FUNDAMENTALS OF QUANTITATIVE MODELING

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Module 2: deterministic models and optimization

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Module 2 content

- Linear models
- Growth and decay in discrete time
- Growth and decay in continuous time
- Classical optimization

Deterministic models

- There are no random/uncertain components (inputs and/or outputs) in these models
- If the inputs to the model are the same then the outputs will always be the same
- The downside of deterministic models: it is hard to assess uncertainty in the outputs

Linear models

- Recall the formula definition of a straight line: $y = mx + b$
- Characterization of a line: the slope is constant
- The change in y for a one-unit change in x is the same, **regardless** of the value of x
- In practice you should ask if the constant slope assertion is realistic
- If it is not realistic, then a straight line model is probably not the way to go

A linear cost function

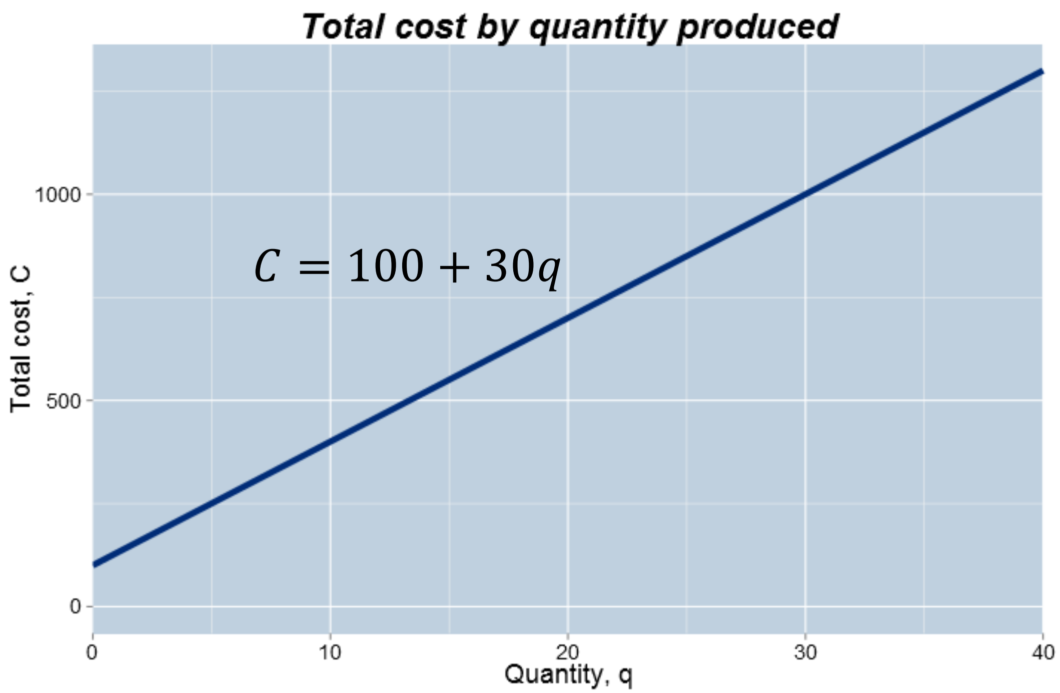
- Call the number of units produced q, and the total cost of producing q units C
- Define:

$$C = 100 + 30q$$

- Calculate some illustrative values:

q	C
0	100
10	400
20	700

Graphing the cost function



Interpretation

- The two coefficients in the line are the intercept and slope: b and m in general, 100 and 30 in this particular instance
- b: the total cost of producing 0 units
  - A better interpretation: that part of total cost that doesn't depend on the quantity produced: the ***fixed*** cost
- m: the slope of the line
  - The change in total cost for an incremental unit of production: the ***variable*** cost

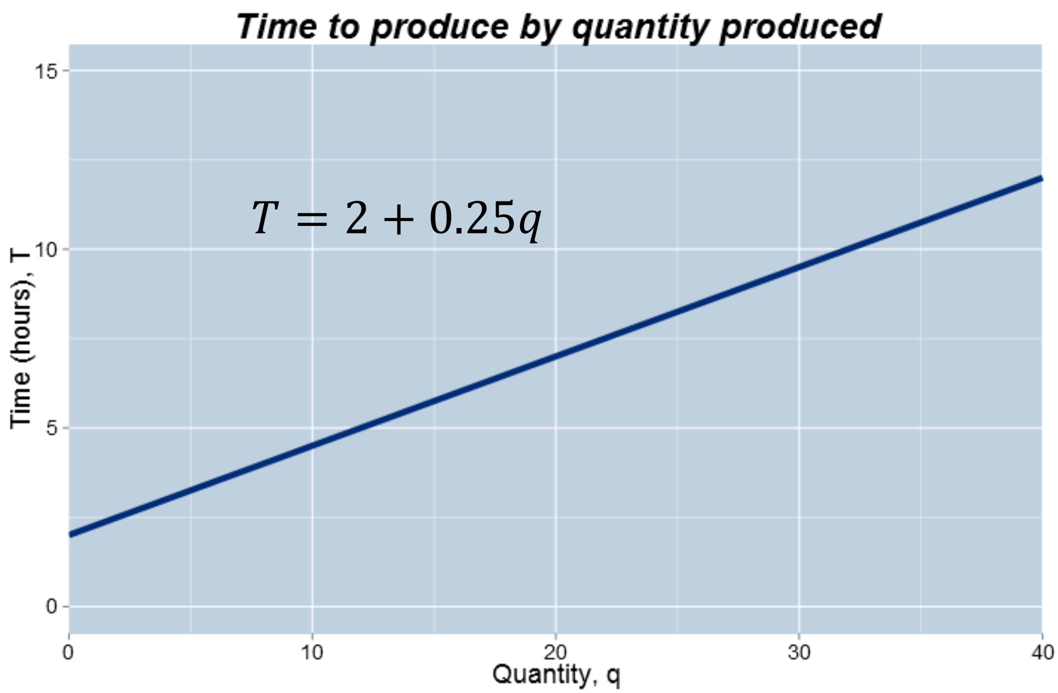
Example with a "time-to-produce" function

- It takes 2 hours to set up a production run, and each incremental unit produced always takes an additional 15 minutes (0.25 hours)
- Call T the time to produce q units, then

$$T = 2 + 0.25q$$

- Interpretation
  - b: the ***setup time*** (2 hours)
  - m: the ***work rate*** (15 minutes per additional item)

Graphing the time-to-produce function



Linear programming

- One of the key uses of linear models is in ***Linear Programming (LP),*** which is a technique to solve certain ***optimization*** problems
- These models incorporate ***constraints*** to make them more realistic
- Linear programming problems can be solved with add-ins for common spreadsheet programs

Growth in discrete time

- Growth is a fundamental business concept
  - The number of customers at time t
  - The revenue in quarter q
  - The value of an investment at some time t in the future
- Sometimes a linear model may be appropriate for a growth process
- But an alternative to a ***linear (additive) growth*** model is a ***proportionate*** one
- Proportionate growth: a constant percent increase (decrease) from one period to the next

Simple interest

- Start off with $100 (***principal***) and at the end of every year earn 10% ***simple interest*** on the initial $100
- Simple interest means that interest is only earned on the principal investment

Year	0	1	2	3	4	10
Value	100	110	120	130	140	200

- Every year the investment grows by the same amount ($10)

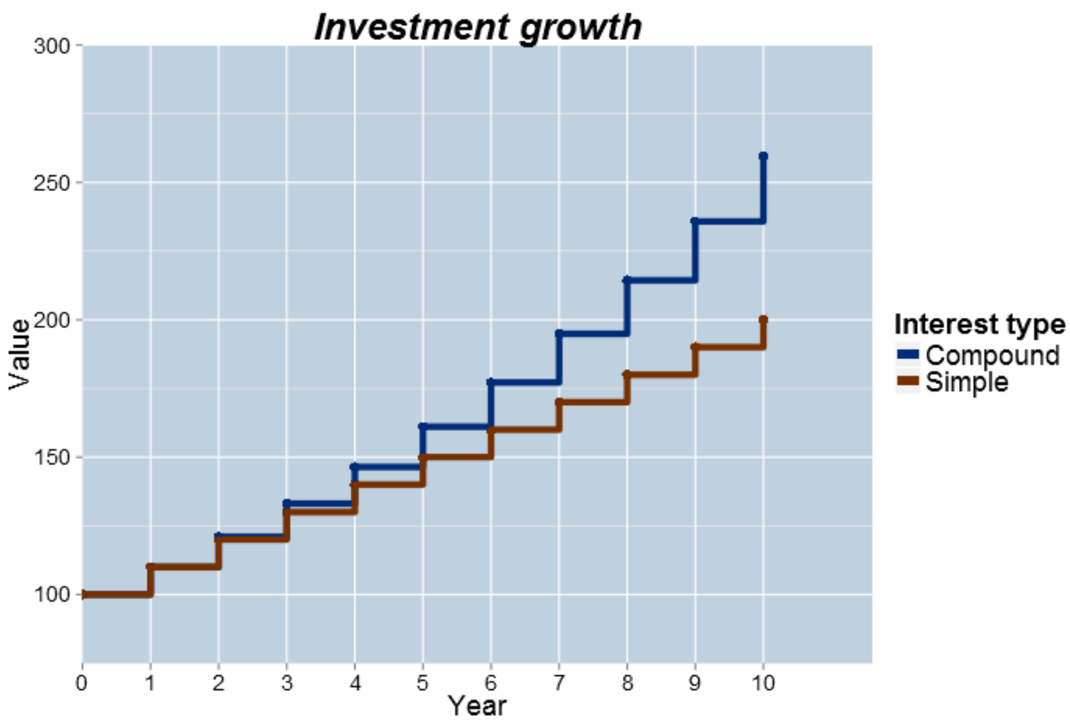
Compound interest

- Start off with $100 (***principal***) and at the end of every year earn 10% ***compound interest*** on the initial $100
- Compound interest means that the interest itself earns interest in subsequent years

Year	0	1	2	3	4	10
Value	100.00	110.00	121.00	133.10	146.41	259.37

- Notice that the growth is no longer the same absolute amount each year, but it is the same proportionate (relative) amount (10%)

The growth of the two investments



Constant proportionate growth

- Denote the initial amount as P_0
- Denote the constant proportionate growth factor by θ (for example 10% growth means multiply by 1.1 at the end of each time period)
- The growth progression is

Time	0	1	2	3	t
Amount, P_t	P_0	$P_0\,\theta$	$P_0\,\theta^2$	$P_0\,\theta^3$	$P_0\,\theta^t$

- θ > 1 means the process is growing
- θ < 1 means the process is declining/decaying
- This type of progression is called a ***geometric progression*** or ***geometric series***

Example

- An Indian Ocean nation caught 200,000 tonnes of fish this year
- The catch is projected to fall by a constant 5% factor each year for the next 10 years

- How many fish are predicted to be caught 5 years from now?
- Including this year, what is the total expected catch over the next 5 years?

The constant multiplier

- For the catch to fall by 5% each year, means that the multiplier is $\theta = 0.95$
- In general, if the process is changing by R% in each time period, then the multiplier is

$$\theta = 1 + \frac{R}{100}$$

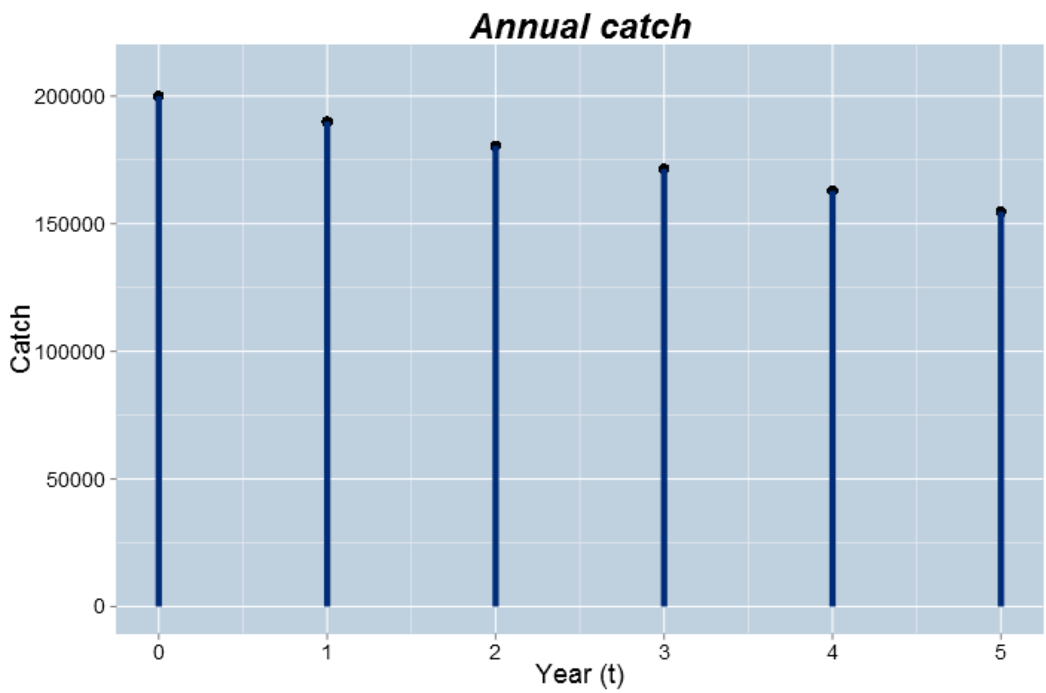
- Example:
  - if the increase is 5% then $\theta = 1.05$
  - If the decrease is 5% then $\theta = 0.95$

Working out the fish catch

- With $P_0 = 200{,}000$ and θ = 0.95, in 5 years the catch will be $200{,}000 \times 0.95^5 = 154{,}756$
- Over the next 5 years:
- We get the total catch summing over the years in question

Year	Catch
0	200,000
1	190,000
2	180,500
3	171,475
4	162,901.25
5	154,756.1875
Total	1,059,632

Graphing the annual fish catch



The sum of the geometric series

- If we denote the sum up to time t as S_t, then

$$S_t = P_0 \frac{1 - \theta^{t+1}}{1 - \theta}$$

- With the fisheries example:

$$S_5 = 200{,}000 \frac{1 - 0.95^{(5+1)}}{1 - 0.95} = 1{,}059{,}632$$

- The mathematical formulation of the model can sometimes provide more direct answers than a spreadsheet

Present and future value

- If there is no inflation and the prevailing interest rate is 4%, then which of the following options would you prefer?
  - $1000 today or $1500 in ten years?
- Either look to see how much $1000 will be worth in ten years, or
- Calculate how much you would have to invest today to get $1500 ten years from now
- The latter approach relies on the concept of ***present value*** – the expected current value of an income stream

The present value calculation

- We know that $P_t = P_0 \theta^{t}$ and making P_0 the subject of the formula means that $P_0 = P_t \theta^{-t}$
- Therefore $1500 in ten years time in a 4% interest rate environment is worth $1500\,(1+0.04)^{-10}$ in today's money
- This equals $1013.346, which is more than $1000, so you should prefer the second investment of $1500 received in ten years
- This straightforward proportionate increase model allows for a simple discounting formula

Uses of present value

- A primary use is in discounting investments to the present time
- An ***annuity*** is a schedule of fixed payments over a specified and finite time period
- The present value of an annuity is the ***sum*** of the present values of each separate payment
- Present value is also used in ***lifetime customer value*** calculations

Continuous compounding

- The compounding period for an investment can be yearly, monthly, weekly, daily etc.
- As the compounding period gets shorter and shorter, in the limit, the process is said to be ***continuously compounded***
- If a principal amount P_0 is continuously compounded at a nominal annual interest rate of R%, then at year t, $P_t = P_0 \, e^{rt}$ where $r = {}^{R}/_{100}$

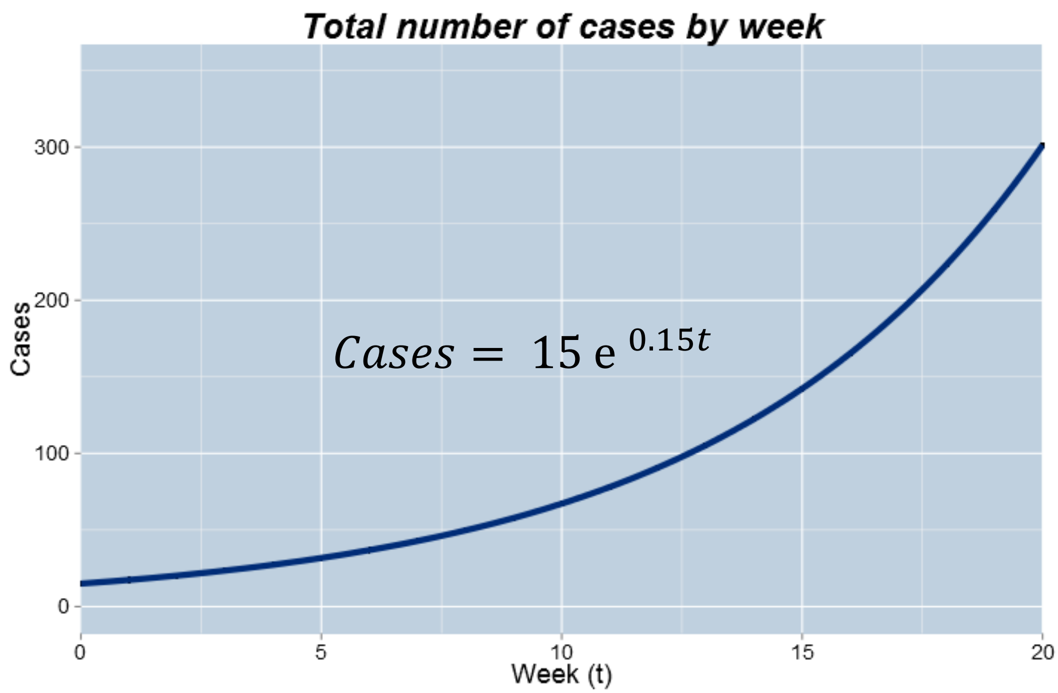
Continuous compounding

- In this model, t can take on any value in an interval, whereas in the discrete model t could only take on distinct values
- $1000 ***continuously compounded*** at a nominal annual interest rate of 4% is worth $1000\,\mathrm{e}^{0.04} = 1040.80$ after one year
- Note that this is a little more than if the 4% was earned at the very end of the time period, in which case you would have exactly $1040 at the end of the year

Modeling an epidemic

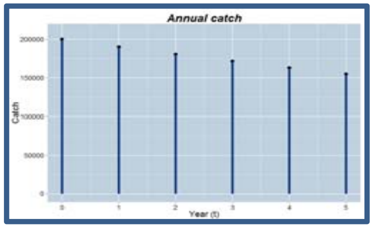
- The model $P_t = P_0 \, \mathrm{e}^{\,rt}$ doesn't just describe money growing
- This model is called exponential growth or decay depending on whether r is positive or negative respectively
- A continuous time model for the initial stages of an epidemic states that the number of cases at week t is $15 \, \mathrm{e}^{\,0.15t}$
- Halfway through week 7, how many cases do you expect?

Graphing the epidemic

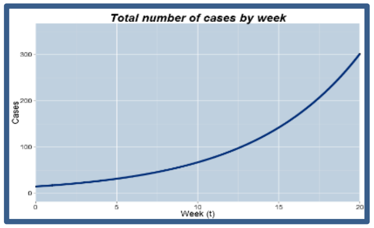


Calculating the expected number of cases

- Halfway through week 7, means t = 7.5
- $15\,e^{0.15 \times 7.5} = 46.2 \approx \mathbf{46\ cases}$
- Interpretation of the 0.15 coefficient:
  - There is an approximate 15% weekly growth rate in cases
- Continuous models allow calculation at any value of t, not just a set of discrete values



Discrete model



Continuous model

Using a model for optimization

- A common modeling objective is to perform a subsequent optimization
- The objective of the optimization is to find the value of an input that maximizes/minimizes an output
- Example: find the price at which profit is maximized

Demand model

- Consider the demand model:

$$\text{Quantity} = 60{,}000\ \text{Price}^{-2.5}$$

- If the price of production is constant at c = 2 for each unit, then at what price is profit maximized?
- Profit = Revenue – Cost
- Revenue = Price × Quantity = p × q
- $\text{Profit} = \text{p q} - \text{c q} = \text{q (p} - \text{c)}$

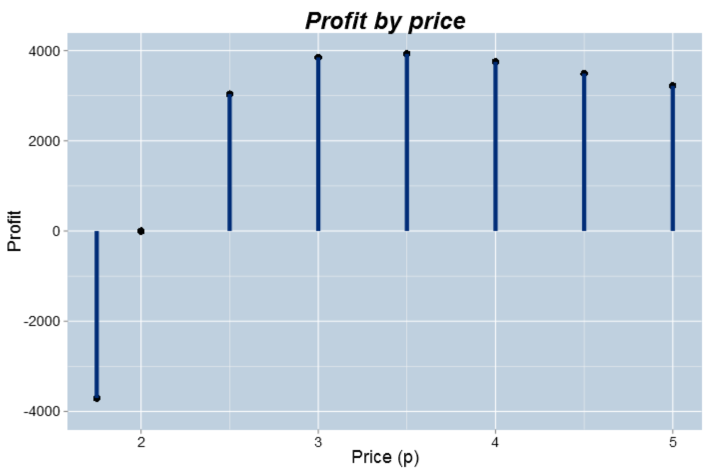
$$= 60{,}000\ \text{p}^{-2.5}\ (p\ - 2)$$

- Goal: Choose p to maximize this equation

Brute force approach

- Choose different values of p and plot profit

Price	Profit
1.75	-3702.509
2.0	0.000
2.5	3035.787
3.0	3849.002
3.5	3927.104
4.0	3750.000
4.5	3491.885
5.0	3219.938



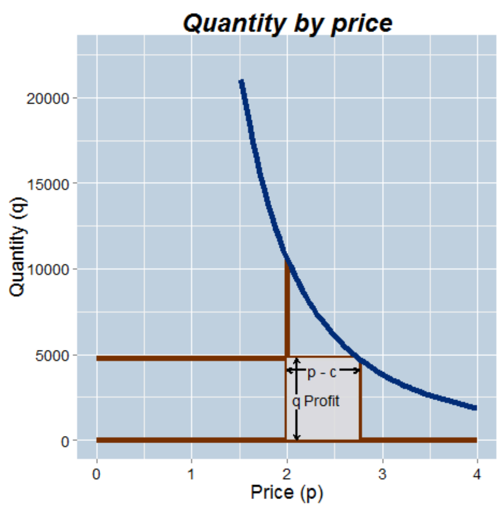
The optimal price is somewhere between 3 and 4

Calculus approach

- Profit is maximized when the ***derivative*** (rate of change) of profit with respect to price equals 0
- Through ***calculus*** one obtains the optimal value of price as $p_{opt}= \frac{c\ b}{1+b}$, where c is the production cost and b is the exponent in the power function
- In this example c = 2 and b = -2.5. This gives $p_{opt}= \frac{2\ \times -2.5}{(1-\ 2.5)}$ ≈ 3.33
- The value (-b) is known as ***the price elasticity of demand***

Visualizing the calculus solution

- The area of the gray shaded box is the profit and the objective is to find the value of price for which the area of the box is largest



Summary

- Linear models
- Growth and decay in discrete time: geometric series models
- Growth and decay in continuous time: exponential growth models
- Present and future value
- Use classical optimization (calculus) to gain additional value from the model

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