FUNDAMENTALS OF QUANTITATIVE MODELING

Richard Waterman

Module 3: Probabilistic models

Wharton
University of Pennsylvania
ONLINE

Module 3 content

- What are probabilistic models?
- Random variables and probability distributions -- the building blocks
- Examples of probabilistic models
- Summaries of probability distributions: means, variances and standard deviation
- Special random variables: Bernoulli, Binomial and Normal
- The Empirical Rule

Probabilistic models

- These are models that incorporate ***random variables*** and ***probability distributions***
- Random variables represent the potential outcomes of an uncertain event
- Probability distributions assign probabilities to the various potential outcomes
- We use probabilistic models in practice because realistic decision making often necessitates recognizing uncertainty (in the inputs and outputs of a process)

Key features of a probabilistic model

- By incorporating ***uncertainty*** explicitly in the model we can measure the uncertainty associated with the outputs, for example by giving a range to a forecast, which is a more realistic goal
- In a business setting incorporating ***uncertainty*** is synonymous with understanding and quantifying the ***risk*** in a business process, and ideally leads to better management decisions

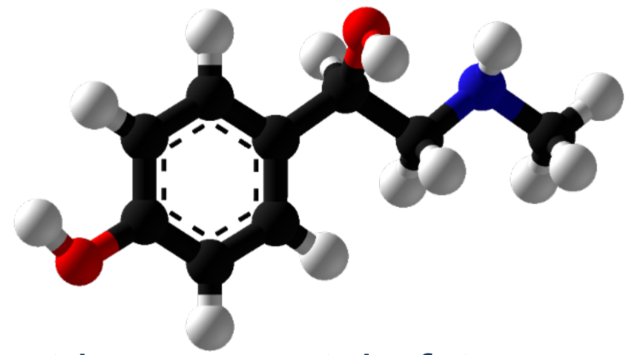
Oil prices

If you run an energy intensive business, an airline for example, then the price of oil is a key determinant of your profitability

For medium or long-term investment planning (buying new planes) the future price of oil is an important consideration

But who knows the price of oil in ten years? No-one. But we may be able to put a probability distribution around the future price and incorporate the uncertainty into the decision making process

Valuing a drug development company



- A company has 10 drugs in a development portfolio
- Given a drug has been approved, you have predicted its revenue
- But whether a drug is approved or not is an uncertain future event (a random variable). You have estimated the probability of approval
- You only wish to invest in the company if the company’s expected total revenue for the portfolio is over $10B in 5 years time
- You need to calculate the ***probability distribution*** of the total revenue to understand the investment risk

Some examples of probabilistic models

- ***Regression models*** (module 4)
- ***Probability trees***
- ***Monte Carlo*** simulation
- ***Markov models***

Regression models

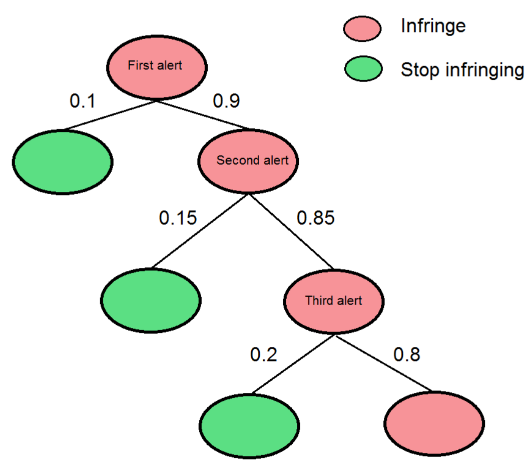
- $E(Price \mid Carats) = -259.6 + 3721 \times Carats$
- The gray band gives a prediction interval for the price of a diamond taken from this population

Regression models

- Regression models use data to estimate the relationship between the mean value of the outcome(Y) and a predictor variable(X)
- The intrinsic variation in the raw data is incorporated into forecasts from the regression model
- The less noise in the underlying data the more precise the forecasts from the regression model will be

Probability trees

- Probability trees allow you to propagate probabilities through a sequence of events



- $P(\text{Stop infringing}) = 0.1 + 0.9\times 0.15 + 0.9 \times 0.85 \times 0.2 = 0.388$.

Monte Carlo simulation

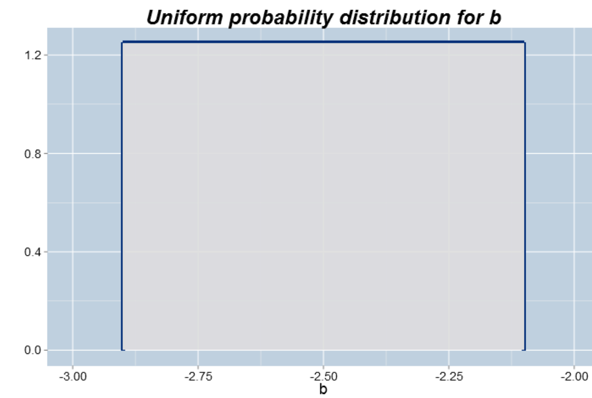
- From the demand model:

  $\text{Quantity} = 60{,}000\ \text{Price}^{-2.5}$

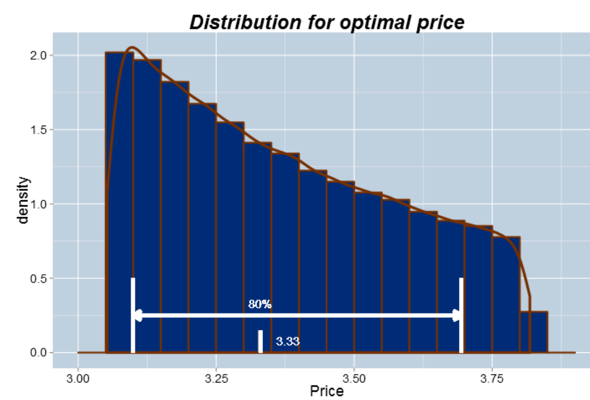
- The optimal price was $p_{opt} = \frac{c\,b}{1+b}$, where b = -2.5, c is the cost, c = 2, and $p_{opt} \approx 3.33$
- But what if b is not known exactly?
- Monte Carlo simulation replaces the number -2.5 with a random variable, and recalculates p_{opt} using different realizations of this random variable from some stated probability distribution

Input and output from a MC simulation

- Input: b from a uniform distribution between -2.9 and -2.1



- Output: $p_{opt} = \frac{c\ b}{1+b}$



- 100,000 replications
- Interval = (3.1,3.7)

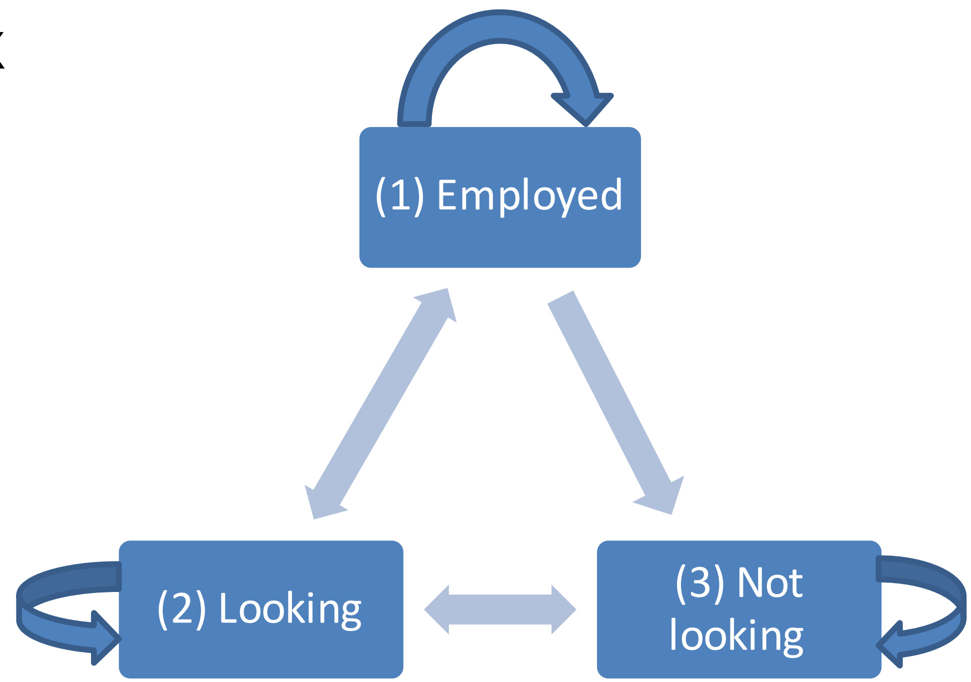
Markov chain models

- Dynamic models for discrete time state space transitions
- Example: employment status (the state of the chain)
- Treat time in 6 month blocks
- Model states:
  1. Employed
  2. Unemployed and looking
  3. Unemployed and not looking

Probability transition matrix

Next state

$$\text{Current state} \begin{pmatrix}1\\2\\3\end{pmatrix} \begin{array}{c} (\;1 \quad 2 \quad 3\;) \\ \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0.3 & 0.7 \end{pmatrix} \end{array}$$



Markov property (lack of memory): transition probabilities only depend on the current state, not on prior states. Given the present, the future does not depend on the past

Building blocks of probability models

- Random variables (discrete and continuous)
- Probability distributions
- Random variables represent the potential outcomes of an uncertain event
- Probability distributions assign probabilities to the various potential outcomes

A discrete random variable

- Roll a fair die

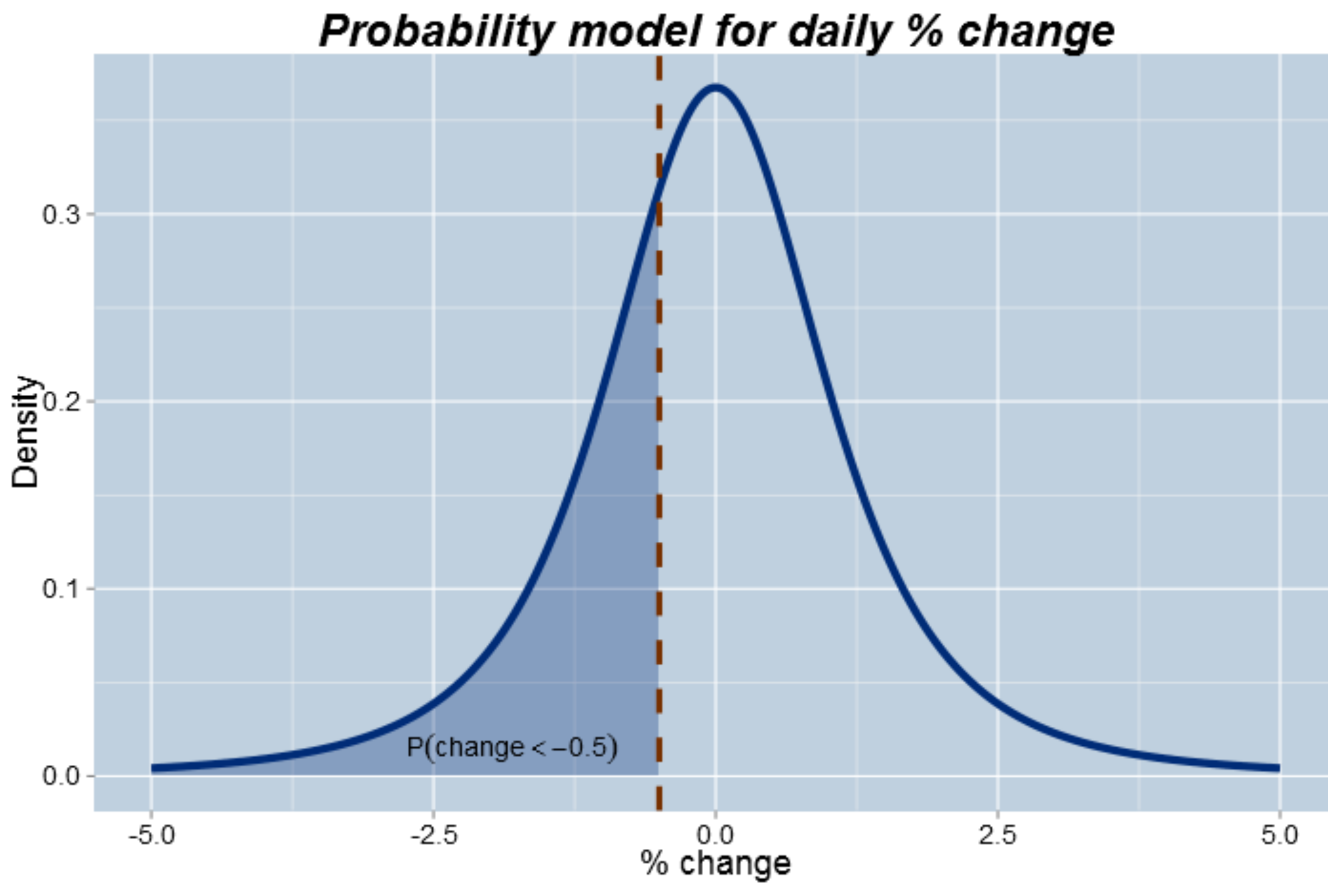
x	⚀	⚁	⚂	⚃	⚄	⚅
P(X = x)	1/6	1/6	1/6	1/6	1/6	1/6

- Probabilities lie between 0 and 1 inclusive
- Probabilities add to 1

A continuous random variable

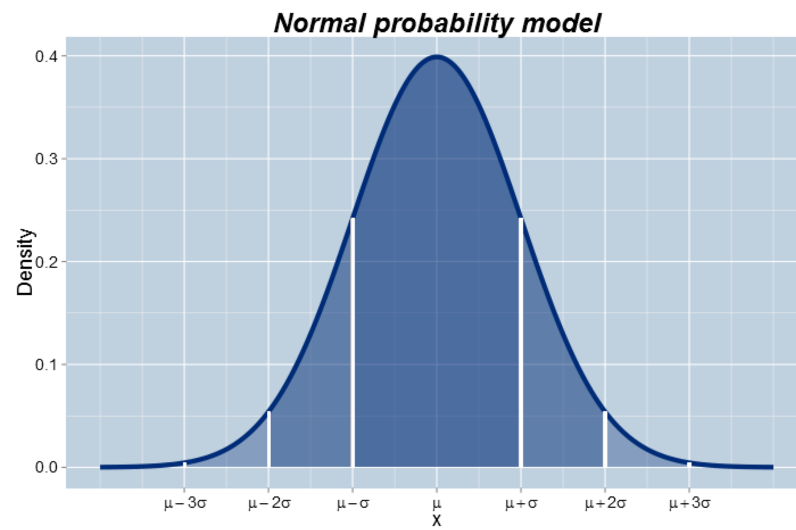
- The ***percent change*** in the S&P500 stock index tomorrow: $100 \times \frac{p_{t+1} - p_t}{p_t}$ where p_t is the closing price on day t
- It can potentially take on **any** number between -100% and infinity
- For a continuous random variable probabilities are computed from areas under the ***probability density function***

Probability distribution of S&P500 daily % change



Key summaries of probability distributions

- Mean (μ) measures centrality
- Two measures of spread:
  - Variance (σ^2)
  - Standard deviation (σ)



The Bernoulli distribution

- The random variable X takes on one of two values:
  - $P(X = 1) = p$
  - $P(X = 0) = 1-p$
- Often viewed as an experiment that takes on two outcomes, success/failure. Success = 1 and failure = 0
- $\mu = E(X) = 1 \times p + 0 \times (1-p) = p$
- $\sigma^2 = E(X-\mu)^2 = (1-p)^2\, p + (0-p)^2\, (1-p) = p(1-p)$
- $\sigma = \sqrt{p(1-p)}$
- For $p = 0.5$, $\mu = 0.5$, $\sigma^2 = 0.25$ and $\sigma = 0.5$

Example: drug development

- Will a drug under development be approved?
- X = $\begin{cases} Yes = 1 \\ No = 0 \end{cases}$
- P(X = Yes) = 0.65
- P(X = No) = 0.35
- If drug is approved then the projected revenue is $500m, 0 otherwise
- Expected(Revenue) = $500m × 0.65 + $0 ×0.35 = $325m

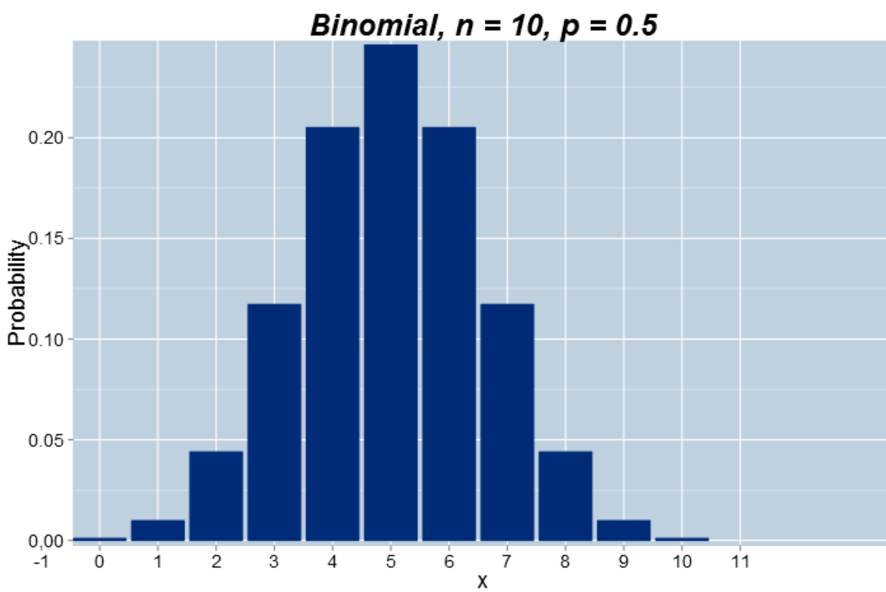
The Binomial distribution

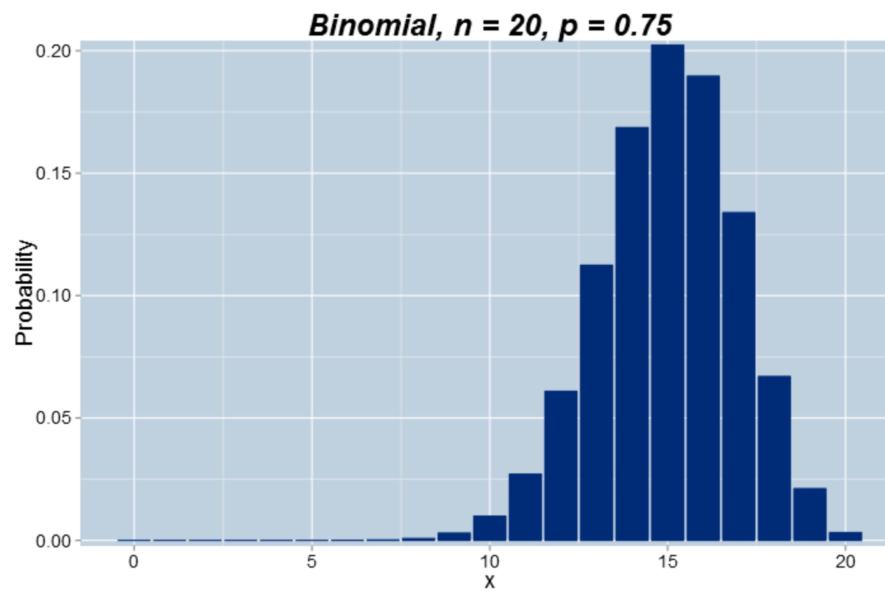
- A Binomial random variable is the number of success in n ***independent*** Bernoulli trials
- Independent means that $P(A \text{ and } B) = P(A) \times P(B)$
- Independence means that knowing that A has occurred provides no information about the occurrence of B
- Independence is a common simplifying assumption in many probability models and makes their construction and subsequent calculations much easier

The Binomial distribution

- Example: toss a fair coin 10 times and count the number of heads (call this X)
- Then X has a Binomial distribution with parameters n = 10 and p = 0.5.
- In general: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, where $\binom{n}{x}$ is the ***binomial coefficient:*** $\frac{n!}{x!(n-x)!}$
- $\mu = E(X) = np, \sigma^2 = E(X - \mu)^2 = np(1-p)$

Binomial probability distributions

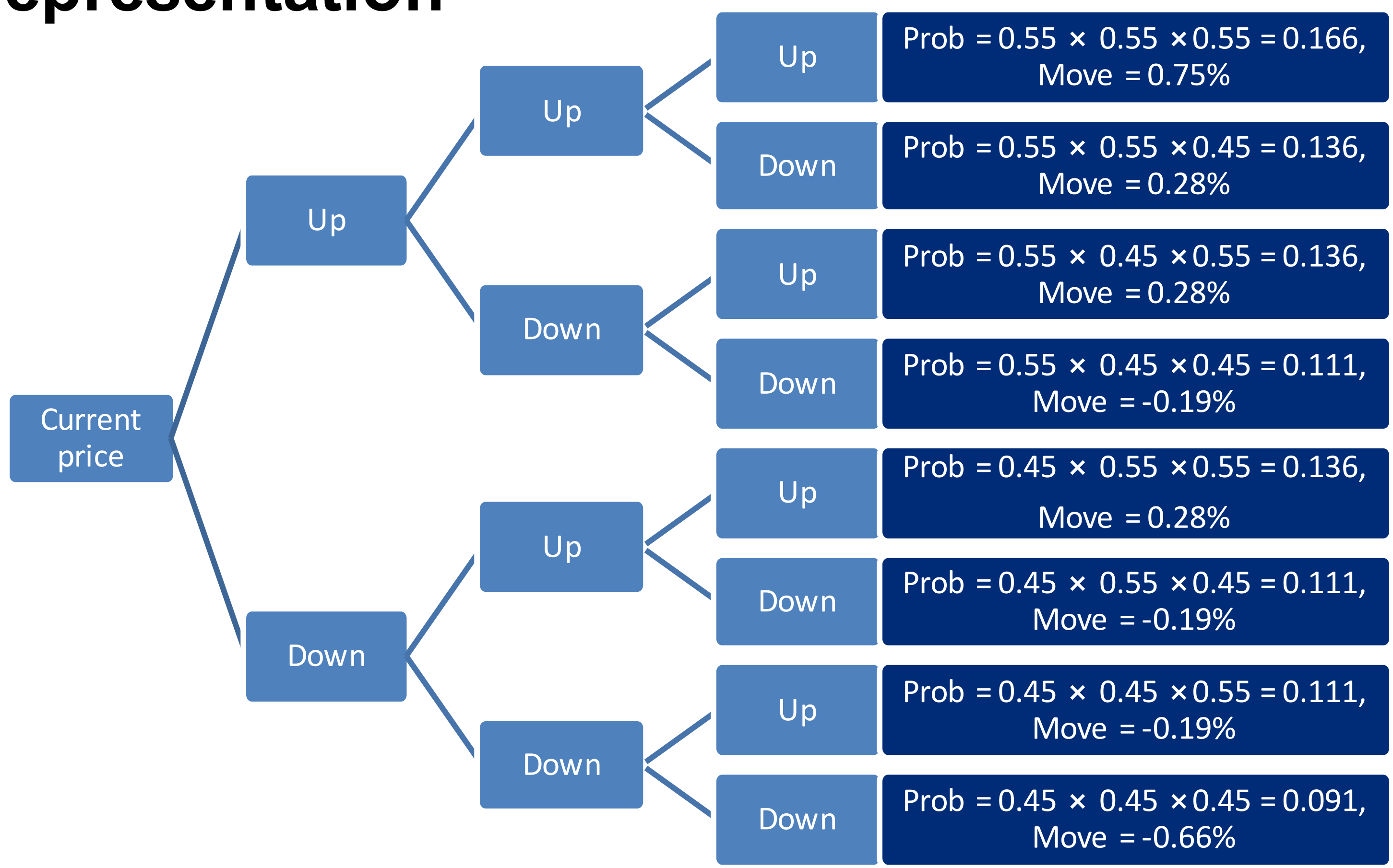




Example: Binomial models for markets (oil for example)

- Assume that the market either goes up or down each day
- It goes up u% with probability p and down d% with probability 1-p
- Assume days are ***independent***
- Example: $p = 0.55$, $1 - p = 0.45$, $u = 0.25\%$, $d = 0.22\%$
- Take a time horizon of 3 days
- There are 8 possible outcomes:
  - {UUU},{UUD},{UDU},{UDD},{DUU},{DUD},{DDU},{DDD}
- For each outcome there will be an associated market move. For example, if we see (U,U,U) then the market goes up by a factor of $1.0025 \times 1.0025 \times 1.0025 = 1.007519$, that is a little over ¾ of a percent.

Tree representation



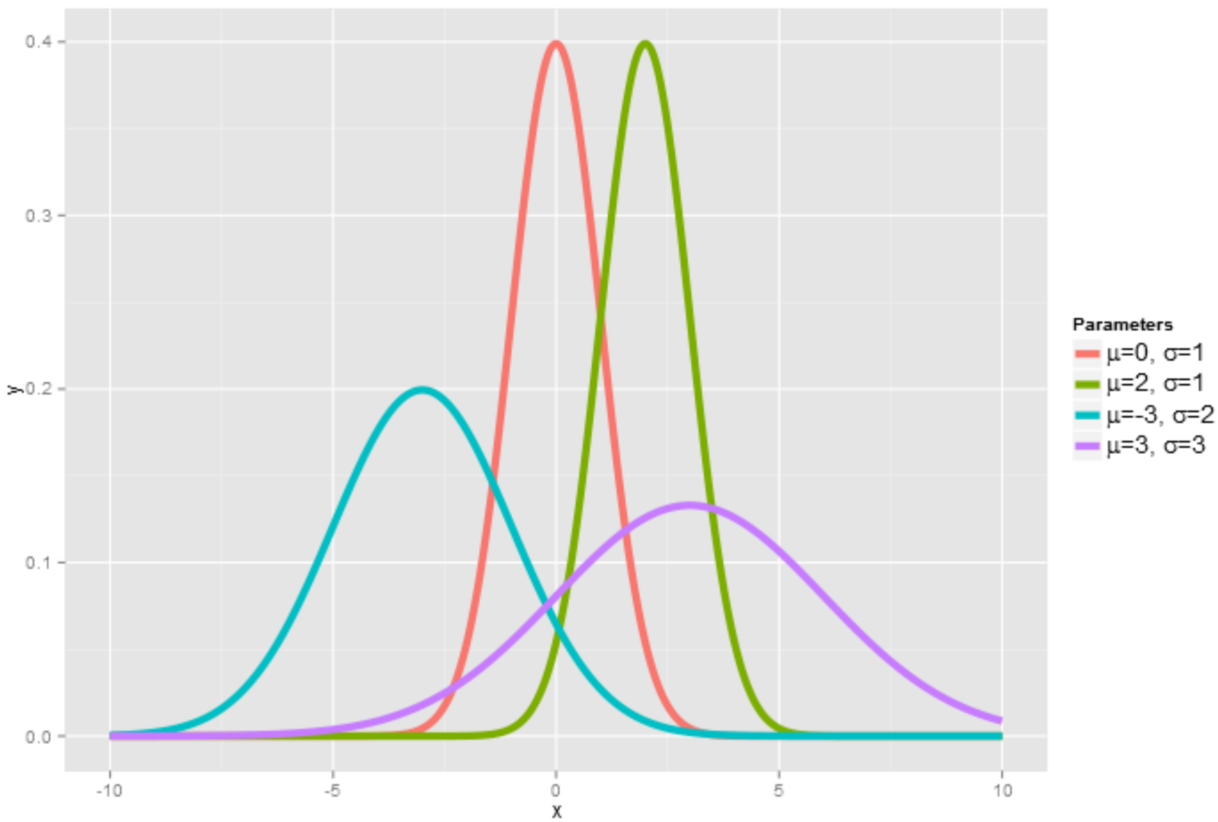
The Normal distribution

- The Normal distribution, colloquially known as the *Bell Curve,* is the most important modeling distribution
- Many disparate processes can be well ***approximated*** by Normal distributions
- There are mathematical theorems (the Central Limit Theorem) that tell us Normal distributions should be expected in many situations
- A Normal distribution is characterized by its mean μ and standard deviation σ. It is symmetric about its mean

Examples

- There is a ***universality*** to the Normal distribution
  - Biological: heights and weights
  - Financial: stock returns
  - Educational: exam scores
  - Manufacturing: the length of an automotive component
- It is therefore often used as a distributional assumption in Monte Carlo simulations (knowing the mean and standard deviation is enough to define a Normal distribution)

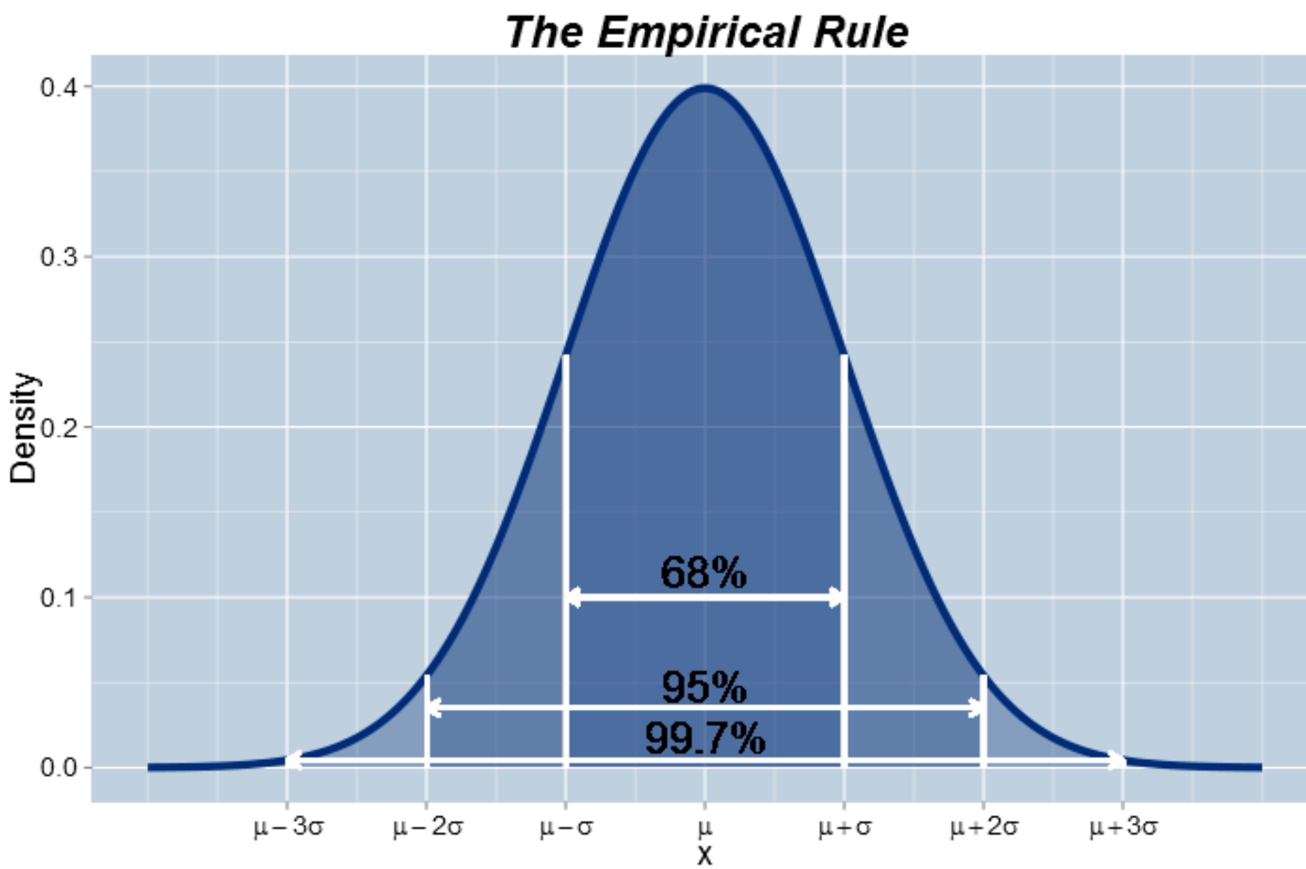
Plots of various Normal distributions



The Empirical Rule

- The Empirical Rule is a rule for calculating probabilities of events when the underlying distribution or observed data is approximately Normally distributed
- It states
  - There is an approximate **68%** chance that an observation falls within **one** standard deviation from the mean
  - There is an approximate **95%** chance that an observation falls within **two** standard deviations from the mean
  - There is an approximate **99.7%** chance that an observation falls within **three** standard deviations from the mean

The Empirical Rule illustrated



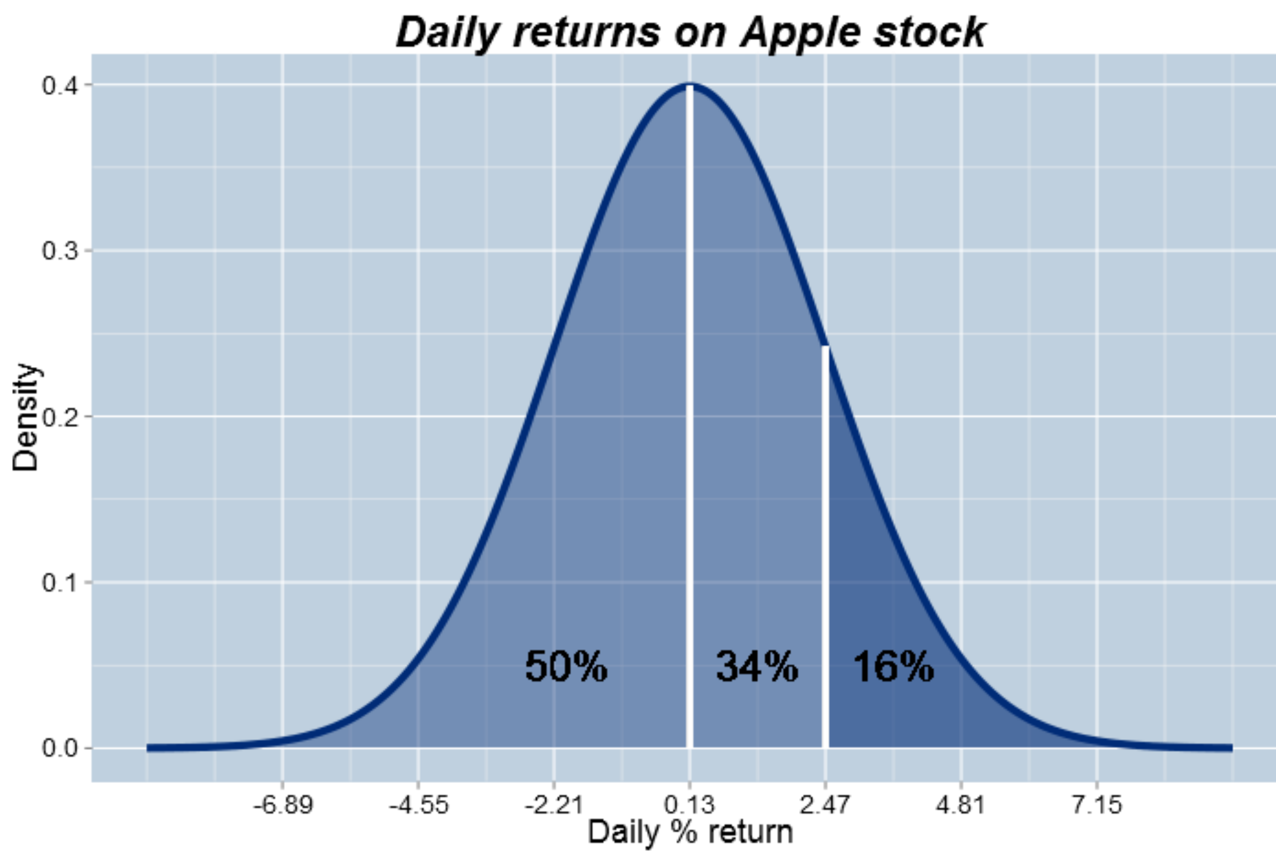
Empirical Rule example

- Assume that the daily **return** on Apple's stock is approximately Normally distributed with mean $\mu = 0.13\%$ and $\sigma = 2.34\%$
- What is the probability that tomorrow Apple's stock price increases by more than 2.47%?
- Technique: count how many standard deviations 2.47% is away from the mean, 0.13%. Call this ***counter*** the ***z-score***

$$Z = \frac{2.47 - 0.13}{2.34} = 1$$

- So, from the Empirical Rule the probability equals approximately 16%

Illustrating the answer



Module 3 Summary

- What are probabilistic models?
- Random variables and probability distributions -- the building blocks
- Examples of probabilistic models
- Summaries of probability distributions: means, variances and standard deviation
- Special random variables: Bernoulli, Binomial and Normal
- The Empirical Rule

Wharton

University *of* Pennsylvania

Wharton
University of Pennsylvania

ONLINE