BERNSTEIN–DOETSCH-TYPE RESULTS FOR GENERAL FUNCTIONAL INEQUALITIES

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Dedicated to the 70th birthday of Professor Zenon Moszner

ABSTRACT. In this paper certain functional inequalities are investigated whose solutions shown to be convex provided that they are locally bounded from above. The results so obtained generalize the classical result of Bernstein and Doetsch for Jensen-convex functions and also that of Ng and Nikodem for δ-Jensen-convex functions.

1. INTRODUCTION

Denote by I a nonempty open real interval throughout this paper. A function $f : I \to \mathbb{R}$ is called *Jensen-convex* if it satisfies

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} \qquad (x, y \in I). \tag{1}$$

It is a well known result (e.g. see the books [HLP52], [Kuc85], [RV73]) that Jensen-convexity yields

$$f\Big(\frac{x+y+z}{3}\Big) \le \frac{f(x)+f(y)+f(z)}{3} \qquad (x, y, z \in I),$$

whence we get that, for any Jensen-convex function $f : I \to \mathbb{R}$,

$$f\Big(\frac{x+y+M(x,y)}{3}\Big) \le \frac{f(x)+f(y)+f(M(x,y))}{3} \qquad (x, y \in I), \tag{2}$$

whenever M is a *two-variable mean on* I, that is, $M : I^2 \to I$ satisfies

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\} \qquad (x, y \in I).$$

If M is the arithmetic mean, then (2) is equivalent to (1). It is interesting to ask if (2) implies (1) also for other two-variable means. This problem was raised during the 7th International Conference on Functional Equations and Inequalities (Złockie Muszyna, Poland, 1999) by the author and Z. Daróczy ([DP99]) in the case when M is the geometric mean and I is the set of positive reals.

Unfortunately, we are not able to answer the above question in this generality. The goal of this paper is to show that under local boundedness assumptions on f and

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continuity assumptions on M, inequalities analogous to (2) imply that f is *convex on* I, that is,

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y) \qquad (x,y \in I,\, t\in[0,1]).$$

The results obtained generalize the classical Bernstein-Doetsch theorem [BD15] (see also [Kuc85]) which states if f is Jensen-convex and locally bounded from above, then it is also convex.

2. Main Results

Consider the functional inequality

$$f\Big(\sum_{i=0}^{n}\lambda_i(x,y)M_i(x,y)\Big) \le \sum_{i=0}^{n}\lambda_i(x,y)f(M_i(x,y))+\delta(x,y) \qquad (x,y\in I), \tag{3}$$

where we make the following assumptions:

(A0) $n \ge 1$ is a fixed integer;

(A1) $\lambda_0,\lambda_n : I^2 \to \mathbb{R}$ are positive and $\lambda_1,\dots,\lambda_{n-1} : I^2 \to \mathbb{R}$ are nonnegative functions such that $\sum_{i=0}^{n}\lambda_i(x,y)=1$ for all $x,y\in I$, and λ_0,λ_n are continuous in both variables, that is, the maps

$$x \mapsto (\lambda_0(x,y_0),\lambda_n(x,y_0)) \qquad \text{and} \qquad y \mapsto (\lambda_0(x_0,y),\lambda_n(x_0,y))$$

are continuous on I for each fixed y_0 and x_0 in I;

(A2) $M_0,M_1,\dots,M_n : I^2 \to I$ are two-variable means and $M_0(x,y)=x$, $M_n(x,y)=y$ for all $x,y\in I$.

(A3) The function $M : I^2 \to \mathbb{R}$ defined by $M(x,y)=\sum_{i=0}^{n}\lambda_i(x,y)M_i(x,y)$ is continuous in both variables, that is, the maps

$$x \mapsto M(x,y_0) \qquad \text{and} \qquad y \mapsto M(x_0,y)$$

are continuous on I for each fixed y_0 and x_0 in I;

(A4) $\delta : I^2 \to \mathbb{R}$ is a nonnegative function.

One can easily see that, due to assumptions (A1) and (A2), the function M defined in (A3) is also a two-variable mean on I.

We can now formulate the main result of this paper. We recall that a function $f : I \to \mathbb{R}$ is called locally bounded from above on I if, for each point of I, there exists a neighbourhood U of this point such that f is bounded from above on U.

Theorem 1. *Suppose that (A0)–(A4) hold and assume that $f : I \to \mathbb{R}$ is a locally bounded from above and satisfies the functional inequality* (3)*. Then f satisfies the following inequality*

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\delta^*(x,y) \qquad (x,y\in I,\, t\in[0,1]), \tag{4}$$

where $\delta^* : I^2 \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$\delta^*(x,y) := \sup_{\min\{x,y\}\le u\le\max\{x,y\}} \max\Big(\frac{\delta(x,u)}{\lambda_0(x,u)},\frac{\delta(u,y)}{\lambda_n(u,y)}\Big) \qquad (x,y\in I,\, x\ne y)$$

and $\delta^*(x,x):=0$ for $x\in I$.

Proof. There is nothing to prove if $x = y$ in (4). Thus, we may assume that $x < y$. The function f being locally bounded from above, we have that it is also bounded from above on the compact interval $[x, y]$. Therefore

$$K(x,y) := \sup_{t\in[0,1]} \big(f(tx + (1-t)y) - tf(x) - (1-t)f(y)\big) \tag{5}$$

is finite. Define

$$g(u) := f(u) - \frac{y-u}{y-x}f(x) - \frac{u-x}{y-x}f(y) \qquad (u \in I).$$

Then g differs from f in an affine function. Hence, it follows from (3) and assumption (A1) that g also satisfies (3), that is,

$$g\Big(\sum_{i=0}^{n} \lambda_i(s,t)M_i(s,t)\Big) \le \sum_{i=0}^{n} \lambda_i(s,t)g(M_i(s,t)) + \delta(s,t) \qquad (s,t \in I). \tag{6}$$

Clearly, $g(x) = g(y) = 0$ and

$$\sup_{z\in[x,y]} g(z) = \sup_{t\in[0,1]} g(tx + (1-t)y) = K(x,y).$$

On the other hand, we always have that

$$\sup_{z\in[x,y]} g(z) = \max\Big(\sup_{z\in[x,M(x,y)]} g(z), \sup_{z\in[M(x,y),y]} g(z)\Big).$$

According to how the supremum is attained, we can distinguish two cases. If

$$K(x,y) = \sup_{z\in[x,y]} g(z) = \sup_{z\in[x,M(x,y)]} g(z),$$

then take $z \in [x, M(x,y)]$ arbitrarily. Due to assumption (A1)–(A3), the codomain of the function $[x,y] \ni u \mapsto M(x,u)$ covers the interval $[x, M(x,y)]$. Hence, there exists $u \in [x,y]$ such that $M(x,u) = z$. Thus, by (6) and $g(M_0(x,y)) = g(x) = 0$, we have

$$\begin{aligned} g(z) &= g(M(x,u)) = g\Big(\sum_{i=0}^{n} \lambda_i(x,u)M_i(x,u)\Big) \\ &\le \sum_{i=0}^{n} \lambda_i(x,u)g(M_i(x,u)) + \delta(x,u) = \sum_{i=1}^{n} \lambda_i(x,u)g(M_i(x,u)) + \delta(x,u) \\ &\le K(x,y)\sum_{i=1}^{n} \lambda_i(x,u) + \delta(x,u) = K(x,y)(1-\lambda_0(x,u)) + \delta(x,u). \end{aligned}$$

Let (z_k) be a sequence in $[x, M(x,y)]$ such that $(g(z_k))$ tends to $K(x,y)$. Then there exists a sequence $u_k \in [x,y]$ such that

$$g(z_k) \le K(x,y)(1-\lambda_0(x,u_k)) + \delta(x,u_k).$$

Taking subsequences if necessary, we may assume that (u_k) is also convergent. It follows from the above inequality that,

$$\frac{g(z_k) - K(x,y)}{\lambda_0(x,u_k)} + K(x,y) \le \frac{\delta(x,u_k)}{\lambda_0(x,u_k)} \le \sup_{u\in[x,y]} \frac{\delta(x,u)}{\lambda_0(x,u)}.$$

Taking the limit $k \to \infty$, we find that

$$K(x,y) \le \sup_{u\in[x,y]} \frac{\delta(x,u)}{\lambda_0(x,u)}. \tag{7}$$

In the case

$$K(x,y) = \sup_{z\in[x,y]} g(z) = \sup_{z\in[M(x,y),y]} g(z)$$

a similar argument yields that

$$K(x,y) \le \sup_{u\in[x,y]} \frac{\delta(u,y)}{\lambda_n(u,y)}. \tag{8}$$

Thus, independently of the underlying case, by (7) and (8) we always have

$$K(x,y) \le \max\Big(\sup_{u\in[x,y]} \frac{\delta(x,u)}{\lambda_0(x,u)}, \sup_{u\in[x,y]} \frac{\delta(u,y)}{\lambda_n(u,y)}\Big) = \delta^*(x,y).$$

Due to the definition (5) of $K(x,y)$, the above inequality leads to (4), which completes the proof. □

Remark 1. If the function f is assumed to be upper semicontinuous, then z in the proof above can be chosen such that $g(z) = K(x,y)$, therefore, the construction of the sequences (z_k) and (u_k) can be avoided, and thus the continuity assumption on λ_0 and λ_n in (A1) can be omitted.

Now we formulate two immediate consequences of the above result.

Corollary 1. *Suppose that (A0)–(A3) hold and assume that $f : I \to \mathbb{R}$ is locally bounded from above and satisfies the functional inequality*

$$f\Big(\sum_{i=0}^{n} \lambda_i(x,y)M_i(x,y)\Big) \le \sum_{i=0}^{n} \lambda_i(x,y) f(M_i(x,y)) \qquad (x,y\in I). \tag{9}$$

Then f is convex on I.

Conversely, if f is a convex function then it satisfies (9)*.*

Proof. Observe that if f is a solution of (9), then (3) is satisfied with $\delta \equiv 0$. Hence, (4) is also satisfied with $\delta^* \equiv 0$, which means that f is convex.

The reversed statement obviously follows from known properties of convex functions (cf. [HLP52], [Kuc85], [RV73]). □

Remark 2. The above result offers a generalization of the Bernstein–Doetsch theorem ([BD15]) even in the case $n = 1$: *If the function $f : I \to \mathbb{R}$ is locally bounded from above and satisfies*

$$f(\lambda(x,y)x + (1-\lambda(x,y))y) \le \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) \qquad (x,y\in I),$$

where $\lambda : I^2 \to]0,1[$ is a continuous function in both variables, then f is convex.

Remark 3. Applying Corollary 1, we can also characterize the regular solutions of (2): *If $M : I^2 \to I$ is a two-variable mean on I which is continuous in both variables and f is locally bounded from above and satisfies* (2)*, then f is convex.*

In order to formulate our second corollary, introduce the following notion: A function $f : I \to \mathbb{R}$ is called *δ-convex on I*, where δ is a nonnegative number, if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \delta \qquad (x, y \in I,\ t \in [0,1]).$$

There is an extensive literature dealing with the connection of δ-convexity and convexity, that is with the stability of convexity (see e.g. [HU52], [Gre52], [Lac99]).

Corollary 2. *Suppose that (A0)–(A3) hold and that λ_0, λ_n are constant functions. Let δ be a nonnegative number and assume that $f : I \to \mathbb{R}$ is locally bounded from above and satisfies the functional inequality*

$$f\Big(\sum_{i=0}^{n} \lambda_i(x,y) M_i(x,y)\Big) \le \sum_{i=0}^{n} \lambda_i(x,y) f(M_i(x,y)) + \delta \qquad (x, y \in I). \tag{10}$$

Then f is $c\delta$-convex on I with $c = \max\{1/\lambda_0, 1/\lambda_n\}$.

Conversely, if f is a δ-convex function then it satisfies (10).

Proof. Observe that if f is a solution of (9), then (3) is satisfied with the constant function δ. Hence, (4) is also satisfied with $\delta^* \equiv \max\{1/\lambda_0, 1/\lambda_n\}\delta = c\delta$, which means that f is $c\delta$-convex.

Now assume that f is δ-convex. We are going to show that

$$f\Big(\sum_{i=0}^{n} t_i x_i\Big) \le \sum_{i=0}^{n} t_i f(x_i) + \delta \tag{11}$$

for $x_0, x_1, \dots, x_n \in I$ and $t_0, t_1 \dots, t_n \ge 0$ with $\sum_{i=1}^{n} t_i = 1$. This inequality obviously yields (10) with the substitutions $t_i := \lambda_i(x,y)$, $x_i := M_i(x,y)$ $(i = 0, 1, \dots, n)$.

In order to prove (11), we may assume that $t_0, \dots, t_n > 0$ and $x_0 < \cdots < x_n$. Denote by u the value $\sum_{i=0}^{n} t_i x_i$ and choose k such that $x_k \le u < x_{k+1}$ holds. Applying the δ-convexity of f with obvious substitutions, we have that

$$f(u) \le \frac{x_j - u}{x_j - x_i} f(x_i) + \frac{u - x_i}{x_j - x_i} f(x_j) + \delta$$

for all (i, j) such that $0 \le i \le k < j \le n$. Multiplying the above inequality by $t_i t_j (x_j - x_i)$ and adding up the inequalities so obtained, we get

$$\begin{aligned}\sum_{i=0}^{k} \sum_{j=k+1}^{n} t_i t_j (x_j - x_i) f(u) \ \le\ & \sum_{i=0}^{k} \sum_{j=k+1}^{n} t_i t_j (x_j - u) f(x_i) \\ & + \sum_{i=0}^{k} \sum_{j=k+1}^{n} t_i t_j (u - x_i) f(x_j) + \sum_{i=0}^{k} \sum_{j=k+1}^{n} t_i t_j (x_j - x_i)\delta.\end{aligned}$$

Now, using the identities

$$\sum_{i=0}^{k} \sum_{j=k+1}^{n} t_i t_j (x_j - x_i) = \sum_{j=k+1}^{n} t_j (x_j - u) = \sum_{i=0}^{k} t_i (u - x_i) =: C > 0,$$

we obtain

$$Cf(u) \le C\sum_{i=0}^{k} f(x_i) + C\sum_{j=k+1}^{n} t_j f(x_j) + C\delta,$$

whence (11) follows. □

Remark 4. If $n = 1$, $\lambda_0 = \lambda_1 = 1/2$, then Corollary 2 reduces to the following statement: *If $f : I \to \mathbb{R}$ is locally bounded from above and, for some nonnegative δ, it is δ-Jensen-convex, i.e.*

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \delta \qquad (x, y \in I),$$

then f is 2δ-convex. This result was obtained first by Ng and Nikodem [NN93, Theorem 1]. Another proof was found by Laczkovich [Lac99, Lemma 4]. We note that the proof of Theorem 1 generalizes the approach of Laczkovich [Lac99].

Remark 5. We can generalize the result concerning (2) mentioned in Remark 3: *If $f : I \to \mathbb{R}$ is locally bounded from above and satisfies*

$$f\Big(\frac{x+y+M(x,y)}{3}\Big) \le \frac{f(x)+f(y)+f(M(x,y))}{3} + \delta \qquad (x, y \in I),$$

where $M : I^2 \to I$ is a two-variable mean which is continuous in both variables, then f is 3δ-convex.

3. Regular Solutions of a Functional Inequality

In this section we investigate the functional inequality

$$g(N(x,y)) \le \sum_{i=0}^{n} \mu_i(x,y) g(N_i(x,y)) \qquad (x, y \in I), \tag{12}$$

where we make the following assumptions:

- (B0) $n \ge 1$ is a fixed integer;
- (B1) $\mu_0, \mu_n : I^2 \to \mathbb{R}$ are positive and $\mu_1, \dots, \mu_{n-1} : I^2 \to \mathbb{R}$ are nonnegative functions such that $\sum_{i=0}^{n} \mu_i(x,y) = 1$ for all $x, y \in I$ and μ_0, μ_n are continuous in both variables;
- (B2) $N_0, N_1, \dots, N_n : I^2 \to I$ are two-variable means and $N_0(x,y) = x$, $N_n(x,y) = y$ for all $x, y \in I$.
- (B3) The function $N : I^2 \to \mathbb{R}$ is a two-variable mean which is continuous in both variables.

In order to describe the solutions of (12), we also consider the related functional equation

$$\varphi(N(x,y)) = \sum_{i=0}^{n} \mu_i(x,y)\varphi(N_i(x,y)) \qquad (x, y \in I). \tag{13}$$

Theorem 2. *Suppose that (B0)–(B3) hold and assume that there exists a continuous strictly monotonic solution $\varphi : I \to \mathbb{R}$ of (13). Let $g : I \to \mathbb{R}$ be locally bounded from above on I. Then g is a solution of (12) if and only if $g \circ \varphi^{-1}$ is convex on $\varphi(I)$.*

Proof. First we prove the necessity of the condition. Denote by J the open interval $\varphi(I)$ and by f the function $g \circ \varphi^{-1}$. Define, for $(s,t) \in J^2$,

$$M(s,t) := \varphi(N(\varphi^{-1}(s), \varphi^{-1}(t))),$$

$$M_i(s,t) := \varphi(N_i(\varphi^{-1}(s), \varphi^{-1}(t))) \qquad (i = 0, 1, \ldots, n)$$

and

$$\lambda_i(s,t) := \mu_i(\varphi^{-1}(s), \varphi^{-1}(t)) \qquad (i = 0, 1, \ldots, n).$$

Then, replacing x and y, by $\varphi^{-1}(s)$ and $\varphi^{-1}(t)$ in (12) and (13), respectively, we get that

$$f(M(s,t)) \leq \sum_{i=0}^{n} \lambda_i(s,t) f(M_i(s,t)) \qquad (s, t \in J) \tag{14}$$

and

$$M(s,t) = \sum_{i=0}^{n} \lambda_i(s,t) M_i(s,t) \qquad (s, t \in J). \tag{15}$$

Hence, f is locally bounded from above and satisfies the functional inequality (9) on J. It is also obvious that all the assumptions (A0)–(A3) are also satisfied. Thus, by Corollary 1, f is a convex function on J.

Conversely, if f is convex on J, then (9) holds on J. On the other hand, φ being a solution of (13), we have (15). This identity together with (9) results that (14) is also valid. The substitutions $s = \varphi(x)$ and $t = \varphi(y)$ now yield that g satisfies (12). □

Remark 6. In order to illustrate the meaning of the above theorem, we consider the following functional inequality:

$$g(\sqrt{xy}) \leq \frac{1}{4}\Big(g(x) + g(y) + g\Big(\frac{x+y}{2}\Big) + g\Big(\frac{2xy}{x+y}\Big)\Big) \qquad (x, y > 0). \tag{16}$$

First, a continuous strictly monotonic solution of the corresponding functional equation

$$\varphi(\sqrt{xy}) = \frac{1}{4}\Big(\varphi(x) + \varphi(y) + \varphi\Big(\frac{x+y}{2}\Big) + \varphi\Big(\frac{2xy}{x+y}\Big)\Big) \qquad (x, y > 0) \tag{17}$$

has to be determined. It is easy to see, that $\varphi(x) = \ln(x)$ is a solution. Thus, by Theorem 2, if g is locally bounded from above, then it is a solution of (16) if and only if $g \circ \exp$ is convex on $\mathbb{R}$.

Remark 7. Results analogous to Theorem 2 concerning the functional inequality

$$g(M(x,y)) + g(N(x,y)) \leq g(x) + g(y) \qquad (x, y \in I)$$

were obtained in [Pal88]. More precisely, if the corresponding functional equation has a continuous strictly monotonic solution $\varphi : I \to \mathbb{R}$, and M, N are continuous two-variable means on I, then a continuous function $g : I \to \mathbb{R}$ is a solution of the above functional inequality if and only if $g \circ \varphi^{-1}$ is convex.

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