System mLJ

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Contents

1 Height preserving admissibility of weakening on the left

- Case(s) rule $\to_R$

$$\frac{\mathtt{h_1}:\Delta_2,\mathtt{F_3}\vdash\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\to\mathtt{F_4}}\to_R \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_2,\mathtt{F_3}\vdash\mathtt{F_4}}\ \text{ax}}{\mathtt{h_1}:\Delta_2,\mathtt{F_3},\mathtt{F_x}\vdash\mathtt{F_4}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_5,\mathtt{F_3}\to\mathtt{F_4}}\to_R$$

- Case(s) rule $\wedge_R$

$$\frac{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\quad \mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\wedge\mathtt{F_4}}\wedge_R \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}}\ \text{ax}}{\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_5,\mathtt{F_3}}\text{IH}\quad \dfrac{\overline{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4}}\ \text{ax}}{\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_5,\mathtt{F_4}}\text{IH}}{\bullet\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_5,\mathtt{F_3}\wedge\mathtt{F_4}}\wedge_R$$

- Case(s) rule $\vee_R$

$$\frac{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\vee\mathtt{F_4}}\vee_R \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4}}\ \text{ax}}{\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_5,\mathtt{F_3}\vee\mathtt{F_4}}\vee_R$$

- Case(s) rule $\perp_R$

$$\frac{\mathtt{h_1}:\Delta_2\vdash\Delta_3}{\bullet\mathtt{h_1}:\Delta_2\vdash\perp,\Delta_3}\perp_R \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_2\vdash\Delta_3}\ \text{ax}}{\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\Delta_3}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\perp,\Delta_3}\perp_R$$

- Case(s) rule $\top_R$

$$\frac{}{\bullet\mathtt{h_1}:\Delta_2\vdash\top,\Delta_3}\top_R \qquad\to\qquad \frac{}{\bullet\mathtt{h_1}:\Delta_2,\mathtt{F_x}\vdash\top,\Delta_3}\top_R$$

- Case(s) rule $\to_L$

$$\frac{\mathtt{h_1}:\Delta_5,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_2}\quad \mathtt{h_1}:\Delta_5,\mathtt{F_3}\vdash\Delta_4}{\bullet\mathtt{h_1}:\Delta_5,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4}\to_L \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_5,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_2}}\ \text{ax}}{\mathtt{h_1}:\Delta_5,\mathtt{F_x},\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_2}}\text{IH}\quad \dfrac{\overline{\mathtt{h_1}:\Delta_5,\mathtt{F_3}\vdash\Delta_4}\ \text{ax}}{\mathtt{h_1}:\Delta_5,\mathtt{F_3},\mathtt{F_x}\vdash\Delta_4}\text{IH}}{\bullet\mathtt{h_1}:\Delta_5,\mathtt{F_x},\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4}\to_L$$

- Case(s) rule $\wedge_L$

$$\frac{\mathtt{h_1}:\Delta_5,\mathtt{F_2},\mathtt{F_3}\vdash\Delta_4}{\bullet\mathtt{h_1}:\Delta_5,\mathtt{F_2}\wedge\mathtt{F_3}\vdash\Delta_4}\wedge_L \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_5,\mathtt{F_2},\mathtt{F_3}\vdash\Delta_4}\ \text{ax}}{\mathtt{h_1}:\Delta_5,\mathtt{F_2},\mathtt{F_3},\mathtt{F_x}\vdash\Delta_4}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_5,\mathtt{F_x},\mathtt{F_2}\wedge\mathtt{F_3}\vdash\Delta_4}\wedge_L$$

- Case(s) rule $\vee_L$

$$\frac{\mathtt{h_1}:\Delta_5,\mathtt{F_2}\vdash\Delta_4\quad \mathtt{h_1}:\Delta_5,\mathtt{F_3}\vdash\Delta_4}{\bullet\mathtt{h_1}:\Delta_5,\mathtt{F_2}\vee\mathtt{F_3}\vdash\Delta_4}\vee_L \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_5,\mathtt{F_2}\vdash\Delta_4}\ \text{ax}}{\mathtt{h_1}:\Delta_5,\mathtt{F_2},\mathtt{F_x}\vdash\Delta_4}\text{IH}\quad \dfrac{\overline{\mathtt{h_1}:\Delta_5,\mathtt{F_3}\vdash\Delta_4}\ \text{ax}}{\mathtt{h_1}:\Delta_5,\mathtt{F_3},\mathtt{F_x}\vdash\Delta_4}\text{IH}}{\bullet\mathtt{h_1}:\Delta_5,\mathtt{F_x},\mathtt{F_2}\vee\mathtt{F_3}\vdash\Delta_4}\vee_L$$

- Case(s) rule $\perp_L$

$$\frac{}{\bullet\mathtt{h_1}:\perp,\Delta_3\vdash\Delta_2}\perp_L \qquad\to\qquad \frac{}{\bullet\mathtt{h_1}:\perp,\Delta_3,\mathtt{F_x}\vdash\Delta_2}\perp_L$$

- Case(s) rule I

$$\frac{}{\bullet\mathtt{h_1}:\Delta_4,\mathtt{p_3}\vdash\Delta_2,\mathtt{p_3}}I \qquad\to\qquad \frac{}{\bullet\mathtt{h_1}:\Delta_4,\mathtt{F_x},\mathtt{p_3}\vdash\Delta_2,\mathtt{p_3}}I$$

- Case(s) rule $\top_L$

$$\frac{\mathtt{h_1}:\Delta_3\vdash\Delta_2}{\bullet\mathtt{h_1}:\top,\Delta_3\vdash\Delta_2}\top_L \qquad\to\qquad \frac{\dfrac{\overline{\mathtt{h_1}:\Delta_3\vdash\Delta_2}\ \text{ax}}{\mathtt{h_1}:\Delta_3,\mathtt{F_x}\vdash\Delta_2}\ \text{IH}}{\bullet\mathtt{h_1}:\top,\Delta_3,\mathtt{F_x}\vdash\Delta_2}\top_L$$

2 Height preserving admissibility of weakening on the right

- Case(s) rule $\to_R$

$$\dfrac{\mathtt{h_1} : \Delta_2, \mathtt{F_3} \vdash \mathtt{F_4}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3} \to \mathtt{F_4}} \ {\to_R} \quad \to \quad \dfrac{\overline{\mathtt{h_1} : \Delta_2, \mathtt{F_3} \vdash \mathtt{F_4}} \ \mathtt{ax}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_x}, \mathtt{F_3} \to \mathtt{F_4}} \ {\to_R}$$

- Case(s) rule $\wedge_R$

$$\dfrac{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3} \quad \mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_4}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3} \wedge \mathtt{F_4}} \ {\wedge_R} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3}} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3}, \mathtt{F_x}} \ \mathtt{IH} \quad \dfrac{\overline{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_4}} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_4}, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_x}, \mathtt{F_3} \wedge \mathtt{F_4}} \ {\wedge_R}$$

- Case(s) rule $\vee_R$

$$\dfrac{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3}, \mathtt{F_4}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3} \vee \mathtt{F_4}} \ {\vee_R} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3}, \mathtt{F_4}} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_3}, \mathtt{F_4}, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \Delta_5, \mathtt{F_x}, \mathtt{F_3} \vee \mathtt{F_4}} \ {\vee_R}$$

- Case(s) rule $\bot_R$

$$\dfrac{\mathtt{h_1} : \Delta_2 \vdash \Delta_3}{\bullet\mathtt{h_1} : \Delta_2 \vdash \bot, \Delta_3} \ {\bot_R} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_2 \vdash \Delta_3} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_2 \vdash \Delta_3, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \Delta_2 \vdash \bot, \Delta_3, \mathtt{F_x}} \ {\bot_R}$$

- Case(s) rule $\top_R$

$$\dfrac{}{\bullet\mathtt{h_1} : \Delta_2 \vdash \top, \Delta_3} \ {\top_R} \quad \to \quad \dfrac{}{\bullet\mathtt{h_1} : \Delta_2 \vdash \top, \Delta_3, \mathtt{F_x}} \ {\top_R}$$

- Case(s) rule $\to_L$

$$\dfrac{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_2} \quad \mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4} \ {\to_L} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_2}} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_2}, \mathtt{F_x}} \ \mathtt{IH} \quad \dfrac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_x}} \ {\to_L}$$

- Case(s) rule $\wedge_L$

$$\dfrac{\mathtt{h_1} : \Delta_5, \mathtt{F_2}, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \wedge \mathtt{F_3} \vdash \Delta_4} \ {\wedge_L} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_2}, \mathtt{F_3} \vdash \Delta_4} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_5, \mathtt{F_2}, \mathtt{F_3} \vdash \Delta_4, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \wedge \mathtt{F_3} \vdash \Delta_4, \mathtt{F_x}} \ {\wedge_L}$$

- Case(s) rule $\vee_L$

$$\dfrac{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vdash \Delta_4 \quad \mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vee \mathtt{F_3} \vdash \Delta_4} \ {\vee_L} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vdash \Delta_4} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vdash \Delta_4, \mathtt{F_x}} \ \mathtt{IH} \quad \dfrac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vee \mathtt{F_3} \vdash \Delta_4, \mathtt{F_x}} \ {\vee_L}$$

- Case(s) rule $\bot_L$

$$\dfrac{}{\bullet\mathtt{h_1} : \bot, \Delta_3 \vdash \Delta_2} \ {\bot_L} \quad \to \quad \dfrac{}{\bullet\mathtt{h_1} : \bot, \Delta_3 \vdash \Delta_2, \mathtt{F_x}} \ {\bot_L}$$

- Case(s) rule I

$$\dfrac{}{\bullet\mathtt{h_1} : \Delta_4, \mathtt{P_3} \vdash \Delta_2, \mathtt{P_3}} \ {I} \quad \to \quad \dfrac{}{\bullet\mathtt{h_1} : \Delta_4, \mathtt{P_3} \vdash \Delta_2, \mathtt{F_x}, \mathtt{P_3}} \ {I}$$

- Case(s) rule $\top_L$

$$\dfrac{\mathtt{h_1} : \Delta_3 \vdash \Delta_2}{\bullet\mathtt{h_1} : \top, \Delta_3 \vdash \Delta_2} \ {\top_L} \quad \to \quad \dfrac{\dfrac{\overline{\mathtt{h_1} : \Delta_3 \vdash \Delta_2} \ \mathtt{ax}}{\mathtt{h_1} : \Delta_3 \vdash \Delta_2, \mathtt{F_x}} \ \mathtt{IH}}{\bullet\mathtt{h_1} : \top, \Delta_3 \vdash \Delta_2, \mathtt{F_x}} \ {\top_L}$$

3 Measure of derivations

- Case(s) rule $\rightarrow_R$

$$\dfrac{h_1 : \Delta_2, F_3 \vdash F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \rightarrow F_4} \rightarrow_R \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_2, F_3 \vdash F_4}\ \text{ax}}{\bullet h_1 : \Delta_2, F_3 \vdash F_4}\ \text{IH}}{\bullet\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \rightarrow F_4} \rightarrow_R$$

- Case(s) rule $\wedge_R$

$$\dfrac{h_1 : \Delta_2 \vdash \Delta_5, F_3 \quad h_1 : \Delta_2 \vdash \Delta_5, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \wedge F_4} \wedge_R \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_2 \vdash \Delta_5, F_3}\ \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_2 \vdash \Delta_5, F_4}\ \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_4}\ \text{IH}}{\bullet\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \wedge F_4} \wedge_R$$

- Case(s) rule $\vee_R$

$$\dfrac{h_1 : \Delta_2 \vdash \Delta_5, F_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \vee F_4} \vee_R \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_2 \vdash \Delta_5, F_3, F_4}\ \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3, F_4}\ \text{IH}}{\bullet\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \vee F_4} \vee_R$$

- Case(s) rule $\perp_R$

$$\dfrac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_2 \vdash \Delta_3}\ \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_3}\ \text{IH}}{\bullet\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R$$

- Case(s) rule $\top_R$

$$\dfrac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \qquad \rightarrow \qquad \dfrac{}{\bullet\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R$$

- Case(s) rule $\rightarrow_L$

$$\dfrac{h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4, F_2 \quad h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4, F_2}\ \text{ax}}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4, F_2}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_3 \vdash \Delta_4}\ \text{ax}}{\bullet h_1 : \Delta_5, F_3 \vdash \Delta_4}\ \text{IH}}{\bullet\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L$$

- Case(s) rule $\wedge_L$

$$\dfrac{h_1 : \Delta_5, F_2, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2, F_3 \vdash \Delta_4}\ \text{ax}}{\bullet h_1 : \Delta_5, F_2, F_3 \vdash \Delta_4}\ \text{IH}}{\bullet\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L$$

- Case(s) rule $\vee_L$

$$\dfrac{h_1 : \Delta_5, F_2 \vdash \Delta_4 \quad h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \vee_L \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2 \vdash \Delta_4}\ \text{ax}}{\bullet h_1 : \Delta_5, F_2 \vdash \Delta_4}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_3 \vdash \Delta_4}\ \text{ax}}{\bullet h_1 : \Delta_5, F_3 \vdash \Delta_4}\ \text{IH}}{\bullet\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \vee_L$$

- Case(s) rule $\perp_L$

$$\dfrac{}{\bullet h_1 : \perp, \Delta_3 \vdash \Delta_2} \perp_L \qquad \rightarrow \qquad \dfrac{}{\bullet\bullet h_1 : \perp, \Delta_3 \vdash \Delta_2} \perp_L$$

- Case(s) rule I

$$\dfrac{}{\bullet h_1 : \Delta_4, p_3 \vdash \Delta_2, p_3} I \qquad \rightarrow \qquad \dfrac{}{\bullet\bullet h_1 : \Delta_4, p_3 \vdash \Delta_2, p_3} I$$

- Case(s) rule $\top_L$

$$\dfrac{h_1 : \Delta_3 \vdash \Delta_2}{\bullet h_1 : \top, \Delta_3 \vdash \Delta_2} \top_L \qquad \rightarrow \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_3 \vdash \Delta_2}\ \text{ax}}{\bullet h_1 : \Delta_3 \vdash \Delta_2}\ \text{IH}}{\bullet\bullet h_1 : \top, \Delta_3 \vdash \Delta_2} \top_L$$

4 Invertibility of Rules

4.1 Status of $\to_R$: : Non invertible

- Case rule $\to_R$

$$\frac{h_3 : \Delta_4, F_5 \vdash F_6}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \to F_2), F_5 \to F_6} \to_R \quad \to \quad \frac{}{\bullet h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{fail}$$

$$\frac{h_1 : \Delta_2, F_3 \vdash F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \to F_4} \to_R \quad \to \quad \frac{\overline{h_1 : \Delta_2, F_3 \vdash F_4} \; \textit{ax}}{\bullet h_1 : \Delta_2, F_3 \vdash F_4} \; \textit{height}$$

- Case rule $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \to F_2 \quad h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \to F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \to F_2), F_5 \wedge F_6} \wedge_R \quad \to \quad \frac{\overline{h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{height}$$

- Case rule $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_6, F_1 \to F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \to F_2), F_5 \vee F_6} \vee_R \quad \to \quad \frac{\overline{h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{height}$$

- Case rule $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \to F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \to F_2} \perp_R \quad \to \quad \frac{\overline{h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{height}$$

- Case rule $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \to F_2} \top_R \quad \to \quad \frac{}{\bullet h_3 : \Delta_4, F_1 \vdash F_2} \; \textit{fail}$$

- Case rule $\to_L$

$$\frac{h_3 : \Delta_7, F_4 \to F_5 \vdash \Delta_6, F_4, F_1 \to F_2 \quad h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1 \to F_2}{\bullet h_3 : \Delta_7, F_4 \to F_5 \vdash \Delta_6, F_1 \to F_2} \to_L \quad \to \quad \frac{\overline{h_3 : \Delta_7, F_1, F_4 \to F_5 \vdash F_2} \; \textit{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_4 \to F_5 \vdash F_2} \; \textit{height}$$

- Case rule $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5 \vdash \Delta_6, F_1 \to F_2}{\bullet h_3 : \Delta_7, F_4 \wedge F_5 \vdash \Delta_6, F_1 \to F_2} \wedge_L \quad \to \quad \frac{\overline{h_3 : \Delta_7, F_1, F_4, F_5 \vdash F_2} \; \textit{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_4 \wedge F_5 \vdash F_2} \wedge_L$$

- Case rule $\vee_L$

$$\frac{h_3 : \Delta_7, F_4 \vdash \Delta_6, F_1 \to F_2 \quad h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1 \to F_2}{\bullet h_3 : \Delta_7, F_4 \vee F_5 \vdash \Delta_6, F_1 \to F_2} \vee_L \quad \to \quad \frac{\overline{h_3 : \Delta_7, F_1, F_4 \vdash F_2} \; \textit{ax/ind} \quad \overline{h_3 : \Delta_7, F_1, F_5 \vdash F_2} \; \textit{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_4 \vee F_5 \vdash F_2} \vee_L$$

- Case rule $\perp_L$

$$\frac{}{\bullet\mathtt{h_3 : \perp, \Delta_5 \vdash \Delta_4, F_1 \to F_2}} \perp_L \quad \to \quad \frac{}{\bullet\mathtt{h_3 : \perp, \Delta_5, F_1 \vdash F_2}} \perp_L$$

- Case rule I

$$\frac{}{\bullet\mathtt{h_3 : p_5, \Delta_6 \vdash p_5, \Delta_4, F_1 \to F_2}} I \quad \to \quad \textcolor{red}{\frac{}{\bullet\mathtt{h_3 : \Delta_6, F_1, p_5 \vdash F_2}} \mathtt{fail}}$$

- Case rule $\top_L$

$$\frac{\mathtt{h_3 : \Delta_5 \vdash \Delta_4, F_1 \to F_2}}{\bullet\mathtt{h_3 : \top, \Delta_5 \vdash \Delta_4, F_1 \to F_2}} \top_L \quad \to \quad \frac{\dfrac{}{\mathtt{h_3 : \Delta_5, F_1 \vdash F_2}} \mathtt{ax/ind}}{\bullet\mathtt{h_3 : \top, \Delta_5, F_1 \vdash F_2}} \top_L$$

4.2 Status of $\wedge_R$: (Left Premise): `Invertible`

- Case rule $\to_R$

$$\frac{\mathtt{h_3 : \Delta_4, F_5 \vdash F_6}}{\bullet\mathtt{h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \to F_6}} \to_R \quad \to \quad \frac{\dfrac{}{\mathtt{h_3 : \Delta_4, F_5 \vdash F_6}} \mathtt{ax}}{\bullet\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \to F_6}} \to_R$$

- Case rule $\wedge_R$

$$\frac{\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \wedge F_2} \quad \mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \wedge F_2}}{\bullet\mathtt{h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \wedge F_6}} \wedge_R \quad \to \quad \frac{\dfrac{}{\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5}} \mathtt{ax/ind} \quad \dfrac{}{\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_6}} \mathtt{ax/ind}}{\bullet\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \wedge F_6}} \wedge_R$$

$$\frac{\mathtt{h_1 : \Delta_2 \vdash \Delta_5, F_3} \quad \mathtt{h_1 : \Delta_2 \vdash \Delta_5, F_4}}{\bullet\mathtt{h_1 : \Delta_2 \vdash \Delta_5, F_3 \wedge F_4}} \wedge_R \quad \to \quad \frac{\dfrac{}{\mathtt{h_1 : \Delta_2 \vdash \Delta_5, F_3}} \mathtt{ax}}{\bullet\mathtt{h_1 : \Delta_2 \vdash \Delta_5, F_3}} \mathtt{height}$$

- Case rule $\vee_R$

$$\frac{\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_6, F_1 \wedge F_2}}{\bullet\mathtt{h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \vee F_6}} \vee_R \quad \to \quad \frac{\dfrac{}{\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5, F_6}} \mathtt{ax/ind}}{\bullet\mathtt{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \vee F_6}} \vee_R$$

- Case rule $\perp_R$

$$\frac{\mathtt{h_3 : \Delta_4 \vdash \Delta_5, F_1 \wedge F_2}}{\bullet\mathtt{h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \wedge F_2}} \perp_R \quad \to \quad \frac{\dfrac{}{\mathtt{h_3 : \Delta_4 \vdash \Delta_5, F_1}} \mathtt{ax/ind}}{\bullet\mathtt{h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1}} \perp_R$$

- Case rule $\top_R$

$$\frac{}{\bullet\mathtt{h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \wedge F_2}} \top_R \quad \to \quad \frac{}{\bullet\mathtt{h_3 : \Delta_4 \vdash \top, \Delta_5, F_1}} \top_R$$

- Case rule $\to_L$

$$\frac{\mathtt{h_3 : \Delta_7, F_4 \to F_5 \vdash \Delta_6, F_4, F_1 \wedge F_2} \quad \mathtt{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1 \wedge F_2}}{\bullet\mathtt{h_3 : \Delta_7, F_4 \to F_5 \vdash \Delta_6, F_1 \wedge F_2}} \to_L \quad \to \quad \frac{\dfrac{}{\mathtt{h_3 : \Delta_7, F_4 \to F_5 \vdash \Delta_6, F_1, F_4}} \mathtt{ax/ind} \quad \dfrac{}{\mathtt{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1}} \mathtt{ax/ind}}{\bullet\mathtt{h_3 : \Delta_7, F_4 \to F_5 \vdash \Delta_6, F_1}} \to_L$$

- Case rule $\wedge_L$

$$\dfrac{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4, \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1 \wedge \mathtt{F}_2}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \wedge \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1 \wedge \mathtt{F}_2} \ \wedge_L \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4, \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1} \ \text{ax/ind}}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \wedge \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1} \ \wedge_L$$

- Case rule $\vee_L$

$$\dfrac{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \vdash \Delta_6, \mathtt{F}_1 \wedge \mathtt{F}_2 \quad \mathtt{h}_3 : \Delta_7, \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1 \wedge \mathtt{F}_2}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \vee \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1 \wedge \mathtt{F}_2} \ \vee_L \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \vdash \Delta_6, \mathtt{F}_1} \ \text{ax/ind} \quad \overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1} \ \text{ax/ind}}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \vee \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_1} \ \vee_L$$

- Case rule $\perp_L$

$$\dfrac{}{\bullet\mathtt{h}_3 : \perp, \Delta_5 \vdash \Delta_4, \mathtt{F}_1 \wedge \mathtt{F}_2} \ \perp_L \qquad \rightarrow \qquad \dfrac{}{\bullet\mathtt{h}_3 : \perp, \Delta_5 \vdash \Delta_4, \mathtt{F}_1} \ \perp_L$$

- Case rule I

$$\dfrac{}{\bullet\mathtt{h}_3 : \mathtt{p}_5, \Delta_6 \vdash \mathtt{p}_5, \Delta_4, \mathtt{F}_1 \wedge \mathtt{F}_2} \ I \qquad \rightarrow \qquad \dfrac{}{\bullet\mathtt{h}_3 : \Delta_6, \mathtt{p}_5 \vdash \Delta_4, \mathtt{F}_1, \mathtt{p}_5} \ I$$

- Case rule $\top_L$

$$\dfrac{\mathtt{h}_3 : \Delta_5 \vdash \Delta_4, \mathtt{F}_1 \wedge \mathtt{F}_2}{\bullet\mathtt{h}_3 : \top, \Delta_5 \vdash \Delta_4, \mathtt{F}_1 \wedge \mathtt{F}_2} \ \top_L \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_5 \vdash \Delta_4, \mathtt{F}_1} \ \text{ax/ind}}{\bullet\mathtt{h}_3 : \top, \Delta_5 \vdash \Delta_4, \mathtt{F}_1} \ \top_L$$

4.3 Status of $\wedge_R$ (Right Premise): : Invertible

- Case rule $\rightarrow_R$

$$\dfrac{\mathtt{h}_3 : \Delta_4, \mathtt{F}_5 \vdash \mathtt{F}_6}{\bullet\mathtt{h}_3 : \Delta_4 \vdash (\Delta_7, \mathtt{F}_1 \wedge \mathtt{F}_2), \mathtt{F}_5 \rightarrow \mathtt{F}_6} \ \rightarrow_R \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_4, \mathtt{F}_5 \vdash \mathtt{F}_6} \ \text{ax}}{\bullet\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_2, \mathtt{F}_5 \rightarrow \mathtt{F}_6} \ \rightarrow_R$$

- Case rule $\wedge_R$

$$\dfrac{\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_5, \mathtt{F}_1 \wedge \mathtt{F}_2 \quad \mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_6, \mathtt{F}_1 \wedge \mathtt{F}_2}{\bullet\mathtt{h}_3 : \Delta_4 \vdash (\Delta_7, \mathtt{F}_1 \wedge \mathtt{F}_2), \mathtt{F}_5 \wedge \mathtt{F}_6} \ \wedge_R \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_2, \mathtt{F}_5} \ \text{ax/ind} \quad \overline{\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_2, \mathtt{F}_6} \ \text{ax/ind}}{\bullet\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_2, \mathtt{F}_5 \wedge \mathtt{F}_6} \ \wedge_R$$

$$\dfrac{\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3 \quad \mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3 \wedge \mathtt{F}_4} \ \wedge_R \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_4} \ \text{ax}}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_4} \ height$$

- Case rule $\vee_R$

$$\dfrac{\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_5, \mathtt{F}_6, \mathtt{F}_1 \wedge \mathtt{F}_2}{\bullet\mathtt{h}_3 : \Delta_4 \vdash (\Delta_7, \mathtt{F}_1 \wedge \mathtt{F}_2), \mathtt{F}_5 \vee \mathtt{F}_6} \ \vee_R \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_2, \mathtt{F}_5, \mathtt{F}_6} \ \text{ax/ind}}{\bullet\mathtt{h}_3 : \Delta_4 \vdash \Delta_7, \mathtt{F}_2, \mathtt{F}_5 \vee \mathtt{F}_6} \ \vee_R$$

- Case rule $\perp_R$

$$\dfrac{\mathtt{h}_3 : \Delta_4 \vdash \Delta_5, \mathtt{F}_1 \wedge \mathtt{F}_2}{\bullet\mathtt{h}_3 : \Delta_4 \vdash \perp, \Delta_5, \mathtt{F}_1 \wedge \mathtt{F}_2} \ \perp_R \qquad \rightarrow \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_4 \vdash \Delta_5, \mathtt{F}_2} \ \text{ax/ind}}{\bullet\mathtt{h}_3 : \Delta_4 \vdash \perp, \Delta_5, \mathtt{F}_2} \ \perp_R$$

- Case rule $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \wedge F_2} \ \top_R \quad \rightarrow \quad \frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_2} \ \top_R$$

- Case rule $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_4 \rightarrow F_5 \vdash \Delta_6, F_4, F_1 \wedge F_2 \quad h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_7, F_4 \rightarrow F_5 \vdash \Delta_6, F_1 \wedge F_2} \ \rightarrow_L \quad \rightarrow \quad \frac{\frac{}{h_3 : \Delta_7, F_4 \rightarrow F_5 \vdash \Delta_6, F_2, F_4} \ \text{ax/ind} \quad \frac{}{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_2} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \rightarrow F_5 \vdash \Delta_6, F_2} \ \rightarrow_L$$

- Case rule $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5 \vdash \Delta_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_7, F_4 \wedge F_5 \vdash \Delta_6, F_1 \wedge F_2} \ \wedge_L \quad \rightarrow \quad \frac{\frac{}{h_3 : \Delta_7, F_4, F_5 \vdash \Delta_6, F_2} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \wedge F_5 \vdash \Delta_6, F_2} \ \wedge_L$$

- Case rule $\vee_L$

$$\frac{h_3 : \Delta_7, F_4 \vdash \Delta_6, F_1 \wedge F_2 \quad h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_7, F_4 \vee F_5 \vdash \Delta_6, F_1 \wedge F_2} \ \vee_L \quad \rightarrow \quad \frac{\frac{}{h_3 : \Delta_7, F_4 \vdash \Delta_6, F_2} \ \text{ax/ind} \quad \frac{}{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_2} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \vee F_5 \vdash \Delta_6, F_2} \ \vee_L$$

- Case rule $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_4, F_1 \wedge F_2} \ \perp_L \quad \rightarrow \quad \frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_4, F_2} \ \perp_L$$

- Case rule I

$$\frac{}{\bullet h_3 : p_5, \Delta_6 \vdash p_5, \Delta_4, F_1 \wedge F_2} \ I \quad \rightarrow \quad \frac{}{\bullet h_3 : \Delta_6, p_5 \vdash \Delta_4, F_2, p_5} \ I$$

- Case rule $\top_L$

$$\frac{h_3 : \Delta_5 \vdash \Delta_4, F_1 \wedge F_2}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_4, F_1 \wedge F_2} \ \top_L \quad \rightarrow \quad \frac{\frac{}{h_3 : \Delta_5 \vdash \Delta_4, F_2} \ \text{ax/ind}}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_4, F_2} \ \top_L$$

4.4   Status of $\vee_R$: :   Invertible

- Case rule $\rightarrow_R$

$$\frac{h_3 : \Delta_4, F_5 \vdash F_6}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \rightarrow F_6} \ \rightarrow_R \quad \rightarrow \quad \frac{\frac{}{h_3 : \Delta_4, F_5 \vdash F_6} \ \text{ax}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \rightarrow F_6} \ \rightarrow_R$$

- Case rule $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \vee F_2 \quad h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \wedge F_6} \ \wedge_R \quad \rightarrow \quad \frac{\frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5} \ \text{ax/ind} \quad \frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_6} \ \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \wedge F_6} \ \wedge_R$$

- Case rule $\vee_R$

$$\dfrac{\texttt{h}_3 : \Delta_4 \vdash \Delta_7, \texttt{F}_5, \texttt{F}_6, \texttt{F}_1 \vee \texttt{F}_2}{\bullet\texttt{h}_3 : \Delta_4 \vdash (\Delta_7, \texttt{F}_1 \vee \texttt{F}_2), \texttt{F}_5 \vee \texttt{F}_6} \ \vee_R \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_3 : \Delta_4 \vdash \Delta_7, \texttt{F}_1, \texttt{F}_2, \texttt{F}_5, \texttt{F}_6}}{\bullet\texttt{h}_3 : \Delta_4 \vdash \Delta_7, \texttt{F}_1, \texttt{F}_2, \texttt{F}_5 \vee \texttt{F}_6}\ ^{\text{ax/ind}}_{\vee_R}$$

$$\dfrac{\texttt{h}_1 : \Delta_2 \vdash \Delta_5, \texttt{F}_3, \texttt{F}_4}{\bullet\texttt{h}_1 : \Delta_2 \vdash \Delta_5, \texttt{F}_3 \vee \texttt{F}_4} \ \vee_R \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_1 : \Delta_2 \vdash \Delta_5, \texttt{F}_3, \texttt{F}_4}}{\bullet\texttt{h}_1 : \Delta_2 \vdash \Delta_5, \texttt{F}_3, \texttt{F}_4}\ ^{\text{ax}}_{height}$$

- Case rule $\perp_R$

$$\dfrac{\texttt{h}_3 : \Delta_4 \vdash \Delta_5, \texttt{F}_1 \vee \texttt{F}_2}{\bullet\texttt{h}_3 : \Delta_4 \vdash \perp, \Delta_5, \texttt{F}_1 \vee \texttt{F}_2} \ \perp_R \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_3 : \Delta_4 \vdash \Delta_5, \texttt{F}_1, \texttt{F}_2}}{\bullet\texttt{h}_3 : \Delta_4 \vdash \perp, \Delta_5, \texttt{F}_1, \texttt{F}_2}\ ^{\text{ax/ind}}_{\perp_R}$$

- Case rule $\top_R$

$$\dfrac{}{\bullet\texttt{h}_3 : \Delta_4 \vdash \top, \Delta_5, \texttt{F}_1 \vee \texttt{F}_2} \ \top_R \qquad \rightarrow \qquad \dfrac{}{\bullet\texttt{h}_3 : \Delta_4 \vdash \top, \Delta_5, \texttt{F}_1, \texttt{F}_2} \ \top_R$$

- Case rule $\rightarrow_L$

$$\dfrac{\texttt{h}_3 : \Delta_7, \texttt{F}_4 \rightarrow \texttt{F}_5 \vdash \Delta_6, \texttt{F}_4, \texttt{F}_1 \vee \texttt{F}_2 \quad \texttt{h}_3 : \Delta_7, \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2}{\bullet\texttt{h}_3 : \Delta_7, \texttt{F}_4 \rightarrow \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2} \ \rightarrow_L \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_3 : \Delta_7, \texttt{F}_4 \rightarrow \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2, \texttt{F}_4}\ ^{\text{ax/ind}} \quad \overline{\texttt{h}_3 : \Delta_7, \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}\ ^{\text{ax/ind}}}{\bullet\texttt{h}_3 : \Delta_7, \texttt{F}_4 \rightarrow \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}\ \rightarrow_L$$

- Case rule $\wedge_L$

$$\dfrac{\texttt{h}_3 : \Delta_7, \texttt{F}_4, \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2}{\bullet\texttt{h}_3 : \Delta_7, \texttt{F}_4 \wedge \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2} \ \wedge_L \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_3 : \Delta_7, \texttt{F}_4, \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}}{\bullet\texttt{h}_3 : \Delta_7, \texttt{F}_4 \wedge \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}\ ^{\text{ax/ind}}_{\wedge_L}$$

- Case rule $\vee_L$

$$\dfrac{\texttt{h}_3 : \Delta_7, \texttt{F}_4 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2 \quad \texttt{h}_3 : \Delta_7, \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2}{\bullet\texttt{h}_3 : \Delta_7, \texttt{F}_4 \vee \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1 \vee \texttt{F}_2} \ \vee_L \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_3 : \Delta_7, \texttt{F}_4 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}\ ^{\text{ax/ind}} \quad \overline{\texttt{h}_3 : \Delta_7, \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}\ ^{\text{ax/ind}}}{\bullet\texttt{h}_3 : \Delta_7, \texttt{F}_4 \vee \texttt{F}_5 \vdash \Delta_6, \texttt{F}_1, \texttt{F}_2}\ \vee_L$$

- Case rule $\perp_L$

$$\dfrac{}{\bullet\texttt{h}_3 : \perp, \Delta_5 \vdash \Delta_4, \texttt{F}_1 \vee \texttt{F}_2} \ \perp_L \qquad \rightarrow \qquad \dfrac{}{\bullet\texttt{h}_3 : \perp, \Delta_5 \vdash \Delta_4, \texttt{F}_1, \texttt{F}_2} \ \perp_L$$

- Case rule I

$$\dfrac{}{\bullet\texttt{h}_3 : \texttt{p}_5, \Delta_6 \vdash \texttt{p}_5, \Delta_4, \texttt{F}_1 \vee \texttt{F}_2} \ I \qquad \rightarrow \qquad \dfrac{}{\bullet\texttt{h}_3 : \Delta_6, \texttt{p}_5 \vdash \Delta_4, \texttt{F}_1, \texttt{F}_2, \texttt{p}_5} \ I$$

- Case rule $\top_L$

$$\dfrac{\texttt{h}_3 : \Delta_5 \vdash \Delta_4, \texttt{F}_1 \vee \texttt{F}_2}{\bullet\texttt{h}_3 : \top, \Delta_5 \vdash \Delta_4, \texttt{F}_1 \vee \texttt{F}_2} \ \top_L \qquad \rightarrow \qquad \dfrac{\overline{\texttt{h}_3 : \Delta_5 \vdash \Delta_4, \texttt{F}_1, \texttt{F}_2}}{\bullet\texttt{h}_3 : \top, \Delta_5 \vdash \Delta_4, \texttt{F}_1, \texttt{F}_2}\ ^{\text{ax/ind}}_{\top_L}$$

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4.5 Status of $\perp_R$: : Invertible

- Case rule $\to_R$

$$\frac{\mathtt{h}_1 : \Delta_2, \mathtt{F}_3 \vdash \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash (\perp, \Delta_5), \mathtt{F}_3 \to \mathtt{F}_4} \to_R \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_2, \mathtt{F}_3 \vdash \mathtt{F}_4} \ ^{\mathtt{ax}}}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3 \to \mathtt{F}_4} \to_R$$

- Case rule $\wedge_R$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \perp, \Delta_5, \mathtt{F}_3 \quad \mathtt{h}_1 : \Delta_2 \vdash \perp, \Delta_5, \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash (\perp, \Delta_5), \mathtt{F}_3 \wedge \mathtt{F}_4} \wedge_R \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3} \ ^{\mathtt{ax/ind}} \quad \overline{\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_4} \ ^{\mathtt{ax/ind}}}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3 \wedge \mathtt{F}_4} \wedge_R$$

- Case rule $\vee_R$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \perp, \Delta_5, \mathtt{F}_3, \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash (\perp, \Delta_5), \mathtt{F}_3 \vee \mathtt{F}_4} \vee_R \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3, \mathtt{F}_4} \ ^{\mathtt{ax/ind}}}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \Delta_5, \mathtt{F}_3 \vee \mathtt{F}_4} \vee_R$$

- Case rule $\perp_R$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \Delta_3}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_2 \vdash \Delta_3} \ ^{\mathtt{ax}}}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \Delta_3} \ _{height}$$

- Case rule $\top_R$

$$\frac{}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \top, \perp, \Delta_3} \top_R \qquad \to \qquad \frac{}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \top, \Delta_3} \top_R$$

- Case rule $\to_L$

$$\frac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \to \mathtt{F}_3 \vdash \perp, \Delta_4, \mathtt{F}_2 \quad \mathtt{h}_1 : \Delta_5, \mathtt{F}_3 \vdash \perp, \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \to \mathtt{F}_3 \vdash \perp, \Delta_4} \to_L \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \to \mathtt{F}_3 \vdash \Delta_4, \mathtt{F}_2} \ ^{\mathtt{ax/ind}} \quad \overline{\mathtt{h}_1 : \Delta_5, \mathtt{F}_3 \vdash \Delta_4} \ ^{\mathtt{ax/ind}}}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \to \mathtt{F}_3 \vdash \Delta_4} \to_L$$

- Case rule $\wedge_L$

$$\frac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2, \mathtt{F}_3 \vdash \perp, \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \wedge \mathtt{F}_3 \vdash \perp, \Delta_4} \wedge_L \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2, \mathtt{F}_3 \vdash \Delta_4} \ ^{\mathtt{ax/ind}}}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \wedge \mathtt{F}_3 \vdash \Delta_4} \wedge_L$$

- Case rule $\vee_L$

$$\frac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \vdash \perp, \Delta_4 \quad \mathtt{h}_1 : \Delta_5, \mathtt{F}_3 \vdash \perp, \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \vee \mathtt{F}_3 \vdash \perp, \Delta_4} \vee_L \qquad \to \qquad \frac{\overline{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \vdash \Delta_4} \ ^{\mathtt{ax/ind}} \quad \overline{\mathtt{h}_1 : \Delta_5, \mathtt{F}_3 \vdash \Delta_4} \ ^{\mathtt{ax/ind}}}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \vee \mathtt{F}_3 \vdash \Delta_4} \vee_L$$

- Case rule $\perp_L$

$$\frac{}{\bullet\mathtt{h}_1 : \perp, \Delta_3 \vdash \perp, \Delta_2} \perp_L \qquad \to \qquad \frac{}{\bullet\mathtt{h}_1 : \perp, \Delta_3 \vdash \Delta_2} \perp_L$$

- Case rule I

$$\frac{}{\bullet\mathtt{h}_1 : \mathtt{p}_3, \Delta_4 \vdash \mathtt{p}_3, \bot, \Delta_2} \; I \qquad \rightarrow \qquad \frac{}{\bullet\mathtt{h}_1 : \Delta_4, \mathtt{p}_3 \vdash \Delta_2, \mathtt{p}_3} \; I$$

- Case rule $\top_L$

$$\frac{\mathtt{h}_1 : \Delta_3 \vdash \bot, \Delta_2}{\bullet\mathtt{h}_1 : \top, \Delta_3 \vdash \bot, \Delta_2} \; \top_L \qquad \rightarrow \qquad \frac{\dfrac{}{\mathtt{h}_1 : \Delta_3 \vdash \Delta_2} \; \mathtt{ax/ind}}{\bullet\mathtt{h}_1 : \top, \Delta_3 \vdash \Delta_2} \; \top_L$$

4.6 Status of $\top_R$: : Invertible

- Case rule $\rightarrow_R$

$$\frac{\mathtt{h}_1 : \Delta_2, \mathtt{F}_3 \vdash \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash (\top, \Delta_5), \mathtt{F}_3 \rightarrow \mathtt{F}_4} \; \rightarrow_R \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\wedge_R$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \top, \Delta_5, \mathtt{F}_3 \quad \mathtt{h}_1 : \Delta_2 \vdash \top, \Delta_5, \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash (\top, \Delta_5), \mathtt{F}_3 \wedge \mathtt{F}_4} \; \wedge_R \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\vee_R$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \top, \Delta_5, \mathtt{F}_3, \mathtt{F}_4}{\bullet\mathtt{h}_1 : \Delta_2 \vdash (\top, \Delta_5), \mathtt{F}_3 \vee \mathtt{F}_4} \; \vee_R \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\bot_R$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \top, \Delta_3}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \bot, \top, \Delta_3} \; \bot_R \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\top_R$

$$\frac{}{\bullet\mathtt{h}_1 : \Delta_2 \vdash \top, \Delta_3} \; \top_R \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\rightarrow_L$

$$\frac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \rightarrow \mathtt{F}_3 \vdash \top, \Delta_4, \mathtt{F}_2 \quad \mathtt{h}_1 : \Delta_5, \mathtt{F}_3 \vdash \top, \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \rightarrow \mathtt{F}_3 \vdash \top, \Delta_4} \; \rightarrow_L \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\wedge_L$

$$\frac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2, \mathtt{F}_3 \vdash \top, \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \wedge \mathtt{F}_3 \vdash \top, \Delta_4} \; \wedge_L \qquad \rightarrow \qquad \texttt{trivial}$$

- Case rule $\vee_L$

$$\frac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \vdash \top, \Delta_4 \quad \mathtt{h}_1 : \Delta_5, \mathtt{F}_3 \vdash \top, \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \vee \mathtt{F}_3 \vdash \top, \Delta_4} \; \vee_L \qquad \rightarrow \qquad \texttt{trivial}$$

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- Case rule $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_3 \vdash \top, \Delta_2} \perp_L \quad \rightarrow \quad \text{trivial}$$

- Case rule I

$$\frac{}{\bullet h_1 : p_3, \Delta_4 \vdash p_3, \top, \Delta_2} I \quad \rightarrow \quad \text{trivial}$$

- Case rule $\top_L$

$$\frac{h_1 : \Delta_3 \vdash \top, \Delta_2}{\bullet h_1 : \top, \Delta_3 \vdash \top, \Delta_2} \top_L \quad \rightarrow \quad \text{trivial}$$

4.7 Status of $\rightarrow_L$: (Left Premise): `Invertible`

- Case rule $\rightarrow_R$

$$\frac{h_3 : \Delta_7, F_4, F_1 \rightarrow F_2 \vdash F_5}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4 \rightarrow F_5} \rightarrow_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_4, F_1 \rightarrow F_2 \vdash F_5} \text{ax}}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_1, F_4 \rightarrow F_5} \rightarrow_R$$

- Case rule $\wedge_R$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_5}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4 \wedge F_5} \wedge_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_1, F_4} \text{ax/ind} \quad \dfrac{}{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_1, F_5} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_1, F_4 \wedge F_5} \wedge_R$$

- Case rule $\vee_R$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4, F_5}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4 \vee F_5} \vee_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_1, F_4, F_5} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_1, F_4 \vee F_5} \vee_R$$

- Case rule $\perp_R$

$$\frac{h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \perp, \Delta_4} \perp_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \perp, \Delta_4, F_1} \perp_R$$

- Case rule $\top_R$

$$\frac{}{\bullet h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \top, \Delta_4} \top_R \quad \rightarrow \quad \frac{}{\bullet h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \top, \Delta_4, F_1} \top_R$$

- Case rule $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2, F_4 \rightarrow F_5 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_1 \rightarrow F_2, F_4 \rightarrow F_5 \vdash \Delta_6, F_1, F_4} \text{ax/ind} \quad \dfrac{}{h_3 : \Delta_7, F_5, F_1 \rightarrow F_2 \vdash \Delta_6, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1 \rightarrow F_2, F_4 \rightarrow F_5 \vdash \Delta_6, F_1} \rightarrow_L$$

$$\frac{h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4, F_2 \quad h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L \quad \rightarrow \quad \frac{\dfrac{}{h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4, F_2} \text{ax}}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4, F_2} \text{height}$$

- Case rule $\wedge_L$

$$\dfrac{\mathtt{h_3}:\Delta_7,\mathtt{F_4},\mathtt{F_5},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6}{\bullet\mathtt{h_3}:(\Delta_7,\mathtt{F_1}\to\mathtt{F_2}),\mathtt{F_4}\wedge\mathtt{F_5}\vdash\Delta_6}\ \wedge_L \qquad\to\qquad \dfrac{\overline{\mathtt{h_3}:\Delta_7,\mathtt{F_4},\mathtt{F_5},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_1}}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2},\mathtt{F_4}\wedge\mathtt{F_5}\vdash\Delta_6,\mathtt{F_1}}\ \wedge_L$$

- Case rule $\vee_L$

$$\dfrac{\mathtt{h_3}:\Delta_7,\mathtt{F_4},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6 \quad \mathtt{h_3}:\Delta_7,\mathtt{F_5},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6}{\bullet\mathtt{h_3}:(\Delta_7,\mathtt{F_1}\to\mathtt{F_2}),\mathtt{F_4}\vee\mathtt{F_5}\vdash\Delta_6}\ \vee_L \quad\to\quad \dfrac{\overline{\mathtt{h_3}:\Delta_7,\mathtt{F_4},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_1}}\ \text{ax/ind} \quad \overline{\mathtt{h_3}:\Delta_7,\mathtt{F_5},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_1}}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2},\mathtt{F_4}\vee\mathtt{F_5}\vdash\Delta_6,\mathtt{F_1}}\ \vee_L$$

- Case rule $\perp_L$

$$\dfrac{}{\bullet\mathtt{h_3}:\perp,\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4}\ \perp_L \qquad\to\qquad \dfrac{}{\bullet\mathtt{h_3}:\perp,\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4,\mathtt{F_1}}\ \perp_L$$

- Case rule I

$$\dfrac{}{\bullet\mathtt{h_3}:\mathtt{p_5},\Delta_6,\mathtt{F_1}\to\mathtt{F_2}\vdash\mathtt{p_5},\Delta_4}\ I \qquad\to\qquad \dfrac{}{\bullet\mathtt{h_3}:\Delta_6,\mathtt{p_5},\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4,\mathtt{F_1},\mathtt{p_5}}\ I$$

- Case rule $\top_L$

$$\dfrac{\mathtt{h_3}:\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4}{\bullet\mathtt{h_3}:\top,\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4}\ \top_L \qquad\to\qquad \dfrac{\overline{\mathtt{h_3}:\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4,\mathtt{F_1}}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\top,\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4,\mathtt{F_1}}\ \top_L$$

4.8   Status of $\to_L$ (Right Premise): :   `Invertible`

- Case rule $\to_R$

$$\dfrac{\mathtt{h_3}:\Delta_7,\mathtt{F_4},\mathtt{F_1}\to\mathtt{F_2}\vdash\mathtt{F_5}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}\to\mathtt{F_5}}\ \to_R \qquad\to\qquad \dfrac{\overline{\mathtt{h_3}:\Delta_7,\mathtt{F_2},\mathtt{F_4}\vdash\mathtt{F_5}}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}\to\mathtt{F_5}}\ \to_R$$

- Case rule $\wedge_R$

$$\dfrac{\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4} \quad \mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_5}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}\wedge\mathtt{F_5}}\ \wedge_R \quad\to\quad \dfrac{\overline{\mathtt{h_3}:\Delta_7,\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}}\ \text{ax/ind} \quad \overline{\mathtt{h_3}:\Delta_7,\mathtt{F_2}\vdash\Delta_6,\mathtt{F_5}}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}\wedge\mathtt{F_5}}\ \wedge_R$$

- Case rule $\vee_R$

$$\dfrac{\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4},\mathtt{F_5}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}\vee\mathtt{F_5}}\ \vee_R \qquad\to\qquad \dfrac{\overline{\mathtt{h_3}:\Delta_7,\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4},\mathtt{F_5}}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\Delta_7,\mathtt{F_2}\vdash\Delta_6,\mathtt{F_4}\vee\mathtt{F_5}}\ \vee_R$$

- Case rule $\perp_R$

$$\dfrac{\mathtt{h_3}:\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\Delta_4}{\bullet\mathtt{h_3}:\Delta_5,\mathtt{F_1}\to\mathtt{F_2}\vdash\perp,\Delta_4}\ \perp_R \qquad\to\qquad \dfrac{\overline{\mathtt{h_3}:\Delta_5,\mathtt{F_2}\vdash\Delta_4}\ \text{ax/ind}}{\bullet\mathtt{h_3}:\Delta_5,\mathtt{F_2}\vdash\perp,\Delta_4}\ \perp_R$$

- Case rule $\top_R$

$$\frac{}{\bullet h_3 : \Delta_5, F_1 \to F_2 \vdash \top, \Delta_4} \ \top_R \quad \to \quad \frac{}{\bullet h_3 : \Delta_5, F_2 \vdash \top, \Delta_4} \ \top_R$$

- Case rule $\to_L$

$$\frac{h_3 : \Delta_7, F_1 \to F_2, F_4 \to F_5 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \to F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \to F_2), F_4 \to F_5 \vdash \Delta_6} \ \to_L \quad \to \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2, F_4 \to F_5 \vdash \Delta_6, F_4} \ \text{ax/ind} \quad \dfrac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \to F_5 \vdash \Delta_6} \ \to_L$$

$$\frac{h_1 : \Delta_5, F_2 \to F_3 \vdash \Delta_4, F_2 \quad h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \to F_3 \vdash \Delta_4} \ \to_L \quad \to \quad \frac{\dfrac{}{h_1 : \Delta_5, F_3 \vdash \Delta_4} \ \text{ax}}{\bullet h_1 : \Delta_5, F_3 \vdash \Delta_4} \ height$$

- Case rule $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5, F_1 \to F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \to F_2), F_4 \wedge F_5 \vdash \Delta_6} \ \wedge_L \quad \to \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2, F_4, F_5 \vdash \Delta_6} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \ \wedge_L$$

- Case rule $\vee_L$

$$\frac{h_3 : \Delta_7, F_4, F_1 \to F_2 \vdash \Delta_6 \quad h_3 : \Delta_7, F_5, F_1 \to F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \to F_2), F_4 \vee F_5 \vdash \Delta_6} \ \vee_L \quad \to \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2, F_4 \vdash \Delta_6} \ \text{ax/ind} \quad \dfrac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \ \vee_L$$

- Case rule $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \to F_2 \vdash \Delta_4} \ \perp_L \quad \to \quad \frac{}{\bullet h_3 : \perp, \Delta_5, F_2 \vdash \Delta_4} \ \perp_L$$

- Case rule I

$$\frac{}{\bullet h_3 : p_5, \Delta_6, F_1 \to F_2 \vdash p_5, \Delta_4} \ I \quad \to \quad \frac{}{\bullet h_3 : \Delta_6, F_2, p_5 \vdash \Delta_4, p_5} \ I$$

- Case rule $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \to F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \to F_2 \vdash \Delta_4} \ \top_L \quad \to \quad \frac{\dfrac{}{h_3 : \Delta_5, F_2 \vdash \Delta_4} \ \text{ax/ind}}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \ \top_L$$

4.9 Status of $\wedge_L$: : Invertible

- Case rule $\to_R$

$$\frac{h_3 : \Delta_7, F_4, F_1 \wedge F_2 \vdash F_5}{\bullet h_3 : \Delta_7, F_1 \wedge F_2 \vdash \Delta_6, F_4 \to F_5} \ \to_R \quad \to \quad \frac{\dfrac{}{h_3 : \Delta_7, F_1, F_2, F_4 \vdash F_5} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2 \vdash \Delta_6, F_4 \to F_5} \ \to_R$$

- Case rule $\wedge_R$

$$\frac{h_3 : \Delta_7, F_1 \wedge F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_1 \wedge F_2 \vdash \Delta_6, F_5}{\bullet h_3 : \Delta_7, F_1 \wedge F_2 \vdash \Delta_6, F_4 \wedge F_5} \ \wedge_R \quad \to \quad \frac{\dfrac{}{h_3 : \Delta_7, F_1, F_2 \vdash \Delta_6, F_4} \ \text{ax/ind} \quad \dfrac{}{h_3 : \Delta_7, F_1, F_2 \vdash \Delta_6, F_5} \ \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2 \vdash \Delta_6, F_4 \wedge F_5} \ \wedge_R$$

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- Case rule $\lor_R$

$$\dfrac{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6, \mathtt{F}_4, \mathtt{F}_5}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6, \mathtt{F}_4 \lor \mathtt{F}_5} \ \lor_R \qquad \to \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2 \vdash \Delta_6, \mathtt{F}_4, \mathtt{F}_5}\ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2 \vdash \Delta_6, \mathtt{F}_4 \lor \mathtt{F}_5} \ \lor_R$$

- Case rule $\bot_R$

$$\dfrac{\mathtt{h}_3 : \Delta_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_4}{\bullet\mathtt{h}_3 : \Delta_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \bot, \Delta_4} \ \bot_R \qquad \to \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_5, \mathtt{F}_1, \mathtt{F}_2 \vdash \Delta_4}\ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_3 : \Delta_5, \mathtt{F}_1, \mathtt{F}_2 \vdash \bot, \Delta_4} \ \bot_R$$

- Case rule $\top_R$

$$\dfrac{}{\bullet\mathtt{h}_3 : \Delta_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \top, \Delta_4} \ \top_R \qquad \to \qquad \dfrac{}{\bullet\mathtt{h}_3 : \Delta_5, \mathtt{F}_1, \mathtt{F}_2 \vdash \top, \Delta_4} \ \top_R$$

- Case rule $\to_L$

$$\dfrac{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4 \to \mathtt{F}_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6, \mathtt{F}_4 \quad \mathtt{h}_3 : \Delta_7, \mathtt{F}_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6}{\bullet\mathtt{h}_3 : (\Delta_7, \mathtt{F}_1 \land \mathtt{F}_2), \mathtt{F}_4 \to \mathtt{F}_5 \vdash \Delta_6} \ \to_L$$

$$\to \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_4 \to \mathtt{F}_5 \vdash \Delta_6, \mathtt{F}_4}\ ^{\mathrm{ax/ind}} \quad \overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_5 \vdash \Delta_6}\ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_4 \to \mathtt{F}_5 \vdash \Delta_6} \ \to_L$$

- Case rule $\land_L$

$$\dfrac{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4, \mathtt{F}_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6}{\bullet\mathtt{h}_3 : (\Delta_7, \mathtt{F}_1 \land \mathtt{F}_2), \mathtt{F}_4 \land \mathtt{F}_5 \vdash \Delta_6} \ \land_L \qquad \to \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_4, \mathtt{F}_5 \vdash \Delta_6}\ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_4 \land \mathtt{F}_5 \vdash \Delta_6} \ \land_L$$

$$\dfrac{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2, \mathtt{F}_3 \vdash \Delta_4}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2 \land \mathtt{F}_3 \vdash \Delta_4} \ \land_L \qquad \to \qquad \dfrac{\overline{\mathtt{h}_1 : \Delta_5, \mathtt{F}_2, \mathtt{F}_3 \vdash \Delta_4}\ ^{\mathrm{ax}}}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{F}_2, \mathtt{F}_3 \vdash \Delta_4} \ _{height}$$

- Case rule $\lor_L$

$$\dfrac{\mathtt{h}_3 : \Delta_7, \mathtt{F}_4, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6 \quad \mathtt{h}_3 : \Delta_7, \mathtt{F}_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_6}{\bullet\mathtt{h}_3 : (\Delta_7, \mathtt{F}_1 \land \mathtt{F}_2), \mathtt{F}_4 \lor \mathtt{F}_5 \vdash \Delta_6} \ \lor_L$$

$$\to \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_4 \vdash \Delta_6}\ ^{\mathrm{ax/ind}} \quad \overline{\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_5 \vdash \Delta_6}\ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_3 : \Delta_7, \mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_4 \lor \mathtt{F}_5 \vdash \Delta_6} \ \lor_L$$

- Case rule $\bot_L$

$$\dfrac{}{\bullet\mathtt{h}_3 : \bot, \Delta_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_4} \ \bot_L \qquad \to \qquad \dfrac{}{\bullet\mathtt{h}_3 : \bot, \Delta_5, \mathtt{F}_1, \mathtt{F}_2 \vdash \Delta_4} \ \bot_L$$

- Case rule I

$$\dfrac{}{\bullet\mathtt{h}_3 : \mathtt{p}_5, \Delta_6, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \mathtt{p}_5, \Delta_4} \ I \qquad \to \qquad \dfrac{}{\bullet\mathtt{h}_3 : \Delta_6, \mathtt{F}_1, \mathtt{F}_2, \mathtt{p}_5 \vdash \Delta_4, \mathtt{p}_5} \ I$$

- Case rule $\top_L$

$$\dfrac{\mathtt{h}_3 : \Delta_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_4}{\bullet\mathtt{h}_3 : \top, \Delta_5, \mathtt{F}_1 \land \mathtt{F}_2 \vdash \Delta_4} \ \top_L \qquad \to \qquad \dfrac{\overline{\mathtt{h}_3 : \Delta_5, \mathtt{F}_1, \mathtt{F}_2 \vdash \Delta_4}\ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_3 : \top, \Delta_5, \mathtt{F}_1, \mathtt{F}_2 \vdash \Delta_4} \ \top_L$$

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4.10 Status of $\vee_L$: (Left Premise): Invertible

- Case rule $\to_R$

$$\frac{\mathrm{h}_3 : \Delta_7, \mathrm{F}_4, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \mathrm{F}_5}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_4 \to \mathrm{F}_5} \to_R \qquad \to \qquad \frac{\overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4 \vdash \mathrm{F}_5} \;\; ax/ind}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vdash \Delta_6, \mathrm{F}_4 \to \mathrm{F}_5} \to_R$$

- Case rule $\wedge_R$

$$\frac{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_4 \quad \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_5}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_4 \wedge \mathrm{F}_5} \wedge_R \quad \to \quad \frac{\overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vdash \Delta_6, \mathrm{F}_4} \; ax/ind \quad \overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vdash \Delta_6, \mathrm{F}_5} \; ax/ind}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vdash \Delta_6, \mathrm{F}_4 \wedge \mathrm{F}_5} \wedge_R$$

- Case rule $\vee_R$

$$\frac{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_4, \mathrm{F}_5}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_4 \vee \mathrm{F}_5} \vee_R \qquad \to \qquad \frac{\overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vdash \Delta_6, \mathrm{F}_4, \mathrm{F}_5} \; ax/ind}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1 \vdash \Delta_6, \mathrm{F}_4 \vee \mathrm{F}_5} \vee_R$$

- Case rule $\perp_R$

$$\frac{\mathrm{h}_3 : \Delta_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_4}{\bullet \mathrm{h}_3 : \Delta_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \perp, \Delta_4} \perp_R \qquad \to \qquad \frac{\overline{\mathrm{h}_3 : \Delta_5, \mathrm{F}_1 \vdash \Delta_4} \; ax/ind}{\bullet \mathrm{h}_3 : \Delta_5, \mathrm{F}_1 \vdash \perp, \Delta_4} \perp_R$$

- Case rule $\top_R$

$$\frac{}{\bullet \mathrm{h}_3 : \Delta_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \top, \Delta_4} \top_R \qquad \to \qquad \frac{}{\bullet \mathrm{h}_3 : \Delta_5, \mathrm{F}_1 \vdash \top, \Delta_4} \top_R$$

- Case rule $\to_L$

$$\frac{\mathrm{h}_3 : \Delta_7, \mathrm{F}_4 \to \mathrm{F}_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6, \mathrm{F}_4 \quad \mathrm{h}_3 : \Delta_7, \mathrm{F}_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6}{\bullet \mathrm{h}_3 : (\Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2), \mathrm{F}_4 \to \mathrm{F}_5 \vdash \Delta_6} \to_L \quad \to \quad \frac{\overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4 \to \mathrm{F}_5 \vdash \Delta_6, \mathrm{F}_4} \; ax/ind \quad \overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_5 \vdash \Delta_6} \; ax/ind}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4 \to \mathrm{F}_5 \vdash \Delta_6} \to_L$$

- Case rule $\wedge_L$

$$\frac{\mathrm{h}_3 : \Delta_7, \mathrm{F}_4, \mathrm{F}_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6}{\bullet \mathrm{h}_3 : (\Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2), \mathrm{F}_4 \wedge \mathrm{F}_5 \vdash \Delta_6} \wedge_L \qquad \to \qquad \frac{\overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4, \mathrm{F}_5 \vdash \Delta_6} \; ax/ind}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4 \wedge \mathrm{F}_5 \vdash \Delta_6} \wedge_L$$

- Case rule $\vee_L$

$$\frac{\mathrm{h}_3 : \Delta_7, \mathrm{F}_4, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6 \quad \mathrm{h}_3 : \Delta_7, \mathrm{F}_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_6}{\bullet \mathrm{h}_3 : (\Delta_7, \mathrm{F}_1 \vee \mathrm{F}_2), \mathrm{F}_4 \vee \mathrm{F}_5 \vdash \Delta_6} \vee_L \quad \to \quad \frac{\overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4 \vdash \Delta_6} \; ax/ind \quad \overline{\mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_5 \vdash \Delta_6} \; ax/ind}{\bullet \mathrm{h}_3 : \Delta_7, \mathrm{F}_1, \mathrm{F}_4 \vee \mathrm{F}_5 \vdash \Delta_6} \vee_L$$

$$\frac{\mathrm{h}_1 : \Delta_5, \mathrm{F}_2 \vdash \Delta_4 \quad \mathrm{h}_1 : \Delta_5, \mathrm{F}_3 \vdash \Delta_4}{\bullet \mathrm{h}_1 : \Delta_5, \mathrm{F}_2 \vee \mathrm{F}_3 \vdash \Delta_4} \vee_L \qquad \to \qquad \frac{\overline{\mathrm{h}_1 : \Delta_5, \mathrm{F}_2 \vdash \Delta_4} \; ax}{\bullet \mathrm{h}_1 : \Delta_5, \mathrm{F}_2 \vdash \Delta_4} \; height$$

- Case rule $\perp_L$

$$\frac{}{\bullet \mathrm{h}_3 : \perp, \Delta_5, \mathrm{F}_1 \vee \mathrm{F}_2 \vdash \Delta_4} \perp_L \qquad \to \qquad \frac{}{\bullet \mathrm{h}_3 : \perp, \Delta_5, \mathrm{F}_1 \vdash \Delta_4} \perp_L$$

- Case rule I

$$\frac{}{\bullet h_3 : p_5, \Delta_6, F_1 \vee F_2 \vdash p_5, \Delta_4} \; I \quad \rightarrow \quad \frac{}{\bullet h_3 : \Delta_6, F_1, p_5 \vdash \Delta_4, p_5} \; I$$

- Case rule $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \; \top_L \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_5, F_1 \vdash \Delta_4} \; ax/ind}{\bullet h_3 : \top, \Delta_5, F_1 \vdash \Delta_4} \; \top_L$$

4.11 Status of $\vee_L$ (Right Premise): : Invertible

- Case rule $\rightarrow_R$

$$\frac{h_3 : \Delta_7, F_4, F_1 \vee F_2 \vdash F_5}{\bullet h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4 \rightarrow F_5} \; \rightarrow_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2, F_4 \vdash F_5} \; ax/ind}{\bullet h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4 \rightarrow F_5} \; \rightarrow_R$$

- Case rule $\wedge_R$

$$\frac{h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_5}{\bullet h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4 \wedge F_5} \; \wedge_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4} \; ax/ind \quad \dfrac{}{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_5} \; ax/ind}{\bullet h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4 \wedge F_5} \; \wedge_R$$

- Case rule $\vee_R$

$$\frac{h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4, F_5}{\bullet h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4 \vee F_5} \; \vee_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4, F_5} \; ax/ind}{\bullet h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4 \vee F_5} \; \vee_R$$

- Case rule $\perp_R$

$$\frac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_5, F_1 \vee F_2 \vdash \perp, \Delta_4} \; \perp_R \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_5, F_2 \vdash \Delta_4} \; ax/ind}{\bullet h_3 : \Delta_5, F_2 \vdash \perp, \Delta_4} \; \perp_R$$

- Case rule $\top_R$

$$\frac{}{\bullet h_3 : \Delta_5, F_1 \vee F_2 \vdash \top, \Delta_4} \; \top_R \quad \rightarrow \quad \frac{}{\bullet h_3 : \Delta_5, F_2 \vdash \top, \Delta_4} \; \top_R$$

- Case rule $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_4 \rightarrow F_5, F_1 \vee F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \; \rightarrow_L \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6, F_4} \; ax/ind \quad \dfrac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \; ax/ind}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \; \rightarrow_L$$

- Case rule $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \wedge F_5 \vdash \Delta_6} \; \wedge_L \quad \rightarrow \quad \frac{\dfrac{}{h_3 : \Delta_7, F_2, F_4, F_5 \vdash \Delta_6} \; ax/ind}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \; \wedge_L$$

- Case rule $\vee_L$

$$\cfrac{h_3 : \Delta_7, F_4, F_1 \vee F_2 \vdash \Delta_6 \quad h_3 : \Delta_7, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \vee F_5 \vdash \Delta_6} \ \vee_L \quad \to \quad \cfrac{\cfrac{}{h_3 : \Delta_7, F_2, F_4 \vdash \Delta_6} \ {}^{ax/ind} \quad \cfrac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \ {}^{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \ \vee_L$$

$$\cfrac{h_1 : \Delta_5, F_2 \vdash \Delta_4 \quad h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \ \vee_L \quad \to \quad \cfrac{\cfrac{}{h_1 : \Delta_5, F_3 \vdash \Delta_4} \ {}^{ax}_{height}}{\bullet h_1 : \Delta_5, F_3 \vdash \Delta_4}$$

- Case rule $\perp_L$

$$\cfrac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \ \perp_L \quad \to \quad \cfrac{}{\bullet h_3 : \perp, \Delta_5, F_2 \vdash \Delta_4} \ \perp_L$$

- Case rule I

$$\cfrac{}{\bullet h_3 : p_5, \Delta_6, F_1 \vee F_2 \vdash p_5, \Delta_4} \ I \quad \to \quad \cfrac{}{\bullet h_3 : \Delta_6, F_2, p_5 \vdash \Delta_4, p_5} \ I$$

- Case rule $\top_L$

$$\cfrac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \ \top_L \quad \to \quad \cfrac{\cfrac{}{h_3 : \Delta_5, F_2 \vdash \Delta_4} \ {}^{ax/ind}}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \ \top_L$$

4.12 Status of $\perp_L$: : Invertible

- Case rule $\to_R$

$$\cfrac{h_1 : \perp, \Delta_5, F_2 \vdash F_3}{\bullet h_1 : \perp, \Delta_5 \vdash \Delta_4, F_2 \to F_3} \ \to_R \quad \to \quad \text{trivial}$$

- Case rule $\wedge_R$

$$\cfrac{h_1 : \perp, \Delta_5 \vdash \Delta_4, F_2 \quad h_1 : \perp, \Delta_5 \vdash \Delta_4, F_3}{\bullet h_1 : \perp, \Delta_5 \vdash \Delta_4, F_2 \wedge F_3} \ \wedge_R \quad \to \quad \text{trivial}$$

- Case rule $\vee_R$

$$\cfrac{h_1 : \perp, \Delta_5 \vdash \Delta_4, F_2, F_3}{\bullet h_1 : \perp, \Delta_5 \vdash \Delta_4, F_2 \vee F_3} \ \vee_R \quad \to \quad \text{trivial}$$

- Case rule $\perp_R$

$$\cfrac{h_1 : \perp, \Delta_3 \vdash \Delta_2}{\bullet h_1 : \perp, \Delta_3 \vdash \perp, \Delta_2} \ \perp_R \quad \to \quad \text{trivial}$$

- Case rule $\top_R$

$$\cfrac{}{\bullet h_1 : \perp, \Delta_3 \vdash \top, \Delta_2} \ \top_R \quad \to \quad \text{trivial}$$

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- Case rule $\to_L$

$$\frac{\mathtt{h_1} : \bot, \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_2} \quad \mathtt{h_1} : \bot, \Delta_5, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : (\bot, \Delta_5), \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4} \to_L \quad \to \quad \texttt{trivial}$$

- Case rule $\wedge_L$

$$\frac{\mathtt{h_1} : \bot, \Delta_5, \mathtt{F_2}, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : (\bot, \Delta_5), \mathtt{F_2} \wedge \mathtt{F_3} \vdash \Delta_4} \wedge_L \quad \to \quad \texttt{trivial}$$

- Case rule $\vee_L$

$$\frac{\mathtt{h_1} : \bot, \Delta_5, \mathtt{F_2} \vdash \Delta_4 \quad \mathtt{h_1} : \bot, \Delta_5, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : (\bot, \Delta_5), \mathtt{F_2} \vee \mathtt{F_3} \vdash \Delta_4} \vee_L \quad \to \quad \texttt{trivial}$$

- Case rule $\bot_L$

$$\frac{}{\bullet\mathtt{h_1} : \bot, \Delta_3 \vdash \Delta_2} \bot_L \quad \to \quad \texttt{trivial}$$

- Case rule I

$$\frac{}{\bullet\mathtt{h_1} : \mathtt{p_3}, \bot, \Delta_4 \vdash \mathtt{p_3}, \Delta_2} I \quad \to \quad \texttt{trivial}$$

- Case rule $\top_L$

$$\frac{\mathtt{h_1} : \bot, \Delta_3 \vdash \Delta_2}{\bullet\mathtt{h_1} : \top, \bot, \Delta_3 \vdash \Delta_2} \top_L \quad \to \quad \texttt{trivial}$$

4.13 Status of I: : Invertible

- Case rule $\to_R$

$$\frac{\mathtt{h_1} : \Delta_6, \mathtt{F_2}, \mathtt{p_5} \vdash \mathtt{F_3}}{\bullet\mathtt{h_1} : \Delta_6, \mathtt{p_5} \vdash (\Delta_4, \mathtt{p_5}), \mathtt{F_2} \to \mathtt{F_3}} \to_R \quad \to \quad \texttt{trivial}$$

- Case rule $\wedge_R$

$$\frac{\mathtt{h_1} : \Delta_6, \mathtt{p_5} \vdash \Delta_4, \mathtt{F_2}, \mathtt{p_5} \quad \mathtt{h_1} : \Delta_6, \mathtt{p_5} \vdash \Delta_4, \mathtt{F_3}, \mathtt{p_5}}{\bullet\mathtt{h_1} : \Delta_6, \mathtt{p_5} \vdash (\Delta_4, \mathtt{p_5}), \mathtt{F_2} \wedge \mathtt{F_3}} \wedge_R \quad \to \quad \texttt{trivial}$$

- Case rule $\vee_R$

$$\frac{\mathtt{h_1} : \Delta_6, \mathtt{p_5} \vdash \Delta_4, \mathtt{F_2}, \mathtt{F_3}, \mathtt{p_5}}{\bullet\mathtt{h_1} : \Delta_6, \mathtt{p_5} \vdash (\Delta_4, \mathtt{p_5}), \mathtt{F_2} \vee \mathtt{F_3}} \vee_R \quad \to \quad \texttt{trivial}$$

- Case rule $\bot_R$

$$\frac{\mathtt{h_1} : \Delta_4, \mathtt{p_3} \vdash \Delta_2, \mathtt{p_3}}{\bullet\mathtt{h_1} : \Delta_4, \mathtt{p_3} \vdash \bot, \Delta_2, \mathtt{p_3}} \bot_R \quad \to \quad \texttt{trivial}$$

- Case rule $\top_R$

$$\frac{}{\bullet h_1 : \Delta_4, p_3 \vdash \top, \Delta_2, p_3} \ \top_R \qquad \to \qquad \text{trivial}$$

- Case rule $\to_L$

$$\frac{h_1 : \Delta_6, p_5, F_2 \to F_3 \vdash \Delta_4, F_2, p_5 \quad h_1 : \Delta_6, F_3, p_5 \vdash \Delta_4, p_5}{\bullet h_1 : (\Delta_6, p_5), F_2 \to F_3 \vdash \Delta_4, p_5} \ \to_L \qquad \to \qquad \text{trivial}$$

- Case rule $\wedge_L$

$$\frac{h_1 : \Delta_6, F_2, F_3, p_5 \vdash \Delta_4, p_5}{\bullet h_1 : (\Delta_6, p_5), F_2 \wedge F_3 \vdash \Delta_4, p_5} \ \wedge_L \qquad \to \qquad \text{trivial}$$

- Case rule $\vee_L$

$$\frac{h_1 : \Delta_6, F_2, p_5 \vdash \Delta_4, p_5 \quad h_1 : \Delta_6, F_3, p_5 \vdash \Delta_4, p_5}{\bullet h_1 : (\Delta_6, p_5), F_2 \vee F_3 \vdash \Delta_4, p_5} \ \vee_L \qquad \to \qquad \text{trivial}$$

- Case rule $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_4, p_3 \vdash \Delta_2, p_3} \ \perp_L \qquad \to \qquad \text{trivial}$$

- Case rule I

$$\frac{}{\bullet h_1 : p_4, \Delta_5, p_3 \vdash p_4, \Delta_2, p_3} \ I \qquad \to \qquad \text{trivial}$$

$$\frac{}{\bullet h_1 : p_3, \Delta_4 \vdash p_3, \Delta_2} \ I \qquad \to \qquad \text{trivial}$$

- Case rule $\top_L$

$$\frac{h_1 : \Delta_4, p_3 \vdash \Delta_2, p_3}{\bullet h_1 : \top, \Delta_4, p_3 \vdash \Delta_2, p_3} \ \top_L \qquad \to \qquad \text{trivial}$$

4.14 Status of $\top_L$: : Invertible

- Case rule $\to_R$

$$\frac{h_1 : \top, \Delta_5, F_2 \vdash F_3}{\bullet h_1 : \top, \Delta_5 \vdash \Delta_4, F_2 \to F_3} \ \to_R \qquad \to \qquad \frac{\dfrac{}{h_1 : \Delta_5, F_2 \vdash F_3} \ \text{ax/ind}}{\bullet h_1 : \Delta_5 \vdash \Delta_4, F_2 \to F_3} \ \to_R$$

- Case rule $\wedge_R$

$$\frac{h_1 : \top, \Delta_5 \vdash \Delta_4, F_2 \quad h_1 : \top, \Delta_5 \vdash \Delta_4, F_3}{\bullet h_1 : \top, \Delta_5 \vdash \Delta_4, F_2 \wedge F_3} \ \wedge_R \qquad \to \qquad \frac{\dfrac{}{h_1 : \Delta_5 \vdash \Delta_4, F_2} \ \text{ax/ind} \quad \dfrac{}{h_1 : \Delta_5 \vdash \Delta_4, F_3} \ \text{ax/ind}}{\bullet h_1 : \Delta_5 \vdash \Delta_4, F_2 \wedge F_3} \ \wedge_R$$

- Case rule $\vee_R$

$$\frac{\mathtt{h_1} : \top, \Delta_5 \vdash \Delta_4, \mathtt{F_2}, \mathtt{F_3}}{\bullet\mathtt{h_1} : \top, \Delta_5 \vdash \Delta_4, \mathtt{F_2} \vee \mathtt{F_3}} \ \vee_R \qquad \to \qquad \frac{\overline{\mathtt{h_1} : \Delta_5 \vdash \Delta_4, \mathtt{F_2}, \mathtt{F_3}} \ \text{ax/ind}}{\bullet\mathtt{h_1} : \Delta_5 \vdash \Delta_4, \mathtt{F_2} \vee \mathtt{F_3}} \ \vee_R$$

- Case rule $\bot_R$

$$\frac{\mathtt{h_1} : \top, \Delta_3 \vdash \Delta_2}{\bullet\mathtt{h_1} : \top, \Delta_3 \vdash \bot, \Delta_2} \ \bot_R \qquad \to \qquad \frac{\overline{\mathtt{h_1} : \Delta_3 \vdash \Delta_2} \ \text{ax/ind}}{\bullet\mathtt{h_1} : \Delta_3 \vdash \bot, \Delta_2} \ \bot_R$$

- Case rule $\top_R$

$$\frac{}{\bullet\mathtt{h_1} : \top, \Delta_3 \vdash \top, \Delta_2} \ \top_R \qquad \to \qquad \frac{}{\bullet\mathtt{h_1} : \Delta_3 \vdash \top, \Delta_2} \ \top_R$$

- Case rule $\to_L$

$$\frac{\mathtt{h_1} : \top, \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_2} \quad \mathtt{h_1} : \top, \Delta_5, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : (\top, \Delta_5), \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4} \ \to_L \quad \to \quad \frac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4, \mathtt{F_2}} \ \text{ax/ind} \quad \overline{\mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4} \ \text{ax/ind}}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \to \mathtt{F_3} \vdash \Delta_4} \ \to_L$$

- Case rule $\wedge_L$

$$\frac{\mathtt{h_1} : \top, \Delta_5, \mathtt{F_2}, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : (\top, \Delta_5), \mathtt{F_2} \wedge \mathtt{F_3} \vdash \Delta_4} \ \wedge_L \qquad \to \qquad \frac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_2}, \mathtt{F_3} \vdash \Delta_4} \ \text{ax/ind}}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \wedge \mathtt{F_3} \vdash \Delta_4} \ \wedge_L$$

- Case rule $\vee_L$

$$\frac{\mathtt{h_1} : \top, \Delta_5, \mathtt{F_2} \vdash \Delta_4 \quad \mathtt{h_1} : \top, \Delta_5, \mathtt{F_3} \vdash \Delta_4}{\bullet\mathtt{h_1} : (\top, \Delta_5), \mathtt{F_2} \vee \mathtt{F_3} \vdash \Delta_4} \ \vee_L \quad \to \quad \frac{\overline{\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vdash \Delta_4} \ \text{ax/ind} \quad \overline{\mathtt{h_1} : \Delta_5, \mathtt{F_3} \vdash \Delta_4} \ \text{ax/ind}}{\bullet\mathtt{h_1} : \Delta_5, \mathtt{F_2} \vee \mathtt{F_3} \vdash \Delta_4} \ \vee_L$$

- Case rule $\bot_L$

$$\frac{}{\bullet\mathtt{h_1} : \bot, \top, \Delta_3 \vdash \Delta_2} \ \bot_L \qquad \to \qquad \frac{}{\bullet\mathtt{h_1} : \bot, \Delta_3 \vdash \Delta_2} \ \bot_L$$

- Case rule I

$$\frac{}{\bullet\mathtt{h_1} : \mathtt{p_3}, \top, \Delta_4 \vdash \mathtt{p_3}, \Delta_2} \ I \qquad \to \qquad \frac{}{\bullet\mathtt{h_1} : \Delta_4, \mathtt{p_3} \vdash \Delta_2, \mathtt{p_3}} \ I$$

- Case rule $\top_L$

$$\frac{\mathtt{h_1} : \Delta_3 \vdash \Delta_2}{\bullet\mathtt{h_1} : \top, \Delta_3 \vdash \Delta_2} \ \top_L \qquad \to \qquad \frac{\overline{\mathtt{h_1} : \Delta_3 \vdash \Delta_2} \ \text{ax}}{\bullet\mathtt{h_1} : \Delta_3 \vdash \Delta_2} \ _{height}$$

5 Height preserving admissibility of contraction on the left

- Case(s) rule $\to_R$

$$\dfrac{h_1 : \Delta_5, F_2, F_6, F_6 \vdash F_3}{\bullet h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2 \to F_3} \to_R \qquad \to \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2, F_6, F_6 \vdash F_3}\ \text{ax}}{h_1 : \Delta_5, F_2, F_6 \vdash F_3}\ \text{IH}}{\bullet h_1 : \Delta_5, F_6 \vdash \Delta_4, F_2 \to F_3} \to_R$$

- Case(s) rule $\wedge_R$

$$\dfrac{h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2 \quad h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_3}{\bullet h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2 \wedge F_3} \wedge_R \quad \to \quad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2}\ \text{ax}}{h_1 : \Delta_5, F_6 \vdash \Delta_4, F_2}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_3}\ \text{ax}}{h_1 : \Delta_5, F_6 \vdash \Delta_4, F_3}\ \text{IH}}{\bullet h_1 : \Delta_5, F_6 \vdash \Delta_4, F_2 \wedge F_3} \wedge_R$$

- Case(s) rule $\vee_R$

$$\dfrac{h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2, F_3}{\bullet h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2 \vee F_3} \vee_R \qquad \to \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_6, F_6 \vdash \Delta_4, F_2, F_3}\ \text{ax}}{h_1 : \Delta_5, F_6 \vdash \Delta_4, F_2, F_3}\ \text{IH}}{\bullet h_1 : \Delta_5, F_6 \vdash \Delta_4, F_2 \vee F_3} \vee_R$$

- Case(s) rule $\perp_R$

$$\dfrac{h_1 : \Delta_3, F_4, F_4 \vdash \Delta_2}{\bullet h_1 : \Delta_3, F_4, F_4 \vdash \perp, \Delta_2} \perp_R \qquad \to \qquad \dfrac{\dfrac{\overline{h_1 : \Delta_3, F_4, F_4 \vdash \Delta_2}\ \text{ax}}{h_1 : \Delta_3, F_4 \vdash \Delta_2}\ \text{IH}}{\bullet h_1 : \Delta_3, F_4 \vdash \perp, \Delta_2} \perp_R$$

- Case(s) rule $\top_R$

$$\dfrac{}{\bullet h_1 : \Delta_3, F_4, F_4 \vdash \top, \Delta_2} \top_R \qquad \to \qquad \dfrac{}{\bullet h_1 : \Delta_3, F_4 \vdash \top, \Delta_2} \top_R$$

- Case(s) rule $\to_L$

$$\dfrac{h_1 : \Delta_5, F_2 \to F_3, F_2 \to F_3 \vdash \Delta_4, F_2 \quad h_1 : \Delta_5, F_3, F_2 \to F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \to F_3, F_2 \to F_3 \vdash \Delta_4} \to_L \quad \to \quad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2 \to F_3, F_2 \to F_3 \vdash \Delta_4, F_2}\ \text{ax}}{h_1 : \Delta_5, F_2 \to F_3 \vdash \Delta_4, F_2}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_3, F_3 \vdash \Delta_4}\ \text{inv-th/ax}}{h_1 : \Delta_5, F_3 \vdash \Delta_4}\ \text{IH}}{\bullet h_1 : \Delta_5, F_2 \to F_3 \vdash \Delta_4} \to_L$$

$$\dfrac{h_1 : \Delta_5, F_6, F_6, F_2 \to F_3 \vdash \Delta_4, F_2 \quad h_1 : \Delta_5, F_3, F_6, F_6 \vdash \Delta_4}{\bullet h_1 : (\Delta_5, F_2 \to F_3), F_6, F_6 \vdash \Delta_4} \to_L \quad \to \quad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_6, F_6, F_2 \to F_3 \vdash \Delta_4, F_2}\ \text{ax}}{h_1 : \Delta_5, F_6, F_2 \to F_3 \vdash \Delta_4, F_2}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_3, F_6, F_6 \vdash \Delta_4}\ \text{ax}}{h_1 : \Delta_5, F_3, F_6 \vdash \Delta_4}\ \text{IH}}{\bullet h_1 : \Delta_5, F_6, F_2 \to F_3 \vdash \Delta_4} \to_L$$

- Case(s) rule $\wedge_L$

$$\dfrac{h_1 : \Delta_5, F_2, F_3, F_2 \wedge F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \wedge F_3, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L \quad \to \quad \dfrac{\dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2, F_2, F_3, F_3 \vdash \Delta_4}\ \text{inv-th/ax}}{h_1 : \Delta_5, F_2, F_2, F_3 \vdash \Delta_4}\ \text{IH}}{h_1 : \Delta_5, F_2, F_3 \vdash \Delta_4}\ \text{IH}}{\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L$$

$$\dfrac{h_1 : \Delta_5, F_2, F_3, F_6, F_6 \vdash \Delta_4}{\bullet h_1 : (\Delta_5, F_2 \wedge F_3), F_6, F_6 \vdash \Delta_4} \wedge_L \quad \to \quad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2, F_3, F_6, F_6 \vdash \Delta_4}\ \text{ax}}{h_1 : \Delta_5, F_2, F_3, F_6 \vdash \Delta_4}\ \text{IH}}{\bullet h_1 : \Delta_5, F_6, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L$$

- Case(s) rule $\vee_L$

$$\dfrac{h_1 : \Delta_5, F_2, F_2 \vee F_3 \vdash \Delta_4 \quad h_1 : \Delta_5, F_3, F_2 \vee F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3, F_2 \vee F_3 \vdash \Delta_4} \vee_L \quad \to \quad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2, F_2 \vdash \Delta_4}\ \text{inv-th/ax}}{h_1 : \Delta_5, F_2 \vdash \Delta_4}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_3, F_3 \vdash \Delta_4}\ \text{inv-th/ax}}{h_1 : \Delta_5, F_3 \vdash \Delta_4}\ \text{IH}}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \vee_L$$

$$\dfrac{h_1 : \Delta_5, F_2, F_6, F_6 \vdash \Delta_4 \quad h_1 : \Delta_5, F_3, F_6, F_6 \vdash \Delta_4}{\bullet h_1 : (\Delta_5, F_2 \vee F_3), F_6, F_6 \vdash \Delta_4} \vee_L \quad \to \quad \dfrac{\dfrac{\overline{h_1 : \Delta_5, F_2, F_6, F_6 \vdash \Delta_4}\ \text{ax}}{h_1 : \Delta_5, F_2, F_6 \vdash \Delta_4}\ \text{IH} \quad \dfrac{\overline{h_1 : \Delta_5, F_3, F_6, F_6 \vdash \Delta_4}\ \text{ax}}{h_1 : \Delta_5, F_3, F_6 \vdash \Delta_4}\ \text{IH}}{\bullet h_1 : \Delta_5, F_6, F_2 \vee F_3 \vdash \Delta_4} \vee_L$$

- Case(s) rule $\perp_L$

$$\dfrac{}{\bullet \mathtt{h}_1 : \Delta_3, \perp, \perp \vdash \Delta_2} \; \perp_L \qquad \rightarrow \qquad \dfrac{}{\bullet \mathtt{h}_1 : \perp, \Delta_3 \vdash \Delta_2} \; \perp_L$$

$$\dfrac{}{\bullet \mathtt{h}_1 : (\perp, \Delta_3), \mathtt{F}_4, \mathtt{F}_4 \vdash \Delta_2} \; \perp_L \qquad \rightarrow \qquad \dfrac{}{\bullet \mathtt{h}_1 : \perp, \Delta_3, \mathtt{F}_4 \vdash \Delta_2} \; \perp_L$$

- Case(s) rule I

$$\dfrac{}{\bullet \mathtt{h}_1 : \Delta_4, \mathtt{p}_3, \mathtt{p}_3 \vdash \Delta_2, \mathtt{p}_3} \; I \qquad \rightarrow \qquad \dfrac{}{\bullet \mathtt{h}_1 : \Delta_4, \mathtt{p}_3 \vdash \Delta_2, \mathtt{p}_3} \; I$$

$$\dfrac{}{\bullet \mathtt{h}_1 : (\Delta_4, \mathtt{p}_3), \mathtt{F}_5, \mathtt{F}_5 \vdash \Delta_2, \mathtt{p}_3} \; I \qquad \rightarrow \qquad \dfrac{}{\bullet \mathtt{h}_1 : \Delta_4, \mathtt{F}_5, \mathtt{p}_3 \vdash \Delta_2, \mathtt{p}_3} \; I$$

- Case(s) rule $\top_L$

$$\dfrac{\dfrac{}{\mathtt{h}_1 : \top, \Delta_3 \vdash \Delta_2}}{\bullet \mathtt{h}_1 : \Delta_3, \top, \top \vdash \Delta_2} \; \top_L \qquad \rightarrow \qquad \dfrac{\dfrac{}{\mathtt{h}_1 : \Delta_3 \vdash \Delta_2} \; \text{inv-th/ax}}{\bullet \mathtt{h}_1 : \top, \Delta_3 \vdash \Delta_2} \; \top_L$$

$$\dfrac{\dfrac{}{\mathtt{h}_1 : \Delta_3, \mathtt{F}_4, \mathtt{F}_4 \vdash \Delta_2}}{\bullet \mathtt{h}_1 : (\top, \Delta_3), \mathtt{F}_4, \mathtt{F}_4 \vdash \Delta_2} \; \top_L \qquad \rightarrow \qquad \dfrac{\dfrac{\dfrac{}{\mathtt{h}_1 : \Delta_3, \mathtt{F}_4, \mathtt{F}_4 \vdash \Delta_2} \; \text{ax}}{\mathtt{h}_1 : \Delta_3, \mathtt{F}_4 \vdash \Delta_2} \; \text{IH}}{\bullet \mathtt{h}_1 : \top, \Delta_3, \mathtt{F}_4 \vdash \Delta_2} \; \top_L$$

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6 Height preserving admissibility of contraction on the Right

- Case(s) rule $\to_R$

$$\cfrac{\mathtt{h_1}:\Delta_2,\mathtt{F_3}\vdash\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\to\mathtt{F_4},\mathtt{F_3}\to\mathtt{F_4}}\ \to_R \qquad\to\qquad \cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2,\mathtt{F_3}\vdash\mathtt{F_4}}\ \text{ax}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\to\mathtt{F_4}}\ \to_R$$

$$\cfrac{\mathtt{h_1}:\Delta_2,\mathtt{F_3}\vdash\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash(\Delta_5,\mathtt{F_3}\to\mathtt{F_4}),\mathtt{F_6},\mathtt{F_6}}\ \to_R \qquad\to\qquad \cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2,\mathtt{F_3}\vdash\mathtt{F_4}}\ \text{ax}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_6},\mathtt{F_3}\to\mathtt{F_4}}\ \to_R$$

- Case(s) rule $\wedge_R$

$$\cfrac{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_3}\wedge\mathtt{F_4}\quad\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4},\mathtt{F_3}\wedge\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\wedge\mathtt{F_4},\mathtt{F_3}\wedge\mathtt{F_4}}\ \wedge_R \quad\to\quad \cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_3}}\ \text{inv-th/ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}}\ \text{IH}\quad\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4},\mathtt{F_4}}\ \text{inv-th/ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\wedge\mathtt{F_4}}\ \wedge_R$$

$$\cfrac{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_6},\mathtt{F_6}\quad\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4},\mathtt{F_6},\mathtt{F_6}}{\bullet\mathtt{h_1}:\Delta_2\vdash(\Delta_5,\mathtt{F_3}\wedge\mathtt{F_4}),\mathtt{F_6},\mathtt{F_6}}\ \wedge_R \quad\to\quad \cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_6},\mathtt{F_6}}\ \text{ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_6}}\ \text{IH}\quad\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4},\mathtt{F_6},\mathtt{F_6}}\ \text{ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_4},\mathtt{F_6}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_6},\mathtt{F_3}\wedge\mathtt{F_4}}\ \wedge_R$$

- Case(s) rule $\vee_R$

$$\cfrac{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4},\mathtt{F_3}\vee\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\vee\mathtt{F_4},\mathtt{F_3}\vee\mathtt{F_4}}\ \vee_R \quad\to\quad \cfrac{\cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_3},\mathtt{F_4},\mathtt{F_4}}\ \text{inv-th/ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_3},\mathtt{F_4}}\ \text{IH}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3}\vee\mathtt{F_4}}\ \vee_R$$

$$\cfrac{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4},\mathtt{F_6},\mathtt{F_6}}{\bullet\mathtt{h_1}:\Delta_2\vdash(\Delta_5,\mathtt{F_3}\vee\mathtt{F_4}),\mathtt{F_6},\mathtt{F_6}}\ \vee_R \quad\to\quad \cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4},\mathtt{F_6},\mathtt{F_6}}\ \text{ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_3},\mathtt{F_4},\mathtt{F_6}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_5,\mathtt{F_6},\mathtt{F_3}\vee\mathtt{F_4}}\ \vee_R$$

- Case(s) rule $\bot_R$

$$\cfrac{\mathtt{h_1}:\Delta_2\vdash\bot,\Delta_3}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_3,\bot,\bot}\ \bot_R \quad\to\quad \cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_3}\ \text{inv-th/ax}}{\bullet\mathtt{h_1}:\Delta_2\vdash\bot,\Delta_3}\ \bot_R$$

$$\cfrac{\mathtt{h_1}:\Delta_2\vdash\Delta_3,\mathtt{F_4},\mathtt{F_4}}{\bullet\mathtt{h_1}:\Delta_2\vdash(\bot,\Delta_3),\mathtt{F_4},\mathtt{F_4}}\ \bot_R \quad\to\quad \cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_2\vdash\Delta_3,\mathtt{F_4},\mathtt{F_4}}\ \text{ax}}{\mathtt{h_1}:\Delta_2\vdash\Delta_3,\mathtt{F_4}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_2\vdash\bot,\Delta_3,\mathtt{F_4}}\ \bot_R$$

- Case(s) rule $\top_R$

$$\cfrac{}{\bullet\mathtt{h_1}:\Delta_2\vdash\Delta_3,\top,\top}\ \top_R \quad\to\quad \cfrac{}{\bullet\mathtt{h_1}:\Delta_2\vdash\top,\Delta_3}\ \top_R$$

$$\cfrac{}{\bullet\mathtt{h_1}:\Delta_2\vdash(\top,\Delta_3),\mathtt{F_4},\mathtt{F_4}}\ \top_R \quad\to\quad \cfrac{}{\bullet\mathtt{h_1}:\Delta_2\vdash\top,\Delta_3,\mathtt{F_4}}\ \top_R$$

- Case(s) rule $\to_L$

$$\cfrac{\mathtt{h_1}:\Delta_6,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_2},\mathtt{F_5},\mathtt{F_5}\quad\mathtt{h_1}:\Delta_6,\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5},\mathtt{F_5}}{\bullet\mathtt{h_1}:\Delta_6,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5},\mathtt{F_5}}\ \to_L \quad\to\quad \cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_6,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_2},\mathtt{F_5},\mathtt{F_5}}\ \text{ax}}{\mathtt{h_1}:\Delta_6,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_2},\mathtt{F_5}}\ \text{IH}\quad\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_6,\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5},\mathtt{F_5}}\ \text{ax}}{\mathtt{h_1}:\Delta_6,\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_6,\mathtt{F_2}\to\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5}}\ \to_L$$

- Case(s) rule $\wedge_L$

$$\cfrac{\mathtt{h_1}:\Delta_6,\mathtt{F_2},\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5},\mathtt{F_5}}{\bullet\mathtt{h_1}:\Delta_6,\mathtt{F_2}\wedge\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5},\mathtt{F_5}}\ \wedge_L \quad\to\quad \cfrac{\cfrac{\cfrac{}{\mathtt{h_1}:\Delta_6,\mathtt{F_2},\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5},\mathtt{F_5}}\ \text{ax}}{\mathtt{h_1}:\Delta_6,\mathtt{F_2},\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5}}\ \text{IH}}{\bullet\mathtt{h_1}:\Delta_6,\mathtt{F_2}\wedge\mathtt{F_3}\vdash\Delta_4,\mathtt{F_5}}\ \wedge_L$$

- Case(s) rule $\vee_L$

$$\dfrac{\mathtt{h}_1 : \Delta_6, \mathtt{F}_2 \vdash \Delta_4, \mathtt{F}_5, \mathtt{F}_5 \quad \mathtt{h}_1 : \Delta_6, \mathtt{F}_3 \vdash \Delta_4, \mathtt{F}_5, \mathtt{F}_5}{\bullet\mathtt{h}_1 : \Delta_6, \mathtt{F}_2 \vee \mathtt{F}_3 \vdash \Delta_4, \mathtt{F}_5, \mathtt{F}_5} \vee_L \quad \rightarrow \quad \dfrac{\dfrac{\overline{\mathtt{h}_1 : \Delta_6, \mathtt{F}_2 \vdash \Delta_4, \mathtt{F}_5, \mathtt{F}_5}}{\mathtt{h}_1 : \Delta_6, \mathtt{F}_2 \vdash \Delta_4, \mathtt{F}_5} \text{ax} \quad \dfrac{\overline{\mathtt{h}_1 : \Delta_6, \mathtt{F}_3 \vdash \Delta_4, \mathtt{F}_5, \mathtt{F}_5}}{\mathtt{h}_1 : \Delta_6, \mathtt{F}_3 \vdash \Delta_4, \mathtt{F}_5} \text{ax IH}}{\bullet\mathtt{h}_1 : \Delta_6, \mathtt{F}_2 \vee \mathtt{F}_3 \vdash \Delta_4, \mathtt{F}_5} \vee_L$$

- Case(s) rule $\bot_L$

$$\dfrac{}{\bullet\mathtt{h}_1 : \bot, \Delta_4 \vdash \Delta_2, \mathtt{F}_3, \mathtt{F}_3} \bot_L \quad \rightarrow \quad \dfrac{}{\bullet\mathtt{h}_1 : \bot, \Delta_4 \vdash \Delta_2, \mathtt{F}_3} \bot_L$$

- Case(s) rule I

$$\dfrac{}{\bullet\mathtt{h}_1 : \Delta_4, \mathtt{p}_3 \vdash \Delta_2, \mathtt{p}_3, \mathtt{p}_3} I \quad \rightarrow \quad \dfrac{}{\bullet\mathtt{h}_1 : \Delta_4, \mathtt{p}_3 \vdash \Delta_2, \mathtt{p}_3} I$$

$$\dfrac{}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{p}_4 \vdash (\Delta_2, \mathtt{p}_4), \mathtt{F}_3, \mathtt{F}_3} I \quad \rightarrow \quad \dfrac{}{\bullet\mathtt{h}_1 : \Delta_5, \mathtt{p}_4 \vdash \Delta_2, \mathtt{F}_3, \mathtt{p}_4} I$$

- Case(s) rule $\top_L$

$$\dfrac{\mathtt{h}_1 : \Delta_4 \vdash \Delta_2, \mathtt{F}_3, \mathtt{F}_3}{\bullet\mathtt{h}_1 : \top, \Delta_4 \vdash \Delta_2, \mathtt{F}_3, \mathtt{F}_3} \top_L \quad \rightarrow \quad \dfrac{\dfrac{\overline{\mathtt{h}_1 : \Delta_4 \vdash \Delta_2, \mathtt{F}_3, \mathtt{F}_3}}{\mathtt{h}_1 : \Delta_4 \vdash \Delta_2, \mathtt{F}_3} \text{ax IH}}{\bullet\mathtt{h}_1 : \top, \Delta_4 \vdash \Delta_2, \mathtt{F}_3} \top_L$$

7 Identity-Expansion

$$
\cfrac{
  \cfrac{
    \cfrac{\overline{-\,:\,\mathbf{F}_0 \vdash \mathbf{F}_0}\ \texttt{IH}}{-\,:\,\mathbf{F}_0 \vdash \mathbf{F}_0, \mathbf{F}_1}\ W
    \qquad
    \cfrac{\overline{-\,:\,\mathbf{F}_1 \vdash \mathbf{F}_1}\ \texttt{IH}}{-\,:\,\mathbf{F}_1 \vdash \mathbf{F}_0, \mathbf{F}_1}\ W
  }{-\,:\,\mathbf{F}_0 \vee \mathbf{F}_1 \vdash \mathbf{F}_0, \mathbf{F}_1}\ \vee_L
}{-\,:\,\mathbf{F}_0 \vee \mathbf{F}_1 \vdash \mathbf{F}_0 \vee \mathbf{F}_1}\ \vee_R
$$

$$
\cfrac{
  \cfrac{
    \cfrac{\overline{-\,:\,\mathbf{F}_0 \vdash \mathbf{F}_0}\ \texttt{IH}}{-\,:\,\mathbf{F}_0, \mathbf{F}_1 \vdash \mathbf{F}_0}\ W
    \qquad
    \cfrac{\overline{-\,:\,\mathbf{F}_1 \vdash \mathbf{F}_1}\ \texttt{IH}}{-\,:\,\mathbf{F}_0, \mathbf{F}_1 \vdash \mathbf{F}_1}\ W
  }{-\,:\,\mathbf{F}_0, \mathbf{F}_1 \vdash \mathbf{F}_0 \wedge \mathbf{F}_1}\ \wedge_R
}{-\,:\,\mathbf{F}_0 \wedge \mathbf{F}_1 \vdash \mathbf{F}_0 \wedge \mathbf{F}_1}\ \wedge_L
$$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{\overline{-\,:\,\mathbf{F}_0 \vdash \mathbf{F}_0}\ \texttt{IH}}{-\,:\,\mathbf{F}_0, \mathbf{F}_0 \to \mathbf{F}_1 \vdash \mathbf{F}_0}\ W
    }{-\,:\,\mathbf{F}_0, \mathbf{F}_0 \to \mathbf{F}_1 \vdash \mathbf{F}_0, \mathbf{F}_1}\ W
    \qquad
    \cfrac{\overline{-\,:\,\mathbf{F}_1 \vdash \mathbf{F}_1}\ \texttt{IH}}{-\,:\,\mathbf{F}_0, \mathbf{F}_1 \vdash \mathbf{F}_1}\ W
  }{-\,:\,\mathbf{F}_0, \mathbf{F}_0 \to \mathbf{F}_1 \vdash \mathbf{F}_1}\ \to_L
}{-\,:\,\mathbf{F}_0 \to \mathbf{F}_1 \vdash \mathbf{F}_0 \to \mathbf{F}_1}\ \to_R
$$

$$
\overline{-\,:\,\top \vdash \top}\ \top_R
$$

$$
\overline{-\,:\,\bot \vdash \bot}\ \bot_L
$$

8 Cut-Elimination

8.1 Status of $\to_R$: OK

- Case rule $\to_R$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{13}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \to F_{11}), F_7 \to F_8), F_{14}} \to_R \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash F_{11}}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \to F_{11}), F_7 \to F_8} \to_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \to F_{11}), F_7 \to F_8} \text{Cut}}{\;}$$
$$\to$$
$$\cfrac{\cfrac{\cfrac{}{- : \Delta_{13}, F_7 \vdash F_8} \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10} \to F_{11}, F_7 \to F_8} \to_R}{\;}$$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{11}, F_8 \vdash F_9}{\bullet h_1 : \Delta_{11} \vdash (\Delta_{10}, F_8 \to F_9), F_{12}} \to_R \quad \cfrac{h_7 : \Delta_{11}, F_8, F_{12} \vdash F_9}{\bullet h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8 \to F_9} \to_R}{- : \Delta_{11} \vdash \Delta_{10}, F_8 \to F_9} \text{Cut}}{\;}$$
$$\to$$
$$\cfrac{\cfrac{\cfrac{}{- : \Delta_{11}, F_8 \vdash F_9} \text{ax/W}}{- : \Delta_{11} \vdash \Delta_{10}, F_8 \to F_9} \to_R}{\;}$$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{12}, F_6 \vdash F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \to F_{10}), F_6 \to F_7} \to_R \quad \cfrac{h_8 : \Delta_{12}, F_9, F_6 \to F_7 \vdash F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_9 \to F_{10}} \to_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \to F_{10}} \text{Cut}}{\;}$$
$$\to$$
$$\cfrac{\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{12}, F_6, F_9 \vdash F_7} \text{ax/W}}{\bullet h_1 : \Delta_{12}, F_9 \vdash F_{10}, F_6 \to F_7} \to_R \quad \cfrac{}{h_8 : \Delta_{12}, F_9, F_6 \to F_7 \vdash F_{10}} \text{ax/W}}{- : \Delta_{12}, F_9 \vdash F_{10}} \text{hCut}}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \to F_{10}} \to_R$$

- Case rule $\wedge_R$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{13}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \wedge F_{11}), F_7 \to F_8), F_{14}} \to_R \quad \cfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_7 \to F_8 \quad h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{11}, F_7 \to F_8}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7 \to F_8} \wedge_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7 \to F_8} \text{Cut}}{\;}$$
$$\to$$
$$\cfrac{\cfrac{\cfrac{}{- : \Delta_{13}, F_7 \vdash F_8} \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8, F_{10} \wedge F_{11}} \to_R}{\;}$$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{12}, F_6 \vdash F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \wedge F_{10}), F_6 \to F_7} \to_R \quad \cfrac{h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_9 \wedge F_{10}} \wedge_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \wedge F_{10}} \text{Cut}}{\;}$$
$$\to$$
$$\cfrac{\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{12}, F_6 \vdash F_7} \text{ax/W}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6 \to F_7} \to_R \; \cfrac{}{h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_9} \text{ax/W}}{- : \Delta_{12} \vdash \Delta_{11}, F_9} \text{hCut} \quad \cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{12}, F_6 \vdash F_7} \text{ax/W}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_6 \to F_7} \to_R \; \cfrac{}{h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_{10}} \text{ax/W}}{- : \Delta_{12} \vdash \Delta_{11}, F_{10}} \text{hCut}}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \wedge F_{10}} \wedge_R$$

- Case rule $\vee_R$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{13}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \vee F_{11}), F_7 \to F_8), F_{14}} \to_R \quad \cfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \to F_8}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7 \to F_8} \vee_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7 \to F_8} \text{Cut}}{\;}$$
$$\to$$
$$\cfrac{\cfrac{\cfrac{}{- : \Delta_{13}, F_7 \vdash F_8} \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8, F_{10} \vee F_{11}} \to_R}{\;}$$

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$$\frac{\dfrac{h_1 : \Delta_{12}, F_6 \vdash F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \vee F_{10}), F_6 \to F_7}\ \to_R \qquad \dfrac{h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_9, F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_9 \vee F_{10}}\ \vee_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \vee F_{10}}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\dfrac{}{h_1 : \Delta_{12}, F_6 \vdash F_7}\ \text{ax/W}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_9, F_6 \to F_7}\ \to_R \qquad \dfrac{}{h_8 : \Delta_{12}, F_6 \to F_7 \vdash \Delta_{11}, F_{10}, F_9}\ \text{ax/W}}{\dfrac{- : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_9}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \vee F_{10}}\ \vee_R}\ \text{hCut}$$

- Case rule $\perp_R$

$$\frac{\dfrac{h_1 : \Delta_{11}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{11} \vdash ((\perp, \Delta_{10}), F_7 \to F_8), F_{12}}\ \to_R \qquad \dfrac{h_9 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_7 \to F_8}{\bullet h_9 : \Delta_{11}, F_{12} \vdash (\perp, \Delta_{10}), F_7 \to F_8}\ \perp_R}{- : \Delta_{11} \vdash (\perp, \Delta_{10}), F_7 \to F_8}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{- : \Delta_{11}, F_7 \vdash F_8}\ \text{ax/W}}{- : \Delta_{11} \vdash \perp, \Delta_{10}, F_7 \to F_8}\ \to_R$$

$$\frac{\dfrac{h_1 : \Delta_{10}, F_6 \vdash F_7}{\bullet h_1 : \Delta_{10} \vdash (\perp, \Delta_9), F_6 \to F_7}\ \to_R \qquad \dfrac{h_8 : \Delta_{10}, F_6 \to F_7 \vdash \Delta_9}{\bullet h_8 : \Delta_{10}, F_6 \to F_7 \vdash \perp, \Delta_9}\ \perp_R}{- : \Delta_{10} \vdash \perp, \Delta_9}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_{10} \vdash \perp, \Delta_9, F_6 \to F_7}\ \text{ax/W} \qquad \dfrac{}{h_8 : \Delta_{10}, F_6 \to F_7 \vdash \perp, \Delta_9}\ \text{ax/W}}{- : \Delta_{10} \vdash \perp, \Delta_9}\ \text{hCut}$$

- Case rule $\top_R$

$$\frac{\dfrac{h_1 : \Delta_{11}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{11} \vdash ((\top, \Delta_{10}), F_7 \to F_8), F_{12}}\ \to_R \qquad \dfrac{}{\bullet h_9 : \Delta_{11}, F_{12} \vdash (\top, \Delta_{10}), F_7 \to F_8}\ \top_R}{- : \Delta_{11} \vdash (\top, \Delta_{10}), F_7 \to F_8}\ \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_{11} \vdash \top, \Delta_{10}, F_7 \to F_8}\ \top_R$$

$$\frac{\dfrac{h_1 : \Delta_{10}, F_6 \vdash F_7}{\bullet h_1 : \Delta_{10} \vdash (\top, \Delta_9), F_6 \to F_7}\ \to_R \qquad \dfrac{}{\bullet h_8 : \Delta_{10}, F_6 \to F_7 \vdash \top, \Delta_9}\ \top_R}{- : \Delta_{10} \vdash \top, \Delta_9}\ \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_{10} \vdash \top, \Delta_9}\ \top_R$$

- Case rule $\to_L$

$$\frac{\dfrac{h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7 \vdash F_8}{\bullet h_1 : \Delta_{13}, F_{10} \to F_{11} \vdash (\Delta_{12}, F_7 \to F_8), F_{14}}\ \to_R \qquad \dfrac{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_7 \to F_8 \quad h_9 : \Delta_{13}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \to F_8}{\bullet h_9 : (\Delta_{13}, F_{10} \to F_{11}), F_{14} \vdash \Delta_{12}, F_7 \to F_8}\ \to_L}{- : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7 \to F_8}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{- : \Delta_{13}, F_7, F_{10} \to F_{11} \vdash F_8}\ \text{ax/W}}{- : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7 \to F_8}\ \to_R$$

$$\frac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \to F_8), F_{10} \to F_{11}}\ \to_R \qquad \dfrac{h_9 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_7 \to F_8 \quad h_9 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_7 \to F_8}{\bullet h_9 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7 \to F_8}\ \to_L}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{- : \Delta_{13}, F_7 \vdash F_8}\ \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8}\ \to_R$$

$$\dfrac{\dfrac{h_1 : (\Delta_{12}, F_9 \to F_{10}), F_6 \vdash F_7}{\bullet h_1 : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_6 \to F_7} \to_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \to F_7, F_9 \to F_{10} \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_{10}, F_6 \to F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \to F_{10}), F_6 \to F_7 \vdash \Delta_{11}} \to_L}{- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\bullet h_1 : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_9, F_6 \to F_7}{- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_9} \ \text{ax/W} \quad \overline{h_8 : \Delta_{12}, F_6 \to F_7, F_9 \to F_{10} \vdash \Delta_{11}, F_9} \ \text{ax/W}}{- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_9} \ \text{hCut} \quad \dfrac{\dfrac{\overline{h_1 : \Delta_{12}, F_{10}, F_6 \vdash F_7} \ \text{inv-th/ax}}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \to F_7} \to_R \quad \overline{h_8 : \Delta_{12}, F_{10}, F_6 \dots}}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}}$$

$$- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11} \quad \to_L$$

$$\dfrac{\dfrac{h_1 : \Delta_{10}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{10} \vdash \Delta_9, F_7 \to F_8} \to_R \quad \dfrac{h_6 : \Delta_{10}, F_7 \to F_8 \vdash \Delta_9, F_7 \quad h_6 : \Delta_{10}, F_8 \vdash \Delta_9}{\bullet h_6 : \Delta_{10}, F_7 \to F_8 \vdash \Delta_9} \to_L}{- : \Delta_{10} \vdash \Delta_9} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\bullet h_1 : \Delta_{10} \vdash \Delta_9, F_7, F_7 \to F_8}{- : \Delta_{10} \vdash \Delta_9, F_7} \ \text{ax/W} \quad \overline{h_6 : \Delta_{10}, F_7 \to F_8 \vdash \Delta_9, F_7} \ \text{ax/W}}{- : \Delta_{10} \vdash \Delta_9, F_7} \ \text{hCut} \quad \dfrac{\overline{- : \Delta_{10}, F_7 \vdash \Delta_9, F_8} \ \text{ax/W} \quad \overline{- : \Delta_{10}, F_8 \vdash \Delta_9} \ \text{ax/W}}{- : \Delta_{10}, F_7 \vdash \Delta_9} \ \text{sCut}}{- : \Delta_{10} \vdash \Delta_9} \ \text{sCut}$$

- Case rule $\wedge_L$

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \vdash F_8}{\bullet h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash (\Delta_{12}, F_7 \to F_8), F_{14}} \to_R \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \to F_8}{\bullet h_9 : (\Delta_{13}, F_{10} \wedge F_{11}), F_{14} \vdash \Delta_{12}, F_7 \to F_8} \wedge_L}{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \to F_8} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{- : \Delta_{13}, F_7, F_{10} \wedge F_{11} \vdash F_8} \ \text{ax/W}}{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \to F_8} \to_R}{}$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \to F_8), F_{10} \wedge F_{11}} \to_R \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_7 \to F_8}{\bullet h_9 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \to F_8} \wedge_L}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{- : \Delta_{13}, F_7 \vdash F_8} \ \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8} \to_R}{}$$

$$\dfrac{\dfrac{h_1 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \vdash F_7}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \to F_7} \to_R \quad \dfrac{h_8 : \Delta_{12}, F_9, F_{10}, F_6 \to F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \to F_7 \vdash \Delta_{11}} \wedge_L}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\dfrac{\overline{h_1 : \Delta_{12}, F_{10}, F_6, F_9 \vdash F_7} \ \text{inv-th/ax}}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \to F_7} \to_R \quad \overline{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \to F_7 \vdash \Delta_{11}} \ \text{ax/W}}{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}} \ \text{hCut}}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \wedge_L$$

- Case rule $\vee_L$

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vdash F_8}{\bullet h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash (\Delta_{12}, F_7 \to F_8), F_{14}} \to_R \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash \Delta_{12}, F_7 \to F_8 \quad h_9 : \Delta_{13}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \to F_8}{\bullet h_9 : (\Delta_{13}, F_{10} \vee F_{11}), F_{14} \vdash \Delta_{12}, F_7 \to F_8} \vee_L}{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \to F_8} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{- : \Delta_{13}, F_7, F_{10} \vee F_{11} \vdash F_8} \ \text{ax/W}}{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \to F_8} \to_R}{}$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \to F_8), F_{10} \vee F_{11}} \to_R \quad \dfrac{h_9 : \Delta_{13}, F_{10} \vdash \Delta_{12}, F_7 \to F_8 \quad h_9 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_7 \to F_8}{\bullet h_9 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \to F_8} \vee_L}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8} \ \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{- : \Delta_{13}, F_7 \vdash F_8} \ \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \to F_8} \to_R}{}$$

$$
\dfrac{\dfrac{h_1 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \vdash F_7}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \to F_7}\ {\scriptstyle \to_R} \quad \dfrac{h_8 : \Delta_{12}, F_9, F_6 \to F_7 \vdash \Delta_{11} \quad h_8 : \Delta_{12}, F_{10}, F_6 \to F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \to F_7 \vdash \Delta_{11}}\ {\scriptstyle \vee_L}}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{h_1 : \Delta_{12}, F_6, F_9 \vdash F_7}{\ }\ {\scriptstyle \text{inv-th/ax}}}{\bullet h_1 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6 \to F_7}\ {\scriptstyle \to_R} \quad \dfrac{h_8 : \Delta_{12}, F_9, F_6 \to F_7 \vdash \Delta_{11}}{\ }\ {\scriptstyle \text{ax/W}}}{- : \Delta_{12}, F_9 \vdash \Delta_{11}}\ {\scriptstyle \text{hCut}} \quad \dfrac{\dfrac{\dfrac{h_1 : \Delta_{12}, F_{10}, F_6 \vdash F_7}{\ }\ {\scriptstyle \text{inv-th/ax}}}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \to F_7}\ {\scriptstyle \to_R} \quad \dfrac{h_8 : \Delta_{12}, F_{10}, F_6 \to F_7 \vdash \Delta_{11}}{\ }\ {\scriptstyle \text{ax/W}}}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}}\ {\scriptstyle \text{hCut}}}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}}\ {\scriptstyle \vee_L}
$$

- Case rule $\perp_L$

$$
\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{11} \vdash (\Delta_{10}, F_7 \to F_8), \perp}\ {\scriptstyle \to_R} \quad \dfrac{}{\bullet h_9 : \Delta_{11}, \perp \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \perp_L}}{- : \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{\dfrac{}{- : \Delta_{11}, F_7 \vdash F_8}\ {\scriptstyle \text{ax/W}}}{- : \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \to_R}
$$

$$
\dfrac{\dfrac{h_1 : (\perp, \Delta_{11}), F_7 \vdash F_8}{\bullet h_1 : \perp, \Delta_{11} \vdash (\Delta_{10}, F_7 \to F_8), F_{12}}\ {\scriptstyle \to_R} \quad \dfrac{}{\bullet h_9 : (\perp, \Delta_{11}), F_{12} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \perp_L}}{- : \perp, \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{}{- : \perp, \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \perp_L}
$$

$$
\dfrac{\dfrac{h_1 : (\perp, \Delta_{10}), F_6 \vdash F_7}{\bullet h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \to F_7}\ {\scriptstyle \to_R} \quad \dfrac{}{\bullet h_8 : (\perp, \Delta_{10}), F_6 \to F_7 \vdash \Delta_9}\ {\scriptstyle \perp_L}}{- : \perp, \Delta_{10} \vdash \Delta_9}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{}{- : \perp, \Delta_{10} \vdash \Delta_9}\ {\scriptstyle \perp_L}
$$

- Case rule I

$$
\dfrac{\dfrac{h_1 : \Delta_{12}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{12} \vdash ((\Delta_{10}, p_{11}), F_7 \to F_8), p_{11}}\ {\scriptstyle \to_R} \quad \dfrac{}{\bullet h_9 : \Delta_{12}, p_{11} \vdash (\Delta_{10}, p_{11}), F_7 \to F_8}\ {\scriptstyle I}}{- : \Delta_{12} \vdash (\Delta_{10}, p_{11}), F_7 \to F_8}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{\dfrac{}{- : \Delta_{12}, F_7 \vdash F_8}\ {\scriptstyle \text{ax/W}}}{- : \Delta_{12} \vdash \Delta_{10}, p_{11}, F_7 \to F_8}\ {\scriptstyle \to_R}
$$

$$
\dfrac{\dfrac{h_1 : (\Delta_{12}, p_{11}), F_7 \vdash F_8}{\bullet h_1 : \Delta_{12}, p_{11} \vdash ((\Delta_{10}, p_{11}), F_7 \to F_8), F_{13}}\ {\scriptstyle \to_R} \quad \dfrac{}{\bullet h_9 : (\Delta_{12}, p_{11}), F_{13} \vdash (\Delta_{10}, p_{11}), F_7 \to F_8}\ {\scriptstyle I}}{- : \Delta_{12}, p_{11} \vdash (\Delta_{10}, p_{11}), F_7 \to F_8}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{}{- : \Delta_{12}, p_{11} \vdash \Delta_{10}, p_{11}, F_7 \to F_8}\ {\scriptstyle I}
$$

$$
\dfrac{\dfrac{h_1 : (\Delta_{11}, p_{10}), F_6 \vdash F_7}{\bullet h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_6 \to F_7}\ {\scriptstyle \to_R} \quad \dfrac{}{\bullet h_8 : (\Delta_{11}, p_{10}), F_6 \to F_7 \vdash \Delta_9, p_{10}}\ {\scriptstyle I}}{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{}{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}}\ {\scriptstyle I}
$$

- Case rule $\top_L$

$$
\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \vdash F_8}{\bullet h_1 : \Delta_{11} \vdash (\Delta_{10}, F_7 \to F_8), \top}\ {\scriptstyle \to_R} \quad \dfrac{h_9 : \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}{\bullet h_9 : \Delta_{11}, \top \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \top_L}}{- : \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \text{Cut}}
$$

$$\to$$

$$
\dfrac{}{- : \Delta_{11} \vdash \Delta_{10}, F_7 \to F_8}\ {\scriptstyle \text{ax/W}}
$$

$$\dfrac{\dfrac{h_1 : (\top, \Delta_{11}), F_7 \vdash F_8}{\bullet h_1 : \top, \Delta_{11} \vdash (\Delta_{10}, F_7 \rightarrow F_8), F_{12}} \rightarrow_R \quad \dfrac{h_9 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_7 \rightarrow F_8}{\bullet h_9 : (\top, \Delta_{11}), F_{12} \vdash \Delta_{10}, F_7 \rightarrow F_8} \top_L}{- : \top, \Delta_{11} \vdash \Delta_{10}, F_7 \rightarrow F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\overline{- : \top, \Delta_{11}, F_7 \vdash F_8}} \text{ax/W}}{- : \top, \Delta_{11} \vdash \Delta_{10}, F_7 \rightarrow F_8} \rightarrow_R$$

$$\dfrac{\dfrac{h_1 : (\top, \Delta_{10}), F_6 \vdash F_7}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7} \rightarrow_R \quad \dfrac{h_8 : \Delta_{10}, F_6 \rightarrow F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9} \top_L}{- : \top, \Delta_{10} \vdash \Delta_9} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7} \text{ax/W} \quad \dfrac{}{h_8 : \top, \Delta_{10}, F_6 \rightarrow F_7 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_{10} \vdash \Delta_9} \text{hCut}$$

8.2 Status of $\wedge_R$: OK

- Case rule $\rightarrow_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7, F_{14} \quad h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_8, F_{14}}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \rightarrow F_{11}), F_7 \wedge F_8), F_{14}} \wedge_R \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash F_{11}}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7 \wedge F_8} \rightarrow_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{14}, F_7, F_{10} \rightarrow F_{11}} \text{ax/W} \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash F_{11}}{\bullet h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_7, F_{10} \rightarrow F_{11}} \text{ax/W}}{} \rightarrow_R}{- : \Delta_{13} \vdash \Delta_{12}, F_7, F_{10} \rightarrow F_{11}} \text{hCut} \quad \dfrac{\dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{14}, F_8, F_{10} \rightarrow F_{11}} \text{ax/W} \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_1}{\bullet h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12},}}{- : \Delta_{13} \vdash \Delta_{12}, F_8, F_{10} \rightarrow F_{11}}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10} \rightarrow F_{11}, F_7 \wedge F_8} \wedge_R$$

$$\dfrac{\dfrac{h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \rightarrow F_{10}), F_6 \quad h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \rightarrow F_{10}), F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \rightarrow F_{10}), F_6 \wedge F_7} \wedge_R \quad \dfrac{h_8 : \Delta_{12}, F_9, F_6 \wedge F_7 \vdash F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9 \rightarrow F_{10}} \rightarrow_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \rightarrow F_{10}} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{}{- : \Delta_{12} \vdash \Delta_{11}, F_6, F_9 \rightarrow F_{10}} \text{ax/W} \quad \dfrac{\dfrac{}{- : \Delta_{12}, F_6 \vdash \Delta_{11}, F_7, F_9 \rightarrow F_{10}} \text{ax/W} \quad \dfrac{\dfrac{}{- : \Delta_{12}, F_6, F_7, F_9 \vdash F_{10}} \text{inv-th/ax}}{- : \Delta_{12}, F_6, F_7 \vdash \Delta_{11}, F_9 \rightarrow F_{10}} \rightarrow_R}{- : \Delta_{12}, F_6 \vdash \Delta_{11}, F_9 \rightarrow F_{10}} \text{sCut}}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \rightarrow F_{10}} \text{sCut}$$

- Case rule $\wedge_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7, F_{14} \quad h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_8, F_{14}}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \wedge F_{11}), F_7 \wedge F_8), F_{14}} \wedge_R \quad \dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10},}{\bullet h_9 : \Delta_{13}, F_1} \wedge_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7 \wedge F_8}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{14}, F_7} \text{inv-th/ax} \quad \dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{14}, F_8} \text{inv-th/ax}}{\bullet h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{14}, F_7 \wedge F_8} \wedge_R \quad \dfrac{}{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_7 \wedge F_8} \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_7 \wedge F_8} \text{hCut} \quad \dfrac{\dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{11}, F_{14}, F_7}}{\bullet h_1 : \Delta_{13} \vdash}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10} \wedge F_{11}, F_7 \wedge F_8}$$

$$\dfrac{\dfrac{h_1 : \Delta_{11} \vdash \Delta_{10}, F_8, F_{12} \quad h_1 : \Delta_{11} \vdash \Delta_{10}, F_9, F_{12}}{\bullet h_1 : \Delta_{11} \vdash (\Delta_{10}, F_8 \wedge F_9), F_{12}} \wedge_R \quad \dfrac{h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8 \quad h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_9}{\bullet h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8 \wedge F_9} \wedge_R}{- : \Delta_{11} \vdash \Delta_{10}, F_8 \wedge F_9} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{11} \vdash \Delta_{10}, F_{12}, F_8} \text{ax/W} \quad \dfrac{h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8}{\bullet h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8} H}{- : \Delta_{11} \vdash \Delta_{10}, F_8} \text{hCut} \quad \dfrac{\dfrac{}{h_1 : \Delta_{11} \vdash \Delta_{10}, F_{12}, F_9} \text{ax/W} \quad \dfrac{h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_9}{\bullet h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_9} H}{- : \Delta_{11} \vdash \Delta_{10}, F_9} \text{hCut}}{- : \Delta_{11} \vdash \Delta_{10}, F_8 \wedge F_9} \wedge_R$$

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$$\dfrac{\dfrac{h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \wedge F_{10}), F_6 \quad h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \wedge F_{10}), F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \wedge F_{10}), F_6 \wedge F_7} \wedge_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9 \wedge F_{10}} \wedge_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \wedge F_{10}} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{}{- : \Delta_{12} \vdash \Delta_{11}, F_6, F_9 \wedge F_{10}} \text{ax/W} \quad \dfrac{\dfrac{}{- : \Delta_{12}, F_6 \vdash \Delta_{11}, F_7, F_9 \wedge F_{10}} \text{ax/W} \quad \dfrac{\dfrac{}{- : \Delta_{12}, F_6, F_7 \vdash \Delta_{11}, F_9} \text{inv-th/ax} \quad \dfrac{}{- : \Delta_{12}, F_6, F_7 \vdash \Delta_{11}, F_{10}} \text{inv-th/ax}}{- : \Delta_{12}, F_6, F_7 \vdash \Delta_{11}, F_9 \wedge F_{10}} \wedge_R}{- : \Delta_{12}, F_6 \vdash \Delta_{11}, F_9 \wedge F_{10}} \text{sCut}}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \wedge F_{10}} \text{sCut}$$

- Case rule $\vee_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7, F_{14} \quad h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_8, F_{14}}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \vee F_{11}), F_7 \wedge F_8), F_{14}} \wedge_R \quad \dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \wedge F_8}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7 \wedge F_8} \vee_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}, F_7} \text{inv-th/ax} \quad \dfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}, F_8} \text{inv-th/ax}}{\bullet h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}, F_7 \wedge F_8} \wedge_R \quad \dfrac{}{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \wedge F_8} \text{ax/W}}{\dfrac{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \wedge F_8}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8, F_{10} \vee F_{11}} \vee_R} \text{hCut}$$

$$\dfrac{\dfrac{h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \vee F_{10}), F_6 \quad h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \vee F_{10}), F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \vee F_{10}), F_6 \wedge F_7} \wedge_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9, F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9 \vee F_{10}} \vee_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \vee F_{10}} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_6, F_9} \text{inv-th/ax} \quad \dfrac{}{h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_7, F_9} \text{inv-th/ax}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_9, F_6 \wedge F_7} \wedge_R \quad \dfrac{}{h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_{10}, F_9} \text{ax/W}}{\dfrac{- : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_9}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \vee F_{10}} \vee_R} \text{hCut}$$

- Case rule $\perp_R$

$$\dfrac{\dfrac{h_1 : \Delta_{11} \vdash (\perp, \Delta_{10}), F_7, F_{12} \quad h_1 : \Delta_{11} \vdash (\perp, \Delta_{10}), F_8, F_{12}}{\bullet h_1 : \Delta_{11} \vdash ((\perp, \Delta_{10}), F_7 \wedge F_8), F_{12}} \wedge_R \quad \dfrac{h_9 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_7 \wedge F_8}{\bullet h_9 : \Delta_{11}, F_{12} \vdash (\perp, \Delta_{10}), F_7 \wedge F_8} \perp_R}{- : \Delta_{11} \vdash (\perp, \Delta_{10}), F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{}{\bullet h_1 : \Delta_{11} \vdash \perp, \Delta_{10}, F_{12}, F_7 \wedge F_8} \text{ax/W} \quad \dfrac{}{h_9 : \Delta_{11}, F_{12} \vdash \perp, \Delta_{10}, F_7 \wedge F_8} \text{ax/W}}{- : \Delta_{11} \vdash \perp, \Delta_{10}, F_7 \wedge F_8} \text{hCut}$$

$$\dfrac{\dfrac{h_1 : \Delta_{10} \vdash (\perp, \Delta_9), F_6 \quad h_1 : \Delta_{10} \vdash (\perp, \Delta_9), F_7}{\bullet h_1 : \Delta_{10} \vdash (\perp, \Delta_9), F_6 \wedge F_7} \wedge_R \quad \dfrac{h_8 : \Delta_{10}, F_6 \wedge F_7 \vdash \Delta_9}{\bullet h_8 : \Delta_{10}, F_6 \wedge F_7 \vdash \perp, \Delta_9} \perp_R}{- : \Delta_{10} \vdash \perp, \Delta_9} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{}{\bullet h_1 : \Delta_{10} \vdash \perp, \Delta_9, F_6 \wedge F_7} \text{ax/W} \quad \dfrac{}{h_8 : \Delta_{10}, F_6 \wedge F_7 \vdash \perp, \Delta_9} \text{ax/W}}{- : \Delta_{10} \vdash \perp, \Delta_9} \text{hCut}$$

- Case rule $\top_R$

$$\dfrac{\dfrac{h_1 : \Delta_{11} \vdash (\top, \Delta_{10}), F_7, F_{12} \quad h_1 : \Delta_{11} \vdash (\top, \Delta_{10}), F_8, F_{12}}{\bullet h_1 : \Delta_{11} \vdash ((\top, \Delta_{10}), F_7 \wedge F_8), F_{12}} \wedge_R \quad \dfrac{}{\bullet h_9 : \Delta_{11}, F_{12} \vdash (\top, \Delta_{10}), F_7 \wedge F_8} \top_R}{- : \Delta_{11} \vdash (\top, \Delta_{10}), F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{}{- : \Delta_{11} \vdash \top, \Delta_{10}, F_7 \wedge F_8} \top_R$$

$$\dfrac{\dfrac{h_1 : \Delta_{10} \vdash (\top, \Delta_9), F_6 \quad h_1 : \Delta_{10} \vdash (\top, \Delta_9), F_7}{\bullet h_1 : \Delta_{10} \vdash (\top, \Delta_9), F_6 \wedge F_7} \wedge_R \quad \dfrac{}{\bullet h_8 : \Delta_{10}, F_6 \wedge F_7 \vdash \top, \Delta_9} \top_R}{- : \Delta_{10} \vdash \top, \Delta_9} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{}{- : \Delta_{10} \vdash \top, \Delta_9} \top_R$$

- Case rule $\to_L$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7, F_{14} \quad h_1 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_8, F_{14}}{\bullet h_1 : \Delta_{13}, F_{10} \to F_{11} \vdash (\Delta_{12}, F_7 \wedge F_8), F_{14}} \wedge_R \quad \cfrac{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_7 \wedge}{\bullet h_9 : (\Delta_{13}, F_{10} \to F_{11})}}{- : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}}{\to}$$

$$\cfrac{\cfrac{\cfrac{}{\bullet h_1 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_{14}, F_7 \wedge F_8} \text{ax/W} \quad \cfrac{}{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_7 \wedge F_8} \text{ax/W}}{- : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_7 \wedge F_8} \text{hCut} \quad \cfrac{\cfrac{h_1 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_1 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_{14}, F_7} \text{inv-th/ax} \quad \cfrac{}{h_1 : \Delta}}{}}{- : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}$$

$$\cfrac{\cfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_{10} \to F_{11} \quad h_1 : \Delta_{13} \vdash \Delta_{12}, F_8, F_{10} \to F_{11}}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \wedge F_8), F_{10} \to F_{11}} \wedge_R \quad \cfrac{h_9 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10},}{\bullet h_9 : \Delta_{13}, F_{10}}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8}$$

$$\cfrac{\cfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_{10} \to F_{11}} \text{ax/W} \quad \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}, F_7}{\bullet h_9 : \Delta_{13}, F_{10} \to F_{11} \vdash \Delta_{12}, F_7} \text{inv-th/ax}}{- : \Delta_{13} \vdash \Delta_{12}, F_7} \text{hCut} \quad \cfrac{h_9 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_7}{} \text{inv-th/ax} \to_L \quad \cfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_8, F_{10} \to F}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8}$$

$$\cfrac{\cfrac{h_1 : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_6 \quad h_1 : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \cfrac{h_8 : \Delta_{12}, F_9 \to F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \to F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \text{Cut}}{- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}}$$

$$\cfrac{\cfrac{}{- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}, F_6} \text{ax/W} \quad \cfrac{\cfrac{}{- : \Delta_{12}, F_6, F_9 \to F_{10} \vdash \Delta_{11}, F_7} \text{ax/W} \quad \cfrac{\cfrac{}{- : \Delta_{12}, F_6, F_7, F_9 \to F_{10} \vdash \Delta_{11}, F_9} \text{inv-th/ax} \quad \cfrac{}{- : \Delta_{12}, F_{10}, F_6, F_7 \vdash \Delta_1}}{- : \Delta_{12}, F_6, F_7, F_9 \to F_{10} \vdash \Delta_{11}} \text{sCut}}{- : \Delta_{12}, F_6, F_9 \to F_{10} \vdash \Delta_{11}} \text{sCut}}{- : \Delta_{12}, F_9 \to F_{10} \vdash \Delta_{11}}$$

- Case rule $\wedge_L$

$$\cfrac{\cfrac{h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7, F_{14} \quad h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_8, F_{14}}{\bullet h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash (\Delta_{12}, F_7 \wedge F_8), F_{14}} \wedge_R \quad \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \wedge F_8}{\bullet h_9 : (\Delta_{13}, F_{10} \wedge F_{11}), F_{14} \vdash \Delta_{12}, F_7 \wedge F_8} \wedge_L}{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}} \text{Cut}$$
$$\to$$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_1 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_{14}, F_7 \wedge F_8} \text{inv-th/ax} \quad \cfrac{h_1 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_{14}, F_8}{} \text{inv-th/ax}}{\bullet h_1 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_{14}, F_7 \wedge F_8} \wedge_R \quad \cfrac{}{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \wedge F_8} \text{ax/W}}{\cfrac{- : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \wedge F_8} \wedge_L} \text{hCut}$$

$$\cfrac{\cfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_{10} \wedge F_{11} \quad h_1 : \Delta_{13} \vdash \Delta_{12}, F_8, F_{10} \wedge F_{11}}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \wedge F_8), F_{10} \wedge F_{11}} \wedge_R \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}{\bullet h_9 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \wedge F_8} \wedge_L}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8} \text{Cut}$$
$$\to$$

$$\cfrac{\cfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_{10} \wedge F_{11}} \text{ax/W} \quad \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_7}{\bullet h_9 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7} \wedge_L}{- : \Delta_{13} \vdash \Delta_{12}, F_7} \text{inv-th/ax} \quad \cfrac{}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_8, F_{10} \wedge F_{11}} \text{ax/W} \quad \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_9 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_8} \text{inv-}}{- : \Delta_{13} \vdash \Delta_{12}, F_8} \text{hC}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8} \wedge_R$$

$$\cfrac{\cfrac{h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \quad h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \cfrac{h_8 : \Delta_{12}, F_9, F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \wedge_L}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \text{Cut}$$
$$\to$$

$$\cfrac{\cfrac{\cfrac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6}{} \text{inv-th/ax} \quad \cfrac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_7}{} \text{inv-th/ax}}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \cfrac{}{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \wedge F_7 \vdash \Delta_{11}} \text{ax/W}}{\cfrac{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \wedge_L} \text{hCut}$$

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$$\dfrac{\dfrac{h_1 : \Delta_{10} \vdash \Delta_9, F_7 \quad h_1 : \Delta_{10} \vdash \Delta_9, F_8}{\bullet h_1 : \Delta_{10} \vdash \Delta_9, F_7 \wedge F_8} \wedge_R \quad \dfrac{h_6 : \Delta_{10}, F_7, F_8 \vdash \Delta_9}{\bullet h_6 : \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \wedge_L}{- : \Delta_{10} \vdash \Delta_9} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\overline{- : \Delta_{10} \vdash \Delta_9, F_7} \; \text{ax/W} \quad \dfrac{\dfrac{\overline{- : \Delta_{10}, F_7 \vdash \Delta_9, F_8}}{} \text{ax/W} \quad \overline{- : \Delta_{10}, F_7, F_8 \vdash \Delta_9} \; \text{ax/W}}{- : \Delta_{10}, F_7 \vdash \Delta_9} \text{sCut}}{- : \Delta_{10} \vdash \Delta_9} \text{sCut}$$

- Case rule $\vee_L$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7, F_{14} \quad h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_8, F_{14}}{\bullet h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash (\Delta_{12}, F_7 \wedge F_8), F_{14}} \wedge_R \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash \Delta_{12}, \mathrm{F}}{\bullet h_9 : (\Delta_{13}, F}}{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}$$

$$\rightarrow$$

$$\dfrac{\overline{h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_{14}, F_7} \; \text{ax/W} \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash \Delta_{12}, F_7}{} \text{inv-th/ax} \quad \dfrac{h_9 : \Delta_{13}, F_{11}, F_{14} \vdash \Delta_{12}, F_7}{} \text{inv-th/ax}}{\dfrac{\bullet h_9 : \Delta_{13}, F_{14}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7}{} \vee_L} \text{hCut} \quad \dfrac{h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, }{}}{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7}$$

$$- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \wedge F_8$$

$$\dfrac{\dfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_{10} \vee F_{11} \quad h_1 : \Delta_{13} \vdash \Delta_{12}, F_8, F_{10} \vee F_{11}}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \wedge F_8), F_{10} \vee F_{11}} \wedge_R \quad \dfrac{h_9 : \Delta_{13}, F_{10} \vdash \Delta_{12}, F_7 \wedge F_8 \quad h_9 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_7 \wedge F_8}{\bullet h_9 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \wedge F_8} \vee_L}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\dfrac{\overline{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_7} \; \text{inv-th/ax} \quad \overline{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_8} \; \text{inv-th/ax}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \wedge F_8} \wedge_R \quad \overline{- : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_{10}, F_7 \wedge F_8} \; \text{ax/W}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_7 \wedge F_8} \text{sCut} \quad \overline{- : \Delta_{13}, F_{10} \vdash \Delta_{12}, F_7 \wedge F_8}}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \wedge F_8}$$

$$\dfrac{\dfrac{h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \quad h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \dfrac{h_8 : \Delta_{12}, F_9, F_6 \wedge F_7 \vdash \Delta_{11} \quad h_8 : \Delta_{12}, F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \vee_L}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\overline{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6} \; \text{ax/W} \quad \dfrac{\overline{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}, F_7} \; \text{ax/W} \quad \dfrac{\dfrac{- : \Delta_{12}, F_6, F_7, F_9 \vdash \Delta_{11}}{} \text{inv-th/ax} \quad \dfrac{- : \Delta_{12}, F_{10}, F_6, F_7 \vdash \Delta_{11}}{} \text{inv-th/ax}}{- : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}} \vee_L}{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}} \text{sCut}}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \text{sCut}$$

- Case rule $\perp_L$

$$\dfrac{\dfrac{h_1 : \Delta_{11} \vdash \Delta_{10}, F_7, \perp \quad h_1 : \Delta_{11} \vdash \Delta_{10}, F_8, \perp}{\bullet h_1 : \Delta_{11} \vdash (\Delta_{10}, F_7 \wedge F_8), \perp} \wedge_R \quad \overline{\bullet h_9 : \Delta_{11}, \perp \vdash \Delta_{10}, F_7 \wedge F_8} \; \perp_L}{- : \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{\dfrac{\overline{h_1 : \Delta_{11} \vdash \perp, \Delta_{10}, F_7} \; \text{ax/W} \quad \overline{\bullet h_9 : \perp, \Delta_{11} \vdash \Delta_{10}, F_7} \; \perp_L}{- : \Delta_{11} \vdash \Delta_{10}, F_7} \text{hCut} \quad \dfrac{\overline{h_1 : \Delta_{11} \vdash \perp, \Delta_{10}, F_8} \; \text{ax/W} \quad \overline{\bullet h_9 : \perp, \Delta_{11} \vdash \Delta_{10}, F_8} \; \perp_L}{- : \Delta_{11} \vdash \Delta_{10}, F_8} \text{hCut}}{- : \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8} \wedge_R$$

$$\dfrac{\dfrac{h_1 : \perp, \Delta_{11} \vdash \Delta_{10}, F_7, F_{12} \quad h_1 : \perp, \Delta_{11} \vdash \Delta_{10}, F_8, F_{12}}{\bullet h_1 : \perp, \Delta_{11} \vdash (\Delta_{10}, F_7 \wedge F_8), F_{12}} \wedge_R \quad \overline{\bullet h_9 : (\perp, \Delta_{11}), F_{12} \vdash \Delta_{10}, F_7 \wedge F_8} \; \perp_L}{- : \perp, \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{}{- : \perp, \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8} \; \perp_L$$

$$\dfrac{\dfrac{h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \quad h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_7}{\bullet h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \wedge F_7} \wedge_R \quad \overline{\bullet h_8 : (\perp, \Delta_{10}), F_6 \wedge F_7 \vdash \Delta_9} \; \perp_L}{- : \perp, \Delta_{10} \vdash \Delta_9} \text{Cut}$$

$$\rightarrow$$

$$\dfrac{}{- : \perp, \Delta_{10} \vdash \Delta_9} \; \perp_L$$

- Case rule I

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$$\dfrac{\dfrac{\mathtt{h_1 : \Delta_{12} \vdash (\Delta_{10}, p_{11}), F_7, p_{11}} \quad \mathtt{h_1 : \Delta_{12} \vdash (\Delta_{10}, p_{11}), F_8, p_{11}}}{\bullet\mathtt{h_1 : \Delta_{12} \vdash ((\Delta_{10}, p_{11}), F_7 \wedge F_8), p_{11}}} \wedge_R \quad \dfrac{}{\bullet\mathtt{h_9 : \Delta_{12}, p_{11} \vdash (\Delta_{10}, p_{11}), F_7 \wedge F_8}} I}{\mathtt{- : \Delta_{12} \vdash (\Delta_{10}, p_{11}), F_7 \wedge F_8}} \mathtt{Cut}$$

$$\dfrac{\rightarrow}{\dfrac{\dfrac{\dfrac{}{\mathtt{h_1 : \Delta_{12} \vdash \Delta_{10}, F_7, p_{11}, p_{11}}} \mathtt{ax/W} \quad \dfrac{}{\bullet\mathtt{h_9 : \Delta_{12}, p_{11} \vdash \Delta_{10}, F_7, p_{11}}} I}{\mathtt{- : \Delta_{12} \vdash \Delta_{10}, F_7, p_{11}}} \mathtt{hCut} \quad \dfrac{\dfrac{}{\mathtt{h_1 : \Delta_{12} \vdash \Delta_{10}, F_8, p_{11}, p_{11}}} \mathtt{ax/W} \quad \dfrac{}{\bullet\mathtt{h_9 : \Delta_{12}, p_{11} \vdash \Delta_{10}, F_8, p_{11}}} I}{\mathtt{- : \Delta_{12} \vdash \Delta_{10}, F_8, p_{11}}} \mathtt{hCut}}{\mathtt{- : \Delta_{12} \vdash \Delta_{10}, p_{11}, F_7 \wedge F_8}} \wedge_R}$$

$$\dfrac{\dfrac{\dfrac{\mathtt{h_1 : \Delta_{12}, p_{11} \vdash (\Delta_{10}, p_{11}), F_7, F_{13}} \quad \mathtt{h_1 : \Delta_{12}, p_{11} \vdash (\Delta_{10}, p_{11}), F_8, F_{13}}}{\bullet\mathtt{h_1 : \Delta_{12}, p_{11} \vdash ((\Delta_{10}, p_{11}), F_7 \wedge F_8), F_{13}}} \wedge_R \quad \dfrac{}{\bullet\mathtt{h_9 : (\Delta_{12}, p_{11}), F_{13} \vdash (\Delta_{10}, p_{11}), F_7 \wedge F_8}} I}{\mathtt{- : \Delta_{12}, p_{11} \vdash (\Delta_{10}, p_{11}), F_7 \wedge F_8}} \mathtt{Cut}}{\dfrac{\rightarrow}{\mathtt{- : \Delta_{12}, p_{11} \vdash \Delta_{10}, p_{11}, F_7 \wedge F_8}} I}$$

$$\dfrac{\dfrac{\dfrac{\mathtt{h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_6} \quad \mathtt{h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_7}}{\bullet\mathtt{h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_6 \wedge F_7}} \wedge_R \quad \dfrac{}{\bullet\mathtt{h_8 : (\Delta_{11}, p_{10}), F_6 \wedge F_7 \vdash \Delta_9, p_{10}}} I}{\mathtt{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}}} \mathtt{Cut}}{\dfrac{\rightarrow}{\mathtt{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}}} I}$$

- Case rule $\top_L$

$$\dfrac{\dfrac{\dfrac{\mathtt{h_1 : \Delta_{11} \vdash \Delta_{10}, F_7, \top} \quad \mathtt{h_1 : \Delta_{11} \vdash \Delta_{10}, F_8, \top}}{\bullet\mathtt{h_1 : \Delta_{11} \vdash (\Delta_{10}, F_7 \wedge F_8), \top}} \wedge_R \quad \dfrac{\mathtt{h_9 : \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8}}{\bullet\mathtt{h_9 : \Delta_{11}, \top \vdash \Delta_{10}, F_7 \wedge F_8}} \top_L}{\mathtt{- : \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8}} \mathtt{Cut}}{\dfrac{\rightarrow}{\mathtt{- : \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8}} \mathtt{ax/W}}$$

$$\dfrac{\dfrac{\dfrac{\mathtt{h_1 : \top, \Delta_{11} \vdash \Delta_{10}, F_7, F_{12}} \quad \mathtt{h_1 : \top, \Delta_{11} \vdash \Delta_{10}, F_8, F_{12}}}{\bullet\mathtt{h_1 : \top, \Delta_{11} \vdash (\Delta_{10}, F_7 \wedge F_8), F_{12}}} \wedge_R \quad \dfrac{\mathtt{h_9 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_7 \wedge F_8}}{\bullet\mathtt{h_9 : (\top, \Delta_{11}), F_{12} \vdash \Delta_{10}, F_7 \wedge F_8}} \top_L}{\mathtt{- : \top, \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8}} \mathtt{Cut}}{\dfrac{\rightarrow}{\dfrac{\dfrac{}{\bullet\mathtt{h_1 : \top, \Delta_{11} \vdash \Delta_{10}, F_{12}, F_7 \wedge F_8}} \mathtt{ax/W} \quad \dfrac{}{\mathtt{h_9 : \top, \Delta_{11}, F_{12} \vdash \Delta_{10}, F_7 \wedge F_8}} \mathtt{ax/W}}{\mathtt{- : \top, \Delta_{11} \vdash \Delta_{10}, F_7 \wedge F_8}} \mathtt{hCut}}}$$

$$\dfrac{\dfrac{\dfrac{\mathtt{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6} \quad \mathtt{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_7}}{\bullet\mathtt{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \wedge F_7}} \wedge_R \quad \dfrac{\mathtt{h_8 : \Delta_{10}, F_6 \wedge F_7 \vdash \Delta_9}}{\bullet\mathtt{h_8 : (\top, \Delta_{10}), F_6 \wedge F_7 \vdash \Delta_9}} \top_L}{\mathtt{- : \top, \Delta_{10} \vdash \Delta_9}} \mathtt{Cut}}{\dfrac{\rightarrow}{\dfrac{\dfrac{}{\bullet\mathtt{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \wedge F_7}} \mathtt{ax/W} \quad \dfrac{}{\mathtt{h_8 : \top, \Delta_{10}, F_6 \wedge F_7 \vdash \Delta_9}} \mathtt{ax/W}}{\mathtt{- : \top, \Delta_{10} \vdash \Delta_9}} \mathtt{hCut}}}$$

8.3 Status of $\vee_R$: OK

- Case rule $\rightarrow_R$

$$\dfrac{\dfrac{\mathtt{h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7, F_8, F_{14}}}{\bullet\mathtt{h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \rightarrow F_{11}), F_7 \vee F_8), F_{14}}} \vee_R \quad \dfrac{\mathtt{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash F_{11}}}{\bullet\mathtt{h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7 \vee F_8}} \rightarrow_R}{\mathtt{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7 \vee F_8}} \mathtt{Cut}$$

$$\dfrac{\rightarrow}{\dfrac{\dfrac{\dfrac{}{\mathtt{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{14}, F_7, F_8, F_{10} \rightarrow F_{11}}} \mathtt{ax/W} \quad \dfrac{\dfrac{}{\mathtt{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash F_{11}}} \mathtt{ax/W}}{\bullet\mathtt{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_7, F_8, F_{10} \rightarrow F_{11}}} \rightarrow_R}{\mathtt{- : \Delta_{13} \vdash \Delta_{12}, F_7, F_8, F_{10} \rightarrow F_{11}}} \mathtt{hCut}}{\mathtt{- : \Delta_{13} \vdash \Delta_{12}, F_{10} \rightarrow F_{11}, F_7 \vee F_8}} \vee_R}$$

$$\frac{\dfrac{h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \to F_{10}), F_6, F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \to F_{10}), F_6 \vee F_7}\ \vee_R \qquad \dfrac{h_8 : \Delta_{12}, F_9, F_6 \vee F_7 \vdash F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9 \to F_{10}}\ \to_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \to F_{10}}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\rule{0pt}{1em}}{- : \Delta_{12} \vdash \Delta_{11}, F_6, F_7, F_9 \to F_{10}}\ \text{ax/W} \qquad \dfrac{\dfrac{- : \Delta_{12}, F_7, F_9 \vdash F_{10}}{- : \Delta_{12}, F_7 \vdash \Delta_{11}, F_6, F_9 \to F_{10}}\ \to_R}{- : \Delta_{12} \vdash \Delta_{11}, F_6, F_9 \to F_{10}}}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \to F_{10}}\ \text{sCut} \qquad \dfrac{\dfrac{- : \Delta_{12}, F_6, F_9 \vdash F_{10}}{- : \Delta_{12}, F_6 \vdash \Delta_{11}, F_9 \to F_{10}}\ \to_R}{}\ \text{sCut}$$

(layout: inv-th/ax on $- : \Delta_{12}, F_7, F_9 \vdash F_{10}$ and inv-th/ax on $- : \Delta_{12}, F_6, F_9 \vdash F_{10}$)

- Case rule $\wedge_R$

$$\frac{\dfrac{h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7, F_8, F_{14}}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \wedge F_{11}), F_7 \vee F_8), F_{14}}\ \vee_R \quad \dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_7 \vee F_8 \quad h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{11}, F_7 \vee F_8}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7 \vee F_8}\ \wedge_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7 \vee F_8}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\rule{0pt}{1em}}{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{14}, F_7, F_8, F_{10} \wedge F_{11}}\ \text{ax/W} \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_7, F_8}{\ }\ \text{inv-th/ax} \quad \dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{11}, F_7, F_8}{\ }\ \text{inv-th/ax}}{\bullet h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_7, F_8, F_{10} \wedge F_{11}}\ \wedge_R}{\dfrac{- : \Delta_{13} \vdash \Delta_{12}, F_7, F_8, F_{10} \wedge F_{11}}{- : \Delta_{13} \vdash \Delta_{12}, F_{10} \wedge F_{11}, F_7 \vee F_8}\ \vee_R}\ \text{hCut}$$

$$\frac{\dfrac{h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \wedge F_{10}), F_6, F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \wedge F_{10}), F_6 \vee F_7}\ \vee_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9 \wedge F_{10}}\ \wedge_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \wedge F_{10}}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\dfrac{h_1 : \Delta_{12} \vdash \Delta_{11}, F_6, F_7, F_9}{\ }\ \text{inv-th/ax}}{\dfrac{h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6, F_7}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6 \vee F_7}\ \vee_R} \quad \dfrac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9}{\ }\ \text{ax/W}}{- : \Delta_{12} \vdash \Delta_{11}, F_9}\ \text{hCut} \qquad \dfrac{\dfrac{\dfrac{h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_6, F_7}{\ }\ \text{inv-th/ax}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_6 \vee F_7}\ \vee_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_{10}}{\ }\ \text{ax/W}}{- : \Delta_{12} \vdash \Delta_{11}, F_{10}}\ \text{hCut}}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \wedge F_{10}}\ \wedge_R$$

- Case rule $\vee_R$

$$\frac{\dfrac{h_1 : \Delta_{13} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7, F_8, F_{14}}{\bullet h_1 : \Delta_{13} \vdash ((\Delta_{12}, F_{10} \vee F_{11}), F_7 \vee F_8), F_{14}}\ \vee_R \quad \dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \vee F_8}{\bullet h_9 : \Delta_{13}, F_{14} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7 \vee F_8}\ \vee_R}{- : \Delta_{13} \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7 \vee F_8}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\dfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}, F_7, F_8}{\ }\ \text{inv-th/ax}}{\bullet h_1 : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}, F_7 \vee F_8}\ \vee_R \quad \dfrac{h_9 : \Delta_{13}, F_{14} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \vee F_8}{\ }\ \text{ax/W}}{\dfrac{- : \Delta_{13} \vdash \Delta_{12}, F_{10}, F_{11}, F_7 \vee F_8}{- : \Delta_{13} \vdash \Delta_{12}, F_{10} \vee F_{11}, F_7 \vee F_8}\ \vee_R}\ \text{hCut}$$

$$\frac{\dfrac{h_1 : \Delta_{11} \vdash \Delta_{10}, F_8, F_9, F_{12}}{\bullet h_1 : \Delta_{11} \vdash (\Delta_{10}, F_8 \vee F_9), F_{12}}\ \vee_R \quad \dfrac{h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8, F_9}{\bullet h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8 \vee F_9}\ \vee_R}{- : \Delta_{11} \vdash \Delta_{10}, F_8 \vee F_9}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\rule{0pt}{1em}}{h_1 : \Delta_{11} \vdash \Delta_{10}, F_{12}, F_8, F_9}\ \text{ax/W} \quad \dfrac{\dfrac{h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8, F_9}{\ }\ \text{ax/W}}{\bullet h_7 : \Delta_{11}, F_{12} \vdash \Delta_{10}, F_8, F_9}\ H}{\dfrac{- : \Delta_{11} \vdash \Delta_{10}, F_8, F_9}{- : \Delta_{11} \vdash \Delta_{10}, F_8 \vee F_9}\ \vee_R}\ \text{hCut}$$

$$\frac{\dfrac{h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \vee F_{10}), F_6, F_7}{\bullet h_1 : \Delta_{12} \vdash (\Delta_{11}, F_9 \vee F_{10}), F_6 \vee F_7}\ \vee_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9, F_{10}}{\bullet h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9 \vee F_{10}}\ \vee_R}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \vee F_{10}}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\dfrac{h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_6, F_7, F_9}{\ }\ \text{inv-th/ax}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_9, F_6 \vee F_7}\ \vee_R \quad \dfrac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_{10}, F_9}{\ }\ \text{ax/W}}{\dfrac{- : \Delta_{12} \vdash \Delta_{11}, F_{10}, F_9}{- : \Delta_{12} \vdash \Delta_{11}, F_9 \vee F_{10}}\ \vee_R}\ \text{hCut}$$

- Case rule $\perp_R$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{11}\vdash(\bot,\Delta_{10}),\mathtt{F_7},\mathtt{F_8},\mathtt{F_{12}}}{\bullet\mathtt{h_1}:\Delta_{11}\vdash((\bot,\Delta_{10}),\mathtt{F_7}\vee\mathtt{F_8}),\mathtt{F_{12}}}\ \vee_R \quad \dfrac{\mathtt{h_9}:\Delta_{11},\mathtt{F_{12}}\vdash\Delta_{10},\mathtt{F_7}\vee\mathtt{F_8}}{\bullet\mathtt{h_9}:\Delta_{11},\mathtt{F_{12}}\vdash(\bot,\Delta_{10}),\mathtt{F_7}\vee\mathtt{F_8}}\ \bot_R}{-:\Delta_{11}\vdash(\bot,\Delta_{10}),\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{\dfrac{}{\bullet\mathtt{h_1}:\Delta_{11}\vdash\bot,\Delta_{10},\mathtt{F_{12}},\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{ax/W}\quad\dfrac{}{\mathtt{h_9}:\Delta_{11},\mathtt{F_{12}}\vdash\bot,\Delta_{10},\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{ax/W}}{-:\Delta_{11}\vdash\bot,\Delta_{10},\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{hCut}$$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{10}\vdash(\bot,\Delta_9),\mathtt{F_6},\mathtt{F_7}}{\bullet\mathtt{h_1}:\Delta_{10}\vdash(\bot,\Delta_9),\mathtt{F_6}\vee\mathtt{F_7}}\ \vee_R \quad \dfrac{\mathtt{h_8}:\Delta_{10},\mathtt{F_6}\vee\mathtt{F_7}\vdash\Delta_9}{\bullet\mathtt{h_8}:\Delta_{10},\mathtt{F_6}\vee\mathtt{F_7}\vdash\bot,\Delta_9}\ \bot_R}{-:\Delta_{10}\vdash\bot,\Delta_9}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{\dfrac{}{\bullet\mathtt{h_1}:\Delta_{10}\vdash\bot,\Delta_9,\mathtt{F_6}\vee\mathtt{F_7}}\ \mathtt{ax/W}\quad\dfrac{}{\mathtt{h_8}:\Delta_{10},\mathtt{F_6}\vee\mathtt{F_7}\vdash\bot,\Delta_9}\ \mathtt{ax/W}}{-:\Delta_{10}\vdash\bot,\Delta_9}\ \mathtt{hCut}$$

- Case rule $\top_R$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{11}\vdash(\top,\Delta_{10}),\mathtt{F_7},\mathtt{F_8},\mathtt{F_{12}}}{\bullet\mathtt{h_1}:\Delta_{11}\vdash((\top,\Delta_{10}),\mathtt{F_7}\vee\mathtt{F_8}),\mathtt{F_{12}}}\ \vee_R \quad \dfrac{}{\bullet\mathtt{h_9}:\Delta_{11},\mathtt{F_{12}}\vdash(\top,\Delta_{10}),\mathtt{F_7}\vee\mathtt{F_8}}\ \top_R}{-:\Delta_{11}\vdash(\top,\Delta_{10}),\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{}{-:\Delta_{11}\vdash\top,\Delta_{10},\mathtt{F_7}\vee\mathtt{F_8}}\ \top_R$$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{10}\vdash(\top,\Delta_9),\mathtt{F_6},\mathtt{F_7}}{\bullet\mathtt{h_1}:\Delta_{10}\vdash(\top,\Delta_9),\mathtt{F_6}\vee\mathtt{F_7}}\ \vee_R \quad \dfrac{}{\bullet\mathtt{h_8}:\Delta_{10},\mathtt{F_6}\vee\mathtt{F_7}\vdash\top,\Delta_9}\ \top_R}{-:\Delta_{10}\vdash\top,\Delta_9}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{}{-:\Delta_{10}\vdash\top,\Delta_9}\ \top_R$$

- Case rule $\rightarrow_L$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8},\mathtt{F_{14}}}{\bullet\mathtt{h_1}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash(\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}),\mathtt{F_{14}}}\ \vee_R \quad \dfrac{\mathtt{h_9}:\Delta_{13},\mathtt{F_{14}},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_{10}},\mathtt{F_7}\vee\mathtt{F_8}\quad \mathtt{h_9}:\Delta_{13},\mathtt{F_{11}},\mathtt{F_{14}}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}{\bullet\mathtt{h_9}:(\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}),\mathtt{F_{14}}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}\ \rightarrow_L}{-:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{\dfrac{}{\mathtt{h_1}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_{14}},\mathtt{F_7},\mathtt{F_8}}\ \mathtt{ax/W}\quad \dfrac{\dfrac{\mathtt{h_9}:\Delta_{13},\mathtt{F_{14}},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_{10}},\mathtt{F_7},\mathtt{F_8}}{}\ \mathtt{inv\text{-}th/ax}\quad\dfrac{\mathtt{h_9}:\Delta_{13},\mathtt{F_{11}},\mathtt{F_{14}}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8}}{}\ \mathtt{inv\text{-}th/ax}}{\bullet\mathtt{h_9}:\Delta_{13},\mathtt{F_{14}},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8}}\ \rightarrow_L}{\dfrac{-:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8}}{-:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}\ \vee_R}\ \mathtt{hCut}$$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{13}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}}{\bullet\mathtt{h_1}:\Delta_{13}\vdash(\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}),\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}}\ \vee_R \quad \dfrac{\mathtt{h_9}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_{10}},\mathtt{F_7}\vee\mathtt{F_8}\quad \mathtt{h_9}:\Delta_{13},\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}{\bullet\mathtt{h_9}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}\ \rightarrow_L}{-:\Delta_{13}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{\dfrac{}{\mathtt{h_1}:\Delta_{13}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}}\ \mathtt{ax/W}\quad \dfrac{\dfrac{\mathtt{h_9}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_{10}},\mathtt{F_7},\mathtt{F_8}}{}\ \mathtt{inv\text{-}th/ax}\quad\dfrac{\mathtt{h_9}:\Delta_{13},\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8}}{}\ \mathtt{inv\text{-}th/ax}}{\bullet\mathtt{h_9}:\Delta_{13},\mathtt{F_{10}}\rightarrow\mathtt{F_{11}}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8}}\ \rightarrow_L}{\dfrac{-:\Delta_{13}\vdash\Delta_{12},\mathtt{F_7},\mathtt{F_8}}{-:\Delta_{13}\vdash\Delta_{12},\mathtt{F_7}\vee\mathtt{F_8}}\ \vee_R}\ \mathtt{hCut}$$

$$\dfrac{\dfrac{\mathtt{h_1}:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}\vdash\Delta_{11},\mathtt{F_6},\mathtt{F_7}}{\bullet\mathtt{h_1}:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}\vdash\Delta_{11},\mathtt{F_6}\vee\mathtt{F_7}}\ \vee_R \quad \dfrac{\mathtt{h_8}:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}},\mathtt{F_6}\vee\mathtt{F_7}\vdash\Delta_{11},\mathtt{F_9}\quad \mathtt{h_8}:\Delta_{12},\mathtt{F_{10}},\mathtt{F_6}\vee\mathtt{F_7}\vdash\Delta_{11}}{\bullet\mathtt{h_8}:(\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}),\mathtt{F_6}\vee\mathtt{F_7}\vdash\Delta_{11}}\ \rightarrow_L}{-:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}\vdash\Delta_{11}}\ \mathtt{Cut}$$
$$\rightarrow$$
$$\dfrac{\dfrac{}{\bullet\mathtt{h_1}:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}\vdash\Delta_{11},\mathtt{F_9},\mathtt{F_6}\vee\mathtt{F_7}}\ \mathtt{ax/W}\quad\dfrac{\mathtt{h_8}:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}},\mathtt{F_6}\vee\mathtt{F_7}\vdash\Delta_{11},\mathtt{F_9}}{}\ \mathtt{ax/W}}{\dfrac{-:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}\vdash\Delta_{11},\mathtt{F_9}}{}\ \mathtt{hCut}}\quad \dfrac{\dfrac{\mathtt{h_1}:\Delta_{12},\mathtt{F_{10}}\vdash\Delta_{11},\mathtt{F_6},\mathtt{F_7}}{\bullet\mathtt{h_1}:\Delta_{12},\mathtt{F_{10}}\vdash\Delta_{11},\mathtt{F_6}\vee\mathtt{F_7}}\ \vee_R\quad\mathtt{h_8}:\Delta_{12},\mathtt{F_{10}},\mathtt{F_6}\dots}{\dfrac{}{\mathtt{inv\text{-}th/ax}}\quad-:\Delta_{12},\mathtt{F_{10}}\vdash\Delta_{11}}}{-:\Delta_{12},\mathtt{F_9}\rightarrow\mathtt{F_{10}}\vdash\Delta_{11}}\ \rightarrow_L$$

- Case rule $\wedge_L$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7, F_8, F_{14}}{\bullet h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash (\Delta_{12}, F_7 \vee F_8), F_{14}} \vee_R
  \qquad
  \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \vee F_8}{\bullet h_9 : (\Delta_{13}, F_{10} \wedge F_{11}), F_{14} \vdash \Delta_{12}, F_7 \vee F_8} \wedge_L
}{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \vee F_8} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{h_1 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_{14}, F_7, F_8}{}\; \mathrm{ax/W}
    \quad
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14} \vdash \Delta_{12}, F_7, F_8}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_9 : \Delta_{13}, F_{14}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7, F_8} \wedge_L
  }{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7, F_8} \; \mathrm{hCut}
}{- : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \vee F_8} \vee_R
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_8, F_{10} \wedge F_{11}}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \vee F_8), F_{10} \wedge F_{11}} \vee_R
  \qquad
  \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_7 \vee F_8}{\bullet h_9 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7 \vee F_8} \wedge_L
}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \vee F_8} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_8, F_{10} \wedge F_{11}}{}\; \mathrm{ax/W}
    \quad
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11} \vdash \Delta_{12}, F_7, F_8}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_9 : \Delta_{13}, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7, F_8} \wedge_L
  }{- : \Delta_{13} \vdash \Delta_{12}, F_7, F_8} \; \mathrm{hCut}
}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \vee F_8} \vee_R
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R
  \qquad
  \cfrac{h_8 : \Delta_{12}, F_9, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \wedge_L
}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{\cfrac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6, F_7}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \vee F_7} \vee_R
    \quad
    \cfrac{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \vee F_7 \vdash \Delta_{11}}{}\; \mathrm{ax/W}
  }{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}} \; \mathrm{hCut}
}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \wedge_L
$$

- Case rule $\vee_L$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7, F_8, F_{14}}{\bullet h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash (\Delta_{12}, F_7 \vee F_8), F_{14}} \vee_R
  \;\;
  \cfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash \Delta_{12}, F_7 \vee F_8 \quad h_9 : \Delta_{13}, F_{11}, F_{14} \vdash \Delta_{12}, F_7 \vee F_8}{\bullet h_9 : (\Delta_{13}, F_{10} \vee F_{11}), F_{14} \vdash \Delta_{12}, F_7 \vee F_8} \vee_L
}{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \vee F_8} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{h_1 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_{14}, F_7, F_8}{}\; \mathrm{ax/W}
    \;\;
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{14} \vdash \Delta_{12}, F_7, F_8}{}\; \mathrm{inv\text{-}th/ax} \quad \cfrac{h_9 : \Delta_{13}, F_{11}, F_{14} \vdash \Delta_{12}, F_7, F_8}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_9 : \Delta_{13}, F_{14}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7, F_8} \vee_L
  }{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7, F_8} \; \mathrm{hCut}
}{- : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \vee F_8} \vee_R
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_8, F_{10} \vee F_{11}}{\bullet h_1 : \Delta_{13} \vdash (\Delta_{12}, F_7 \vee F_8), F_{10} \vee F_{11}} \vee_R
  \;\;
  \cfrac{h_9 : \Delta_{13}, F_{10} \vdash \Delta_{12}, F_7 \vee F_8 \quad h_9 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_7 \vee F_8}{\bullet h_9 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \vee F_8} \vee_L
}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \vee F_8} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{h_1 : \Delta_{13} \vdash \Delta_{12}, F_7, F_8, F_{10} \vee F_{11}}{}\; \mathrm{ax/W}
    \;\;
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10} \vdash \Delta_{12}, F_7, F_8}{}\; \mathrm{inv\text{-}th/ax} \quad \cfrac{h_9 : \Delta_{13}, F_{11} \vdash \Delta_{12}, F_7, F_8}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_9 : \Delta_{13}, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7, F_8} \vee_L
  }{- : \Delta_{13} \vdash \Delta_{12}, F_7, F_8} \; \mathrm{hCut}
}{- : \Delta_{13} \vdash \Delta_{12}, F_7 \vee F_8} \vee_R
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R
  \;\;
  \cfrac{h_8 : \Delta_{12}, F_9, F_6 \vee F_7 \vdash \Delta_{11} \quad h_8 : \Delta_{12}, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \vee_L
}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{\cfrac{h_1 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6, F_7}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_1 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6 \vee F_7} \vee_R
    \;\;
    \cfrac{h_8 : \Delta_{12}, F_9, F_6 \vee F_7 \vdash \Delta_{11}}{}\; \mathrm{ax/W}
  }{- : \Delta_{12}, F_9 \vdash \Delta_{11}} \; \mathrm{hCut}
  \qquad
  \cfrac{
    \cfrac{\cfrac{h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6, F_7}{}\; \mathrm{inv\text{-}th/ax}}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R
    \;\;
    \cfrac{h_8 : \Delta_{12}, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{}\; \mathrm{ax/W}
  }{- : \Delta_{12}, F_{10} \vdash \Delta_{11}} \; \mathrm{hCut}
}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \vee_L
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{10} \vdash \Delta_9, F_7, F_8}{\bullet h_1 : \Delta_{10} \vdash \Delta_9, F_7 \vee F_8} \vee_R
  \qquad
  \cfrac{h_6 : \Delta_{10}, F_7 \vdash \Delta_9 \quad h_6 : \Delta_{10}, F_8 \vdash \Delta_9}{\bullet h_6 : \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \vee_L
}{- : \Delta_{10} \vdash \Delta_9} \; \mathrm{Cut}
$$

$\rightarrow$

$$
\cfrac{
  \cfrac{
    \cfrac{- : \Delta_{10} \vdash \Delta_9, F_7, F_8}{}\; \mathrm{ax/W}
    \quad
    \cfrac{- : \Delta_{10}, F_8 \vdash \Delta_9, F_7}{}\; \mathrm{ax/W}
  }{- : \Delta_{10} \vdash \Delta_9, F_7} \; \mathrm{sCut}
  \qquad
  \cfrac{- : \Delta_{10}, F_7 \vdash \Delta_9}{}\; \mathrm{ax/W}
}{- : \Delta_{10} \vdash \Delta_9} \; \mathrm{sCut}
$$

- Case rule $\perp_L$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \perp}{\bullet\mathtt{h_1} : \Delta_{11} \vdash (\Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}), \perp} \vee_R \qquad \dfrac{}{\bullet\mathtt{h_9} : \Delta_{11}, \perp \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \perp_L
}{- : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{
\dfrac{\dfrac{}{\mathtt{h_1} : \Delta_{11} \vdash \perp, \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}} \texttt{ax/W} \qquad \dfrac{}{\bullet\mathtt{h_9} : \perp, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}} \perp_L}{- : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}} \text{hCut}
}{- : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \vee_R
$$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \perp, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \mathtt{F_{12}}}{\bullet\mathtt{h_1} : \perp, \Delta_{11} \vdash (\Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}), \mathtt{F_{12}}} \vee_R \qquad \dfrac{}{\bullet\mathtt{h_9} : (\perp, \Delta_{11}), \mathtt{F_{12}} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \perp_L
}{- : \perp, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{}{- : \perp, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \perp_L
$$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \perp, \Delta_{10} \vdash \Delta_9, \mathtt{F_6}, \mathtt{F_7}}{\bullet\mathtt{h_1} : \perp, \Delta_{10} \vdash \Delta_9, \mathtt{F_6} \vee \mathtt{F_7}} \vee_R \qquad \dfrac{}{\bullet\mathtt{h_8} : (\perp, \Delta_{10}), \mathtt{F_6} \vee \mathtt{F_7} \vdash \Delta_9} \perp_L
}{- : \perp, \Delta_{10} \vdash \Delta_9} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{}{- : \perp, \Delta_{10} \vdash \Delta_9} \perp_L
$$

- Case rule I

$$
\dfrac{
\dfrac{\mathtt{h_1} : \Delta_{12} \vdash (\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7}, \mathtt{F_8}, \mathtt{p_{11}}}{\bullet\mathtt{h_1} : \Delta_{12} \vdash ((\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7} \vee \mathtt{F_8}), \mathtt{p_{11}}} \vee_R \qquad \dfrac{}{\bullet\mathtt{h_9} : \Delta_{12}, \mathtt{p_{11}} \vdash (\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7} \vee \mathtt{F_8}} I
}{- : \Delta_{12} \vdash (\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7} \vee \mathtt{F_8}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{
\dfrac{\dfrac{}{\mathtt{h_1} : \Delta_{12} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \mathtt{p_{11}}, \mathtt{p_{11}}} \texttt{ax/W} \qquad \dfrac{}{\bullet\mathtt{h_9} : \Delta_{12}, \mathtt{p_{11}} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \mathtt{p_{11}}} I}{- : \Delta_{12} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \mathtt{p_{11}}} \text{hCut}
}{- : \Delta_{12} \vdash \Delta_{10}, \mathtt{p_{11}}, \mathtt{F_7} \vee \mathtt{F_8}} \vee_R
$$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \Delta_{12}, \mathtt{p_{11}} \vdash (\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7}, \mathtt{F_8}, \mathtt{F_{13}}}{\bullet\mathtt{h_1} : \Delta_{12}, \mathtt{p_{11}} \vdash ((\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7} \vee \mathtt{F_8}), \mathtt{F_{13}}} \vee_R \qquad \dfrac{}{\bullet\mathtt{h_9} : (\Delta_{12}, \mathtt{p_{11}}), \mathtt{F_{13}} \vdash (\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7} \vee \mathtt{F_8}} I
}{- : \Delta_{12}, \mathtt{p_{11}} \vdash (\Delta_{10}, \mathtt{p_{11}}), \mathtt{F_7} \vee \mathtt{F_8}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{}{- : \Delta_{12}, \mathtt{p_{11}} \vdash \Delta_{10}, \mathtt{p_{11}}, \mathtt{F_7} \vee \mathtt{F_8}} I
$$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \Delta_{11}, \mathtt{p_{10}} \vdash (\Delta_9, \mathtt{p_{10}}), \mathtt{F_6}, \mathtt{F_7}}{\bullet\mathtt{h_1} : \Delta_{11}, \mathtt{p_{10}} \vdash (\Delta_9, \mathtt{p_{10}}), \mathtt{F_6} \vee \mathtt{F_7}} \vee_R \qquad \dfrac{}{\bullet\mathtt{h_8} : (\Delta_{11}, \mathtt{p_{10}}), \mathtt{F_6} \vee \mathtt{F_7} \vdash \Delta_9, \mathtt{p_{10}}} I
}{- : \Delta_{11}, \mathtt{p_{10}} \vdash \Delta_9, \mathtt{p_{10}}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{}{- : \Delta_{11}, \mathtt{p_{10}} \vdash \Delta_9, \mathtt{p_{10}}} I
$$

- Case rule $\top_L$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \top}{\bullet\mathtt{h_1} : \Delta_{11} \vdash (\Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}), \top} \vee_R \qquad \dfrac{\mathtt{h_9} : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}}{\bullet\mathtt{h_9} : \Delta_{11}, \top \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \top_L
}{- : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{}{- : \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \texttt{ax/W}
$$

$$
\dfrac{
\dfrac{\mathtt{h_1} : \top, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7}, \mathtt{F_8}, \mathtt{F_{12}}}{\bullet\mathtt{h_1} : \top, \Delta_{11} \vdash (\Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}), \mathtt{F_{12}}} \vee_R \qquad \dfrac{\mathtt{h_9} : \Delta_{11}, \mathtt{F_{12}} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}}{\bullet\mathtt{h_9} : (\top, \Delta_{11}), \mathtt{F_{12}} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \top_L
}{- : \top, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \text{Cut}
$$

$$\rightarrow$$

$$
\dfrac{
\dfrac{}{\bullet\mathtt{h_1} : \top, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_{12}}, \mathtt{F_7} \vee \mathtt{F_8}} \texttt{ax/W} \qquad \dfrac{}{\mathtt{h_9} : \top, \Delta_{11}, \mathtt{F_{12}} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \texttt{ax/W}
}{- : \top, \Delta_{11} \vdash \Delta_{10}, \mathtt{F_7} \vee \mathtt{F_8}} \text{hCut}
$$

$$\frac{\dfrac{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6, F_7}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7}\ \vee_R \qquad \dfrac{h_8 : \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \vee F_7 \vdash \Delta_9}\ \top_L}{- : \top, \Delta_{10} \vdash \Delta_9}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{\overline{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7}\ \text{ax/W} \qquad \overline{h_8 : \top, \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9}\ \text{ax/W}}{- : \top, \Delta_{10} \vdash \Delta_9}\ \text{hCut}$$

8.4 Status of $\bot_R$: OK

- Case rule $\rightarrow_R$

$$\frac{\dfrac{h_1 : \Delta_8 \vdash \Delta_7, F_5 \rightarrow F_6}{\bullet h_1 : \Delta_8 \vdash (\Delta_7, F_5 \rightarrow F_6), \bot}\ \bot_R \qquad \dfrac{h_4 : \bot, \Delta_8, F_5 \vdash F_6}{\bullet h_4 : \Delta_8, \bot \vdash \Delta_7, F_5 \rightarrow F_6}\ \rightarrow_R}{- : \Delta_8 \vdash \Delta_7, F_5 \rightarrow F_6}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{}{- : \Delta_8 \vdash \Delta_7, F_5 \rightarrow F_6}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_9 \vdash (\Delta_8, F_6 \rightarrow F_7), F_{10}}{\bullet h_1 : \Delta_9 \vdash (\bot, \Delta_8, F_6 \rightarrow F_7), F_{10}}\ \bot_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_{10} \vdash F_7}{\bullet h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \rightarrow F_7}\ \rightarrow_R}{- : \Delta_9 \vdash \bot, \Delta_8, F_6 \rightarrow F_7}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{\overline{h_1 : \Delta_9 \vdash \bot, \Delta_8, F_{10}, F_6 \rightarrow F_7}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \rightarrow F_7}\ \text{ax/W}}{- : \Delta_9 \vdash \bot, \Delta_8, F_6 \rightarrow F_7}\ \text{hCut}$$

- Case rule $\wedge_R$

$$\frac{\dfrac{h_1 : \Delta_8 \vdash \Delta_7, F_5 \wedge F_6}{\bullet h_1 : \Delta_8 \vdash (\Delta_7, F_5 \wedge F_6), \bot}\ \bot_R \qquad \dfrac{h_4 : \bot, \Delta_8 \vdash \Delta_7, F_5 \quad h_4 : \bot, \Delta_8 \vdash \Delta_7, F_6}{\bullet h_4 : \Delta_8, \bot \vdash \Delta_7, F_5 \wedge F_6}\ \wedge_R}{- : \Delta_8 \vdash \Delta_7, F_5 \wedge F_6}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{}{- : \Delta_8 \vdash \Delta_7, F_5 \wedge F_6}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_9 \vdash (\Delta_8, F_6 \wedge F_7), F_{10}}{\bullet h_1 : \Delta_9 \vdash (\bot, \Delta_8, F_6 \wedge F_7), F_{10}}\ \bot_R \qquad \dfrac{h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \quad h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_7}{\bullet h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \wedge F_7}\ \wedge_R}{- : \Delta_9 \vdash \bot, \Delta_8, F_6 \wedge F_7}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{\overline{h_1 : \Delta_9 \vdash \bot, \Delta_8, F_{10}, F_6 \wedge F_7}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \wedge F_7}\ \text{ax/W}}{- : \Delta_9 \vdash \bot, \Delta_8, F_6 \wedge F_7}\ \text{hCut}$$

- Case rule $\vee_R$

$$\frac{\dfrac{h_1 : \Delta_8 \vdash \Delta_7, F_5 \vee F_6}{\bullet h_1 : \Delta_8 \vdash (\Delta_7, F_5 \vee F_6), \bot}\ \bot_R \qquad \dfrac{h_4 : \bot, \Delta_8 \vdash \Delta_7, F_5, F_6}{\bullet h_4 : \Delta_8, \bot \vdash \Delta_7, F_5 \vee F_6}\ \vee_R}{- : \Delta_8 \vdash \Delta_7, F_5 \vee F_6}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{}{- : \Delta_8 \vdash \Delta_7, F_5 \vee F_6}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_9 \vdash (\Delta_8, F_6 \vee F_7), F_{10}}{\bullet h_1 : \Delta_9 \vdash (\bot, \Delta_8, F_6 \vee F_7), F_{10}}\ \bot_R \qquad \dfrac{h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6, F_7}{\bullet h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \vee F_7}\ \vee_R}{- : \Delta_9 \vdash \bot, \Delta_8, F_6 \vee F_7}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{\overline{h_1 : \Delta_9 \vdash \bot, \Delta_8, F_{10}, F_6 \vee F_7}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_9, F_{10} \vdash \bot, \Delta_8, F_6 \vee F_7}\ \text{ax/W}}{- : \Delta_9 \vdash \bot, \Delta_8, F_6 \vee F_7}\ \text{hCut}$$

- Case rule $\bot_R$

$$\frac{\dfrac{h_1 : \Delta_6 \vdash \bot, \Delta_5}{\bullet h_1 : \Delta_6 \vdash (\bot, \Delta_5), \bot}\ \bot_R \qquad \dfrac{h_4 : \bot, \Delta_6 \vdash \Delta_5}{\bullet h_4 : \Delta_6, \bot \vdash \bot, \Delta_5}\ \bot_R}{- : \Delta_6 \vdash \bot, \Delta_5}\ \text{Cut}$$

$$\rightarrow$$

$$\frac{}{- : \Delta_6 \vdash \bot, \Delta_5}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_7 \vdash \Delta_6, F_8}{\bullet h_1 : \Delta_7 \vdash (\bot, \Delta_6), F_8}\ \bot_R \qquad \dfrac{h_5 : \Delta_7, F_8 \vdash \Delta_6}{\bullet h_5 : \Delta_7, F_8 \vdash \bot, \Delta_6}\ \bot_R}{- : \Delta_7 \vdash \bot, \Delta_6}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{h_1 : \Delta_7 \vdash \bot, \Delta_6, F_8}\ \text{ax/W} \qquad \dfrac{}{\bullet h_5 : \Delta_7, F_8 \vdash \bot, \Delta_6}\ \text{ax/W}}{- : \Delta_7 \vdash \bot, \Delta_6}\ \text{hCut}$$

- Case rule $\top_R$

$$\frac{\dfrac{h_1 : \Delta_6 \vdash \top, \Delta_5}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_5), \bot}\ \bot_R \qquad \dfrac{}{\bullet h_4 : \Delta_6, \bot \vdash \top, \Delta_5}\ \top_R}{- : \Delta_6 \vdash \top, \Delta_5}\ \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_6 \vdash \top, \Delta_5}\ \top_R$$

$$\frac{\dfrac{h_1 : \Delta_7 \vdash (\top, \Delta_6), F_8}{\bullet h_1 : \Delta_7 \vdash (\bot, \top, \Delta_6), F_8}\ \bot_R \qquad \dfrac{}{\bullet h_5 : \Delta_7, F_8 \vdash \bot, \top, \Delta_6}\ \top_R}{- : \Delta_7 \vdash \bot, \top, \Delta_6}\ \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_7 \vdash \bot, \top, \Delta_6}\ \top_R$$

- Case rule $\to_L$

$$\frac{\dfrac{h_1 : \Delta_8, F_5 \to F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \to F_6 \vdash \Delta_7, \bot}\ \bot_R \qquad \dfrac{h_4 : \bot, \Delta_8, F_5 \to F_6 \vdash \Delta_7, F_5 \quad h_4 : \bot, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \to F_6), \bot \vdash \Delta_7}\ \text{Cut}}{- : \Delta_8, F_5 \to F_6 \vdash \Delta_7}\ \to_L$$

$$\to$$

$$\frac{}{- : \Delta_8, F_5 \to F_6 \vdash \Delta_7}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_9, F_6 \to F_7 \vdash \Delta_8, F_{10}}{\bullet h_1 : \Delta_9, F_6 \to F_7 \vdash (\bot, \Delta_8), F_{10}}\ \bot_R \qquad \dfrac{h_5 : \Delta_9, F_{10}, F_6 \to F_7 \vdash \bot, \Delta_8, F_6 \quad h_5 : \Delta_9, F_7, F_{10} \vdash \bot, \Delta_8}{\bullet h_5 : (\Delta_9, F_6 \to F_7), F_{10} \vdash \bot, \Delta_8}\ \text{Cut}}{- : \Delta_9, F_6 \to F_7 \vdash \bot, \Delta_8}\ \to_L$$

$$\to$$

$$\frac{\dfrac{}{h_1 : \Delta_9, F_6 \to F_7 \vdash \bot, \Delta_8, F_{10}}\ \text{ax/W} \qquad \dfrac{}{\bullet h_5 : \Delta_9, F_{10}, F_6 \to F_7 \vdash \bot, \Delta_8}\ \text{ax/W}}{- : \Delta_9, F_6 \to F_7 \vdash \bot, \Delta_8}\ \text{hCut}$$

$$\frac{\dfrac{h_1 : \Delta_9 \vdash \Delta_8, F_6 \to F_7}{\bullet h_1 : \Delta_9 \vdash (\bot, \Delta_8), F_6 \to F_7}\ \bot_R \qquad \dfrac{h_5 : \Delta_9, F_6 \to F_7 \vdash \bot, \Delta_8, F_6 \quad h_5 : \Delta_9, F_7 \vdash \bot, \Delta_8}{\bullet h_5 : \Delta_9, F_6 \to F_7 \vdash \bot, \Delta_8}\ \text{Cut}}{- : \Delta_9 \vdash \bot, \Delta_8}\ \to_L$$

$$\to$$

$$\frac{\dfrac{}{h_1 : \Delta_9 \vdash \bot, \Delta_8, F_6 \to F_7}\ \text{ax/W} \qquad \dfrac{}{\bullet h_5 : \Delta_9, F_6 \to F_7 \vdash \bot, \Delta_8}\ \text{ax/W}}{- : \Delta_9 \vdash \bot, \Delta_8}\ \text{hCut}$$

- Case rule $\wedge_L$

$$\frac{\dfrac{h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7, \bot}\ \bot_R \qquad \dfrac{h_4 : \bot, \Delta_8, F_5, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \wedge F_6), \bot \vdash \Delta_7}\ \wedge_L}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}\ \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_9, F_6 \wedge F_7 \vdash \Delta_8, F_{10}}{\bullet h_1 : \Delta_9, F_6 \wedge F_7 \vdash (\bot, \Delta_8), F_{10}}\ \bot_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_7, F_{10} \vdash \bot, \Delta_8}{\bullet h_5 : (\Delta_9, F_6 \wedge F_7), F_{10} \vdash \bot, \Delta_8}\ \wedge_L}{- : \Delta_9, F_6 \wedge F_7 \vdash \bot, \Delta_8}\ \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{h_1 : \Delta_9, F_6 \wedge F_7 \vdash \bot, \Delta_8, F_{10}}\ \text{ax/W} \qquad \dfrac{}{\bullet h_5 : \Delta_9, F_{10}, F_6 \wedge F_7 \vdash \bot, \Delta_8}\ \text{ax/W}}{- : \Delta_9, F_6 \wedge F_7 \vdash \bot, \Delta_8}\ \text{hCut}$$

$$\frac{\dfrac{h_1 : \Delta_9 \vdash \Delta_8, F_6 \wedge F_7}{\bullet h_1 : \Delta_9 \vdash (\bot, \Delta_8), F_6 \wedge F_7}\ \perp_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_7 \vdash \bot, \Delta_8}{\bullet h_5 : \Delta_9, F_6 \wedge F_7 \vdash \bot, \Delta_8}\ \wedge_L}{- : \Delta_9 \vdash \bot, \Delta_8}\ \text{Cut}$$

$$\to$$

$$\frac{\overline{h_1 : \Delta_9 \vdash \bot, \Delta_8, F_6 \wedge F_7}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_9, F_6 \wedge F_7 \vdash \bot, \Delta_8}\ \text{ax/W}}{- : \Delta_9 \vdash \bot, \Delta_8}\ \text{hCut}$$

- Case rule $\vee_L$

$$\frac{\dfrac{h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7, \bot}\ \perp_R \qquad \dfrac{h_4 : \bot, \Delta_8, F_5 \vdash \Delta_7 \quad h_4 : \bot, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \vee F_6), \bot \vdash \Delta_7}\ \vee_L}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}\ \text{Cut}$$

$$\to$$

$$\overline{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_9, F_6 \vee F_7 \vdash \Delta_8, F_{10}}{\bullet h_1 : \Delta_9, F_6 \vee F_7 \vdash (\bot, \Delta_8), F_{10}}\ \perp_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_{10} \vdash \bot, \Delta_8 \quad h_5 : \Delta_9, F_7, F_{10} \vdash \bot, \Delta_8}{\bullet h_5 : (\Delta_9, F_6 \vee F_7), F_{10} \vdash \bot, \Delta_8}\ \vee_L}{- : \Delta_9, F_6 \vee F_7 \vdash \bot, \Delta_8}\ \text{Cut}$$

$$\to$$

$$\frac{\overline{h_1 : \Delta_9, F_6 \vee F_7 \vdash \bot, \Delta_8, F_{10}}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_9, F_{10}, F_6 \vee F_7 \vdash \bot, \Delta_8}\ \text{ax/W}}{- : \Delta_9, F_6 \vee F_7 \vdash \bot, \Delta_8}\ \text{hCut}$$

$$\frac{\dfrac{h_1 : \Delta_9 \vdash \Delta_8, F_6 \vee F_7}{\bullet h_1 : \Delta_9 \vdash (\bot, \Delta_8), F_6 \vee F_7}\ \perp_R \qquad \dfrac{h_5 : \Delta_9, F_6 \vdash \bot, \Delta_8 \quad h_5 : \Delta_9, F_7 \vdash \bot, \Delta_8}{\bullet h_5 : \Delta_9, F_6 \vee F_7 \vdash \bot, \Delta_8}\ \vee_L}{- : \Delta_9 \vdash \bot, \Delta_8}\ \text{Cut}$$

$$\to$$

$$\frac{\overline{h_1 : \Delta_9 \vdash \bot, \Delta_8, F_6 \vee F_7}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_9, F_6 \vee F_7 \vdash \bot, \Delta_8}\ \text{ax/W}}{- : \Delta_9 \vdash \bot, \Delta_8}\ \text{hCut}$$

- Case rule $\perp_L$

$$\frac{\dfrac{h_1 : \Delta_6 \vdash \Delta_5}{\bullet h_1 : \Delta_6 \vdash \Delta_5, \bot}\ \perp_R \qquad \overline{\bullet h_4 : \Delta_6, \bot \vdash \Delta_5}\ \perp_L}{- : \Delta_6 \vdash \Delta_5}\ \text{Cut}$$

$$\to$$

$$\overline{- : \Delta_6 \vdash \Delta_5}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \Delta_7 \vdash \Delta_6, \bot}{\bullet h_1 : \Delta_7 \vdash (\bot, \Delta_6), \bot}\ \perp_R \qquad \overline{\bullet h_5 : \Delta_7, \bot \vdash \bot, \Delta_6}\ \perp_L}{- : \Delta_7 \vdash \bot, \Delta_6}\ \text{Cut}$$

$$\to$$

$$\overline{- : \Delta_7 \vdash \bot, \Delta_6}\ \text{ax/W}$$

$$\frac{\dfrac{h_1 : \bot, \Delta_7 \vdash \Delta_6, F_8}{\bullet h_1 : \bot, \Delta_7 \vdash (\bot, \Delta_6), F_8}\ \perp_R \qquad \overline{\bullet h_5 : (\bot, \Delta_7), F_8 \vdash \bot, \Delta_6}\ \perp_L}{- : \bot, \Delta_7 \vdash \bot, \Delta_6}\ \text{Cut}$$

$$\to$$

$$\overline{- : \bot, \Delta_7 \vdash \bot, \Delta_6}\ \perp_L$$

- Case rule I

$$\frac{\dfrac{h_1 : \Delta_7, p_6 \vdash \Delta_5, p_6}{\bullet h_1 : \Delta_7, p_6 \vdash (\Delta_5, p_6), \bot}\ \perp_R \qquad \overline{\bullet h_4 : (\Delta_7, p_6), \bot \vdash \Delta_5, p_6}\ I}{- : \Delta_7, p_6 \vdash \Delta_5, p_6}\ \text{Cut}$$

$$\to$$

$$\overline{- : \Delta_7, p_6 \vdash \Delta_5, p_6}\ I$$

$$\frac{\dfrac{h_1 : \Delta_8 \vdash (\Delta_6, p_7), p_7}{\bullet h_1 : \Delta_8 \vdash (\bot, \Delta_6, p_7), p_7}\ \perp_R \qquad \overline{\bullet h_5 : \Delta_8, p_7 \vdash \bot, \Delta_6, p_7}\ I}{- : \Delta_8 \vdash \bot, \Delta_6, p_7}\ \text{Cut}$$

$$\to$$

$$\frac{\overline{h_1 : \Delta_8 \vdash \bot, \Delta_6, p_7, p_7}\ \text{ax/W} \qquad \overline{\bullet h_5 : \Delta_8, p_7 \vdash \bot, \Delta_6, p_7}\ I}{- : \Delta_8 \vdash \bot, \Delta_6, p_7}\ \text{hCut}$$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_8, p_7 \vdash (\Delta_6, p_7), F_9}
{\bullet h_1 : \Delta_8, p_7 \vdash (\bot, \Delta_6, p_7), F_9}\ \perp_R
\qquad
\cfrac{}{\bullet h_5 : (\Delta_8, p_7), F_9 \vdash \bot, \Delta_6, p_7}\ I
}
{- : \Delta_8, p_7 \vdash \bot, \Delta_6, p_7}\ \text{Cut}
}
{\cfrac{\rightarrow}{- : \Delta_8, p_7 \vdash \bot, \Delta_6, p_7}}\ I
$$

- Case rule $\top_L$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \top, \Delta_6 \vdash \Delta_5}{\bullet h_1 : \top, \Delta_6 \vdash \Delta_5, \bot}\ \perp_R
\qquad
\cfrac{h_4 : \bot, \Delta_6 \vdash \Delta_5}{\bullet h_4 : (\top, \Delta_6), \bot \vdash \Delta_5}\ \top_L
}
{- : \top, \Delta_6 \vdash \Delta_5}\ \text{Cut}
}
{\cfrac{\rightarrow}{- : \top, \Delta_6 \vdash \Delta_5}}\ \text{ax/W}
$$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_7 \vdash \Delta_6, \top}{\bullet h_1 : \Delta_7 \vdash (\bot, \Delta_6), \top}\ \perp_R
\qquad
\cfrac{h_5 : \Delta_7 \vdash \bot, \Delta_6}{\bullet h_5 : \Delta_7, \top \vdash \bot, \Delta_6}\ \top_L
}
{- : \Delta_7 \vdash \bot, \Delta_6}\ \text{Cut}
}
{\cfrac{\rightarrow}{- : \Delta_7 \vdash \bot, \Delta_6}}\ \text{ax/W}
$$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \top, \Delta_7 \vdash \Delta_6, F_8}{\bullet h_1 : \top, \Delta_7 \vdash (\bot, \Delta_6), F_8}\ \perp_R
\qquad
\cfrac{h_5 : \Delta_7, F_8 \vdash \bot, \Delta_6}{\bullet h_5 : (\top, \Delta_7), F_8 \vdash \bot, \Delta_6}\ \top_L
}
{- : \top, \Delta_7 \vdash \bot, \Delta_6}\ \text{Cut}
}
{\cfrac{\rightarrow}
{\cfrac{\cfrac{}{h_1 : \top, \Delta_7 \vdash \bot, \Delta_6, F_8}\ \text{ax/W} \qquad \cfrac{}{\bullet h_5 : \top, \Delta_7, F_8 \vdash \bot, \Delta_6}\ \text{ax/W}}
{- : \top, \Delta_7 \vdash \bot, \Delta_6}\ \text{hCut}}
}
$$

8.5 Status of $\top_R$: OK

- Case rule $\rightarrow_R$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet h_1 : \Delta_8 \vdash (\Delta_7, F_5 \rightarrow F_6), \top}\ \top_R
\qquad
\cfrac{h_4 : \top, \Delta_8, F_5 \vdash F_6}{\bullet h_4 : \Delta_8, \top \vdash \Delta_7, F_5 \rightarrow F_6}\ \rightarrow_R
}
{- : \Delta_8 \vdash \Delta_7, F_5 \rightarrow F_6}\ \text{Cut}
}
{\cfrac{\rightarrow}
{\cfrac{\cfrac{\cfrac{}{\bullet h_1 : \Delta_8, F_5 \vdash \top, F_6}\ \top_R \qquad \cfrac{}{h_4 : \top, \Delta_8, F_5 \vdash F_6}\ \text{ax/W}}{- : \Delta_8, F_5 \vdash F_6}\ \text{hCut}}
{- : \Delta_8 \vdash \Delta_7, F_5 \rightarrow F_6}\ \rightarrow_R}
}
$$

$$
\cfrac{
\cfrac{}{\bullet h_1 : \Delta_9 \vdash (\top, \Delta_8, F_6 \rightarrow F_7), F_{10}}\ \top_R
\qquad
\cfrac{h_5 : \Delta_9, F_6, F_{10} \vdash F_7}{\bullet h_5 : \Delta_9, F_{10} \vdash \top, \Delta_8, F_6 \rightarrow F_7}\ \rightarrow_R
}
{\cfrac{- : \Delta_9 \vdash \top, \Delta_8, F_6 \rightarrow F_7}{\cfrac{\rightarrow}{- : \Delta_9 \vdash \top, \Delta_8, F_6 \rightarrow F_7}\ \top_R}}\ \text{Cut}
$$

- Case rule $\wedge_R$

$$
\cfrac{
\cfrac{}{\bullet h_1 : \Delta_8 \vdash (\Delta_7, F_5 \wedge F_6), \top}\ \top_R
\qquad
\cfrac{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5 \quad h_4 : \top, \Delta_8 \vdash \Delta_7, F_6}{\bullet h_4 : \Delta_8, \top \vdash \Delta_7, F_5 \wedge F_6}\ \wedge_R
}
{\cfrac{- : \Delta_8 \vdash \Delta_7, F_5 \wedge F_6}{\cfrac{\rightarrow}{\cdots}}}\ \text{Cut}
$$

$$
\cfrac{
\cfrac{\cfrac{}{\bullet h_1 : \Delta_8 \vdash \top, \Delta_7, F_5}\ \top_R \quad \cfrac{}{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5}\ \text{ax/W}}{- : \Delta_8 \vdash \Delta_7, F_5}\ \text{hCut}
\qquad
\cfrac{\cfrac{}{\bullet h_1 : \Delta_8 \vdash \top, \Delta_7, F_6}\ \top_R \quad \cfrac{}{h_4 : \top, \Delta_8 \vdash \Delta_7, F_6}\ \text{ax/W}}{- : \Delta_8 \vdash \Delta_7, F_6}\ \text{hCut}
}
{- : \Delta_8 \vdash \Delta_7, F_5 \wedge F_6}\ \wedge_R
$$

$$
\cfrac{
\cfrac{}{\bullet h_1 : \Delta_9 \vdash (\top, \Delta_8, F_6 \wedge F_7), F_{10}}\ \top_R
\qquad
\cfrac{h_5 : \Delta_9, F_{10} \vdash \top, \Delta_8, F_6 \quad h_5 : \Delta_9, F_{10} \vdash \top, \Delta_8, F_7}{\bullet h_5 : \Delta_9, F_{10} \vdash \top, \Delta_8, F_6 \wedge F_7}\ \wedge_R
}
{\cfrac{- : \Delta_9 \vdash \top, \Delta_8, F_6 \wedge F_7}{\cfrac{\rightarrow}{- : \Delta_9 \vdash \top, \Delta_8, F_6 \wedge F_7}\ \top_R}}\ \text{Cut}
$$

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- Case rule $\vee_R$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_8 \vdash (\Delta_7, F_5 \vee F_6), \top}\ \top_R \qquad \cfrac{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5, F_6}{\bullet h_4 : \Delta_8, \top \vdash \Delta_7, F_5 \vee F_6}\ \vee_R}{- : \Delta_8 \vdash \Delta_7, F_5 \vee F_6}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{\bullet h_1 : \Delta_8 \vdash \top, \Delta_7, F_5, F_6}\ \top_R \qquad \cfrac{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5, F_6}{}\ \text{ax/W}}{- : \Delta_8 \vdash \Delta_7, F_5, F_6}\ \text{hCut}}{- : \Delta_8 \vdash \Delta_7, F_5 \vee F_6}\ \vee_R$$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_9 \vdash (\top, \Delta_8, F_6 \vee F_7), F_{10}}\ \top_R \qquad \cfrac{h_5 : \Delta_9, F_{10} \vdash \top, \Delta_8, F_6, F_7}{\bullet h_5 : \Delta_9, F_{10} \vdash \top, \Delta_8, F_6 \vee F_7}\ \vee_R}{- : \Delta_9 \vdash \top, \Delta_8, F_6 \vee F_7}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{}{- : \Delta_9 \vdash \top, \Delta_8, F_6 \vee F_7}\ \top_R$$

- Case rule $\bot_R$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_6 \vdash (\bot, \Delta_5), \top}\ \top_R \qquad \cfrac{h_4 : \top, \Delta_6 \vdash \Delta_5}{\bullet h_4 : \Delta_6, \top \vdash \bot, \Delta_5}\ \bot_R}{- : \Delta_6 \vdash \bot, \Delta_5}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_6 \vdash \bot, \top, \Delta_5}\ \top_R \qquad \cfrac{h_4 : \top, \Delta_6 \vdash \bot, \Delta_5}{}\ \text{ax/W}}{- : \Delta_6 \vdash \bot, \Delta_5}\ \text{hCut}$$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_7 \vdash (\top, \bot, \Delta_6), F_8}\ \top_R \qquad \cfrac{h_5 : \Delta_7, F_8 \vdash \top, \Delta_6}{\bullet h_5 : \Delta_7, F_8 \vdash \top, \bot, \Delta_6}\ \bot_R}{- : \Delta_7 \vdash \top, \bot, \Delta_6}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{}{- : \Delta_7 \vdash \bot, \top, \Delta_6}\ \top_R$$

- Case rule $\top_R$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_5), \top}\ \top_R \qquad \cfrac{}{\bullet h_4 : \Delta_6, \top \vdash \top, \Delta_5}\ \top_R}{- : \Delta_6 \vdash \top, \Delta_5}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{}{- : \Delta_6 \vdash \top, \Delta_5}\ \top_R$$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_7 \vdash (\top, \Delta_6), F_8}\ \top_R \qquad \cfrac{}{\bullet h_5 : \Delta_7, F_8 \vdash \top, \Delta_6}\ \top_R}{- : \Delta_7 \vdash \top, \Delta_6}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{}{- : \Delta_7 \vdash \top, \Delta_6}\ \top_R$$

- Case rule $\to_L$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_8, F_5 \to F_6 \vdash \Delta_7, \top}\ \top_R \quad \cfrac{h_4 : \top, \Delta_8, F_5 \to F_6 \vdash \Delta_7, F_5 \quad h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \to F_6), \top \vdash \Delta_7}\ \to_L}{- : \Delta_8, F_5 \to F_6 \vdash \Delta_7}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{\bullet h_1 : \Delta_8, F_5 \to F_6 \vdash \top, \Delta_7, F_5}\ \text{ax/W} \quad \cfrac{h_4 : \top, \Delta_8, F_5 \to F_6 \vdash \Delta_7, F_5}{}}{- : \Delta_8, F_5 \to F_6 \vdash \Delta_7, F_5}\ \text{hCut} \quad \cfrac{\cfrac{}{\bullet h_1 : \Delta_8, F_6 \vdash \top, \Delta_7}\ \top_R \quad \cfrac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{}\ \text{ax/W}}{- : \Delta_8, F_6 \vdash \Delta_7}\ \text{hCut}}{- : \Delta_8, F_5 \to F_6 \vdash \Delta_7}\ \to_L$$

$$\cfrac{\cfrac{}{\bullet h_1 : \Delta_9, F_6 \to F_7 \vdash (\top, \Delta_8), F_{10}}\ \top_R \quad \cfrac{h_5 : \Delta_9, F_{10}, F_6 \to F_7 \vdash \top, \Delta_8, F_6 \quad h_5 : \Delta_9, F_7, F_{10} \vdash \top, \Delta_8}{\bullet h_5 : (\Delta_9, F_6 \to F_7), F_{10} \vdash \top, \Delta_8}\ \to_L}{- : \Delta_9, F_6 \to F_7 \vdash \top, \Delta_8}\ \text{Cut}$$

$$\rightarrow$$

$$\cfrac{}{- : \Delta_9, F_6 \to F_7 \vdash \top, \Delta_8}\ \top_R$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9 \vdash (\top, \Delta_8), F_6 \to F_7} \ \top_R \qquad \dfrac{h_5 : \Delta_9, F_6 \to F_7 \vdash \top, \Delta_8, F_6 \quad h_5 : \Delta_9, F_7 \vdash \top, \Delta_8}{\bullet h_5 : \Delta_9, F_6 \to F_7 \vdash \top, \Delta_8} \to_L}{- : \Delta_9 \vdash \top, \Delta_8} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \Delta_9 \vdash \top, \Delta_8} \ \top_R$$

- Case rule $\wedge_L$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7, \top} \ \top_R \qquad \dfrac{h_4 : \top, \Delta_8, F_5, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \wedge F_6), \top \vdash \Delta_7} \ \wedge_L}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \ \text{Cut}$$
$$\to$$
$$\frac{\dfrac{\dfrac{}{\bullet h_1 : \Delta_8, F_5, F_6 \vdash \top, \Delta_7} \ \top_R \quad \dfrac{h_4 : \top, \Delta_8, F_5, F_6 \vdash \Delta_7}{} \ \text{ax/W}}{- : \Delta_8, F_5, F_6 \vdash \Delta_7} \ \text{hCut}}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \ \wedge_L$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9, F_6 \wedge F_7 \vdash (\top, \Delta_8), F_{10}} \ \top_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_7, F_{10} \vdash \top, \Delta_8}{\bullet h_5 : (\Delta_9, F_6 \wedge F_7), F_{10} \vdash \top, \Delta_8} \ \wedge_L}{- : \Delta_9, F_6 \wedge F_7 \vdash \top, \Delta_8} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \Delta_9, F_6 \wedge F_7 \vdash \top, \Delta_8} \ \top_R$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9 \vdash (\top, \Delta_8), F_6 \wedge F_7} \ \top_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_7 \vdash \top, \Delta_8}{\bullet h_5 : \Delta_9, F_6 \wedge F_7 \vdash \top, \Delta_8} \ \wedge_L}{- : \Delta_9 \vdash \top, \Delta_8} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \Delta_9 \vdash \top, \Delta_8} \ \top_R$$

- Case rule $\vee_L$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7, \top} \ \top_R \qquad \dfrac{h_4 : \top, \Delta_8, F_5 \vdash \Delta_7 \quad h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \vee F_6), \top \vdash \Delta_7} \ \vee_L}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \ \text{Cut}$$
$$\to$$
$$\frac{\dfrac{\dfrac{}{\bullet h_1 : \Delta_8, F_5 \vdash \top, \Delta_7} \ \top_R \quad \dfrac{h_4 : \top, \Delta_8, F_5 \vdash \Delta_7}{} \ \text{ax/W}}{- : \Delta_8, F_5 \vdash \Delta_7} \ \text{hCut} \qquad \dfrac{\dfrac{}{\bullet h_1 : \Delta_8, F_6 \vdash \top, \Delta_7} \ \top_R \quad \dfrac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{} \ \text{ax/W}}{- : \Delta_8, F_6 \vdash \Delta_7} \ \text{hCut}}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \ \vee_L$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9, F_6 \vee F_7 \vdash (\top, \Delta_8), F_{10}} \ \top_R \qquad \dfrac{h_5 : \Delta_9, F_6, F_{10} \vdash \top, \Delta_8 \quad h_5 : \Delta_9, F_7, F_{10} \vdash \top, \Delta_8}{\bullet h_5 : (\Delta_9, F_6 \vee F_7), F_{10} \vdash \top, \Delta_8} \ \vee_L}{- : \Delta_9, F_6 \vee F_7 \vdash \top, \Delta_8} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \Delta_9, F_6 \vee F_7 \vdash \top, \Delta_8} \ \top_R$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9 \vdash (\top, \Delta_8), F_6 \vee F_7} \ \top_R \qquad \dfrac{h_5 : \Delta_9, F_6 \vdash \top, \Delta_8 \quad h_5 : \Delta_9, F_7 \vdash \top, \Delta_8}{\bullet h_5 : \Delta_9, F_6 \vee F_7 \vdash \top, \Delta_8} \ \vee_L}{- : \Delta_9 \vdash \top, \Delta_8} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \Delta_9 \vdash \top, \Delta_8} \ \top_R$$

- Case rule $\perp_L$

$$\frac{\dfrac{}{\bullet h_1 : \perp, \Delta_6 \vdash \Delta_5, \top} \ \top_R \qquad \dfrac{}{\bullet h_4 : (\perp, \Delta_6), \top \vdash \Delta_5} \ \perp_L}{- : \perp, \Delta_6 \vdash \Delta_5} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \perp, \Delta_6 \vdash \Delta_5} \ \perp_L$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_7 \vdash (\top, \Delta_6), \perp} \ \top_R \qquad \dfrac{}{\bullet h_5 : \Delta_7, \perp \vdash \top, \Delta_6} \ \perp_L}{- : \Delta_7 \vdash \top, \Delta_6} \ \text{Cut}$$
$$\to$$
$$\frac{}{- : \Delta_7 \vdash \top, \Delta_6} \ \top_R$$

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$$\frac{\dfrac{}{\bullet h_1 : \bot, \Delta_7 \vdash (\top, \Delta_6), F_8} \ \top_R \qquad \dfrac{}{\bullet h_5 : (\bot, \Delta_7), F_8 \vdash \top, \Delta_6} \ \bot_L}{\dfrac{- : \bot, \Delta_7 \vdash \top, \Delta_6}{\dfrac{\rightarrow}{- : \bot, \Delta_7 \vdash \top, \Delta_6} \ \top_R}} \ \text{Cut}$$

- Case rule I

$$\frac{\dfrac{}{\bullet h_1 : \Delta_7, p_6 \vdash (\Delta_5, p_6), \top} \ \top_R \qquad \dfrac{}{\bullet h_4 : (\Delta_7, p_6), \top \vdash \Delta_5, p_6} \ I}{\dfrac{- : \Delta_7, p_6 \vdash \Delta_5, p_6}{\dfrac{\rightarrow}{- : \Delta_7, p_6 \vdash \Delta_5, p_6} \ I}} \ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_8 \vdash (\top, \Delta_6, p_7), p_7} \ \top_R \qquad \dfrac{}{\bullet h_5 : \Delta_8, p_7 \vdash \top, \Delta_6, p_7} \ I}{\dfrac{- : \Delta_8 \vdash \top, \Delta_6, p_7}{\dfrac{\rightarrow}{- : \Delta_8 \vdash \top, \Delta_6, p_7} \ \top_R}} \ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_8, p_7 \vdash (\top, \Delta_6, p_7), F_9} \ \top_R \qquad \dfrac{}{\bullet h_5 : (\Delta_8, p_7), F_9 \vdash \top, \Delta_6, p_7} \ I}{\dfrac{- : \Delta_8, p_7 \vdash \top, \Delta_6, p_7}{\dfrac{\rightarrow}{- : \Delta_8, p_7 \vdash \top, \Delta_6, p_7} \ \top_R}} \ \text{Cut}$$

- Case rule $\top_L$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_6 \vdash \Delta_5, \top} \ \top_R \qquad \dfrac{h_4 : \Delta_6 \vdash \Delta_5}{\bullet h_4 : \Delta_6, \top \vdash \Delta_5} \ \top_L}{\dfrac{- : \Delta_6 \vdash \Delta_5}{\dfrac{\rightarrow}{- : \Delta_6 \vdash \Delta_5} \ \text{ax/W}}} \ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_7 \vdash (\top, \Delta_6), \top} \ \top_R \qquad \dfrac{h_5 : \Delta_7 \vdash \top, \Delta_6}{\bullet h_5 : \Delta_7, \top \vdash \top, \Delta_6} \ \top_L}{\dfrac{- : \Delta_7 \vdash \top, \Delta_6}{\dfrac{\rightarrow}{- : \Delta_7 \vdash \top, \Delta_6} \ \top_R}} \ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \top, \Delta_7 \vdash (\top, \Delta_6), F_8} \ \top_R \qquad \dfrac{h_5 : \Delta_7, F_8 \vdash \top, \Delta_6}{\bullet h_5 : (\top, \Delta_7), F_8 \vdash \top, \Delta_6} \ \top_L}{\dfrac{- : \top, \Delta_7 \vdash \top, \Delta_6}{\dfrac{\rightarrow}{- : \top, \Delta_7 \vdash \top, \Delta_6} \ \top_R}} \ \text{Cut}$$

8.6   Status of $\rightarrow_L$: OK

- Case rule $\rightarrow_R$

$$\frac{\dfrac{h_1 : \Delta_{13}, F_7 \rightarrow F_8 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_7, F_{14} \quad h_1 : \Delta_{13}, F_8 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \rightarrow F_8 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), F_{14}} \ \rightarrow_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \rightarrow F_8 \vdash}{\bullet h_9 : (\Delta_{13}, F_7 \rightarrow F_8), F_{14} \vdash \Delta_{12}, F}}{\dfrac{- : \Delta_{13}, F_7 \rightarrow F_8 \vdash \Delta_{12}, F_{10} \rightarrow F_{11}}{\rightarrow}}$$

$$\frac{\dfrac{h_1 : \Delta_{13}, F_7 \rightarrow F_8 \vdash \Delta_{12}, F_{14}, F_7, F_{10} \rightarrow F_{11}}{\text{ax/W}} \quad \dfrac{\bullet h_9 : \Delta_{13}, F_{14}, F_7 \rightarrow F_8 \vdash \Delta_{12}, F_7, F_{10} \rightarrow F_{11}}{\text{ax/W}}}{\dfrac{- : \Delta_{13}, F_7 \rightarrow F_8 \vdash \Delta_{12}, F_7, F_{10} \rightarrow F_{11}}{\text{hCut}}} \qquad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{14}, F_{10} \rightarrow F_{11}}{\text{ax/W}} \quad \dfrac{\bullet h_9}{- : \Delta_{13}, F_8 \vdash \Delta_{12}, F_1}$$

$$- : \Delta_{13}, F_7 \rightarrow F_8 \vdash \Delta_{12}, F_{10} \rightarrow F_{11}$$

- Case rule $\wedge_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_7, F_{14} \quad h_1 : \Delta_{13}, F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \to F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \to F_8 \vdash \Delta_{12},}{\bullet h_9 : (\Delta_{13}, F_7 \to}}{{-} : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \to$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{14}, F_7, F_{10} \wedge F_{11}}{} \text{ax/W} \quad \dfrac{\bullet h_9 : \Delta_{13}, F_{14}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \wedge F_{11}}{-: \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \wedge F_{11}} \text{hCut} \quad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{14}, F_{10} \wedge F_{11}}{} \text{ax/W} \quad \dfrac{h_9 : \Delta}{-: \Delta_{13}, F_8 \vdash \Delta}}{-: \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}$$

- Case rule $\vee_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_7, F_{14} \quad h_1 : \Delta_{13}, F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \to F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{\bullet h_9 : (\Delta_{13}, F_7 \to F_8), F_{14} \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_R}{{-} : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \to \text{Cut}$$

$$\dfrac{\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}, F_7}{} \text{inv-th/ax} \quad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}}{} \text{inv-th/ax}}{\dfrac{\bullet h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}}{-: \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{} \text{ax/W}}{\dfrac{-: \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{-: \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_R} \text{hCut}$$

- Case rule $\perp_R$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \to F_8 \vdash (\perp, \Delta_{10}), F_7, F_{12} \quad h_1 : \Delta_{11}, F_8 \vdash (\perp, \Delta_{10}), F_{12}}{\bullet h_1 : \Delta_{11}, F_7 \to F_8 \vdash (\perp, \Delta_{10}), F_{12}} \to_L \quad \dfrac{h_9 : \Delta_{11}, F_{12}, F_7 \to F_8 \vdash \Delta_{10}}{\bullet h_9 : (\Delta_{11}, F_7 \to F_8), F_{12} \vdash \perp, \Delta_{10}} \perp_R}{{-} : \Delta_{11}, F_7 \to F_8 \vdash \perp, \Delta_{10}} \to \text{Cut}$$

$$\dfrac{\dfrac{\bullet h_1 : \Delta_{11}, F_7 \to F_8 \vdash \perp, \Delta_{10}, F_{12}}{} \text{ax/W} \quad \dfrac{h_9 : \Delta_{11}, F_{12}, F_7 \to F_8 \vdash \perp, \Delta_{10}}{} \text{ax/W}}{{-} : \Delta_{11}, F_7 \to F_8 \vdash \perp, \Delta_{10}} \text{hCut}$$

- Case rule $\top_R$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \to F_8 \vdash (\top, \Delta_{10}), F_7, F_{12} \quad h_1 : \Delta_{11}, F_8 \vdash (\top, \Delta_{10}), F_{12}}{\bullet h_1 : \Delta_{11}, F_7 \to F_8 \vdash (\top, \Delta_{10}), F_{12}} \to_L \quad \dfrac{}{\bullet h_9 : (\Delta_{11}, F_7 \to F_8), F_{12} \vdash \top, \Delta_{10}} \top_R}{{-} : \Delta_{11}, F_7 \to F_8 \vdash \top, \Delta_{10}} \to \text{Cut}$$

$$\dfrac{}{{-} : \Delta_{11}, F_7 \to F_8 \vdash \top, \Delta_{10}} \top_R$$

- Case rule $\to_L$

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{14} \quad h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7 \to F_8 \vdash \Delta_{12}, F_{14}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \to F_8, F_{10} \to}{\bullet h_9 : ((\Delta_{13}}}{{-} : (\Delta_{13}, F_{10} \to F_{11}), F_7 \to F_8 \vdash \Delta_{12}} \to$$

$$\dfrac{\dfrac{\bullet h_1 : \Delta_{13}, F_{10} \to F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{14}}{} \text{ax/W} \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}}{} \text{ax/W}}{{-} : \Delta_{13}, F_{10} \to F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}} \text{hCut} \quad \dfrac{\dfrac{h_1 : \Delta_{13}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{14}, F_7}{} \text{inv-t}}{\bullet h_1 : \Delta_{13}, F_{11}, F_7 \to}$$

$${-} : \Delta_{13}, F_{10} \to F_{11}, F_7 \to F_8 \vdash \Delta_{12}$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \to F_{11} \quad h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_7 \to F_8, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}}{\bullet h_9 : (\Delta_{13}, F_7 \to F_8), F_{10}}}{{-} : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}} \to$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \to F_{11}}{} \text{ax/W} \quad \dfrac{\bullet h_9 : \Delta_{13}, F_{10} \to F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_7}{-: \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7} \text{hCut} \quad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{} \text{ax/W} \quad \dfrac{h_9 : \Delta_{13}, F_8, F_{10}}{-: \Delta_{13}, F_8 \vdash \Delta_{12}}}{{-} : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}}$$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_8 \to F_9 \vdash \Delta_{10}, F_8, F_{12} \quad h_1 : \Delta_{11}, F_9 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : \Delta_{11}, F_8 \to F_9 \vdash \Delta_{10}, F_{12}} \to_L \quad \dfrac{h_7 : \Delta_{11}, F_{12}, F_8 \to F_9 \vdash \Delta_{10}, F_8 \quad h_7 : \Delta_{11}, F_9, F_{12} \vdash \Delta_{10}}{\bullet h_7 : (\Delta_{11}, F_8 \to F_9), F_{12} \vdash \Delta_{10}} \to_L}{- : \Delta_{11}, F_8 \to F_9 \vdash \Delta_{10}} \;\text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_8 \to F_9 \vdash \Delta_{10}, F_{12}, F_8}{- : \Delta_{11}, F_8 \to F_9 \vdash \Delta_{10}, F_8}\;\text{ax/W} \quad \dfrac{\bullet h_7 : \Delta_{11}, F_{12}, F_8 \to F_9 \vdash \Delta_{10}, F_8}{}\;\text{ax/W}}{\vphantom{x}}\;\text{hCut} \quad \dfrac{h_1 : \Delta_{11}, F_9 \vdash \Delta_{10}, F_{12}}{}\;\text{ax/W} \quad \dfrac{\dfrac{h_7 : \Delta_{11}, F_{12}, F_9 \vdash \Delta_{10}}{\bullet h_7 : \Delta_{11}, F_{12}, F_9 \vdash \Delta_{10}}\;\text{ax/W}}{- : \Delta_{11}, F_9 \vdash \Delta_{10}}\;H$$
$$- : \Delta_{11}, F_8 \to F_9 \vdash \Delta_{10}$$

- **Case rule $\wedge_L$**

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{14} \quad h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \to F_8 \vdash \Delta_{12}, F_{14}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14}, F_7 \to F_8 \vdash \Delta_{12}}{\bullet h_9 : ((\Delta_{13}, F_{10} \wedge F_{11}), F_7 \to F_8), F_{14} \vdash \Delta_{12}} \wedge_L}{- : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \to F_8 \vdash \Delta_{12}}\;\text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\dfrac{h_1 : \Delta_{13}, F_{10}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_1 : \Delta_{13}, F_{10}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{14}}\;\text{inv-th/ax} \quad \dfrac{h_1 : \Delta_{13}, F_{10}, F_{11}, F_8 \vdash \Delta_{12}, F_{14}}{}\;\text{inv-th/ax}}{- : \Delta_{13}, F_{10}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}}\to_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14}, F_7 \to F_8 \vdash \Delta_{12}}{}\;\text{ax/W}}{- : \Delta_{13}, F_7 \to F_8, F_{10} \wedge F_{11} \vdash \Delta_{12}}\wedge_L$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \wedge F_{11} \quad h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \to F_8), F_{10} \wedge F_{11} \vdash \Delta_{12}} \wedge_L}{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}}\;\text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \wedge F_{11}}{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7}\;\text{ax/W} \quad \dfrac{\bullet h_9 : \Delta_{13}, F_7 \to F_8, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7}{}\;\text{ax/W}}{\vphantom{x}}\;\text{hCut} \quad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}{}\;\text{ax/W} \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11},}{\bullet h_9 : \Delta_{13}, F_8, F_{10} \wedge}$$
$$- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7 \qquad\qquad - : \Delta_{13}, F_8 \vdash \Delta_{12}$$
$$- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}$$

- **Case rule $\vee_L$**

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{14} \quad h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \to F_8 \vdash \Delta_{12}, F_{14}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \to F_8 \vdash \Delta}{\bullet h_9 : ((\Delta_{13}, F_{10} \vee}}{- : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \to F_8 \vdash \Delta_{12}}$$

$$\to$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_{14}, F_7}{- : \Delta_{13}, F_7 \to F_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7}\;\text{ax/W} \quad \dfrac{\bullet h_9 : \Delta_{13}, F_{14}, F_7 \to F_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7}{}\;\text{ax/W}}{\vphantom{x}}\;\text{hCut} \quad \dfrac{h_1 : \Delta_{13}, F_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_{14}}{}\;\text{ax/W} \quad \dfrac{h_9 : \Delta}{}$$
$$- : \Delta_{13}, F_7 \to F_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7 \qquad\qquad - : \Delta_{13}, F_8, F_{10}$$
$$- : \Delta_{13}, F_7 \to F_8, F_{10} \vee F_{11} \vdash \Delta_{12}$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_7, F_{10} \vee F_{11} \quad h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \to_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_7 \to F_8 \vdash \Delta_{12} \quad h_9 : \Delta_{13}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \to F_8), F_{10} \vee F_{11} \vdash \Delta_{12}}}{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}}\;\text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\dfrac{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_7}{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}}\;\text{inv-th/ax} \quad \dfrac{- : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{}\;\text{inv-th/ax}}{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}, F_{11}}\to_L \quad \dfrac{- : \Delta_{13}, F_{11}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}}{}\;\text{ax/W} \quad \dfrac{- : \Delta_{13}, F_{10}, F_7 \to}{}}{- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}, F_{10}}\;\text{sCut}$$
$$- : \Delta_{13}, F_7 \to F_8 \vdash \Delta_{12}$$

- **Case rule $\perp_L$**

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_7, \perp \quad h_1 : \Delta_{11}, F_8 \vdash \Delta_{10}, \perp}{\bullet h_1 : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, \perp} \to_L \quad \dfrac{\bullet h_9 : (\Delta_{11}, F_7 \to F_8), \perp \vdash \Delta_{10}}{}\;\perp_L}{- : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}}\;\text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \to F_8 \vdash \perp, \Delta_{10}, F_7}{- : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_7}\;\text{ax/W} \quad \dfrac{\bullet h_9 : \perp, \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_7}{}\;\text{ax/W}}{\vphantom{x}}\;\text{hCut} \quad \dfrac{h_1 : \Delta_{11}, F_8 \vdash \perp, \Delta_{10}}{- : \Delta_{11}, F_8 \vdash \Delta_{10}}\;\text{ax/W} \quad \dfrac{\bullet h_9 : \perp, \Delta_{11}, F_8 \vdash \Delta_{10}}{}\;\perp_L$$
$$- : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_7 \qquad\qquad - : \Delta_{11}, F_8 \vdash \Delta_{10}$$
$$- : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}$$

$$\dfrac{\dfrac{h_1 : (\bot, \Delta_{11}), F_7 \to F_8 \vdash \Delta_{10}, F_7, F_{12} \quad h_1 : (\bot, \Delta_{11}), F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : (\bot, \Delta_{11}), F_7 \to F_8 \vdash \Delta_{10}, F_{12}} \to_L \quad \dfrac{}{\bullet h_9 : ((\bot, \Delta_{11}), F_7 \to F_8), F_{12} \vdash \Delta_{10}} \bot_L}{\dfrac{-\ :\ (\bot, \Delta_{11}), F_7 \to F_8 \vdash \Delta_{10}}{\dfrac{\to}{-\ :\ \bot, \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}} \bot_L} \text{Cut}}$$

- Case rule I

$$\dfrac{\dfrac{h_1 : \Delta_{12}, F_7 \to F_8 \vdash (\Delta_{10}, p_{11}), F_7, p_{11} \quad h_1 : \Delta_{12}, F_8 \vdash (\Delta_{10}, p_{11}), p_{11}}{\bullet h_1 : \Delta_{12}, F_7 \to F_8 \vdash (\Delta_{10}, p_{11}), p_{11}} \to_L \quad \dfrac{}{\bullet h_9 : (\Delta_{12}, F_7 \to F_8), p_{11} \vdash \Delta_{10}, p_{11}} I}{\dfrac{-\ :\ \Delta_{12}, F_7 \to F_8 \vdash \Delta_{10}, p_{11}}{\to} \text{Cut}}$$

$$\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{12}, F_7 \to F_8 \vdash \Delta_{10}, F_7, p_{11}, p_{11}} \text{ax/W} \quad \dfrac{}{\bullet h_9 : \Delta_{12}, p_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_7, p_{11}} I}{-\ :\ \Delta_{12}, F_7 \to F_8 \vdash \Delta_{10}, F_7, p_{11}} \text{hCut} \quad \dfrac{\dfrac{}{h_1 : \Delta_{12}, F_8 \vdash \Delta_{10}, p_{11}, p_{11}} \text{ax/W} \quad \dfrac{}{\bullet h_9 : \Delta_{12}, F_8, p_{11} \vdash \Delta_{10}}}{-\ :\ \Delta_{12}, F_8 \vdash \Delta_{10}, p_{11}}}{-\ :\ \Delta_{12}, F_7 \to F_8 \vdash \Delta_{10}, p_{11}} \to_L$$

$$\dfrac{\dfrac{h_1 : (\Delta_{12}, p_{11}), F_7 \to F_8 \vdash (\Delta_{10}, p_{11}), F_7, F_{13} \quad h_1 : (\Delta_{12}, p_{11}), F_8 \vdash (\Delta_{10}, p_{11}), F_{13}}{\bullet h_1 : (\Delta_{12}, p_{11}), F_7 \to F_8 \vdash (\Delta_{10}, p_{11}), F_{13}} \to_L \quad \dfrac{}{\bullet h_9 : ((\Delta_{12}, p_{11}), F_7 \to F_8), F_{13} \vdash \Delta_{10}, p_{11}} I}{\dfrac{-\ :\ (\Delta_{12}, p_{11}), F_7 \to F_8 \vdash \Delta_{10}, p_{11}}{\dfrac{\to}{-\ :\ \Delta_{12}, p_{11}, F_7 \to F_8 \vdash \Delta_{10}, p_{11}} I} \text{Cut}}$$

- Case rule $\top_L$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_7, \top \quad h_1 : \Delta_{11}, F_8 \vdash \Delta_{10}, \top}{\bullet h_1 : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, \top} \to_L \quad \dfrac{h_9 : \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}}{\bullet h_9 : (\Delta_{11}, F_7 \to F_8), \top \vdash \Delta_{10}} \top_L}{\dfrac{-\ :\ \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}}{\dfrac{\to}{-\ :\ \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}} \text{ax/W}} \text{Cut}}$$

$$\dfrac{\dfrac{h_1 : (\top, \Delta_{11}), F_7 \to F_8 \vdash \Delta_{10}, F_7, F_{12} \quad h_1 : (\top, \Delta_{11}), F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : (\top, \Delta_{11}), F_7 \to F_8 \vdash \Delta_{10}, F_{12}} \to_L \quad \dfrac{h_9 : \Delta_{11}, F_{12}, F_7 \to F_8 \vdash \Delta_{10}}{\bullet h_9 : ((\top, \Delta_{11}), F_7 \to F_8), F_{12} \vdash \Delta_{10}} \top_L}{\dfrac{-\ :\ (\top, \Delta_{11}), F_7 \to F_8 \vdash \Delta_{10}}{\dfrac{\to}{\dfrac{\dfrac{}{\bullet h_1 : \top, \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W} \quad \dfrac{}{h_9 : \top, \Delta_{11}, F_{12}, F_7 \to F_8 \vdash \Delta_{10}} \text{ax/W}}{-\ :\ \top, \Delta_{11}, F_7 \to F_8 \vdash \Delta_{10}} \text{hCut}}} \text{Cut}}$$

8.7   Status of $\wedge_L$: `OK`

- Case rule $\to_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash (\Delta_{12}, F_{10} \to F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash (\Delta_{12}, F_{10} \to F_{11}), F_{14}} \wedge_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \wedge F_8 \vdash F_{11}}{\bullet h_9 : (\Delta_{13}, F_7 \wedge F_8), F_{14} \vdash \Delta_{12}, F_{10} \to F_{11}} \to_R}{\dfrac{-\ :\ \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{\to}} \text{Cut}$$

$$\dfrac{\dfrac{\dfrac{}{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{14}, F_{10} \to F_{11}} \text{ax/W} \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7, F_8 \vdash F_{11}}{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7, F_8 \vdash F_{11}} \text{inv-th/ax}}{\bullet h_9 : \Delta_{13}, F_{14}, F_7, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \to_R}{\dfrac{-\ :\ \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{-\ :\ \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \wedge_L} \text{hCut}$$

- Case rule $\wedge_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14}} \wedge_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \quad h_9 : \Delta_{13}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{11}}{\bullet h_9 : (\Delta_{13}, F_7 \wedge F_8), F_{14} \vdash \Delta_{12}, F_{10} \wedge F_{11}} \wedge_R}{\dfrac{-\ :\ \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}{\to}} \text{Cut}$$

$$\dfrac{\dfrac{}{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{14}, F_{10} \wedge F_{11}} \text{ax/W} \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_{14}, F_7, F_8 \vdash \Delta_{12}, F_{10}}{h_9 : \Delta_{13}, F_{14}, F_7, F_8 \vdash \Delta_{12}, F_{10}} \text{inv-th/ax} \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7, F_8 \vdash \Delta_{12}, F_{11}}{h_9 : \Delta_{13}, F_{14}, F_7, F_8 \vdash \Delta_{12}, F_{11}} \text{inv-th/ax}}{\bullet h_9 : \Delta_{13}, F_{14}, F_7, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \wedge_R}{\dfrac{\dfrac{-\ :\ \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}{-\ :\ \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \wedge_L}{} \text{hCut}}$$

- Case rule $\vee_R$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14}} \wedge_L
\quad
\cfrac{h_9 : \Delta_{13}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{\bullet h_9 : (\Delta_{13}, F_7 \wedge F_8), F_{14} \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_R
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \text{Cut}
}{\to}
$$

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}} \text{inv-th/ax}}{} \wedge_L
\quad
\cfrac{h_9 : \Delta_{13}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{} \text{ax/W}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10}, F_{11}} \text{hCut}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_R
$$

- Case rule $\bot_R$

$$
\cfrac{
\cfrac{h_1 : \Delta_{11}, F_7, F_8 \vdash (\bot, \Delta_{10}), F_{12}}{\bullet h_1 : \Delta_{11}, F_7 \wedge F_8 \vdash (\bot, \Delta_{10}), F_{12}} \wedge_L
\quad
\cfrac{h_9 : \Delta_{11}, F_{12}, F_7 \wedge F_8 \vdash \Delta_{10}}{\bullet h_9 : (\Delta_{11}, F_7 \wedge F_8), F_{12} \vdash \bot, \Delta_{10}} \bot_R
}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \bot, \Delta_{10}} \text{Cut}
$$

$$
\to
$$

$$
\cfrac{
\cfrac{\bullet h_1 : \Delta_{11}, F_7 \wedge F_8 \vdash \bot, \Delta_{10}, F_{12}}{} \text{ax/W}
\quad
\cfrac{h_9 : \Delta_{11}, F_{12}, F_7 \wedge F_8 \vdash \bot, \Delta_{10}}{} \text{ax/W}
}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \bot, \Delta_{10}} \text{hCut}
$$

- Case rule $\top_R$

$$
\cfrac{
\cfrac{h_1 : \Delta_{11}, F_7, F_8 \vdash (\top, \Delta_{10}), F_{12}}{\bullet h_1 : \Delta_{11}, F_7 \wedge F_8 \vdash (\top, \Delta_{10}), F_{12}} \wedge_L
\quad
\cfrac{}{\bullet h_9 : (\Delta_{11}, F_7 \wedge F_8), F_{12} \vdash \top, \Delta_{10}} \top_R
}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \top, \Delta_{10}} \text{Cut}
$$

$$
\to
$$

$$
\cfrac{}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \top, \Delta_{10}} \top_R
$$

- Case rule $\to_L$

$$
\cfrac{
\cfrac{h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7, F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7 \wedge F_8 \vdash \Delta_{12}, F_{14}} \wedge_L
\quad
\cfrac{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \quad h_9 : \Delta_{13}, F_{11}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}}{\bullet h_9 : ((\Delta_{13}, F_{10} \to F_{11}), F_7 \wedge F_8), F_{14} \vdash \Delta_{12}} \to_L
}{- : (\Delta_{13}, F_{10} \to F_{11}), F_7 \wedge F_8 \vdash \Delta_{12}} \text{Cut}
$$

$$
\to
$$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_{13}, F_7, F_8, F_{10} \to F_{11} \vdash \Delta_{12}, F_{14}}{} \text{ax/W}
\quad
\cfrac{
\cfrac{h_9 : \Delta_{13}, F_{14}, F_7, F_8, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}}{\bullet h_9 : \Delta_{13}, F_{14}, F_7, F_8, F_{10} \to F_{11} \vdash \Delta_{12}} \text{inv-th/ax}
\quad
\cfrac{h_9 : \Delta_{13}, F_{11}, F_{14}, F_7, F_8 \vdash \Delta_{12}}{} \text{inv-th/ax}
}{} \to_L
}{- : \Delta_{13}, F_7, F_8, F_{10} \to F_{11} \vdash \Delta_{12}} \text{hCut}
}{- : \Delta_{13}, F_{10} \to F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}} \wedge_L
$$

$$
\cfrac{
\cfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \wedge_L
\quad
\cfrac{h_9 : \Delta_{13}, F_{10} \to F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \quad h_9 : \Delta_{13}, F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \wedge F_8), F_{10} \to F_{11} \vdash \Delta_{12}} \to_L
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}} \text{Cut}
$$

$$
\to
$$

$$
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{} \text{ax/W}
\quad
\cfrac{
\cfrac{h_9 : \Delta_{13}, F_7, F_8, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}}{\bullet h_9 : \Delta_{13}, F_7, F_8, F_{10} \to F_{11} \vdash \Delta_{12}} \text{inv-th/ax}
\quad
\cfrac{h_9 : \Delta_{13}, F_{11}, F_7, F_8 \vdash \Delta_{12}}{} \text{inv-th/ax}
}{} \to_L
}{- : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}} \text{hCut}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}} \wedge_L
$$

- Case rule $\wedge_L$

$$
\cfrac{
\cfrac{h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7, F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8 \vdash \Delta_{12}, F_{14}} \wedge_L
\quad
\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}}{\bullet h_9 : ((\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8), F_{14} \vdash \Delta_{12}} \wedge_L
}{- : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8 \vdash \Delta_{12}} \text{Cut}
$$

$$
\to
$$

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{h_1 : \Delta_{13}, F_{10}, F_{11}, F_7, F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : \Delta_{13}, F_{10}, F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{14}} \text{inv-th/ax}}{} \wedge_L
\quad
\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}}{} \text{ax/W}
}{- : \Delta_{13}, F_{10}, F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}} \text{hCut}
}{- : \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}} \wedge_L
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}\ {\wedge_L}
  \quad
  \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \wedge F_8), F_{10} \wedge F_{11} \vdash \Delta_{12}}\ {\wedge_L}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}}\ \text{Cut}
$$

$$\rightarrow$$

$$
\cfrac{
  \cfrac{
    \cfrac{}{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}\ \texttt{ax/W}
    \quad
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_7, F_8 \vdash \Delta_{12}}{\bullet h_9 : \Delta_{13}, F_7, F_8, F_{10} \wedge F_{11} \vdash \Delta_{12}}\ \texttt{inv-th/ax}}{}\ {\wedge_L}
  }{- : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}}\ \text{hCut}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}}\ {\wedge_L}
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{11}, F_8, F_9 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : \Delta_{11}, F_8 \wedge F_9 \vdash \Delta_{10}, F_{12}}\ {\wedge_L}
  \quad
  \cfrac{h_7 : \Delta_{11}, F_8, F_9, F_{12} \vdash \Delta_{10}}{\bullet h_7 : (\Delta_{11}, F_8 \wedge F_9), F_{12} \vdash \Delta_{10}}\ {\wedge_L}
}{- : \Delta_{11}, F_8 \wedge F_9 \vdash \Delta_{10}}\ \text{Cut}
$$

$$\rightarrow$$

$$
\cfrac{
  \cfrac{
    \cfrac{}{h_1 : \Delta_{11}, F_8, F_9 \vdash \Delta_{10}, F_{12}}\ \texttt{ax/W}
    \quad
    \cfrac{\cfrac{h_7 : \Delta_{11}, F_{12}, F_8, F_9 \vdash \Delta_{10}}{\bullet h_7 : \Delta_{11}, F_{12}, F_8, F_9 \vdash \Delta_{10}}\ \texttt{ax/W}}{}\ H
  }{- : \Delta_{11}, F_8, F_9 \vdash \Delta_{10}}\ \text{hCut}
}{- : \Delta_{11}, F_8 \wedge F_9 \vdash \Delta_{10}}\ {\wedge_L}
$$

- Case rule $\vee_L$

$$
\cfrac{
  \cfrac{h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7, F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \wedge F_8 \vdash \Delta_{12}, F_{14}}\ {\wedge_L}
  \quad
  \cfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12} \quad h_9 : \Delta_{13}, F_{11}, F_{14}, F_7 \wedge F_8 \vdash \Delta_{12}}{\bullet h_9 : ((\Delta_{13}, F_{10} \vee F_{11}), F_7 \wedge F_8), F_{14} \vdash \Delta_{12}}\ {\vee_L}
}{- : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \wedge F_8 \vdash \Delta_{12}}\ \text{Cut}
$$

$$\rightarrow$$

$$
\cfrac{
  \cfrac{
    \cfrac{}{h_1 : \Delta_{13}, F_7, F_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_{14}}\ \texttt{ax/W}
    \quad
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7, F_8 \vdash \Delta_{12}}{}\ \texttt{inv-th/ax} \quad \cfrac{h_9 : \Delta_{13}, F_{11}, F_{14}, F_7, F_8 \vdash \Delta_{12}}{}\ \texttt{inv-th/ax}}{\bullet h_9 : \Delta_{13}, F_{14}, F_7, F_8, F_{10} \vee F_{11} \vdash \Delta_{12}}\ {\vee_L}
  }{- : \Delta_{13}, F_7, F_8, F_{10} \vee F_{11} \vdash \Delta_{12}}\ \text{hCut}
}{- : \Delta_{13}, F_7 \wedge F_8, F_{10} \vee F_{11} \vdash \Delta_{12}}\ {\wedge_L}
$$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}}\ {\wedge_L}
  \quad
  \cfrac{h_9 : \Delta_{13}, F_{10}, F_7 \wedge F_8 \vdash \Delta_{12} \quad h_9 : \Delta_{13}, F_{11}, F_7 \wedge F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \wedge F_8), F_{10} \vee F_{11} \vdash \Delta_{12}}\ {\vee_L}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}}\ \text{Cut}
$$

$$\rightarrow$$

$$
\cfrac{
  \cfrac{
    \cfrac{}{h_1 : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}}\ \texttt{ax/W}
    \quad
    \cfrac{\cfrac{h_9 : \Delta_{13}, F_{10}, F_7, F_8 \vdash \Delta_{12}}{}\ \texttt{inv-th/ax} \quad \cfrac{h_9 : \Delta_{13}, F_{11}, F_7, F_8 \vdash \Delta_{12}}{}\ \texttt{inv-th/ax}}{\bullet h_9 : \Delta_{13}, F_7, F_8, F_{10} \vee F_{11} \vdash \Delta_{12}}\ {\vee_L}
  }{- : \Delta_{13}, F_7, F_8 \vdash \Delta_{12}}\ \text{hCut}
}{- : \Delta_{13}, F_7 \wedge F_8 \vdash \Delta_{12}}\ {\wedge_L}
$$

- Case rule $\bot_L$

$$
\cfrac{
  \cfrac{h_1 : \Delta_{11}, F_7, F_8 \vdash \Delta_{10}, \bot}{\bullet h_1 : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, \bot}\ {\wedge_L}
  \quad
  \cfrac{}{\bullet h_9 : (\Delta_{11}, F_7 \wedge F_8), \bot \vdash \Delta_{10}}\ {\bot_L}
}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}}\ \text{Cut}
$$

$$\rightarrow$$

$$
\cfrac{
  \cfrac{
    \cfrac{}{h_1 : \Delta_{11}, F_7, F_8 \vdash \bot, \Delta_{10}}\ \texttt{ax/W}
    \quad
    \cfrac{}{\bullet h_9 : \bot, \Delta_{11}, F_7, F_8 \vdash \Delta_{10}}\ {\bot_L}
  }{- : \Delta_{11}, F_7, F_8 \vdash \Delta_{10}}\ \text{hCut}
}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}}\ {\wedge_L}
$$

$$
\cfrac{
  \cfrac{h_1 : (\bot, \Delta_{11}), F_7, F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : (\bot, \Delta_{11}), F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}}\ {\wedge_L}
  \quad
  \cfrac{}{\bullet h_9 : ((\bot, \Delta_{11}), F_7 \wedge F_8), F_{12} \vdash \Delta_{10}}\ {\bot_L}
}{- : (\bot, \Delta_{11}), F_7 \wedge F_8 \vdash \Delta_{10}}\ \text{Cut}
$$

$$\rightarrow$$

$$
\cfrac{}{- : \bot, \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}}\ {\bot_L}
$$

- Case rule I

51

$$\frac{\dfrac{h_1 : \Delta_{12}, F_7, F_8 \vdash (\Delta_{10}, p_{11}), p_{11}}{\bullet h_1 : \Delta_{12}, F_7 \wedge F_8 \vdash (\Delta_{10}, p_{11}), p_{11}} \wedge_L \qquad \dfrac{}{\bullet h_9 : (\Delta_{12}, F_7 \wedge F_8), p_{11} \vdash \Delta_{10}, p_{11}} I}{- : \Delta_{12}, F_7 \wedge F_8 \vdash \Delta_{10}, p_{11}} \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\dfrac{}{h_1 : \Delta_{12}, F_7, F_8 \vdash \Delta_{10}, p_{11}, p_{11}} ax/W \qquad \dfrac{}{\bullet h_9 : \Delta_{12}, F_7, F_8, p_{11} \vdash \Delta_{10}, p_{11}} I}{- : \Delta_{12}, F_7, F_8 \vdash \Delta_{10}, p_{11}} \text{hCut}}{- : \Delta_{12}, F_7 \wedge F_8 \vdash \Delta_{10}, p_{11}} \wedge_L$$

$$\frac{\dfrac{h_1 : (\Delta_{12}, p_{11}), F_7, F_8 \vdash (\Delta_{10}, p_{11}), F_{13}}{\bullet h_1 : (\Delta_{12}, p_{11}), F_7 \wedge F_8 \vdash (\Delta_{10}, p_{11}), F_{13}} \wedge_L \qquad \dfrac{}{\bullet h_9 : ((\Delta_{12}, p_{11}), F_7 \wedge F_8), F_{13} \vdash \Delta_{10}, p_{11}} I}{- : (\Delta_{12}, p_{11}), F_7 \wedge F_8 \vdash \Delta_{10}, p_{11}} \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_{12}, p_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, p_{11}} I$$

- Case rule $\top_L$

$$\frac{\dfrac{h_1 : \Delta_{11}, F_7, F_8 \vdash \Delta_{10}, \top}{\bullet h_1 : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, \top} \wedge_L \qquad \dfrac{h_9 : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}}{\bullet h_9 : (\Delta_{11}, F_7 \wedge F_8), \top \vdash \Delta_{10}} \top_L}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}} \text{Cut}$$

$$\to$$

$$\frac{}{- : \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}} ax/W$$

$$\frac{\dfrac{h_1 : (\top, \Delta_{11}), F_7, F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : (\top, \Delta_{11}), F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}} \wedge_L \qquad \dfrac{h_9 : \Delta_{11}, F_{12}, F_7 \wedge F_8 \vdash \Delta_{10}}{\bullet h_9 : ((\top, \Delta_{11}), F_7 \wedge F_8), F_{12} \vdash \Delta_{10}} \top_L}{- : (\top, \Delta_{11}), F_7 \wedge F_8 \vdash \Delta_{10}} \text{Cut}$$

$$\to$$

$$\frac{\dfrac{}{\bullet h_1 : \top, \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}} ax/W \qquad \dfrac{}{h_9 : \top, \Delta_{11}, F_{12}, F_7 \wedge F_8 \vdash \Delta_{10}} ax/W}{- : \top, \Delta_{11}, F_7 \wedge F_8 \vdash \Delta_{10}} \text{hCut}$$

8.8   Status of $\vee_L$: OK

- Case rule $\to_R$

$$\frac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash (\Delta_{12}, F_{10} \to F_{11}), F_{14} \quad h_1 : \Delta_{13}, F_8 \vdash (\Delta_{12}, F_{10} \to F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash (\Delta_{12}, F_{10} \to F_{11}), F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \vee F_8 \vdash F_{11}}{\bullet h_9 : (\Delta_{13}, F_7 \vee F_8), F_{14} \vdash \Delta_{12}, F_{10} \to F_{11}} \to_R}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \text{Cut}$$

$$\to$$

$$\frac{\dfrac{\dfrac{}{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{14}, F_{10} \to F_{11}} ax/W \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \vdash F_{11}}{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \vdash F_{11}} inv\text{-}th/ax}{\bullet h_9 : \Delta_{13}, F_{14}, F_7 \vdash \Delta_{12}, F_{10} \to F_{11}} \to_R}{- : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10} \to F_{11}} \text{hCut} \quad \dfrac{\dfrac{}{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{14}, F_{10} \to F_{11}} ax/W \quad \dfrac{}{\bullet h_9 : \Delta_{13}, F_{14}, F_8 \vdash \varDelta}{}}{- : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \vee_L$$

- Case rule $\wedge_R$

$$\frac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14} \quad h_1 : \Delta_{13}, F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \vee F_8 \vdash \varDelta}{\bullet h_9 : (\Delta_{13}, F}{}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} $$

$$\to$$

$$\frac{\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10}, F_{14}}{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10}, F_{14}} inv\text{-}th/ax \quad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10}, F_{14}}{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10}, F_{14}} inv\text{-}th/ax}{\dfrac{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{14}}{\vee_L} \quad \dfrac{}{h_9 : \Delta_{13}, F_{14}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}} ax/W}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}} \text{hCut} \quad \dfrac{}{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{11}, F_{14}} \dfrac{}{\bullet h_1 : \Delta_{13}, }{}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}$$

- Case rule $\vee_R$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14} \quad h_1 : \Delta_{13}, F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{\bullet h_9 : (\Delta_{13}, F_7 \vee F_8), F_{14} \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_R}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \text{Cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}}{} \text{inv-th/ax} \quad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}}{} \text{inv-th/ax}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{11}, F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{} \text{ax/W}}{\dfrac{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{11}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_R} \text{hCut}$$

- Case rule $\bot_R$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \vdash (\bot, \Delta_{10}), F_{12} \quad h_1 : \Delta_{11}, F_8 \vdash (\bot, \Delta_{10}), F_{12}}{\bullet h_1 : \Delta_{11}, F_7 \vee F_8 \vdash (\bot, \Delta_{10}), F_{12}} \vee_L \quad \dfrac{h_9 : \Delta_{11}, F_{12}, F_7 \vee F_8 \vdash \Delta_{10}}{\bullet h_9 : (\Delta_{11}, F_7 \vee F_8), F_{12} \vdash \bot, \Delta_{10}} \bot_R}{- : \Delta_{11}, F_7 \vee F_8 \vdash \bot, \Delta_{10}} \text{Cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{}{\bullet h_1 : \Delta_{11}, F_7 \vee F_8 \vdash \bot, \Delta_{10}, F_{12}} \text{ax/W} \quad \dfrac{h_9 : \Delta_{11}, F_{12}, F_7 \vee F_8 \vdash \bot, \Delta_{10}}{} \text{ax/W}}{- : \Delta_{11}, F_7 \vee F_8 \vdash \bot, \Delta_{10}} \text{hCut}$$

- Case rule $\top_R$

$$\dfrac{\dfrac{h_1 : \Delta_{11}, F_7 \vdash (\top, \Delta_{10}), F_{12} \quad h_1 : \Delta_{11}, F_8 \vdash (\top, \Delta_{10}), F_{12}}{\bullet h_1 : \Delta_{11}, F_7 \vee F_8 \vdash (\top, \Delta_{10}), F_{12}} \vee_L \quad \dfrac{}{\bullet h_9 : (\Delta_{11}, F_7 \vee F_8), F_{12} \vdash \top, \Delta_{10}} \top_R}{- : \Delta_{11}, F_7 \vee F_8 \vdash \top, \Delta_{10}} \text{Cut}$$

$$\downarrow$$

$$\dfrac{}{- : \Delta_{11}, F_7 \vee F_8 \vdash \top, \Delta_{10}} \top_R$$

- Case rule $\to_L$

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7 \vdash \Delta_{12}, F_{14} \quad h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \to F_{11}), F_7 \vee F_8 \vdash \Delta_{12}, F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11}, F_7 \vee F_8 \vdash \Delta_{12}}{\bullet h_9 : ((\Delta_{13}, F_{10} \to F_{11}))} }{- : (\Delta_{13}, F_{10} \to F_{11}), F_7 \vee F_8 \vdash \Delta_{12}}$$

$$\downarrow$$

$$\dfrac{\dfrac{}{\bullet h_1 : \Delta_{13}, F_{10} \to F_{11}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{14}} \text{ax/W} \quad \dfrac{h_9 : \Delta_{13}, F_{14}, F_{10} \to F_{11}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}}{} \text{ax/W}}{\dfrac{- : \Delta_{13}, F_{10} \to F_{11}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}}{- : \Delta_{13}, F_{10} \to F_{11}, F_7 \vee F_8 \vdash \Delta_{12}}} \text{hCut} \qquad \dfrac{\dfrac{h_1 : \Delta_{13}, F_{11}, F_7 \vdash \Delta_{12}, F_{14}}{} \text{inv-th/ax} \quad \dfrac{h_1 : \Delta_{13}, F_{11}, F_7 \vdash \Delta_{12}}{}}{}$$

$$\dfrac{\dfrac{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10} \to F_{11} \quad h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \to F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \to F_{11}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{10} \to F_{11}, F_7 \vee F_8 \vdash \Delta}{\bullet h_9 : (\Delta_{13}, F_7 \vee }}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}}$$

$$\downarrow$$

$$\dfrac{\dfrac{}{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10} \to F_{11}} \text{ax/W} \quad \dfrac{\dfrac{h_9 : \Delta_{13}, F_7, F_{10} \to F_{11} \vdash \Delta_{12}, F_{10}}{} \text{inv-th/ax} \quad \dfrac{h_9 : \Delta_{13}, F_{11}, F_7 \vdash \Delta_{12}}{} \text{inv-th/ax}}{\dfrac{\bullet h_9 : \Delta_{13}, F_7, F_{10} \to F_{11} \vdash \Delta_{12}}{} \to_L}}{\dfrac{- : \Delta_{13}, F_7 \vdash \Delta_{12}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}}} \text{hCut} \qquad \dfrac{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \to F}{}$$

- Case rule $\wedge_L$

$$\dfrac{\dfrac{h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \vdash \Delta_{12}, F_{14} \quad h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \vee F_8 \vdash \Delta_{12}, F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14}, F_7 \vee F_8 \vdash \Delta_{12}}{\bullet h_9 : ((\Delta_{13}, F_{10} \wedge F_{11}), F_7 \vee F_8), F_{14} \vdash \Delta_{12}} \wedge_L}{- : (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \vee F_8 \vdash \Delta_{12}} \text{Cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{h_1 : \Delta_{13}, F_{10}, F_{11}, F_7 \vdash \Delta_{12}, F_{14}}{} \text{inv-th/ax} \quad \dfrac{h_1 : \Delta_{13}, F_{10}, F_{11}, F_8 \vdash \Delta_{12}, F_{14}}{} \text{inv-th/ax}}{\bullet h_1 : \Delta_{13}, F_{10}, F_{11}, F_7 \vee F_8 \vdash \Delta_{12}, F_{14}} \vee_L \quad \dfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_{14}, F_7 \vee F_8 \vdash \Delta_{12}}{} \text{ax/W}}{\dfrac{- : \Delta_{13}, F_{10}, F_{11}, F_7 \vee F_8 \vdash \Delta_{12}}{- : \Delta_{13}, F_{10} \wedge F_{11}, F_7 \vee F_8 \vdash \Delta_{12}} \wedge_L} \text{hCut}$$

$$\cfrac{\cfrac{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10} \wedge F_{11} \quad h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \vee_L \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_7 \vee F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \vee F_8), F_{10} \wedge F_{11} \vdash \Delta_{12}} \wedge_L}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}} \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \text{ax/W} \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_7 \vdash \Delta_{12}}{\bullet h_9 : \Delta_{13}, F_7, F_{10} \wedge F_{11} \vdash \Delta_{12}} \begin{array}{l}\text{inv-th/ax}\\ \wedge_L\end{array}}{- : \Delta_{13}, F_7 \vdash \Delta_{12}} \text{hCut} \quad \cfrac{\cfrac{}{h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \wedge F_{11}} \text{ax/W} \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{11}, F_8 \vdash \Delta_{12}}{\bullet h_9 : \Delta_{13}, F_8, F_{10} \wedge F_{11} \vdash \Delta_{12}} \begin{array}{l}\text{inv-}\\ \wedge_L\end{array}}{- : \Delta_{13}, F_8 \vdash \Delta_{12}} \text{hC}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}} \vee_L$$

- Case rule $\vee_L$

$$\cfrac{\cfrac{h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vdash \Delta_{12}, F_{14} \quad h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_8 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vee F_8 \vdash \Delta_{12}, F_{14}} \vee_L \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \vee F_8}{\bullet h_9 : ((\Delta_{13}, F}{- : (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vee F_8 \vdash \Delta_{12}}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{13}, F_{10}, F_7 \vdash \Delta_{12}, F_{14}} \text{inv-th/ax} \quad \cfrac{}{h_1 : \Delta_{13}, F_{10}, F_8 \vdash \Delta_{12}, F_{14}} \begin{array}{l}\text{inv-th/ax}\\ \vee_L\end{array}}{\bullet h_1 : \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash \Delta_{12}, F_{14}} \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_{14}, F_7 \vee F_8 \vdash \Delta_{12}}{} \text{ax/W} \quad \cfrac{h_1 : \Delta_{13}, F_{11}, F_7 \vdash \Delta_{12}, F_{14}}{\bullet h_1 : \Delta_{13},}}{- : \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash \Delta_{12}} \text{hCut}$$

$$- : \Delta_{13}, F_{10} \vee F_{11}, F_7 \vee F_8 \vdash \Delta_{12}$$

$$\cfrac{\cfrac{h_1 : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10} \vee F_{11} \quad h_1 : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}}{\bullet h_1 : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10} \vee F_{11}} \vee_L \quad \cfrac{h_9 : \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash \Delta_{12} \quad h_9 : \Delta_{13}, F_{11}, F_7 \vee F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_{13}, F_7 \vee F_8), F_{10} \vee F_{11} \vdash \Delta_{12}} \vee_L}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}} \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{- : \Delta_{13}, F_7 \vdash \Delta_{12}, F_{10}, F_{11}} \text{inv-th/ax} \quad \cfrac{}{- : \Delta_{13}, F_8 \vdash \Delta_{12}, F_{10}, F_{11}} \begin{array}{l}\text{inv-th/ax}\\ \vee_L\end{array}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}, F_{11}} \quad \cfrac{\cfrac{}{- : \Delta_{13}, F_{11}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}} \text{ax/W}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}, F_{10}} \text{sCut} \quad \cfrac{}{- : \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash \Delta_{12}}}{- : \Delta_{13}, F_7 \vee F_8 \vdash \Delta_{12}}$$

$$\cfrac{\cfrac{h_1 : \Delta_{11}, F_8 \vdash \Delta_{10}, F_{12} \quad h_1 : \Delta_{11}, F_9 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : \Delta_{11}, F_8 \vee F_9 \vdash \Delta_{10}, F_{12}} \vee_L \quad \cfrac{h_7 : \Delta_{11}, F_8, F_{12} \vdash \Delta_{10} \quad h_7 : \Delta_{11}, F_9, F_{12} \vdash \Delta_{10}}{\bullet h_7 : (\Delta_{11}, F_8 \vee F_9), F_{12} \vdash \Delta_{10}} \vee_L}{- : \Delta_{11}, F_8 \vee F_9 \vdash \Delta_{10}} \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{11}, F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W} \quad \cfrac{h_7 : \Delta_{11}, F_{12}, F_8 \vdash \Delta_{10}}{\bullet h_7 : \Delta_{11}, F_{12}, F_8 \vdash \Delta_{10}} \begin{array}{l}\text{ax/W}\\ H\end{array}}{- : \Delta_{11}, F_8 \vdash \Delta_{10}} \text{hCut} \quad \cfrac{\cfrac{}{h_1 : \Delta_{11}, F_9 \vdash \Delta_{10}, F_{12}} \text{ax/W} \quad \cfrac{h_7 : \Delta_{11}, F_{12}, F_9 \vdash \Delta_{10}}{\bullet h_7 : \Delta_{11}, F_{12}, F_9 \vdash \Delta_{10}} \begin{array}{l}\text{ax/W}\\ H\end{array}}{- : \Delta_{11}, F_9 \vdash \Delta_{10}} \text{hCut}}{- : \Delta_{11}, F_8 \vee F_9 \vdash \Delta_{10}} \vee_L$$

- Case rule $\perp_L$

$$\cfrac{\cfrac{h_1 : \Delta_{11}, F_7 \vdash \Delta_{10}, \perp \quad h_1 : \Delta_{11}, F_8 \vdash \Delta_{10}, \perp}{\bullet h_1 : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}, \perp} \vee_L \quad \cfrac{}{\bullet h_9 : (\Delta_{11}, F_7 \vee F_8), \perp \vdash \Delta_{10}} \perp_L}{- : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}} \text{Cut}$$

$$\rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{11}, F_7 \vdash \perp, \Delta_{10}} \text{ax/W} \quad \cfrac{}{\bullet h_9 : \perp, \Delta_{11}, F_7 \vdash \Delta_{10}} \perp_L}{- : \Delta_{11}, F_7 \vdash \Delta_{10}} \text{hCut} \quad \cfrac{\cfrac{}{h_1 : \Delta_{11}, F_8 \vdash \perp, \Delta_{10}} \text{ax/W} \quad \cfrac{}{\bullet h_9 : \perp, \Delta_{11}, F_8 \vdash \Delta_{10}} \perp_L}{- : \Delta_{11}, F_8 \vdash \Delta_{10}} \text{hCut}}{- : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}} \vee_L$$

$$\cfrac{\cfrac{h_1 : (\perp, \Delta_{11}), F_7 \vdash \Delta_{10}, F_{12} \quad h_1 : (\perp, \Delta_{11}), F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : (\perp, \Delta_{11}), F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \vee_L \quad \cfrac{}{\bullet h_9 : ((\perp, \Delta_{11}), F_7 \vee F_8), F_{12} \vdash \Delta_{10}} \perp_L}{- : (\perp, \Delta_{11}), F_7 \vee F_8 \vdash \Delta_{10}} \text{Cut}$$

$$\rightarrow$$

$$\cfrac{}{- : \perp, \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}} \perp_L$$

- Case rule I

$$\cfrac{\cfrac{h_1 : \Delta_{12}, F_7 \vdash (\Delta_{10}, p_{11}), p_{11} \quad h_1 : \Delta_{12}, F_8 \vdash (\Delta_{10}, p_{11}), p_{11}}{\bullet h_1 : \Delta_{12}, F_7 \vee F_8 \vdash (\Delta_{10}, p_{11}), p_{11}} \vee_L \quad \cfrac{}{\bullet h_9 : (\Delta_{12}, F_7 \vee F_8), p_{11} \vdash \Delta_{10}, p_{11}} I}{- : \Delta_{12}, F_7 \vee F_8 \vdash \Delta_{10}, p_{11}} \text{Cut}$$

$$\to$$

$$\cfrac{\cfrac{\cfrac{}{h_1 : \Delta_{12}, F_7 \vdash \Delta_{10}, p_{11}, p_{11}} \text{ax/W} \quad \cfrac{}{\bullet h_9 : \Delta_{12}, F_7, p_{11} \vdash \Delta_{10}, p_{11}} I}{- : \Delta_{12}, F_7 \vdash \Delta_{10}, p_{11}} \text{hCut} \quad \cfrac{\cfrac{}{h_1 : \Delta_{12}, F_8 \vdash \Delta_{10}, p_{11}, p_{11}} \text{ax/W} \quad \cfrac{}{\bullet h_9 : \Delta_{12}, F_8, p_{11} \vdash \Delta_{10}, p_{11}} I}{- : \Delta_{12}, F_8 \vdash \Delta_{10}, p_{11}} \text{hCut}}{- : \Delta_{12}, F_7 \vee F_8 \vdash \Delta_{10}, p_{11}} \vee_L$$

$$\cfrac{\cfrac{h_1 : (\Delta_{12}, p_{11}), F_7 \vdash (\Delta_{10}, p_{11}), F_{13} \quad h_1 : (\Delta_{12}, p_{11}), F_8 \vdash (\Delta_{10}, p_{11}), F_{13}}{\bullet h_1 : (\Delta_{12}, p_{11}), F_7 \vee F_8 \vdash (\Delta_{10}, p_{11}), F_{13}} \vee_L \quad \cfrac{}{\bullet h_9 : ((\Delta_{12}, p_{11}), F_7 \vee F_8), F_{13} \vdash \Delta_{10}, p_{11}} I}{- : (\Delta_{12}, p_{11}), F_7 \vee F_8 \vdash \Delta_{10}, p_{11}} \text{Cut}$$

$$\to$$

$$\cfrac{}{- : \Delta_{12}, p_{11}, F_7 \vee F_8 \vdash \Delta_{10}, p_{11}} I$$

- Case rule $\top_L$

$$\cfrac{\cfrac{h_1 : \Delta_{11}, F_7 \vdash \Delta_{10}, \top \quad h_1 : \Delta_{11}, F_8 \vdash \Delta_{10}, \top}{\bullet h_1 : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}, \top} \vee_L \quad \cfrac{h_9 : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}}{\bullet h_9 : (\Delta_{11}, F_7 \vee F_8), \top \vdash \Delta_{10}} \top_L}{- : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}} \text{Cut}$$

$$\to$$

$$\cfrac{}{- : \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}} \text{ax/W}$$

$$\cfrac{\cfrac{h_1 : (\top, \Delta_{11}), F_7 \vdash \Delta_{10}, F_{12} \quad h_1 : (\top, \Delta_{11}), F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_1 : (\top, \Delta_{11}), F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \vee_L \quad \cfrac{h_9 : \Delta_{11}, F_{12}, F_7 \vee F_8 \vdash \Delta_{10}}{\bullet h_9 : ((\top, \Delta_{11}), F_7 \vee F_8), F_{12} \vdash \Delta_{10}} \top_L}{- : (\top, \Delta_{11}), F_7 \vee F_8 \vdash \Delta_{10}} \text{Cut}$$

$$\to$$

$$\cfrac{\cfrac{}{\bullet h_1 : \top, \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W} \quad \cfrac{}{h_9 : \top, \Delta_{11}, F_{12}, F_7 \vee F_8 \vdash \Delta_{10}} \text{ax/W}}{- : \top, \Delta_{11}, F_7 \vee F_8 \vdash \Delta_{10}} \text{hCut}$$

8.9 Status of $\perp_L$: OK

- Case rule $\to_R$

$$\cfrac{\cfrac{}{\bullet h_1 : \perp, \Delta_9 \vdash (\Delta_8, F_6 \to F_7), F_{10}} \perp_L \quad \cfrac{h_5 : \perp, \Delta_9, F_6, F_{10} \vdash F_7}{\bullet h_5 : (\perp, \Delta_9), F_{10} \vdash \Delta_8, F_6 \to F_7} \to_R}{- : \perp, \Delta_9 \vdash \Delta_8, F_6 \to F_7} \text{Cut}$$

$$\to$$

$$\cfrac{}{- : \perp, \Delta_9 \vdash \Delta_8, F_6 \to F_7} \perp_L$$

- Case rule $\wedge_R$

$$\cfrac{\cfrac{}{\bullet h_1 : \perp, \Delta_9 \vdash (\Delta_8, F_6 \wedge F_7), F_{10}} \perp_L \quad \cfrac{h_5 : \perp, \Delta_9, F_{10} \vdash \Delta_8, F_6 \quad h_5 : \perp, \Delta_9, F_{10} \vdash \Delta_8, F_7}{\bullet h_5 : (\perp, \Delta_9), F_{10} \vdash \Delta_8, F_6 \wedge F_7} \wedge_R}{- : \perp, \Delta_9 \vdash \Delta_8, F_6 \wedge F_7} \text{Cut}$$

$$\to$$

$$\cfrac{}{- : \perp, \Delta_9 \vdash \Delta_8, F_6 \wedge F_7} \perp_L$$

- Case rule $\vee_R$

$$\cfrac{\cfrac{}{\bullet h_1 : \perp, \Delta_9 \vdash (\Delta_8, F_6 \vee F_7), F_{10}} \perp_L \quad \cfrac{h_5 : \perp, \Delta_9, F_{10} \vdash \Delta_8, F_6, F_7}{\bullet h_5 : (\perp, \Delta_9), F_{10} \vdash \Delta_8, F_6 \vee F_7} \vee_R}{- : \perp, \Delta_9 \vdash \Delta_8, F_6 \vee F_7} \text{Cut}$$

$$\to$$

$$\cfrac{}{- : \perp, \Delta_9 \vdash \Delta_8, F_6 \vee F_7} \perp_L$$

- Case rule $\perp_R$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_7 \vdash (\bot, \Delta_6), \mathtt{F}_8} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_7, \mathtt{F}_8 \vdash \Delta_6}{\bullet \mathtt{h}_5 : (\bot, \Delta_7), \mathtt{F}_8 \vdash \bot, \Delta_6} \ {}^{\bot_R}
}{- : \bot, \Delta_7 \vdash \bot, \Delta_6} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_7 \vdash \bot, \Delta_6} \ {}^{\bot_L}
}
$$

- Case rule $\top_R$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_7 \vdash (\top, \Delta_6), \mathtt{F}_8} \ {}^{\bot_L}
\qquad
\cfrac{}{\bullet \mathtt{h}_5 : (\bot, \Delta_7), \mathtt{F}_8 \vdash \top, \Delta_6} \ {}^{\top_R}
}{- : \bot, \Delta_7 \vdash \top, \Delta_6} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_7 \vdash \top, \Delta_6} \ {}^{\top_R}
}
$$

- Case rule $\rightarrow_L$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_9, \mathtt{F}_6 \rightarrow \mathtt{F}_7 \vdash \Delta_8, \mathtt{F}_{10}} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_{10}, \mathtt{F}_6 \rightarrow \mathtt{F}_7 \vdash \Delta_8, \mathtt{F}_6 \quad \mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_7, \mathtt{F}_{10} \vdash \Delta_8}{\bullet \mathtt{h}_5 : (\bot, \Delta_9, \mathtt{F}_6 \rightarrow \mathtt{F}_7), \mathtt{F}_{10} \vdash \Delta_8} \ {}^{\rightarrow_L}
}{- : \bot, \Delta_9, \mathtt{F}_6 \rightarrow \mathtt{F}_7 \vdash \Delta_8} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_9, \mathtt{F}_6 \rightarrow \mathtt{F}_7 \vdash \Delta_8} \ {}^{\bot_L}
}
$$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_9 \vdash \Delta_8, \mathtt{F}_6 \rightarrow \mathtt{F}_7} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_6 \rightarrow \mathtt{F}_7 \vdash \Delta_8, \mathtt{F}_6 \quad \mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_7 \vdash \Delta_8}{\bullet \mathtt{h}_5 : (\bot, \Delta_9), \mathtt{F}_6 \rightarrow \mathtt{F}_7 \vdash \Delta_8} \ {}^{\rightarrow_L}
}{- : \bot, \Delta_9 \vdash \Delta_8} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_9 \vdash \Delta_8} \ {}^{\bot_L}
}
$$

- Case rule $\wedge_L$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_9, \mathtt{F}_6 \wedge \mathtt{F}_7 \vdash \Delta_8, \mathtt{F}_{10}} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_6, \mathtt{F}_7, \mathtt{F}_{10} \vdash \Delta_8}{\bullet \mathtt{h}_5 : (\bot, \Delta_9, \mathtt{F}_6 \wedge \mathtt{F}_7), \mathtt{F}_{10} \vdash \Delta_8} \ {}^{\wedge_L}
}{- : \bot, \Delta_9, \mathtt{F}_6 \wedge \mathtt{F}_7 \vdash \Delta_8} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_9, \mathtt{F}_6 \wedge \mathtt{F}_7 \vdash \Delta_8} \ {}^{\bot_L}
}
$$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_9 \vdash \Delta_8, \mathtt{F}_6 \wedge \mathtt{F}_7} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_6, \mathtt{F}_7 \vdash \Delta_8}{\bullet \mathtt{h}_5 : (\bot, \Delta_9), \mathtt{F}_6 \wedge \mathtt{F}_7 \vdash \Delta_8} \ {}^{\wedge_L}
}{- : \bot, \Delta_9 \vdash \Delta_8} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_9 \vdash \Delta_8} \ {}^{\bot_L}
}
$$

- Case rule $\vee_L$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_9, \mathtt{F}_6 \vee \mathtt{F}_7 \vdash \Delta_8, \mathtt{F}_{10}} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_6, \mathtt{F}_{10} \vdash \Delta_8 \quad \mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_7, \mathtt{F}_{10} \vdash \Delta_8}{\bullet \mathtt{h}_5 : (\bot, \Delta_9, \mathtt{F}_6 \vee \mathtt{F}_7), \mathtt{F}_{10} \vdash \Delta_8} \ {}^{\vee_L}
}{- : \bot, \Delta_9, \mathtt{F}_6 \vee \mathtt{F}_7 \vdash \Delta_8} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_9, \mathtt{F}_6 \vee \mathtt{F}_7 \vdash \Delta_8} \ {}^{\bot_L}
}
$$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_9 \vdash \Delta_8, \mathtt{F}_6 \vee \mathtt{F}_7} \ {}^{\bot_L}
\qquad
\cfrac{\mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_6 \vdash \Delta_8 \quad \mathtt{h}_5 : \bot, \Delta_9, \mathtt{F}_7 \vdash \Delta_8}{\bullet \mathtt{h}_5 : (\bot, \Delta_9), \mathtt{F}_6 \vee \mathtt{F}_7 \vdash \Delta_8} \ {}^{\vee_L}
}{- : \bot, \Delta_9 \vdash \Delta_8} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_9 \vdash \Delta_8} \ {}^{\bot_L}
}
$$

- Case rule $\bot_L$

$$
\cfrac{
\cfrac{
\cfrac{}{\bullet \mathtt{h}_1 : \bot, \Delta_7 \vdash \Delta_6, \mathtt{F}_8} \ {}^{\bot_L}
\qquad
\cfrac{}{\bullet \mathtt{h}_5 : (\bot, \Delta_7), \mathtt{F}_8 \vdash \Delta_6} \ {}^{\bot_L}
}{- : \bot, \Delta_7 \vdash \Delta_6} \ {}^{\text{Cut}}
}{
\cfrac{\rightarrow}{- : \bot, \Delta_7 \vdash \Delta_6} \ {}^{\bot_L}
}
$$

- Case rule I

$$\frac{\dfrac{}{\bullet h_1 : \bot, \Delta_8 \vdash (\Delta_6, p_7), p_7}\ \bot_L \qquad \dfrac{}{\bullet h_5 : (\bot, \Delta_8), p_7 \vdash \Delta_6, p_7}\ I}{\begin{array}{c} - : \bot, \Delta_8 \vdash \Delta_6, p_7 \\ \to \\ \hline - : \bot, \Delta_8 \vdash \Delta_6, p_7 \end{array}\ \bot_L}\ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \bot, \Delta_8, p_7 \vdash (\Delta_6, p_7), F_9}\ \bot_L \qquad \dfrac{}{\bullet h_5 : (\bot, \Delta_8, p_7), F_9 \vdash \Delta_6, p_7}\ I}{\begin{array}{c} - : \bot, \Delta_8, p_7 \vdash \Delta_6, p_7 \\ \to \\ \hline - : \bot, \Delta_8, p_7 \vdash \Delta_6, p_7 \end{array}\ \bot_L}\ \text{Cut}$$

- Case rule $\top_L$

$$\frac{\dfrac{}{\bullet h_1 : \bot, \Delta_7 \vdash \Delta_6, \top}\ \bot_L \qquad \dfrac{h_5 : \bot, \Delta_7 \vdash \Delta_6}{\bullet h_5 : (\bot, \Delta_7), \top \vdash \Delta_6}\ \top_L}{\begin{array}{c} - : \bot, \Delta_7 \vdash \Delta_6 \\ \to \\ \hline - : \bot, \Delta_7 \vdash \Delta_6 \end{array}\ \bot_L}\ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \bot, \top, \Delta_7 \vdash \Delta_6, F_8}\ \bot_L \qquad \dfrac{h_5 : \bot, \Delta_7, F_8 \vdash \Delta_6}{\bullet h_5 : (\bot, \top, \Delta_7), F_8 \vdash \Delta_6}\ \top_L}{\begin{array}{c} - : \bot, \top, \Delta_7 \vdash \Delta_6 \\ \to \\ \hline - : \bot, \top, \Delta_7 \vdash \Delta_6 \end{array}\ \bot_L}\ \text{Cut}$$

8.10 Status of I: `OK`

- Case rule $\to_R$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_{11}, p_{10} \vdash ((\Delta_9, F_7 \to F_8), p_{10}), F_{12}}\ I \qquad \dfrac{h_6 : \Delta_{11}, F_7, F_{12}, p_{10} \vdash F_8}{\bullet h_6 : (\Delta_{11}, p_{10}), F_{12} \vdash (\Delta_9, F_7 \to F_8), p_{10}}\ \to_R}{\begin{array}{c} - : \Delta_{11}, p_{10} \vdash (\Delta_9, F_7 \to F_8), p_{10} \\ \to \\ \hline - : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}, F_7 \to F_8 \end{array}\ I}\ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9, p_{10} \vdash (\Delta_8, F_6 \to F_7), p_{10}}\ I \qquad \dfrac{h_5 : \Delta_9, F_6, p_{10}, p_{10} \vdash F_7}{\bullet h_5 : (\Delta_9, p_{10}), p_{10} \vdash \Delta_8, F_6 \to F_7}\ \to_R}{\begin{array}{c} - : \Delta_9, p_{10} \vdash \Delta_8, F_6 \to F_7 \\ \to \\ \dfrac{\dfrac{}{\bullet h_1 : \Delta_9, F_6, p_{10} \vdash F_7, p_{10}}\ I \quad \dfrac{}{h_5 : \Delta_9, F_6, p_{10}, p_{10} \vdash F_7}\ \text{ax/W}}{- : \Delta_9, F_6, p_{10} \vdash F_7}\ \text{hCut} \\ \hline - : \Delta_9, p_{10} \vdash \Delta_8, F_6 \to F_7 \end{array}\ \to_R}\ \text{Cut}$$

- Case rule $\wedge_R$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_{11}, p_{10} \vdash ((\Delta_9, F_7 \wedge F_8), p_{10}), F_{12}}\ I \quad \dfrac{h_6 : \Delta_{11}, F_{12}, p_{10} \vdash \Delta_9, F_7, p_{10} \quad h_6 : \Delta_{11}, F_{12}, p_{10} \vdash \Delta_9, F_8, p_{10}}{\bullet h_6 : (\Delta_{11}, p_{10}), F_{12} \vdash (\Delta_9, F_7 \wedge F_8), p_{10}}\ \wedge_R}{\begin{array}{c} - : \Delta_{11}, p_{10} \vdash (\Delta_9, F_7 \wedge F_8), p_{10} \\ \to \\ \hline - : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}, F_7 \wedge F_8 \end{array}\ I}\ \text{Cut}$$

$$\frac{\dfrac{}{\bullet h_1 : \Delta_9, p_{10} \vdash (\Delta_8, F_6 \wedge F_7), p_{10}}\ I \quad \dfrac{h_5 : \Delta_9, p_{10}, p_{10} \vdash \Delta_8, F_6 \quad h_5 : \Delta_9, p_{10}, p_{10} \vdash \Delta_8, F_7}{\bullet h_5 : (\Delta_9, p_{10}), p_{10} \vdash \Delta_8, F_6 \wedge F_7}\ \wedge_R}{\begin{array}{c} - : \Delta_9, p_{10} \vdash \Delta_8, F_6 \wedge F_7 \\ \to \\ \dfrac{\dfrac{}{\bullet h_1 : \Delta_9, p_{10} \vdash \Delta_8, F_6, p_{10}}\ I \ \dfrac{}{h_5 : \Delta_9, p_{10}, p_{10} \vdash \Delta_8, F_6}\ \text{ax/W}}{- : \Delta_9, p_{10} \vdash \Delta_8, F_6}\ \text{hCut} \quad \dfrac{\dfrac{}{\bullet h_1 : \Delta_9, p_{10} \vdash \Delta_8, F_7, p_{10}}\ I \ \dfrac{}{h_5 : \Delta_9, p_{10}, p_{10} \vdash \Delta_8, F_7}\ \text{ax/W}}{- : \Delta_9, p_{10} \vdash \Delta_8, F_7}\ \text{hCut} \\ \hline - : \Delta_9, p_{10} \vdash \Delta_8, F_6 \wedge F_7 \end{array}\ \wedge_R}\ \text{Cut}$$

- Case rule $\vee_R$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_{11}, p_{10} \vdash ((\Delta_9, F_7 \vee F_8), p_{10}), F_{12}} \ I \quad
\dfrac{h_6 : \Delta_{11}, F_{12}, p_{10} \vdash \Delta_9, F_7, F_8, p_{10}}{\bullet h_6 : (\Delta_{11}, p_{10}), F_{12} \vdash (\Delta_9, F_7 \vee F_8), p_{10}} \ \vee_R
}{
\dfrac{- : \Delta_{11}, p_{10} \vdash (\Delta_9, F_7 \vee F_8), p_{10}}{\begin{array}{c}\to\\ \hline - : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}, F_7 \vee F_8\end{array}} \ I
} \ \text{Cut}
$$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_9, p_{10} \vdash (\Delta_8, F_6 \vee F_7), p_{10}} \ I \quad
\dfrac{h_5 : \Delta_9, p_{10}, p_{10} \vdash \Delta_8, F_6, F_7}{\bullet h_5 : (\Delta_9, p_{10}), p_{10} \vdash \Delta_8, F_6 \vee F_7} \ \vee_R
}{
\dfrac{- : \Delta_9, p_{10} \vdash \Delta_8, F_6 \vee F_7}{\begin{array}{c}\to\\ \hline \dfrac{\dfrac{}{\bullet h_1 : \Delta_9, p_{10} \vdash \Delta_8, F_6, F_7, p_{10}} \ I \quad \dfrac{}{h_5 : \Delta_9, p_{10}, p_{10} \vdash \Delta_8, F_6, F_7} \ \text{ax/W}}{\dfrac{- : \Delta_9, p_{10} \vdash \Delta_8, F_6, F_7}{- : \Delta_9, p_{10} \vdash \Delta_8, F_6 \vee F_7} \ \vee_R} \ \text{hCut}\end{array}}
} \ \text{Cut}
$$

- Case rule $\bot_R$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_9, p_8 \vdash ((\bot, \Delta_7), p_8), F_{10}} \ I \quad
\dfrac{h_6 : \Delta_9, F_{10}, p_8 \vdash \Delta_7, p_8}{\bullet h_6 : (\Delta_9, p_8), F_{10} \vdash (\bot, \Delta_7), p_8} \ \bot_R
}{
\dfrac{- : \Delta_9, p_8 \vdash (\bot, \Delta_7), p_8}{\begin{array}{c}\to\\ \hline - : \Delta_9, p_8 \vdash \bot, \Delta_7, p_8\end{array}} \ I
} \ \text{Cut}
$$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_7, p_8 \vdash (\bot, \Delta_6), p_8} \ I \quad
\dfrac{h_5 : \Delta_7, p_8, p_8 \vdash \Delta_6}{\bullet h_5 : (\Delta_7, p_8), p_8 \vdash \bot, \Delta_6} \ \bot_R
}{
\dfrac{- : \Delta_7, p_8 \vdash \bot, \Delta_6}{\begin{array}{c}\to\\ \hline \dfrac{\dfrac{}{\bullet h_1 : \Delta_7, p_8 \vdash \bot, \Delta_6, p_8} \ I \quad \dfrac{}{h_5 : \Delta_7, p_8, p_8 \vdash \bot, \Delta_6} \ \text{ax/W}}{- : \Delta_7, p_8 \vdash \bot, \Delta_6} \ \text{hCut}\end{array}}
} \ \text{Cut}
$$

- Case rule $\top_R$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_9, p_8 \vdash ((\top, \Delta_7), p_8), F_{10}} \ I \quad
\dfrac{}{\bullet h_6 : (\Delta_9, p_8), F_{10} \vdash (\top, \Delta_7), p_8} \ \top_R
}{
\dfrac{- : \Delta_9, p_8 \vdash (\top, \Delta_7), p_8}{\begin{array}{c}\to\\ \hline - : \Delta_9, p_8 \vdash \top, \Delta_7, p_8\end{array}} \ \top_R
} \ \text{Cut}
$$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_7, p_8 \vdash (\top, \Delta_6), p_8} \ I \quad
\dfrac{}{\bullet h_5 : (\Delta_7, p_8), p_8 \vdash \top, \Delta_6} \ \top_R
}{
\dfrac{- : \Delta_7, p_8 \vdash \top, \Delta_6}{\begin{array}{c}\to\\ \hline - : \Delta_7, p_8 \vdash \top, \Delta_6\end{array}} \ \top_R
} \ \text{Cut}
$$

- Case rule $\to_L$

$$
\dfrac{
\dfrac{}{\bullet h_1 : (\Delta_{11}, F_7 \to F_8), p_{10} \vdash (\Delta_9, p_{10}), F_{12}} \ I \quad
\dfrac{h_6 : \Delta_{11}, F_{12}, p_{10}, F_7 \to F_8 \vdash \Delta_9, F_7, p_{10} \quad h_6 : \Delta_{11}, F_8, F_{12}, p_{10} \vdash \Delta_9, p_{10}}{\bullet h_6 : ((\Delta_{11}, F_7 \to F_8), p_{10}), F_{12} \vdash \Delta_9, p_{10}} \ \to_L
}{
\dfrac{- : (\Delta_{11}, F_7 \to F_8), p_{10} \vdash \Delta_9, p_{10}}{\begin{array}{c}\to\\ \hline - : \Delta_{11}, p_{10}, F_7 \to F_8 \vdash \Delta_9, p_{10}\end{array}} \ I
} \ \text{Cut}
$$

$$
\dfrac{
\dfrac{}{\bullet h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_7 \to F_8} \ I \quad
\dfrac{h_6 : \Delta_{11}, p_{10}, F_7 \to F_8 \vdash \Delta_9, F_7, p_{10} \quad h_6 : \Delta_{11}, F_8, p_{10} \vdash \Delta_9, p_{10}}{\bullet h_6 : (\Delta_{11}, p_{10}), F_7 \to F_8 \vdash \Delta_9, p_{10}} \ \to_L
}{
\dfrac{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}}{\begin{array}{c}\to\\ \hline - : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}\end{array}} \ I
} \ \text{Cut}
$$

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$$\dfrac{\overline{\bullet h_1 : (\Delta_9, F_6 \to F_7), p_{10} \vdash \Delta_8, p_{10}} \ I \quad \dfrac{h_5 : \Delta_9, p_{10}, p_{10}, F_6 \to F_7 \vdash \Delta_8, F_6 \quad h_5 : \Delta_9, F_7, p_{10}, p_{10} \vdash \Delta_8}{\bullet h_5 : ((\Delta_9, F_6 \to F_7), p_{10}), p_{10} \vdash \Delta_8} \to_L}{- : (\Delta_9, F_6 \to F_7), p_{10} \vdash \Delta_8} \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{\bullet h_1 : \Delta_9, p_{10}, F_6 \to F_7 \vdash \Delta_8, F_6, p_{10}} \ \text{ax/W} \quad \overline{h_5 : \Delta_9, p_{10}, p_{10}, F_6 \to F_7 \vdash \Delta_8, F_6} \ \text{ax/W}}{- : \Delta_9, p_{10}, F_6 \to F_7 \vdash \Delta_8, F_6} \text{hCut} \quad \dfrac{\overline{\bullet h_1 : \Delta_9, F_7, p_{10} \vdash \Delta_8, p_{10}} \ I \quad \overline{h_5 : \Delta_9, F_7, p_{10}, p_{10} \vdash \Delta_8} \ \text{ax/W}}{- : \Delta_9, F_7, p_{10} \vdash \Delta_8} \text{hCut}}{- : \Delta_9, p_{10}, F_6 \to F_7 \vdash \Delta_8} \to_L$$

- Case rule $\wedge_L$

$$\dfrac{\overline{\bullet h_1 : (\Delta_{11}, F_7 \wedge F_8), p_{10} \vdash (\Delta_9, p_{10}), F_{12}} \ I \quad \dfrac{h_6 : \Delta_{11}, F_7, F_8, F_{12}, p_{10} \vdash \Delta_9, p_{10}}{\bullet h_6 : ((\Delta_{11}, F_7 \wedge F_8), p_{10}), F_{12} \vdash \Delta_9, p_{10}} \wedge_L}{- : (\Delta_{11}, F_7 \wedge F_8), p_{10} \vdash \Delta_9, p_{10}} \text{Cut}$$

$$\to$$

$$\dfrac{}{- : \Delta_{11}, p_{10}, F_7 \wedge F_8 \vdash \Delta_9, p_{10}} \ I$$

$$\dfrac{\overline{\bullet h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_7 \wedge F_8} \ I \quad \dfrac{h_6 : \Delta_{11}, F_7, F_8, p_{10} \vdash \Delta_9, p_{10}}{\bullet h_6 : (\Delta_{11}, p_{10}), F_7 \wedge F_8 \vdash \Delta_9, p_{10}} \wedge_L}{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}} \text{Cut}$$

$$\to$$

$$\dfrac{}{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}} \ I$$

$$\dfrac{\overline{\bullet h_1 : (\Delta_9, F_6 \wedge F_7), p_{10} \vdash \Delta_8, p_{10}} \ I \quad \dfrac{h_5 : \Delta_9, F_6, F_7, p_{10}, p_{10} \vdash \Delta_8}{\bullet h_5 : ((\Delta_9, F_6 \wedge F_7), p_{10}), p_{10} \vdash \Delta_8} \wedge_L}{- : (\Delta_9, F_6 \wedge F_7), p_{10} \vdash \Delta_8} \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{\bullet h_1 : \Delta_9, F_6, F_7, p_{10} \vdash \Delta_8, p_{10}} \ I \quad \overline{h_5 : \Delta_9, F_6, F_7, p_{10}, p_{10} \vdash \Delta_8} \ \text{ax/W}}{- : \Delta_9, F_6, F_7, p_{10} \vdash \Delta_8} \text{hCut}}{- : \Delta_9, p_{10}, F_6 \wedge F_7 \vdash \Delta_8} \wedge_L$$

- Case rule $\vee_L$

$$\dfrac{\overline{\bullet h_1 : (\Delta_{11}, F_7 \vee F_8), p_{10} \vdash (\Delta_9, p_{10}), F_{12}} \ I \quad \dfrac{h_6 : \Delta_{11}, F_7, F_{12}, p_{10} \vdash \Delta_9, p_{10} \quad h_6 : \Delta_{11}, F_8, F_{12}, p_{10} \vdash \Delta_9, p_{10}}{\bullet h_6 : ((\Delta_{11}, F_7 \vee F_8), p_{10}), F_{12} \vdash \Delta_9, p_{10}} \vee_L}{- : (\Delta_{11}, F_7 \vee F_8), p_{10} \vdash \Delta_9, p_{10}} \text{Cut}$$

$$\to$$

$$\dfrac{}{- : \Delta_{11}, p_{10}, F_7 \vee F_8 \vdash \Delta_9, p_{10}} \ I$$

$$\dfrac{\overline{\bullet h_1 : \Delta_{11}, p_{10} \vdash (\Delta_9, p_{10}), F_7 \vee F_8} \ I \quad \dfrac{h_6 : \Delta_{11}, F_7, p_{10} \vdash \Delta_9, p_{10} \quad h_6 : \Delta_{11}, F_8, p_{10} \vdash \Delta_9, p_{10}}{\bullet h_6 : (\Delta_{11}, p_{10}), F_7 \vee F_8 \vdash \Delta_9, p_{10}} \vee_L}{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}} \text{Cut}$$

$$\to$$

$$\dfrac{}{- : \Delta_{11}, p_{10} \vdash \Delta_9, p_{10}} \ I$$

$$\dfrac{\overline{\bullet h_1 : (\Delta_9, F_6 \vee F_7), p_{10} \vdash \Delta_8, p_{10}} \ I \quad \dfrac{h_5 : \Delta_9, F_6, p_{10}, p_{10} \vdash \Delta_8 \quad h_5 : \Delta_9, F_7, p_{10}, p_{10} \vdash \Delta_8}{\bullet h_5 : ((\Delta_9, F_6 \vee F_7), p_{10}), p_{10} \vdash \Delta_8} \vee_L}{- : (\Delta_9, F_6 \vee F_7), p_{10} \vdash \Delta_8} \text{Cut}$$

$$\to$$

$$\dfrac{\dfrac{\overline{\bullet h_1 : \Delta_9, F_6, p_{10} \vdash \Delta_8, p_{10}} \ I \quad \overline{h_5 : \Delta_9, F_6, p_{10}, p_{10} \vdash \Delta_8} \ \text{ax/W}}{- : \Delta_9, F_6, p_{10} \vdash \Delta_8} \text{hCut} \quad \dfrac{\overline{\bullet h_1 : \Delta_9, F_7, p_{10} \vdash \Delta_8, p_{10}} \ I \quad \overline{h_5 : \Delta_9, F_7, p_{10}, p_{10} \vdash \Delta_8} \ \text{ax/W}}{- : \Delta_9, F_7, p_{10} \vdash \Delta_8} \text{hCut}}{- : \Delta_9, p_{10}, F_6 \vee F_7 \vdash \Delta_8} \vee_L$$

- Case rule $\bot_L$

$$\dfrac{\overline{\bullet h_1 : \Delta_9, p_8 \vdash (\Delta_7, p_8), \bot} \ I \quad \overline{\bullet h_6 : (\Delta_9, p_8), \bot \vdash \Delta_7, p_8} \ \bot_L}{- : \Delta_9, p_8 \vdash \Delta_7, p_8} \text{Cut}$$

$$\to$$

$$\dfrac{}{- : \Delta_9, p_8 \vdash \Delta_7, p_8} \ I$$

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$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : (\bot, \Delta_9), \mathtt{p}_8 \vdash (\Delta_7, \mathtt{p}_8), \mathtt{F}_{10}}\ I \quad \overline{\bullet\mathtt{h}_6 : ((\bot, \Delta_9), \mathtt{p}_8), \mathtt{F}_{10} \vdash \Delta_7, \mathtt{p}_8}\ \bot_L}{- : (\bot, \Delta_9), \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \bot, \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ \bot_L$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : (\bot, \Delta_7), \mathtt{p}_8 \vdash \Delta_6, \mathtt{p}_8}\ I \quad \overline{\bullet\mathtt{h}_5 : ((\bot, \Delta_7), \mathtt{p}_8), \mathtt{p}_8 \vdash \Delta_6}\ \bot_L}{- : (\bot, \Delta_7), \mathtt{p}_8 \vdash \Delta_6}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \bot, \Delta_7, \mathtt{p}_8 \vdash \Delta_6}\ \bot_L$$

- Case rule I

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : \Delta_{10}, \mathtt{p}_8 \vdash ((\Delta_7, \mathtt{p}_9), \mathtt{p}_8), \mathtt{p}_9}\ I \quad \overline{\bullet\mathtt{h}_6 : (\Delta_{10}, \mathtt{p}_8), \mathtt{p}_9 \vdash (\Delta_7, \mathtt{p}_9), \mathtt{p}_8}\ I}{- : \Delta_{10}, \mathtt{p}_8 \vdash (\Delta_7, \mathtt{p}_9), \mathtt{p}_8}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \Delta_{10}, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8, \mathtt{p}_9}\ I$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : (\Delta_{10}, \mathtt{p}_9), \mathtt{p}_8 \vdash ((\Delta_7, \mathtt{p}_9), \mathtt{p}_8), \mathtt{F}_{11}}\ I \quad \overline{\bullet\mathtt{h}_6 : ((\Delta_{10}, \mathtt{p}_9), \mathtt{p}_8), \mathtt{F}_{11} \vdash (\Delta_7, \mathtt{p}_9), \mathtt{p}_8}\ I}{- : (\Delta_{10}, \mathtt{p}_9), \mathtt{p}_8 \vdash (\Delta_7, \mathtt{p}_9), \mathtt{p}_8}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \Delta_{10}, \mathtt{p}_8, \mathtt{p}_9 \vdash \Delta_7, \mathtt{p}_8, \mathtt{p}_9}\ I$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : \Delta_9, \mathtt{p}_8 \vdash (\Delta_7, \mathtt{p}_8), \mathtt{F}_{10}}\ I \quad \overline{\bullet\mathtt{h}_6 : (\Delta_9, \mathtt{p}_8), \mathtt{F}_{10} \vdash \Delta_7, \mathtt{p}_8}\ I}{- : \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ I$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : \Delta_8, \mathtt{p}_7 \vdash (\Delta_6, \mathtt{p}_7), \mathtt{p}_7}\ I \quad \overline{\bullet\mathtt{h}_5 : (\Delta_8, \mathtt{p}_7), \mathtt{p}_7 \vdash \Delta_6, \mathtt{p}_7}\ I}{- : \Delta_8, \mathtt{p}_7 \vdash \Delta_6, \mathtt{p}_7}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \Delta_8, \mathtt{p}_7 \vdash \Delta_6, \mathtt{p}_7}\ I$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : (\Delta_8, \mathtt{p}_7), \mathtt{p}_9 \vdash (\Delta_6, \mathtt{p}_7), \mathtt{p}_9}\ I \quad \overline{\bullet\mathtt{h}_5 : ((\Delta_8, \mathtt{p}_7), \mathtt{p}_9), \mathtt{p}_9 \vdash \Delta_6, \mathtt{p}_7}\ I}{- : (\Delta_8, \mathtt{p}_7), \mathtt{p}_9 \vdash \Delta_6, \mathtt{p}_7}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \Delta_8, \mathtt{p}_7, \mathtt{p}_9 \vdash \Delta_6, \mathtt{p}_7}\ I$$

- Case rule $\top_L$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : \Delta_9, \mathtt{p}_8 \vdash (\Delta_7, \mathtt{p}_8), \top}\ I \quad \cfrac{\mathtt{h}_6 : \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}{\bullet\mathtt{h}_6 : (\Delta_9, \mathtt{p}_8), \top \vdash \Delta_7, \mathtt{p}_8}\ \top_L}{- : \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ I$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : (\top, \Delta_9), \mathtt{p}_8 \vdash (\Delta_7, \mathtt{p}_8), \mathtt{F}_{10}}\ I \quad \cfrac{\mathtt{h}_6 : \Delta_9, \mathtt{F}_{10}, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}{\bullet\mathtt{h}_6 : ((\top, \Delta_9), \mathtt{p}_8), \mathtt{F}_{10} \vdash \Delta_7, \mathtt{p}_8}\ \top_L}{- : (\top, \Delta_9), \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{}{- : \top, \Delta_9, \mathtt{p}_8 \vdash \Delta_7, \mathtt{p}_8}\ I$$

$$\cfrac{\cfrac{\overline{\bullet\mathtt{h}_1 : (\top, \Delta_7), \mathtt{p}_8 \vdash \Delta_6, \mathtt{p}_8}\ I \quad \cfrac{\mathtt{h}_5 : \Delta_7, \mathtt{p}_8, \mathtt{p}_8 \vdash \Delta_6}{\bullet\mathtt{h}_5 : ((\top, \Delta_7), \mathtt{p}_8), \mathtt{p}_8 \vdash \Delta_6}\ \top_L}{- : (\top, \Delta_7), \mathtt{p}_8 \vdash \Delta_6}\ \text{Cut}}{\xrightarrow{}}$$
$$\cfrac{\cfrac{}{\bullet\mathtt{h}_1 : \top, \Delta_7, \mathtt{p}_8 \vdash \Delta_6, \mathtt{p}_8}\ \texttt{ax/W} \quad \cfrac{}{\mathtt{h}_5 : \top, \Delta_7, \mathtt{p}_8, \mathtt{p}_8 \vdash \Delta_6}\ \texttt{ax/W}}{- : \top, \Delta_7, \mathtt{p}_8 \vdash \Delta_6}\ \texttt{hCut}$$

8.11 Status of $\top_L$: OK

- Case rule $\to_R$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_9 \vdash (\Delta_8, \mathtt{F_6} \to \mathtt{F_7}), \mathtt{F_{10}}}{\bullet\mathtt{h_1} : \top, \Delta_9 \vdash (\Delta_8, \mathtt{F_6} \to \mathtt{F_7}), \mathtt{F_{10}}} \ \top_L
\qquad
\cfrac{\mathtt{h_5} : \top, \Delta_9, \mathtt{F_6}, \mathtt{F_{10}} \vdash \mathtt{F_7}}{\bullet\mathtt{h_5} : (\top, \Delta_9), \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}} \ \to_R
}{- : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}} \ \mathtt{Cut}
}{
\cfrac{
\cfrac{}{\mathtt{h_1} : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_{10}}, \mathtt{F_6} \to \mathtt{F_7}} \ \mathtt{ax/W}
\qquad
\cfrac{}{\bullet\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}} \ \mathtt{ax/W}
}{- : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}} \ \mathtt{hCut}
} \ \to
$$

- Case rule $\wedge_R$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_9 \vdash (\Delta_8, \mathtt{F_6} \wedge \mathtt{F_7}), \mathtt{F_{10}}}{\bullet\mathtt{h_1} : \top, \Delta_9 \vdash (\Delta_8, \mathtt{F_6} \wedge \mathtt{F_7}), \mathtt{F_{10}}} \ \top_L
\qquad
\cfrac{\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \quad \mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_7}}{\bullet\mathtt{h_5} : (\top, \Delta_9), \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \wedge \mathtt{F_7}} \ \wedge_R
}{- : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \wedge \mathtt{F_7}} \ \mathtt{Cut}
}{
\cfrac{
\cfrac{}{\mathtt{h_1} : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_{10}}, \mathtt{F_6} \wedge \mathtt{F_7}} \ \mathtt{ax/W}
\qquad
\cfrac{}{\bullet\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \wedge \mathtt{F_7}} \ \mathtt{ax/W}
}{- : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \wedge \mathtt{F_7}} \ \mathtt{hCut}
} \ \to
$$

- Case rule $\vee_R$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_9 \vdash (\Delta_8, \mathtt{F_6} \vee \mathtt{F_7}), \mathtt{F_{10}}}{\bullet\mathtt{h_1} : \top, \Delta_9 \vdash (\Delta_8, \mathtt{F_6} \vee \mathtt{F_7}), \mathtt{F_{10}}} \ \top_L
\qquad
\cfrac{\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6}, \mathtt{F_7}}{\bullet\mathtt{h_5} : (\top, \Delta_9), \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \vee \mathtt{F_7}} \ \vee_R
}{- : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \vee \mathtt{F_7}} \ \mathtt{Cut}
}{
\cfrac{
\cfrac{}{\mathtt{h_1} : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_{10}}, \mathtt{F_6} \vee \mathtt{F_7}} \ \mathtt{ax/W}
\qquad
\cfrac{}{\bullet\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}} \vdash \Delta_8, \mathtt{F_6} \vee \mathtt{F_7}} \ \mathtt{ax/W}
}{- : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \vee \mathtt{F_7}} \ \mathtt{hCut}
} \ \to
$$

- Case rule $\perp_R$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_7 \vdash (\perp, \Delta_6), \mathtt{F_8}}{\bullet\mathtt{h_1} : \top, \Delta_7 \vdash (\perp, \Delta_6), \mathtt{F_8}} \ \top_L
\qquad
\cfrac{\mathtt{h_5} : \top, \Delta_7, \mathtt{F_8} \vdash \Delta_6}{\bullet\mathtt{h_5} : (\top, \Delta_7), \mathtt{F_8} \vdash \perp, \Delta_6} \ \perp_R
}{- : \top, \Delta_7 \vdash \perp, \Delta_6} \ \mathtt{Cut}
}{
\cfrac{
\cfrac{}{\mathtt{h_1} : \top, \Delta_7 \vdash \perp, \Delta_6, \mathtt{F_8}} \ \mathtt{ax/W}
\qquad
\cfrac{}{\bullet\mathtt{h_5} : \top, \Delta_7, \mathtt{F_8} \vdash \perp, \Delta_6} \ \mathtt{ax/W}
}{- : \top, \Delta_7 \vdash \perp, \Delta_6} \ \mathtt{hCut}
} \ \to
$$

- Case rule $\top_R$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_7 \vdash (\top, \Delta_6), \mathtt{F_8}}{\bullet\mathtt{h_1} : \top, \Delta_7 \vdash (\top, \Delta_6), \mathtt{F_8}} \ \top_L
\qquad
\cfrac{}{\bullet\mathtt{h_5} : (\top, \Delta_7), \mathtt{F_8} \vdash \top, \Delta_6} \ \top_R
}{- : \top, \Delta_7 \vdash \top, \Delta_6} \ \mathtt{Cut}
}{
\cfrac{}{- : \top, \Delta_7 \vdash \top, \Delta_6} \ \top_R
} \ \to
$$

- Case rule $\to_L$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8, \mathtt{F_{10}}}{\bullet\mathtt{h_1} : \top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8, \mathtt{F_{10}}} \ \top_L
\qquad
\cfrac{\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}}, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8, \mathtt{F_6} \quad \mathtt{h_5} : \top, \Delta_9, \mathtt{F_7}, \mathtt{F_{10}} \vdash \Delta_8}{\bullet\mathtt{h_5} : (\top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7}), \mathtt{F_{10}} \vdash \Delta_8} \ \to_L
}{- : \top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8} \ \mathtt{Cut}
}{
\cfrac{
\cfrac{}{\mathtt{h_1} : \top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8, \mathtt{F_{10}}} \ \mathtt{ax/W}
\qquad
\cfrac{}{\bullet\mathtt{h_5} : \top, \Delta_9, \mathtt{F_{10}}, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8} \ \mathtt{ax/W}
}{- : \top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8} \ \mathtt{hCut}
} \ \to
$$

$$
\cfrac{
\cfrac{
\cfrac{\mathtt{h_1} : \Delta_9 \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}}{\bullet\mathtt{h_1} : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}} \ \top_L
\qquad
\cfrac{\mathtt{h_5} : \top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8, \mathtt{F_6} \quad \mathtt{h_5} : \top, \Delta_9, \mathtt{F_7} \vdash \Delta_8}{\bullet\mathtt{h_5} : (\top, \Delta_9), \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8} \ \to_L
}{- : \top, \Delta_9 \vdash \Delta_8} \ \mathtt{Cut}
}{
\cfrac{
\cfrac{}{\mathtt{h_1} : \top, \Delta_9 \vdash \Delta_8, \mathtt{F_6} \to \mathtt{F_7}} \ \mathtt{ax/W}
\qquad
\cfrac{}{\bullet\mathtt{h_5} : \top, \Delta_9, \mathtt{F_6} \to \mathtt{F_7} \vdash \Delta_8} \ \mathtt{ax/W}
}{- : \top, \Delta_9 \vdash \Delta_8} \ \mathtt{hCut}
} \ \to
$$

- Case rule $\wedge_L$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_9, F_6 \wedge F_7 \vdash \Delta_8, F_{10}}{\bullet h_1 : \top, \Delta_9, F_6 \wedge F_7 \vdash \Delta_8, F_{10}} \ \top_L
\qquad
\dfrac{h_5 : \top, \Delta_9, F_6, F_7, F_{10} \vdash \Delta_8}{\bullet h_5 : (\top, \Delta_9, F_6 \wedge F_7), F_{10} \vdash \Delta_8} \ \wedge_L
}{- : \top, \Delta_9, F_6 \wedge F_7 \vdash \Delta_8} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{
\dfrac{}{h_1 : \top, \Delta_9, F_6 \wedge F_7 \vdash \Delta_8, F_{10}} \ \text{ax/W}
\qquad
\dfrac{}{\bullet h_5 : \top, \Delta_9, F_{10}, F_6 \wedge F_7 \vdash \Delta_8} \ \text{ax/W}
}{- : \top, \Delta_9, F_6 \wedge F_7 \vdash \Delta_8} \ \text{hCut}
$$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_9 \vdash \Delta_8, F_6 \wedge F_7}{\bullet h_1 : \top, \Delta_9 \vdash \Delta_8, F_6 \wedge F_7} \ \top_L
\qquad
\dfrac{h_5 : \top, \Delta_9, F_6, F_7 \vdash \Delta_8}{\bullet h_5 : (\top, \Delta_9), F_6 \wedge F_7 \vdash \Delta_8} \ \wedge_L
}{- : \top, \Delta_9 \vdash \Delta_8} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{
\dfrac{}{h_1 : \top, \Delta_9 \vdash \Delta_8, F_6 \wedge F_7} \ \text{ax/W}
\qquad
\dfrac{}{\bullet h_5 : \top, \Delta_9, F_6 \wedge F_7 \vdash \Delta_8} \ \text{ax/W}
}{- : \top, \Delta_9 \vdash \Delta_8} \ \text{hCut}
$$

- Case rule $\vee_L$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_9, F_6 \vee F_7 \vdash \Delta_8, F_{10}}{\bullet h_1 : \top, \Delta_9, F_6 \vee F_7 \vdash \Delta_8, F_{10}} \ \top_L
\qquad
\dfrac{h_5 : \top, \Delta_9, F_6, F_{10} \vdash \Delta_8 \qquad h_5 : \top, \Delta_9, F_7, F_{10} \vdash \Delta_8}{\bullet h_5 : (\top, \Delta_9, F_6 \vee F_7), F_{10} \vdash \Delta_8} \ \vee_L
}{- : \top, \Delta_9, F_6 \vee F_7 \vdash \Delta_8} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{
\dfrac{}{h_1 : \top, \Delta_9, F_6 \vee F_7 \vdash \Delta_8, F_{10}} \ \text{ax/W}
\qquad
\dfrac{}{\bullet h_5 : \top, \Delta_9, F_{10}, F_6 \vee F_7 \vdash \Delta_8} \ \text{ax/W}
}{- : \top, \Delta_9, F_6 \vee F_7 \vdash \Delta_8} \ \text{hCut}
$$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_9 \vdash \Delta_8, F_6 \vee F_7}{\bullet h_1 : \top, \Delta_9 \vdash \Delta_8, F_6 \vee F_7} \ \top_L
\qquad
\dfrac{h_5 : \top, \Delta_9, F_6 \vdash \Delta_8 \qquad h_5 : \top, \Delta_9, F_7 \vdash \Delta_8}{\bullet h_5 : (\top, \Delta_9), F_6 \vee F_7 \vdash \Delta_8} \ \vee_L
}{- : \top, \Delta_9 \vdash \Delta_8} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{
\dfrac{}{h_1 : \top, \Delta_9 \vdash \Delta_8, F_6 \vee F_7} \ \text{ax/W}
\qquad
\dfrac{}{\bullet h_5 : \top, \Delta_9, F_6 \vee F_7 \vdash \Delta_8} \ \text{ax/W}
}{- : \top, \Delta_9 \vdash \Delta_8} \ \text{hCut}
$$

- Case rule $\perp_L$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_7 \vdash \Delta_6, \perp}{\bullet h_1 : \top, \Delta_7 \vdash \Delta_6, \perp} \ \top_L
\qquad
\dfrac{}{\bullet h_5 : (\top, \Delta_7), \perp \vdash \Delta_6} \ \perp_L
}{- : \top, \Delta_7 \vdash \Delta_6} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{
\dfrac{}{h_1 : \top, \Delta_7 \vdash \perp, \Delta_6} \ \text{ax/W}
\qquad
\dfrac{}{\bullet h_5 : \perp, \top, \Delta_7 \vdash \Delta_6} \ \perp_L
}{- : \top, \Delta_7 \vdash \Delta_6} \ \text{hCut}
$$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \perp, \Delta_7 \vdash \Delta_6, F_8}{\bullet h_1 : \top, \perp, \Delta_7 \vdash \Delta_6, F_8} \ \top_L
\qquad
\dfrac{}{\bullet h_5 : (\top, \perp, \Delta_7), F_8 \vdash \Delta_6} \ \perp_L
}{- : \top, \perp, \Delta_7 \vdash \Delta_6} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{}{- : \perp, \top, \Delta_7 \vdash \Delta_6} \ \perp_L
$$

- Case rule I

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_8 \vdash (\Delta_6, p_7), p_7}{\bullet h_1 : \top, \Delta_8 \vdash (\Delta_6, p_7), p_7} \ \top_L
\qquad
\dfrac{}{\bullet h_5 : (\top, \Delta_8), p_7 \vdash \Delta_6, p_7} \ I
}{- : \top, \Delta_8 \vdash \Delta_6, p_7} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{
\dfrac{}{h_1 : \top, \Delta_8 \vdash \Delta_6, p_7, p_7} \ \text{ax/W}
\qquad
\dfrac{}{\bullet h_5 : \top, \Delta_8, p_7 \vdash \Delta_6, p_7} \ I
}{- : \top, \Delta_8 \vdash \Delta_6, p_7} \ \text{hCut}
$$

$$
\dfrac{
\dfrac{
\dfrac{h_1 : \Delta_8, p_7 \vdash (\Delta_6, p_7), F_9}{\bullet h_1 : \top, \Delta_8, p_7 \vdash (\Delta_6, p_7), F_9} \ \top_L
\qquad
\dfrac{}{\bullet h_5 : (\top, \Delta_8, p_7), F_9 \vdash \Delta_6, p_7} \ I
}{- : \top, \Delta_8, p_7 \vdash \Delta_6, p_7} \ \text{Cut}
}{}
$$

$$
\to
$$

$$
\dfrac{}{- : \top, \Delta_8, p_7 \vdash \Delta_6, p_7} \ I
$$

- Case rule $\top_L$

$$
\dfrac{
  \dfrac{\mathtt{h_1} : \Delta_7 \vdash \Delta_6, \mathtt{F_8}}{\bullet\mathtt{h_1} : \top, \Delta_7 \vdash \Delta_6, \mathtt{F_8}} \ \top_L
  \qquad
  \dfrac{\mathtt{h_5} : \Delta_7, \mathtt{F_8} \vdash \Delta_6}{\bullet\mathtt{h_5} : (\top, \Delta_7), \mathtt{F_8} \vdash \Delta_6} \ \top_L
}{
  - : \top, \Delta_7 \vdash \Delta_6
} \ \mathtt{Cut}
$$

$$\to$$

$$
\dfrac{
  \dfrac{}{\mathtt{h_1} : \top, \Delta_7 \vdash \Delta_6, \mathtt{F_8}} \ \mathtt{ax/W}
  \qquad
  \dfrac{}{\bullet\mathtt{h_5} : \top, \Delta_7, \mathtt{F_8} \vdash \Delta_6} \ \mathtt{ax/W}
}{
  - : \top, \Delta_7 \vdash \Delta_6
} \ \mathtt{hCut}
$$