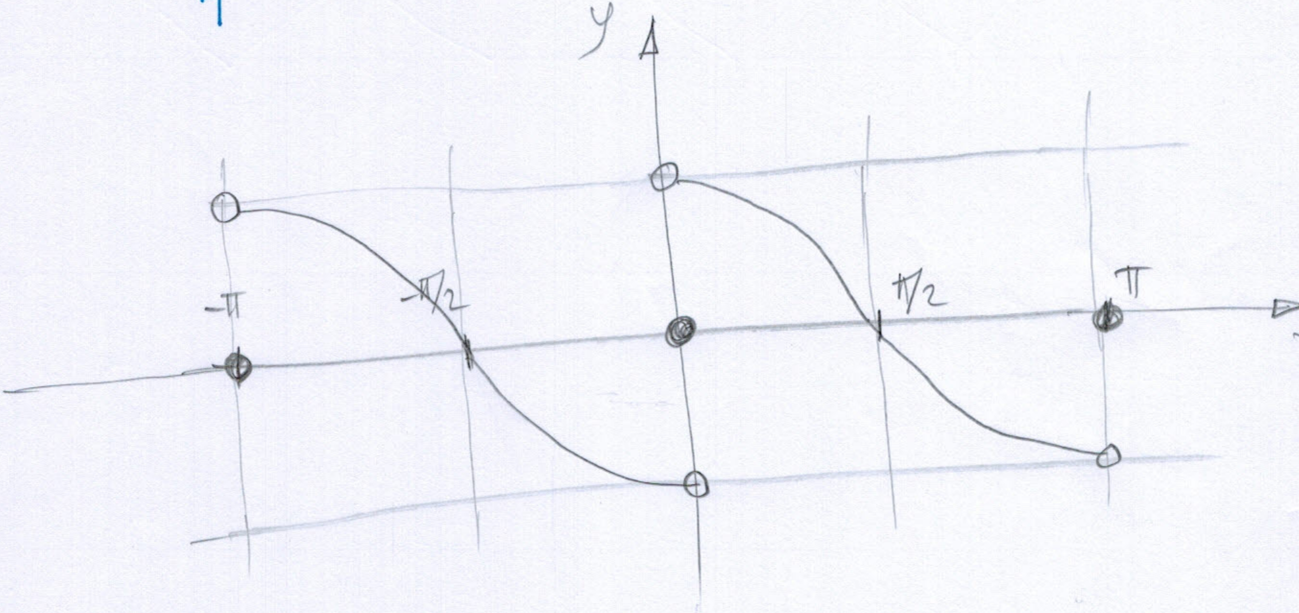
Exercício de Fourier

Considere a função f em $[-\pi, \pi]$ definida por

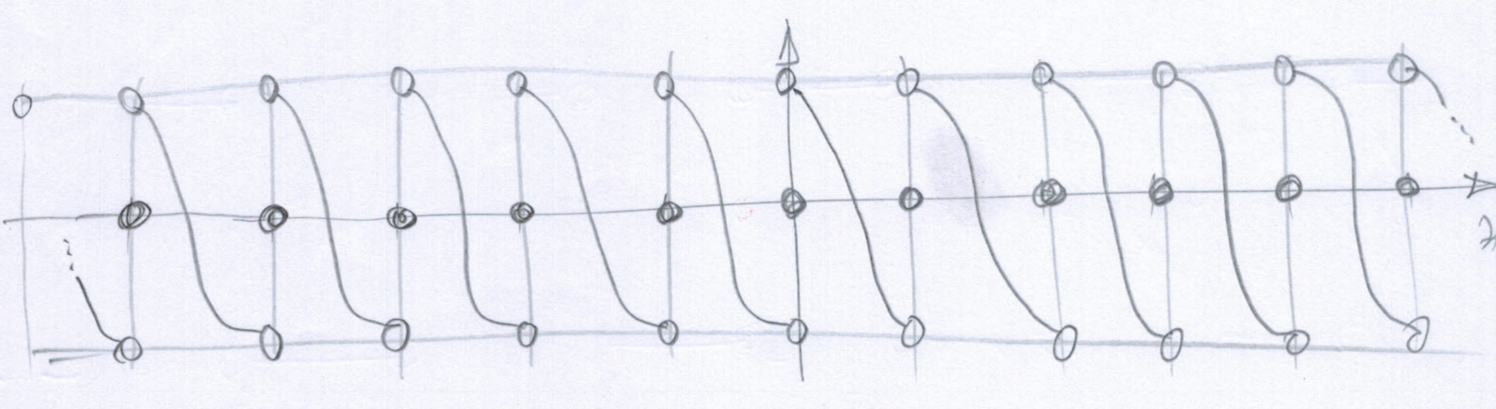
$$f(x) = \begin{cases} \cos x & \text{se } x \in]0, \pi[\\ -\cos x & \text{se } x \in]-\pi, 0[\\ 0 & \text{se } x \in \{0, \pi, -\pi\} \end{cases}$$

$x = k\pi,\ k \in \mathbb{Z}$

a) Faça um esboço do gráfico de f



Extensão 2π-periódica a todo o $\mathbb{R}$



b) Mostre que a série de Fourier associada a f é uma série de senos, ou seja é da forma

$$\sum_{n=1}^{+\infty} b_n \sin(nx)$$

A função f é impar:

– Se $x \in]0, \pi[$

$$\left.\begin{array}{l} f(x) = \cos x \\ -f(-x) = -(-\cos(-x)) = \cos x \end{array}\right\rangle =$$

– Se $x \in]-\pi, 0[$

$$\left.\begin{array}{l} f(x) = -\cos x \\ -f(-x) = -\cos(-x) = -\cos x \end{array}\right\rangle =$$

– Se $x = \pi$

$$\left.\begin{array}{l} f(x) = \cos \pi = -1 \\ -f(-\pi) = -(-\cos(-\pi)) = -1 \end{array}\right\rangle =$$

$$f(x) = -f(-x) \quad , \forall x \in [-\pi, \pi]$$

c) Calcule a série de Fourier associada a f.

Para $n = 1, 2, 3, \ldots$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \overbrace{\underbrace{f(x)}_{\text{ímpar}} \underbrace{\sin(nx)}_{\text{ímpar}}}^{\text{par}} \, dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx$$

Se $n = 1$

$$P \cos x \sin x = P \frac{1}{2} \sin 2x$$

$$= -\frac{1}{4} \cos(2x)$$

$$\int_0^{\pi} \cos x \sin x \, dx =$$

$$= \left[-\frac{1}{4} \cos 2x\right]_0^{\pi}$$

$$= -\frac{1}{4} \cos(2\pi) + \frac{1}{4} \cos 0$$

$$= -\frac{1}{4} + \frac{1}{4} = 0$$

//

$$P\left[\underbrace{\cos x}_{u'} \underbrace{\sin nx}_{v}\right]$$

$$u' = \cos x \qquad v = \sin nx$$

$$u = \sin x \qquad v' = n \cos nx$$

$$Pu'v = uv - Puv'$$

$$\boxed{Pu'v = \sin x \sin nx - nP \underbrace{\sin x}_{u_1'} \underbrace{\cos nx}_{v_1}} \text{ — P1}$$

$$u_1' = \sin x \qquad v_1 = \cos nx$$

$$u_1 = -\cos x \qquad v_1' = -n \sin nx$$

$$P u_1' v_1 = u_1 v_1 - P u_1 v_1'$$

$$P \sin x \cos nx = -\cos x \cos nx - P(-\cos x)(-n \sin nx)$$

$$\boxed{P \sin x \cos nx = -\cos x \cos nx - n P \cos x \sin nx} \quad P2$$

Substituo P2 em P1

$$P u'v = \sin x \sin nx - n \left[-\cos x \cos nx - n P \underbrace{\cos x}_{u'} \underbrace{\sin nx}_{v} \right.$$

$$P u'v = \sin x \sin nx + n \cos x \cos nx + n^2 P u'v$$

$$(1-n^2) P u'v = \sin x \sin nx + n \cos x \cos nx \quad n>1$$

$$P u'v = \frac{1}{1-n^2} \cdot \left[\sin x \sin nx + n \cos x \cos nx \right]$$

Então, para $n = 2, 3, 4, 5, \ldots$

$$b_n = \frac{2}{\pi} \left[\frac{1}{1-n^2} \left(\sin x \sin nx + n \overbrace{\cos x \cos nx}^{(-1)^n} \right) \right]_0^{\pi}$$

$$b_n = \frac{2}{\pi} \left[\frac{1}{1-n^2} \left(n \overbrace{\cos \pi}^{-1} \overbrace{\cos n\pi}^{(-1)^n} \right) - \frac{1}{1-n^2} \left(n \overbrace{\cos 0}^{1} \overbrace{\cos 0}^{1} \right) \right]$$

$$b_n = \frac{2}{\pi} \left[\frac{1}{1-n^2} \left(n (-1)^{n+1} \right) - \frac{n}{1-n^2} \right]$$

$$b_n = \frac{2n}{\pi} \frac{(-1)^{n+1} - 1}{1-n^2}$$

$$(-1)^{n+1} - 1 = \begin{cases} 0 & \text{se } n \text{ ímpar} \\ -2 & \text{se } n \text{ par} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{se } n \text{ ímpar} \\ \dfrac{-4n}{\pi(1-n^2)} = \dfrac{4n}{\pi(n^2-1)} & \text{se } n \text{ par} \end{cases}$$

$$f(x) \sim \sum_{n=1}^{+\infty} \frac{4(2n)}{\pi((2n)^2-1)} \sin(2nx)$$

$$f(x) \sim \frac{8}{\pi} \sum_{n=1}^{+\infty} \frac{n}{4n^2-1} \sin(2nx)$$

$$f(x) \sim \frac{8}{\pi}\left[\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right]$$

d) Calcule os coeficientes b_1 e b_2 da série de Fourier

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{f(x)\,\text{sin}\,x}^{\text{par}}}_{\text{ímpar}\quad\text{ímpar}}\,dx$$

$$b_1 = \frac{2}{\pi}\int_{0}^{\pi} \cos x\,\text{sin}\,x\,dx$$

$$b_1 = \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}\,\text{sin}\,2x\,dx$$

$$b_1 = \frac{1}{\pi}\int_{0}^{\pi} \text{sin}\,2x\,dx$$

$$b_1 = \frac{1}{\pi}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$b_1 = \frac{1}{\pi}\left[-\frac{1}{2}\overbrace{\cos 2\pi}^{1} + \frac{1}{2}\overbrace{\cos 0}^{1}\right]$$

$$b_1 = \frac{1}{\pi}\left(-\frac{1}{2}+\frac{1}{2}\right) = 0$$

$$b_2 = \frac{2}{\pi} \int_0^{\pi} \cos x \sin 2x \, dx$$

$$b_2 = \frac{2}{\pi} \int_0^{\pi} \cos x \, (2 \sin x \cos x) \, dx$$

$$b_2 = \frac{2}{\pi} \int_0^{\pi} 2 \sin x \cos^2 x \, dx$$

$$b_2 = \frac{4}{\pi} \int_0^{\pi} \sin x \cos^2 x \, dx$$

$$b_2 = \frac{4}{\pi} \left[-\frac{\cos^3 x}{3} \right]_0^{\pi}$$

$$b_2 = \frac{4}{\pi} \left[-\frac{1}{3} \cos^3 \pi + \frac{1}{3} \cos^3 0 \right]$$

$$b_2 = \frac{4}{\pi} \left[-\frac{1}{3} (-1)^3 + \frac{1}{3} (1) \right]$$

$$b_2 = \frac{4}{3\pi} (1+1) = \frac{8}{3\pi} \;//$$

e) Sabendo que a série referida na alínea a) pode ser escrita na forma

$$\frac{8}{\pi} \sum_{n=1}^{+\infty} \frac{n}{4n^2-1} \sin(2nx)$$

justifique que converge para $f(x)$ no intervalo $[-\pi, \pi]$.

Com vista à aplicação do T Dirichlet constatamos que

① f é seccionalmente contínua em $[-\pi,\pi]$ pois é contínua em todos os pontos à excepção de $x=-\pi$, $x=0$ e $x=\pi$. Nesses pontos os limites laterais são finitos

$$\lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 0^-} f(x) = -1$$

$$\lim_{x\to \pi^-} f(x) = -1$$

$$\lim_{x\to (-\pi)^+} f(x) = 1$$

② f' é ~~seccionalmente~~ contínua em $[-\pi,\pi]$

$$f'(x) = \begin{cases} -\sin x & \text{se } x \in]0,\pi[\\ +\sin x & \text{se } x \in]-\pi,0[\end{cases}$$

É contínua em todos os pontos à excepção de $x=-\pi$ e $x=+\pi$. Nesses pontos os limites laterais são finitos

$$\lim_{x\to \pi^-} f'(x) = 0$$

$$\lim_{x\to (-\pi)^+} f'(x) = 0$$

Assim, pelo T. Dirichlet a série converge para

$$S(x)=\begin{cases} f(x) & \text{se } x\in]-\pi,0[\cup]0,\pi[\\ 0 & \text{se } x = k\pi \;, k=-1,0,1 \end{cases}$$

$$\frac{f(x^+)+f(x^-)}{2}$$

Esboçar o gráfico de $S(x)$?

d) Calcule, justificando, a soma das seguintes séries

$$\sum_{n=1}^{+\infty} \frac{n}{4n^2-1}\sin\left(\frac{n\pi}{2}\right) = \frac{\pi\sqrt{2}}{16}$$

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{+\infty}\frac{n}{4n^2-1}\sin(2nx)$$

<u>Se $x=0$</u>:

$$0 = \frac{8}{\pi}\sum_{n=1}^{+\infty}\frac{n}{4n^2-1}\sin(0) \Leftrightarrow 0=0$$

Se $x = \frac{\pi}{2}$:

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{8}{\pi} \sum_{n=1}^{+\infty} \frac{n}{4n^2-1} \sin\left(\overbrace{2n\frac{\pi}{2}}^{n\pi}\right) = 0$$

Se $x = \pi$:

$$f(\pi) = 0 = \frac{8}{\pi} \ \ldots \ 0 = 0$$

Se $x = \pi/4$:

$$f(\pi/4) = \frac{\sqrt{2}}{2} = \frac{8}{\pi} \sum_{n=1}^{+\infty} \frac{n}{4n^2-1} \sin\left(2n\frac{\pi}{4}\right)$$

$$\frac{\sqrt{2}}{2} = \frac{8}{\pi} \sum_{n=1}^{+\infty} \frac{n}{4n^2-1} \sin\left(\frac{n\pi}{2}\right) = (-1)^{n+1}$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & \text{se } n = 1, 5, 9, \ldots \\ -1 & \text{se } n = 3, 7, 11 \\ 0 & \text{se } n \text{ par} \end{cases}$$

$$\cancel{\sin\left(\frac{n\pi}{2}\right) = (-1)^{4n-1}}$$

$$\boxed{\sum_{n=1}^{+\infty} \frac{n}{4n^2-1} \sin\left(\frac{n\pi}{2}\right) = \frac{\pi\sqrt{2}}{16}}$$

e) Prove que a série de Fourier de f não converge uniformemente.

Como $\left| \frac{n}{4n^2-1} \sin(2nx) \right| \leq \frac{n}{4n^2-1}$

e como $\sum \frac{n}{4n^2-1}$ diverge

o C.W. é inconclusivo.

No entanto, $f(x)$ é descontínua

mas

$$f_n(x) = \frac{n}{4n^2-1} \sin(2nx) \quad , \begin{array}{l} n \in \mathbb{N} \\ x \in \mathbb{R} \end{array}$$

são funções contínuas.

Ora, havendo conv. uniforme

$\sum_{n=1}^{+\infty} f_n(x)$ teria de ser uma

funç. contínua. Não é!

Então a conv. não é uniforme. //