FUNDAMENTALS OF QUANTITATIVE MODELING

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Module 4: Regression models

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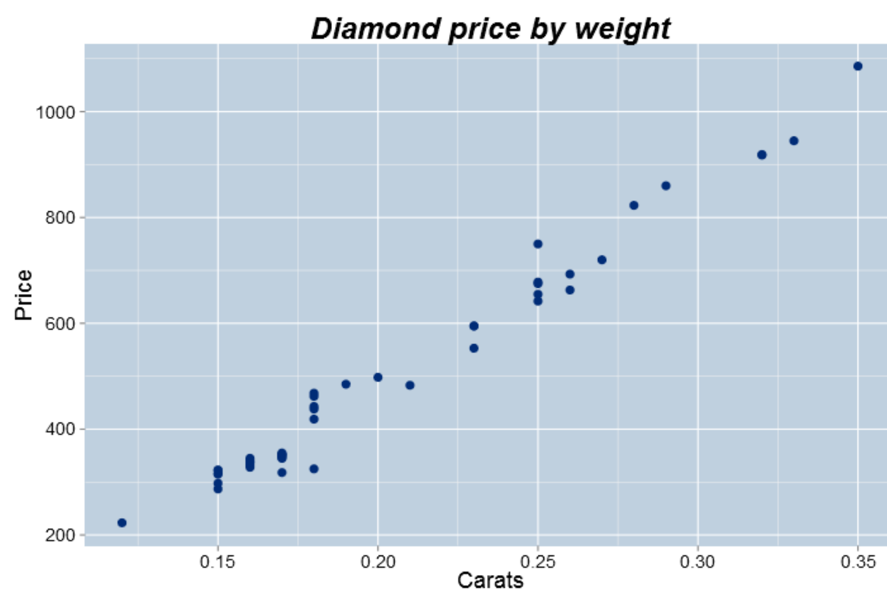
Module 4 content

- What is a regression model?
- Questions that a regression can answer
- Correlation and linear association
- Fitting a line to data
- Interpretation of the regression coefficients
- Prediction intervals in regression
- Multiple regression – many predictor variables
- Logistic regression -- what to do when the outcome variable is dichotomous

Regression models

- A ***simple regression*** model uses a single predictor variable X to estimate the **mean** of an outcome variable Y, as a function of X



Example

- Using the diamonds data: the predictor variable is the diamond's weight in carats and the outcome variable is the price of the diamond
- Heavier stones tend to cost more money (positive association) but a regression formalizes this idea into a model that reveals how the expected price varies with weight
- If the relationship is modeled with a straight line we call it a ***linear*** regression: $E(Y|X) = b_0 + b_1 X$

A linear regression model for the diamonds data

$$E(Price \mid Weight) = -260 + 3721 \times Weight$$

Correlation

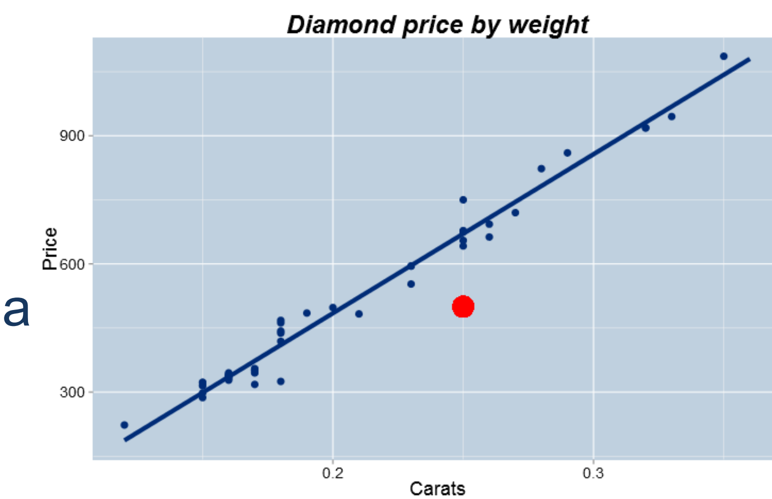
- ***Correlation*** is a measure of the strength of ***linear association*** between two variables
- It is denoted by the letter r. Fact: $-1 \leq r \leq +1$
- Negative values of the correlation indicate negative association and positive values indicate positive association
- A correlation of 0 means no linear association between the variables
- For the diamonds data, $r = 0.989$ which is an extremely strong positive correlation

Questions that can be answered with a regression

- In a business setting regression is most often used as a ***prediction*** tool. It is a core ***predictive analytics*** methodology
  - What price do you expect to pay for a diamond that weighs 0.3 carats?
  - Give me a ***prediction interval*** in which the price is likely to fall
- Interpreting coefficients from the model
  - How much on average do you expect to pay for diamonds that weigh 0.3 carats v. diamonds that weigh 0.2 carats? (ans. = 372)
- How much of the variability in price is accounted for by the weight of the diamond?

Questions that can be answered with a regression

- Prospecting for opportunities (new customers, investments etc.)
- If you found a diamond for sale that weighed 0.25 carats but cost only $500, would you be interested?
- The key idea is that this point is below the regression line
- Maybe it is mispriced and a great opportunity or maybe it is a flawed diamond, but it is certainly worth a look!



Fitting a model to data using least squares

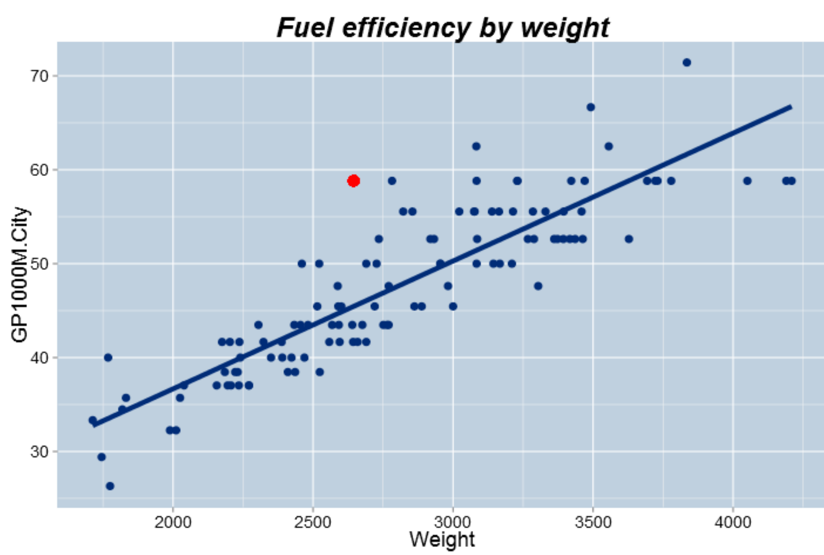
- Fitting a model requires an optimality criteria
- Most regression models are fit using ***least squares***
  - Find the line that minimizes the sum of the squares of the vertical distance from the points to the line

Residuals and fitted values

- Key insight:
  - The regression line decomposes the observed data into two components
    1. The fitted values (the predictions)
    2. The residuals (the vertical distance from point to line)
- Both are useful:
  - The fitted values are the forecasts
  - The residuals allow us to assess the quality of the fit. If a point has a large residual it is not well fit by the regression. If we can explain why, we have learnt something new

Example: fuel economy v. weight

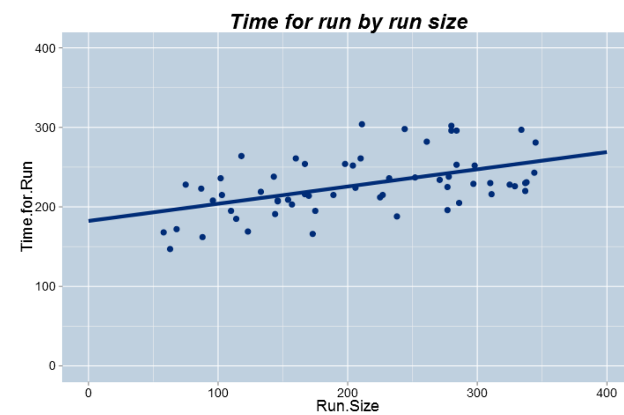
- Y = GP1000M (City), X = weight
- The point with the biggest residual is identified in red



Mazda RX-7 with rotary engine

Interpretation of regression coefficients

- $E(Y|X) = 182 + 0.22\,X$
- Equate units on each side
- Intercept is measured in units of Y
- Slope is measured in units of Y/X
- Intercept = Setup time in minutes
- Slope = Work rate in minutes per additional item



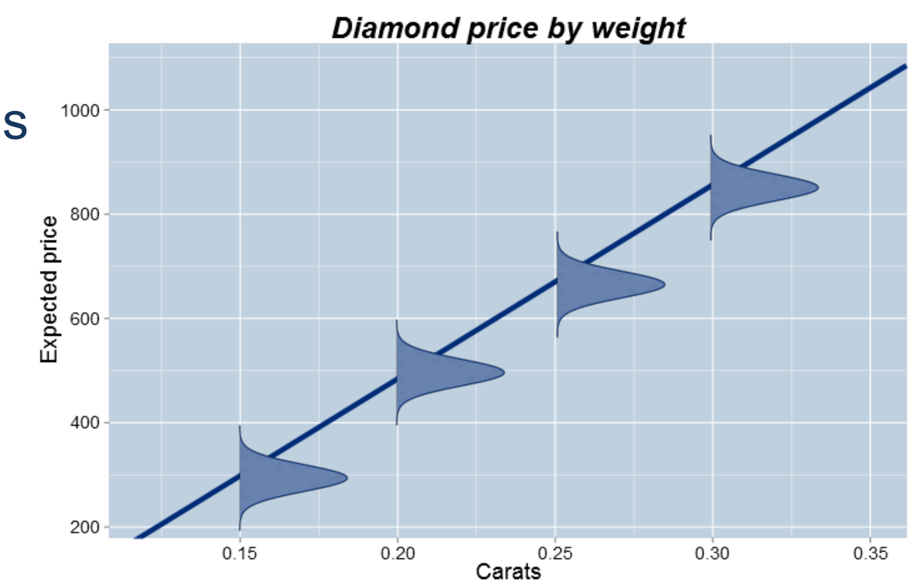
R^2 and Root Mean Squared Error (RMSE)

- R^2 measures the proportion of variability in Y explained by the regression model. It is the square of the correlation, r
- RMSE measures the standard deviation of the residuals (the spread of the points about the fitted regression line)

Example	R^2	RMSE
Diamonds	98%	31.84
Fuel economy	77%	4.23
Production time	26%	32.11

Using Root Mean Squared Error

- Assumption: at a fixed value of X, the distribution of points about the true regression line follows a Normal distribution, centered on the regression line
- These normal distributions all have the same standard deviation σ, which is estimated by RMSE

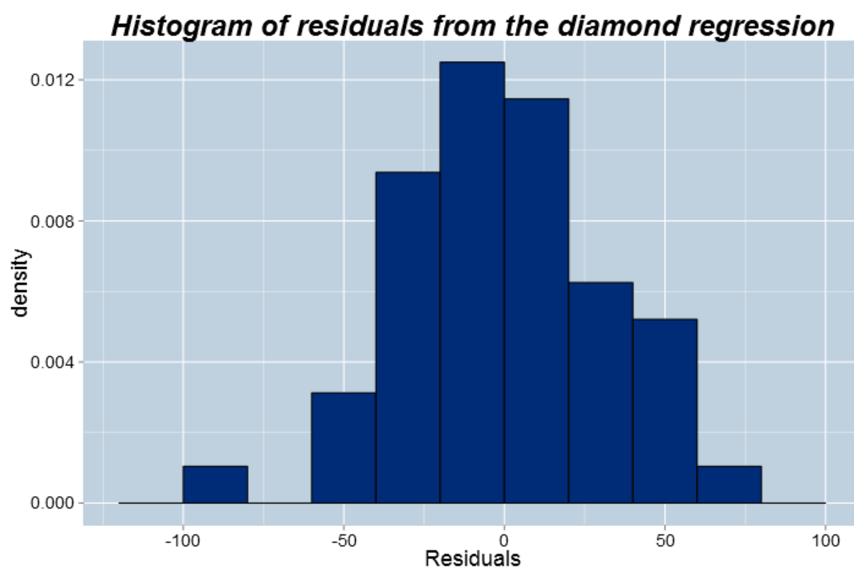


An approximate 95% prediction interval for a new observation

- Using the Normality assumption and the Empirical Rule, (within the range of the observed data) an approximate 95% prediction interval for a new observation is given by:
  - $Forecast \pm 2 \times RMSE$
- For the diamonds data the RMSE is approximately 32
- Therefore under the Normality assumption the width of the approximate 95% prediction interval is $\pm$ \$64
- An approximate 95% PI for the price of a diamond that weighs 0.25 carats is $-260 + 3721 \times 0.25 \pm 64 = (606, 734)$

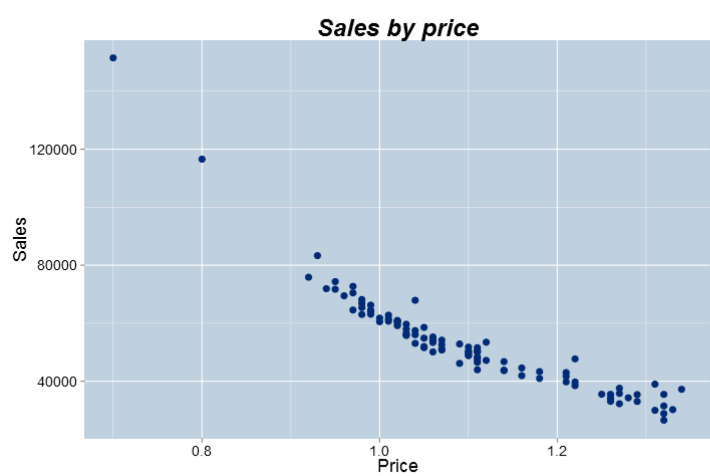
Residual diagnostics – checking the Normality assumption

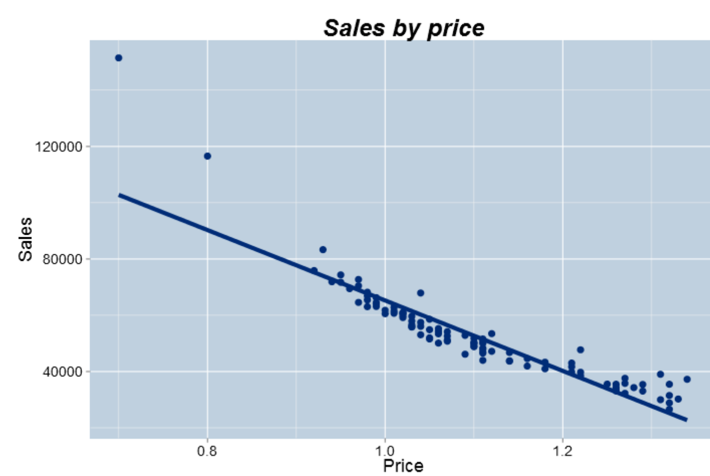
- The histogram of residuals from the diamonds regression is approximately Normally distributed, providing no strong evidence ***against*** the Normality assumption



Fitting curves to data

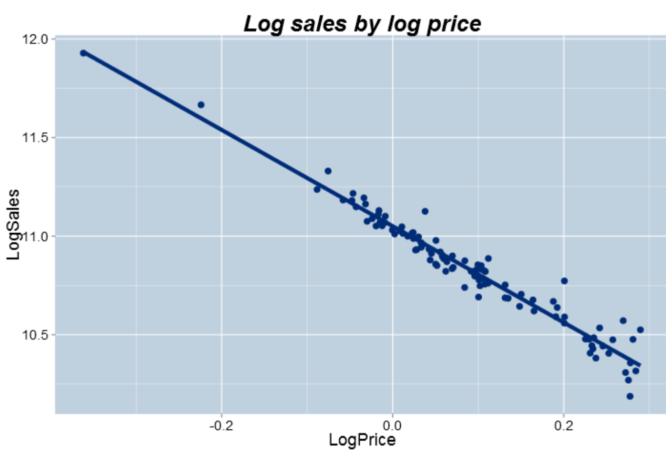
- Often relationships are non-linear
- Demand for a pet food (measured in cases sold) against average price. A line is a bad fit to the data





On observing curvature, transform

- This is where the basic math functions discussed in module 1 come in very useful
- Look at the pet food data after having taken the log transform



The regression equation for the log-log model

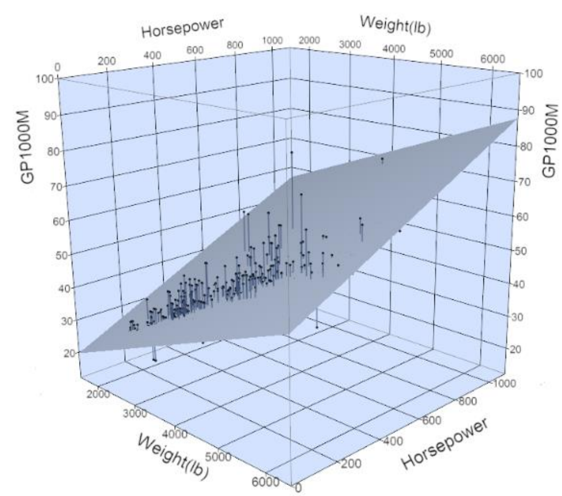
- The regression equation is now
  - $E(\log(Sales) \mid Price)) = b_0 + b_1 \log(Price)$
- In this instance we have:
  - $E(\log(Sales) \mid Price)) = 11.015 - 2.442 \log(Price)$
- This process shows how we could actually estimate the demand model that was the subject of the optimization in module 2

Multiple regression

- ***Multiple regression*** models allow for the inclusion of many predictor variables
  - In the fuel economy dataset we might add the horsepower of a car as an additional predictor
  - In the diamonds data set we might add in the color of the diamond to improve the model
- With two predictors, X_1 and X_2 the regression model becomes
  - $E(Y|X_1, X_2) = b_0 + b_1 X_1 + b_2 X_2$

Weight and horsepower as predictors of fuel economy

- Fitting a multiple regression model of fuel economy as a function of weight and horsepower gives:
  - $E(GP1000M|Weight, Horsepower) = 11.68 + 0.0089\ Weight + 0.0884\ Horsepower$
- The model is now a plane rather than a line
- For this model R^2 = 84% and RMSE= 3.45, an improvement over the simple regression model with only weight included



Logistic regression

- Linear regression is most appropriate when the outcome variable Y is continuous
- In many business problems, the outcome variable is **not** continuous but rather, ***discrete***
  - Purchase a product: Yes/No
  - Medical outcome: Live/Die
  - Website activity: Sign up/Don’t sign up
- These outcomes can be viewed as Bernoulli random variables

Logistic regression

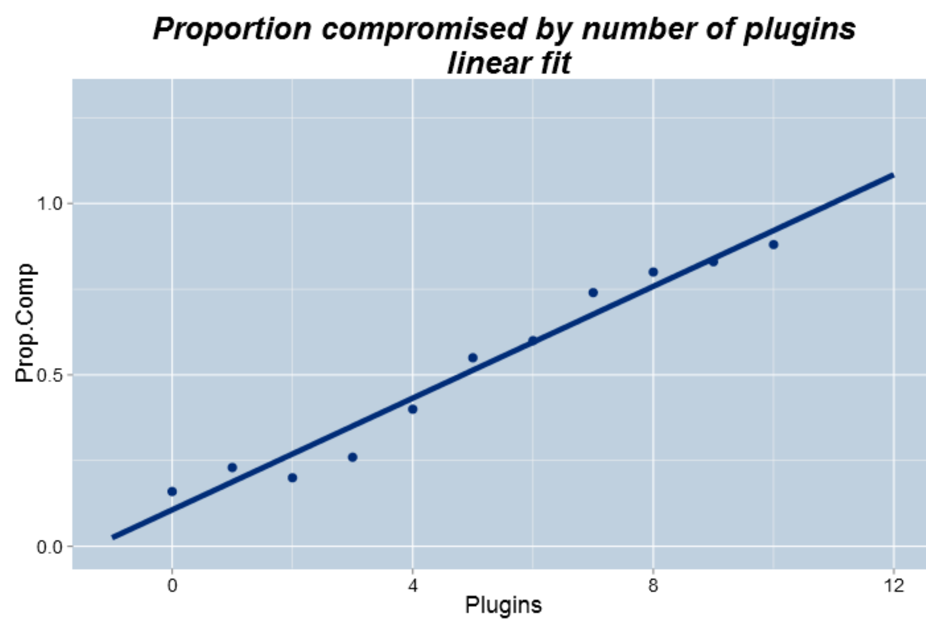
- Logistic regression is used to estimate the probability that a Bernoulli random variable is a *success*, as a function of predictor variables
- For example, how does the probability that a website is compromised vary as a function of the number of plugins that the site has installed?

Website compromise study

#Plugins	0	1	2	3	4	5	6	7	8	9	10
Compromised	16	23	20	26	40	55	60	74	80	83	88
Not compromised	84	77	80	74	60	45	40	26	20	17	12
Proportion compromised	0.16	0.23	0.20	0.26	0.40	0.55	0.60	0.74	0.80	0.83	0.88

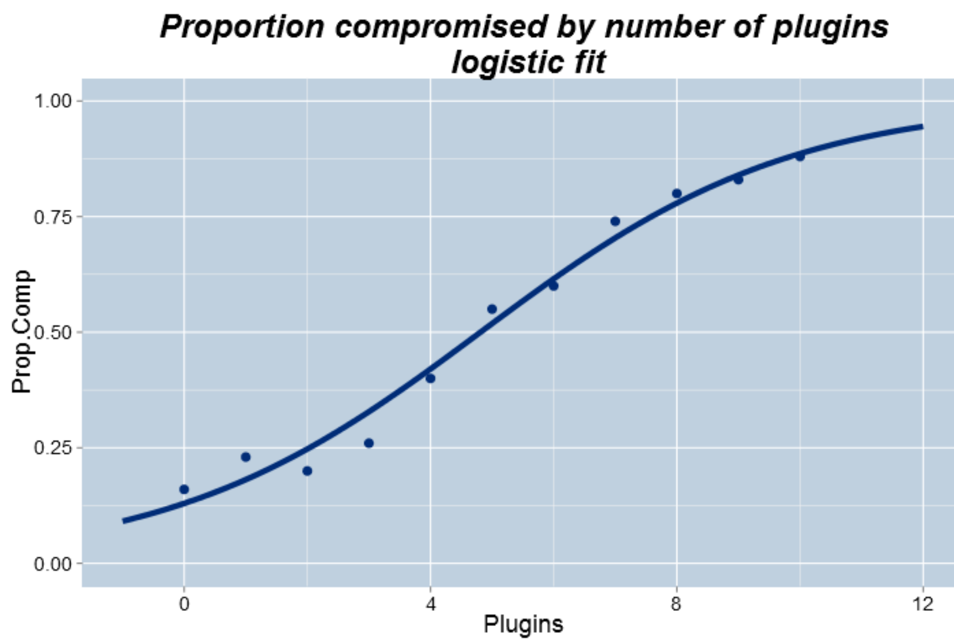
Linear fit

- The linear fit does not extrapolate well, predicting proportions greater than 1



Logistic regression fit

- The logistic regression fit is more appropriate, always predicting probabilities between 0 and 1



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