

# Benchmarking Linear Logic: sequents inspired from Kleene's intuitionistic theorems\*

October 10, 2018

## 1 General Information

- Test run on a QEMU Virtual CPU, 2GHz, 64 bits, 2GB of RAM running Ubuntu.
- Timeout in all the cases was 2 minutes.
- Maude version: 2.7.1 built: Oct 2 2017.
- The search procedure considers proofs using up to 4 times the copy rule (focusing on one of the formulas of the classical context). Benchmarks 23 and 25 required the limit to be 5.

## 2 Translations

(1)  $\cdot \vdash A \rightarrow A$

LJ (28ms)

$$\frac{}{\cdot \vdash A \rightarrow A} \star$$

MULTIPLICATIVE encoding (28ms)

$$\frac{\overline{\cdot : A \vdash A}}{\cdot : \cdot \vdash A \multimap A} \star$$

CALL-BY-NAME encoding (28ms)

$$\frac{\overline{A : \cdot \vdash A}}{\cdot : \cdot \vdash !(A) \multimap A} \star$$

CALL-BY-VALUE encoding (41ms)

$$\frac{\frac{\overline{A : \cdot \vdash A}}{A : \cdot \vdash !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} \star$$
$$\frac{}{\cdot : \cdot \vdash !(A) \multimap !(A)} !$$

01-ENC encoding (35ms)

$$\frac{\overline{A : \cdot \vdash A}}{\cdot : \cdot \vdash !(A) \multimap A} \star$$
$$\frac{}{\cdot : \cdot \vdash !(A) \multimap A} !$$

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\*See <https://github.com/meta-logic/lltp> for details on the encodings used.

LJ (46ms)

MULTIPLICATIVE encoding (49ms)

CALL-BY-NAME encoding (123ms)

CALL-BY-VALUE encoding (165ms)

01-ENC encoding (146ms)

LJ (46ms)

MULTIPLICATIVE encoding (47ms)

CALL-BY-NAME encoding (71ms)

CALL-BY-VALUE encoding (162ms)

01-ENC encoding (158ms)

[illegible]

$$(4) \ A \rightarrow B \rightarrow C \vdash A \wedge B \rightarrow C$$

LJ (47ms)

$$\frac{\frac{\frac{A, B, A \rightarrow B \rightarrow C \vdash A}{A, B, A \rightarrow B \rightarrow C \vdash C} \star}{A, B, A \rightarrow B \rightarrow C \vdash C} \supset_L \quad \frac{\frac{\frac{A, B, B \rightarrow C \vdash B}{A, B, B \rightarrow C \vdash C} \star \quad \frac{A, B, C \vdash C}{A, B, B \rightarrow C \vdash C} \star}{A, B, B \rightarrow C \vdash C} \supset_L}{A \rightarrow B \rightarrow C \vdash A \wedge B \rightarrow C} \star$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\frac{\cdot \vdash A \vdash A}{\cdot \vdash A, B, A \multimap B \multimap C \vdash C} \multimap}{\cdot \vdash A \multimap B \multimap C \vdash A \otimes B \multimap C} \star}{\cdot \vdash B, B \multimap C \vdash C} \multimap$$

CALL-BY-NAME encoding (71ms)

$$\begin{array}{c}
\frac{\frac{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash A}{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(A)} \quad \frac{\frac{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash B}{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(B)}}{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(A \& B)} \quad \frac{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash C}{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : C \multimap C} \quad \multimap \\
\frac{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(A \& B) \quad A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(B) \multimap C \multimap C}{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(A \& B) \multimap C \multimap C} \quad \multimap \\
\frac{A, B, \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C : \vdash \mathfrak{!}(A \& B) \multimap C \multimap C}{\vdash : \mathfrak{!}(A) \multimap \mathfrak{!}(B) \multimap C \vdash \mathfrak{!}(A \& B) \multimap C} \quad D_C \quad *
\end{array}$$

CALL-BY-VALUE encoding (156ms)

[illegible]

01-ENC encoding (151ms)

[illegible]

$$(5) \quad A \wedge B \rightarrow C \vdash A \rightarrow B \rightarrow C$$

LJ (41ms)

$$\frac{\frac{\frac{A, B, A \wedge B \rightarrow C \vdash A \wedge B}{A, B, C \vdash C}^*}{A, B, A \wedge B \rightarrow C \vdash C}}{A \wedge B \rightarrow C \vdash A \rightarrow B \rightarrow C}^* \supset_L$$

MULTIPLICATIVE encoding (54ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : A, B \vdash A \otimes B} \otimes \frac{\cdot : C \vdash C}{\cdot : A, B, A \otimes B \multimap C \vdash C} \multimap}{\cdot : A \otimes B \multimap C \vdash A \multimap B \multimap C} \star$$

CALL-BY-NAME encoding (73ms)

$$\frac{\frac{\frac{A, B, !(A \& B) \multimap C : \multimap A \quad A, B, !(A \& B) \multimap C : \multimap B}{A, B, !(A \& B) \multimap C : \multimap A \& B} \quad \frac{A, B, !(A \& B) \multimap C : \multimap A \& B}{A, B, !(A \& B) \multimap C : \multimap !(A \& B)} !}{\frac{A, B, !(A \& B) \multimap C : \multimap !(A \& B) \multimap C \multimap C}{A, B, !(A \& B) \multimap C : \multimap !(A \& B) \multimap C \multimap C} D_C} \multimap$$

CALL-BY-VALUE encoding (205ms)

[illegible]

01-ENC encoding (201ms)

[illegible]

$$(6) \ A \rightarrow B \vdash B \rightarrow C \rightarrow A \rightarrow C$$

LJ (47ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow C \vdash A}{A, A \rightarrow B, B \rightarrow C \vdash C} \star \quad \frac{\frac{A, B, B \rightarrow C \vdash B}{A, B, B \rightarrow C \vdash C} \star}{A \rightarrow B \vdash B \rightarrow C \rightarrow A \rightarrow C} \star}{A, A \rightarrow B, B \rightarrow C \vdash C} \star \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\vdots : A \vdash A}{\vdots : A, A \multimap B, B \multimap C \vdash C} \multimap \quad \frac{\vdots : B \vdash B \quad \vdots : C \vdash C}{\vdots : B, B \multimap C \vdash C} \multimap}{\vdots : A, A \multimap B, B \multimap C \vdash C} \multimap}{\vdots : A \multimap B \vdash B \multimap C \multimap A \multimap C} \star$$

CALL-BY-NAME encoding (120ms)

$$\frac{\frac{\frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash A}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash \lambda(A)} \vdash \quad \frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash B}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash \lambda(B)} \vdash}{\frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash B}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash C} \multimap} \multimap$$

CALL-BY-VALUE encoding (168ms)

$$\frac{\frac{\frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash A}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash \lambda(A)} \vdash \quad \frac{A, B, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash B}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash \lambda(B)} \vdash}{\frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash B}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash C} \multimap} \multimap$$

01-ENC encoding (159ms)

$$\frac{\frac{\frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash A}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash \lambda(A)} \vdash \quad \frac{A, B, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash B}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash \lambda(B)} \vdash}{\frac{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash B}{A, \lambda(A) \multimap B, \lambda(B) \multimap C : \vdash C} \multimap} \multimap$$

$$(7) \ A \rightarrow B \vdash C \rightarrow A \rightarrow C \rightarrow B$$

LJ (47ms)

$$\frac{\frac{\frac{C, A \rightarrow B, C \rightarrow A \vdash C}{C, A \rightarrow B, C \rightarrow A \vdash A} \star \quad \frac{A, C, A \rightarrow B \vdash A}{C, A \rightarrow B, C \rightarrow A \vdash A} \star}{\frac{C, A \rightarrow B, C \rightarrow A \vdash B}{A \rightarrow B \vdash C \rightarrow A \rightarrow C \rightarrow B} \star} \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\vdots : A \vdash A}{\vdots : C \vdash C} \multimap \quad \frac{\vdots : B \vdash B}{\vdots : A, A \multimap B \vdash B} \multimap}{\vdots : C, A \multimap B, C \multimap A \vdash B} \multimap}{\vdots : A \multimap B \vdash C \multimap A \multimap C \multimap B} \star$$

CALL-BY-NAME encoding (120ms)

$$\frac{\frac{\frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash C}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash \lambda(C)} \vdash \quad \frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash A}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash \lambda(A)} \vdash}{\frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash A}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash C} \multimap} \multimap$$

CALL-BY-VALUE encoding (167ms)

$$\frac{\frac{\frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash C}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash \lambda(C)} \vdash \quad \frac{A, C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash A}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash \lambda(A)} \vdash}{\frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash A}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash C} \multimap} \multimap$$

01-ENC encoding (156ms)

$$\frac{\frac{\frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash C}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash \lambda(C)} \vdash \quad \frac{A, C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash A}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash \lambda(A)} \vdash}{\frac{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash A}{C, \lambda(A) \multimap B, \lambda(C) \multimap A : \vdash C} \multimap} \multimap$$

$$(8) \ A \rightarrow B \vdash A \wedge C \rightarrow B \wedge C$$

LJ (37ms)

$$\frac{\frac{\overline{A, C, A \rightarrow B \vdash A} \star \overline{A, B, C \vdash B}}{A, C, A \rightarrow B \vdash B} \star}{A \rightarrow B \vdash A \wedge C \rightarrow B \wedge C} \star \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\overline{\cdot : B \vdash B} \quad \overline{\cdot : C \vdash C}}{\cdot : A \vdash A \quad \cdot : B, C \vdash B \otimes C} \otimes}{\cdot : A, C, A \multimap B \vdash B \otimes C} \multimap}{\cdot : A \multimap B \vdash A \otimes C \multimap B \otimes C} \star$$

CALL-BY-NAME encoding (73ms)

$$\frac{\frac{\overline{A, C, !!(A) \multimap B : \vdash A} \quad \overline{A, C, !(A) \multimap B : B \vdash B}}{\overline{A, C, !(A) \multimap B : !(A) \multimap B \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap B : \vdash B} \quad \overline{A, C, !(A) \multimap B : B \vdash B} \quad D_C}{\overline{A, C, !(A) \multimap B : \vdash C} \star} \star$$

CALL-BY-VALUE encoding (135ms)

$$\frac{\frac{\overline{A, C, !(A) \multimap !(B) : \vdash A} \quad \overline{A, B, C, !(A) \multimap !(B) : \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : \vdash !(A)} \quad \overline{A, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : !(A) \multimap !(B) \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap !(B) : \vdash B} \quad \overline{A, C, !(A) \multimap !(B) : B \vdash B} \quad D_C}{\overline{A, C, !(A) \multimap !(B) : \vdash C} \star} \star$$

01-ENC encoding (130ms)

$$\frac{\frac{\overline{A, C, !(A) \multimap !(B) : \vdash A} \quad \overline{A, B, C, !(A) \multimap !(B) : \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : \vdash !(A)} \quad \overline{A, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : !(A) \multimap !(B) \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap !(B) : \vdash B} \quad \overline{A, C, !(A) \multimap !(B) : B \vdash B} \quad D_C}{\overline{A, C, !(A) \multimap !(B) : \vdash C} \star} \star$$

$$(9) \ A \rightarrow B \vdash C \wedge A \rightarrow C \wedge B$$

LJ (37ms)

$$\frac{\frac{\overline{A, C, A \rightarrow B \vdash A} \star \overline{A, B, C \vdash B}}{A, C, A \rightarrow B \vdash B} \star}{A \rightarrow B \vdash C \wedge A \rightarrow C \wedge B} \star \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\overline{\cdot : C \vdash C} \quad \overline{\cdot : B \vdash B}}{\cdot : A \vdash A \quad \cdot : B, C \vdash C \otimes B} \otimes}{\cdot : A, C, A \multimap B \vdash C \otimes B} \multimap}{\cdot : A \multimap B \vdash C \otimes A \multimap C \otimes B} \star$$

CALL-BY-NAME encoding (60ms)

$$\frac{\frac{\overline{A, C, !(A) \multimap B : \vdash A} \quad \overline{A, C, !(A) \multimap B : B \vdash B}}{\overline{A, C, !(A) \multimap B : !(A) \multimap B \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap B : \vdash C} \quad \overline{A, C, !(A) \multimap B : B \vdash B} \quad D_C}{\overline{A, C, !(A) \multimap B : \vdash B} \star} \star$$

CALL-BY-VALUE encoding (137ms)

$$\frac{\frac{\overline{A, B, C, !(A) \multimap !(B) : \vdash C} \quad \overline{A, B, C, !(A) \multimap !(B) : \vdash B} \star}{\overline{A, B, C, !(A) \multimap !(B) : \vdash !(C)} \quad \overline{A, B, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, B, C, !(A) \multimap !(B) : !(A) \multimap !(B) \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap !(B) : \vdash A} \quad \overline{A, B, C, !(A) \multimap !(B) : \vdash !(C)} \quad \overline{A, B, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : \vdash !(A)} \quad \overline{A, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : !(A) \multimap !(B) \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap !(B) : \vdash B} \quad \overline{A, C, !(A) \multimap !(B) : B \vdash B} \quad D_C}{\overline{A, C, !(A) \multimap !(B) : \vdash C} \star} \star$$

01-ENC encoding (131ms)

$$\frac{\frac{\overline{A, B, C, !(A) \multimap !(B) : \vdash C} \quad \overline{A, B, C, !(A) \multimap !(B) : \vdash B} \star}{\overline{A, B, C, !(A) \multimap !(B) : \vdash !(C)} \quad \overline{A, B, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, B, C, !(A) \multimap !(B) : !(A) \multimap !(B) \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap !(B) : \vdash A} \quad \overline{A, B, C, !(A) \multimap !(B) : \vdash !(C)} \quad \overline{A, B, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : \vdash !(A)} \quad \overline{A, C, !(A) \multimap !(B) : !(B) \vdash B} \star}{\overline{A, C, !(A) \multimap !(B) : !(A) \multimap !(B) \vdash B} \multimap} \multimap}{\overline{A, C, !(A) \multimap !(B) : \vdash B} \quad \overline{A, C, !(A) \multimap !(B) : B \vdash B} \quad D_C}{\overline{A, C, !(A) \multimap !(B) : \vdash C} \star} \star$$

$$(10) \neg A \vdash A \rightarrow B$$

LJ (37ms)

$$\frac{\frac{A, A \rightarrow \perp \vdash A}{A, A \rightarrow \perp \vdash B} \star}{A \rightarrow \perp \vdash A \rightarrow B} \star \supset_L$$

MULTIPLICATIVE encoding (41ms)

**Not provable**

CALL-BY-NAME encoding (66ms)

$$\frac{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash B}{A, !(A) \multimap \mathbf{0} : \cdot \vdash B} \star}{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash B}{\cdot : !(A) \multimap \mathbf{0} \vdash !(A) \multimap B} D_C} \multimap$$

CALL-BY-VALUE encoding (117ms)

$$\frac{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash !(B)}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(B)} \star}{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(B)}{!(A) \multimap \mathbf{0} : \cdot \vdash !(A) \multimap !(B)} D_C} \multimap$$

$$\frac{\frac{!(A) \multimap \mathbf{0} : \cdot \vdash !(A) \multimap !(B)}{!(A) \multimap \mathbf{0} : \cdot \vdash !(!(A) \multimap !(B))} !}{\cdot : !(A) \multimap \mathbf{0} \vdash !(A) \multimap !(B)} \star$$

01-ENC encoding (118ms)

$$\frac{\frac{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash A}{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap !(\mathbf{0}) : !(\mathbf{0}) \vdash B}{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash B} \star}{\frac{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash B}{!(A) \multimap !(\mathbf{0}) : \cdot \vdash !(A) \multimap B} D_C} \multimap$$

$$\frac{\frac{!(A) \multimap !(\mathbf{0}) : \cdot \vdash !(A) \multimap B}{!(A) \multimap !(\mathbf{0}) : \cdot \vdash !(!(A) \multimap B)} !}{\cdot : !(A) \multimap \mathbf{0} \vdash !(A) \multimap B} \star$$

$$(11) A \vdash \neg A \rightarrow B$$

LJ (37ms)

$$\frac{\frac{A, A \rightarrow \perp \vdash A}{A, A \rightarrow \perp \vdash B} \star}{A \vdash A \rightarrow \perp \rightarrow B} \star \supset_L$$

MULTIPLICATIVE encoding (28ms)

**Not provable**

CALL-BY-NAME encoding (65ms)

$$\frac{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash B}{A, !(A) \multimap \mathbf{0} : \cdot \vdash B} \star}{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash B}{\cdot : !(A) \vdash !(A) \multimap B} D_C} \multimap$$

CALL-BY-VALUE encoding (78ms)

$$\frac{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash !(B)}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(B)} \star}{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(B)}{!(A) \multimap \mathbf{0} : \cdot \vdash !(A) \multimap !(B)} D_C} \multimap$$

$$\frac{\frac{!(A) \multimap \mathbf{0} : \cdot \vdash !(A) \multimap !(B)}{!(A) \multimap \mathbf{0} : \cdot \vdash !(!(A) \multimap !(B))} !}{\cdot : !(A) \vdash !(A) \multimap !(A) \multimap B} \star$$

01-ENC encoding (76ms)

$$\frac{\frac{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash A}{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap !(\mathbf{0}) : !(\mathbf{0}) \vdash B}{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash B} \star}{\frac{A, !(A) \multimap !(\mathbf{0}) : \cdot \vdash B}{!(A) \multimap !(\mathbf{0}) : \cdot \vdash !(A) \multimap B} D_C} \multimap$$

$$\frac{\frac{!(A) \multimap !(\mathbf{0}) : \cdot \vdash !(A) \multimap B}{!(A) \multimap !(\mathbf{0}) : \cdot \vdash !(!(A) \multimap B)} !}{\cdot : !(A) \vdash !(A) \multimap !(A) \multimap B} \star$$

$$(12) \ B \vdash A \rightarrow B$$

LJ (21ms)

$$\overline{B \vdash A \rightarrow B} \star$$

MULTIPLICATIVE encoding (21ms)

**Not provable**

CALL-BY-NAME encoding (27ms)

$$\frac{\overline{A, B : \cdot \vdash B}}{\cdot : !(B) \vdash !(A) \multimap B} \star$$

CALL-BY-VALUE encoding (47ms)

$$\frac{\frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash !(B)} !}{B : \cdot \vdash !(A) \multimap !(B)} \star$$

$$\frac{\frac{B : \cdot \vdash !(A) \multimap !(B)}{B : \cdot \vdash !(A) \multimap !(B)} !}{\cdot : !(B) \vdash !(A) \multimap !(B)} \star$$

01-ENC encoding (39ms)

$$\frac{\overline{A, B : \cdot \vdash B}}{B : \cdot \vdash !(A) \multimap B} \star$$

$$\frac{\frac{B : \cdot \vdash !(A) \multimap B}{B : \cdot \vdash !(A) \multimap B} !}{\cdot : !(B) \vdash !(A) \multimap B} \star$$

$$(13) \ A \rightarrow B \vdash \neg B \rightarrow \neg A$$

LJ (47ms)

$$\frac{\frac{\overline{A, B, B \rightarrow \perp \vdash A} \star \quad \frac{\overline{A, B, B \rightarrow \perp \vdash B} \star \quad \overline{A, B, \perp \vdash \perp} \star}{A, B, B \rightarrow \perp \vdash \perp} \supset_L}{A \rightarrow B \vdash B \rightarrow \perp \rightarrow A \rightarrow \perp} \star$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\cdot : A \vdash A \quad \frac{\cdot : B \vdash B \quad \cdot : \perp \vdash \perp}{\cdot : B, B \multimap \perp \vdash \perp} \multimap}{\cdot : A, A \multimap B, B \multimap \perp \vdash \perp} \multimap}{\cdot : A \multimap B \vdash B \multimap \perp \multimap A \multimap \perp} \star$$

CALL-BY-NAME encoding (143ms)

CALL-BY-VALUE encoding (174ms)

01-ENC encoding (178ms)

$$(14) \quad A \rightarrow \neg B \vdash \neg\neg B \rightarrow \neg A$$

LJ (64ms)

$$\frac{\frac{A, A \rightarrow B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A}^* \quad \frac{\frac{\frac{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^* \quad \frac{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^*}{\frac{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^*} \supset_k \quad \frac{\frac{A, A \rightarrow B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, A \rightarrow B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^* \quad \frac{\frac{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^* \quad \frac{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^*}{\frac{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}{A, B \rightarrow \perp, B \rightarrow \perp \rightarrow \perp}^*} \supset_k$$

MULTIPLICATIVE encoding (75ms)

$$\begin{array}{c}
\frac{\vdots : B \vdash B \quad \vdots : \perp \vdash \perp}{\vdots : A \vdash A} \multimap \\
\frac{\vdots : A, B, A \multimap B \multimap \perp \vdash \perp}{\vdots : A, A \multimap B \multimap \perp \vdash B \multimap \perp} \multimap \\
\frac{\vdots : A, A \multimap B \multimap \perp, B \multimap \perp \multimap \perp \vdash \perp}{\vdots : A \multimap B \multimap \perp \vdash B \multimap \perp \multimap \perp \multimap A \multimap \perp} \multimap
\end{array}$$

CALL-BY-NAME encoding (180ms)

CALL-BY-VALUE encoding (217ms)

Figure 1 is a phylogenetic tree showing the relationships between 10 isolates (1-10) based on 16S rDNA sequences. The tree is rooted at the bottom left. The isolates are numbered 1 through 10. The tree shows several clusters of isolates, with some isolates showing high similarity to each other. Bootstrap values are indicated at the nodes.

01-ENC encoding (226ms)

[illegible]

$$(15) \quad A \rightarrow B, B \rightarrow A \vdash (A) \leftrightarrow (B)$$

LJ (51ms)

$$\frac{\frac{\overline{A, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \overline{A, B, B \rightarrow A \vdash B}^*}{A, A \rightarrow B, B \rightarrow A \vdash B} \supset_L \quad \frac{\overline{B, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \overline{A, B, A \rightarrow B \vdash A}^*}{B, A \rightarrow B, B \rightarrow A \vdash A} \supset_L}{A \rightarrow B, B \rightarrow A \vdash A \rightarrow B \wedge B \rightarrow A}^*$$

MULTIPLICATIVE encoding (75ms)

$$\frac{\frac{\frac{\vdots : A \vdash A \quad \vdots : B \vdash B}{\vdots : A, A \multimap B \vdash B} \multimap \quad \frac{\frac{\vdots : B \vdash B \quad \vdots : A \vdash A}{\vdots : B, B \multimap A \vdash A} \multimap}{\vdots : A \multimap B \vdash A \multimap B} \star \quad \frac{\vdots : B \multimap A \vdash B \multimap A}{\vdots : B \multimap A \vdash B \multimap A} \star}{\vdots : A \multimap B, B \multimap A \vdash A \multimap B \otimes B \multimap A} \otimes$$

CALL-BY-NAME encoding (138ms)

[illegible]

CALL-BY-VALUE encoding (205ms)

01-ENC encoding (191ms)

[illegible]



$$(16) (A) \leftrightarrow (B) \vdash A \rightarrow B$$

LJ (37ms)

$$\frac{\overline{A, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \overline{A, B, B \rightarrow A \vdash B}^*}{\frac{A, A \rightarrow B, B \rightarrow A \vdash B}{A \rightarrow B \wedge B \rightarrow A \vdash A \rightarrow B}^*} \supset_L$$

MULTIPLICATIVE encoding (29ms)

**Not provable**

CALL-BY-NAME encoding (102ms)

$$\frac{\overline{A, !!(A) \multimap B, !(B) \multimap A : \cdot \vdash A} \quad \overline{A, !(A) \multimap B, !(B) \multimap A : B \vdash B}}{\frac{A, !(A) \multimap B, !(B) \multimap A : !(A) \multimap B \vdash B}{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash B} D_C} \multimap$$

$$\vdash : !(!(A) \multimap B \& !(B) \multimap A) \vdash !(A) \multimap B^*$$

CALL-BY-VALUE encoding (132ms)

$$\frac{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A} \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B} \quad \overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash !(B)}^*}{\frac{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash !(B)}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} D_C} \multimap$$

$$\frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A) \multimap !(B)}{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(A) \multimap !(B))}^*$$

$$\vdash : !(!(A) \multimap !(B)) \otimes !(!(B) \multimap !(A)) \vdash !(!(A) \multimap !(B))^*$$

01-ENC encoding (126ms)

$$\frac{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A} \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B}^*}{\frac{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A) \quad \overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \vdash B}}{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash B} D_C} \multimap$$

$$\frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A) \multimap B}{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(A) \multimap B)}^*$$

$$\vdash : !(!(!(A) \multimap !(B)) \& !(!(B) \multimap !(A))) \vdash !(!(A) \multimap B)^*$$

$$(17) (A) \leftrightarrow (B) \vdash B \rightarrow A$$

LJ (37ms)

$$\frac{\overline{B, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \overline{A, B, A \rightarrow B \vdash A}^*}{\frac{B, A \rightarrow B, B \rightarrow A \vdash A}{A \rightarrow B \wedge B \rightarrow A \vdash B \rightarrow A}^*} \supset_L$$

MULTIPLICATIVE encoding (29ms)

**Not provable**

CALL-BY-NAME encoding (105ms)

$$\frac{\overline{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash B} \quad \overline{B, !(A) \multimap B, !(B) \multimap A : A \vdash A}}{\frac{B, !(A) \multimap B, !(B) \multimap A : !(B) \multimap A \vdash A}{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash A} D_C} \multimap$$

$$\vdash : !(!(A) \multimap B \& !(B) \multimap A) \vdash !(B) \multimap A^*$$

CALL-BY-VALUE encoding (140ms)

$$\frac{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B} \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A} \quad \overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) : !(A) \vdash !(A)}^*}{\frac{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) \vdash !(A)}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)} D_C} \multimap$$

$$\frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A) \multimap B}{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(A) \multimap B)}^*$$

$$\vdash : !(!(A) \multimap !(B)) \otimes !(!(B) \multimap !(A)) \vdash !(!(B) \multimap A)^*$$

01-ENC encoding (122ms)

$$\frac{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B} \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A}^*}{\frac{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B) \quad \overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \vdash A}}{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) \vdash A} D_C} \multimap$$

$$\frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B) \multimap A}{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(B) \multimap A)}^*$$

$$\vdash : !(!(!(A) \multimap !(B)) \& !(!(B) \multimap !(A))) \vdash !(!(B) \multimap A)^*$$

$$(18) (A) \leftrightarrow (B), A \vdash B$$

LJ (38ms)

$$\frac{\overline{A, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \overline{A, B, B \rightarrow A \vdash B}^*}{\overline{A, A \rightarrow B, B \rightarrow A \vdash B}^*} \supset_L$$

$$\frac{}{A, A \rightarrow B \wedge B \rightarrow A \vdash B}^*$$

MULTIPLICATIVE encoding (29ms)

**Not provable**

CALL-BY-NAME encoding (102ms)

$$\frac{\overline{A, !!(A) \multimap B, !(B) \multimap A : \cdot \vdash A}^*}{\overline{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash !(A)}^*} \quad \frac{}{A, !(A) \multimap B, !(B) \multimap A : B \vdash B}^* \multimap$$

$$\frac{\overline{A, !(A) \multimap B, !(B) \multimap A : !(A) \multimap B \vdash B}^*}{\overline{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash B}^*} D_C$$

$$\vdash : !(A), !(!(A) \multimap B \& !(B) \multimap A) \vdash B^*$$

CALL-BY-VALUE encoding (119ms)

$$\frac{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A}^* \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B}^*}{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)}^*} \quad \frac{}{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \vdash !(B)}^* \multimap$$

$$\frac{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash !(B)}^*}{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)}^*} D_C$$

$$\vdash : !(A), !(!(A) \multimap !(B)) \otimes !(!(B) \multimap !(A)) \vdash !(B)^*$$

01-ENC encoding (111ms)

$$\frac{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A}^* \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B}^*}{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)}^*} \quad \frac{}{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \vdash B}^* \multimap$$

$$\frac{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash B}^*}{\overline{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B}^*} D_C$$

$$\vdash : !(A), !(!(A) \multimap !(B)) \& !(!(B) \multimap !(A)) \vdash B^*$$

$$(19) (A) \leftrightarrow (B), B \vdash A$$

LJ (38ms)

$$\frac{\overline{B, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \overline{A, B, A \rightarrow B \vdash A}^*}{\overline{B, A \rightarrow B, B \rightarrow A \vdash A}^*} \supset_L$$

$$\frac{}{B, A \rightarrow B \wedge B \rightarrow A \vdash A}^*$$

MULTIPLICATIVE encoding (32ms)

**Not provable**

CALL-BY-NAME encoding (102ms)

$$\frac{\overline{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash B}^*}{\overline{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash !(B)}^*} \quad \frac{}{B, !(A) \multimap B, !(B) \multimap A : A \vdash A}^* \multimap$$

$$\frac{\overline{B, !(A) \multimap B, !(B) \multimap A : !(B) \multimap A \vdash A}^*}{\overline{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash A}^*} D_C$$

$$\vdash : !(B), !(!(A) \multimap B \& !(B) \multimap A) \vdash A^*$$

CALL-BY-VALUE encoding (118ms)

$$\frac{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B}^* \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A}^*}{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)}^*} \quad \frac{}{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \vdash !(A)}^* \multimap$$

$$\frac{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) \vdash !(A)}^*}{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)}^*} D_C$$

$$\vdash : !(B), !(!(A) \multimap !(B)) \otimes !(!(B) \multimap !(A)) \vdash !(A)^*$$

01-ENC encoding (108ms)

$$\frac{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B}^* \quad \overline{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A}^*}{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)}^*} \quad \frac{}{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \vdash A}^* \multimap$$

$$\frac{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) \vdash A}^*}{\overline{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A}^*} D_C$$

$$\vdash : !(B), !(!(A) \multimap !(B)) \& !(!(B) \multimap !(A)) \vdash A^*$$

$$(20) \cdot \vdash (A) \leftrightarrow (A)$$

LJ (21ms)

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$$\cdot \vdash A \rightarrow A \wedge A \rightarrow A \quad \star$$

MULTIPLICATIVE encoding (34ms)

$$\frac{\frac{\overline{\cdot \vdash A \multimap A}}{\cdot \vdash \cdot \vdash A \multimap A} \star \frac{\overline{\cdot \vdash A \multimap A}}{\cdot \vdash \cdot \vdash A \multimap A} \star}{\cdot \vdash \cdot \vdash A \multimap A \otimes A \multimap A} \otimes$$

CALL-BY-NAME encoding (27ms)

$$\frac{\overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A}}{\cdot : \cdot \vdash !(A) \multimap A \& !(A) \multimap A} \star$$

CALL-BY-VALUE encoding (47ms)

$$\frac{\frac{\frac{A : \cdot \vdash A}{A : \cdot \vdash !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} ! \quad \star \quad \frac{\frac{\frac{A : \cdot \vdash A}{A : \cdot \vdash !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} ! \quad \star \quad \frac{\frac{\frac{A : \cdot \vdash A}{A : \cdot \vdash !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} ! \quad \otimes \quad \frac{\frac{\frac{A : \cdot \vdash A}{A : \cdot \vdash !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} !}{\cdot : \cdot \vdash !(A) \multimap !(A)} ! \quad \otimes$$

01-ENC encoding (47ms)

$$\frac{\frac{\overline{A} : \cdot \vdash \overline{A}}{\cdot : \cdot \vdash !(A) \multimap A}^* \quad \frac{\overline{A} : \cdot \vdash \overline{A}}{\cdot : \cdot \vdash !(A) \multimap A}^*}{\frac{\cdot : \cdot \vdash !(A) \multimap A}{\cdot : \cdot \vdash !(A) \multimap A}!}^* \quad \frac{\cdot : \cdot \vdash !(A) \multimap A}{\cdot : \cdot \vdash !(A) \multimap A}!}{\frac{\cdot : \cdot \vdash !(A) \multimap A \& !(A) \multimap A}{\cdot : \cdot \vdash !(A) \multimap A \& !(A) \multimap A}!}^*$$

(21)  $(A) \leftrightarrow (B) \vdash (B) \leftrightarrow (A)$

LJ (50ms)

$$\frac{\frac{B, A \rightarrow B, B \rightarrow A \vdash B}{B, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \frac{A, B, A \rightarrow B \vdash A}{A, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \frac{A, A \rightarrow B, B \rightarrow A \vdash A}{A, A \rightarrow B, B \rightarrow A \vdash A \vdash A}^* \quad \frac{A, B, B \rightarrow A \vdash B}{A, A \rightarrow B, B \rightarrow A \vdash B}^*}{A, A \rightarrow B, B \rightarrow A \vdash A \vdash A} \supset_L \quad \frac{A, A \rightarrow B, B \rightarrow A \vdash A \vdash A}{A, A \rightarrow B, B \rightarrow A \vdash A \vdash B} \supset_L \quad \frac{A, A \rightarrow B, B \rightarrow A \vdash A \vdash B}{A \rightarrow B \wedge B \rightarrow A \vdash B \rightarrow A \wedge A \rightarrow B}^*$$

MULTIPLICATIVE encoding (81ms)

$$\frac{\frac{\frac{\vdots : B \vdash B}{\vdots : B, B \multimap A \vdash A} \multimap \quad \frac{\vdots : A \vdash A}{\vdots : A, A \multimap B \vdash B} \multimap}{\vdots : B \multimap A \vdash B \multimap A} \star \quad \frac{\frac{\vdots : A \vdash A}{\vdots : A \multimap B \vdash A \multimap B} \multimap \quad \frac{\vdots : B \vdash B}{\vdots : A \multimap B \vdash A \multimap B} \multimap}{\vdots : A \multimap B \vdash A \multimap B} \otimes}{\vdots : A \multimap B \otimes B \multimap A \vdash B \multimap A \otimes A \multimap B} \star$$

CALL-BY-NAME encoding (138ms)

[illegible]

CALL-BY-VALUE encoding (203ms)

[illegible]

01-ENC encoding (188ms)

[illegible]

$$(22) (A \leftrightarrow (B), (B \leftrightarrow (C) \vdash (A \leftrightarrow (C))$$

LJ (75ms)

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MULTIPLICATIVE encoding (115ms)

$$\frac{\frac{\frac{\vdash B \vdash B \quad \vdash C \vdash C}{\vdash A \vdash A} \multimap \quad \frac{\frac{\vdash B \vdash B \quad \vdash A \vdash A}{\vdash C \vdash C} \multimap}{\vdash A, A \multimap B, B \multimap C \vdash C} \multimap \quad \frac{\frac{\vdash C \vdash C \quad \vdash B, B \multimap A \vdash A}{\vdash C, B \multimap A, C \multimap B \vdash A} \multimap}{\vdash A \multimap B, B \multimap C \vdash A \multimap C} * \quad \frac{\vdash B \multimap A, C \multimap B \vdash A}{\vdash A \multimap B, B \multimap C \vdash A \multimap C} \otimes}{\vdash A \multimap B, B \multimap A, B \multimap C, C \multimap B \vdash A \multimap C \otimes C \multimap C \multimap A} * \quad \vdash A \multimap B \otimes B \multimap A, B \multimap C \otimes C \multimap B \vdash A \multimap C \otimes C \multimap A} *$$

CALL-BY-NAME encoding (218ms)

=====

CALL-BY-VALUE encoding (331ms)

=====

01-ENC encoding (291ms)

=====

$$(23) A \rightarrow B \rightarrow C, \neg\neg A, \neg\neg B \vdash \neg\neg C$$

LJ (206ms)

=====

MULTIPLICATIVE encoding (124ms)

$$\frac{\frac{\frac{\vdash C \vdash C \quad \vdash \bot \vdash \bot}{\vdash B \vdash B} \multimap \quad \frac{\vdash C \multimap \bot \vdash \bot}{\vdash C, C \multimap \bot \vdash \bot} \multimap}{\vdash A \vdash A} \multimap \quad \frac{\frac{\vdash B, B \multimap C, C \multimap \bot \vdash \bot}{\vdash A, B, A \multimap B \multimap C, C \multimap \bot \vdash \bot} \multimap}{\vdash A, A \multimap B \multimap C, C \multimap \bot \vdash B \multimap \bot} * \quad \vdash \bot \vdash \bot}{\vdash A, A \multimap B \multimap C, C \multimap \bot, B \multimap \bot \multimap \bot \vdash \bot} \multimap \quad \frac{\vdash A \multimap B \multimap C, C \multimap \bot, A \multimap \bot \multimap \bot \vdash A \multimap \bot} * \quad \vdash \bot \vdash \bot}{\vdash A \multimap B \multimap C, C \multimap \bot, A \multimap \bot \multimap \bot, B \multimap \bot \multimap \bot \vdash \bot} \multimap \quad \vdash A \multimap B \multimap C, A \multimap \bot \multimap \bot, B \multimap \bot \multimap \bot \vdash C \multimap \bot \multimap \bot} *$$

CALL-BY-NAME encoding (221ms)

=====

CALL-BY-VALUE encoding (324ms)

=====

01-ENC encoding (380ms)

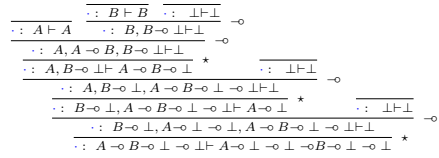
=====

$$(24) \neg\neg A \rightarrow B \vdash \neg\neg A \rightarrow \neg\neg B$$

LJ (94ms)



MULTIPLICATIVE encoding (106ms)



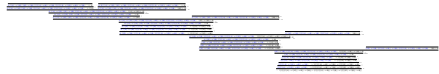
CALL-BY-NAME encoding (199ms)



CALL-BY-VALUE encoding (241ms)



01-ENC encoding (269ms)

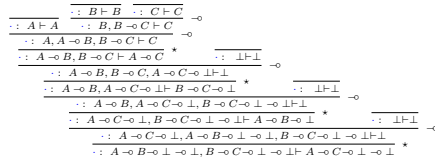


(25)  $\neg\neg A \rightarrow B, \neg\neg B \rightarrow C \vdash \neg\neg A \rightarrow C$

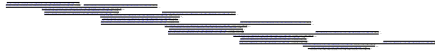
LJ (324ms)



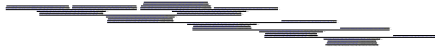
MULTIPLICATIVE encoding (145ms)



CALL-BY-NAME encoding (283ms)



CALL-BY-VALUE encoding (375ms)



01-ENC encoding (421ms)



$$(26) \cdot \vdash (\neg\neg A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$$

LJ (137ms)



MULTIPLICATIVE encoding (66ms)

**Not provable**

CALL-BY-NAME encoding (601ms)



CALL-BY-VALUE encoding (584ms)



01-ENC encoding (646ms)



$$(27) \cdot \vdash (\neg\neg(A) \leftrightarrow (B)) \leftrightarrow (\neg\neg A \rightarrow B \wedge \neg\neg B \rightarrow A)$$

LJ (2703ms)



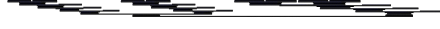
MULTIPLICATIVE encoding (84ms)

**Not provable**

CALL-BY-NAME encoding (661ms)



CALL-BY-VALUE encoding (841ms)



01-ENC encoding (1132ms)

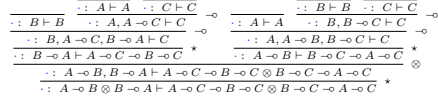


$$(28) (A) \leftrightarrow (B) \vdash (A \rightarrow C) \leftrightarrow (B \rightarrow C)$$

LJ (75ms)



MULTIPLICATIVE encoding (95ms)



CALL-BY-NAME encoding (173ms)



CALL-BY-VALUE encoding (288ms)



01-ENC encoding (268ms)

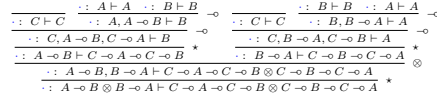


$$(29) (A) \leftrightarrow (B) \vdash (C \rightarrow A) \leftrightarrow (C \rightarrow B)$$

LJ (72ms)



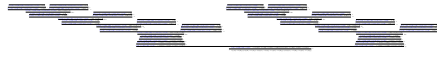
MULTIPLICATIVE encoding (96ms)



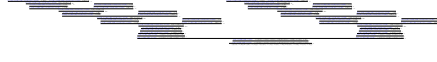
CALL-BY-NAME encoding (250ms)



CALL-BY-VALUE encoding (387ms)



01-ENC encoding (356ms)



$$(30) \quad (A) \leftrightarrow (B) \vdash (A \wedge C) \leftrightarrow (B \wedge C)$$

LJ (51ms)

$$\frac{\frac{\overline{A, C, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \overline{A, B, C, B \rightarrow A \vdash B}^*}{A, C, A \rightarrow B, B \rightarrow A \vdash B} \supset_L \quad \frac{\overline{B, C, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \overline{A, B, C, A \rightarrow B \vdash A}^*}{B, C, A \rightarrow B, B \rightarrow A \vdash A} \supset_L}{A \rightarrow B \wedge B \rightarrow A \vdash A \wedge C \rightarrow B \wedge C \wedge B \wedge C \rightarrow A \wedge C}^*$$

MULTIPLICATIVE encoding (99ms)

[illegible]

CALL-BY-NAME encoding (138ms)

$\frac{A_1 \cup A_2 \cup \dots \cup A_n \cup A_{n+1} \cup \dots \cup A_{n+m}}{A_1 \cup A_2 \cup \dots \cup A_n \cup A_{n+1} \cup \dots \cup A_{n+m}} = \frac{A_1 \cup A_2 \cup \dots \cup A_n \cup A_{n+1} \cup \dots \cup A_{n+m}}{A_1 \cup A_2 \cup \dots \cup A_n \cup A_{n+1} \cup \dots \cup A_{n+m}}$

CALL-BY-VALUE encoding (238ms)

Figure 1. The structure of the proposed model. The model is divided into three main parts: the input layer, the hidden layer, and the output layer. The input layer consists of a set of input variables  $X_1, X_2, \dots, X_n$ . The hidden layer consists of a set of hidden variables  $h_1, h_2, \dots, h_m$ . The output layer consists of a set of output variables  $Y_1, Y_2, \dots, Y_p$ . The model is trained using a set of training data  $D$  and a set of test data  $T$ . The model is evaluated using a set of evaluation metrics  $M$ .

01-ENC encoding (234ms)

[illegible]

(31)  $(A) \leftrightarrow (B) \vdash (C \wedge A) \leftrightarrow (C \wedge B)$

LJ (51ms)

$$\frac{\frac{\overline{A, C, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \overline{A, B, C, B \rightarrow A \vdash B}^*}{A, C, A \rightarrow B, B \rightarrow A \vdash B} \supset_L \quad \frac{\overline{B, C, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \overline{A, B, C, A \rightarrow B \vdash A}^*}{B, C, A \rightarrow B, B \rightarrow A \vdash A} \supset_L}{A \rightarrow B \wedge B \rightarrow A \vdash A \vdash C \wedge A \rightarrow C \wedge B \wedge C \wedge B \rightarrow C \wedge A}^*$$

MULTIPLICATIVE encoding (98ms)

[illegible]

CALL-BY-NAME encoding (142ms)

[illegible]

CALL-BY-VALUE encoding (227ms)

[illegible]

01-ENC encoding (229ms)

[illegible]



$$(32) \quad (A) \leftrightarrow (B) \vdash (\neg A) \leftrightarrow (\neg B)$$

LJ (74ms)

$$\frac{\overbrace{A \vee B, A \vee C, B \vee C}^{\text{S}} \quad \overbrace{A \vee B, A \vee C, B \vee C}^{\text{S}}}{\overbrace{A \vee B, A \vee C, B \vee C}^{\text{S}}} \text{S}_4 \quad \frac{\overbrace{A \vee B, A \vee C, B \vee C}^{\text{S}} \quad \overbrace{A \vee B, A \vee C, B \vee C}^{\text{S}}}{\overbrace{A \vee B, A \vee C, B \vee C}^{\text{S}}} \text{S}_4$$

MULTIPLICATIVE encoding (97ms)

[illegible]

CALL-BY-NAME encoding (223ms)

CALL-BY-VALUE encoding (304ms)

01-ENC encoding (320ms)

$$(33) \cdot \vdash (A \wedge B \wedge C) \leftrightarrow (A \wedge B \wedge C)$$

LJ (21ms)

$$\overline{\cdot \vdash A \wedge B \wedge C \rightarrow A \wedge B \wedge C \wedge A \wedge B \wedge C \rightarrow A \wedge B \wedge C}^*$$

MULTIPLICATIVE encoding (66ms)

$$\frac{\frac{\frac{\vdots}{\vdots} A \vdash A \quad \frac{\vdots}{\vdots} B \vdash B}{\vdots, A, B \vdash A \otimes B} \otimes \quad \frac{\vdots}{\vdots} C \vdash C}{\vdots, A, B, C \vdash A \otimes B \otimes C} \otimes \quad \frac{\frac{\frac{\vdots}{\vdots} A \vdash A \quad \frac{\vdots}{\vdots} B \vdash B}{\vdots, A, B \vdash A \otimes B} \otimes \quad \frac{\vdots}{\vdots} C \vdash C}{\vdots, A, B, C \vdash A \otimes B \otimes C} \otimes}{\vdots, \vdash A \otimes B \otimes C \multimap A \otimes B \otimes C} \star \quad \frac{\frac{\frac{\frac{\vdots}{\vdots} A \vdash A \quad \frac{\vdots}{\vdots} B \vdash B}{\vdots, A, B \vdash A \otimes B} \otimes \quad \frac{\vdots}{\vdots} C \vdash C}{\vdots, A, B, C \vdash A \otimes B \otimes C} \otimes \quad \frac{\vdots}{\vdots} A \otimes B \otimes C \multimap A \otimes B \otimes C}{\vdots, \vdash A \otimes B \otimes C \multimap A \otimes B \otimes C} \star}{\vdots, \vdash A \otimes B \otimes C \multimap A \otimes B \otimes C \multimap A \otimes B \otimes C \multimap A \otimes B \otimes C} \otimes$$

CALL-BY-NAME encoding (34ms)

$$\frac{\overline{A, B, C : \vdash A} \quad \overline{A, B, C : \vdash B} \quad \overline{A, B, C : \vdash C} \quad \overline{A, B, C : \vdash A} \quad \overline{A, B, C : \vdash B} \quad \overline{A, B, C : \vdash C}}{\vdash : \vdash !(A \& B \& C) \multimap A \& B \& C \& !(A \& B \& C) \multimap A \& B \& C} *$$

CALL-BY-VALUE encoding (99ms)

[illegible]

01-ENC encoding (124ms)

[illegible]

$$(34) \cdot \vdash (A \wedge B) \leftrightarrow (B \wedge A)$$

LJ (21ms)

$$\overline{\cdot \vdash A \wedge B \rightarrow B \wedge A \wedge B \wedge A \rightarrow A \wedge B}^{\star}$$

MULTIPLICATIVE encoding (73ms)

$$\frac{\frac{\frac{\vdots : B \vdash B \quad \vdots : A \vdash A}{\vdots : A, B \vdash B \otimes A} \otimes \quad \frac{\frac{\vdots : A \vdash A \quad \vdots : B \vdash B}{\vdots : A, B \vdash A \otimes B} \otimes}{\vdots : \vdash A \otimes B \multimap B \otimes A} * \quad \frac{\frac{\vdots : \vdash B \otimes A \multimap A \otimes B}{} *}{\vdots : \vdash A \otimes B \multimap B \otimes A \otimes B \otimes A \multimap A \otimes B} \otimes$$

CALL-BY-NAME encoding (35ms)

$$\frac{A, B : \cdot \vdash B \quad A, B : \cdot \vdash A \quad A, B : \cdot \vdash A \quad A, B : \cdot \vdash B}{\cdot : \cdot \vdash !(A \& B) \multimap B \& A \& !(B \& A) \multimap A \& B} \star$$

CALL-BY-VALUE encoding (107ms)

$$\begin{array}{c}
\frac{A, B : \vdash \neg B}{A, B : \vdash \neg ! (B)} \quad \frac{A, B : \vdash \neg A}{A, B : \vdash \neg ! (A)} \quad \frac{A, B : \vdash \neg A}{A, B : \vdash \neg ! (A)} \quad \frac{A, B : \vdash \neg B}{A, B : \vdash \neg ! (B)} \\
\frac{A, B : \vdash ! (B) \otimes ! (A)}{\vdash ! (A) \otimes ! (B) \multimap ! (B) \otimes ! (A)} \quad \frac{A, B : \vdash ! (A) \otimes ! (B)}{\vdash ! (B) \otimes ! (A) \multimap ! (A) \otimes ! (B)} \\
\vdash ! (A) \otimes ! (B) \multimap ! (B) \otimes ! (A) \quad \vdash ! (B) \otimes ! (A) \multimap ! (A) \otimes ! (B) \\
\vdash ! (A) \otimes ! (B) \multimap ! (B) \otimes ! (A) \quad \vdash ! (B) \otimes ! (A) \multimap ! (A) \otimes ! (B) \\
\vdash ! (A) \otimes ! (B) \multimap ! (B) \otimes ! (A) \quad \vdash ! (B) \otimes ! (A) \multimap ! (A) \otimes ! (B)
\end{array}$$

01-ENC encoding (102ms)

[illegible]

$$(35) \cdot \vdash (A \wedge A) \leftrightarrow (A)$$

LJ (21ms)

$$\cdot \vdash A \wedge A \rightarrow A \wedge A \rightarrow A \wedge A \quad \star$$

MULTIPLICATIVE encoding (34ms)

Not provable

CALL-BY-NAME encoding (28ms)

$$\frac{\overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A}}{\cdot : \cdot \vdash !(A \& A) \multimap A \& !(A) \multimap A \& A} \star$$

CALL-BY-VALUE encoding (76ms)

$$\frac{\frac{\overline{A : \vdash A}}{A : \vdash ! (A)} !}{\vdots, \vdash ! (A) \otimes ! (A) \multimap ! (A)} * \quad \frac{\frac{\overline{A : \vdash A} ! \quad \overline{A : \vdash A} !}{A : \vdash ! (A) \otimes ! (A)} !}{\vdots, \vdash ! (A) \multimap ! (A) \otimes ! (A)} * \quad \frac{\vdots, \vdash ! (! (A) \otimes ! (A) \multimap ! (A)) !}{\vdots, \vdash ! (! (A) \otimes ! (A) \multimap ! (A)) \otimes ! (! (A) \multimap ! (A) \otimes ! (A))} !$$

01-ENC encoding (81ms)

[illegible]

$$(36) A \vdash (A \rightarrow B) \leftrightarrow (B)$$

LJ (37ms)

$$\frac{\frac{\overline{A, A \rightarrow B \vdash A} \star \overline{A, B \vdash B}}{A, A \rightarrow B \vdash B} \supset_L}{A \vdash A \rightarrow B \rightarrow B \wedge B \rightarrow A \rightarrow B} \star$$

MULTIPLICATIVE encoding (42ms)

**Not provable**

CALL-BY-NAME encoding (73ms)

$$\frac{\frac{\overline{A, ! (A) \multimap B : \cdot \vdash A}}{A, ! (A) \multimap B : \cdot \vdash ! (A)} ! \quad \frac{\overline{A, ! (A) \multimap B : B \vdash B}}{A, ! (A) \multimap B : ! (A) \multimap B \vdash B} \multimap}{\frac{A, ! (A) \multimap B : ! (A) \multimap B \vdash B}{\cdot : ! (A) \vdash ! ( ! (A) \multimap B) \multimap B \& ! (B) \multimap ! (A) \multimap B} D_C} \star$$

CALL-BY-VALUE encoding (129ms)

$$\frac{\frac{\frac{\overline{A, ! (A) \multimap ! (B) : \cdot \vdash A}}{A, ! (A) \multimap ! (B) : \cdot \vdash ! (A)} ! \quad \frac{\overline{A, B, ! (A) \multimap ! (B) : \cdot \vdash ! (B)} !}{\frac{A, ! (A) \multimap ! (B) : ! (A) \multimap ! (B) \vdash ! (B)}{A, ! (A) \multimap ! (B) : \cdot \vdash ! (B)} D_C} \star \quad \frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash ! (A) \multimap ! (B)} !}{\frac{\frac{A, ! (A) \multimap ! (B) : \cdot \vdash ! (B)}{A : \cdot \vdash ! ( ! (A) \multimap ! (B)) \multimap ! (B)} \star \quad \frac{\overline{A, B : \cdot \vdash ! (A) \multimap ! (B)} !}{A : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B))} !}{\frac{A : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B)) )}{\cdot : ! (A) \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B)) )} \otimes} \star$$

01-ENC encoding (123ms)

$$\frac{\frac{\frac{\overline{A, ! (A) \multimap ! (B) : \cdot \vdash A}}{A, ! (A) \multimap ! (B) : \cdot \vdash ! (A)} ! \quad \frac{\overline{A, B, ! (A) \multimap ! (B) : \cdot \vdash B}}{A, ! (A) \multimap ! (B) : ! (A) \multimap ! (B) \vdash B} \star}{\frac{A, ! (A) \multimap ! (B) : ! (A) \multimap ! (B) \vdash B}{A, ! (A) \multimap ! (B) : \cdot \vdash B} D_C} \star \quad \frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash ! (A) \multimap B} !}{\frac{\frac{A, ! (A) \multimap ! (B) : \cdot \vdash B}{A : \cdot \vdash ! ( ! (A) \multimap ! (B)) \multimap B} \star \quad \frac{\overline{A, B : \cdot \vdash ! (A) \multimap B}}{A : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap B)} !}{\frac{A : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B) )}{A : \cdot \vdash ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B) )} \star} \star$$

$$(37) B \vdash (A \rightarrow B) \leftrightarrow (B)$$

LJ (21ms)

$$\overline{B \vdash A \rightarrow B \rightarrow B \wedge B \rightarrow A \rightarrow B} \star$$

MULTIPLICATIVE encoding (40ms)

**Not provable**

CALL-BY-NAME encoding (39ms)

$$\frac{\overline{B, ! (A) \multimap B : \cdot \vdash B} \quad \overline{A, B : \cdot \vdash B}}{\cdot : ! (B) \vdash ! ( ! (A) \multimap B) \multimap B \& ! (B) \multimap ! (A) \multimap B} \star$$

CALL-BY-VALUE encoding (152ms)

$$\frac{\frac{\overline{B, ! (A) \multimap ! (B) : \cdot \vdash B}}{B : \cdot \vdash ! ( ! (A) \multimap ! (B)) \multimap ! (B)} ! \quad \frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash ! (A) \multimap ! (B)} !}{\frac{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \star \quad \frac{\overline{B : \cdot \vdash ! (A) \multimap ! (B)} !}{B : \cdot \vdash ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B)) )} !}{\frac{B : \cdot \vdash ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B)) )}{\cdot : ! (B) \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B)) )} \otimes} \star$$

01-ENC encoding (88ms)

$$\frac{\frac{\overline{B, ! (A) \multimap ! (B) : \cdot \vdash B}}{B : \cdot \vdash ! ( ! (A) \multimap ! (B)) \multimap B} \star \quad \frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash ! (A) \multimap B} !}{\frac{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \star \quad \frac{\overline{B : \cdot \vdash ! (B) \multimap ! ( ! (A) \multimap B) } !}{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap B) } !}{\frac{B : \cdot \vdash ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B) )}{B : \cdot \vdash ! ( ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B) )} \star} \star$$

$$(38) \neg A \vdash (A \rightarrow B) \leftrightarrow (\neg A)$$

LJ (52ms)

$$\frac{\frac{\overline{A, A \rightarrow B, A \rightarrow \perp \vdash A}^* \quad \overline{A, \perp, A \rightarrow B \vdash \perp}^*}{A, A \rightarrow B, A \rightarrow \perp \vdash \perp} \supset_L \quad \frac{\overline{A, A \rightarrow \perp \vdash A}^* \quad \overline{A, \perp \vdash B}^*}{A, A \rightarrow \perp \vdash B} \supset_L}{A \rightarrow \perp \vdash A \rightarrow B \rightarrow A \rightarrow \perp \wedge A \rightarrow \perp \rightarrow A \rightarrow B}^*$$

MULTIPLICATIVE encoding (42ms)

Not provable

CALL-BY-NAME encoding (101ms)

[illegible]

CALL-BY-VALUE encoding (201ms)

[illegible]

01-ENC encoding (203ms)

[illegible]

$$(39) \neg B \vdash (A \rightarrow B) \leftrightarrow (\neg A)$$

LJ (61ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} *}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} *}{\frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp} *}{B \rightarrow \perp \vdash A \rightarrow B \rightarrow A \rightarrow \perp \wedge A \rightarrow \perp \rightarrow A \rightarrow B} * \circ_L \circ_L \circ_L$$

MULTIPLICATIVE encoding (43ms)

Not provable

CALL-BY-NAME encoding (185ms)

CALL-BY-VALUE encoding (256ms)

[illegible]

01-ENC encoding (259ms)

Figure 1 illustrates a hierarchical tree diagram representing the classification of 1000 genes into 10 clusters. The tree structure shows the relationships between the genes and the clusters. The genes are listed in a column on the left, and the clusters are listed in a column on the right. The tree structure shows the relationships between the genes and the clusters.

$$(40) \quad B \vdash (A \wedge B) \leftrightarrow (A)$$

LJ (21ms)

$$\overline{B \vdash A \wedge B \rightarrow A \wedge A \rightarrow A \wedge B}^{\star}$$

MULTIPLICATIVE encoding (34ms)

Not provable

CALL-BY-NAME encoding (34ms)

$$\frac{\overline{A, B : \cdot \vdash A} \quad \overline{A, B : \cdot \vdash A} \quad \overline{A, B : \cdot \vdash B}}{\cdot : !(B) \vdash !(A \& B) \multimap A \& !(A) \multimap A \& B} \star$$

CALL-BY-VALUE encoding (88ms)

[illegible]

01-ENC encoding (88ms)

[illegible]

$$(41) \neg B \vdash (A \wedge B) \leftrightarrow (B)$$

LJ (37ms)

$$\frac{\frac{\overline{B, B \rightarrow \perp \vdash B}^* \quad \overline{B, \perp \vdash A}^*}{B, B \rightarrow \perp \vdash A} \supset_L}{B \rightarrow \perp \vdash A \wedge B \rightarrow B \wedge B \rightarrow A \wedge B}^*$$

MULTIPLICATIVE encoding (41ms)

Not provable

CALL-BY-NAME encoding (78ms)

$$\frac{\frac{\frac{B, \mathfrak{!}(B) \multimap 0 : \vdash B}{B, \mathfrak{!}(B) \multimap 0 : \vdash \mathfrak{!}(B)} \quad \frac{B, \mathfrak{!}(B) \multimap 0 : 0 \vdash A}{B, \mathfrak{!}(B) \multimap 0 : \mathfrak{!}(B) \multimap 0 \vdash A} \quad \frac{B, \mathfrak{!}(B) \multimap 0 : \mathfrak{!}(B) \multimap 0 \vdash A}{B, \mathfrak{!}(B) \multimap 0 : \vdash A} \quad \frac{A, B, \mathfrak{!}(B) \multimap 0 : \vdash B}{\vdash : \mathfrak{!}(B) \multimap 0 \vdash \mathfrak{!}(A \& B) \multimap B \& \mathfrak{!}(B) \multimap A \& B} \quad D_C}{\vdash : \mathfrak{!}(B) \multimap 0 : \vdash B} \quad *$$

CALL-BY-VALUE encoding (172ms)

[illegible]

01-ENC encoding (194ms)

[illegible]

$$(42) \cdot \vdash A \rightarrow \neg\neg A$$

LJ (38ms)

$$\frac{\frac{\frac{A, A \rightarrow \perp \vdash A}{\vdash A}^* \quad \frac{A, \perp \vdash \perp}{\vdash \perp}^*}{A, A \rightarrow \perp \vdash \perp} \quad \supset_L}{\vdash A \rightarrow A \rightarrow \perp \rightarrow \perp}^*$$

MULTIPLICATIVE encoding (41ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : A, A \multimap \perp \vdash \perp} \multimap}{\cdot : \cdot \vdash A \multimap A \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (66ms)

$$\frac{\frac{\frac{A, !(A) \multimap 0 : \cdot \vdash A}{A, !(A) \multimap 0 : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap 0 : 0 \vdash 0}{A, !(A) \multimap 0 : 0 \vdash 0} *}{\frac{A, !(A) \multimap 0 : !(A) \multimap 0 \vdash 0}{A, !(A) \multimap 0 : \cdot \vdash 0} D_C} \frac{\cdot : \cdot \vdash !(A) \multimap !(A) \multimap 0 \vdash 0}{\cdot : \cdot \vdash !(A) \multimap !(A) \multimap 0 \vdash 0} *$$

CALL-BY-VALUE encoding (84ms)

[illegible]

01-ENC encoding (81ms)

$$\begin{array}{c}
\frac{A, !(\mathbf{A}) \multimap !(\mathbf{0}) : \cdot \vdash A}{A, !(\mathbf{A}) \multimap !(\mathbf{0}) : \cdot \vdash !(\mathbf{A})} ! \quad \frac{A, !(\mathbf{A}) \multimap !(\mathbf{0}) : !(\mathbf{0}) \vdash \mathbf{0}}{\cdot} \star \\
\frac{A, !(\mathbf{A}) \multimap !(\mathbf{0}) : !(\mathbf{A}) \multimap !(\mathbf{0}) : \cdot \vdash !(\mathbf{A}) \multimap !(\mathbf{0}) \vdash \mathbf{0}}{A, !(\mathbf{A}) \multimap !(\mathbf{0}) : \cdot \vdash \mathbf{0}} D_C \\
\frac{A : \cdot \vdash !(!(\mathbf{A}) \multimap !(\mathbf{0})) \multimap \mathbf{0}}{A : \cdot \vdash !(!(!(\mathbf{A}) \multimap !(\mathbf{0})) \multimap \mathbf{0})} \star \\
\frac{\cdot : \cdot \vdash !(!(\mathbf{A}) \multimap !(!(!(\mathbf{A}) \multimap !(\mathbf{0})) \multimap \mathbf{0}))}{\cdot : \cdot \vdash !(!(\mathbf{A}) \multimap !(!(!(\mathbf{A}) \multimap !(\mathbf{0})) \multimap \mathbf{0}))} \star \\
\vdots : \cdot \vdash !(!(\mathbf{A}) \multimap !(!(!(\mathbf{A}) \multimap !(\mathbf{0})) \multimap \mathbf{0})) !
\end{array}$$

$$(43) \cdot \vdash (\neg\neg\neg A) \leftrightarrow (\neg A)$$

LJ (81ms)

[illegible]

MULTIPLICATIVE encoding (105ms)

[illegible]

CALL-BY-NAME encoding (244ms)

CALL-BY-VALUE encoding (295ms)

01-ENC encoding (343ms)

$$(44) \cdot \vdash \neg A \wedge \neg A$$

LJ (38ms)

$$\frac{\frac{\frac{A, A \rightarrow \perp \vdash A}{\vdash A \wedge A \rightarrow \perp \rightarrow \perp}^*}{A, A \rightarrow \perp \vdash \perp} \quad \frac{A, \perp \vdash \perp}{\vdash A \wedge A \rightarrow \perp \rightarrow \perp}^* \supset_L$$

MULTIPLICATIVE encoding (41ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : A, A \multimap \perp \vdash \perp} \multimap}{\cdot : \cdot \vdash A \otimes A \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (65ms)

$$\frac{\frac{\frac{A, !(A) \multimap \mathbf{0} : \vdash A}{A, !(A) \multimap \mathbf{0} : \vdash !(A)} ! \quad \frac{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash \mathbf{0}}{} *}{\frac{A, !(A) \multimap \mathbf{0} : !(A) \multimap \mathbf{0} \vdash \mathbf{0}}{A, !(A) \multimap \mathbf{0} : \vdash \mathbf{0}} D_C} \multimap$$

CALL-BY-VALUE encoding (70ms)

$$\frac{\frac{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash \mathbf{0}}{A, !(A) \multimap \mathbf{0} : \cdot \vdash \mathbf{0}} \star}{\frac{A, !(A) \multimap \mathbf{0} : \cdot \vdash \mathbf{0}}{A, !(A) \multimap \mathbf{0} : \cdot \vdash \mathbf{0}} D_C} \star$$

01-ENC encoding (70ms)

$$\frac{\frac{\frac{A, ! (A) \multimap ! (0) : \vdash A}{A, ! (A) \multimap ! (0) : \vdash ! (A)} ! \quad \frac{A, ! (A) \multimap ! (0) : ! (0) \vdash 0}{A, ! (A) \multimap ! (0) : ! (0) \vdash 0} \star}{\frac{A, ! (A) \multimap ! (0) : ! (A) \multimap ! (0) \vdash 0}{A, ! (A) \multimap ! (0) : \vdash 0} D_C} \star$$

$$\frac{\vdash : \vdash ! ( ! (A) \& ! ( ! (A) \multimap ! (0) ) ) \multimap 0}{\vdash : \vdash ! ( ! ( ! (A) \& ! ( ! (A) \multimap ! (0) ) ) \multimap 0 )} !$$

$$(45) \cdot \vdash \neg(A) \leftrightarrow (\neg A)$$

LJ (80ms)

$$\frac{\frac{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp \rightarrow \perp}{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp} \supset_2 \quad \frac{\frac{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp \rightarrow \perp}{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp} \supset_2 \quad \frac{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp}{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp} \supset_2}{\frac{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp \rightarrow \perp}{A \rightarrow A \rightarrow \perp \rightarrow \perp \rightarrow \perp} \supset_2} \supset_2$$

MULTIPLICATIVE encoding (41ms)

Not provable

CALL-BY-NAME encoding (231ms)

[illegible]

CALL-BY-VALUE encoding (282ms)

Phylogenetic tree of the 16S rDNA sequences of the 16 isolates. The tree shows relationships between various bacterial strains, with bootstrap values indicated at the nodes. The strains are grouped into several clusters, with some labeled as 'E. coli' and others as 'Shigella'.

01-ENC encoding (300ms)

Phylogenetic tree showing relationships between 12 species. The tree is rooted at the bottom and branches upwards. Bootstrap values are indicated at the nodes. The species names are listed on the right side of the tree.

$$(46) \cdot \vdash \neg\neg\neg\neg A \rightarrow A$$

LJ (66ms)

[illegible]

MULTIPLICATIVE encoding (41ms)

Not provable

CALL-BY-NAME encoding (198ms)

Figure 1: Schematic representation of the 1200 bp DNA fragment. The diagram shows a horizontal line representing the DNA sequence with various restriction enzyme sites marked by vertical bars and labels. The sites include NotI, XbaI, PstI, KpnI, SmaI, BamHI, EcoRI, HindIII, SalI, XhoI, and ClaI. The fragment is divided into several segments, with the largest segment being approximately 1000 bp. The restriction sites are distributed throughout the fragment, with some sites appearing multiple times.

CALL-BY-VALUE encoding (210ms)

[illegible]

01-ENC encoding (217ms)

[illegible]

$$(47) \cdot \vdash (A \wedge B \wedge \neg B) \leftrightarrow (B \wedge \neg B)$$

LJ (61ms)

$$\frac{\frac{\frac{A, B, B \rightarrow \perp \vdash B^*}{A, B, B \rightarrow \perp \vdash \perp}^* \supset_L \quad \frac{\frac{B, B \rightarrow \perp \vdash B^*}{B, B \rightarrow \perp \vdash A}^* \supset_L \quad \frac{\frac{B, B \rightarrow \perp \vdash B^*}{B, \perp \vdash \perp}^* \supset_L}{\vdash A \wedge B \wedge B \rightarrow \perp \rightarrow B \wedge B \rightarrow \perp \wedge B \wedge B \rightarrow \perp \rightarrow A \wedge B \wedge B \rightarrow \perp}^*}$$

MULTIPLICATIVE encoding (42ms)

Not provable

CALL-BY-NAME encoding (121ms)

[illegible]

CALL-BY-VALUE encoding (140ms)

[illegible]

01-ENC encoding (132ms)



$$(48) \cdot \vdash A \rightarrow B \rightarrow \neg A \wedge \neg B$$

LJ (49ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \perp A}{A, B, B \rightarrow \perp \perp A}^* \quad \frac{A, B, B \rightarrow \perp \perp B}{A, B, \perp \perp \perp}^*}{A, B, B \rightarrow \perp \perp \perp} \supset_L \quad \frac{A, B, \perp \perp \perp}{\vdash A \rightarrow B \rightarrow A \wedge B \rightarrow \perp \rightarrow \perp}^* \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\cdot \vdash A \multimap A}{\cdot \vdash A, A \multimap B, B \multimap \perp \perp \perp} \multimap}{\cdot \vdash \cdot \vdash A \multimap B \multimap A \otimes B \multimap \perp \multimap \perp} \star}{\cdot \vdash B \vdash B \quad \cdot \vdash \perp \perp \perp} \multimap$$

CALL-BY-NAME encoding (143ms)

[illegible]

CALL-BY-VALUE encoding (167ms)

The diagram illustrates the decomposition of a polynomial expression into linear and quadratic terms. The root expression is a polynomial in  $x, y, z, w, v$ . It branches into two main parts, each of which is further decomposed into linear and quadratic terms. The diagram uses color-coding: blue for linear terms and red for quadratic terms. The final result is a sum of several terms, each representing a different component of the polynomial.

The root expression is:

$$x^2y^2z^2w^2v^2 + x^2y^2z^2w^2v + x^2y^2z^2w + x^2y^2z^2 + x^2y^2z + x^2y^2 + x^2y + x^2 + x + y^2z^2w^2v^2 + y^2z^2w^2v + y^2z^2w^2 + y^2z^2w + y^2z^2 + y^2z + y^2 + y + z^2w^2v^2 + z^2w^2v + z^2w^2 + z^2w + z^2 + z + w^2v^2 + w^2v + w^2 + w + v^2 + v + 1$$

The diagram shows the following decomposition steps:

- Root expression branches into two main parts:
  - Left part:  $x^2y^2z^2w^2v^2 + x^2y^2z^2w^2v + x^2y^2z^2w^2 + x^2y^2z^2w + x^2y^2z^2 + x^2y^2z + x^2y^2 + x^2y + x^2 + x + y^2z^2w^2v^2 + y^2z^2w^2v + y^2z^2w^2 + y^2z^2w + y^2z^2 + y^2z + y^2 + y + z^2w^2v^2 + z^2w^2v + z^2w^2 + z^2w + z^2 + z + w^2v^2 + w^2v + w^2 + w + v^2 + v + 1$
  - Right part:  $x^2y^2z^2w^2v^2 + x^2y^2z^2w^2v + x^2y^2z^2w^2 + x^2y^2z^2w + x^2y^2z^2 + x^2y^2z + x^2y^2 + x^2y + x^2 + x + y^2z^2w^2v^2 + y^2z^2w^2v + y^2z^2w^2 + y^2z^2w + y^2z^2 + y^2z + y^2 + y + z^2w^2v^2 + z^2w^2v + z^2w^2 + z^2w + z^2 + z + w^2v^2 + w^2v + w^2 + w + v^2 + v + 1$
- Each main part branches into linear and quadratic terms. For example, the left part branches into:
  - Linear terms:  $x^2y^2z^2w^2v^2 + x^2y^2z^2w^2v + x^2y^2z^2w^2 + x^2y^2z^2w + x^2y^2z^2 + x^2y^2z + x^2y^2 + x^2y + x^2 + x + y^2z^2w^2v^2 + y^2z^2w^2v + y^2z^2w^2 + y^2z^2w + y^2z^2 + y^2z + y^2 + y + z^2w^2v^2 + z^2w^2v + z^2w^2 + z^2w + z^2 + z + w^2v^2 + w^2v + w^2 + w + v^2 + v + 1$
  - Quadratic terms:  $x^2y^2z^2w^2v^2 + x^2y^2z^2w^2v + x^2y^2z^2w^2 + x^2y^2z^2w + x^2y^2z^2 + x^2y^2z + x^2y^2 + x^2y + x^2 + x + y^2z^2w^2v^2 + y^2z^2w^2v + y^2z^2w^2 + y^2z^2w + y^2z^2 + y^2z + y^2 + y + z^2w^2v^2 + z^2w^2v + z^2w^2 + z^2w + z^2 + z + w^2v^2 + w^2v + w^2 + w + v^2 + v + 1$
- The final result is a sum of several terms, each representing a different component of the polynomial.

01-ENC encoding (169ms)

```

graph TD
    A[2019-2020 Academic Year] --> B[Fall Semester]
    A --> C[Spring Semester]
    B --> D[Mathematics]
    B --> E[Science]
    C --> F[Mathematics]
    C --> G[Science]
    D --> H[Algebra]
    D --> I[Geometry]
    E --> J[Biology]
    E --> K[Chemistry]
    F --> L[Algebra]
    F --> M[Geometry]
    G --> N[Biology]
    G --> O[Chemistry]
    H --> P[Classroom]
    H --> Q[Lab]
    I --> R[Classroom]
    I --> S[Lab]
    J --> T[Classroom]
    J --> U[Lab]
    K --> V[Classroom]
    K --> W[Lab]
    P --> X[Theory]
    P --> Y[Practice]
    Q --> Z[Experiment]
    Q --> AA[Theory]
    R --> AB[Theory]
    R --> AC[Practice]
    S --> AD[Experiment]
    S --> AE[Theory]
    T --> AF[Theory]
    T --> AG[Practice]
    U --> AH[Experiment]
    U --> AI[Theory]
    V --> AJ[Theory]
    V --> AK[Practice]
    W --> AL[Experiment]
    W --> AM[Theory]
    
```

The diagram illustrates the hierarchical structure of the 2019-2020 academic year. It starts with the '2019-2020 Academic Year' at the top, which branches into 'Fall Semester' and 'Spring Semester'. Each semester further branches into 'Mathematics' and 'Science'. 'Mathematics' branches into 'Algebra' and 'Geometry', while 'Science' branches into 'Biology' and 'Chemistry'. Each of these branches further into 'Classroom' and 'Lab'. 'Classroom' branches into 'Theory' and 'Practice', while 'Lab' branches into 'Experiment' and 'Theory'. The diagram shows a total of 10 classrooms, 10 labs, 10 theory sessions, and 10 practice sessions.

(49)  $\vdash (A \rightarrow \neg B) \leftrightarrow (\neg A \wedge B)$

LJ (58ms)

[illegible]

MULTIPLICATIVE encoding (88ms)

$$\begin{array}{c}
\frac{\frac{\frac{\vdots}{\vdots} B \vdash B \quad \vdots}{\vdots} \perp \vdash \perp}{\vdots} \quad \frac{\vdots}{\vdots} A \vdash A \quad \frac{\vdots}{\vdots} B, B \rightarrow \perp \vdash \perp}{\vdots} \quad \frac{\vdots}{\vdots} A \vdash A \quad \frac{\vdots}{\vdots} B \vdash B}{\vdots} \quad \frac{\vdots}{\vdots} A, B \vdash A \otimes B \quad \frac{\vdots}{\vdots} \perp \vdash \perp}{\vdots} \\
\frac{\vdots}{\vdots} A, B, A \rightarrow B \rightarrow \perp \vdash \perp \quad \frac{\vdots}{\vdots} A, B, A \otimes B \rightarrow \perp \vdash \perp}{\vdots} \quad \frac{\vdots}{\vdots} A \rightarrow B \rightarrow \perp \rightarrow A \otimes B \rightarrow \perp \quad \frac{\vdots}{\vdots} A \rightarrow B \rightarrow \perp \rightarrow A \otimes B \rightarrow \perp}{\vdots} \quad \frac{\vdots}{\vdots} A \rightarrow B \rightarrow \perp \rightarrow A \otimes B \rightarrow \perp \quad \frac{\vdots}{\vdots} A \otimes B \rightarrow \perp \rightarrow A \rightarrow B \rightarrow \perp}{\vdots}
\end{array}$$

CALL-BY-NAME encoding (118ms)

[illegible]

CALL-BY-VALUE encoding (281ms)

Figure 1 illustrates the relationships between various mathematical concepts, likely related to the study of the  $3x+1$  problem. The diagram shows a hierarchy of theorems and lemmas, with arrows indicating dependencies or implications. Key components include:

- Top Level:** Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, Theorem 1.5, Theorem 1.6, Theorem 1.7, Theorem 1.8, Theorem 1.9, Theorem 1.10, Theorem 1.11, Theorem 1.12, Theorem 1.13, Theorem 1.14, Theorem 1.15, Theorem 1.16, Theorem 1.17, Theorem 1.18, Theorem 1.19, Theorem 1.20, Theorem 1.21, Theorem 1.22, Theorem 1.23, Theorem 1.24, Theorem 1.25, Theorem 1.26, Theorem 1.27, Theorem 1.28, Theorem 1.29, Theorem 1.30, Theorem 1.31, Theorem 1.32, Theorem 1.33, Theorem 1.34, Theorem 1.35, Theorem 1.36, Theorem 1.37, Theorem 1.38, Theorem 1.39, Theorem 1.40, Theorem 1.41, Theorem 1.42, Theorem 1.43, Theorem 1.44, Theorem 1.45, Theorem 1.46, Theorem 1.47, Theorem 1.48, Theorem 1.49, Theorem 1.50, Theorem 1.51, Theorem 1.52, Theorem 1.53, Theorem 1.54, Theorem 1.55, Theorem 1.56, Theorem 1.57, Theorem 1.58, Theorem 1.59, Theorem 1.60, Theorem 1.61, Theorem 1.62, Theorem 1.63, Theorem 1.64, Theorem 1.65, Theorem 1.66, Theorem 1.67, Theorem 1.68, Theorem 1.69, Theorem 1.70, Theorem 1.71, Theorem 1.72, Theorem 1.73, Theorem 1.74, Theorem 1.75, Theorem 1.76, Theorem 1.77, Theorem 1.78, Theorem 1.79, Theorem 1.80, Theorem 1.81, Theorem 1.82, Theorem 1.83, Theorem 1.84, Theorem 1.85, Theorem 1.86, Theorem 1.87, Theorem 1.88, Theorem 1.89, Theorem 1.90, Theorem 1.91, Theorem 1.92, Theorem 1.93, Theorem 1.94, Theorem 1.95, Theorem 1.96, Theorem 1.97, Theorem 1.98, Theorem 1.99, Theorem 1.100.
- Intermediate Levels:** Theorems 1.1 through 1.100 are organized into several groups, with some theorems (e.g., 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23, 1.24, 1.25, 1.26, 1.27, 1.28, 1.29, 1.30, 1.31, 1.32, 1.33, 1.34, 1.35, 1.36, 1.37, 1.38, 1.39, 1.40, 1.41, 1.42, 1.43, 1.44, 1.45, 1.46, 1.47, 1.48, 1.49, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.56, 1.57, 1.58, 1.59, 1.60, 1.61, 1.62, 1.63, 1.64, 1.65, 1.66, 1.67, 1.68, 1.69, 1.70, 1.71, 1.72, 1.73, 1.74, 1.75, 1.76, 1.77, 1.78, 1.79, 1.80, 1.81, 1.82, 1.83, 1.84, 1.85, 1.86, 1.87, 1.88, 1.89, 1.90, 1.91, 1.92, 1.93, 1.94, 1.95, 1.96, 1.97, 1.98, 1.99, 1.100) being grouped together.
- Bottom Level:** Theorems 1.1 through 1.100 are organized into several groups, with some theorems (e.g., 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23, 1.24, 1.25, 1.26, 1.27, 1.28, 1.29, 1.30, 1.31, 1.32, 1.33, 1.34, 1.35, 1.36, 1.37, 1.38, 1.39, 1.40, 1.41, 1.42, 1.43, 1.44, 1.45, 1.46, 1.47, 1.48, 1.49, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.56, 1.57, 1.58, 1.59, 1.60, 1.61, 1.62, 1.63, 1.64, 1.65, 1.66, 1.67, 1.68, 1.69, 1.70, 1.71, 1.72, 1.73, 1.74, 1.75, 1.76, 1.77, 1.78, 1.79, 1.80, 1.81, 1.82, 1.83, 1.84, 1.85, 1.86, 1.87, 1.88, 1.89, 1.90, 1.91, 1.92, 1.93, 1.94, 1.95, 1.96, 1.97, 1.98, 1.99, 1.100) being grouped together.

01-ENC encoding (318ms)

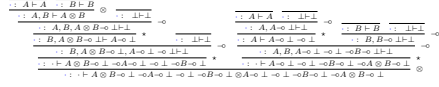
A phylogenetic tree showing the relationships between various bacterial taxa. The tree is rooted at the bottom and branches upwards. The taxa are labeled with their full names, including the genus and species, and are color-coded: blue for Proteobacteria, green for Bacteroidetes, yellow for Actinobacteria, and red for Firmicutes. The tree shows a high degree of diversity, with many branches and a complex network of relationships. The scale bar at the bottom indicates a distance of 0.01 substitutions per site.

$$(50) \cdot \vdash (\neg A \wedge B) \leftrightarrow (\neg \neg A \rightarrow \neg B)$$

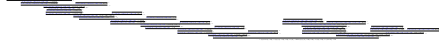
LJ (93ms)



MULTIPLICATIVE encoding (134ms)



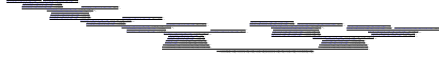
CALL-BY-NAME encoding (333ms)



CALL-BY-VALUE encoding (421ms)



01-ENC encoding (470ms)

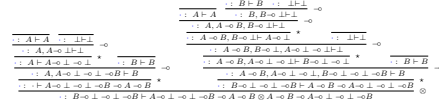


$$(51) \neg \neg B \rightarrow B \vdash (\neg \neg A \rightarrow B) \leftrightarrow (A \rightarrow B)$$

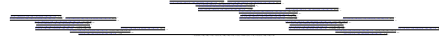
LJ (135ms)



MULTIPLICATIVE encoding (156ms)



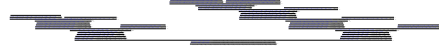
CALL-BY-NAME encoding (278ms)



CALL-BY-VALUE encoding (414ms)

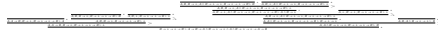


01-ENC encoding (449ms)



$$(52) \neg\neg B \rightarrow B \vdash (A \rightarrow B) \leftrightarrow (\neg A \wedge \neg B)$$

LJ (94ms)



MULTIPLICATIVE encoding (134ms)



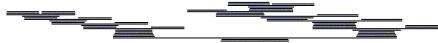
CALL-BY-NAME encoding (296ms)



CALL-BY-VALUE encoding (457ms)

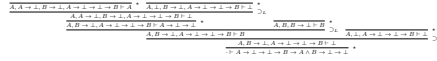


01-ENC encoding (489ms)

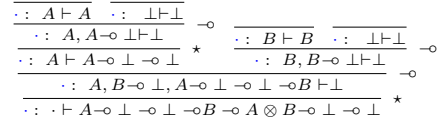


$$(53) \vdash \neg\neg A \rightarrow B \rightarrow \neg A \wedge \neg B$$

LJ (65ms)



MULTIPLICATIVE encoding (75ms)



CALL-BY-NAME encoding (178ms)



CALL-BY-VALUE encoding (204ms)



01-ENC encoding (215ms)



$$(54) \cdot \vdash A \wedge B \rightarrow \neg A \rightarrow \neg B$$

LJ (47ms)

$$\frac{\frac{\frac{A, B, A \rightarrow B \rightarrow \perp \vdash A}{A, B, A \rightarrow B \rightarrow \perp \vdash \perp} \star \quad \frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp} \star \quad \frac{A, B, \perp \vdash \perp}{A, B, A \rightarrow B \rightarrow \perp \vdash \perp} \star}{\vdash A \wedge B \rightarrow A \rightarrow B \rightarrow \perp \vdash \perp} \star$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\vdash : A \vdash A}{\vdash : A, B, A \multimap B \multimap \perp \vdash \perp} \star \quad \frac{\vdash : B \vdash B}{\vdash : B, B \multimap \perp \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (77ms)

$$\frac{\frac{\frac{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash A}{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash \perp} \star \quad \frac{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash B}{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

CALL-BY-VALUE encoding (105ms)

$$\frac{\frac{\frac{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash A}{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash \perp} \star \quad \frac{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash B}{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

01-ENC encoding (106ms)

$$\frac{\frac{\frac{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash A}{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash \perp} \star \quad \frac{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash B}{A, B, \lambda(A) \multimap \lambda(B) \multimap \mathbf{0} : \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

$$(55) \cdot \vdash A \wedge \neg B \rightarrow \neg A \rightarrow B$$

LJ (48ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} \star \quad \frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp} \star \quad \frac{A, A \rightarrow B, B \rightarrow \perp \vdash \perp}{\vdash A \wedge B \rightarrow \perp \vdash A \rightarrow B \rightarrow \perp} \star}{\vdash A \wedge \neg B \rightarrow \perp \vdash A \rightarrow B \rightarrow \perp} \star$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\frac{\vdash : A \vdash A}{\vdash : A, A \multimap B, B \multimap \perp \vdash \perp} \star \quad \frac{\vdash : B \vdash B}{\vdash : B, B \multimap \perp \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (144ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} \star \quad \frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

CALL-BY-VALUE encoding (168ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} \star \quad \frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

01-ENC encoding (168ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} \star \quad \frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp} \star}{\vdash : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

$$(56) \cdot \vdash \neg\neg A \wedge B \rightarrow \neg A \rightarrow \neg B$$

LJ (62ms)

$$\frac{\frac{\frac{A, B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash A}{A, B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash A} *}{\frac{A, B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash A}{A, B, A \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash \perp} \supset_L} *}{\frac{\frac{A, B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash \perp}{B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash \perp} *}{\frac{B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash \perp}{\vdash A \rightarrow \perp \rightarrow \perp \wedge B \rightarrow A \rightarrow B \rightarrow \perp \rightarrow \perp} *} \supset_L} *}{\frac{B, A \rightarrow B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \vdash \perp}{B, \perp, A \rightarrow B \rightarrow \perp \vdash \perp \vdash \perp} *} \supset_L} *$$

MULTIPLICATIVE encoding (77ms)

$$\begin{array}{c}
\frac{\frac{\frac{\vdots : B \vdash \bar{B}}{} \quad \vdots : \perp \vdash \perp}{\vdots : A \vdash A} \quad \vdots : B, B \multimap \perp \vdash \perp}{\vdots : A, B, A \multimap B \multimap \perp \vdash \perp} \multimap \\
\frac{\vdots : B, A \multimap B \multimap \perp \vdash A \multimap \perp \quad \vdots : \perp \vdash \perp}{\vdots : B, A \multimap B \multimap \perp, A \multimap \perp \multimap \perp \vdash \perp} \multimap \\
\vdots : \vdash A \multimap \perp \multimap \perp \otimes B \multimap A \multimap B \multimap \perp \multimap \perp \quad *
\end{array}$$

CALL-BY-NAME encoding (182ms)

CALL-BY-VALUE encoding (212ms)

[illegible]

01-ENC encoding (210ms)

(57)  $\vdash (\neg\neg A \wedge \neg B) \leftrightarrow (\neg A \rightarrow B)$

LJ (114ms)

Figure 1. Schematic representation of the experimental design. The figure shows a sequence of events: a stimulus (a word) is presented, followed by a response (a word), and then a feedback (a word). The response is either correct or incorrect. The feedback is either positive (correct) or negative (incorrect). The sequence of events is: Stimulus (word) → Response (word) → Feedback (word). The response is either correct or incorrect. The feedback is either positive (correct) or negative (incorrect). The sequence of events is: Stimulus (word) → Response (word) → Feedback (word).

MULTIPLICATIVE encoding (58ms)

Not provable

CALL-BY-NAME encoding (326ms)

CALL-BY-VALUE encoding (421ms)

01-ENC encoding (454ms)

$$(58) \cdot \vdash (\neg A \rightarrow B) \leftrightarrow (\neg\neg A \wedge \neg B)$$

LJ (109ms)



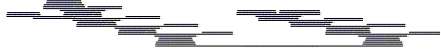
MULTIPLICATIVE encoding (41ms)

**Not provable**

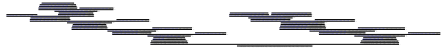
CALL-BY-NAME encoding (597ms)



CALL-BY-VALUE encoding (481ms)



01-ENC encoding (501ms)



$$(59) \cdot \vdash (\neg\neg A \rightarrow B) \leftrightarrow (\neg A \wedge \neg B)$$

LJ (112ms)



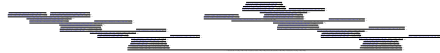
MULTIPLICATIVE encoding (57ms)

**Not provable**

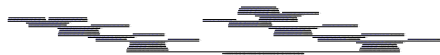
CALL-BY-NAME encoding (594ms)



CALL-BY-VALUE encoding (468ms)



01-ENC encoding (498ms)

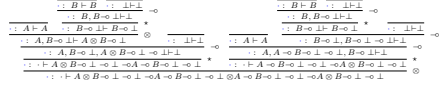


$$(60) \cdot \vdash (\neg A \wedge \neg B) \leftrightarrow (A \rightarrow \neg \neg B)$$

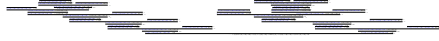
LJ (92ms)



MULTIPLICATIVE encoding (124ms)



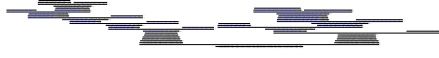
CALL-BY-NAME encoding (317ms)



CALL-BY-VALUE encoding (436ms)



01-ENC encoding (468ms)

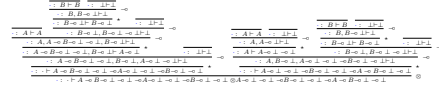


$$(61) \cdot \vdash (A \rightarrow \neg \neg B) \leftrightarrow (\neg \neg A \rightarrow \neg \neg B)$$

LJ (169ms)



MULTIPLICATIVE encoding (163ms)



CALL-BY-NAME encoding (412ms)



CALL-BY-VALUE encoding (479ms)



01-ENC encoding (552ms)



### 3 Alternative Translations

$$(10) \quad \cdot : A \multimap \mathbf{0} \vdash A \multimap B$$

encoding (55ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : A, A \multimap \mathbf{0} \vdash B} \multimap}{\cdot : A \multimap \mathbf{0} \vdash A \multimap B} \star}{\cdot : A \vdash A \quad \cdot : \mathbf{0} \vdash B} \star$$

$$(11) \quad \cdot : A \vdash A \multimap \mathbf{0} \multimap B$$

encoding (41ms)

$$\frac{\frac{\frac{\cdot : A \vdash A \quad \cdot : \mathbf{0} \vdash B}{\cdot : A, A \multimap \mathbf{0} \vdash B} \quad \star}{\cdot : A \vdash A \multimap \mathbf{0} \multimap B} \quad \star \quad \multimap$$

$$(12) \quad \cdot : B \vdash !(A) \multimap B$$

encoding (22ms)

$$\frac{\overline{A : B \vdash B}}{\cdot : B \vdash !(A) \multimap B} \star$$

(16)  $\cdot : A \multimap B \otimes !(B \multimap A) \vdash A \multimap B$

encoding (35ms)

$$\frac{\frac{B \multimap A : A \vdash A \quad B \multimap A : B \vdash B}{B \multimap A : A, A \multimap B \vdash B} \multimap \quad \cdot : A \multimap B \otimes !(B \multimap A) \vdash A \multimap B}{\cdot : A \multimap B \otimes !(B \multimap A) \vdash A \multimap B} \star$$

$$(17) \cdot : !(A \multimap B) \otimes B \multimap A \vdash B \multimap A$$

encoding (38ms)

$$\frac{\frac{A \multimap B : B \vdash B \quad A \multimap B : A \vdash A}{A \multimap B : B, B \multimap A \vdash A} \multimap \text{C} \quad \cdot : !(A \multimap B) \otimes B \multimap A \vdash B \multimap A}{\cdot : !(A \multimap B) \otimes B \multimap A \vdash B \multimap A} \star$$

$$(18) \cdot : A, A \multimap B \otimes B \multimap A \vdash B \otimes B \multimap A$$

encoding (56ms)

$$\begin{array}{c}
\frac{\frac{\frac{\vdots: B \vdash B \quad \vdots: A \vdash A}{\vdots: B, B \multimap A \vdash A} \multimap}{\vdots: A \vdash A} \multimap}{\vdots: B, B \multimap A \vdash B \multimap A \vdash B \multimap A} \star \\
\frac{\vdots: A, A \multimap B, B \multimap A \vdash B \multimap B \multimap A}{\vdots: A, A \multimap B \multimap B \multimap A \vdash B \multimap B \multimap A} \multimap
\end{array}$$

$$(19) \cdot : B, A \multimap B \otimes B \multimap A \vdash A \otimes A \multimap B$$

encoding (55ms)

$$\begin{array}{c}
\frac{\frac{\frac{\vdots \vdash A \vdash A}{\vdots \vdash A, A \multimap B \vdash B} \multimap}{\vdots \vdash B \vdash B} \multimap}{\vdots \vdash B, A \multimap B, B \multimap A \vdash A \otimes A \multimap B} \multimap \\
\frac{\vdots \vdash B, A \multimap B \otimes B \multimap A \vdash A \vdash A \otimes A \multimap B}{\vdots \vdash B, A \multimap B \otimes B \multimap A \vdash A \vdash A \otimes A \multimap B} \star
\end{array}$$

$$(26a) \quad \therefore \vdash A \& B \multimap \perp \multimap \perp \multimap A \multimap \perp \multimap \perp \& B \multimap \perp \multimap \perp$$

encoding (107ms)

$$\begin{array}{l} \frac{\vdots, A \vdash A}{\vdots, A, A \rightarrow \perp \vdash \perp} \rightarrow \\ \vdots, A, A \rightarrow \perp \vdash \perp \quad \&_1 \\ \vdots, A \& B, A \rightarrow \perp \vdash \perp \\ \vdots, A \rightarrow \perp \& A \& B \rightarrow \perp \quad * \quad \vdots, \perp \vdash \perp \\ \vdots, A \rightarrow \perp, A \& B \rightarrow \perp \rightarrow \perp \vdash \perp \quad \rightarrow \\ \vdots, A \& B \rightarrow \perp \rightarrow \perp \rightarrow \perp \vdash A \& B \rightarrow \perp \rightarrow \perp \quad * \end{array} \quad \begin{array}{l} \frac{\vdots, B \vdash B}{\vdots, B, B \rightarrow \perp \vdash \perp} \rightarrow \\ \vdots, B, B \rightarrow \perp \vdash \perp \quad \&_2 \\ \vdots, A \& B, B \rightarrow \perp \vdash \perp \\ \vdots, B \rightarrow \perp \& A \& B \rightarrow \perp \quad * \quad \vdots, \perp \vdash \perp \\ \vdots, B \rightarrow \perp, A \& B \rightarrow \perp \rightarrow \perp \vdash \perp \quad \rightarrow \\ \vdots, B \rightarrow \perp \& A \& B \rightarrow \perp \rightarrow \perp \vdash A \& B \rightarrow \perp \rightarrow \perp \quad * \end{array}$$

$$(26b) \quad \therefore \vdash A \multimap \perp \multimap \perp \otimes B \multimap \perp \multimap \perp \multimap A \otimes B \multimap \perp \multimap \perp$$

encoding (109ms)

[illegible]

$$(27a) \vdash \vdash ! (A \multimap B) \otimes ! (B \multimap A) \multimap \perp \multimap \perp \multimap A \multimap B \multimap \perp \multimap \perp \otimes B \multimap A \multimap \perp \multimap \perp$$

encoding (155ms)

Figure 1. The proof of Lemma 1. The figure shows a sequence of 12 steps, each with a diagram and a corresponding inequality. The diagrams consist of horizontal bars of varying lengths and colors (blue, green, yellow, red) representing different components of the proof. The inequalities involve terms like  $\frac{1}{2} \log \frac{1}{2}$ ,  $\frac{1}{2} \log \frac{1}{4}$ ,  $\frac{1}{2} \log \frac{1}{8}$ ,  $\frac{1}{2} \log \frac{1}{16}$ ,  $\frac{1}{2} \log \frac{1}{32}$ ,  $\frac{1}{2} \log \frac{1}{64}$ ,  $\frac{1}{2} \log \frac{1}{128}$ ,  $\frac{1}{2} \log \frac{1}{256}$ ,  $\frac{1}{2} \log \frac{1}{512}$ ,  $\frac{1}{2} \log \frac{1}{1024}$ ,  $\frac{1}{2} \log \frac{1}{2048}$ , and  $\frac{1}{2} \log \frac{1}{4096}$ . The final result is  $\frac{1}{2} \log \frac{1}{4096}$ .

$$(27b) \quad \vdash A \multimap B \multimap \perp \multimap \perp \otimes B \multimap A \multimap \perp \multimap \perp \multimap A \multimap B \otimes B \multimap A \multimap \perp \multimap \perp$$

encoding (168ms)

$$\begin{array}{l}
 \vdash A \supset A, \vdash B \supset B \quad \circ, \quad \vdash B \supset B, \vdash A \supset A \quad \circ \\
 \vdash A, A \supset B \supset B \quad \circ, \quad \vdash B, B \supset A \supset A \quad \circ \\
 \vdash A \supset B \supset A \supset B \quad *, \quad \vdash B \supset A \supset B \supset A \quad * \\
 \vdash A \supset B, B \supset A \supset A \supset B \supset A \quad \circ, \quad \vdash \perp \vdash \perp \quad \circ \\
 \vdash A \supset B, B \supset A, A \supset B \supset B \supset A \supset A \supset \perp \perp \quad \circ \\
 \vdash A \supset A, B \supset B \supset B \supset A \supset A \supset \perp \perp \supset B \supset A \quad * \\
 \vdash \vdash \perp \vdash \perp \quad \circ \\
 \vdash A \supset B, A \supset B \supset B \supset A \supset \perp, B \supset A \supset \perp \supset \perp \supset \perp \perp \quad \circ \\
 \vdash A \supset B \supset B \supset A \supset \perp, B \supset A \supset \perp \supset A \supset \perp \supset A \supset B \supset \perp \quad * \\
 \vdash \vdash \perp \vdash \perp \quad \circ \\
 \vdash A \supset B \supset B \supset A \supset \perp, A \supset B \supset \perp \supset \perp, B \supset A \supset \perp \supset \perp \perp \perp \quad \circ \\
 \vdash A \supset B \supset \perp \supset \perp \supset \perp \supset B \supset A \supset \perp \supset \perp \supset A \supset B \supset A \supset \perp \supset \perp \perp \quad \circ
 \end{array}$$

(35)  $\vdash \vdash ! (A) \otimes ! (A) \multimap ! (A) \otimes ! (A) \multimap ! (A) \otimes ! (A)$

encoding (59ms)



$$(36) \therefore A \vdash A \multimap B \multimap B \otimes B \multimap !(A) \multimap B$$

(37)  $\therefore B \vdash !(A \multimap B) \multimap B \otimes B \multimap !(A) \multimap B$

$$(38) \cdot : A \multimap \perp \vdash ! (A \multimap B) \multimap A \multimap \perp \otimes A \multimap \mathbf{0} \multimap A \multimap B$$

(39)  $\vdash : B \multimap \mathbf{0} \vdash A \multimap B \multimap A \multimap \mathbf{0} \otimes A \multimap \mathbf{0} \multimap A \multimap B$

$$(40) \cdot : B \vdash A \otimes ! (B) \multimap A \otimes A \multimap A \otimes B$$

$$(41) \cdot : B \multimap \mathbf{0} \vdash !(A) \otimes B \multimap B \otimes B \multimap A \otimes B$$

$$(45) \quad \cdot : \cdot \vdash !(A \multimap A \multimap \perp) \otimes !(A) \multimap \perp \multimap !(A) \multimap \perp$$

(46)  $\cdot : \cdot \vdash !(A \multimap \perp \multimap 0) \multimap A \multimap \perp$

$$(47) \quad \vdash A \otimes B \otimes B \multimap 0 \multimap B \otimes B \multimap 0 \otimes B \otimes B \multimap 0 \multimap A \otimes B \otimes B \multimap 0$$

(57a)  $\cdot : \cdot \vdash A \multimap \perp \multimap \otimes B \multimap \perp \multimap A \multimap B \multimap \perp$

$$(57b) \quad \cdot : \cdot \vdash !(A) \multimap B \multimap \perp \multimap A \multimap \mathbf{0} \multimap \perp \& B \multimap \perp$$

(58a)  $\vdash \vdash !(! (A) \multimap B \multimap \perp) \multimap A \otimes B \multimap \perp \multimap \mathbf{0} \multimap \perp$

(58b)  $\cdot : \cdot \vdash A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap A \multimap B \multimap \perp$

$$\begin{array}{c}
\frac{\vdots : A \vdash A}{\vdots : A \vdash A} \quad \frac{\vdots : B \vdash B \quad \vdots : \perp \vdash \perp}{\vdots : B, B \multimap \perp \vdash \perp} \multimap \\
\frac{\vdots : A, A \multimap B, B \multimap \perp \vdash \perp}{\vdots : A \multimap B \vdash A \otimes B \multimap \perp \multimap \perp} \multimap \\
\frac{\vdots : A \multimap B, A \otimes B \multimap \perp \multimap \perp \vdash \perp \quad \vdots : \perp \vdash \perp}{\vdots : A \multimap B \vdash A \otimes B \multimap \perp \multimap \perp \multimap \perp} \star \\
\frac{\vdots : A \multimap B \vdash A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap A \multimap B \multimap \perp}{\vdots : A \multimap B \vdash A \otimes B \multimap \perp \multimap \perp \multimap A \multimap B \multimap \perp} \star
\end{array}$$

$$(59a) \quad \cdot : \cdot \vdash A \multimap B \multimap \perp \multimap \perp \multimap A \otimes B \multimap \perp \multimap \perp$$

encoding (71ms)

$$\begin{array}{c}
\frac{\vdots : A \vdash A}{\vdots : A \vdash A} \quad \frac{\vdots : B \vdash B \quad \vdots : \perp \vdash \perp}{\vdots : B, B \multimap \perp \vdash \perp} \multimap \\
\frac{\vdots : A, A \multimap B, B \multimap \perp \vdash \perp}{\vdots : A, B \multimap \perp \vdash A \multimap B \multimap \perp} \multimap \\
\frac{\vdots : A, B \multimap \perp, A \multimap B \multimap \perp \vdash \perp}{\vdots : A, B \multimap \perp, A \multimap B \multimap \perp \vdash \perp} \star \\
\frac{\vdots : A, B \multimap \perp, A \multimap B \multimap \perp \vdash \perp}{\vdots : A, B \multimap \perp, A \multimap B \multimap \perp \vdash \perp} \multimap \\
\frac{\vdots : A, B \multimap \perp, A \multimap B \multimap \perp \vdash \perp}{\vdots : A, B \multimap \perp, A \multimap B \multimap \perp \vdash \perp} \star
\end{array}$$

$$(59b) \quad \cdot : \cdot \vdash A \otimes B \multimap \perp \multimap 0 \multimap !(A \multimap B \multimap \perp) \multimap \perp$$

encoding (163ms)

[illegible]