Hopfield Networks

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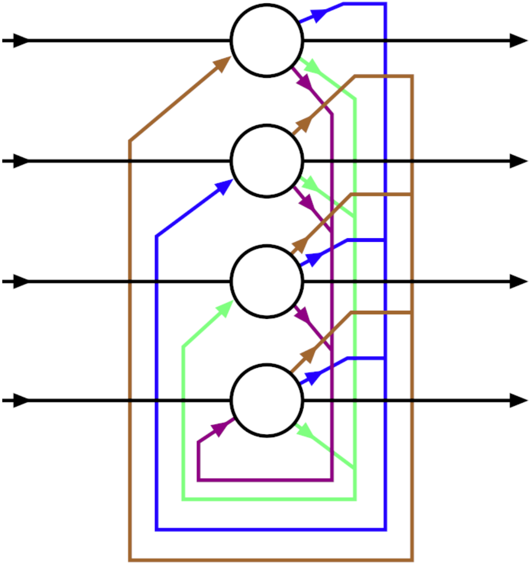
May 5, 2022

Model –Why?, Neurophysiology, etc.

- Often referred to as "content-addressable memory" or (more recently) "dense associative memory" system.
- Similarities to Ising's model: overall state of the model is defined by individual states pointing in some direction (-1 or 1), and each individual state depends on its neighbors – explicit in the original paper
- Proposed by Hopfield in 1982 with "Neural networks and physical systems with emergent collective computational abilities" [1]. Hopfield (1984) [2] considered the continuous case
- Hebbian Learning: $w_{ij} = V_i^s V_j^s$
  - "Neurons that fire together wire together" - attributed to Hebb [3]
  - A neurophysiological intuition is exploited so *association* between neurons that can be used to determine connections among them

Model



Model – Creating memories

At a given state s the weights of the network are

$$w_{ij} = V_i^s V_j^s \tag{1}$$

We can "learn" a set of target network states (memories) $V^{(1)}, ..., V^{(M)}$ by setting weights of the network as the average over these memories:

$$w_{ij} = \frac{1}{M} \sum_{1}^{M} V_i V_j \tag{2}$$

with $w_{ii} = 0$. Or alternatively,

$$W = VV^T - I \tag{3}$$

Model – Retrieving memories

Each node is updated as

$$V_i = \left\{ \begin{array}{cc} 1 & \text{if } \sum_j w_{ij} V_i > 0 \\ -1 & \text{otherwise} \end{array} \right\} \tag{4}$$

Energy of the system (entropy?):

$$E = -\sum_i s_i b_i - \sum_{i \neq j} s_i s_j w_{ij} \tag{5}$$

Marginal changes for each node:

$$\Delta E_i = b_i + \sum s_j w_{ij} \tag{6}$$

- Memories considered minima in the energy space of the network
- Ideally, since the system will consistently move towards a lower energy state, it should eventually retrieve a memory

Hopfield (1984):

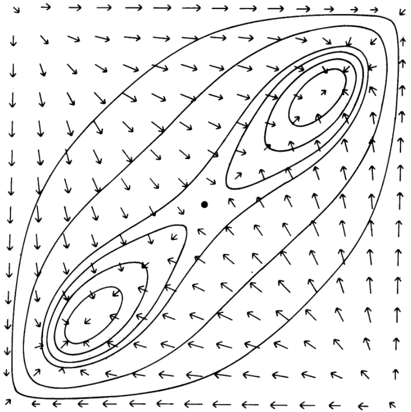


FIG. 3. An energy contour map for a two-neuron, two-stable-state system. The ordinate and abscissa are the outputs of the two neurons. Stable states are located near the lower left and upper right corners, and unstable extrema at the other two corners. The arrows show the motion of the state from Eq. **5**. This motion is not in general perpendicular to the energy contours. The system parameters are $T_{12} = T_{21} = 1$, $\lambda = 1.4$, and $g(u) = (2/\pi)\tan^{-1}(\pi\lambda u/2)$. Energy contours are 0.449, 0.156, 0.017, −0.003, −0.023, and −0.041.

Experiments (so far...) – Hamming Distance

Hamming distance:

$$\sum_{i=1}^{N} I(x_i \neq y_i) \tag{7}$$

- Distance between network state and an original memory.
- Distance between perturbed memories/states and original memories
- i.e.: Any two binary states

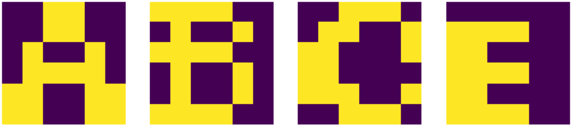
Experiments (so far...) – Simple state vectors

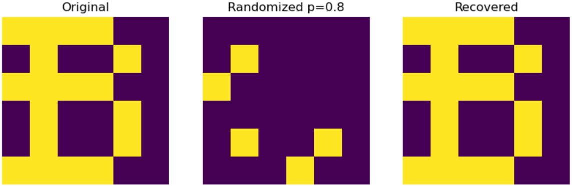
```
(base) cimv@mwc-064127 src % python3 cam.py

Original vector:    1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
Randomized vector:  1 -1  1  1 -1  1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
Hamming Distance:   4
Euclidean Distance: 16.000

Iter | Energy          | Hamming | X/State
----------------------------------------------------------------------------
   0 |              -5 |       4 |  1 -1  1  1 -1  1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
   1 |             -13 |       3 |  1  1  1  1  1  1  1  1 -1  1  1  1 -1 -1 -1 -1
   2 |             -21 |       1 |  1  1  1  1 -1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   3 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   4 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   5 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   6 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   7 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   8 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
   9 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
  10 |             -28 |       0 |  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1

Network state vs memory states
Hamming distance to memory 0: 0
Hamming distance to memory 1: 8
Hamming distance to memory 2: 8
Hamming distance to memory 3: 8
(base) cimv@mwc-064127 src %
```

Experiments (so far...) – Simple state vectors

```
Original vector:   [ 1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1]
Randomized vector: [-1 -1  1  1  1 -1 -1  1 -1 -1 -1 -1 -1 -1 -1 -1]
Hamming Distance:   4
Euclidean Distance: 16.000

Iter | Energy | Hamming | X/State
---------------------------------------------------------------------------
   0 |     -3 |       4 | [-1 -1  1  1  1 -1 -1  1 -1 -1 -1 -1 -1 -1 -1 -1]
   1 |     -9 |       5 | [ 1  1  1  1 -1  1  1  1 -1 -1  1  1 -1 -1  1  1]
   2 |    -15 |       7 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1 -1  1]
   3 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
   4 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
   5 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
   6 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
   7 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
   8 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
   9 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
  10 |    -20 |       8 | [-1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1]
```

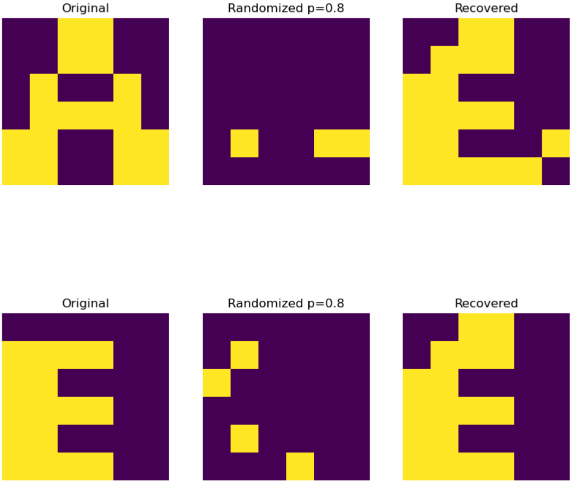
Experiments (so far...)



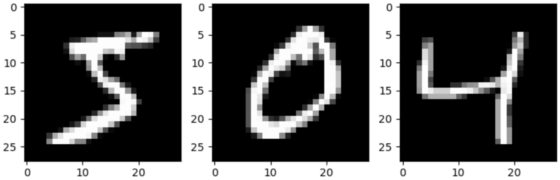


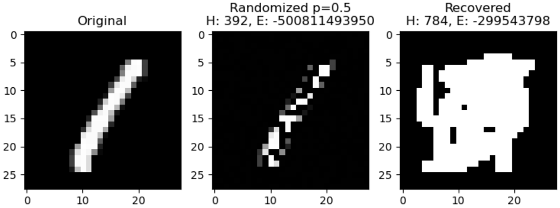

```
Original vector:    1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
Randomized vector: -1 -1 -1 -1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
Hamming Distance:   13
Euclidean Distance: 52.000

Iter | Energy | Hamming | X/State
---------------------------------------------------------------------------------
   0 |    -18 |      13 | -1 -1 -1 -1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   1 |   -107 |       6 |  1  1  1  1 -1 -1  1  1  1 -1  ...  1 -1 -1
   2 |   -142 |       1 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   3 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   4 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   5 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   6 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   7 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   8 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
   9 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
  10 |   -156 |       0 |  1  1  1  1 -1 -1 -1  1 -1 -1  ...  1 -1 -1
```

Experiments (so far...)

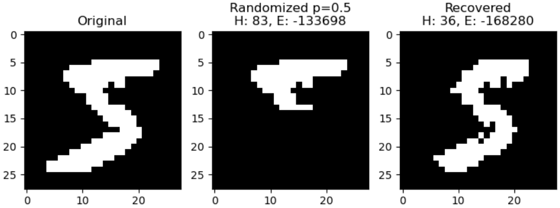


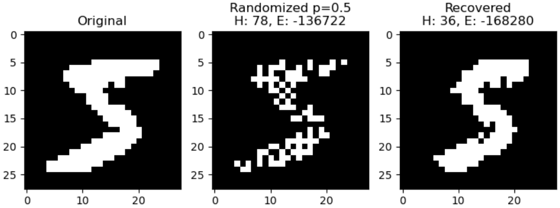
Experiments (so far...) – MNIST



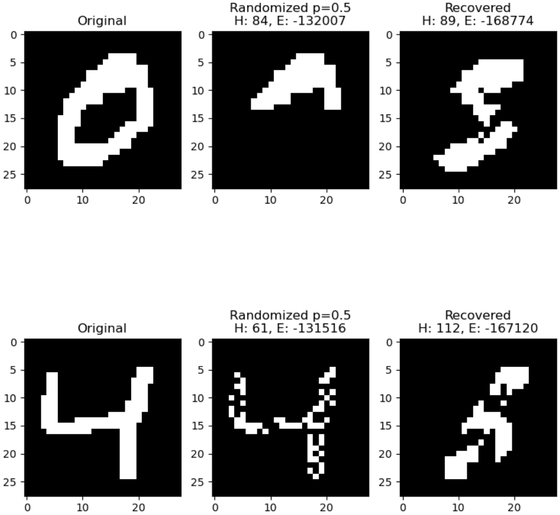


Experiments (so far...) – MNIST





Experiments (so far...) – MNIST



Next Steps

- Simplest forms of Hopfield Networks are too prone to local minima or spurious states
- Good approximation for optimization problems that use other graphical models [4].
- Two (maybe three) routes:
    - Discrete "Modern Hopfield Networks": far greater capacity, more general (in their mathematical expression) [5]
    - Application of Ising models to a particular dataset later to be tested against Hopfield Net.
    - Stochastic activation function on traditional Hopfield
    - Last two options will likely emphasize the model as a tool for inference and/or comparison with models studied in class

References

[1] J. J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," *Proceedings of the national academy of sciences*, vol. 79, no. 8, pp. 2554–2558, 1982.

[2] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proceedings of the national academy of sciences*, vol. 81, no. 10, pp. 3088–3092, 1984.

[3] D. O. Hebb, *The organization of behavior: A neuropsychological theory*. Psychology Press, 2005.

[4] D. J. MacKay, D. J. Mac Kay, *et al.*, *Information theory, inference and learning algorithms*. Cambridge university press, 2003.

[5] D. Krotov and J. J. Hopfield, "Dense associative memory for pattern recognition," *Advances in neural information processing systems*, vol. 29, 2016.