

# Benchmarking Linear Logic: sequents inspired from Kleene's intuitionistic theorems\*

October 18, 2018

## 1 General Information

- Test run on a QEMU Virtual CPU, 2GHz, 64 bits, 2GB of RAM running Ubuntu.
- Maude version: 2.7.1 built: Oct 2 2017.
- The search procedure considers proofs using up to 4 times the copy rule (focusing on one of the formulas of the classical context). Benchmarks 23 and 25 required the limit to be 5.

## 2 Translations

$$(1) \cdot \vdash A \rightarrow A$$

LJ (28ms)

$$\cdot \vdash A \rightarrow A \quad \star$$

MULTIPLICATIVE encoding (28ms)

$$\frac{\cdot \vdash A \vdash A}{\cdot \vdash \cdot \vdash A \multimap A} \star$$

CALL-BY-NAME encoding (28ms)

$$\frac{\overline{A : \cdot \vdash A}}{\cdot \vdash \cdot \vdash !(A) \multimap A} \star$$

CALL-BY-VALUE encoding (41ms)

$$\frac{\frac{\overline{A : \cdot \vdash A}}{A : \cdot \vdash !(A)} !}{\cdot \vdash \cdot \vdash !(A) \multimap !(A)} \star$$

01-ENC encoding (35ms)

$$\frac{\overline{A : \cdot \vdash A}}{\cdot \vdash \cdot \vdash !(A) \multimap A} \star$$

$$(2) A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$$

LJ (46ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow C \vdash A}{A, A \rightarrow B, B \rightarrow C \vdash C} \star}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \star}{\frac{A, B, B \rightarrow C \vdash B}{A, B, B \rightarrow C \vdash C} \star} \multimap_L$$

MULTIPLICATIVE encoding (49ms)

$$\frac{\frac{\cdot \vdash A \vdash A}{\cdot \vdash A, A \multimap B, B \multimap C \vdash C} \star}{\cdot \vdash A \multimap B, B \multimap C \vdash A \multimap C} \star$$

CALL-BY-NAME encoding (123ms)

$$\frac{\frac{\frac{\overline{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash A}}{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash B} !}{\frac{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash B}{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash C} D_C}{\cdot \vdash !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash C} \multimap$$

CALL-BY-VALUE encoding (165ms)

$$\frac{\frac{\frac{\frac{\overline{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash A}}{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash B} !}{\frac{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash B}{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash C} D_C}{\cdot \vdash !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash C} \multimap$$

01-ENC encoding (146ms)

$$\frac{\frac{\frac{\frac{\overline{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash A}}{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash B} !}{\frac{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash B}{A, !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash C} D_C}{\cdot \vdash !!(A) \multimap B, !!(B) \multimap C : \cdot \vdash C} \multimap$$

\*See <https://github.com/meta-logic/lltp> for details on the encodings used.

(3)  $A \rightarrow B \rightarrow C \vdash B \rightarrow A \rightarrow C$

LJ (46ms)

$$\frac{\frac{\frac{A, B, A \rightarrow B \rightarrow C \vdash A}{A, B, A \rightarrow B \rightarrow C \vdash C} \star}{A \rightarrow B \rightarrow C \vdash B \rightarrow A \rightarrow C} \star}{\frac{\frac{A, B, B \rightarrow C \vdash B}{A, B, B \rightarrow C \vdash C} \star}{A, B, C \vdash C} \star} \supset_L$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : B, B \multimap C \vdash C} \multimap \quad \frac{\cdot : B \vdash B \quad \cdot : C \vdash C}{\cdot : B, B \multimap C \vdash C} \multimap}{\cdot : A, B, A \multimap B \multimap C \vdash C} \multimap \quad \frac{\cdot : A, B, A \multimap B \multimap C \vdash C}{\cdot : A \multimap B \multimap C \vdash B \multimap A \multimap C} \star$$

CALL-BY-NAME encoding (71ms)

$$\frac{\frac{\frac{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash A}{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash \mathfrak{l}(A)} \quad \frac{\frac{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash \mathfrak{l}(B)}{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash \mathfrak{l}(B)} \quad \frac{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash C}{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash C} \quad \multimap}{\frac{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash C}{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash C} \quad D_C} \quad \multimap \quad \frac{A, B, \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C : \vdash C}{\vdash : \mathfrak{l}(A) \multimap \mathfrak{l}(B) \multimap C \vdash \mathfrak{l}(A) \multimap C} \quad *$$

CALL-BY-VALUE encoding (162ms)

01-ENC encoding (158ms)

[illegible]

$$(4) \ A \rightarrow B \rightarrow C \vdash A \wedge B \rightarrow C$$

LJ (47ms)

$$\frac{\frac{\overline{A, B, A \rightarrow B \rightarrow C \vdash A}^* \quad \frac{\overline{A, B, B \rightarrow C \vdash B}^* \quad \overline{A, B, C \vdash C}^*}{A, B, B \rightarrow C \vdash C} \supset_L}{\frac{A, B, A \rightarrow B \rightarrow C \vdash C}{A \rightarrow B \rightarrow C \vdash A \wedge B \rightarrow C}^*} \supset_L$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : B, B \multimap C \vdash C} \multimap \quad \frac{\cdot : B \vdash B \quad \cdot : C \vdash C}{\cdot : B, B \multimap C \vdash C} \multimap}{\cdot : A, B, A \multimap B \multimap C \vdash C} \multimap \quad \frac{\cdot : A, B, A \multimap B \multimap C \vdash C}{\cdot : A \multimap B \multimap C \vdash A \otimes B \multimap C} \star$$

CALL-BY-NAME encoding (71ms)

[illegible]

CALL-BY-VALUE encoding (156ms)

[illegible]

01-ENC encoding (151ms)

[illegible]

$$(5) \quad A \wedge B \rightarrow C \vdash A \rightarrow B \rightarrow C$$

LJ (41ms)

$$\frac{\frac{A, B, A \wedge B \rightarrow C \vdash A \wedge B}{A, B, A \wedge B \rightarrow C \vdash C}^*}{A \wedge B \rightarrow C \vdash A \rightarrow B \rightarrow C}^* \supset_L$$

MULTIPLICATIVE encoding (54ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : A, B \vdash A \otimes B} \otimes \frac{\cdot : B \vdash B}{\cdot : C \vdash C}}{\cdot : A, B, A \otimes B \multimap C \vdash C} \multimap}{\cdot : A \otimes B \multimap C \vdash A \multimap B \multimap C} \star$$

CALL-BY-NAME encoding (73ms)

$$\frac{\begin{array}{c} \overline{A, B, [(A \& B) \multimap C] : \neg A} \quad \overline{A, B, [(A \& B) \multimap C] : \neg B} \\ \overline{A, B, [(A \& B) \multimap C] : \neg A \wedge B} \\ \overline{A, B, [(A \& B) \multimap C] : \neg ![(A \& B)]} \end{array}}{\overline{A, B, [(A \& B) \multimap C] : C \vdash C}}^* \quad \text{D}_C$$

CALL-BY-VALUE encoding (205ms)

[illegible]

01-ENC encoding (201ms)

[illegible]

$$(6) \quad A \rightarrow B \vdash B \rightarrow C \rightarrow A \rightarrow C$$

LJ (47ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow C \vdash A}{A, B, B \rightarrow C \vdash A} \star \quad \frac{\frac{A, B, B \rightarrow C \vdash B}{A, B, B \rightarrow C \vdash B} \star \quad \frac{A, B, C \vdash C}{A, B, C \vdash C} \star}{A, B, B \rightarrow C \vdash C} \supset_L}{\frac{A, A \rightarrow B, B \rightarrow C \vdash C}{A \rightarrow B \vdash B \rightarrow C \rightarrow A \rightarrow C} \star} \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\cdot \vdash A \vdash A}{\cdot \vdash A \vdash A} \quad \frac{\cdot \vdash B \vdash B \quad \cdot \vdash C \vdash C}{\cdot \vdash B, B \multimap C \vdash C} \multimap}{\cdot \vdash A, A \multimap B, B \multimap C \vdash C} \multimap}{\cdot \vdash A \multimap B \vdash B \multimap C \multimap A \multimap C} \star$$

CALL-BY-NAME encoding (120ms)

[illegible]

CALL-BY-VALUE encoding (168ms)

[illegible]

01-ENC encoding (159ms)

[illegible]





$$(11) \ A \vdash \neg A \rightarrow B$$

LJ (37ms)

$$\frac{\frac{\overline{A, A \rightarrow \perp \vdash A} \star \overline{A, \perp \vdash B} \star}{\overline{A, A \rightarrow \perp \vdash B}} \supset_L}{\overline{A \vdash A \rightarrow \perp \rightarrow B} \star}$$

MULTIPLICATIVE encoding (28ms)

**Not provable**

CALL-BY-NAME encoding (65ms)

$$\frac{\frac{\overline{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}}{\overline{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)}} ! \frac{\overline{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash B} \star}{\overline{A, !(A) \multimap \mathbf{0} : \cdot \vdash B} D_C} \multimap}{\overline{\cdot : !(A) \vdash !(A) \multimap \mathbf{0} \rightarrow B} \star}$$

CALL-BY-VALUE encoding (78ms)

$$\frac{\frac{\overline{A, !(A) \multimap \mathbf{0} : \cdot \vdash A}}{\overline{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(A)}} ! \frac{\overline{A, !(A) \multimap \mathbf{0} : \mathbf{0} \vdash !(B)} \star}{\overline{A, !(A) \multimap \mathbf{0} : \cdot \vdash !(B)} D_C} \multimap}{\frac{\overline{A : \cdot \vdash !(A) \multimap \mathbf{0} \rightarrow !(B)} \star}{\overline{A : \cdot \vdash !(A) \multimap \mathbf{0} \rightarrow !(B)} !} \star}$$

01-ENC encoding (76ms)

$$\frac{\frac{\overline{A, !(A) \multimap !(0) : \cdot \vdash A}}{\overline{A, !(A) \multimap !(0) : \cdot \vdash !(A)}} ! \frac{\overline{A, !(A) \multimap !(0) : !(0) \vdash B} \star}{\overline{A, !(A) \multimap !(0) : \cdot \vdash B} D_C} \multimap}{\frac{\overline{A : \cdot \vdash !(A) \multimap !(0) \rightarrow B} \star}{\overline{A : \cdot \vdash !(A) \multimap !(0) \rightarrow B} !} \star}$$

$$(12) \ B \vdash A \rightarrow B$$

LJ (21ms)

$$\overline{B \vdash A \rightarrow B} \star$$

MULTIPLICATIVE encoding (21ms)

**Not provable**

CALL-BY-NAME encoding (27ms)

$$\frac{\overline{A, B : \cdot \vdash B}}{\overline{\cdot : !(B) \vdash !(A) \multimap B} \star}$$

CALL-BY-VALUE encoding (47ms)

$$\frac{\frac{\overline{A, B : \cdot \vdash B}}{\overline{A, B : \cdot \vdash !(B)}} !}{\frac{\overline{B : \cdot \vdash !(A) \multimap !(B)} \star}{\overline{B : \cdot \vdash !(A) \multimap !(B)} !} \star}$$

01-ENC encoding (39ms)

$$\frac{\overline{A, B : \cdot \vdash B}}{\overline{B : \cdot \vdash !(A) \multimap B} \star} \frac{\overline{B : \cdot \vdash !(A) \multimap B} !}{\overline{\cdot : !(B) \vdash !(A) \multimap B} \star}$$

$$(13) \quad A \rightarrow B \vdash \neg B \rightarrow \neg A$$

LJ (47ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp}^* \quad \frac{\frac{A, B, B \rightarrow \perp \vdash B}{A, B, \perp \vdash \perp}^*}{A, B, B \rightarrow \perp \vdash \perp}^* \supset_L}{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash \perp}{A \rightarrow B \vdash B \rightarrow \perp \rightarrow A \rightarrow \perp}^*} \supset_L$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : B \vdash B} \quad \frac{\cdot : B \vdash B \quad \cdot : \perp \vdash \perp}{\cdot : B, B \multimap \perp \vdash \perp} \multimap}{\cdot : A, A \multimap B, B \multimap \perp \vdash \perp} \multimap}{\cdot : A \multimap B \vdash B \multimap \perp \multimap A \multimap \perp} \star$$

CALL-BY-NAME encoding (143ms)

[illegible]

CALL-BY-VALUE encoding (174ms)

[illegible]

01-ENC encoding (178ms)

[illegible]

$$(14) \quad A \rightarrow \neg B \vdash \neg\neg B \rightarrow \neg A$$

LJ (64ms)

$$\frac{\frac{A, B \rightarrow \perp, B \rightarrow \perp \vdash B}{A, B \rightarrow \perp, B \rightarrow \perp \vdash \perp}^* \quad \frac{A, B, \perp, B \rightarrow \perp \vdash \perp}{A, B \rightarrow \perp, B \rightarrow \perp \vdash \perp}}{\frac{A, A \rightarrow B \rightarrow \perp, B \rightarrow \perp \vdash A}{A, A \rightarrow B \rightarrow \perp, B \rightarrow \perp \vdash \perp}^*} \supset_L$$

MULTIPLICATIVE encoding (75ms)

$$\begin{array}{c}
\frac{\vdots : B \vdash B \quad \vdots : \perp \vdash \perp}{\vdots : A \vdash A} \multimap \quad \vdots : B, B \multimap \perp \vdash \perp \\
\vdots : A, B, A \multimap B \multimap \perp \vdash \perp \\
\vdots : A, A \multimap B \multimap \perp \vdash B \multimap \perp \quad * \quad \vdots : \perp \vdash \perp \\
\hline
\vdots : A, A \multimap B \multimap \perp, B \multimap \perp \multimap \perp \vdash \perp \quad \multimap \\
\vdots : A \multimap B \multimap \perp \vdash B \multimap \perp \multimap \perp \multimap A \multimap \perp \quad *
\end{array}$$

CALL-BY-NAME encoding (180ms)

Figure 1: Schematic representation of the experimental design. The diagram shows a sequence of steps: 1. Pre-test (N=100) leading to 2. Main test (N=100) leading to 3. Post-test (N=100) leading to 4. Follow-up (N=100) leading to 5. Final test (N=100). Each step is represented by a box, and the flow is indicated by arrows. The boxes are labeled with the step number, the name of the test, and the number of participants.

CALL-BY-VALUE encoding (217ms)

[illegible]

01-ENC encoding (226ms)

(15)  $A \rightarrow B, B \rightarrow A \vdash (A) \leftrightarrow (B)$

LJ (51ms)

$$\frac{\frac{A, A \rightarrow B, B \rightarrow A \vdash A}{A, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \frac{\frac{A, B, B \rightarrow A \vdash B}{B, A \rightarrow B, B \rightarrow A \vdash A}^*}{A \rightarrow B, B \rightarrow A \vdash A \rightarrow B \wedge B \rightarrow A}^* \supset_L$$

MULTIPLICATIVE encoding (75ms)

$$\frac{\frac{\frac{\cdot : A \vdash A \quad \cdot : B \vdash B}{\cdot : A, A \multimap B \vdash B} \multimap \quad \frac{\cdot : B \vdash B \quad \cdot : A \vdash A}{\cdot : B, B \multimap A \vdash A} \multimap}{\cdot : A \multimap B \vdash A \multimap B} \star \quad \frac{\cdot : B \multimap A \vdash B \multimap A}{\cdot : B \multimap A \vdash B \multimap A} \star}{\cdot : A \multimap B, B \multimap A \vdash A \multimap B \otimes B \multimap A} \otimes$$

CALL-BY-NAME encoding (138ms)

[illegible]

CALL-BY-VALUE encoding (205ms)

[illegible]

01-ENC encoding (191ms)

[illegible]

$$(16) \quad (A) \leftrightarrow (B) \vdash A \rightarrow B$$

LJ (37ms)

$$\frac{\frac{A, A \rightarrow B, B \rightarrow A \vdash A \quad \star \quad A, B, B \rightarrow A \vdash B \quad \star}{A, A \rightarrow B, B \rightarrow A \vdash B} \quad \star}{A \rightarrow B \wedge B \rightarrow A \vdash A \rightarrow B} \quad \star \quad \supset_L$$

MULTIPLICATIVE encoding (29ms)

Not provable

CALL-BY-NAME encoding (102ms)

$$\frac{\frac{\overline{A, ! (A) \multimap B, ! (B) \multimap A : \vdash A}}{A, ! (A) \multimap B, ! (B) \multimap A : \vdash ! (A)} \quad ! \quad \frac{A, ! (A) \multimap B, ! (B) \multimap A : B \vdash B}{A, ! (A) \multimap B, ! (B) \multimap A : ! (A) \multimap B \vdash B} \multimap \quad \frac{\frac{A, ! (A) \multimap B, ! (B) \multimap A : ! (A) \multimap B \vdash B}{A, ! (A) \multimap B, ! (B) \multimap A : \vdash B} \quad D_C}{\vdash : ! ( ! (A) \multimap B \& ! (B) \multimap A) \vdash ! (A) \multimap B} *$$

CALL-BY-VALUE encoding (132ms)

[illegible]

01-ENC encoding (126ms)

[illegible]



$$(17) (A) \leftrightarrow (B) \vdash B \rightarrow A$$

LJ (37ms)

$$\frac{\frac{B, A \rightarrow B, B \rightarrow A \vdash \bar{B} \quad \star \quad A, B, A \rightarrow B \vdash A \quad \star}{B, A \rightarrow B, B \rightarrow A \vdash A} \supset_L}{A \rightarrow B \wedge B \rightarrow A \vdash A} \star$$

MULTIPLICATIVE encoding (29ms)

**Not provable**

CALL-BY-NAME encoding (105ms)

$$\frac{\frac{B, !!(A) \multimap B, !(B) \multimap A : \cdot \vdash \bar{B}}{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash !(B)} ! \quad \frac{B, !(A) \multimap B, !(B) \multimap A : A \vdash A}{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash A} \star}{B, !(A) \multimap B, !(B) \multimap A : \cdot \vdash A} D_C \multimap \quad \vdash : !(!(A) \multimap B \& !(B) \multimap A) \vdash !(B) \multimap A \quad \star$$

CALL-BY-VALUE encoding (140ms)

$$\frac{\frac{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{B}}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} ! \quad \frac{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{A}}{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)} ! \quad \star}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} \star \quad \frac{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) \vdash !(A)}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)} D_C \multimap \quad \frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B) \multimap !(A)}{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(B) \multimap !(A))} ! \quad \star}{\vdash : !(!(A) \multimap !(B)) \otimes !(!(B) \multimap !(A)) \vdash !(!(B) \multimap !(A))} \star$$

01-ENC encoding (122ms)

$$\frac{\frac{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{B}}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} ! \quad \frac{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{A}}{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)} ! \quad \star}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} \star \quad \frac{B, !(A) \multimap !(B), !(B) \multimap !(A) : !(B) \multimap !(A) \vdash A}{B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash A} D_C \multimap \quad \frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B) \multimap A}{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(B) \multimap A)} \star \quad \frac{!(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(!(B) \multimap A)}{\vdash : !(!(A) \multimap !(B)) \& !(!(B) \multimap !(A)) \vdash !(!(B) \multimap A)} \star$$

$$(18) (A) \leftrightarrow (B), A \vdash B$$

LJ (38ms)

$$\frac{\frac{A, A \rightarrow B, B \rightarrow A \vdash \bar{A} \quad \star \quad A, B, B \rightarrow A \vdash B \quad \star}{A, A \rightarrow B, B \rightarrow A \vdash B} \supset_L}{A, A \rightarrow B \wedge B \rightarrow A \vdash B} \star$$

MULTIPLICATIVE encoding (29ms)

**Not provable**

CALL-BY-NAME encoding (102ms)

$$\frac{\frac{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash \bar{A}}{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \multimap B, !(B) \multimap A : B \vdash B}{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash B} \star}{A, !(A) \multimap B, !(B) \multimap A : \cdot \vdash B} D_C \multimap \quad \vdash : !(A), !(!(A) \multimap B \& !(B) \multimap A) \vdash B \quad \star$$

CALL-BY-VALUE encoding (119ms)

$$\frac{\frac{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{A}}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)} ! \quad \frac{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{B}}{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} ! \quad \star}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} \star \quad \frac{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash !(B)}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} D_C \multimap \quad \vdash : !(A), !(!(A) \multimap !(B)) \otimes !(!(B) \multimap !(A)) \vdash !(B) \quad \star$$

01-ENC encoding (111ms)

$$\frac{\frac{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{A}}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(A)} ! \quad \frac{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash \bar{B}}{A, B, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} ! \quad \star}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash !(B)} \star \quad \frac{A, !(A) \multimap !(B), !(B) \multimap !(A) : !(A) \multimap !(B) \vdash B}{A, !(A) \multimap !(B), !(B) \multimap !(A) : \cdot \vdash B} D_C \multimap \quad \vdash : !(A), !(!(A) \multimap !(B)) \& !(!(B) \multimap !(A)) \vdash B \quad \star$$

$$(19) (A) \leftrightarrow (B), B \vdash A$$

LJ (38ms)

$$\frac{\frac{B, A \rightarrow B, B \rightarrow A \vdash B}{B, A \rightarrow B, B \rightarrow A \vdash A} \star}{B, A \rightarrow B \wedge B \rightarrow A \vdash A} \star \supset_L$$

MULTIPLICATIVE encoding (32ms)

**Not provable**

CALL-BY-NAME encoding (102ms)

$$\frac{\frac{B, ! (A) \multimap B, ! (B) \multimap A : \cdot \vdash B}{B, ! (A) \multimap B, ! (B) \multimap A : \cdot \vdash ! (B)} !}{\frac{B, ! (A) \multimap B, ! (B) \multimap A : ! (B) \multimap A \vdash A}{B, ! (A) \multimap B, ! (B) \multimap A : \cdot \vdash A} D_C}{\cdot : ! (B), ! ( ! (A) \multimap B \& ! (B) \multimap A) \vdash A} \star \multimap$$

CALL-BY-VALUE encoding (118ms)

$$\frac{\frac{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash B}{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash ! (B)} !}{\frac{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : ! (B) \multimap ! (A) \vdash ! (A)}{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash ! (A)} \star}{\frac{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash ! (A)}{\cdot : ! (B), ! ( ! (A) \multimap ! (B) ) \otimes ! ( ! (B) \multimap ! (A) ) \vdash ! (A)} \star} D_C$$

01-ENC encoding (108ms)

$$\frac{\frac{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash B}{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash ! (B)} !}{\frac{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : ! (B) \multimap ! (A) \vdash A}{B, ! (A) \multimap ! (B), ! (B) \multimap ! (A) : \cdot \vdash A} D_C}{\cdot : ! (B), ! ( ! ( ! (A) \multimap ! (B) ) \& ! ( ! (B) \multimap ! (A) ) ) \vdash A} \star \multimap$$

$$(20) \cdot \vdash (A) \leftrightarrow (A)$$

LJ (21ms)

$$\frac{}{\cdot \vdash A \rightarrow A \wedge A \rightarrow A} \star$$

MULTIPLICATIVE encoding (34ms)

$$\frac{\frac{\cdot : A \vdash A}{\cdot : \cdot \vdash A \multimap A} \star}{\cdot : \cdot \vdash A \multimap A \otimes A \multimap A} \star \otimes$$

CALL-BY-NAME encoding (27ms)

$$\frac{\frac{A : \cdot \vdash A}{\cdot : \cdot \vdash ! (A) \multimap A} \star}{\cdot : \cdot \vdash ! (A) \multimap A \& ! (A) \multimap A} \star$$

CALL-BY-VALUE encoding (47ms)

$$\frac{\frac{\frac{A : \cdot \vdash A}{A : \cdot \vdash ! (A)} !}{\cdot : \cdot \vdash ! (A) \multimap ! (A)} \star}{\cdot : \cdot \vdash ! ( ! (A) \multimap ! (A) )} ! \star}{\frac{\frac{A : \cdot \vdash A}{A : \cdot \vdash ! (A)} !}{\cdot : \cdot \vdash ! (A) \multimap ! (A)} \star}{\cdot : \cdot \vdash ! ( ! (A) \multimap ! (A) ) \otimes ! ( ! (A) \multimap ! (A) )} ! \otimes$$

01-ENC encoding (47ms)

$$\frac{\frac{\frac{A : \cdot \vdash A}{\cdot : \cdot \vdash ! (A) \multimap A} \star}{\cdot : \cdot \vdash ! ( ! (A) \multimap A )} !}{\cdot : \cdot \vdash ! ( ! (A) \multimap A ) \& ! ( ! (A) \multimap A )} ! \star}{\cdot : \cdot \vdash ! ( ! ( ! (A) \multimap A ) \& ! ( ! (A) \multimap A ) )} ! \star$$

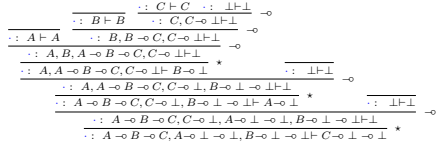


(23)  $A \rightarrow B \rightarrow C, \neg A, \neg B \vdash \neg C$

LJ (206ms)



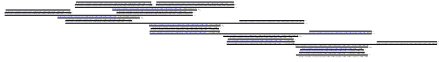
MULTIPLICATIVE encoding (124ms)



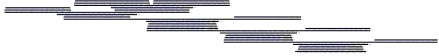
CALL-BY-NAME encoding (221ms)



CALL-BY-VALUE encoding (324ms)



01-ENC encoding (380ms)

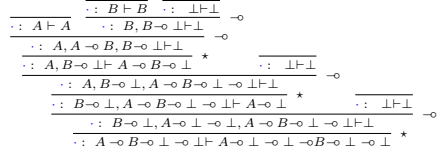


(24)  $\neg A \rightarrow B \vdash \neg A \rightarrow \neg B$

LJ (94ms)



MULTIPLICATIVE encoding (106ms)



CALL-BY-NAME encoding (199ms)



CALL-BY-VALUE encoding (241ms)



01-ENC encoding (269ms)

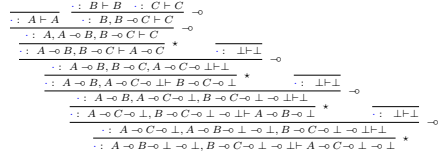


$$(25) \neg\neg A \rightarrow B, \neg\neg B \rightarrow C \vdash \neg\neg A \rightarrow C$$

LJ (324ms)



MULTIPLICATIVE encoding (145ms)



CALL-BY-NAME encoding (283ms)



CALL-BY-VALUE encoding (375ms)



01-ENC encoding (421ms)



$$(26) \cdot \vdash (\neg\neg A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$$

LJ (137ms)



MULTIPLICATIVE encoding (66ms)

**Not provable**

CALL-BY-NAME encoding (601ms)



CALL-BY-VALUE encoding (584ms)



01-ENC encoding (646ms)



$$(27) \cdot \vdash (\neg\neg(A) \leftrightarrow (B)) \leftrightarrow (\neg\neg A \rightarrow B \wedge \neg\neg B \rightarrow A)$$

LJ (2703ms)

\_\_\_\_\_

MULTIPLICATIVE encoding (84ms)

**Not provable**

CALL-BY-NAME encoding (661ms)

\_\_\_\_\_

CALL-BY-VALUE encoding (841ms)

\_\_\_\_\_

01-ENC encoding (1132ms)

\_\_\_\_\_

$$(28) (A) \leftrightarrow (B) \vdash (A \rightarrow C) \leftrightarrow (B \rightarrow C)$$

LJ (75ms)

\_\_\_\_\_

MULTIPLICATIVE encoding (95ms)

$$\frac{\frac{\vdash A \vdash A \quad \vdash C \vdash C}{\vdash B \vdash B} \multimap \quad \frac{\vdash B \vdash B \quad \vdash C \vdash C}{\vdash A, A \multimap C \vdash C} \multimap}{\vdash B, A \multimap C, B \multimap A \vdash C} \multimap \quad \frac{\vdash A \vdash A \quad \vdash B, B \multimap C \vdash C}{\vdash A, A \multimap B, B \multimap C \vdash C} \multimap}{\vdash B \multimap A \vdash A \multimap C \multimap B \multimap C} * \quad \frac{\vdash A \vdash A \quad \vdash B, B \multimap C \vdash C}{\vdash A \multimap B \vdash B \multimap C \multimap A \multimap C} *}{\vdash A \multimap B, B \multimap A \vdash A \multimap C \multimap B \multimap C \otimes B \multimap C \multimap A \multimap C} \otimes}{\vdash A \multimap B \otimes B \multimap A \vdash A \multimap C \multimap B \multimap C \otimes B \multimap C \multimap A \multimap C} *$$

CALL-BY-NAME encoding (173ms)

\_\_\_\_\_

CALL-BY-VALUE encoding (288ms)

\_\_\_\_\_

01-ENC encoding (268ms)

\_\_\_\_\_

$$(29) (A) \leftrightarrow (B) \vdash (C \rightarrow A) \leftrightarrow (C \rightarrow B)$$

LJ (72ms)

MULTIPLICATIVE encoding (96ms)

CALL-BY-NAME encoding (250ms)

CALL-BY-VALUE encoding (387ms)

01-ENC encoding (356ms)

$$(30) (A) \leftrightarrow (B) \vdash (A \wedge C) \leftrightarrow (B \wedge C)$$

LJ (51ms)

MULTIPLICATIVE encoding (99ms)

CALL-BY-NAME encoding (138ms)

CALL-BY-VALUE encoding (238ms)

01-ENC encoding (234ms)

$$(31) (A) \leftrightarrow (B) \vdash (C \wedge A) \leftrightarrow (C \wedge B)$$

LJ (51ms)

$$\frac{\frac{A, C, A \rightarrow B, B \rightarrow A \vdash A}{A, C, A \rightarrow B, B \rightarrow A \vdash B}^* \quad \frac{A, B, C, B \rightarrow A \vdash B}{B, C, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \frac{B, C, A \rightarrow B, B \rightarrow A \vdash B}{B, C, A \rightarrow B, B \rightarrow A \vdash A}^* \quad \frac{A, B, C, A \rightarrow B \vdash A}{B, C, A \rightarrow B, B \rightarrow A \vdash A}^*}{A \rightarrow B \wedge B \rightarrow A \vdash A \vdash C \wedge A \rightarrow C \wedge B \wedge C \wedge B \rightarrow C \wedge A}^* \supset_L$$

MULTIPLICATIVE encoding (98ms)

$$\begin{array}{c}
\frac{\vdots \quad C \vdash \overline{C} \quad \vdots \quad B \vdash \overline{B}}{\vdots \quad A \vdash A} \otimes \quad \frac{\vdots \quad B \vdash \overline{C} \quad \vdots \quad C \otimes B}{\vdots \quad B \vdash \overline{C} \quad \vdots \quad A \vdash \overline{C} \otimes A} \otimes \\
\frac{\vdots \quad A, C, A \multimap \overline{B} \multimap C \otimes B}{\vdots \quad A \multimap \overline{B} \vdash C \otimes A \multimap \overline{C} \otimes \overline{B}} * \quad \frac{\vdots \quad B, C, B \multimap A \multimap C \otimes A}{\vdots \quad B \multimap A \vdash C \otimes B \multimap \overline{C} \otimes A} * \\
\frac{\vdots \quad A \multimap B, B \multimap A \vdash C \otimes A \multimap \overline{C} \otimes B \quad C \otimes B \quad C \otimes B \multimap \overline{C} \otimes A}{\vdots \quad A \multimap B \quad B \multimap A \vdash C \otimes A \multimap \overline{C} \otimes B \quad C \otimes B \quad C \otimes B \multimap \overline{C} \otimes A} *
\end{array}$$

CALL-BY-NAME encoding (142ms)

[illegible]

CALL-BY-VALUE encoding (227ms)

01-ENC encoding (229ms)

Figure 1. The structure of the proposed model. The model is composed of three main parts: the input layer, the hidden layer, and the output layer. The input layer consists of 10 nodes, representing the input features. The hidden layer consists of 10 nodes, representing the hidden features. The output layer consists of 10 nodes, representing the output features. The model is trained using a supervised learning approach, where the input features are used to predict the output features. The model is evaluated using a set of test data, and the performance is measured using the mean squared error (MSE) and the coefficient of determination (R-squared).

$$(32) (A) \leftrightarrow (B) \vdash (\neg A) \leftrightarrow (\neg B)$$

LJ (74ms)

[illegible]

MULTIPLICATIVE encoding (97ms)

[illegible]

CALL-BY-NAME encoding (223ms)

The diagram illustrates the experimental design for two groups. It shows two parallel timelines. The left timeline (Group 1) starts with a 10-minute rest, followed by a 10-minute walk, then a 10-minute rest, and finally a 10-minute walk. The right timeline (Group 2) starts with a 10-minute rest, followed by a 10-minute walk, then a 10-minute rest, and finally a 10-minute walk. The diagram uses arrows to indicate the flow of the experiment and labels for each phase.

CALL-BY-VALUE encoding (304ms)

01-ENC encoding (320ms)

The diagram illustrates the experimental design for two groups: Control and Experimental. Both groups follow a similar sequence of phases: Baseline, Training, Transfer, and Retention. The Control group's performance is shown as a continuous line across all phases. The Experimental group's performance is shown as a line that drops significantly during the Transfer phase, marked with a red 'X' and a red arrow pointing to the Retention phase, indicating a failure to maintain performance during transfer.



$$(33) \cdot \vdash (A \wedge B \wedge C) \leftrightarrow (A \wedge B \wedge C)$$

LJ (21ms)

$$\cdot \vdash A \wedge B \wedge C \rightarrow A \wedge B \wedge C \wedge A \wedge B \wedge C \rightarrow A \wedge B \wedge C \quad \star$$

MULTIPLICATIVE encoding (66ms)

$$\frac{\frac{\frac{\vdots, A \vdash A \quad \vdots, B \vdash B}{\vdots, A, B \vdash A \otimes B} \otimes \quad \frac{\vdots, C \vdash C}{\vdots, C \vdash C}}{\vdots, A, B, C \vdash A \otimes B \otimes C} \otimes \quad \frac{\frac{\frac{\vdots, A \vdash A \quad \vdots, B \vdash B}{\vdots, A, B \vdash A \otimes B} \otimes \quad \frac{\vdots, C \vdash C}{\vdots, C \vdash C}}{\vdots, A, B, C \vdash A \otimes B \otimes C} \otimes \quad \frac{\frac{\vdots, A \otimes B \otimes C \multimap A \otimes B \otimes C}{\vdots, A \otimes B \otimes C \multimap A \otimes B \otimes C} \star \quad \frac{\vdots, A \otimes B \otimes C \multimap A \otimes B \otimes C}{\vdots, A \otimes B \otimes C \multimap A \otimes B \otimes C} \star}{\vdots, \vdots \vdash A \otimes B \otimes C \multimap A \otimes B \otimes C \otimes A \otimes B \otimes C \multimap A \otimes B \otimes C} \otimes$$

CALL-BY-NAME encoding (34ms)

$$\frac{A, B, C : \vdash A \quad A, B, C : \vdash B \quad A, B, C : \vdash C \quad A, B, C : \vdash A \quad A, B, C : \vdash B \quad A, B, C : \vdash C}{\therefore \vdash !(A \& B \& C) \multimap A \& B \& C \& !(A \& B \& C) \multimap A \& B \& C}$$

CALL-BY-VALUE encoding (99ms)

[illegible]

01-ENC encoding (124ms)

$$(34) \cdot \vdash (A \wedge B) \leftrightarrow (B \wedge A)$$

LJ (21ms)

$$\cdot \vdash A \wedge B \rightarrow B \wedge A \wedge B \wedge A \rightarrow A \wedge B \quad \star$$

MULTIPLICATIVE encoding (73ms)

$$\frac{\frac{\frac{\vdots : B \vdash B}{\vdots : A, B \vdash B \otimes A} \otimes \quad \frac{\frac{\vdots : A \vdash A}{\vdots : A, B \vdash A \otimes B} \otimes}{\vdots : \vdash A \otimes B \multimap B \otimes A} \star \quad \frac{\frac{\frac{\vdots : A \vdash A}{\vdots : A, B \vdash A \otimes B} \otimes \quad \frac{\vdots : B \vdash B}{\vdots : A, B \vdash B \otimes B} \otimes}{\vdots : \vdash B \otimes A \multimap A \otimes B} \star}{\vdots : \vdash A \otimes B \multimap B \otimes A \otimes B \otimes A \multimap A \otimes B} \otimes$$

CALL-BY-NAME encoding (35ms)

$$\frac{A, B : \cdot \vdash B \quad A, B : \cdot \vdash A \quad A, B : \cdot \vdash A \quad A, B : \cdot \vdash B}{\cdot : \cdot \vdash !(A \& B) \multimap B \& A \& !(B \& A) \multimap A \& B} \star$$

CALL-BY-VALUE encoding (107ms)

$$\begin{array}{c}
\frac{A, B : \vdash B \quad A, B : \vdash A}{A, B : \vdash ! (B) \quad A, B : \vdash ! (A)} ! \quad \frac{A, B : \vdash A \quad A, B : \vdash B}{A, B : \vdash ! (A) \quad A, B : \vdash ! (B)} ! \\
\frac{A, B : \vdash ! (B) \otimes ! (A)}{\vdash : \vdash ! (A) \otimes ! (B) \rightarrow ! (B) \otimes ! (A)} * \quad \frac{A, B : \vdash ! (A) \otimes ! (B)}{\vdash : \vdash ! (B) \otimes ! (A) \rightarrow ! (A) \otimes ! (B)} * \\
\frac{\vdash : \vdash ! (A) \otimes ! (B) \rightarrow ! (B) \otimes ! (A) \quad \vdash : \vdash ! (B) \otimes ! (A) \rightarrow ! (A) \otimes ! (B)}{\vdash : \vdash ! (A) \otimes ! (B) \rightarrow ! (B) \otimes ! (A) \rightarrow ! (A) \otimes ! (B)} * \\
\frac{\vdash : \vdash ! (A) \otimes ! (B) \rightarrow ! (B) \otimes ! (A) \quad \vdash : \vdash ! (B) \otimes ! (A) \rightarrow ! (A) \otimes ! (B)}{\vdash : \vdash ! (A) \otimes ! (B) \rightarrow ! (B) \otimes ! (A) \rightarrow ! (B) \otimes ! (A) \rightarrow ! (A) \otimes ! (B)} *
\end{array}$$

01-ENC encoding (102ms)

$$\begin{array}{c}
\frac{\frac{A, B : \vdash B \quad A, B : \vdash A}{A, B : \vdash B \& A} \quad *}{A, B : \vdash ! (B \& A)} \quad ! \\
\vdots : \vdash ! ( ! (A) \& ! (B) ) \rightarrow ! (B \& A) \quad * \\
\vdots : \vdash ! ( ! ( ! (A) \& ! (B) ) \rightarrow ! (B \& A) ) \quad * \\
\vdots : \vdash ! ( ! ( ! ( ! (A) \& ! (B) ) \rightarrow ! (B \& A) ) \& ! ( ! ( ! (B) \& ! (A) ) \rightarrow ! (A \& B) ) ) \quad * \\
\vdots : \vdash ! ( ! ( ! ( ! ( ! (A) \& ! (B) ) \rightarrow ! (B \& A) ) \& ! ( ! ( ! (B) \& ! (A) ) \rightarrow ! (A \& B) ) ) \rightarrow ! ( ! ( ! (B) \& ! (A) ) \rightarrow ! (A \& B) ) ) \quad !
\end{array}$$

$$(35) \cdot \vdash (A \wedge A) \leftrightarrow (A)$$

LJ (21ms)

$$\frac{}{\cdot \vdash A \wedge A \rightarrow A \wedge A \rightarrow A \wedge A} \star$$

MULTIPLICATIVE encoding (34ms)

**Not provable**

CALL-BY-NAME encoding (28ms)

$$\frac{\overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A}}{\cdot : \cdot \vdash !(A \& A) \multimap A \& !(A) \multimap A \& A} \star$$

CALL-BY-VALUE encoding (76ms)

$$\frac{\frac{\overline{A : \cdot \vdash A}}{A : \cdot \vdash !(A)} ! \quad \frac{\overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A}}{A : \cdot \vdash !(A)} ! \quad \frac{\overline{A : \cdot \vdash A}}{A : \cdot \vdash !(A)} !}{\cdot : \cdot \vdash !(!(A) \otimes !(A) \multimap !(A)) \star \quad \cdot : \cdot \vdash !(!(A) \multimap !(A) \otimes !(A)) \star} \otimes$$

01-ENC encoding (81ms)

$$\frac{\frac{\overline{A : \cdot \vdash A}}{\cdot : \cdot \vdash !(!(A) \& !(A)) \multimap A} \star \quad \frac{\overline{A : \cdot \vdash A} \quad \overline{A : \cdot \vdash A}}{A : \cdot \vdash !(A \& A)} !}{\cdot : \cdot \vdash !(!(!(A) \& !(A)) \multimap A) ! \quad \cdot : \cdot \vdash !(!(A) \multimap !(A \& A)) \star} \star$$

$$(36) A \vdash (A \rightarrow B) \leftrightarrow (B)$$

LJ (37ms)

$$\frac{\frac{\overline{A, A \rightarrow B \vdash A} \star \quad \overline{A, B \vdash B} \star}{A, A \rightarrow B \vdash B} \supset_L}{A \vdash A \rightarrow B \rightarrow B \wedge B \rightarrow A \rightarrow B} \star$$

MULTIPLICATIVE encoding (42ms)

**Not provable**

CALL-BY-NAME encoding (73ms)

$$\frac{\frac{\overline{A, !(A) \multimap B : \cdot \vdash A}}{A, !(A) \multimap B : \cdot \vdash !(A)} ! \quad \frac{\overline{A, !(A) \multimap B : B \vdash B}}{A, !(A) \multimap B : B \vdash B} \multimap}{\frac{A, !(A) \multimap B : \cdot \vdash B}{\cdot : !(A) \vdash !(!(A) \multimap B) \multimap B \& !(B) \multimap !(A) \multimap B} D_C} \star$$

CALL-BY-VALUE encoding (129ms)

$$\frac{\frac{\overline{A, !(A) \multimap !(B) : \cdot \vdash A}}{A, !(A) \multimap !(B) : \cdot \vdash !(A)} ! \quad \frac{\overline{A, B, !(A) \multimap !(B) : \cdot \vdash B}}{A, B, !(A) \multimap !(B) : \cdot \vdash !(B)} ! \quad \frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash !(B)} !}{\frac{A, !(A) \multimap !(B) : \cdot \vdash !(B)}{A : \cdot \vdash !(!(A) \multimap !(B)) \star} \star \quad \frac{A, B, !(A) \multimap !(B) : \cdot \vdash !(B)}{A, B : \cdot \vdash !(!(A) \multimap !(B)) \star} \star} \otimes$$

01-ENC encoding (123ms)

$$\frac{\frac{\overline{A, !(A) \multimap !(B) : \cdot \vdash A}}{A, !(A) \multimap !(B) : \cdot \vdash !(A)} ! \quad \frac{\overline{A, B, !(A) \multimap !(B) : \cdot \vdash B}}{A, B, !(A) \multimap !(B) : \cdot \vdash !(B)} ! \quad \frac{\overline{A, B : \cdot \vdash B}}{A, B : \cdot \vdash !(B)} !}{\frac{A, !(A) \multimap !(B) : \cdot \vdash B}{A : \cdot \vdash !(!(A) \multimap !(B)) \star} \star \quad \frac{A, B, !(A) \multimap !(B) : \cdot \vdash !(B)}{A, B : \cdot \vdash !(!(A) \multimap !(B)) \star} \star} \otimes$$

$$(37) \ B \vdash (A \rightarrow B) \leftrightarrow (B)$$

LJ (21ms)

$$\overline{B \vdash A \rightarrow B \rightarrow B \wedge B \rightarrow A \rightarrow B}^*$$

MULTIPLICATIVE encoding (40ms)

**Not provable**

CALL-BY-NAME encoding (39ms)

$$\frac{\overline{B, ! (A) \multimap B : \cdot \vdash B} \quad \overline{A, B : \cdot \vdash B}}{\cdot : ! (B) \vdash ! ( ! (A) \multimap B) \multimap B \& ! (B) \multimap ! (A) \multimap B}^*$$

CALL-BY-VALUE encoding (152ms)

$$\frac{\frac{\overline{B, ! (A) \multimap ! (B) : \cdot \vdash B} \quad \frac{\overline{B : \cdot \vdash ! ( ! (A) \multimap ! (B))}^!}{B : \cdot \vdash ! ( ! (A) \multimap ! (B)) \multimap ! (B)}^*}{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B))}^! \quad \frac{\overline{A, B : \cdot \vdash B}^! \quad \frac{\overline{B : \cdot \vdash ! (A) \multimap ! (B)}^!}{B : \cdot \vdash ! ( ! (A) \multimap ! (B))}^*}{B : \cdot \vdash ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B))}^!}{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B))}^!} \otimes$$

$$\frac{\cdot : ! (B) \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B))}^*}{\cdot : ! (B) \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap ! (B)) \otimes ! ( ! (B) \multimap ! ( ! (A) \multimap ! (B))}^*}$$

01-ENC encoding (88ms)

$$\frac{\frac{\overline{B, ! (A) \multimap ! (B) : \cdot \vdash B} \quad \frac{\overline{B : \cdot \vdash ! ( ! (A) \multimap ! (B)) \multimap B}^*}{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap B)}^!}{B : \cdot \vdash ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B)}^!}{B : \cdot \vdash ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B)}^!}^*$$

$$\frac{\cdot : ! (B) \vdash ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B)}^*}{\cdot : ! (B) \vdash ! ( ! ( ! ( ! (A) \multimap ! (B)) \multimap B) \& ! ( ! (B) \multimap ! ( ! (A) \multimap B)}^*}$$

$$(38) \ \neg A \vdash (A \rightarrow B) \leftrightarrow (\neg A)$$

LJ (52ms)

$$\frac{\overline{A, A \rightarrow B, A \rightarrow \perp \vdash A}^* \quad \overline{A, \perp, A \rightarrow B \vdash \perp}^* \quad \overline{A, A \rightarrow \perp \vdash A}^* \quad \overline{A, \perp \vdash B}^*}{\frac{A, A \rightarrow B, A \rightarrow \perp \vdash \perp}{A \rightarrow \perp \vdash A \rightarrow B \rightarrow A \rightarrow \perp \wedge A \rightarrow \perp \rightarrow A \rightarrow B}^*} \supset_L$$

MULTIPLICATIVE encoding (42ms)

**Not provable**

CALL-BY-NAME encoding (101ms)

$$\frac{\overline{A, ! (A) \multimap B, ! (A) \multimap \emptyset : \vdash A} \quad \overline{A, ! (A) \multimap B, ! (A) \multimap \emptyset : \emptyset \vdash \emptyset} \quad \overline{A, ! (A) \multimap \emptyset : \vdash A} \quad \overline{A, ! (A) \multimap \emptyset : \emptyset \vdash B}^*}{\frac{\overline{A, ! (A) \multimap B, ! (A) \multimap \emptyset : ! (A)}^! \quad \overline{A, ! (A) \multimap B, ! (A) \multimap \emptyset : \emptyset \vdash \emptyset}^!}{\frac{A, ! (A) \multimap B, ! (A) \multimap \emptyset : ! (A) \multimap \emptyset \vdash \emptyset}{\vdash ! ( ! (A) \multimap \emptyset) \vdash ! ( ! (A) \multimap B) \multimap ! (A) \multimap \emptyset \& ! ( ! (A) \multimap \emptyset) \multimap ! (A) \multimap B}^*}^!}_{D_C}$$

CALL-BY-VALUE encoding (201ms)

$$\frac{\overline{A, ! (A) \multimap \emptyset, ! (A) \multimap ! (B) : \vdash A} \quad \overline{A, ! (A) \multimap \emptyset, ! (A) \multimap ! (B) : ! (A)}^! \quad \overline{A, ! (A) \multimap \emptyset, ! (A) \multimap ! (B) : \emptyset \vdash \emptyset}^*}{\frac{\overline{A, ! (A) \multimap \emptyset, ! (A) \multimap ! (B) : ! (A)}^! \quad \overline{A, ! (A) \multimap \emptyset, ! (A) \multimap ! (B) : \emptyset \vdash \emptyset}^!}{\frac{A, ! (A) \multimap \emptyset, ! (A) \multimap ! (B) : ! (A) \multimap \emptyset}{\vdash ! ( ! (A) \multimap \emptyset) \vdash ! ( ! (A) \multimap ! (B)) \multimap ! (A) \multimap \emptyset \& ! ( ! (A) \multimap \emptyset) \multimap ! (A) \multimap ! (B)}^*}^!}_{D_C}$$

01-ENC encoding (203ms)

$$\frac{\overline{A, ! (A) \multimap ! (B), ! (A) \multimap ! (B) : \vdash A} \quad \overline{A, ! (A) \multimap ! (B), ! (A) \multimap ! (B) : ! (A)}^! \quad \overline{A, ! (A) \multimap ! (B), ! (A) \multimap ! (B) : ! (B) \vdash B}^*}{\frac{\overline{A, ! (A) \multimap ! (B), ! (A) \multimap ! (B) : ! (A)}^! \quad \overline{A, ! (A) \multimap ! (B), ! (A) \multimap ! (B) : ! (B) \vdash B}^!}{\frac{A, ! (A) \multimap ! (B), ! (A) \multimap ! (B) : ! (A) \multimap ! (B)}{\vdash ! ( ! (A) \multimap ! (B)) \vdash ! ( ! (A) \multimap ! (B)) \multimap ! (A) \multimap ! (B)}^*}^!}_{D_C}$$

$$(39) \neg B \vdash (A \rightarrow B) \leftrightarrow (\neg A)$$

LJ (61ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \vdash A}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} *}{A, A \rightarrow B, B \rightarrow \perp \vdash \perp} *}{B \rightarrow \perp \vdash A \rightarrow B \rightarrow A \rightarrow \perp \wedge A \rightarrow \perp \vdash A \rightarrow B} * \supset_L \supset_L \frac{\frac{\frac{A, B, B \rightarrow \perp \vdash \perp}{A, B, B \rightarrow \perp \vdash \perp} *}{A, B, B \rightarrow \perp \vdash \perp} *}{A, A \rightarrow \perp, B \rightarrow \perp \vdash A} * \frac{\frac{A, \perp, B \rightarrow \perp \vdash B}{A, \perp, B \rightarrow \perp \vdash \perp} *}{A, A \rightarrow \perp, B \rightarrow \perp \vdash \perp} * \supset_L$$

MULTIPLICATIVE encoding (43ms)

Not provable

CALL-BY-NAME encoding (185ms)

[illegible]

CALL-BY-VALUE encoding (256ms)

[illegible]

01-ENC encoding (259ms)

Figure 1 illustrates a hierarchical tree diagram representing the classification of 1000 genes into 10 clusters. The tree structure shows the relationships between the genes and the clusters. The genes are listed in a column on the left, and the clusters are listed in a column on the right. The tree structure shows the relationships between the genes and the clusters.

$$(40) \quad B \vdash (A \wedge B) \leftrightarrow (A)$$

LJ (21ms)

$$\overline{B \vdash A \wedge B \rightarrow A \wedge A \rightarrow A \wedge B}^{\star}$$

MULTIPLICATIVE encoding (34ms)

Not provable

CALL-BY-NAME encoding (34ms)

$$\frac{\overline{A, B : \cdot \vdash A} \quad \overline{A, B : \cdot \vdash A} \quad \overline{A, B : \cdot \vdash B}}{\cdot : !(B) \vdash !(A \& B) \multimap A \& !(A) \multimap A \& B}^*$$

CALL-BY-VALUE encoding (88ms)

$$\begin{array}{c}
\frac{\overline{A, B : \vdash A}}{A, B : \vdash ! (A)} ! \\
\frac{\overline{B : \vdash ! (A) \otimes ! (B) \rightarrow ! (A)}}{B : \vdash ! (! (A) \otimes ! (B) \rightarrow ! (A))} ! \\
\frac{\overline{B : \vdash ! (! (A) \otimes ! (B) \rightarrow ! (A))}}{B : \vdash ! (! (! (A) \otimes ! (B) \rightarrow ! (A)) \rightarrow ! (! (A) \rightarrow ! (! (A) \otimes ! (B)))} ! \\
\frac{\overline{A, B : \vdash A} \quad \overline{A, B : \vdash ! (A)} ! \quad \overline{A, B : \vdash ! (B)}}{A, B : \vdash ! (A) \rightarrow ! (A) \otimes ! (B)} \otimes \\
\frac{\overline{B : \vdash ! (! (A) \rightarrow ! (A) \otimes ! (B))} \quad \overline{B : \vdash ! (! (A) \rightarrow ! (A) \otimes ! (B))} \quad \overline{B : \vdash ! (! (A) \rightarrow ! (A) \otimes ! (B))}}{B : \vdash ! (! (A) \rightarrow ! (A) \otimes ! (B))} \otimes
\end{array}$$

01-ENC encoding (88ms)

$$\begin{array}{c}
\frac{}{A, B : \vdash A \quad A, B : \vdash B} \star \\
\frac{}{A, B : \vdash A \& B} ! \\
\frac{B : \vdash \mathbb{I}!(A) \& \mathbb{I}(B) \rightarrow A}{B : \vdash \mathbb{I}!(\mathbb{I}!(A) \& \mathbb{I}(B)) \rightarrow A} \star \\
\frac{B : \vdash \mathbb{I}!(\mathbb{I}!(A) \& \mathbb{I}(B)) \rightarrow A \& \mathbb{I}!(A) \rightarrow \mathbb{I}(A \& B)}{B : \vdash \mathbb{I}!(\mathbb{I}!(A) \& \mathbb{I}(B)) \rightarrow A \& \mathbb{I}!(A) \rightarrow \mathbb{I}(A \& B)} \star \\
\frac{}{\vdash \mathbb{I}(B) \rightarrow \mathbb{I}!(\mathbb{I}!(A) \& \mathbb{I}(B)) \rightarrow A \& \mathbb{I}!(A) \rightarrow \mathbb{I}(A \& B))} \star
\end{array}$$

$$(41) \neg B \vdash (A \wedge B) \leftrightarrow (B)$$

LJ (37ms)

$$\frac{\frac{\overline{B, B \rightarrow \perp \vdash B}^* \quad \overline{B, \perp \vdash A}^*}{B, B \rightarrow \perp \vdash A} \supset_L}{B \rightarrow \perp \vdash A \wedge B \rightarrow B \wedge B \rightarrow A \wedge B}^*$$

MULTIPLICATIVE encoding (41ms)

Not provable

CALL-BY-NAME encoding (78ms)

$$\frac{\frac{\frac{B, ! (B) \multimap 0 : \vdash B}{B, ! (B) \multimap 0 : \vdash ! (B)} \quad ! \quad \frac{B, ! (B) \multimap 0 : \vdash B}{B, ! (B) \multimap 0 : \vdash A} \quad *}{\frac{B, ! (B) \multimap 0 : \vdash ! (B) \quad ! (B) \multimap 0 \vdash A}{B, ! (B) \multimap 0 : \vdash A} \quad D_C} \multimap \quad \frac{\frac{A, B, ! (B) \multimap 0 : \vdash B}{\vdash : ! (B) \multimap 0 \vdash ! (A \& B) \multimap 0 \vdash B \& ! (B) \multimap 0 \vdash A \& B} \quad \frac{B, ! (B) \multimap 0 : \vdash B}{B, ! (B) \multimap 0 : \vdash B} \quad ,$$

CALL-BY-VALUE encoding (172ms)

$$\begin{array}{c}
\frac{A, B, \ulcorner B \urcorner \rightarrow 0 : \vdash \bar{B}}{A, B, \ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner B \urcorner} \quad \frac{B, \ulcorner B \urcorner \rightarrow 0 : \vdash \bar{B}}{B, \ulcorner B \urcorner \rightarrow 0 : \vdash 0 \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner} \quad \frac{B, \ulcorner B \urcorner \rightarrow 0 : \ulcorner B \urcorner \rightarrow 0 \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner}{B, \ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner} \quad D_C \\
\frac{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner}{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner} \quad \frac{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner B \urcorner \rightarrow \ulcorner A \urcorner \otimes \ulcorner B \urcorner}{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner B \urcorner \rightarrow \ulcorner A \urcorner \otimes \ulcorner B \urcorner} \quad \otimes \\
\frac{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner}{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner} \quad \frac{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner}{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner} \quad \otimes \\
\frac{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner}{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner} \quad \frac{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner}{\ulcorner B \urcorner \rightarrow 0 : \vdash \ulcorner A \urcorner \otimes \ulcorner B \urcorner \rightarrow \ulcorner B \urcorner} \quad \otimes
\end{array}$$

01-ENC encoding (194ms)

[illegible]

(42)  $\vdash A \rightarrow \neg\neg A$

LJ (38ms)

$$\frac{\frac{\frac{A, A \rightarrow \perp \vdash A}{\vdash A}^* \quad \frac{A, \perp \vdash \perp}{\vdash \perp}^*}{A, A \rightarrow \perp \vdash \perp} \supset_L}{\vdash A \rightarrow A \rightarrow \perp \rightarrow \perp}^*$$

MULTIPLICATIVE encoding (41ms)

$$\frac{\frac{\cdot : A \vdash A \quad \cdot : \perp \vdash \perp}{\cdot : A, A \multimap \perp \vdash \perp} \multimap}{\cdot : \cdot \vdash A \multimap A \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (66ms)

$$\frac{\frac{\frac{A, !(A) \multimap 0 : \cdot \vdash A}{A, !(A) \multimap 0 : \cdot \vdash !(A)} ! \quad \frac{}{A, !(A) \multimap 0 : 0 \vdash 0} \star}{\frac{\frac{A, !(A) \multimap 0 : !(A) \multimap 0 \vdash 0}{A, !(A) \multimap 0 : \cdot \vdash 0} D_C}{\cdot : \cdot \vdash !(A) \multimap !(A) \multimap 0 \multimap 0} \star} \multimap$$

CALL-BY-VALUE encoding (84ms)

$$\frac{\frac{\frac{A, !(A) \rightarrow \mathbf{0} : \cdot \vdash A}{A, !(A) \rightarrow \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \rightarrow \mathbf{0} : \mathbf{0} \vdash \mathbf{0}}{A, !(A) \rightarrow \mathbf{0} : \mathbf{0} \vdash \mathbf{0}} \star}{\frac{A, !(A) \rightarrow \mathbf{0} : \cdot \vdash A}{A, !(A) \rightarrow \mathbf{0} : \cdot \vdash !(A)} ! \quad \frac{A, !(A) \rightarrow \mathbf{0} : \mathbf{0} \vdash \mathbf{0}}{A, !(A) \rightarrow \mathbf{0} : \mathbf{0} \vdash \mathbf{0}} \star} D_C \quad \star$$

01-ENC encoding (81ms)

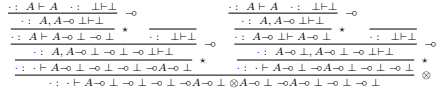
[illegible]

$$(43) \cdot \vdash (\neg \neg \neg A) \leftrightarrow (\neg A)$$

LJ (81ms)



MULTIPLICATIVE encoding (105ms)



CALL-BY-NAME encoding (244ms)



CALL-BY-VALUE encoding (295ms)



01-ENC encoding (343ms)



$$(44) \cdot \vdash \neg A \wedge \neg A$$

LJ (38ms)

$$\frac{\frac{\frac{}{A, A \rightarrow \perp \vdash A} \star \quad \frac{}{A, \perp \vdash \perp} \star}{A, A \rightarrow \perp \vdash \perp} \supset_L \quad \cdot \vdash A \wedge A \rightarrow \perp \rightarrow \perp} \star$$

MULTIPLICATIVE encoding (41ms)

$$\frac{\frac{\cdot : A \vdash A \quad \cdot : \perp \vdash \perp}{\cdot : A, A \multimap \perp \vdash \perp} \multimap \quad \cdot : \cdot \vdash A \otimes A \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (65ms)

$$\frac{\frac{\frac{A, !(A) \multimap 0 : \cdot \vdash A}{A, !(A) \multimap 0 : \cdot \vdash !(A)} ! \quad \frac{}{A, !(A) \multimap 0 : 0 \vdash 0} \star}{\frac{A, !(A) \multimap 0 : !(A) \multimap 0 \vdash 0}{A, !(A) \multimap 0 : \cdot \vdash 0} D_C} \multimap \quad \cdot : \cdot \vdash !(A \& !(A) \multimap 0) \multimap 0} \star$$

CALL-BY-VALUE encoding (70ms)

$$\frac{\frac{\frac{A, !(A) \multimap 0 : \cdot \vdash A}{A, !(A) \multimap 0 : \cdot \vdash !(A)} ! \quad \frac{}{A, !(A) \multimap 0 : 0 \vdash 0} \star}{\frac{A, !(A) \multimap 0 : !(A) \multimap 0 \vdash 0}{A, !(A) \multimap 0 : \cdot \vdash 0} D_C} \multimap \quad \cdot : \cdot \vdash !(A) \otimes !(A) \multimap 0 \multimap 0} \star \quad \cdot : \cdot \vdash !(A) \otimes !(A) \multimap 0 \multimap 0} !$$

01-ENC encoding (70ms)

$$\frac{\frac{\frac{A, !(A) \multimap !(0) : \cdot \vdash A}{A, !(A) \multimap !(0) : \cdot \vdash !(A)} ! \quad \frac{}{A, !(A) \multimap !(0) : !(0) \vdash 0} \star}{\frac{A, !(A) \multimap !(0) : !(A) \multimap !(0) \vdash 0}{A, !(A) \multimap !(0) : \cdot \vdash 0} D_C} \multimap \quad \cdot : \cdot \vdash !(A) \& !(A) \multimap !(0) \multimap 0} \star \quad \cdot : \cdot \vdash !(A) \& !(A) \multimap !(0) \multimap 0} !$$

(45)  $\vdash \neg(A) \leftrightarrow (\neg A)$

LJ (80ms)

[illegible]

MULTIPLICATIVE encoding (41ms)

Not provable

CALL-BY-NAME encoding (231ms)

[illegible]

CALL-BY-VALUE encoding (282ms)

Phylogenetic tree of the 16S rDNA sequences of the 16 isolates. The tree shows relationships between various bacterial strains, with bootstrap values indicated at the nodes. The strains are grouped into several clusters, with some labeled as 'E. coli' and others as 'Shigella'.

01-ENC encoding (300ms)

$$(46) \cdot \vdash \neg\neg\neg\neg A \rightarrow A$$

LJ (66ms)

[illegible]

MULTIPLICATIVE encoding (41ms)

Not provable

CALL-BY-NAME encoding (198ms)

CALL-BY-VALUE encoding (210ms)

The diagram illustrates the hierarchical structure of the 2019-2020 COVID-19 dataset. It is organized into a grid of boxes, each representing a different level of data aggregation. The top row shows the full dataset (1,000,000 records). Subsequent rows show the dataset split by country (e.g., 1,000,000 records for the USA, 1,000,000 records for the UK, etc.). Further rows show the dataset split by region (e.g., 1,000,000 records for the USA, 1,000,000 records for the UK, etc.). The bottom row shows the dataset split by hospital (e.g., 1,000,000 records for the USA, 1,000,000 records for the UK, etc.). The diagram illustrates the flow of data from the full dataset down to individual hospital records.

01-ENC encoding (217ms)

$$(47) \cdot \vdash (A \wedge B \wedge \neg B) \leftrightarrow (B \wedge \neg B)$$

LJ (61ms)

$$\frac{\frac{\frac{A, B, B \rightarrow \perp \vdash B}{A, B, B \rightarrow \perp \vdash \perp}^* \quad \frac{A, B, \perp \vdash \perp}{A, B, B \rightarrow \perp \vdash \perp}^* \quad \frac{\frac{B, B \rightarrow \perp \vdash B}{B, B \rightarrow \perp \vdash A}^* \quad \frac{B, \perp \vdash A}{B, B \rightarrow \perp \vdash A}^* \quad \frac{\frac{B, B \rightarrow \perp \vdash B}{B, B \rightarrow \perp \vdash \perp}^* \quad \frac{B, \perp \vdash \perp}{B, B \rightarrow \perp \vdash \perp}^*}{\vdash \vdash A \wedge B \wedge B \rightarrow \perp \rightarrow B \wedge B \rightarrow \perp \wedge B \wedge B \rightarrow \perp \rightarrow A \wedge B \wedge B \rightarrow \perp}^* \quad \supset_L$$

MULTIPLICATIVE encoding (42ms)

Not provable

CALL-BY-NAME encoding (121ms)

[illegible]

CALL-BY-VALUE encoding (140ms)

[illegible]

01-ENC encoding (132ms)

[illegible]

$$(48) \cdot \vdash A \rightarrow B \rightarrow \neg A \wedge \neg B$$

LJ (49ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \perp A}{A, A \rightarrow B, B \rightarrow \perp \perp \perp} \star}{\vdash A \rightarrow B \rightarrow A \wedge B \rightarrow \perp \rightarrow \perp} \star}{\frac{\frac{\frac{A, B, B \rightarrow \perp \perp B}{A, B, B \rightarrow \perp \perp \perp} \star}{A, B, B \rightarrow \perp \perp \perp} \star}{\vdash A \rightarrow B \rightarrow A \wedge B \rightarrow \perp \rightarrow \perp} \star} \supset_L$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\cdot : A \vdash A}{\cdot : A, A \multimap B, B \multimap \perp \vdash \perp} \multimap \quad \frac{\frac{\cdot : B \vdash B \quad \cdot : \perp \vdash \perp}{\cdot : B, B \multimap \perp \vdash \perp} \multimap}{\cdot : \vdash A \multimap B \multimap A \otimes B \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (143ms)

CALL-BY-VALUE encoding (167ms)

[illegible]

01-ENC encoding (169ms)

```

graph TD
    A[2019-2020 Academic Year] --> B[Fall Semester]
    A --> C[Spring Semester]
    B --> D[Mathematics]
    B --> E[Science]
    C --> F[Mathematics]
    C --> G[Science]
    D --> H[Algebra]
    D --> I[Geometry]
    E --> J[Biology]
    E --> K[Chemistry]
    F --> L[Algebra]
    F --> M[Geometry]
    G --> N[Biology]
    G --> O[Chemistry]
    H --> P[Classroom]
    H --> Q[Lab]
    I --> R[Classroom]
    I --> S[Lab]
    J --> T[Classroom]
    J --> U[Lab]
    K --> V[Classroom]
    K --> W[Lab]
    P --> X[Theory]
    P --> Y[Practice]
    Q --> Z[Experiment]
    Q --> AA[Observation]
    R --> AB[Theory]
    R --> AC[Practice]
    S --> AD[Experiment]
    S --> AE[Observation]
    T --> AF[Theory]
    T --> AG[Practice]
    U --> AH[Experiment]
    U --> AI[Observation]
    V --> AJ[Theory]
    V --> AK[Practice]
    W --> AL[Experiment]
    W --> AM[Observation]
  
```





$$(51) \neg\neg B \rightarrow B \vdash (\neg\neg A \rightarrow B) \leftrightarrow (A \rightarrow B)$$

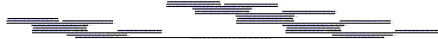
LJ (135ms)



MULTIPLICATIVE encoding (156ms)



CALL-BY-NAME encoding (278ms)



CALL-BY-VALUE encoding (414ms)



01-ENC encoding (449ms)

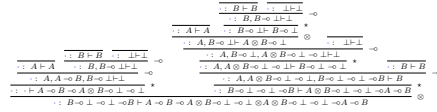


$$(52) \neg\neg B \rightarrow B \vdash (A \rightarrow B) \leftrightarrow (\neg A \wedge \neg B)$$

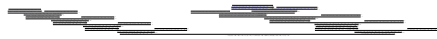
LJ (94ms)



MULTIPLICATIVE encoding (134ms)



CALL-BY-NAME encoding (296ms)



CALL-BY-VALUE encoding (457ms)



01-ENC encoding (489ms)



$$(53) \cdot \vdash \neg\neg A \rightarrow B \rightarrow \neg A \wedge \neg B$$

LJ (65ms)

$$\frac{\frac{\frac{A, A \rightarrow \perp, B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \rightarrow B \vdash B \vdash \perp}{A, A \rightarrow \perp, B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \rightarrow B \vdash \perp}^*}{\frac{A, B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \rightarrow B \vdash B \vdash \perp}{A, B, B \rightarrow \perp \vdash B}^*}^* \supset_L \frac{\frac{A, B, B \rightarrow \perp \vdash B}^*}{\frac{A, B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \rightarrow B \vdash B}{A, B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \rightarrow B \vdash B}^*}^* \supset_L \frac{A, B \rightarrow \perp, A \rightarrow \perp \rightarrow \perp \rightarrow B \vdash B}{\vdash A \rightarrow \perp \rightarrow \perp \rightarrow B \rightarrow A \wedge B \rightarrow \perp \rightarrow \perp}^* \supset_L$$

MULTIPLICATIVE encoding (75ms)

$$\frac{\frac{\frac{\vdots}{A \vdash A} \quad \vdots}{A, A \multimap \perp \multimap \perp} \multimap \quad \frac{\vdots}{B \vdash B} \quad \vdots}{B, B \multimap \perp \multimap \perp} \multimap}{\frac{\vdots}{A, B \multimap \perp, A \multimap \perp \multimap \perp \multimap B \perp} \star} \star$$

CALL-BY-NAME encoding (178ms)

[illegible]

CALL-BY-VALUE encoding (204ms)

Figure 1 illustrates the 10-step process for developing a new product, presented in a descending staircase format. The steps are as follows:

1. Determine the market need
2. Develop a concept
3. Develop a business plan
4. Develop a prototype
5. Develop a marketing plan
6. Develop a distribution plan
7. Develop a financial plan
8. Develop a legal plan
9. Develop a production plan
10. Develop a sales plan

01-ENC encoding (215ms)

Figure 1: Schematic representation of the 1000 Genomes Project. The diagram shows a hierarchical structure of genomic data. At the top, '1000 Genomes Project' is written. Below it, 'Genomes' are listed, followed by 'Regions', 'Variants', and 'Annotations'. The diagram illustrates the flow of data from the project to individual genomes, then to specific regions, and finally to the identification of variants and their associated annotations.

$$(54) \cdot \vdash A \wedge B \rightarrow \neg A \rightarrow \neg B$$

LJ (47ms)

$$\frac{\frac{\frac{}{A, B, A \rightarrow B \rightarrow \perp \perp A} \star}{A, B, A \rightarrow B \rightarrow \perp \perp \perp} \star \quad \frac{\frac{\frac{}{A, B, B \rightarrow \perp \perp B} \star \quad \frac{}{A, B, \perp \perp \perp} \star}{A, B, B \rightarrow \perp \perp \perp} \star}{A, B, A \rightarrow B \rightarrow \perp \perp \perp} \supset_L}{\perp A \wedge B \rightarrow A \rightarrow B \rightarrow \perp \rightarrow \perp} \star$$

MULTIPLICATIVE encoding (48ms)

$$\frac{\frac{\frac{\vdots : A \vdash A}{\vdots : B \vdash B} \quad \frac{\vdots : B \vdash B \quad \vdots : \perp \vdash \perp}{\vdots : B, B \multimap \perp \vdash \perp} \multimap}{\vdots : A, B, A \multimap B \multimap \perp \vdash \perp} \multimap}{\vdots : \vdash A \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star$$

CALL-BY-NAME encoding (77ms)

[illegible]

CALL-BY-VALUE encoding (105ms)

[illegible]

01-ENC encoding (106ms)

[illegible]

$$(55) \cdot \vdash A \wedge \neg B \rightarrow \neg A \rightarrow B$$

LJ (48ms)

$$\frac{\frac{\frac{A, A \rightarrow B, B \rightarrow \perp \perp A}{A, A \rightarrow B, B \rightarrow \perp \perp A}^* \quad \frac{\frac{A, B, B \rightarrow \perp \perp B}{A, B, \perp \perp \perp}^*}{A, B, B \rightarrow \perp \perp \perp}^* \supset_L}{\frac{A, A \rightarrow B, B \rightarrow \perp \perp \perp}{\vdash A \wedge B \rightarrow \perp \rightarrow A \rightarrow B \rightarrow \perp}^*} \supset_L$$

MULTIPLICATIVE encoding (47ms)

$$\frac{\frac{\frac{\vdots \vdash A \otimes B \multimap \perp \multimap A \multimap B \multimap \perp}{\vdots \vdash A, A \multimap B, B \multimap \perp \perp} \star}{\vdots \vdash A \vdash A} \multimap}{\vdots \vdash B \vdash B \quad \vdots \vdash \perp \perp \perp} \multimap$$

CALL-BY-NAME encoding (144ms)

[illegible]

CALL-BY-VALUE encoding (168ms)

[illegible]

01-ENC encoding (168ms)

[illegible]

$$(56) \cdot \vdash \neg\neg A \wedge B \rightarrow \neg A \rightarrow \neg B$$

LJ (62ms)

[illegible]

MULTIPLICATIVE encoding (77ms)

$$\begin{array}{c}
\frac{\frac{\vdots : A \multimap A}{\vdots : A \multimap A} \quad \frac{\vdots : B \multimap B \quad \vdots : \perp \multimap \perp}{\vdots : B, B \multimap \perp \multimap \perp} \multimap}{\vdots : A, B, A \multimap B \multimap \perp \multimap \perp} \multimap \\
\frac{\vdots : B, A \multimap B \multimap \perp \multimap \perp \quad \vdots : \perp \multimap \perp}{\vdots : B, A \multimap B \multimap \perp \multimap \perp, A \multimap \perp \multimap \perp} \star \\
\frac{\vdots : B, A \multimap B \multimap \perp \multimap \perp, A \multimap \perp \multimap \perp \multimap \perp \quad \vdots : \perp \multimap \perp}{\vdots : \multimap A \multimap \perp \multimap \perp \multimap \perp \otimes B \multimap A \multimap B \multimap \perp \multimap \perp} \star
\end{array}$$

CALL-BY-NAME encoding (182ms)

CALL-BY-VALUE encoding (212ms)

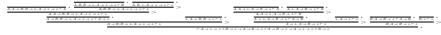
[illegible]

01-ENC encoding (210ms)

Figure 1: Schematic representation of the experimental design. The diagram illustrates the flow of participants through the study. It starts with 'Randomization' into two groups: 'Control' and 'Intervention'. The 'Control' group receives a 'Placebo' and the 'Intervention' group receives a 'Specific Intervention'. Both groups undergo a 'Baseline Assessment' and a 'Post-Intervention Assessment' at the end of the 12-week period. The diagram also shows the timeline for 'Data Collection' and 'Data Analysis'.

$$(57) \cdot \vdash (\neg\neg A \wedge \neg B) \leftrightarrow (\neg A \rightarrow B)$$

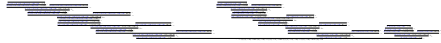
LJ (114ms)



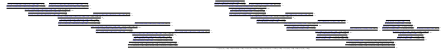
MULTIPLICATIVE encoding (58ms)

**Not provable**

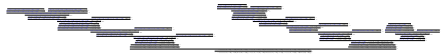
CALL-BY-NAME encoding (326ms)



CALL-BY-VALUE encoding (421ms)



01-ENC encoding (454ms)



$$(58) \cdot \vdash (\neg A \rightarrow B) \leftrightarrow (\neg\neg A \wedge \neg B)$$

LJ (109ms)



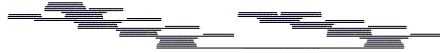
MULTIPLICATIVE encoding (41ms)

**Not provable**

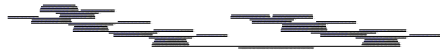
CALL-BY-NAME encoding (597ms)



CALL-BY-VALUE encoding (481ms)



01-ENC encoding (501ms)



$$(59) \cdot \vdash (\neg \neg A \rightarrow B) \leftrightarrow (\neg A \wedge \neg B)$$

LJ (112ms)



MULTIPLICATIVE encoding (57ms)

**Not provable**

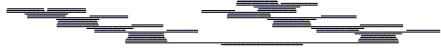
CALL-BY-NAME encoding (594ms)



CALL-BY-VALUE encoding (468ms)

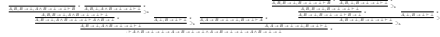


01-ENC encoding (498ms)

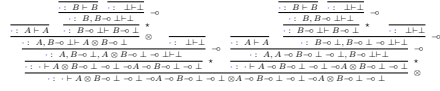


$$(60) \cdot \vdash (\neg A \wedge \neg B) \leftrightarrow (A \rightarrow \neg \neg B)$$

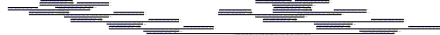
LJ (92ms)



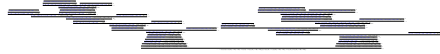
MULTIPLICATIVE encoding (124ms)



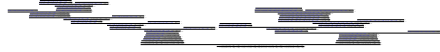
CALL-BY-NAME encoding (317ms)



CALL-BY-VALUE encoding (436ms)



01-ENC encoding (468ms)

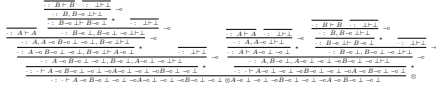


$$(61) \cdot \vdash (A \rightarrow \neg \neg B) \leftrightarrow (\neg \neg A \rightarrow \neg \neg B)$$

LJ (169ms)



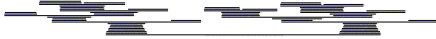
MULTIPLICATIVE encoding (163ms)



CALL-BY-NAME encoding (412ms)



CALL-BY-VALUE encoding (479ms)



01-ENC encoding (552ms)



### 3 Alternative Translations

$$(10) \cdot : A \multimap \mathbf{0} \vdash A \multimap B$$

encoding (55ms)

$$\frac{\frac{\cdot : A \vdash A \quad \cdot : \mathbf{0} \vdash B}{\cdot : A, A \multimap \mathbf{0} \vdash B} \multimap \quad \star}{\cdot : A \multimap \mathbf{0} \vdash A \multimap B} \star$$

$$(11) \cdot : A \vdash A \multimap \mathbf{0} \multimap B$$

encoding (41ms)

$$\frac{\frac{\cdot : A \vdash A \quad \cdot : \mathbf{0} \vdash B}{\cdot : A, A \multimap \mathbf{0} \vdash B} \multimap \quad \star}{\cdot : A \vdash A \multimap \mathbf{0} \multimap B} \star$$

$$(12) \cdot : B \vdash !(A) \multimap B$$

encoding (22ms)

$$\frac{A : B \vdash B}{\cdot : B \vdash !(A) \multimap B} \star$$

$$(16) \cdot : A \multimap B \otimes !(B \multimap A) \vdash A \multimap B$$

encoding (35ms)

$$\frac{\frac{B \multimap A : A \vdash A \quad B \multimap A : B \vdash B}{B \multimap A : A, A \multimap B \vdash B} \multimap \quad \star}{\cdot : A \multimap B \otimes !(B \multimap A) \vdash A \multimap B} \star$$

$$(17) \cdot : !(A \multimap B) \otimes B \multimap A \vdash B \multimap A$$

encoding (38ms)

$$\frac{\frac{A \multimap B : B \vdash B \quad A \multimap B : A \vdash A}{A \multimap B : B, B \multimap A \vdash A} \multimap \quad \star}{\cdot : !(A \multimap B) \otimes B \multimap A \vdash B \multimap A} \star$$

$$(18) \cdot : A, A \multimap B \otimes B \multimap A \vdash B \otimes B \multimap A$$

encoding (56ms)

(19)  $\therefore B, A \multimap B \otimes B \multimap A \vdash A \otimes A \multimap B$

$$(26a) \quad \therefore \cdot \vdash A \& B \multimap \perp \multimap \perp \multimap A \multimap \perp \multimap \perp \& B \multimap \perp \multimap \perp$$

(26b)  $\vdash A \multimap \perp \multimap \perp \otimes B \multimap \perp \multimap \perp \multimap A \otimes B \multimap \perp \multimap \perp$

$$(27a) \vdash : \vdash ! (A \multimap B) \otimes ! (B \multimap A) \multimap \perp \multimap \perp \multimap \neg A \multimap B \multimap \perp \multimap \perp \multimap \& B \multimap A \multimap \perp \multimap \perp$$
$$(27b) \quad \vdash \vdash A \multimap B \multimap \perp \multimap \perp \otimes B \multimap A \multimap \perp \multimap \perp \multimap A \multimap B \otimes B \multimap A \multimap \perp \multimap \perp$$

$$(35) \quad \vdash \vdash ! (A) \otimes ! (A) \multimap ! (A) \otimes ! (A) \multimap ! (A) \otimes ! (A)$$

$$(36) \cdot : A \vdash A \multimap B \multimap B \otimes B \multimap !(A) \multimap B$$

$$(37) \quad \therefore B \vdash !(A \multimap B) \multimap B \otimes B \multimap !(A) \multimap B$$

$$(38) \therefore A \multimap \perp \vdash ! (A \multimap B) \multimap A \multimap \perp \otimes A \multimap \mathbf{0} \multimap A \multimap B$$

(39)  $\therefore B \multimap \mathbf{0} \vdash A \multimap B \multimap A \multimap \mathbf{0} \otimes A \multimap \mathbf{0} \multimap A \multimap B$

$$(40) \cdot : B \vdash A \otimes ! (B) \multimap A \otimes A \multimap A \otimes B$$

$$(41) \quad \cdot : B \multimap \mathbf{0} \vdash !(A) \otimes B \multimap B \otimes B \multimap A \otimes B$$

$$(45) \quad \cdot : \cdot \vdash !(A \multimap A \multimap \perp) \otimes !(A) \multimap \perp \multimap !(A) \multimap \perp$$



$$\begin{array}{c}
\frac{\frac{\frac{A, \perp, A \multimap \perp, A \multimap A \multimap \perp : \perp \vdash \perp}{A, \perp, A \multimap \perp, A \multimap A \multimap \perp : \vdash \perp} \quad D_C}{A \multimap A \multimap \perp : \vdash \vdash (A) \multimap \perp} \quad * \\
\frac{\frac{\frac{A, \perp, A \multimap \perp, A \multimap A \multimap \perp : \perp \vdash \perp}{A, \perp, A \multimap \perp, A \multimap A \multimap \perp : \vdash \perp} \quad D_C}{A \multimap A \multimap \perp : \vdash \vdash (A) \vdash \perp} \quad * \\
\frac{\frac{A \multimap A \multimap \perp : \vdash (A) \multimap \perp \multimap (A) \vdash \perp}{\vdash \vdash \vdash (A \multimap A \multimap \perp) \otimes \vdash (A) \multimap \perp \multimap (A) \multimap \perp} \quad *}{\vdash \vdash \vdash (A \multimap A \multimap \perp) \otimes \vdash (A) \multimap \perp \multimap (A) \multimap \perp} \quad -\circ
\end{array}$$

encoding (138ms)

$$(47) \quad \vdash : \vdash A \otimes B \otimes B \multimap 0 \multimap B \otimes B \multimap 0 \otimes B \otimes B \multimap 0 \multimap A \otimes B \otimes B \multimap 0$$

[illegible]

encoding (71ms)

$$(57b) \quad \therefore \cdot \vdash !(A) \multimap B \multimap \perp \multimap A \multimap \mathbf{0} \multimap \perp \& B \multimap \perp$$

[illegible]

encoding (162ms)

(58b)  $\vdash A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap A \multimap B \multimap \perp$

$$\begin{array}{c}
\frac{}{\vdots : A \vdash A} \quad \frac{\vdots : B \vdash B \quad \vdots : \perp \vdash \perp}{\vdots : B, B \multimap \perp \vdash \perp} \multimap \\
\frac{\vdots : A, A \multimap B, B \multimap \perp \vdash \perp}{\vdots : A \multimap B \vdash A \otimes B \multimap \perp \multimap \perp} \multimap \quad \frac{}{\vdots : \perp \vdash \perp} \multimap \\
\frac{\vdots : A \multimap B, A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap \perp \vdash \perp \multimap \perp \multimap \perp \vdash \perp}{\vdots : A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap \perp \multimap \perp \vdash \perp \multimap \perp \multimap \perp} \multimap \quad \frac{}{\vdots : \perp \vdash \perp} \multimap \\
\frac{\vdots : A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap \perp \multimap \perp \vdash \perp \multimap \perp \multimap \perp}{\vdots : A \otimes B \multimap \perp \multimap \perp \multimap \perp \multimap \perp \multimap \perp \vdash \perp \multimap \perp \multimap \perp} \multimap
\end{array}$$

encoding (71ms)

$$(59b) \quad \cdot : \cdot \vdash A \otimes B \multimap \perp \multimap 0 \multimap (! (A) \multimap B \multimap \perp) \multimap \perp$$

encoding (163ms)