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A GENERALIZATION OF BERNSTEIN-DOETSCH THEOREM

Abstract. Let V be an open convex subset of a nontrivial real normed space X. In the paper we give a partial generalization of Bernstein-Doetsch Theorem. We prove that if there exist a base $\mathcal{B}$ of X and a point $x \in V$ such that a midconvex function $f : X \to \mathbb{R}$ is locally bounded above on b-ray at x for each $b \in \mathcal{B}$, then f is convex. Moreover, we show that under the above assumption, f is also continuous in case $X = \mathbb{R}^N$, but not in general.

Let X be a real normed space and V be a convex subset of X. A function $f : V \to \mathbb{R}$ is called convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad \text{for } x, y \in V,\ t \in [0,1].$$

If the above inequality holds for $t = \frac{1}{2}$, then f is said to be midconvex (or Jensen convex).

F. Bernstein and G. Doetsch [1] proved that every midconvex function $f : (a,b) \to \mathbb{R}$ locally bounded above at a point is continuous (clearly a continuous midconvex function must be convex). The above statement has many generalizations (see e. g. [2]–[10]). Nowadays by Bernstein-Doetsch Theorem we usually mean the following one.

Bernstein-Doetsch Theorem. *Let V be an open convex subset of X and let $f : V \to \mathbb{R}$ be midconvex. If f is locally bounded above at a point, then f is continuous and convex.*

In the paper we give a generalization of Bernstein-Doetsch Theorem. To formulate this generalization, we introduce the following definition.

Definition 1. Let V be an open subset of a nontrivial normed space X and $x \in V$. For each $u \in V$ we define the set D_u by

$$D_u = \{t \in \mathbb{R} : \ x + tu \in V\}.$$

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A function $f : V \to \mathbb{R}$ is called locally bounded above on u-ray at a point x if the function $f_u : D_u \ni t \to f(x + tu)$ is locally bounded above at zero.

Lemma 1. *Let V be an open convex subset of $\mathbb{R}^N$ and $f : V \to \mathbb{R}$ a midconvex function. Let $\mathcal{B} = \{b_1, \ldots, b_N\}$ be an algebraic base of $\mathbb{R}^N$ and let $x \in V$. If the function f is locally bounded above on b_n-ray at x for each $n \in \{1, \ldots, N\}$, then f is continuous and convex.*

Proof. Take an arbitrary $n \in \{1, \ldots, N\}$. We show that the function f_{b_n} is continuous. From the definition of the function f_{b_n}, for every $t_1, t_2 \in D_{b_n}$, we get

$$\begin{aligned} f_{b_n}\left(\frac{t_1 + t_2}{2}\right) &= f\left(x + \frac{t_1 + t_2}{2} b_n\right) = f\left(\frac{(x + t_1 b_n) + (x + t_2 b_n)}{2}\right) \\ &\leq \frac{f(x + t_1 b_n) + f(x + t_2 b_n)}{2} = \frac{f_{b_n}(t_1) + f_{b_n}(t_2)}{2}. \end{aligned}$$

This means that f_{b_n} is midconvex. Moreover, by assumption, f_{b_n} is locally bounded above at zero. So, according to Bernstein-Doetsch Theorem, the function f_{b_n} is continuous.

Now, we prove that f is locally bounded above at a point x. Since V is an open set, there exists $r > 0$ such that $B(x, r) \subset V$. The continuity of the functions f_{b_n} implies that they are bounded on the interval $[-\frac{r}{2}, \frac{r}{2}]$. Consequently, there exists a constant $M \in \mathbb{R}$ with

$$(1) \qquad f_{b_n}(t) \leq M \quad \text{for } t \in \left[-\frac{r}{2}, \frac{r}{2}\right],\ n \in \{1, \ldots N\}.$$

It is well known that $\beta_1 b_1 + \ldots \beta_N b_N \to 0$ if and only if $\beta_n \to 0$ for each $n \in \{0, \ldots, N\}$, where $\beta_1, \ldots, \beta_N \in \mathbb{R}$. Consequently, there exists $r_1 \in (0, r)$ such that if $\|\beta_1 b_1 + \ldots + \beta_N b_N\| < r_1$, then $|\beta_1|, \ldots, |\beta_N| \in [0, \frac{r}{2N})$.

Fix $z \in B(x, r_1)$. There exist $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ such that

$$z - x = \alpha_1 b_1 + \ldots + \alpha_N b_N.$$

Whence we get

$$(2) \qquad z = \frac{(x + N\alpha_1 b_1) + \ldots + (x + N\alpha_N b_N)}{N}.$$

Since $\|z - x\| = \|\alpha_1 b_1 + \ldots + \alpha_N b_N\| < r_1$, we have $|\alpha_1|, \ldots, |\alpha_N| \in [0, \frac{r}{2N})$. Hence, applying (1) and (2) we obtain

$$\begin{aligned} f(z) &= f\left(\frac{(x + N\alpha_1 b_1) + \ldots (x + N\alpha_N b_N)}{N}\right) \\ &\leq \frac{f(x + N\alpha_1 b_1) + \ldots + f(x + N\alpha_N b_N)}{N} \\ &= \frac{f_{b_1}(N\alpha_1) + \ldots + f_{b_N}(N\alpha_N)}{N} \leq \frac{NM}{N} = M. \end{aligned}$$

This means that f is bounded on the ball $B(x, r_1)$. Using Bernstein-Doetsch Theorem, we conclude that f is continuous and convex. This ends the proof. ∎

Theorem 1. *Let V be an open convex subset of a nontrivial real normed space X and $f : V \to \mathbb{R}$ a midconvex function. Let $\mathcal{B}$ be an algebraic base of X and let $x \in V$. If f is locally bounded above on b-ray at x for each $b \in \mathcal{B}$, then f is convex.*

Moreover in case $X = \mathbb{R}^N$, f is also continuous.

Proof. Using the translation in X we may assume that $x = 0$. Take $w, z \in V$ such that $w \neq 0$ or $z \neq 0$. There exist $b_1, \ldots, b_N \in \mathcal{B}$ such that w and z can be written as linear combinations of $b_1, \ldots, b_N$. Put $A := \mathrm{lin}(b_1, \ldots, b_N)$, $V_A := V \cap A$ and $f_A := f|_{V_A}$. Notice that the set V_A is open and convex in the normed space A. Furthermore $w, z \in V_A$ and f_A is locally bounded above on b_n-ray for each $n \in \{1, \ldots, N\}$. Of course the function f_A is midconvex, hence according to Lemma 1, f_A is convex. In particular we have

$$f_A(tw + (1-t)z) \leq t f_A(w) + (1-t) f_A(z) \qquad \text{for } t \in [0,1].$$

Consequently

$$f(tw + (1-t)z) \leq t f(w) + (1-t) f(z) \qquad \text{for } t \in [0,1].$$

Since w and z are arbitrarily chosen, it means that the function f is convex.

According to Lemma 1, if $X = \mathbb{R}^N$, then f is continuous. ∎

It is easy to notice that in case $\dim X = \infty$, under assumptions of the above theorem, the function f need not be continuous. As it is well known in such a case there exists a discontinuous linear functional. Such a functional is obviously midconvex and continuous on rays, and hence bounded on rays.

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