_0 + \Delta u'$ with respect to these places by the construction of $\tau_{\mathcal{N}}$) and we have

$$\Delta u''(p) \geqq 0 \quad \text{for} \quad p \in P \setminus \Omega(s(\bar{u}))\,.$$

Now let $\alpha \in \mathbb{N}$ such that u'' is firable in $(\alpha, \dots, \alpha)$. By Lemma 4 we can find a firing sequence $\bar{u}(\nu\alpha)$ with

$$\begin{aligned}&(m_0 + \Delta\bar{u}(\nu\alpha))(p) = (m_0 + \Delta\bar{u})(p) && \text{for } p \in P \setminus \Omega(s(\bar{u}))\,,\\ &(m_0 + \Delta\bar{u}(\nu\alpha))(p) \geqq (m_0 + \Delta\bar{u})(p) + \nu\alpha && \text{for } p \in \Omega(s(\bar{u}))\,.\end{aligned}$$

Then the sequence $\bar{u}(\nu\alpha)\,t$ is a firing sequence (since $\bar{u}t$ is a firing sequence, i.e., $t^- \leqq m_0 + \Delta\bar{u}$) with

$$\begin{aligned}&(m_0 + \Delta\bar{u}(\nu\alpha)\,t)(p) = (m_0 + \Delta\bar{u}t)(p) && \text{for } p \in P \setminus \Omega(s(\bar{u}))\,,\\ &(m_0 + \Delta\bar{u}(\nu\alpha)\,t)(p) \geqq (m_0 + \Delta\bar{u}t)(p) + \nu\alpha && \text{for } p \in \Omega(s(\bar{u}))\,.\end{aligned}$$

Hence $\bar{u}(\nu\alpha)\,t(u'')^\nu$ is a firing sequence with

$$(m_0 + \Delta\bar{u}(\nu\alpha)\,t(u'')^\nu)(p) \geqq (m_0 + \Delta\bar{u}t)(p) = (m_0 + \Delta u'u'')(p)$$

for all $p \in P$ and therefore

$$\bar{u}(\nu\alpha)\,t(u'')^\nu\,u'''$$

is a firing sequence for all $\nu \in \mathbb{N}$.
Since all additional ω-sequences in $\bar{u}(\nu\alpha)$ are labelled by the empty word (cf. (c)), we get

$$\begin{aligned}h(\bar{u}(\nu\alpha)\,t(u'')^\nu\,u''') &= h(\bar{u}(\nu\alpha))\,h(t)\,h((u'')^\nu)\,h(u''')\\ &= h(\bar{u})\,h(t)\,(h(u''))^\nu\,h(u''')\\ &= h(u'u'')\,(h(u''))^\nu\,h(u''')\end{aligned}$$

is in $L_{\mathcal{N}}$ for all $\nu \in \mathbb{N}$. Now, for $x := h(u')$, $y := h(u'')$ and $z := h(u''')$ we have $r = xyz$ and $xyy^\nu z \in L_{\mathcal{N}}$ for all $\nu \in \mathbb{N}$. □

Remark. The small difference to the pumping lemma for regular languages cannot be overcome as shown in Fig. 5, where we have

$$L_{\mathcal{N}} = \{A^k C B^l \mid k \geqq l \geqq 0\}$$

$\big(h(t) := t$ for all $t \in T\big)$.

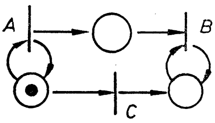

Fig. 5

Especially the sequence $u = A^k C B^k \in L_{\mathcal{N}}$ can be decomposed only by $x = A^l$, $y = A^m$, $z = A^n C B^k$ with $l + m + n = k$, $0 < m \leqq k$. Then the sequences $A^l A^{\nu m} A^n C B^k$ are in $L_{\mathcal{N}}$ for $\nu = 1, 2, 3, \ldots$ but not for $\nu = 0$.

4. Final remarks

For terminal Petri net languages $L_{\mathcal{N}}^{m_f}$ of labelled Petri nets with an endmarking m_f, defined by

$$L_{\mathcal{N}}^{m_f} := \{h(u) \mid u \text{ is a firing sequence with } m_0 + \Delta u = m_f\} ,$$

the pumping lemma does not hold (since $\{a^n b^n \mid n \in \mathbb{N}\}$ is such a language). While the terminal Petri net languages build up a substantially richer class than the nonterminal Petri net languages, the related problem for Petri net predicates is still open. The nonterminal Petri net predicate $M_{\mathcal{N}}$ of $\mathcal{N}$ with respect to a subset $X \subseteqq P$ of places is the projection of the reachability set $R_{\mathcal{N}}$ to X:

$$M_{\mathcal{N}} := \Pi_X (R_{\mathcal{N}}) = \{x \mid x \in \mathbb{N}^X \wedge \exists y \in \mathbb{N}^{P \setminus X} : [x, y] \in R_{\mathcal{N}}\} ,$$

and the terminal Petri net predicate $M_{\mathcal{N}}^y$ of $\mathcal{N}$ with respect to X and a given final submarking $y \in \mathbb{N}^{P \setminus X}$ is given by

$$M_{\mathcal{N}}^y := \Pi_X \big(R_{\mathcal{N}} \cap (\mathbb{N}^X \times \{y\})\big) = \{x \mid x \in \mathbb{N}^X \wedge [x, y] \in R_{\mathcal{N}}\} .$$

If M is infinite, it must contain certain linear subsets according to the corollary to Theorem 1. Thus the class $\mathfrak{M}$ of all nonterminally generable Petri net predicates is a small subset of the class $\mathfrak{M}_{re}$ of all recursively enumerable predicates. On the other hand, it is unknown whether the class $\mathfrak{M}_0$ of all terminal Petri net predicates equals $\mathfrak{M}_{re}$ (this question is related to the open question whether all recursively enumerable languages are terminal Petri net languages). But, in difference to the situation in the case of languages, it is also unknown whether or not $\mathfrak{M} = \mathfrak{M}_0 \setminus \{\emptyset\}$.

Moreover, it can be shown that the reachability problem would be decidable if it was possible to construct a net $\mathcal{N}'$ which nonterminally generates $M_{\mathcal{N}'} = M_{\mathcal{N}}^y$ to each Petri net $\mathcal{N}$ terminally generating a nonempty set $M_{\mathcal{N}}^y$.

To show this let $\mathcal{N} = (P, T, F, m_0)$ be a Petri net where we have to decide whether a certain marking $m \neq m_0$ is reachable. We add a "transition count place" p^+ to the net $\mathcal{N}$ such that each transition $t \in T$ adds one token to p^+, in the beginning the place p^+ is clean. Then for $X := \{p^+\}$ and $y := m$ the new net $\mathcal{N}^+$ terminally generates the predicate

$$M_{\mathcal{N}^+}^y = \{n \mid \exists u \in T^n : m_0 + \Delta u = m \quad \text{and } u \text{ is a firing sequence}\} .$$

Since $\mathfrak{M}_0$ is (constructively) closed under union, we can construct a Petri net $\mathcal{N}^*$ such that $\mathcal{N}^*$ terminally generates

$$M_{\mathcal{N}^*}^{y^*} = M_{\mathcal{N}^+}^y \cup \{0\} .$$

If we now were able to construct a net $\mathcal{N}'$ which on a place p' nonterminally generates the same set $M_{\mathcal{N}^*}^{y^*} = M_{\mathcal{N}'}$, we would only have to decide whether or not the place p'

is 0-bounded in $\mathcal{N}'$ and then we would know whether $M^y_{\mathcal{N}^+}$ was empty, i.e. whether m was reachable in the net $\mathcal{N}$ or not.

It is not known, whether conversely the decidability of the reachability problem would imply $\mathfrak{M} = \mathfrak{M}_0 \setminus \{\emptyset\}$.

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Kurzfassung

Eine sorgfältige Analyse des Überdeckbarkeitsbaumes für Petri-Netze zeigt, daß jede unendliche Erreichbarkeitsmenge unendliche lineare Teilmengen enthält. Das kann in Form eines "pumping lemma" für Markierungen beschrieben werden. Ferner gilt für die nicht-terminalen Petri-Netz-Sprachen ein "pumping lemma" ähnlich dem für reguläre Sprachen.

Резюме

Тщательный анализ деревьев покрываемости для сетей Петри показывает, что каждое бесконечное множество достижимости содержит бесконечные линейные подмножества. Это можно сформулировать в форме леммы Бар-Хиллела для маркировок. Кроме того, для нетерминальных языков сетей Петри имеет место некоторая лемма Бар-Хиллела, подобная соответственной лемме для регулярных языков.

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