Rocket science notes

IN104, Year 2020

Project 9: Rocket design with python

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1 Introduction

This booklet is an introduction to rocket science. It contains the necessary material that you will use for developing your project. The content can be updated as the course advanced to make explanations more clear or to add more material that could be useful for certain implementations in your code.

1.1 Assumptions

Througout this document, some assumptions are done in order to simplify the derivations of the equations. These assumptions are:

- Net lift acting on the rocket is null.
- Atmospheric effects (i.e. drag, temperature changes) are not considered.

1.2 How to use this document

At the beginning of each chapter there is a short description on its contents and what you will. There are some demonstrations and derivations that I consider to be interesting for knowing where some concepts come from. However, these derivations are not at all necessary for carrying on the project, and you can skip them if you wish. The only necessary equations that you will need are contained within boxes; see, for instance, Equation (18).

If you don't understand something that you think it is essential for advancing, write me an email as soon as possible. The physics contained in this document must not suppose a barrier for you to proceed, as they are not the objective but a tool for developing your code.

2 Fundamentals

This chapter presents the background for any rocket scientist. It is organised in four different sections with the necessary information for analyzing single-stage rockets.

Section 2.1 contains the derivation of Newton's second law applied to rockets, which is useful for the definitions stated in the following sections. However, a total understanding of this derivation is not essential, and you can skip this section if you wish. You can jump to section 2.2 and further for getting the formulas you will use, and go back to 2.1 if you are curious or want more extra information.

2.1 Newton's second law

Rockets are **variable-mass systems** that produce propulsion by expelling mass (the mass of burnt propellants). Their motion is dominated by Newton's second law. In its most fundamental form, this law takes the following form

$$\sum \boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \tag{1}$$

where $\boldsymbol{F}$ are the external forces applied to the rocket and $\boldsymbol{p}$ is the momentum.

Momentum change

Figure 1 shows a rocket at two time consequtive time instants: t and $t + dt$. At the first one, the rocket with mass m is moving at a velocity $\boldsymbol{v}$. After a time some time dt, some mass dm has been expelled at velocity $\boldsymbol{u}$, producing a velocity increase in the rocket to $\boldsymbol{v} + d\boldsymbol{v}$ and a mass decrease to $m - dm$.

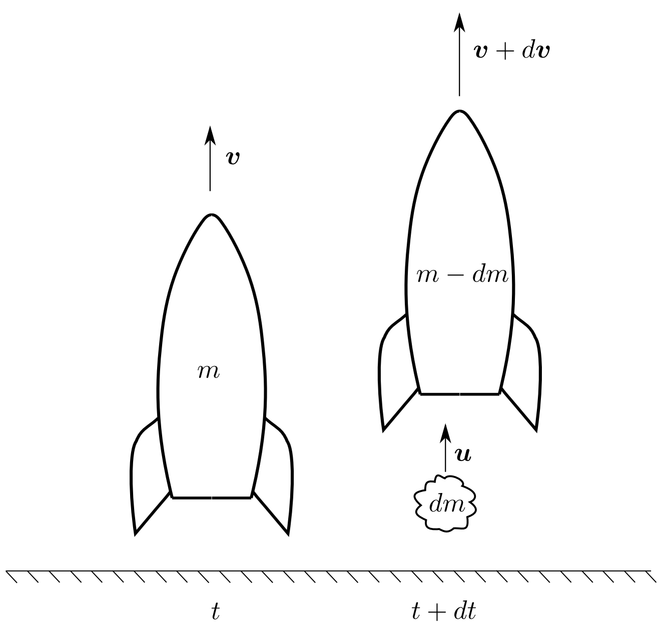

Figure 1: Momentum change in a variable-mass system.

The momentum of the system at the first time instant is:

$$\boldsymbol{p}_t = m\boldsymbol{v} \tag{2}$$

At the second time instant, the momentum is:

$$\boldsymbol{p}_{t+dt} = (m - dm)\,(\boldsymbol{v} + d\boldsymbol{v}) + dm\boldsymbol{u} \tag{3}$$

The difference in momentum between both time instants is:

$$d\boldsymbol{p} = \boldsymbol{p}_{t+dt} - \boldsymbol{p}_t = m \cdot d\boldsymbol{v} + (\boldsymbol{u} - \boldsymbol{v})\, dm \tag{4}$$

To simplify this expression, another velocity can be defined: the **relative velocity of expelled propellants with respect to the rocket**, denoted by $\mathbf{c}$ and defined as $\mathbf{c} = \mathbf{v} - \mathbf{u}$. With this definition, the last expression is:

$$d\boldsymbol{p} = md\boldsymbol{v} - \boldsymbol{c}dm \tag{5}$$

And finally the derivative with respect to time can be taken, yielding the right-hand side of Equation (1):

$$\frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} - \boldsymbol{c}\frac{dm}{dt} \tag{6}$$

In order to produce motion, rockets burn propellant and expell exhaust gases. This means that the mass reduction of the rocket is the mass expulsion of propellants. In this way, the **propellant mass flow rate** (i.e. the rate of mass expulsion of the rocket) $\dot{m}$ can be defined as follows:

$$\dot{m} = \frac{dm}{dt} \tag{7}$$

So the derivative of momentum with respect to time can finally be rewritten as follows:

$$\frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} - \dot{m}\boldsymbol{c} \tag{8}$$

External forces

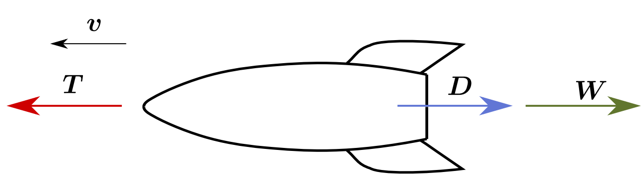

Figure 2: External forces acting on rocket: thrust $\mathbf{T}$, drag $\mathbf{D}$ and weight $\mathbf{W}$.

For determining the left-hand side of Equation (1), the external forces are divided into two contributions (see Figure 2):

1. Aerodynamic forces $\boldsymbol{F_a}$. Assuming that net lift acting on the rocket is zero, external forces are reduced to two contributions: **drag** ($\boldsymbol{D}$) and a net thrust force acting on the nozzle exit, called **pressure thrust** ($\boldsymbol{T_p}$):

$$\boldsymbol{F_a} = \boldsymbol{T_p} + \boldsymbol{D} = A_e\,(p_e - p_0)\,\boldsymbol{n} + \boldsymbol{D} \tag{9}$$

   with $\boldsymbol{T_p} = A_e\,(p_e - p_0)\,\boldsymbol{n}$ where: A_e is the exit area of the nozzle, p_e is the exit pressre of the nozzle, p_0 is the atmospheric pressure and $\boldsymbol{n}$ is the normal vector to the nozzle exit surface. Details on drag and on the derivation of pressure thrust are not given in this document.

2. Weight $\boldsymbol{W}$. It is given by the following expression:

$$\boldsymbol{W} = m\boldsymbol{g} \tag{10}$$

So the forces' term is given by:

$$\sum \boldsymbol{F} = \boldsymbol{F_a} + \boldsymbol{W} = A_e\,(p_e - p_0)\,\boldsymbol{n} + \boldsymbol{D} + m\boldsymbol{g} \tag{11}$$

And eventually, both expressions (8) and (11) can be plugged into (1) to yield Newton's second law:

$$m\frac{d\boldsymbol{v}}{dt} - \dot{m}\boldsymbol{c} = A_e\left(p_e - p_0\right)\boldsymbol{n} + \boldsymbol{D} + m\boldsymbol{g} \tag{12}$$

This equation can be rearranged by putting the term $\dot{m}\boldsymbol{c}$ into the right hand side:

$$m\frac{d\boldsymbol{v}}{dt} = \underbrace{\dot{m}\boldsymbol{c} + A_e\left(p_e - p_0\right)\boldsymbol{n}}_{\text{Thrust}} + \boldsymbol{D} + m\boldsymbol{g} \tag{13}$$

In the last equation, there are two terms that generate propulsive force known as **thrust**: the **impulse thrust** produced by mass expulsion and given by the term $\dot{m}\boldsymbol{c}$, and the **pressure thrust** already defined. If thrust is denoted by $\boldsymbol{T}$, then Newton's second law is:

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{T} + \boldsymbol{D} + m\boldsymbol{g} \tag{14}$$

Simplifications

The equations derived previously are vectorial, so they can be applied to any direction in space. For the purpose of this project, the main direction of interest is the direction of motion, represented by the vertical coordinate z as in Figures 1 and 2. Therefore, for the sake of simplicity the bold symbols denoting vectors will be removed, and all quantities will refer to direction z unless otherwise stated. With these considerations, Newton's second law is expressed as:

$$m\frac{dv}{dt} = T + D + mg \tag{15}$$

When calculating flight trajectories, two or three directions must be considered (for 2D or 3D trajectories respectively). In this project, the trajectories we will calculate are 2D. Therefore, distinction in terms of horizontal and vertical directions (x and z coordinates) has to be done. We will deal with this in Chapter 4.

2.2 Thrust and effective exhaust velocity

Thrust is the propulsive force of the rocket. According to Equation (13), it can be defined as (eliminating vectors):

$$T = \dot{m}c + A_e\left(p_e - p_0\right) \tag{16}$$

In this project, we will not deal neither with pressures (p_e, p_0) nor with geometric characteristics of the nozzles (A_e). For getting rid of these magnitudes, a magnitude name **effective exhaust velocity** (c_{eff}) can be defined:

$$c_{\text{eff}} = c + \frac{A_e\left(p_e - p_0\right)}{\dot{m}} \tag{17}$$

This magnitude can be substituted in Equation (16) to produce a more simple definition of thrust:

$$\boxed{T = \dot{m}c_{\text{eff}}} \tag{18}$$

Thrust will usually be expressed in kN, $\dot{m}$ in kg/s and c_{eff} in m/s.

2.3 Specific impulse

The **specific impulse** is a very important definition in rockets. It can be defined from thrust:

$$\boxed{I_{\text{sp}} = \frac{T}{\dot{m}g_0} = \frac{c_{\text{eff}}}{g_0}} \tag{19}$$

where g_0 is the gravity acceleration at sea level: $g_0 = 9.81\ m/s^2$. I_{sp}, which is expressed in s, is a figure of merit that indicates how efficiently a rocket burns its propellant.

2.4 The rocket equation

One of the most important relations in rocket propulsion is the **rocket equation**, also called Tsiolkovsky's equation on behalf of its author. This expression relates the change of velocity of a variable-mass system with the mass difference:

$$\boxed{\Delta v = v_f - v_0 = I_{\mathrm{sp}} g_0 \ln\left(\frac{m_0}{m_f}\right)} \tag{20}$$

where:

- v_0 if the initial velocity of the rocket.
- v_f is the final velocity of the rocket after all the propellant has been burnt.
- m_0 if the initial mass of the rocket.
- m_f is the final mass of the rocket after all the propellant has been burnt

The rocket equation (20) shows that the higher the I_{sp} of a rocket, the larger its final velocity.

2.5 Final mass and burning time

Let the propellant mass be m_p, which is a known parameter. Thus, the final mass of the rocket is simply the subtraction of the initial and propellant masses:

$$\boxed{m_f = m_0 - m_p} \tag{21}$$

The propellant mass will be consumed at a rate $\dot{m}$. Then, the **burning time** can be calculated as:

$$\boxed{t_b = \frac{m_p}{\dot{m}}} \tag{22}$$

3 Rockets characteristics

This chapter contains the geometric and mass definitions needed for defining and sketching rockets. All the concepts in this chapter appear in the rocket database which is at your disposal in github.

For simplifying the visualization of the geometric characteristics and mass distribution inside rockets, Figure 3 is used. In the left a rocket with a single stage is shown, while the rocket in the right shows a two-stages rocket.

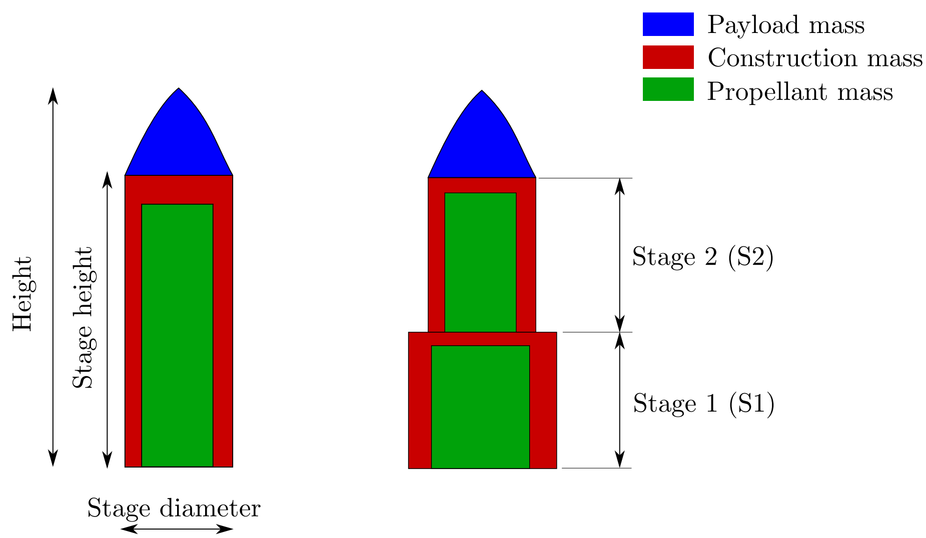

Figure 3: Simplified geometry of rockets with a schematic of mass distribution. *Left*: single-staged rocket. *Right*: multi-staged rocket with two stages.

3.1 Mass distribution in rockets

Before launch, a rocket has an initial mass m_0, also called **lift-off mass**. This mass is the addition of three contributions:

$$\boxed{m_0 = m_c + m_p + m_u} \tag{23}$$

where:

- m_c is the **construction mass**: the mass of the structure and equipment that belongs to the rocket. This mass is not expelled during flight.
- m_p is the **propellant mass**: the mass that is expelled during flight.
- m_u is the **payload mass** (also called useful mass): the mass carried by the rocket for achieving a given mission. Examples of payload masses are satellites which are put into orbit, astronauts, probes which are sent to the outer space, etc.

These three masses are depicted in Figure 3 in different colours. In the mission, the propellant mass is expelled at a rate $\dot{m}$ during a time t_b. For calculating these values, you can do the following:

1. As you know the specific impulse I_{sp} and the thrust T (they are defined in the database), then you can calculate $\dot{m}$ from Equation (19).
2. Then, Equation (22) is applied for obtaining t_b.

3.2 Multi-staged rockets

In reality there are not many single-staged rockets being used, as they can not reach velocities which are high enough to overcome the effect of gravity and leave the Earth. Most rockets are **multi-staged**.

The idea of multi-staging is that staged are stacked one onto each other. Each staged has its own construction and propellant masses: once a stage has consumed all its propellant, its corresponding construction mass is expelled and the next stage can start to burn its fuel. On top of the last stage there is the payload mass of the rocket. Therefore, in a rocket there will be as many propellant and construction masses as stages, but there will be only one payload mass.

In this project, we will work with rockets of up to two stages. Therefore, the **lift-off mass for a two-stages rocket** is given by:

$$\boxed{m_0 = m_{c_1} + m_{p_1} + m_{c_2} + m_{p_2} + m_u} \tag{24}$$

where each subindex refers to each stage. The bottom stage will be stage 1 (**S1**), while the upper stage will be stage 2 (**S2**), see Figure 3.

All rockets contained in the database have two stages except one, the rocket Miura 1.

3.3 Geometry

Rockets will be sketched as shown in Figure 3: they will be considered as a rectangle plus a triangle. The rectangle will be the stage containing the construction and propellant masses, and the triangle will represent the contained of the useful mass (this part is known as nosecone).

The rockets will have a **diameter** and a **height**. A **stage height** is defined too, so that the difference between this value and the height is the length of the nosecone.

For two-stages rockets, each stage is defined by its own length and diameter: **S1 length**, **S1 diameter**, **S2 length** and **S2 diameter**. One way of testing the coherence of the rocket's geometry is by checking that the sum of both lenghts is not larger than the height of the rocket: in other words, that the length of the nosecone is negative. You can implement this functionality as a test that is performed when a new rocket is defined and added into the database.

For single-staged rockets (Miura 1 and the ones that you add), all the characteristics from stage 2 are not defined. In the database, these attributes are empty values.

4 Flight trajectories

Each body launched from Earth or moving through space follows a path known as trajectory. Trajectories will be calculated according to the planned mission to be achieved. This means that the first step before designing a rocket is to define the mission and the trajectory to achieve this mission.

When reaching high altitudes or for interplanetary missions, rocket's motion will follow elliptic, parabolic or hyperbolic trajectories in **Kepler orbits**. These trajectories are governed by the laws of astrodynamics and need the introduction of orbital parameters.

A more simple version of rocket's paths can be done by considering **flat trajectories**. These kind of trajectories will not take into account the Earth's curvature, and can be a very good representation of the movement of rockets intended to reach low altitudes.

For the sake of simplicity, in this work only flat trajectories will be considered. Even that only flat-Earthers would like them, they will considerably simplify the physics and will not need the definition of orbital paremeters.

4.1 Assumptions

Flat trajectories develop in this chapter consider the following assumptions:

- Earth curvature is neglected, so rocket moves in a cartesian $x - z$ framework.
- Gravity acceleration is constant with altitude and equal to its value at sea level: $g = g_0 = 9.81\ m/s^2$. This simplification follows from ignoring the Earth's curvature.
- Rocket moves in vacuum. Therefore, drag force is neglected.
- During powered flight (§4.3), thrust T, propellant mass flow rate $\dot{m}$ and pitch angle θ (see Figure 4) are constant.

4.2 Phases

The trajectories will be split into two main phases:

1. **Powered flight**, §4.3. In this phase the rocket is burning and expelling fuel, hence producing thrust. This phase goes from take-off until all fuel is burnt. There will be as many powered flight phases as rocket stages.
2. **Ballistic flight after extinction**, §4.4. Once all fuel has been burnt, the rocket will follow a ballistic trajectory until it falls back to ground.

4.3 Powered flight

Powered flight is the phase of the trajectory that extends from take-off until the **extinction point**, which is the point at which all the fuel has been burnt. During this phase, mass expulsion produces a thrust force in the rocket.

Figure 4 shows the definitions and external forces applied to the rocket in a 2D trajectory. As movement is assumed to take place in vacuum, drag is neglected and thrust and weight act in the body.

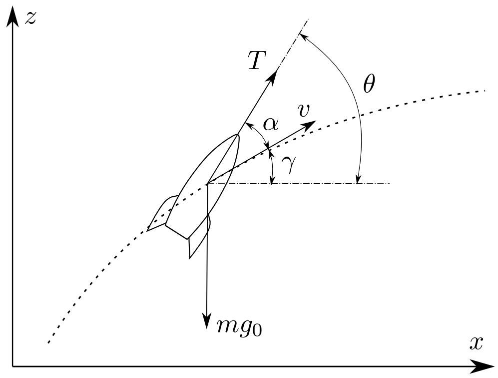

Figure 4: System of reference and angles definition for 2D rocket motion. Dashed line indicates the trajectory.

where x and z are horizontal and vertical coordinates, T is thrust, v is the rocket velocity (which is parallel to the trajectory, represented by the dashed line), and θ, α and γ are the **pitch angle**, **flight path angle** and **angle of attack** respectively.

The equations of motion can be obtained by applying Newton's second law to the scenario presented in Figure 4:

$$m\frac{dv_x}{dt} = T\cos\theta \tag{25a}$$

$$m\frac{dv_z}{dt} = T\sin\theta - mg_0 \tag{25b}$$

Velocity can be obtained by solving Eqs. (25) numerically. However, if the pitch angle θ is assumed to be constant during flight, the equations of motion can be integrated to get an analytical expression for the velocities:

$$v_x(t) = v_{x_0} + I_{\text{sp}} g_0 \ln\left(\frac{m_0}{m(t)}\right)\cos\theta \tag{26a}$$

$$v_z(t) = v_{z_0} + I_{\text{sp}} g_0 \ln\left(\frac{m_0}{m(t)}\right)\sin\theta - g_0 t \tag{26b}$$

where v_{x_0} and v_{z_0} are the rocket velocities at $t = 0$. If the rocket takes-off from ground, their value is 0. Note that these expressions are equivalent to applying rocket equation (20) considering two dimensions and gravity.

The mass with time can be obtained from the following expression:

$$\boxed{m(t) = m_0 - \dot{m}t} \tag{27}$$

For obtaining the attitude of the rocket, the flight path angle γ can be obtained from the horizontal and vertical velocities along the trajectory:

$$\gamma(t) = \arctan\left(\frac{v_z(t)}{v_x(t)}\right) \tag{28}$$

After some mathematical development (not shown here), γ is given by the following expression:

$$\boxed{\gamma\left(t\right) = \arctan\left(\tan\theta - \frac{t}{I_{\text{sp}}\ln\left(\frac{m_0}{m(t)}\right)\cos\theta}\right)} \tag{29}$$

The initial flight path angle cannot be obtained by substituting $t = 0$ in the previous equation, as it would produce an indeterminate form. Instead, it can be calculated by applying the following relation:

$$\boxed{\gamma_0 = \gamma\left(t = 0\right) = \arctan\left(\tan\theta - \frac{m_0 g_0}{T\cos\theta}\right)} \tag{30}$$

Finally, the angle of attack α can by easily obtained from the other angles:

$$\boxed{\alpha\left(t\right) = \theta - \gamma\left(t\right)} \tag{31}$$

It follows that at $t = 0$, $\alpha_0 = \theta - \gamma_0$.

For obtaining the path of the trajectory, Equations (26) can be integrated with time:

$$x\left(t\right) = x_0 + v_{x_0} t + \frac{m_0 I_{\text{sp}}^2 g_0^2}{T}\left[1 - \frac{m\left(t\right)}{m_0}\left(\ln\left(\frac{m_0}{m\left(t\right)}\right) + 1\right)\right]\cos\theta \tag{32a}$$

$$z\left(t\right) = z_0 + v_{z_0} t + \frac{m_0 I_{\text{sp}}^2 g_0^2}{T}\left[1 - \frac{m\left(t\right)}{m_0}\left(\ln\left(\frac{m_0}{m\left(t\right)}\right) + 1\right)\right]\sin\theta - \frac{1}{2}g_0 t^2 \tag{32b}$$

Extinction point

The extinction point is reached when all the fuel has been burnt; that is, at the burning time t_b. At this stage, the mass is is equal to the final mass m_f. Therefore, the properties at the extinction point can be obtained by substituting $t = t_b$ and $m\left(t = t_b\right) = m_f$ in the previous equations:

$$x_e = x_0 + v_{x_0} t_b + \frac{m_0 I_{\text{sp}}^2 g_0^2}{T}\left[1 - \frac{m_f}{m_0}\left(\ln\left(\frac{m_0}{m_f}\right) + 1\right)\right]\cos\theta \tag{33a}$$

$$z_e = z_0 + v_{z_0} t_b + \frac{m_0 I_{\text{sp}}^2 g_0^2}{T}\left[1 - \frac{m_f}{m_0}\left(\ln\left(\frac{m_0}{m_f}\right) + 1\right)\right]\sin\theta - \frac{1}{2}g_0 t_b^2 \tag{33b}$$

$$v_{x_e} = v_{x_0} + I_{\text{sp}} g_0 \ln\left(\frac{m_0}{m_f}\right)\cos\theta \tag{33c}$$

$$v_{z_e} = v_{z_0} + I_{\text{sp}} g_0 \ln\left(\frac{m_0}{m_f}\right)\sin\theta - g_0 t_b \tag{33d}$$

$$\gamma_e = \arctan\left(\tan\theta - \frac{t_b}{I_{\text{sp}}\ln\left(\frac{m_0}{m_f}\right)\cos\theta}\right) \tag{33e}$$

$$\alpha_e = \theta - \gamma_e \tag{33f}$$

4.4 Ballistic flight after extinction

After extinction, no more thrust is produced and the only external force is the gravity. The rocket follows a ballistic trajectory from the extinction point until it impacts the Earth again.

The equations of motion for ballistic flight are given by Eqs. (25) with $T = 0$. Integrating them, and considering as initial condition the extinction point of powered flight, the velocities of the ballistic trajectory are:

$$v_x(t) = v_{x_e} \tag{34a}$$
$$v_z(t) = v_{z_e} - g_0(t - t_b) \tag{34b}$$

The flight path angle γ is obtained by applying Eq. (28) with the previous expressions for the velocities. The position is easily obtained by integrating the velocity:

$$\boxed{x(t) = x_e + v_{x_e}(t - t_b)} \tag{35a}$$
$$\boxed{z(t) = z_e + v_{z_e}(t - t_b) - \frac{1}{2}g_0(t - t_b)^2} \tag{35b}$$

Two important point of ballistic flight are the **culmination point** and the **impact point**. The **culmination point** which is the highest point of the trajectory: following a ballistic trajectory, the rocket will rise until arriving to the the culmination point. Once it has been reached, the rocket will start to fall until the **impact point**, i.e. the moment when the rocket goes back to the ground. This is the last moment of the trajectory.

At the **culmination point**, the vertical velocity is zero: $v_z = 0$. By substituting this into Eq. (34b), the time to reach the culmination point t_c can be obtained:

$$\boxed{t_c = I_{\text{sp}} \ln\left(\frac{m_0}{m_f}\right)\sin\theta} \tag{36}$$

And the coordinates of the culmination point are given by subsituting $t = t_c$ into (35):

$$\boxed{x_c = x_e + v_{x_e}(t_c - t_b)} \tag{37a}$$
$$\boxed{z_c = z_e + v_{z_e}(t_c - t_b) - \frac{1}{2}g_0(t_c - t_b)^2} \tag{37b}$$

Finally, the **impact point** can be calculated as the point in which the vertical coordinate is null: $z_i = 0$. Then, the time to reach the impact point t_i is:

$$\boxed{t_i = t_c + \sqrt{\frac{2z_c}{g_0}}} \tag{38}$$

And its horizontal position x_i, which is also the **range** of the trajectory, is:

$$\boxed{x_i = x_c + v_{x_e}\sqrt{\frac{2z_c}{g_0}}} \tag{39}$$

4.5 Example of trajectory

An example of flat trajectory followed by the rocket Miura 1 is shown in Figure 5. This path has been obtained by applying the equations introduced in this chapter with a pitch angle $\theta = 60°$. The top figure shows the path, the bottom graph shows the evolution of the flight path angle γ along the trajectory.

The trajectory can be animated by plotting its spatial evolution with time. The sketches of the rockets can be imported into the figure and its orientation can be set according to the value of the flight path angle γ. An example of the trajectory from Figure 5 is shown in Figure 6.

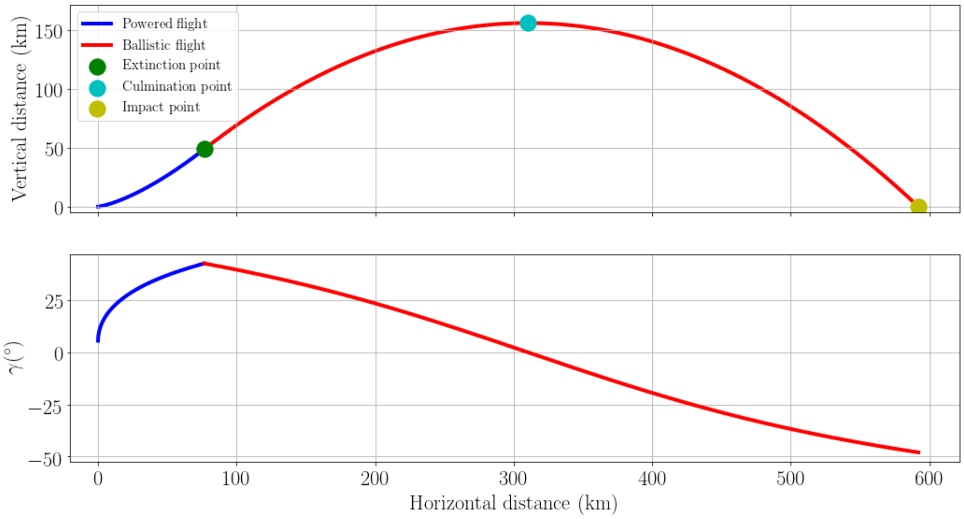

Figure 5: Flat trajectory followed by rocket Miura 1 with pitch angle $\theta = 60°$.

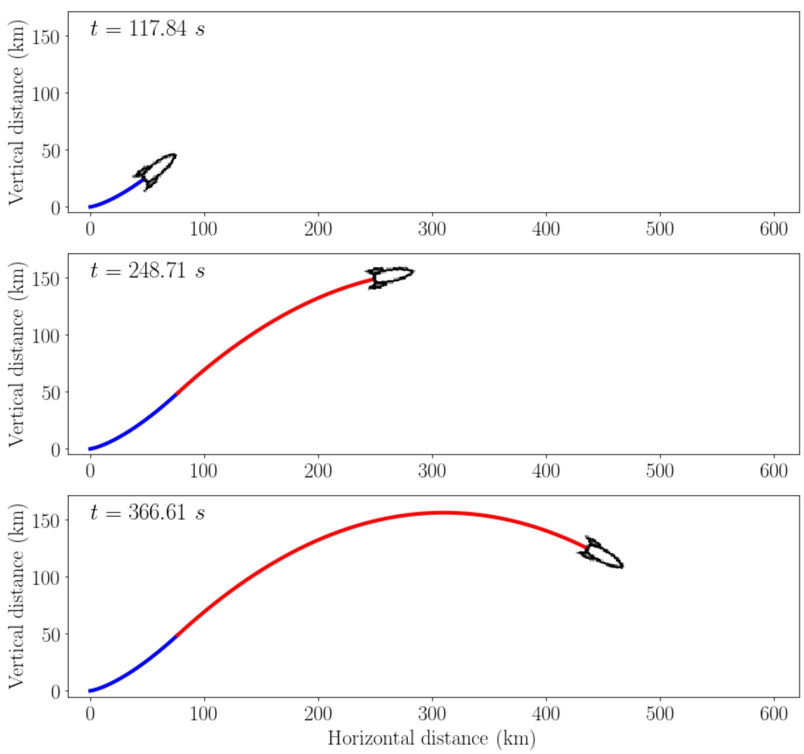

Figure 6: Three instants along the trajectory followed by rocket Miura 1 with pitch angle $\theta = 60°$.