

# System G3C

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9 Admissibility of  $A \rightarrow A \wedge A$

65

10 Admissibility of  $A \wedge A \rightarrow A$

67

# 1 Height preserving admissibility of weakening on the left

- Case(s) rule  $\rightarrow_R$

$$\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_5} \text{ ax} \quad \frac{h_1 : \Delta_2, F_W \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \rightarrow F_5} \text{ IH}}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4} \text{ ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_5} \text{ ax}}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \wedge F_5} \text{ IH} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4, F_5} \text{ ax} \quad \frac{h_1 : \Delta_2, F_W \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \vee F_5} \text{ IH}}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \vee F_5} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3} \text{ ax} \quad \frac{h_1 : \Delta_2, F_W \vdash \Delta_3}{\bullet h_1 : \Delta_2, F_W \vdash \perp, \Delta_3} \text{ IH}}{\bullet h_1 : \Delta_2, F_W \vdash \perp, \Delta_3} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \rightsquigarrow \frac{}{\bullet h_1 : \Delta_2, F_W \vdash \top, \Delta_3} \top_R$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_5, F_3} \text{ ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4, F_W \vdash \Delta_5} \text{ ax}}{\bullet h_1 : \Delta_2, F_W, F_3 \rightarrow F_4 \vdash \Delta_5} \text{ IH} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3, F_4, F_W \vdash \Delta_5} \text{ ax} \quad \frac{h_1 : \Delta_2, F_3, F_4, F_W \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_W, F_3 \wedge F_4 \vdash \Delta_5} \text{ IH}}{\bullet h_1 : \Delta_2, F_W, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3, F_W \vdash \Delta_5} \text{ ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4, F_W \vdash \Delta_5} \text{ ax}}{\bullet h_1 : \Delta_2, F_W, F_3 \vee F_4 \vdash \Delta_5} \text{ IH} \vee_L$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \rightsquigarrow \frac{}{\bullet h_1 : \perp, \Delta_2, F_W \vdash \Delta_3} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \quad \rightsquigarrow \quad \frac{}{\bullet h_1 : \Delta_2, F_W, p_3 \vdash \Delta_4, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{h_1 : \Delta_2, F_W \vdash \Delta_3} \text{IH}}{\bullet h_1 : \top, \Delta_2, F_W \vdash \Delta_3} \top_L$$

## 2 Height preserving admissibility of weakening on the right

- Case(s) rule  $\rightarrow_R$

$$\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \text{ ax}}{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \rightarrow F_5} \text{ IH}} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_W} \text{ ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{h_1 : \Delta_2 \vdash \Delta_3, F_5, F_W} \text{ ax}}{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \wedge F_5} \text{ IH}} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5, F_W} \text{ ax}}{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \vee F_5} \text{ IH}} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{h_1 : \Delta_2 \vdash \Delta_3, F_W} \text{ ax}}{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_W}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3, F_W} \text{ IH}} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \rightsquigarrow \frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3, F_W} \top_R$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{h_1 : \Delta_2 \vdash \Delta_5, F_3, F_W} \text{ ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{h_1 : \Delta_2, F_4 \vdash \Delta_5, F_W} \text{ ax}}{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3, F_W}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5, F_W} \text{ IH}} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5, F_W} \text{ ax}}{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5, F_W}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5, F_W} \text{ IH}} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{h_1 : \Delta_2, F_3 \vdash \Delta_5, F_W} \text{ ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{h_1 : \Delta_2, F_4 \vdash \Delta_5, F_W} \text{ ax}}{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5, F_W}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5, F_W} \text{ IH}} \vee_L$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \rightsquigarrow \frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3, F_W} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \rightsquigarrow \frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, F_W, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{h_1 : \Delta_2 \vdash \Delta_3, F_W} \text{ ax}}{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_W}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3, F_W} \text{ IH}} \top_L$$

### 3 Measure of derivations

- Case(s) rule  $\rightarrow_R$

$$\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \text{IH}}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \text{ax} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4} \text{IH} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_5} \text{IH}}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \text{ax} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \text{IH}}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \text{ax} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{IH}}{\bullet \bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \text{ax} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \quad \rightsquigarrow \quad \frac{}{\bullet \bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{IH} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{IH}}{\bullet \bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \text{ax} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \text{IH}}{\bullet \bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \text{ax} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vdash \Delta_5} \text{IH} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{IH}}{\bullet \bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \text{ax} \vee_L$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \quad \rightsquigarrow \quad \frac{}{\bullet \bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \quad \rightsquigarrow \quad \frac{}{\bullet \bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{IH}}{\bullet \bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \text{ax} \top_L$$

## 4 Invertibility of Rules

### 4.1 Status of $\rightarrow_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_6, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \rightarrow F_2), F_5 \rightarrow F_6} \rightarrow_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4, F_1, F_5 \vdash \Delta_7, F_2, F_6}}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \rightarrow F_6} \text{ax/ind} \rightarrow_R$$

$$\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5} \text{ax} \text{height}$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \rightarrow F_2 \quad h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \rightarrow F_2), F_5 \wedge F_6} \wedge_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5} \text{ax/ind} \quad \overline{h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \wedge F_6} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_6, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \rightarrow F_2), F_5 \vee F_6} \vee_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \vee F_6} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \rightarrow F_2} \perp_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4, F_1 \vdash \Delta_5, F_2} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \perp, \Delta_5, F_2} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \rightarrow F_2} \top_R \rightsquigarrow \frac{}{\bullet h_3 : \Delta_4, F_1 \vdash \top, \Delta_5, F_2} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_6, F_2 \rightarrow F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \rightarrow F_3} \rightarrow_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_2 \vdash \Delta_1, F_3, F_6} \text{ax/ind} \quad \overline{h_4 : \Delta_5, F_2, F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_2, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \rightarrow F_3} \wedge_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_2, F_6, F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_2, F_6 \wedge F_7 \vdash \Delta_1, F_3} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2 \rightarrow F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \rightarrow F_3} \vee_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_2, F_6 \vdash \Delta_1, F_3} \text{ax/ind} \quad \overline{h_4 : \Delta_5, F_2, F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_2, F_6 \vee F_7 \vdash \Delta_1, F_3} \vee_L$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3} \perp_L \quad \rightsquigarrow \quad \frac{}{\bullet h_4 : \perp, \Delta_5, F_2 \vdash \Delta_1, F_3} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \rightarrow F_2} I \quad \rightsquigarrow \quad \frac{}{\bullet h_3 : \Delta_4, F_1, p_5 \vdash \Delta_6, F_2, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3} \top_L \quad \rightsquigarrow \quad \frac{\frac{}{h_4 : \Delta_5, F_2 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \top, \Delta_5, F_2 \vdash \Delta_1, F_3} \top_L$$

## 4.2 Status of $\wedge_R$ : (Left Premise): Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \rightarrow F_6} \rightarrow_R \quad \rightsquigarrow \quad \frac{\frac{}{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_1, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \rightarrow F_6} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \wedge F_2 \quad h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \wedge F_6} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5} \text{ax/ind} \quad \frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \wedge F_6} \wedge_R$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{}{h_1 : \Delta_2 \vdash \Delta_3, F_4} \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4} \text{height}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \vee F_6} \vee_R \quad \rightsquigarrow \quad \frac{\frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \vee F_6} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \wedge F_2} \perp_R \quad \rightsquigarrow \quad \frac{\frac{}{h_3 : \Delta_4 \vdash \Delta_5, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \wedge F_2} \top_R \quad \rightsquigarrow \quad \frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_6, F_2 \wedge F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \wedge F_3} \rightarrow_L \quad \rightsquigarrow \quad \frac{\frac{}{h_4 : \Delta_5 \vdash \Delta_1, F_2, F_6} \text{ax/ind} \quad \frac{}{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \wedge F_3} \wedge_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2}}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2} \text{ax/ind} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2 \wedge F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \wedge F_3} \vee_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2} \quad \overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2}}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2} \text{ax/ind} \vee_L$$

- Case rule  $\perp_L$

$$\overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \perp_L \rightsquigarrow \overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2} \perp_L$$

- Case rule  $I$

$$\overline{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \wedge F_2} I \rightsquigarrow \overline{\bullet h_3 : \Delta_4, p_5 \vdash \Delta_6, F_1, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \top_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_2}}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2} \text{ax/ind} \top_L$$

### 4.3 Status of $\wedge_R$ (Right Premise): : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \rightarrow F_6} \rightarrow_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_2, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \rightarrow F_6} \text{ax/ind} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \wedge F_2 \quad h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \wedge F_6} \wedge_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5} \quad \overline{h_3 : \Delta_4 \vdash \Delta_7, F_2, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \wedge F_6} \text{ax/ind} \wedge_R$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_3, F_5}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_5} \text{ax/height}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_6, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \vee F_6} \vee_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \vee F_6} \text{ax/ind} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \wedge F_2} \perp_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_5, F_2}}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_2} \text{ax/ind} \perp_R$$



- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \wedge F_2} \top_R \rightsquigarrow \frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_2} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_6, F_2 \wedge F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \wedge F_3} \rightarrow_L \rightsquigarrow \frac{\frac{}{h_4 : \Delta_5 \vdash \Delta_1, F_3, F_6} \text{ax/ind} \quad \frac{}{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \wedge F_3} \wedge_L \rightsquigarrow \frac{\frac{}{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_3} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2 \wedge F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \wedge F_3} \vee_L \rightsquigarrow \frac{\frac{}{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_3} \text{ax/ind} \quad \frac{}{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_3} \vee_L$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \perp_L \rightsquigarrow \frac{}{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_3} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \wedge F_2} I \rightsquigarrow \frac{}{\bullet h_3 : \Delta_4, p_5 \vdash \Delta_6, F_2, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \top_L \rightsquigarrow \frac{\frac{}{h_4 : \Delta_5 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_3} \top_L$$

#### 4.4 Status of $\vee_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_6, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \rightarrow F_6} \rightarrow_R \rightsquigarrow \frac{\frac{}{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_1, F_2, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \rightarrow F_6} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_1 \vee F_2 \quad h_3 : \Delta_4 \vdash \Delta_7, F_6, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \wedge F_6} \wedge_R \rightsquigarrow \frac{\frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5} \text{ax/ind} \quad \frac{}{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \wedge F_6} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_7, F_5, F_6, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \vee F_6} \vee_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5, F_6} \text{ ax/ind}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \vee F_6} \vee_R$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5} \text{ ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5} \text{ height}$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \vee F_2} \perp_R \rightsquigarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_5, F_1, F_2} \text{ ax/ind}}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1, F_2} \perp_R$$

- Case rule  $\top_R$

$$\overline{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \vee F_2} \top_R \rightsquigarrow \overline{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1, F_2} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_6, F_2 \vee F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \vee F_3} \rightarrow_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_2, F_3, F_6} \text{ ax/ind} \quad \overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2, F_3} \text{ ax/ind}}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2, F_3} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \vee F_3} \wedge_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2, F_3} \text{ ax/ind}}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2, F_3} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2 \vee F_3 \quad h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \vee F_3} \vee_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2, F_3} \text{ ax/ind} \quad \overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2, F_3} \text{ ax/ind}}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2, F_3} \vee_L$$

- Case rule  $\perp_L$

$$\overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \vee F_3} \perp_L \rightsquigarrow \overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2, F_3} \perp_L$$

- Case rule  $I$

$$\overline{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \vee F_2} I \rightsquigarrow \overline{\bullet h_3 : \Delta_4, p_5 \vdash \Delta_6, F_1, F_2, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \vee F_3} \top_L \rightsquigarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_2, F_3} \text{ ax/ind}}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2, F_3} \top_L$$

## 4.5 Status of $\perp_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_1 : \Delta_2, F_3 \vdash \perp, \Delta_5, F_4}{\bullet h_1 : \Delta_2 \vdash (\perp, \Delta_5), F_3 \rightarrow F_4} \rightarrow_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2, F_3 \vdash \Delta_5, F_4}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \rightarrow F_4} \text{ax/ind} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash \perp, \Delta_5, F_3 \quad h_1 : \Delta_2 \vdash \perp, \Delta_5, F_4}{\bullet h_1 : \Delta_2 \vdash (\perp, \Delta_5), F_3 \wedge F_4} \wedge_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{ax/ind} \quad \overline{h_1 : \Delta_2 \vdash \Delta_5, F_4} \text{ax/ind}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \wedge F_4} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \perp, \Delta_5, F_3, F_4}{\bullet h_1 : \Delta_2 \vdash (\perp, \Delta_5), F_3 \vee F_4} \vee_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_5, F_3, F_4} \text{ax/ind}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \vee F_4} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightsquigarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_3} \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{height}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \perp, \Delta_3} \top_R \rightsquigarrow \frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_2 : \Delta_3 \vdash \perp, \Delta_1, F_4 \quad h_2 : \Delta_3, F_5 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \perp, \Delta_1} \rightarrow_L \rightsquigarrow \frac{\overline{h_2 : \Delta_3 \vdash \Delta_1, F_4} \text{ax/ind} \quad \overline{h_2 : \Delta_3, F_5 \vdash \Delta_1} \text{ax/ind}}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \Delta_1} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_2 : \Delta_3, F_4, F_5 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \wedge F_5 \vdash \perp, \Delta_1} \wedge_L \rightsquigarrow \frac{\overline{h_2 : \Delta_3, F_4, F_5 \vdash \Delta_1} \text{ax/ind}}{\bullet h_2 : \Delta_3, F_4 \wedge F_5 \vdash \Delta_1} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_2 : \Delta_3, F_4 \vdash \perp, \Delta_1 \quad h_2 : \Delta_3, F_5 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \perp, \Delta_1} \vee_L \rightsquigarrow \frac{\overline{h_2 : \Delta_3, F_4 \vdash \Delta_1} \text{ax/ind} \quad \overline{h_2 : \Delta_3, F_5 \vdash \Delta_1} \text{ax/ind}}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \Delta_1} \vee_L$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_2 : \perp, \Delta_3 \vdash \perp, \Delta_1} \perp_L \rightsquigarrow \frac{}{\bullet h_2 : \perp, \Delta_3 \vdash \Delta_1} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_3, \Delta_2 \vdash p_3, \perp, \Delta_4} I \quad \rightsquigarrow \quad \frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I$$

- Case rule  $\top_L$

$$\frac{h_2 : \Delta_3 \vdash \perp, \Delta_1}{\bullet h_2 : \top, \Delta_3 \vdash \perp, \Delta_1} \top_L \quad \rightsquigarrow \quad \frac{\frac{}{h_2 : \Delta_3 \vdash \Delta_1} \text{ax/ind}}{\bullet h_2 : \top, \Delta_3 \vdash \Delta_1} \top_L$$

## 4.6 Status of $\top_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_1 : \Delta_2, F_3 \vdash \top, \Delta_5, F_4}{\bullet h_1 : \Delta_2 \vdash (\top, \Delta_5), F_3 \rightarrow F_4} \rightarrow_R \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash \top, \Delta_5, F_3 \quad h_1 : \Delta_2 \vdash \top, \Delta_5, F_4}{\bullet h_1 : \Delta_2 \vdash (\top, \Delta_5), F_3 \wedge F_4} \wedge_R \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \top, \Delta_5, F_3, F_4}{\bullet h_1 : \Delta_2 \vdash (\top, \Delta_5), F_3 \vee F_4} \vee_R \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \top, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \top, \Delta_3} \perp_R \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\rightarrow_L$

$$\frac{h_2 : \Delta_3 \vdash \top, \Delta_1, F_4 \quad h_2 : \Delta_3, F_5 \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \top, \Delta_1} \rightarrow_L \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\wedge_L$

$$\frac{h_2 : \Delta_3, F_4, F_5 \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \wedge F_5 \vdash \top, \Delta_1} \wedge_L \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\vee_L$

$$\frac{h_2 : \Delta_3, F_4 \vdash \top, \Delta_1 \quad h_2 : \Delta_3, F_5 \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \top, \Delta_1} \vee_L \quad \rightsquigarrow \quad \text{trivial}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_2 : \perp, \Delta_3 \vdash \top, \Delta_1} \perp_L \rightsquigarrow \text{trivial}$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_3, \Delta_2 \vdash p_3, \top, \Delta_4} I \rightsquigarrow \text{trivial}$$

- Case rule  $\top_L$

$$\frac{h_2 : \Delta_3 \vdash \top, \Delta_1}{\bullet h_2 : \top, \Delta_3 \vdash \top, \Delta_1} \top_L \rightsquigarrow \text{trivial}$$

## 4.7 Status of $\rightarrow_L$ : (Left Premise): Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : \Delta_1, F_6, F_2 \rightarrow F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_6 \vdash \Delta_5, F_2, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \quad h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R}{\frac{\frac{}{h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6} \text{ax/ind} \quad \frac{}{h_4 : \Delta_1 \vdash \Delta_5, F_2, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6 \wedge F_7} \wedge_R} \rightsquigarrow$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \perp, \Delta_5} \perp_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1 \vdash \Delta_5, F_2} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \perp, \Delta_5, F_2} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \top, \Delta_5} \top_R \rightsquigarrow \frac{}{\bullet h_4 : \Delta_1 \vdash \top, \Delta_5, F_2} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_7 \vdash \Delta_6, F_1, F_4} \text{ax/ind} \quad \frac{}{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \rightarrow F_5 \vdash \Delta_6, F_1} \rightarrow_L$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightsquigarrow \frac{\frac{}{h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{height}$$

- Case rule  $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \quad \rightsquigarrow \quad \frac{\overline{h_3 : \Delta_7, F_4, F_5 \vdash \Delta_6, F_1} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_4 \wedge F_5 \vdash \Delta_6, F_1} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : \Delta_7, F_4, F_1 \rightarrow F_2 \vdash \Delta_6 \quad h_3 : \Delta_7, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \quad \rightsquigarrow \quad \frac{\overline{h_3 : \Delta_7, F_4 \vdash \Delta_6, F_1} \text{ ax/ind} \quad \overline{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_4 \vee F_5 \vdash \Delta_6, F_1} \vee_L$$

- Case rule  $\perp_L$

$$\overline{\bullet h_3 : \perp, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \perp_L \quad \rightsquigarrow \quad \overline{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_4, F_1} \perp_L$$

- Case rule  $I$

$$\overline{\bullet h_3 : p_4, \Delta_6, F_1 \rightarrow F_2 \vdash p_4, \Delta_5} I \quad \rightsquigarrow \quad \overline{\bullet h_3 : \Delta_6, p_4 \vdash \Delta_5, F_1, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \top_L \quad \rightsquigarrow \quad \frac{\overline{h_3 : \Delta_5 \vdash \Delta_4, F_1} \text{ ax/ind}}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_4, F_1} \top_L$$

#### 4.8 Status of $\rightarrow_L$ (Right Premise): : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : \Delta_1, F_6, F_2 \rightarrow F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \quad \rightsquigarrow \quad \frac{\overline{h_4 : \Delta_1, F_3, F_6 \vdash \Delta_5, F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \quad h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \quad \rightsquigarrow \quad \frac{\overline{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6} \text{ ax/ind} \quad \overline{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \quad \rightsquigarrow \quad \frac{\overline{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6, F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \perp, \Delta_5} \perp_R \quad \rightsquigarrow \quad \frac{\overline{h_4 : \Delta_1, F_3 \vdash \Delta_5} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \perp, \Delta_5} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \top, \Delta_5} \top_R \rightsquigarrow \frac{}{\bullet h_4 : \Delta_1, F_3 \vdash \top, \Delta_5} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4} \text{ax/ind} \quad \frac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightsquigarrow \frac{\frac{}{h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{ax}}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{height}$$

- Case rule  $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_7, F_2, F_4, F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : \Delta_7, F_4, F_1 \rightarrow F_2 \vdash \Delta_6 \quad h_3 : \Delta_7, F_5, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_7, F_2, F_4 \vdash \Delta_6} \text{ax/ind} \quad \frac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \perp_L \rightsquigarrow \frac{}{\bullet h_3 : \perp, \Delta_5, F_2 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \rightarrow F_2 \vdash p_4, \Delta_5} I \rightsquigarrow \frac{}{\bullet h_3 : \Delta_6, F_2, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \top_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_5, F_2 \vdash \Delta_4} \text{ax/ind}}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \top_L$$

## 4.9 Status of $\wedge_L$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : \Delta_1, F_6, F_2 \wedge F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_2, F_3, F_6 \vdash \Delta_5, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \quad h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6} \text{ax/ind} \quad \frac{}{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{\frac{h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \vee F_7} \text{ ax/ind} \rightsquigarrow \frac{\frac{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \vee F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{\frac{h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \perp, \Delta_5} \perp_R}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \perp, \Delta_5} \text{ ax/ind} \rightsquigarrow \frac{\frac{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \perp, \Delta_5} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \perp, \Delta_5} \perp_R$$

- Case rule  $\top_R$

$$\frac{\overline{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \top, \Delta_5}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \top, \Delta_5} \top_R \rightsquigarrow \frac{\overline{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \top, \Delta_5}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \top, \Delta_5} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{\frac{h_3 : \Delta_7, F_1 \wedge F_2 \vdash \Delta_6, F_4}{\bullet h_3 : (\Delta_7, F_1 \wedge F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L}{\bullet h_3 : (\Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ ax/ind} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_7, F_1, F_2 \vdash \Delta_6, F_4}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{\frac{h_3 : \Delta_7, F_4, F_5, F_1 \wedge F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \wedge F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L}{\bullet h_3 : (\Delta_7, F_1, F_2, F_4 \wedge F_5 \vdash \Delta_6} \text{ ax/ind} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_7, F_1, F_2, F_4, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \wedge F_5 \vdash \Delta_6} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

$$\frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{ ax} \rightsquigarrow \frac{\frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{ ax}}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{ height}}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{ height}$$

- Case rule  $\vee_L$

$$\frac{\frac{h_3 : \Delta_7, F_4, F_1 \wedge F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \wedge F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L}{\bullet h_3 : (\Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ ax/ind} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_7, F_1, F_2, F_4 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

- Case rule  $\perp_L$

$$\frac{\overline{\bullet h_3 : \perp, \Delta_5, F_1 \wedge F_2 \vdash \Delta_4}}{\bullet h_3 : \perp, \Delta_5, F_1, F_2 \vdash \Delta_4} \perp_L \rightsquigarrow \frac{\overline{\bullet h_3 : \perp, \Delta_5, F_1, F_2 \vdash \Delta_4}}{\bullet h_3 : \perp, \Delta_5, F_1, F_2 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{\overline{\bullet h_3 : p_4, \Delta_6, F_1 \wedge F_2 \vdash p_4, \Delta_5}}{\bullet h_3 : p_4, \Delta_6, F_1, F_2 \vdash p_4, \Delta_5} I \rightsquigarrow \frac{\overline{\bullet h_3 : \Delta_6, F_1, F_2, p_4 \vdash \Delta_5, p_4}}{\bullet h_3 : \Delta_6, F_1, F_2, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{\frac{h_3 : \Delta_5, F_1 \wedge F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \wedge F_2 \vdash \Delta_4} \top_L}{\bullet h_3 : \top, \Delta_5, F_1, F_2 \vdash \Delta_4} \text{ ax/ind} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_5, F_1, F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1, F_2 \vdash \Delta_4} \text{ ax/ind}}{\bullet h_3 : \top, \Delta_5, F_1, F_2 \vdash \Delta_4} \top_L}{\bullet h_3 : \top, \Delta_5, F_1, F_2 \vdash \Delta_4} \top_L$$



#### 4.10 Status of $\vee_L$ : (Left Premise): Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : \Delta_1, F_6, F_2 \vee F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightsquigarrow \frac{\overline{h_4 : \Delta_1, F_2, F_6 \vdash \Delta_5, F_7}}{\bullet h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6 \rightarrow F_7} \xrightarrow{\text{ax/ind}} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \quad h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \rightsquigarrow \frac{\overline{h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6} \quad \overline{h_4 : \Delta_1, F_2 \vdash \Delta_5, F_7}}{\bullet h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6 \wedge F_7} \xrightarrow{\text{ax/ind}} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \rightsquigarrow \frac{\overline{h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6, F_7}}{\bullet h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6 \vee F_7} \xrightarrow{\text{ax/ind}} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \perp, \Delta_5} \perp_R \rightsquigarrow \frac{\overline{h_4 : \Delta_1, F_2 \vdash \Delta_5}}{\bullet h_4 : \Delta_1, F_2 \vdash \perp, \Delta_5} \xrightarrow{\text{ax/ind}} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \top, \Delta_5} \top_R \rightsquigarrow \frac{}{\bullet h_4 : \Delta_1, F_2 \vdash \top, \Delta_5} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightsquigarrow \frac{\overline{h_3 : \Delta_7, F_1 \vdash \Delta_6, F_4} \quad \overline{h_3 : \Delta_7, F_1, F_5 \vdash \Delta_6}}{\bullet h_3 : \Delta_7, F_1, F_4 \rightarrow F_5 \vdash \Delta_6} \xrightarrow{\text{ax/ind}} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightsquigarrow \frac{\overline{h_3 : \Delta_7, F_1, F_4, F_5 \vdash \Delta_6}}{\bullet h_3 : \Delta_7, F_1, F_4 \wedge F_5 \vdash \Delta_6} \xrightarrow{\text{ax/ind}} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : \Delta_7, F_4, F_1 \vee F_2 \vdash \Delta_6 \quad h_3 : \Delta_7, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightsquigarrow \frac{\overline{h_3 : \Delta_7, F_1, F_4 \vdash \Delta_6} \quad \overline{h_3 : \Delta_7, F_1, F_5 \vdash \Delta_6}}{\bullet h_3 : \Delta_7, F_1, F_4 \vee F_5 \vdash \Delta_6} \xrightarrow{\text{ax/ind}} \vee_L$$

$$\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightsquigarrow \frac{\overline{h_1 : \Delta_2, F_3 \vdash \Delta_5}}{\bullet h_1 : \Delta_2, F_3 \vdash \Delta_5} \xrightarrow{\text{ax/height}} \vee_L$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \perp_L \rightsquigarrow \frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \vee F_2 \vdash p_4, \Delta_5} I \rightsquigarrow \frac{}{\bullet h_3 : \Delta_6, F_1, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \top_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_5, F_1 \vdash \Delta_4} \text{ax/ind}}{\bullet h_3 : \top, \Delta_5, F_1 \vdash \Delta_4} \top_L$$

#### 4.11 Status of $\vee_L$ (Right Premise): : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \quad h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6} \text{ax/ind} \quad \frac{}{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \perp, \Delta_5} \perp_R \rightsquigarrow \frac{\frac{}{h_4 : \Delta_1, F_3 \vdash \Delta_5} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \perp, \Delta_5} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \top, \Delta_5} \top_R \rightsquigarrow \frac{}{\bullet h_4 : \Delta_1, F_3 \vdash \top, \Delta_5} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \vee F_2 \vdash \Delta_6, F_4 \quad h_3 : \Delta_7, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4} \text{ax/ind} \quad \frac{}{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_3 : \Delta_7, F_4, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightsquigarrow \frac{\frac{}{h_3 : \Delta_7, F_2, F_4, F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : \Delta_7, F_4, F_1 \vee F_2 \vdash \Delta_6 \quad h_3 : \Delta_7, F_5, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightsquigarrow \frac{\frac{h_3 : \Delta_7, F_2, F_4 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ax/ind} \quad \frac{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

$$\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5 \quad h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{ax}}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{height}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \perp_L \rightsquigarrow \frac{}{\bullet h_3 : \perp, \Delta_5, F_2 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \vee F_2 \vdash p_4, \Delta_5} I \rightsquigarrow \frac{}{\bullet h_3 : \Delta_6, F_2, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \top_L \rightsquigarrow \frac{\frac{h_3 : \Delta_5, F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \text{ax/ind}}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \top_L$$

## 4.12 Status of $\perp_L$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_2 : \perp, \Delta_1, F_4 \vdash \Delta_3, F_5}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightsquigarrow \text{trivial}$$

- Case rule  $\wedge_R$

$$\frac{h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \quad h_2 : \perp, \Delta_1 \vdash \Delta_3, F_5}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightsquigarrow \text{trivial}$$

- Case rule  $\vee_R$

$$\frac{h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4, F_5}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightsquigarrow \text{trivial}$$

- Case rule  $\perp_R$

$$\frac{h_2 : \perp, \Delta_1 \vdash \Delta_3}{\bullet h_2 : \perp, \Delta_1 \vdash \perp, \Delta_3} \perp_R \rightsquigarrow \text{trivial}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_2 : \perp, \Delta_1 \vdash \top, \Delta_3} \top_R \rightsquigarrow \text{trivial}$$

- Case rule  $\rightarrow_L$

$$\frac{h_1 : \perp, \Delta_5 \vdash \Delta_4, F_2 \quad h_1 : \perp, \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : (\perp, \Delta_5), F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L \rightsquigarrow \text{trivial}$$

- Case rule  $\wedge_L$

$$\frac{h_1 : \perp, \Delta_5, F_2, F_3 \vdash \Delta_4}{\bullet h_1 : (\perp, \Delta_5), F_2 \wedge F_3 \vdash \Delta_4} \wedge_L \rightsquigarrow \text{trivial}$$

- Case rule  $\vee_L$

$$\frac{h_1 : \perp, \Delta_5, F_2 \vdash \Delta_4 \quad h_1 : \perp, \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : (\perp, \Delta_5), F_2 \vee F_3 \vdash \Delta_4} \vee_L \rightsquigarrow \text{trivial}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \rightsquigarrow \text{trivial}$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_2, \perp, \Delta_4 \vdash p_2, \Delta_3} I \rightsquigarrow \text{trivial}$$

- Case rule  $\top_L$

$$\frac{h_1 : \perp, \Delta_3 \vdash \Delta_2}{\bullet h_1 : \top, \perp, \Delta_3 \vdash \Delta_2} \top_L \rightsquigarrow \text{trivial}$$

#### 4.13 Status of $I$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : \Delta_1, F_4, p_2 \vdash \Delta_6, F_5, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash (\Delta_6, p_2), F_4 \rightarrow F_5} \rightarrow_R \rightsquigarrow \text{trivial}$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_1, p_2 \vdash \Delta_6, F_4, p_2 \quad h_3 : \Delta_1, p_2 \vdash \Delta_6, F_5, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash (\Delta_6, p_2), F_4 \wedge F_5} \wedge_R \rightsquigarrow \text{trivial}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_1, p_2 \vdash \Delta_6, F_4, F_5, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash (\Delta_6, p_2), F_4 \vee F_5} \vee_R \rightsquigarrow \text{trivial}$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_1, p_2 \vdash \Delta_4, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash \perp, \Delta_4, p_2} \perp_R \rightsquigarrow \text{trivial}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_1, p_2 \vdash \top, \Delta_4, p_2} \top_R \rightsquigarrow \text{trivial}$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_6, p_1 \vdash \Delta_2, F_4, p_1 \quad h_3 : \Delta_6, F_5, p_1 \vdash \Delta_2, p_1}{\bullet h_3 : (\Delta_6, p_1), F_4 \rightarrow F_5 \vdash \Delta_2, p_1} \rightarrow_L \rightsquigarrow \text{trivial}$$

- Case rule  $\wedge_L$

$$\frac{h_3 : \Delta_6, F_4, F_5, p_1 \vdash \Delta_2, p_1}{\bullet h_3 : (\Delta_6, p_1), F_4 \wedge F_5 \vdash \Delta_2, p_1} \wedge_L \rightsquigarrow \text{trivial}$$

- Case rule  $\vee_L$

$$\frac{h_3 : \Delta_6, F_4, p_1 \vdash \Delta_2, p_1 \quad h_3 : \Delta_6, F_5, p_1 \vdash \Delta_2, p_1}{\bullet h_3 : (\Delta_6, p_1), F_4 \vee F_5 \vdash \Delta_2, p_1} \vee_L \rightsquigarrow \text{trivial}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_4, p_1 \vdash \Delta_2, p_1} \perp_L \rightsquigarrow \text{trivial}$$

- Case rule  $I$

$$\frac{}{\bullet h_2 : p_3, \Delta_5, p_1 \vdash p_3, \Delta_4, p_1} I \rightsquigarrow \text{trivial}$$

$$\frac{}{\bullet h_1 : p_3, \Delta_2 \vdash p_3, \Delta_4} I \rightsquigarrow \text{trivial}$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_4, p_1 \vdash \Delta_2, p_1}{\bullet h_3 : \top, \Delta_4, p_1 \vdash \Delta_2, p_1} \top_L \rightsquigarrow \text{trivial}$$

#### 4.14 Status of $\top_L$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_2 : \top, \Delta_1, F_4 \vdash \Delta_3, F_5}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightsquigarrow \frac{\frac{}{h_2 : \Delta_1, F_4 \vdash \Delta_3, F_5} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \quad h_2 : \top, \Delta_1 \vdash \Delta_3, F_5}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightsquigarrow \frac{\frac{}{h_2 : \Delta_1 \vdash \Delta_3, F_4} \text{ax/ind} \quad \frac{}{h_2 : \Delta_1 \vdash \Delta_3, F_5} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_2 : \top, \Delta_1 \vdash \Delta_3, F_4, F_5}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightsquigarrow \frac{\frac{h_2 : \Delta_1 \vdash \Delta_3, F_4, F_5}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_2 : \top, \Delta_1 \vdash \Delta_3}{\bullet h_2 : \top, \Delta_1 \vdash \perp, \Delta_3} \perp_R \rightsquigarrow \frac{\frac{h_2 : \Delta_1 \vdash \Delta_3}{\bullet h_2 : \Delta_1 \vdash \perp, \Delta_3} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \perp, \Delta_3} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_2 : \top, \Delta_1 \vdash \top, \Delta_3} \top_R \rightsquigarrow \frac{}{\bullet h_2 : \Delta_1 \vdash \top, \Delta_3} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{h_1 : \top, \Delta_5 \vdash \Delta_4, F_2 \quad h_1 : \top, \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : (\top, \Delta_5), F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L \rightsquigarrow \frac{\frac{h_1 : \Delta_5 \vdash \Delta_4, F_2}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \text{ax/ind} \quad \frac{h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \text{ax/ind}}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_1 : \top, \Delta_5, F_2, F_3 \vdash \Delta_4}{\bullet h_1 : (\top, \Delta_5), F_2 \wedge F_3 \vdash \Delta_4} \wedge_L \rightsquigarrow \frac{\frac{h_1 : \Delta_5, F_2, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \text{ax/ind}}{\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_1 : \top, \Delta_5, F_2 \vdash \Delta_4 \quad h_1 : \top, \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : (\top, \Delta_5), F_2 \vee F_3 \vdash \Delta_4} \vee_L \rightsquigarrow \frac{\frac{h_1 : \Delta_5, F_2 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \text{ax/ind} \quad \frac{h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \text{ax/ind}}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \vee_L$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \top, \Delta_3 \vdash \Delta_2} \perp_L \rightsquigarrow \frac{}{\bullet h_1 : \perp, \Delta_3 \vdash \Delta_2} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_2, \top, \Delta_4 \vdash p_2, \Delta_3} I \rightsquigarrow \frac{}{\bullet h_1 : \Delta_4, p_2 \vdash \Delta_3, p_2} I$$

- Case rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \rightsquigarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{height}$$

## 5 Height preserving admissibility of contraction on the left

- Case(s) rule  $\rightarrow_R$

$$\frac{h_3 : \Delta_1, F_2, F_2, F_5 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5 \rightarrow F_6} \rightarrow_R \rightsquigarrow \frac{\frac{h_3 : \Delta_1, F_2, F_2, F_5 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5} \xrightarrow{\text{IH}} \frac{h_3 : \Delta_1, F_2, F_2, F_5 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5 \rightarrow F_6} \xrightarrow{\text{ax}} \frac{h_3 : \Delta_1, F_2, F_2, F_5 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5 \rightarrow F_6} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5 \quad h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5 \wedge F_6} \wedge_R \rightsquigarrow \frac{\frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5} \xrightarrow{\text{IH}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5} \xrightarrow{\text{ax}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5} \xrightarrow{\text{IH}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5 \wedge F_6} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5, F_6}{\bullet h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5 \vee F_6} \vee_R \rightsquigarrow \frac{\frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5, F_6}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5, F_6} \xrightarrow{\text{IH}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5, F_6}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5 \vee F_6} \xrightarrow{\text{ax}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4, F_5, F_6}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4, F_5 \vee F_6} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, F_2, F_2 \vdash \perp, \Delta_4} \perp_R \rightsquigarrow \frac{\frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4} \xrightarrow{\text{IH}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, F_2 \vdash \Delta_4} \xrightarrow{\text{ax}} \frac{h_3 : \Delta_1, F_2, F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, F_2 \vdash \perp, \Delta_4} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_1, F_2, F_2 \vdash \top, \Delta_4} \top_R \rightsquigarrow \frac{}{\bullet h_3 : \Delta_1, F_2 \vdash \top, \Delta_4} \top_R$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_2 : \Delta_1, F_3 \rightarrow F_4 \vdash \Delta_5, F_3 \quad h_2 : \Delta_1, F_4, F_3 \rightarrow F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \rightarrow F_4, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightsquigarrow \frac{\frac{h_2 : \Delta_1 \vdash \Delta_5, F_3, F_3}{\bullet h_2 : \Delta_1 \vdash \Delta_5, F_3} \xrightarrow{\text{IH-Mutual}} \frac{h_2 : \Delta_1 \vdash \Delta_5, F_3}{\bullet h_2 : \Delta_1, F_3 \rightarrow F_4 \vdash \Delta_5} \xrightarrow{\text{inv-th/ax}} \frac{h_2 : \Delta_1 \vdash \Delta_5, F_3, F_3}{\bullet h_2 : \Delta_1, F_3 \rightarrow F_4 \vdash \Delta_5} \xrightarrow{\text{inv-th/ax}} \frac{h_2 : \Delta_1, F_4, F_3 \rightarrow F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \rightarrow F_4, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L$$

$$\frac{h_2 : \Delta_6, F_1, F_1 \vdash \Delta_5, F_3 \quad h_2 : \Delta_6, F_1, F_1, F_4 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, F_3 \rightarrow F_4), F_1, F_1 \vdash \Delta_5} \rightarrow_L \rightsquigarrow \frac{\frac{h_2 : \Delta_6, F_1, F_1 \vdash \Delta_5, F_3}{\bullet h_2 : \Delta_6, F_1 \vdash \Delta_5, F_3} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_6, F_1, F_1 \vdash \Delta_5, F_3}{\bullet h_2 : \Delta_6, F_1 \vdash \Delta_5, F_3} \xrightarrow{\text{ax}} \frac{h_2 : \Delta_6, F_1, F_1 \vdash \Delta_5, F_3}{\bullet h_2 : \Delta_6, F_1 \vdash \Delta_5, F_3} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_6, F_1, F_1, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_2 : \Delta_1, F_3, F_4, F_3 \wedge F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \wedge F_4, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightsquigarrow \frac{\frac{h_2 : \Delta_1, F_3, F_4, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3, F_3, F_4 \vdash \Delta_5} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_1, F_3, F_3, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \wedge F_4 \vdash \Delta_5} \xrightarrow{\text{inv-th/ax}} \frac{h_2 : \Delta_1, F_3, F_4, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L$$

$$\frac{h_2 : \Delta_6, F_1, F_1, F_3, F_4 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, F_3 \wedge F_4), F_1, F_1 \vdash \Delta_5} \wedge_L \rightsquigarrow \frac{\frac{h_2 : \Delta_6, F_1, F_1, F_3, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3, F_4 \vdash \Delta_5} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_6, F_1, F_3, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \wedge F_4 \vdash \Delta_5} \xrightarrow{\text{ax}} \frac{h_2 : \Delta_6, F_1, F_1, F_3, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_2 : \Delta_1, F_3, F_3 \vee F_4 \vdash \Delta_5 \quad h_2 : \Delta_1, F_4, F_3 \vee F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \vee F_4, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightsquigarrow \frac{\frac{h_2 : \Delta_1, F_3, F_3 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \vdash \Delta_5} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_1, F_3, F_3 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \vee F_4 \vdash \Delta_5} \xrightarrow{\text{inv-th/ax}} \frac{h_2 : \Delta_1, F_3, F_3 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \vee F_4 \vdash \Delta_5} \xrightarrow{\text{inv-th/ax}} \frac{h_2 : \Delta_1, F_4, F_3 \vee F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, F_3 \vee F_4 \vdash \Delta_5} \vee_L$$

$$\frac{h_2 : \Delta_6, F_1, F_1, F_3 \vdash \Delta_5 \quad h_2 : \Delta_6, F_1, F_1, F_4 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, F_3 \vee F_4), F_1, F_1 \vdash \Delta_5} \vee_L \rightsquigarrow \frac{\frac{h_2 : \Delta_6, F_1, F_1, F_3 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \vdash \Delta_5} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_6, F_1, F_3 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \vee F_4 \vdash \Delta_5} \xrightarrow{\text{ax}} \frac{h_2 : \Delta_6, F_1, F_1, F_3 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \vee F_4 \vdash \Delta_5} \xrightarrow{\text{IH}} \frac{h_2 : \Delta_6, F_1, F_1, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_3 \vee F_4 \vdash \Delta_5} \vee_L$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_2 : \Delta_1, \perp, \perp \vdash \Delta_3} \perp_L \rightsquigarrow \frac{}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3} \perp_L$$

$$\frac{}{\bullet h_2 : (\perp, \Delta_4), F_1, F_1 \vdash \Delta_3} \perp_L \rightsquigarrow \frac{}{\bullet h_2 : \perp, \Delta_4, F_1 \vdash \Delta_3} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_2 : \Delta_1, p_3, p_3 \vdash \Delta_4, p_3} I \rightsquigarrow \frac{}{\bullet h_2 : \Delta_1, p_3 \vdash \Delta_4, p_3} I$$

$$\frac{}{\bullet h_2 : (\Delta_5, p_3), F_1, F_1 \vdash \Delta_4, p_3} I \rightsquigarrow \frac{}{\bullet h_2 : \Delta_5, F_1, p_3 \vdash \Delta_4, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_2 : \top, \Delta_1 \vdash \Delta_3}{\bullet h_2 : \Delta_1, \top, \top \vdash \Delta_3} \top_L \rightsquigarrow \frac{\frac{}{h_2 : \Delta_1 \vdash \Delta_3} \text{inv-th/ax}}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3} \top_L$$

$$\frac{h_2 : \Delta_4, F_1, F_1 \vdash \Delta_3}{\bullet h_2 : (\top, \Delta_4), F_1, F_1 \vdash \Delta_3} \top_L \rightsquigarrow \frac{\frac{\frac{}{h_2 : \Delta_4, F_1, F_1 \vdash \Delta_3} \text{ax}}{h_2 : \Delta_4, F_1 \vdash \Delta_3} \text{IH}}{\bullet h_2 : \top, \Delta_4, F_1 \vdash \Delta_3} \top_L$$



## 6 Height preserving admissibility of contraction on the Right

- Case(s) rule  $\rightarrow_R$

$$\begin{array}{c}
 \frac{h_2 : \Delta_3, F_4 \vdash \Delta_1, F_5, F_4 \rightarrow F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \rightarrow F_5, F_4 \rightarrow F_5} \rightarrow_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3, F_4, F_4 \vdash \Delta_1, F_5, F_5}{h_2 : \Delta_3, F_4 \vdash \Delta_1, F_5, F_5} \text{IH-Mutual} \quad \frac{h_2 : \Delta_3, F_4 \vdash \Delta_1, F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \rightarrow F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \rightarrow F_5} \rightarrow_R \\
 \\
 \frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_1, F_5}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \rightarrow F_5), F_1, F_1} \rightarrow_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_1, F_5}{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_5} \text{ax} \quad \frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \rightarrow F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \rightarrow F_5} \rightarrow_R
 \end{array}$$

- Case(s) rule  $\wedge_R$

$$\begin{array}{c}
 \frac{h_2 : \Delta_3 \vdash \Delta_1, F_4, F_4 \wedge F_5 \quad h_2 : \Delta_3 \vdash \Delta_1, F_5, F_4 \wedge F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \wedge F_5, F_4 \wedge F_5} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, F_4, F_4}{h_2 : \Delta_3 \vdash \Delta_1, F_4} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, F_5, F_5}{h_2 : \Delta_3 \vdash \Delta_1, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \wedge F_5} \wedge_R \\
 \\
 \frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_1, F_4 \quad h_2 : \Delta_3 \vdash \Delta_6, F_1, F_1, F_5}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \wedge F_5), F_1, F_1} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_1, F_4}{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4} \text{ax} \quad \frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_1, F_5}{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_5} \text{ax}}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \wedge F_5} \wedge_R
 \end{array}$$

- Case(s) rule  $\vee_R$

$$\begin{array}{c}
 \frac{h_2 : \Delta_3 \vdash \Delta_1, F_4, F_5, F_4 \vee F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \vee F_5, F_4 \vee F_5} \vee_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, F_4, F_4, F_5, F_5}{h_2 : \Delta_3 \vdash \Delta_1, F_4, F_4, F_5} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, F_4, F_5}{h_2 : \Delta_3 \vdash \Delta_1, F_4 \vee F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, F_4 \vee F_5} \vee_R \\
 \\
 \frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_1, F_4, F_5}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \vee F_5), F_1, F_1} \vee_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_1, F_4, F_5}{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4, F_5} \text{ax} \quad \frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4, F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \vee F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \vee F_5} \vee_R
 \end{array}$$

- Case(s) rule  $\perp_R$

$$\begin{array}{c}
 \frac{h_2 : \Delta_3 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \perp, \perp} \perp_R \quad \rightsquigarrow \quad \frac{h_2 : \Delta_3 \vdash \perp}{\bullet h_2 : \Delta_3 \vdash \perp, \Delta_1} \text{inv-th/ax} \perp_R \\
 \\
 \frac{h_2 : \Delta_3 \vdash \Delta_4, F_1, F_1}{\bullet h_2 : \Delta_3 \vdash (\perp, \Delta_4), F_1, F_1} \perp_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_4, F_1, F_1}{h_2 : \Delta_3 \vdash \Delta_4, F_1} \text{ax} \quad \frac{h_2 : \Delta_3 \vdash \Delta_4, F_1}{\bullet h_2 : \Delta_3 \vdash \perp, \Delta_4, F_1} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \perp, \Delta_4, F_1} \perp_R
 \end{array}$$

- Case(s) rule  $\top_R$

$$\begin{array}{c}
 \frac{}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \top, \top} \top_R \quad \rightsquigarrow \quad \frac{}{\bullet h_2 : \Delta_3 \vdash \top, \Delta_1} \top_R \\
 \\
 \frac{}{\bullet h_2 : \Delta_3 \vdash (\top, \Delta_4), F_1, F_1} \top_R \quad \rightsquigarrow \quad \frac{}{\bullet h_2 : \Delta_3 \vdash \top, \Delta_4, F_1} \top_R
 \end{array}$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_2, F_5 \quad h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, F_2, F_2} \rightarrow_L \quad \rightsquigarrow \quad \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_2, F_5}{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_5} \text{ax} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2, F_2}{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2} \text{ax}}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, F_2} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2, F_2}}{\wedge_L} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2}}{\wedge_L} \text{ ax} \quad \frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2} \text{ IH}}{\wedge_L}$$

- Case(s) rule  $\vee_L$

$$\frac{\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2, F_2} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2, F_2}{\vee_L}}{\vee_L} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2} \text{ ax} \quad \frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2} \text{ IH}}{\vee_L} \quad \frac{\frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2} \text{ ax} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2} \text{ IH}}{\vee_L}$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, F_2, F_2} \perp_L \rightsquigarrow \frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, F_2} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_1, p_4, p_4} I \rightsquigarrow \frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_1, p_4} I$$

$$\frac{}{\bullet h_2 : \Delta_3, p_4 \vdash (\Delta_5, p_4), F_1, F_1} I \rightsquigarrow \frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_5, F_1, p_4} I$$

- Case(s) rule  $\top_L$

$$\frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2, F_2}}{\top_L} \rightsquigarrow \frac{\frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_2}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2} \text{ ax} \quad \frac{h_3 : \Delta_4 \vdash \Delta_1, F_2}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2} \text{ IH}}{\top_L}}{\top_L}$$

## 7 Identity-Expansion

$$\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0 \vdash F_0, F_1} W \quad \frac{\frac{\overline{- : F_1 \vdash F_1} \text{ IH}}{- : F_1 \vdash F_0, F_1} W}{- : F_0 \vee F_1 \vdash F_0, F_1} \vee_L}{- : F_0 \vee F_1 \vdash F_0 \vee F_1} \vee_R$$

$$\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0, F_1 \vdash F_0} W \quad \frac{\frac{\overline{- : F_1 \vdash F_1} \text{ IH}}{- : F_0, F_1 \vdash F_1} W}{- : F_0, F_1 \vdash F_0 \wedge F_1} \wedge_R}{- : F_0 \wedge F_1 \vdash F_0 \wedge F_1} \wedge_L$$

$$\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0 \vdash F_0, F_1} W \quad \frac{\frac{\overline{- : F_1 \vdash F_1} \text{ IH}}{- : F_0, F_1 \vdash F_1} W}{- : F_0, F_0 \rightarrow F_1 \vdash F_1} \rightarrow_L}{- : F_0 \rightarrow F_1 \vdash F_0 \rightarrow F_1} \rightarrow_R$$

$$\frac{}{- : \top \vdash \top} \top_R$$

$$\frac{}{- : \perp \vdash \perp} \perp_L$$

## 8 Cut-Elimination

### 8.1 Status of $\rightarrow_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7, F_9 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_8, F_{10} \quad \bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}), F_8}{- : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}} \rightarrow_R \quad \frac{h_{11} : \Delta_7, F_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \rightarrow F_{10} \quad \bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}}{- : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}} \rightarrow_R \\
\text{Cut} \\
\frac{\frac{h_2 : \Delta_7, F_{12}, F_9 \vdash \Delta_{14}, F_{10}, F_{13}, F_8}{\bullet h_2 : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_8, F_9 \rightarrow F_{10}} \xrightarrow{\text{inv-th/ax}} \frac{h_{11} : \Delta_7, F_{12}, F_8 \vdash \Delta_{14}, F_{13}, F_9 \rightarrow F_{10}}{- : \Delta_7 \vdash \Delta_{14}, F_{12} \rightarrow F_{13}, F_9 \rightarrow F_{10}} \xrightarrow{\text{ax/W}} \frac{- : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \rightarrow F_{10}}{- : \Delta_7 \vdash \Delta_{14}, F_{12} \rightarrow F_{13}, F_9 \rightarrow F_{10}} \xrightarrow{\text{hCut}} \rightarrow_R \\
\frac{h_2 : \Delta_7, F_{11} \vdash \Delta_{10}, F_8, F_{12} \quad \bullet h_2 : \Delta_7 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_8}{- : \Delta_7 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \quad \frac{h_9 : \Delta_7, F_8, F_{11} \vdash \Delta_{10}, F_{12} \quad \bullet h_9 : \Delta_7, F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}}{- : \Delta_7 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \\
\text{Cut} \\
\frac{\frac{h_2 : \Delta_7, F_{11} \vdash \Delta_{10}, F_{12}, F_8}{\bullet h_2 : \Delta_7, F_{11} \vdash \Delta_{10}, F_{12}, F_8} \xrightarrow{\text{ax/W}} \frac{h_9 : \Delta_7, F_{11}, F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_7, F_{11}, F_8 \vdash \Delta_{10}, F_{12}} \xrightarrow{\text{height}} \frac{- : \Delta_7, F_{11} \vdash \Delta_{10}, F_{12}}{- : \Delta_7 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \xrightarrow{\text{hCut}} \rightarrow_R \\
\frac{h_1 : \Delta_6, F_7 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_8 \quad \bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \rightarrow F_8}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_{11}, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{12} \quad \bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \\
\text{Cut} \\
\frac{\frac{h_1 : \Delta_6, F_{11}, F_7 \vdash \Delta_{10}, F_{12}, F_8}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \rightarrow F_8} \xrightarrow{\text{inv-th/ax}} \frac{h_9 : \Delta_6, F_{11}, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{12}}{- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}} \xrightarrow{\text{ax/W}} \frac{- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \xrightarrow{\text{hCut}} \rightarrow_R
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7, F_9 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_8, F_{10} \quad \bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \wedge F_{13}), F_9 \rightarrow F_{10}), F_8}{- : \Delta_7 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \rightarrow F_{10}} \rightarrow_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{12}, F_9 \rightarrow F_{10} \quad h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{13}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \rightarrow F_{10}} \wedge_R \\
\text{Cut} \\
\frac{\frac{h_2 : \Delta_7, F_9 \vdash \Delta_{14}, F_{10}, F_8, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_7, F_9 \vdash \Delta_{14}, F_{10}, F_8, F_{12} \wedge F_{13}} \xrightarrow{\text{ax/W}} \frac{h_{11} : \Delta_7, F_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12}}{\bullet h_{11} : \Delta_7, F_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \wedge F_{13}} \xrightarrow{\text{inv-th/ax}} \frac{h_{11} : \Delta_7, F_8, F_9 \vdash \Delta_{14}, F_{10}, F_{13}}{\bullet h_{11} : \Delta_7, F_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \wedge F_{13}} \xrightarrow{\text{hCut}} \rightarrow_R \\
\frac{h_1 : \Delta_6, F_7 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_8 \quad \bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7 \rightarrow F_8}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \quad h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R \\
\text{Cut} \\
\frac{\frac{h_1 : \Delta_6, F_7 \vdash \Delta_{10}, F_{11}, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_7 \rightarrow F_8} \xrightarrow{\text{inv-th/ax}} \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11}}{- : \Delta_6 \vdash \Delta_{10}, F_{11}} \xrightarrow{\text{ax/W}} \frac{h_1 : \Delta_6, F_7 \vdash \Delta_{10}, F_{12}, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{12}, F_7 \rightarrow F_8} \xrightarrow{\text{inv-th/ax}} \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{12}}{- : \Delta_6 \vdash \Delta_{10}, F_{12}} \xrightarrow{\text{hCut}} \rightarrow_R \\
\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11} \quad - : \Delta_6 \vdash \Delta_{10}, F_{12}}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7, F_9 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_8, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \vee F_{13}), F_9 \rightarrow F_{10}), F_8} \rightarrow_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \rightarrow F_{10}} \vee_R \\
\hline
- : \Delta_7 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \rightarrow F_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{\frac{h_2 : \Delta_7, F_9 \vdash \Delta_{14}, F_{10}, F_8, F_{12} \vee F_{13}}{\rightarrow_R} \quad \frac{h_{11} : \Delta_7, F_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12}, F_{13}}{\bullet h_{11} : \Delta_7, F_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \vee F_{13}} \vee_R}{- : \Delta_7, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \vee F_{13}} \text{ax/W} \quad \frac{\sim}{- : \Delta_7 \vdash \Delta_{14}, F_9 \rightarrow F_{10}, F_{12} \vee F_{13}} \rightarrow_R \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_9 \rightarrow F_{10}, F_{12} \vee F_{13} \quad \text{hCut} \\
\hline
\sim \\
\frac{h_1 : \Delta_6, F_7 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_8}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \text{Cut} \\
\hline
\sim \\
\frac{h_1 : \Delta_6, F_7 \vdash \Delta_{10}, F_{11}, F_{12}, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7 \rightarrow F_8} \text{inv-th/ax} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12} \quad \vee_R \quad \text{hCut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7, F_9 \vdash (\perp, \Delta_{12}), F_8, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\perp, \Delta_{12}), F_9 \rightarrow F_{10}), F_8} \rightarrow_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{12}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\perp, \Delta_{12}), F_9 \rightarrow F_{10}} \perp_R \\
\hline
- : \Delta_7 \vdash (\perp, \Delta_{12}), F_9 \rightarrow F_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{\frac{h_2 : \Delta_7 \vdash \perp, \Delta_{12}, F_8, F_9 \rightarrow F_{10}}{\rightarrow_R} \quad \frac{h_{11} : \Delta_7, F_8 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10}} \text{ax/W}}{- : \Delta_7 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10}} \text{hCut} \\
\hline
\sim \\
\frac{h_1 : \Delta_6, F_7 \vdash (\perp, \Delta_{10}), F_8}{\bullet h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \perp, \Delta_{10}} \perp_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{\frac{h_1 : \Delta_6 \vdash \perp, \Delta_{10}, F_7 \rightarrow F_8}{\rightarrow_R} \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \perp, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \perp, \Delta_{10}} \text{ax/W}}{- : \Delta_6 \vdash \perp, \Delta_{10}} \text{hCut} \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7, F_9 \vdash (\top, \Delta_{12}), F_8, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\top, \Delta_{12}), F_9 \rightarrow F_{10}), F_8} \rightarrow_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash (\top, \Delta_{12}), F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\top, \Delta_{12}), F_9 \rightarrow F_{10}} \top_R \\
\hline
- : \Delta_7 \vdash (\top, \Delta_{12}), F_9 \rightarrow F_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{\sim}{- : \Delta_7 \vdash \top, \Delta_{12}, F_9 \rightarrow F_{10}} \top_R \\
\hline
\sim \\
\frac{h_1 : \Delta_6, F_7 \vdash (\top, \Delta_{10}), F_8}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \top, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \top, \Delta_{10}} \top_R \\
\hline
- : \Delta_6 \vdash \top, \Delta_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{\sim}{- : \Delta_6 \vdash \top, \Delta_{10}} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_2 : (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : \Delta_{14}, F_7 \vdash \Delta_8, F_{12}, F_9 \rightarrow F_{10} \quad h_{11} : \Delta_{14}, F_7, F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \rightarrow F_{13}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_L \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{\frac{h_{11} : \Delta_{14}, F_7, F_9 \vdash \Delta_8, F_{10}, F_{12}}{\rightarrow_R} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7, F_9 \vdash \Delta_8, F_{10}}{\bullet h_{11} : \Delta_{14}, F_7, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}} \text{inv-th/ax}}{- : \Delta_{14}, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}} \text{ax/W} \quad \frac{\sim}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R \\
\hline
- : \Delta_{14}, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10} \quad \text{hCut} \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_{12} \rightarrow F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), F_{12} \rightarrow F_{13}} \rightarrow_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_{12}, F_8 \rightarrow F_9 \quad h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8 \rightarrow F_9} \rightarrow_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \rightarrow F_{13}}{\bullet h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \rightarrow F_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_8 \vdash \Delta_7, F_{12}, F_9}{\bullet h_{10} : \Delta_{11}, F_8, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_9} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{11}, F_{13}, F_8 \vdash \Delta_7, F_9}{\bullet h_{10} : \Delta_{11}, F_{13}, F_8 \vdash \Delta_7, F_9} \text{inv-th/ax}}{- : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \rightarrow_R}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \text{hCut} \\
\sim \\
\frac{\frac{h_1 : (\Delta_{12}, F_9 \rightarrow F_{10}), F_6 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_6 \rightarrow F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \rightarrow F_{10}), F_6 \rightarrow F_7 \vdash \Delta_{11}} \rightarrow_L}{- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}} \text{Cut} \\
\sim \\
\frac{\frac{h_1 : \Delta_{12}, F_6 \vdash \Delta_{11}, F_7, F_9}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6 \rightarrow F_7} \text{inv-th/ax} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_6 \rightarrow F_7 \vdash \Delta_{11}, F_9}{\bullet h_8 : \Delta_{12}, F_6 \rightarrow F_7 \vdash \Delta_{11}, F_9} \text{ax/W}}{- : \Delta_{12} \vdash \Delta_{11}, F_9} \text{hCut} \\
\sim \\
\frac{\frac{h_1 : \Delta_{12}, F_{10}, F_6 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \text{inv-th/ax} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : \Delta_{12}, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}} \text{ax/W}}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}} \rightarrow_L \\
\sim \\
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11} \\
\sim \\
\frac{\frac{h_1 : \Delta_7, F_8 \vdash \Delta_{10}, F_9}{\bullet h_1 : \Delta_7 \vdash \Delta_{10}, F_8 \rightarrow F_9} \rightarrow_R \quad \frac{h_6 : \Delta_7 \vdash \Delta_{10}, F_8 \quad h_6 : \Delta_7, F_9 \vdash \Delta_{10}}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{10}} \rightarrow_L}{- : \Delta_7 \vdash \Delta_{10}} \text{Cut} \\
\sim \\
\frac{\frac{- : \Delta_7, F_8 \vdash \Delta_{10}, F_9}{- : \Delta_7 \vdash \Delta_{10}, F_8} \text{ax/W} \quad \frac{- : \Delta_7, F_8, F_9 \vdash \Delta_{10}}{- : \Delta_7, F_8 \vdash \Delta_{10}} \text{ax/W}}{- : \Delta_7 \vdash \Delta_{10}} \text{sCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{h_2 : (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : \Delta_{14}, F_7, F_{12}, F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \wedge F_{13}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \wedge_L}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{14}, F_9, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_7}{\bullet h_2 : \Delta_{14}, F_9, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_7} \text{ax/W} \quad \frac{\frac{h_{11} : \Delta_{14}, F_{12}, F_{13}, F_7, F_9 \vdash \Delta_8, F_{10}}{\bullet h_{11} : \Delta_{14}, F_7, F_9, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}} \text{inv-th/ax} \wedge_L}{- : \Delta_{14}, F_9, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}} \rightarrow_R}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{hCut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_{12} \wedge F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), F_{12} \wedge F_{13}} \rightarrow_R \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8 \rightarrow F_9} \wedge_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \wedge F_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_{12}, F_{13}, F_8 \vdash \Delta_7, F_9}{\bullet h_{10} : \Delta_{11}, F_8, F_{12} \wedge F_{13} \vdash \Delta_7, F_9} \text{inv-th/ax} \wedge_L}{- : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \rightarrow_R}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \text{hCut} \\
\sim \\
\frac{\frac{h_1 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_9, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \rightarrow F_7 \vdash \Delta_{11}} \wedge_L}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \text{Cut} \\
\sim \\
\frac{\frac{h_1 : \Delta_{12}, F_{10}, F_6, F_9 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \rightarrow F_7} \text{inv-th/ax} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : \Delta_{12}, F_{10}, F_9, F_6 \rightarrow F_7 \vdash \Delta_{11}} \text{ax/W}}{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}} \wedge_L \\
\sim \\
- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{h_2 : (\Delta_{14}, F_{12} \vee F_{13}), F_9 \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : \Delta_{14}, F_7, F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \quad h_{11} : \Delta_{14}, F_7, F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \vee F_{13}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \vee_L}{- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{14}, F_9, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_7}{\bullet h_2 : \Delta_{14}, F_9, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_7} \text{ax/W} \quad \frac{\frac{h_{11} : \Delta_{14}, F_{12}, F_7, F_9 \vdash \Delta_8, F_{10}}{\bullet h_{11} : \Delta_{14}, F_7, F_9, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7, F_9 \vdash \Delta_8, F_{10}}{\bullet h_{11} : \Delta_{14}, F_{13}, F_7, F_9 \vdash \Delta_8, F_{10}} \text{inv-th/ax}}{\bullet h_{11} : \Delta_{14}, F_7, F_9, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}} \vee_L}{- : \Delta_{14}, F_9, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}} \text{hCut} \\
\rightarrow_R \\
- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_{12} \vee F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), F_{12} \vee F_{13}} \rightarrow_R \quad \frac{h_{10} : \Delta_{11}, F_{12} \vdash \Delta_7, F_8 \rightarrow F_9 \quad h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \vee F_{13} \vdash \Delta_7, F_8 \rightarrow F_9} \vee_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \vee F_{13}}{\bullet h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \vee F_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_{12}, F_8 \vdash \Delta_7, F_9}{\bullet h_{10} : \Delta_{11}, F_8, F_{12} \vee F_{13} \vdash \Delta_7, F_9} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{11}, F_{13}, F_8 \vdash \Delta_7, F_9}{\bullet h_{10} : \Delta_{11}, F_{13}, F_8 \vdash \Delta_7, F_9} \text{inv-th/ax}}{\bullet h_{10} : \Delta_{11}, F_8, F_{12} \vee F_{13} \vdash \Delta_7, F_9} \vee_L}{- : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \text{hCut} \\
\rightarrow_R \\
- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9 \\
\sim \\
\frac{\frac{h_1 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_9, F_6 \rightarrow F_7 \vdash \Delta_{11} \quad h_8 : \Delta_{12}, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \rightarrow F_7 \vdash \Delta_{11}} \vee_L}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \text{Cut} \\
\sim \\
\frac{\frac{h_1 : \Delta_{12}, F_6, F_9 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6 \rightarrow F_7} \text{inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_9, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6 \rightarrow F_7} \rightarrow_R}{- : \Delta_{12}, F_9 \vdash \Delta_{11}} \text{hCut} \\
\text{ax/W} \\
\frac{\frac{h_1 : \Delta_{12}, F_{10}, F_6 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \text{inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \rightarrow_R}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}} \text{hCut} \\
\vee_L \\
- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, \perp, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), \perp} \rightarrow_R \quad \frac{\bullet h_{10} : \Delta_{11}, \perp \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, \perp \vdash \Delta_7, F_8 \rightarrow F_9} \perp_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \perp, \Delta_7, F_9}{\bullet h_2 : \Delta_{11}, F_8 \vdash \perp, \Delta_7, F_9} \text{ax/W} \quad \frac{\bullet h_{10} : \perp, \Delta_{11}, F_8 \vdash \Delta_7, F_9}{\bullet h_{10} : \perp, \Delta_{11}, F_8 \vdash \Delta_7, F_9} \perp_L}{- : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \text{hCut} \\
\rightarrow_R \\
- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9 \\
\sim \\
\frac{\frac{h_2 : (\perp, \Delta_{12}), F_9 \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \perp, \Delta_{12} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{\bullet h_{11} : (\perp, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\perp, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \perp_L}{- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut} \\
\sim \\
- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \\
\perp_L \\
- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \\
\sim \\
\frac{\frac{h_1 : (\perp, \Delta_{10}), F_6 \vdash \Delta_9, F_7}{\bullet h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{\bullet h_8 : (\perp, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9}{\bullet h_8 : (\perp, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9} \perp_L}{- : \perp, \Delta_{10} \vdash \Delta_9} \text{Cut} \\
\sim \\
- : \perp, \Delta_{10} \vdash \Delta_9 \\
\perp_L \\
- : \perp, \Delta_{10} \vdash \Delta_9
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_2 : \Delta_{10}, F_7 \vdash (\Delta_{12}, p_{11}), p_{11}, F_8}{\bullet h_2 : \Delta_{10} \vdash ((\Delta_{12}, p_{11}), F_7 \rightarrow F_8), p_{11}} \rightarrow_R \quad \frac{\bullet h_9 : \Delta_{10}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7 \rightarrow F_8}{\bullet h_9 : \Delta_{10}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7 \rightarrow F_8} I}{- : \Delta_{10} \vdash (\Delta_{12}, p_{11}), F_7 \rightarrow F_8} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{10}, F_7 \vdash \Delta_{12}, F_8, p_{11}, p_{11}}{\bullet h_2 : \Delta_{10}, F_7 \vdash \Delta_{12}, F_8, p_{11}, p_{11}} \text{ax/W} \quad \frac{\bullet h_9 : \Delta_{10}, F_7, p_{11} \vdash \Delta_{12}, F_8, p_{11}}{\bullet h_9 : \Delta_{10}, F_7, p_{11} \vdash \Delta_{12}, F_8, p_{11}} I}{- : \Delta_{10}, F_7 \vdash \Delta_{12}, F_8, p_{11}} \text{hCut} \\
\rightarrow_R \\
- : \Delta_{10} \vdash \Delta_{12}, p_{11}, F_7 \rightarrow F_8
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : (\Delta_{13}, p_{11}), F_8 \vdash (\Delta_{12}, p_{11}), F_7, F_9}{\bullet h_2 : \Delta_{13}, p_{11} \vdash ((\Delta_{12}, p_{11}), F_8 \rightarrow F_9), F_7} \rightarrow_R \quad \frac{h_{10} : (\Delta_{13}, p_{11}), F_7 \vdash (\Delta_{12}, p_{11}), F_8 \rightarrow F_9}{\bullet h_{10} : (\Delta_{13}, p_{11}), F_7 \vdash (\Delta_{12}, p_{11}), F_8 \rightarrow F_9} I \\
\hline
- : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_8 \rightarrow F_9 \quad \text{Cut} \\
\hline
- : \Delta_{13}, p_{11} \vdash \Delta_{12}, p_{11}, F_8 \rightarrow F_9 \quad I
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : (\Delta_{11}, p_9), F_6 \vdash (\Delta_{10}, p_9), F_7}{\bullet h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : (\Delta_{11}, p_9), F_6 \rightarrow F_7 \vdash \Delta_{10}, p_9}{\bullet h_8 : (\Delta_{11}, p_9), F_6 \rightarrow F_7 \vdash \Delta_{10}, p_9} I \\
\hline
- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9 \quad \text{Cut} \\
\hline
- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9 \quad I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{11}, F_8 \vdash \top, F_9}{\bullet h_2 : \Delta_{11} \vdash (\top, F_8 \rightarrow F_9), \top} \rightarrow_R \quad \frac{h_{10} : \Delta_{11} \vdash \top, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, \top \vdash \top, F_8 \rightarrow F_9} \top_L \\
\hline
- : \Delta_{11} \vdash \top, F_8 \rightarrow F_9 \quad \text{Cut} \\
\hline
- : \Delta_{11} \vdash \top, F_8 \rightarrow F_9 \quad \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : (\top, \Delta_{12}), F_9 \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \top, \Delta_{12} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : \Delta_{12}, F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\top, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \top_L \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \quad \text{Cut} \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \quad \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{\bullet h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9 \rightarrow F_{10} \quad \text{ax/W} \quad h_{11} : \top, \Delta_{12}, F_7 \vdash \Delta_8, F_9 \rightarrow F_{10} \quad \text{ax/W}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : (\top, \Delta_{10}), F_6 \vdash \Delta_9, F_7}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Delta_{10}, F_6 \rightarrow F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9} \top_L \\
\hline
- : \top, \Delta_{10} \vdash \Delta_9 \quad \text{Cut} \\
\hline
- : \top, \Delta_{10} \vdash \Delta_9 \quad \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7 \quad \text{ax/W} \quad h_8 : \top, \Delta_{10}, F_6 \rightarrow F_7 \vdash \Delta_9 \quad \text{ax/W}}{- : \top, \Delta_{10} \vdash \Delta_9} \text{hCut}
\end{array}$$

## 8.2 Status of $\wedge_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_8, F_9 \quad h_2 : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_8, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10}), F_8} \wedge_R \quad \frac{h_{11} : \Delta_7, F_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10}} \rightarrow_R \\
\hline
- : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10} \quad \text{Cut} \\
\hline
- : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_8, F_9 \quad \text{inv-th/ax} \quad h_2 : \Delta_7, F_{12} \vdash \Delta_{14}, F_{10}, F_{13}, F_8 \quad \text{inv-th/ax}}{\bullet h_2 : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_8, F_9 \wedge F_{10}} \wedge_R \quad \frac{h_{11} : \Delta_7, F_{12}, F_8 \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_7, F_{12}, F_8 \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10}} \text{ax/W} \\
\hline
- : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10} \quad \text{hCut} \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_{12} \rightarrow F_{13}, F_9 \wedge F_{10} \quad \rightarrow_R
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \quad h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_8}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12} \quad \text{Cut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \quad \text{inv-th/ax} \quad h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_8 \quad \text{inv-th/ax}}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12} \quad \text{hCut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12} \quad \rightarrow_R
\end{array}$$

- Case rule  $\wedge_R$





$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \quad h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_8}{\bullet h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \perp, \Delta_{10}} \perp_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \text{Cut} \\
\hline
\frac{}{\bullet h_1 : \Delta_6 \vdash \perp, \Delta_{10}, F_7 \wedge F_8} \text{ax/W} \quad \frac{}{h_9 : \Delta_6, F_7 \wedge F_8 \vdash \perp, \Delta_{10}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\top, \Delta_{12}), F_8, F_9 \quad h_2 : \Delta_7 \vdash (\top, \Delta_{12}), F_8, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\top, \Delta_{12}), F_9 \wedge F_{10}), F_8} \wedge_R \quad \frac{}{\bullet h_{11} : \Delta_7, F_8 \vdash (\top, \Delta_{12}), F_9 \wedge F_{10}} \top_R \\
\hline
- : \Delta_7 \vdash (\top, \Delta_{12}), F_9 \wedge F_{10} \quad \text{Cut} \\
\hline
- : \Delta_7 \vdash \top, \Delta_{12}, F_9 \wedge F_{10} \quad \top_R \\
\hline
\frac{h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \quad h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_8}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \wedge F_8} \wedge_R \quad \frac{}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \top, \Delta_{10}} \top_R \\
\hline
- : \Delta_6 \vdash \top, \Delta_{10} \quad \text{Cut} \\
\hline
- : \Delta_6 \vdash \top, \Delta_{10} \quad \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_7, F_9 \quad h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash (\Delta_8, F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : \Delta_{14}, F_7 \vdash \Delta_8, F_{12}, F_9 \wedge F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \wedge F_{10}), F_7} \rightarrow_L \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \wedge F_{10} \quad \text{Cut} \\
\hline
\frac{}{h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_7, F_9} \text{ax/W} \quad \frac{h_{11} : \Delta_{14}, F_7 \vdash \Delta_8, F_{12}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_9}{\bullet h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \quad \text{hCut} \quad \rightarrow_L \quad h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10} \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \wedge F_{10} \\
\hline
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \rightarrow F_{13}, F_8 \quad h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \rightarrow F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), F_{12} \rightarrow F_{13}} \wedge_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_{12}, F_8 \wedge F_9 \quad h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8 \wedge F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8 \wedge F_9} \rightarrow_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9 \quad \text{Cut} \\
\hline
- : \Delta_{11}, F_{12} \vdash \Delta_7, F_{13}, F_8 \quad \text{inv-th/ax} \quad - : \Delta_{11}, F_{12} \vdash \Delta_7, F_{13}, F_9 \quad \text{inv-th/ax} \\
\hline
- : \Delta_{11}, F_{12} \vdash \Delta_7, F_{13}, F_8 \wedge F_9 \quad \wedge_R \quad - : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8 \wedge F_9 \quad \text{ax/W} \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_{12}, F_8 \wedge F_9 \quad \text{ax/W} \quad - : \Delta_{11}, F_{12} \vdash \Delta_7, F_8 \wedge F_9 \quad \text{sCut} \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9 \\
\hline
\frac{h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \quad h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \frac{h_8 : \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \rightarrow F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \rightarrow_L \\
\hline
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\hline
- : \Delta_{12}, F_6, F_7 \vdash \Delta_{11}, F_9 \quad \text{inv-th/ax} \quad - : \Delta_{12}, F_{10}, F_6, F_7 \vdash \Delta_{11} \quad \text{inv-th/ax} \\
\hline
- : \Delta_{12}, F_6, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_7 \quad \text{ax/W} \quad - : \Delta_{12}, F_6, F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \text{sCut} \\
\hline
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \quad \text{ax/W} \quad - : \Delta_{12}, F_6, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \text{sCut} \\
\hline
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_7, F_9 \quad h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash (\Delta_8, F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : \Delta_{14}, F_7, F_{12}, F_{13} \vdash \Delta_8, F_9 \wedge F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \wedge F_{13}), F_7 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_L \\
\hline
- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \wedge F_{10} \quad \text{Cut} \\
\hline
\frac{}{h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_7, F_9} \text{inv-th/ax} \quad \frac{}{h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_{10}, F_7} \text{inv-th/ax} \\
\hline
\bullet h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_7, F_9 \wedge F_{10} \quad \wedge_R \quad h_{11} : \Delta_{14}, F_{12}, F_{13}, F_7 \vdash \Delta_8, F_9 \wedge F_{10} \\
\hline
- : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_9 \wedge F_{10} \quad \text{ax/W} \\
\hline
- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \wedge F_{10} \quad \wedge_L \quad \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \wedge F_{13}, F_8 \quad h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \wedge F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), F_{12} \wedge F_{13}} \wedge_R \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8 \wedge F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8 \wedge F_9} \wedge_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9 \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_{12} \wedge F_{13}}{- : \Delta_{11} \vdash \Delta_7, F_8} \text{ ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8} \text{ inv-th/ax} \quad \frac{h_2 : \Delta_{11} \vdash \Delta_7, F_9, F_{12} \wedge F_{13}}{- : \Delta_{11} \vdash \Delta_7, F_9} \text{ ax/W}}{\frac{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \wedge_L} \text{ hCut} \\
\hline
\frac{h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \quad h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \frac{h_8 : \Delta_{12}, F_9, F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \wedge_L \\
\hline
- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \wedge F_7} \text{ inv-th/ax} \quad \frac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \wedge F_7} \text{ inv-th/ax}}{\frac{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \wedge_L} \wedge_R \\
\hline
\frac{h_1 : \Delta_7 \vdash \Delta_{10}, F_8 \quad h_1 : \Delta_7 \vdash \Delta_{10}, F_9}{\bullet h_1 : \Delta_7 \vdash \Delta_{10}, F_8 \wedge F_9} \wedge_R \quad \frac{h_6 : \Delta_7, F_8, F_9 \vdash \Delta_{10}}{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{10}} \wedge_L \\
\hline
- : \Delta_7 \vdash \Delta_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{- : \Delta_7 \vdash \Delta_{10}, F_8}{- : \Delta_7 \vdash \Delta_{10}, F_8} \text{ ax/W} \quad \frac{- : \Delta_7, F_8 \vdash \Delta_{10}, F_9}{- : \Delta_7, F_8 \vdash \Delta_{10}} \text{ ax/W} \quad \frac{- : \Delta_7, F_8, F_9 \vdash \Delta_{10}}{- : \Delta_7, F_8 \vdash \Delta_{10}} \text{ ax/W}}{\frac{- : \Delta_7 \vdash \Delta_{10}}{- : \Delta_7 \vdash \Delta_{10}} \text{ sCut}} \text{ sCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_7, F_9 \quad h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash (\Delta_8, F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : \Delta_{14}, F_7, F_{12} \vdash \Delta_8, F_9 \wedge F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \vee F_{13}) \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_L \\
\hline
- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \wedge F_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_7, F_9}{- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9} \text{ ax/W} \quad \frac{\frac{h_{11} : \Delta_{14}, F_{12}, F_7 \vdash \Delta_8, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \vee F_{13} \vdash \Delta_8, F_9} \text{ inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_9}{\bullet h_{11} : (\Delta_{14}, F_{12} \vee F_{13}) \vdash \Delta_8, F_9} \text{ inv-th/ax}}{\frac{- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9}{- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9} \vee_L} \text{ hCut} \\
\hline
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \vee F_{13}, F_8 \quad h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \vee F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), F_{12} \vee F_{13}} \wedge_R \quad \frac{h_{10} : \Delta_{11}, F_{12} \vdash \Delta_7, F_8 \wedge F_9 \quad h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8 \wedge F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \vee F_{13} \vdash \Delta_7, F_8 \wedge F_9} \vee_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9 \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \text{ inv-th/ax} \quad \frac{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_9}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \text{ inv-th/ax}}{\frac{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \wedge_R} \wedge_R \\
\hline
\frac{\frac{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \text{ ax/W} \quad \frac{- : \Delta_{11}, F_{13} \vdash \Delta_7, F_{12}, F_8 \wedge F_9}{- : \Delta_{11}, F_{12} \vdash \Delta_7, F_8 \wedge F_9} \text{ ax/W}}{\frac{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{ sCut}} \text{ sCut} \\
\hline
\frac{h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \quad h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \frac{h_8 : \Delta_{12}, F_9, F_6 \wedge F_7 \vdash \Delta_{11} \quad h_8 : \Delta_{12}, F_{10}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \wedge_L \\
\hline
- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}, F_7}{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}, F_7} \text{ ax/W} \quad \frac{- : \Delta_{12}, F_6, F_7, F_9 \vdash \Delta_{11}}{- : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}} \text{ inv-th/ax}}{\frac{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}}{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}} \vee_L} \text{ sCut} \\
\hline
\frac{\frac{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6} \text{ ax/W} \quad \frac{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}, F_7}{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}} \text{ ax/W} \quad \frac{- : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}}{- : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}} \text{ sCut}}{\frac{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \text{ sCut}} \text{ sCut}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{11} \vdash \Delta_7, \perp, F_8 \quad h_2 : \Delta_{11} \vdash \Delta_7, \perp, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), \perp} \wedge_R \quad \frac{}{\bullet h_{10} : \Delta_{11}, \perp \vdash \Delta_7, F_8 \wedge F_9} \perp_L \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{11} \vdash \perp, \Delta_7, F_8}{- : \Delta_{11} \vdash \Delta_7, F_8} \text{ax/W} \quad \frac{\bullet h_{10} : \perp, \Delta_{11} \vdash \Delta_7, F_8}{- : \Delta_{11} \vdash \Delta_7, F_8} \perp_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{hCut} \quad \frac{\frac{h_2 : \Delta_{11} \vdash \perp, \Delta_7, F_9}{- : \Delta_{11} \vdash \Delta_7, F_9} \text{ax/W} \quad \frac{\bullet h_{10} : \perp, \Delta_{11} \vdash \Delta_7, F_9}{- : \Delta_{11} \vdash \Delta_7, F_9} \perp_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{hCut} \quad \wedge_R \\
\sim \\
\frac{h_2 : \perp, \Delta_{12} \vdash \Delta_8, F_7, F_9 \quad h_2 : \perp, \Delta_{12} \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \perp, \Delta_{12} \vdash (\Delta_8, F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{}{\bullet h_{11} : (\perp, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \wedge F_{10}} \perp_L \\
\frac{}{- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut} \\
\sim \\
\frac{}{- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \wedge F_{10}} \perp_L \\
\sim \\
\frac{h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \quad h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_7}{\bullet h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \wedge F_7} \wedge_R \quad \frac{}{\bullet h_8 : (\perp, \Delta_{10}), F_6 \wedge F_7 \vdash \Delta_9} \perp_L \\
\frac{}{- : \perp, \Delta_{10} \vdash \Delta_9} \text{Cut} \\
\sim \\
\frac{}{- : \perp, \Delta_{10} \vdash \Delta_9} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_2 : \Delta_{10} \vdash (\Delta_{12}, p_{11}), p_{11}, F_7 \quad h_2 : \Delta_{10} \vdash (\Delta_{12}, p_{11}), p_{11}, F_8}{\bullet h_2 : \Delta_{10} \vdash ((\Delta_{12}, p_{11}), F_7 \wedge F_8), p_{11}} \wedge_R \quad \frac{}{\bullet h_9 : \Delta_{10}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7 \wedge F_8} I \\
\frac{}{- : \Delta_{10} \vdash (\Delta_{12}, p_{11}), F_7 \wedge F_8} \text{Cut} \\
\sim \\
\frac{\frac{h_2 : \Delta_{10} \vdash \Delta_{12}, F_7, p_{11}, p_{11}}{- : \Delta_{10} \vdash \Delta_{12}, F_7, p_{11}} \text{ax/W} \quad \frac{\bullet h_9 : \Delta_{10}, p_{11} \vdash \Delta_{12}, F_7, p_{11}}{- : \Delta_{10} \vdash \Delta_{12}, F_7, p_{11}} I}{- : \Delta_{10} \vdash \Delta_{12}, p_{11}, F_7 \wedge F_8} \text{hCut} \quad \frac{\frac{h_2 : \Delta_{10} \vdash \Delta_{12}, F_8, p_{11}, p_{11}}{- : \Delta_{10} \vdash \Delta_{12}, F_8, p_{11}} \text{ax/W} \quad \frac{\bullet h_9 : \Delta_{10}, p_{11} \vdash \Delta_{12}, F_8, p_{11}}{- : \Delta_{10} \vdash \Delta_{12}, F_8, p_{11}} I}{- : \Delta_{10} \vdash \Delta_{12}, p_{11}, F_7 \wedge F_8} \text{hCut} \quad \wedge_R \\
\sim \\
\frac{h_2 : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7, F_8 \quad h_2 : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7, F_9}{\bullet h_2 : \Delta_{13}, p_{11} \vdash ((\Delta_{12}, p_{11}), F_8 \wedge F_9), F_7} \wedge_R \quad \frac{}{\bullet h_{10} : (\Delta_{13}, p_{11}), F_7 \vdash (\Delta_{12}, p_{11}), F_8 \wedge F_9} I \\
\frac{}{- : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_8 \wedge F_9} \text{Cut} \\
\sim \\
\frac{}{- : \Delta_{13}, p_{11} \vdash \Delta_{12}, p_{11}, F_8 \wedge F_9} I \\
\sim \\
\frac{h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6 \quad h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_7}{\bullet h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6 \wedge F_7} \wedge_R \quad \frac{}{\bullet h_8 : (\Delta_{11}, p_9), F_6 \wedge F_7 \vdash \Delta_{10}, p_9} I \\
\frac{}{- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9} \text{Cut} \\
\sim \\
\frac{}{- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{11} \vdash \Delta_7, \top, F_8 \quad h_2 : \Delta_{11} \vdash \Delta_7, \top, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), \top} \wedge_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9}{\bullet h_{10} : \Delta_{11}, \top \vdash \Delta_7, F_8 \wedge F_9} \top_L \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{Cut} \\
\sim \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{ax/W} \\
\sim \\
\frac{h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9 \quad h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_{10}}{\bullet h_2 : \top, \Delta_{12} \vdash (\Delta_8, F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : \Delta_{12}, F_7 \vdash \Delta_8, F_9 \wedge F_{10}}{\bullet h_{11} : (\top, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \wedge F_{10}} \top_L \\
\frac{}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut} \\
\sim \\
\frac{\frac{\bullet h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9 \wedge F_{10}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W} \quad \frac{h_{11} : \top, \Delta_{12}, F_7 \vdash \Delta_8, F_9 \wedge F_{10}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \wedge F_{10}} \text{hCut} \\
\sim \\
\frac{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \quad h_1 : \top, \Delta_{10} \vdash \Delta_9, F_7}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \wedge F_7} \wedge_R \quad \frac{h_8 : \Delta_{10}, F_6 \wedge F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \wedge F_7 \vdash \Delta_9} \top_L \\
\frac{}{- : \top, \Delta_{10} \vdash \Delta_9} \text{Cut} \\
\sim \\
\frac{\frac{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \wedge F_7}{- : \top, \Delta_{10} \vdash \Delta_9} \text{ax/W} \quad \frac{h_8 : \top, \Delta_{10}, F_6 \wedge F_7 \vdash \Delta_9}{- : \top, \Delta_{10} \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_{10} \vdash \Delta_9} \text{hCut}
\end{array}$$

### 8.3 Status of $\vee_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_8, F_9, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vee F_{10}), F_8} \vee_R \quad \frac{h_{11} : \Delta_7, F_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vee F_{10}} \rightarrow_R \\
\hline
- : \Delta_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_2 : \Delta_7, F_{12} \vdash \Delta_{14}, F_{10}, F_{13}, F_8, F_9}{\bullet h_2 : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_8, F_9 \vee F_{10}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_7, F_{12}, F_8 \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_{12}, F_8 \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}} \text{ax/W} \\
\hline
- : \Delta_7, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10} \quad \text{hCut} \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_{12} \rightarrow F_{13}, F_9 \vee F_{10} \quad \rightarrow_R
\end{array} \\
\hline
\frac{h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7, F_8}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_{11}, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7, F_8}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \vee F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_{11}, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_{11}, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12} \quad \text{hCut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12} \quad \rightarrow_R
\end{array}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_8, F_9, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vee F_{10}), F_8} \vee_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{12}, F_9 \vee F_{10} \quad h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vee F_{10}} \wedge_R \\
\hline
- : \Delta_7 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash \Delta_{14}, F_{10}, F_8, F_9, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_7 \vdash \Delta_{14}, F_{10}, F_8, F_9, F_{12} \wedge F_{13}} \text{ax/W} \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{10}, F_{12}, F_9}{\bullet h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{10}, F_{12}, F_9} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{10}, F_{13}, F_9}{\bullet h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{10}, F_{13}, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_{10}, F_9, F_{12} \wedge F_{13} \quad \text{hCut} \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_{12} \wedge F_{13}, F_9 \vee F_{10} \quad \vee_R
\end{array} \\
\hline
\frac{h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7, F_8}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \quad h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_7, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_7 \vee F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}} \text{ax/W} \quad \frac{h_1 : \Delta_6 \vdash \Delta_{10}, F_{12}, F_7, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{12}, F_7 \vee F_8} \text{inv-th/ax} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \quad \text{hCut} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \text{hCut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12} \quad \wedge_R
\end{array}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_8, F_9, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\Delta_{14}, F_{12} \vee F_{13}), F_9 \vee F_{10}), F_8} \vee_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \vee F_{10}} \vee_R \\
\hline
- : \Delta_7 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash \Delta_{14}, F_{10}, F_{12}, F_{13}, F_8, F_9}{\bullet h_2 : \Delta_7 \vdash \Delta_{14}, F_{12}, F_{13}, F_8, F_9 \vee F_{10}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10}} \text{ax/W} \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10} \quad \text{hCut} \\
\hline
- : \Delta_7 \vdash \Delta_{14}, F_{12} \vee F_{13}, F_9 \vee F_{10} \quad \vee_R
\end{array}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash \Delta_{10}, F_8, F_{11}, F_{12}}{\bullet h_2 : \Delta_7 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_8} \vee_R \quad \frac{h_9 : \Delta_7, F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_7, F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_7 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{}{h_2 : \Delta_7 \vdash \Delta_{10}, F_{11}, F_{12}, F_8} \text{ax/W} \quad \frac{h_9 : \Delta_7, F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_7, F_8 \vdash \Delta_{10}, F_{11}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_7 \vdash \Delta_{10}, F_{11}, F_{12} \quad \text{height} \\
\hline
- : \Delta_7 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \vee_R \quad \text{hCut} \\
\hline
\frac{h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7, F_8}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7 \vee F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12} \quad \vee_R \quad \text{hCut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\perp, \Delta_{12}), F_8, F_9, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\perp, \Delta_{12}), F_9 \vee F_{10}), F_8} \vee_R \quad \frac{h_{11} : \Delta_7, F_8 \vdash \Delta_{12}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash (\perp, \Delta_{12}), F_9 \vee F_{10}} \perp_R \\
\hline
- : \Delta_7 \vdash (\perp, \Delta_{12}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\bullet h_2 : \Delta_7 \vdash \perp, \Delta_{12}, F_8, F_9 \vee F_{10}}{- : \Delta_7 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}} \text{ax/W} \quad \frac{h_{11} : \Delta_7, F_8 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_7, F_8 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}} \text{ax/W} \\
\hline
- : \Delta_7 \vdash \perp, \Delta_{12}, F_9 \vee F_{10} \quad \text{hCut} \\
\hline
\frac{h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7, F_8}{\bullet h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \perp, \Delta_{10}} \perp_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\bullet h_1 : \Delta_6 \vdash \perp, \Delta_{10}, F_7 \vee F_8}{- : \Delta_6 \vdash \perp, \Delta_{10}} \text{ax/W} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \perp, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \perp, \Delta_{10}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash (\top, \Delta_{12}), F_8, F_9, F_{10}}{\bullet h_2 : \Delta_7 \vdash ((\top, \Delta_{12}), F_9 \vee F_{10}), F_8} \vee_R \quad \frac{}{\bullet h_{11} : \Delta_7, F_8 \vdash (\top, \Delta_{12}), F_9 \vee F_{10}} \top_R \\
\hline
- : \Delta_7 \vdash (\top, \Delta_{12}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{}{- : \Delta_7 \vdash \top, \Delta_{12}, F_9 \vee F_{10}} \top_R \\
\hline
\frac{h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7, F_8}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \vee F_8} \vee_R \quad \frac{}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \top, \Delta_{10}} \top_R \\
\hline
- : \Delta_6 \vdash \top, \Delta_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{}{- : \Delta_6 \vdash \top, \Delta_{10}} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_7, F_9, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : \Delta_{14}, F_7 \vdash \Delta_8, F_{12}, F_9 \vee F_{10} \quad h_{11} : \Delta_{14}, F_7, F_{13} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \rightarrow F_{13}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \rightarrow_L \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{h_{11} : \Delta_{14}, F_7 \vdash \Delta_8, F_{10}, F_{12}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_9 \quad \text{ax/W} \quad \text{hCut} \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \vee F_{10} \quad \vee_R
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \rightarrow F_{13}, F_8, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), F_{12} \rightarrow F_{13}} \vee_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_{12}, F_8 \vee F_9 \quad h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8 \vee F_9} \rightarrow_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad \text{Cut} \\
\hline
\sim \\
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, F_{12} \rightarrow F_{13}}{- : \Delta_{11} \vdash \Delta_7, F_8, F_9} \text{ax/W} \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_{12}, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8, F_9 \quad \text{hCut} \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad \vee_R \\
\hline
\frac{h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9 \quad h_8 : \Delta_{12}, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \rightarrow F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \rightarrow_L \\
\hline
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\sim \\
\frac{h_1 : \Delta_{12} \vdash \Delta_{11}, F_6, F_7, F_9}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6 \vee F_7} \text{inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9}{- : \Delta_{12} \vdash \Delta_{11}, F_9} \vee_R \quad \frac{h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \text{inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}} \vee_R \\
\hline
- : \Delta_{12} \vdash \Delta_{11}, F_9 \quad \text{hCut} \quad - : \Delta_{12}, F_{10} \vdash \Delta_{11} \quad \text{ax/W} \quad \text{hCut} \\
\hline
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \rightarrow_L
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_7, F_9, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : \Delta_{14}, F_7, F_{12}, F_{13} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \wedge F_{13}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \wedge_L \\
\hline
- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_7, F_9}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9} \text{ax/W} \quad \frac{h_{11} : \Delta_{14}, F_{12}, F_{13}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9 \quad \wedge_L \\
\hline
- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{hCut} \\
\hline
\vee_R \\
\hline
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12} \wedge F_{13}, F_8, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), F_{12} \wedge F_{13}} \vee_R \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8 \vee F_9} \wedge_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad \text{Cut} \\
\hline
\sim \\
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, F_{12} \wedge F_{13}}{- : \Delta_{11} \vdash \Delta_7, F_8, F_9} \text{ax/W} \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8, F_9 \quad \wedge_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad \text{hCut} \\
\hline
\vee_R \\
\hline
\frac{h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{12}, F_9, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \wedge_L \\
\hline
- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\sim \\
\frac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \vee F_7} \text{inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \vee F_7 \vdash \Delta_{11}}{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}} \text{ax/W} \\
\hline
- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11} \quad \text{hCut} \\
\hline
- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11} \quad \wedge_L
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_7, F_9, F_{10}}{\bullet h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : \Delta_{14}, F_7, F_{12} \vdash \Delta_8, F_9 \vee F_{10} \quad h_{11} : \Delta_{14}, F_7, F_{13} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \vee F_{13}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \vee_L \\
\hline
- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\sim \\
\frac{h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_7, F_9}{- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9} \text{ax/W} \quad \frac{h_{11} : \Delta_{14}, F_{12}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax} \\
\hline
- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9 \quad \vee_R \quad \text{hCut} \\
\hline
- : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \vee F_{10}
\end{array}$$





$$\begin{array}{c}
\frac{h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6, F_7}{\bullet h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6 \vee F_7} \vee_R \quad \frac{}{\bullet h_8 : (\Delta_{11}, p_9), F_6 \vee F_7 \vdash \Delta_{10}, p_9} I \\
\hline
- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9 \quad \text{Cut} \\
\hline
\sim \\
\hline
- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9 \quad I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_{11} \vdash \Delta_7, \top, F_8, F_9}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), \top} \vee_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, \top \vdash \Delta_7, F_8 \vee F_9} \top_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad \text{Cut} \\
\hline
\sim \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad \text{ax/W} \\
\\
\frac{h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9, F_{10}}{\bullet h_2 : \top, \Delta_{12} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : \Delta_{12}, F_7 \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\top, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \top_L \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\sim \\
\hline
\frac{\bullet h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9 \vee F_{10}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10}} \text{ax/W} \quad \frac{h_{11} : \top, \Delta_{12}, F_7 \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : \top, \Delta_{12}, F_7 \vdash \Delta_8, F_9 \vee F_{10}} \text{ax/W} \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{hCut} \\
\\
\frac{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6, F_7}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \vee F_7 \vdash \Delta_9} \top_L \\
\hline
- : \top, \Delta_{10} \vdash \Delta_9 \quad \text{Cut} \\
\hline
\sim \\
\hline
\frac{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7}{- : \top, \Delta_{10} \vdash \Delta_9} \text{ax/W} \quad \frac{h_8 : \top, \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9}{\bullet h_8 : \top, \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9} \text{ax/W} \\
\hline
- : \top, \Delta_{10} \vdash \Delta_9 \quad \text{hCut}
\end{array}$$

## 8.4 Status of $\perp_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \rightarrow F_8), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4, F_7 \vdash \Delta_6, F_8}{\bullet h_5 : \Delta_4, \perp \vdash \Delta_6, F_7 \rightarrow F_8} \rightarrow_R \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8 \quad \text{Cut} \\
\hline
\sim \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8 \quad \text{ax/W} \\
\\
\frac{h_2 : \Delta_5 \vdash (\Delta_{10}, F_8 \rightarrow F_9), F_6}{\bullet h_2 : \Delta_5 \vdash (\perp, \Delta_{10}, F_8 \rightarrow F_9), F_6} \perp_R \quad \frac{h_7 : \Delta_5, F_6, F_8 \vdash \perp, \Delta_{10}, F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9} \rightarrow_R \\
\hline
- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9 \quad \text{Cut} \\
\hline
\sim \\
\hline
\frac{h_2 : \Delta_5 \vdash \perp, \Delta_{10}, F_6, F_8 \rightarrow F_9}{- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9} \text{ax/W} \\
\hline
- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9 \quad \text{hCut}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \wedge F_8), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash \Delta_6, F_7 \quad h_5 : \perp, \Delta_4 \vdash \Delta_6, F_8}{\bullet h_5 : \Delta_4, \perp \vdash \Delta_6, F_7 \wedge F_8} \wedge_R \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8 \quad \text{Cut} \\
\hline
\sim \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8 \quad \text{ax/W} \\
\\
\frac{h_2 : \Delta_5 \vdash (\Delta_{10}, F_8 \wedge F_9), F_6}{\bullet h_2 : \Delta_5 \vdash (\perp, \Delta_{10}, F_8 \wedge F_9), F_6} \perp_R \quad \frac{h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \quad h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \wedge F_9} \wedge_R \\
\hline
- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \wedge F_9 \quad \text{Cut} \\
\hline
\sim \\
\hline
\frac{h_2 : \Delta_5 \vdash \perp, \Delta_{10}, F_6, F_8 \wedge F_9}{- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \wedge F_9} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \wedge F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \wedge F_9} \text{ax/W} \\
\hline
- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \wedge F_9 \quad \text{hCut}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_4 \vdash \Delta_6, F_7 \vee F_8}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \vee F_8), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash \Delta_6, F_7, F_8}{\bullet h_5 : \Delta_4, \perp \vdash \Delta_6, F_7 \vee F_8} \vee_R}{- : \Delta_4 \vdash \Delta_6, F_7 \vee F_8} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \Delta_4 \vdash \Delta_6, F_7 \vee F_8} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_5 \vdash (\Delta_{10}, F_8 \vee F_9), F_6}{\bullet h_2 : \Delta_5 \vdash (\perp, \Delta_{10}, F_8 \vee F_9), F_6} \perp_R \quad \frac{h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8, F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \vee_R}{- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_2 : \Delta_5 \vdash \perp, \Delta_{10}, F_8, F_9}{\bullet h_2 : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \text{ax/W} \quad \frac{h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \vee F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \text{ax/W}}{- : \Delta_5 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \text{hCut}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_4 \vdash \perp, \Delta_6}{\bullet h_1 : \Delta_4 \vdash (\perp, \Delta_6), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash \Delta_6}{\bullet h_5 : \Delta_4, \perp \vdash \perp, \Delta_6} \perp_R}{- : \Delta_4 \vdash \perp, \Delta_6} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \Delta_4 \vdash \perp, \Delta_6} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_5 \vdash \Delta_8, F_6}{\bullet h_2 : \Delta_5 \vdash (\perp, \Delta_8), F_6} \perp_R \quad \frac{h_7 : \Delta_5, F_6 \vdash \Delta_8}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_8} \perp_R}{- : \Delta_5 \vdash \perp, \Delta_8} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_2 : \Delta_5 \vdash \perp, \Delta_8, F_6}{\bullet h_2 : \Delta_5 \vdash \perp, \Delta_8, F_6} \text{ax/W} \quad \frac{h_7 : \Delta_5, F_6 \vdash \perp, \Delta_8}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \Delta_8} \text{ax/W}}{- : \Delta_5 \vdash \perp, \Delta_8} \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_4 \vdash \top, \Delta_6}{\bullet h_1 : \Delta_4 \vdash (\top, \Delta_6), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash \top, \Delta_6}{\bullet h_5 : \Delta_4, \perp \vdash \top, \Delta_6} \top_R}{- : \Delta_4 \vdash \top, \Delta_6} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \Delta_4 \vdash \top, \Delta_6} \top_R \\
\frac{\frac{h_2 : \Delta_5 \vdash (\top, \Delta_8), F_6}{\bullet h_2 : \Delta_5 \vdash (\perp, \top, \Delta_8), F_6} \perp_R \quad \frac{h_7 : \Delta_5, F_6 \vdash \perp, \top, \Delta_8}{\bullet h_7 : \Delta_5, F_6 \vdash \perp, \top, \Delta_8} \top_R}{- : \Delta_5 \vdash \perp, \top, \Delta_8} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \Delta_5 \vdash \perp, \top, \Delta_8} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7, \perp} \perp_R \quad \frac{h_4 : \perp, \Delta_8 \vdash \Delta_7, F_5 \quad h_4 : \perp, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \rightarrow F_6), \perp \vdash \Delta_7} \rightarrow_L}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash \Delta_6, F_5}{\bullet h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : \Delta_{10}, F_5 \vdash \perp, \Delta_6, F_8 \quad h_7 : \Delta_{10}, F_5, F_9 \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \rightarrow F_9), F_5 \vdash \perp, \Delta_6} \rightarrow_L}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6, F_5}{\bullet h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6, F_5} \text{ax/W} \quad \frac{h_7 : \Delta_{10}, F_5, F_8 \rightarrow F_9 \vdash \perp, \Delta_6}{\bullet h_7 : \Delta_{10}, F_5, F_8 \rightarrow F_9 \vdash \perp, \Delta_6} \text{ax/W}}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6} \text{hCut} \\
\frac{\frac{h_2 : \Delta_7 \vdash \Delta_5, F_8 \rightarrow F_9}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), F_8 \rightarrow F_9} \perp_R \quad \frac{h_6 : \Delta_7 \vdash \perp, \Delta_5, F_8 \quad h_6 : \Delta_7, F_9 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \perp, \Delta_5} \rightarrow_L}{- : \Delta_7 \vdash \perp, \Delta_5} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \rightarrow F_9}{\bullet h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \rightarrow F_9} \text{ax/W} \quad \frac{h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \perp, \Delta_5} \text{ax/W}}{- : \Delta_7 \vdash \perp, \Delta_5} \text{hCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7, \perp} \perp_R \quad \frac{h_4 : \perp, \Delta_8, F_5, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \wedge F_6), \perp \vdash \Delta_7} \wedge_L}{\frac{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}{\sim} \text{Cut}} \text{ax/W} \\
\frac{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}{\sim} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash \Delta_6, F_5}{\bullet h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : \Delta_{10}, F_5, F_8, F_9 \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \wedge F_9), F_5 \vdash \perp, \Delta_6} \wedge_L}{\frac{- : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6}{\sim} \text{Cut}} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6, F_5}{\sim} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_{10}, F_5, F_8 \wedge F_9 \vdash \perp, \Delta_6}{\sim} \text{ax/W}}{\frac{- : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6}{\sim} \text{hCut}} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_7 \vdash \Delta_5, F_8 \wedge F_9}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), F_8 \wedge F_9} \perp_R \quad \frac{h_6 : \Delta_7, F_8, F_9 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \perp, \Delta_5} \wedge_L}{\frac{- : \Delta_7 \vdash \perp, \Delta_5}{\sim} \text{Cut}} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \wedge F_9}{\sim} \text{ax/W} \quad \frac{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \perp, \Delta_5}{\sim} \text{ax/W}}{\frac{- : \Delta_7 \vdash \perp, \Delta_5}{\sim} \text{hCut}} \text{ax/W}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7, \perp} \perp_R \quad \frac{h_4 : \perp, \Delta_8, F_5 \vdash \Delta_7 \quad h_4 : \perp, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \vee F_6), \perp \vdash \Delta_7} \vee_L}{\frac{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{\sim} \text{Cut}} \text{ax/W} \\
\frac{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{\sim} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \vee F_9 \vdash \Delta_6, F_5}{\bullet h_2 : \Delta_{10}, F_8 \vee F_9 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : \Delta_{10}, F_5, F_8 \vdash \perp, \Delta_6 \quad h_7 : \Delta_{10}, F_5, F_9 \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \vee F_9), F_5 \vdash \perp, \Delta_6} \vee_L}{\frac{- : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6}{\sim} \text{Cut}} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6, F_5}{\sim} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_{10}, F_5, F_8 \vee F_9 \vdash \perp, \Delta_6}{\sim} \text{ax/W}}{\frac{- : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6}{\sim} \text{hCut}} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_7 \vdash \Delta_5, F_8 \vee F_9}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), F_8 \vee F_9} \perp_R \quad \frac{h_6 : \Delta_7, F_8 \vdash \perp, \Delta_5 \quad h_6 : \Delta_7, F_9 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \perp, \Delta_5} \vee_L}{\frac{- : \Delta_7 \vdash \perp, \Delta_5}{\sim} \text{Cut}} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \vee F_9}{\sim} \text{ax/W} \quad \frac{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \perp, \Delta_5}{\sim} \text{ax/W}}{\frac{- : \Delta_7 \vdash \perp, \Delta_5}{\sim} \text{hCut}} \text{ax/W}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_5 \vdash \Delta_6}{\bullet h_1 : \Delta_5 \vdash \Delta_6, \perp} \perp_R \quad \frac{}{\bullet h_4 : \Delta_5, \perp \vdash \Delta_6} \perp_L}{\frac{- : \Delta_5 \vdash \Delta_6}{\sim} \text{Cut}} \text{ax/W} \\
\frac{- : \Delta_5 \vdash \Delta_6}{\sim} \text{ax/W} \\
\frac{\frac{h_2 : \Delta_7 \vdash \Delta_5, \perp}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), \perp} \perp_R \quad \frac{}{\bullet h_6 : \Delta_7, \perp \vdash \perp, \Delta_5} \perp_L}{\frac{- : \Delta_7 \vdash \perp, \Delta_5}{\sim} \text{Cut}} \text{ax/W} \\
\frac{- : \Delta_7 \vdash \perp, \Delta_5}{\sim} \text{ax/W} \\
\frac{\frac{h_2 : \perp, \Delta_8 \vdash \Delta_6, F_5}{\bullet h_2 : \perp, \Delta_8 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{}{\bullet h_7 : (\perp, \Delta_8), F_5 \vdash \perp, \Delta_6} \perp_L}{\frac{- : \perp, \Delta_8 \vdash \perp, \Delta_6}{\sim} \text{Cut}} \perp_L \\
\frac{- : \perp, \Delta_8 \vdash \perp, \Delta_6}{\sim} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_1 : \Delta_7, p_5 \vdash \Delta_6, p_5}{\bullet h_1 : \Delta_7, p_5 \vdash (\Delta_6, p_5), \perp} \perp_R \quad \frac{}{\bullet h_4 : (\Delta_7, p_5), \perp \vdash \Delta_6, p_5} I \\
\hline
- : \Delta_7, p_5 \vdash \Delta_6, p_5 \quad \text{Cut} \\
\hline
\frac{}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} \rightsquigarrow I \\
\hline
\frac{h_2 : \Delta_6 \vdash (\Delta_8, p_7), p_7}{\bullet h_2 : \Delta_6 \vdash (\perp, \Delta_8, p_7), p_7} \perp_R \quad \frac{}{\bullet h_5 : \Delta_6, p_7 \vdash \perp, \Delta_8, p_7} I \\
\hline
- : \Delta_6 \vdash \perp, \Delta_8, p_7 \quad \text{Cut} \\
\hline
\frac{}{- : \Delta_6 \vdash \perp, \Delta_8, p_7} \rightsquigarrow \\
\frac{h_2 : \Delta_6 \vdash \perp, \Delta_8, p_7, p_7}{\bullet h_2 : \Delta_6 \vdash \perp, \Delta_8, p_7, p_7} \text{ax/W} \quad \frac{}{\bullet h_5 : \Delta_6, p_7 \vdash \perp, \Delta_8, p_7} I \\
\hline
- : \Delta_6 \vdash \perp, \Delta_8, p_7 \quad \text{hCut} \\
\hline
\frac{h_2 : \Delta_9, p_7 \vdash (\Delta_8, p_7), F_5}{\bullet h_2 : \Delta_9, p_7 \vdash (\perp, \Delta_8, p_7), F_5} \perp_R \quad \frac{}{\bullet h_6 : (\Delta_9, p_7), F_5 \vdash \perp, \Delta_8, p_7} I \\
\hline
- : \Delta_9, p_7 \vdash \perp, \Delta_8, p_7 \quad \text{Cut} \\
\hline
\frac{}{- : \Delta_9, p_7 \vdash \perp, \Delta_8, p_7} \rightsquigarrow I \\
\hline
\frac{}{- : \Delta_9, p_7 \vdash \perp, \Delta_8, p_7} \rightsquigarrow I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_1 : \top, \Delta_6 \vdash \Delta_5}{\bullet h_1 : \top, \Delta_6 \vdash \Delta_5, \perp} \perp_R \quad \frac{h_4 : \perp, \Delta_6 \vdash \Delta_5}{\bullet h_4 : (\top, \Delta_6), \perp \vdash \Delta_5} \top_L \\
\hline
- : \top, \Delta_6 \vdash \Delta_5 \quad \text{Cut} \\
\hline
\frac{}{- : \top, \Delta_6 \vdash \Delta_5} \rightsquigarrow \text{ax/W} \\
\hline
\frac{h_2 : \Delta_7 \vdash \Delta_5, \top}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), \top} \perp_R \quad \frac{h_6 : \Delta_7 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, \top \vdash \perp, \Delta_5} \top_L \\
\hline
- : \Delta_7 \vdash \perp, \Delta_5 \quad \text{Cut} \\
\hline
\frac{}{- : \Delta_7 \vdash \perp, \Delta_5} \rightsquigarrow \text{ax/W} \\
\hline
\frac{h_2 : \top, \Delta_8 \vdash \Delta_6, F_5}{\bullet h_2 : \top, \Delta_8 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : \Delta_8, F_5 \vdash \perp, \Delta_6}{\bullet h_7 : (\top, \Delta_8), F_5 \vdash \perp, \Delta_6} \top_L \\
\hline
- : \top, \Delta_8 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\frac{}{- : \top, \Delta_8 \vdash \perp, \Delta_6} \rightsquigarrow \text{ax/W} \\
\frac{h_2 : \top, \Delta_8 \vdash \perp, \Delta_6, F_5}{\bullet h_2 : \top, \Delta_8 \vdash \perp, \Delta_6, F_5} \text{ax/W} \quad \frac{h_7 : \Delta_8, F_5 \vdash \perp, \Delta_6}{\bullet h_7 : \top, \Delta_8, F_5 \vdash \perp, \Delta_6} \text{ax/W} \\
\hline
- : \top, \Delta_8 \vdash \perp, \Delta_6 \quad \text{hCut} \\
\hline
\frac{}{- : \top, \Delta_8 \vdash \perp, \Delta_6} \rightsquigarrow
\end{array}$$

## 8.5 Status of $\top_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \rightarrow F_8), \top} \top_R \quad \frac{h_5 : \top, \Delta_4, F_7 \vdash \Delta_6, F_8}{\bullet h_5 : \Delta_4, \top \vdash \Delta_6, F_7 \rightarrow F_8} \rightarrow_R \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8 \quad \text{Cut} \\
\hline
\frac{}{- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8} \rightsquigarrow \\
\frac{\bullet h_1 : \Delta_4, F_7 \vdash \top, \Delta_6, F_8}{\bullet h_1 : \Delta_4, F_7 \vdash \top, \Delta_6, F_8} \top_R \quad \frac{h_5 : \top, \Delta_4, F_7 \vdash \Delta_6, F_8}{\bullet h_5 : \top, \Delta_4, F_7 \vdash \Delta_6, F_8} \text{ax/W} \\
\hline
- : \Delta_4, F_7 \vdash \Delta_6, F_8 \quad \text{hCut} \\
\hline
\frac{}{- : \Delta_4, F_7 \vdash \Delta_6, F_8} \rightarrow_R \\
\hline
\frac{}{- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8} \rightarrow_R \\
\hline
\frac{h_2 : \Delta_5 \vdash (\top, \Delta_{10}, F_8 \rightarrow F_9), F_6}{\bullet h_2 : \Delta_5 \vdash (\top, \Delta_{10}, F_8 \rightarrow F_9), F_6} \top_R \quad \frac{h_7 : \Delta_5, F_6, F_8 \vdash \top, \Delta_{10}, F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \top, \Delta_{10}, F_8 \rightarrow F_9} \rightarrow_R \\
\hline
- : \Delta_5 \vdash \top, \Delta_{10}, F_8 \rightarrow F_9 \quad \text{Cut} \\
\hline
\frac{}{- : \Delta_5 \vdash \top, \Delta_{10}, F_8 \rightarrow F_9} \rightsquigarrow \\
\hline
\frac{}{- : \Delta_5 \vdash \top, \Delta_{10}, F_8 \rightarrow F_9} \top_R
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \wedge F_8), \top}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8} \top_R \quad \frac{\frac{h_5 : \top, \Delta_4 \vdash \Delta_6, F_7 \quad h_5 : \top, \Delta_4 \vdash \Delta_6, F_8}{\bullet h_5 : \Delta_4, \top \vdash \Delta_6, F_7 \wedge F_8} \wedge_R}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8} \text{Cut} \\
\sim \\
\frac{\frac{\frac{\bullet h_1 : \Delta_4 \vdash \top, \Delta_6, F_7}{\vdash : \Delta_4 \vdash \Delta_6, F_7} \top_R \quad \frac{\frac{h_5 : \top, \Delta_4 \vdash \Delta_6, F_7}{\vdash : \Delta_4 \vdash \Delta_6, F_7} \text{ax/W} \quad \frac{\frac{\bullet h_1 : \Delta_4 \vdash \top, \Delta_6, F_8}{\vdash : \Delta_4 \vdash \Delta_6, F_8} \top_R \quad \frac{h_5 : \top, \Delta_4 \vdash \Delta_6, F_8}{\vdash : \Delta_4 \vdash \Delta_6, F_8} \text{ax/W}}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8} \text{hCut}}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8} \wedge_R \\
\sim \\
\frac{\frac{\frac{\bullet h_2 : \Delta_5 \vdash (\top, \Delta_{10}, F_8 \wedge F_9), F_6}{\vdash : \Delta_5 \vdash \top, \Delta_{10}, F_8 \wedge F_9} \top_R \quad \frac{\frac{h_7 : \Delta_5, F_6 \vdash \top, \Delta_{10}, F_8 \quad h_7 : \Delta_5, F_6 \vdash \top, \Delta_{10}, F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \top, \Delta_{10}, F_8 \wedge F_9} \wedge_R}{\vdash : \Delta_5 \vdash \top, \Delta_{10}, F_8 \wedge F_9} \text{Cut} \\
\sim \\
\vdash : \Delta_5 \vdash \top, \Delta_{10}, F_8 \wedge F_9 \top_R
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \vee F_8), \top}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \vee F_8} \top_R \quad \frac{\frac{h_5 : \top, \Delta_4 \vdash \Delta_6, F_7, F_8}{\bullet h_5 : \Delta_4, \top \vdash \Delta_6, F_7 \vee F_8} \vee_R}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \vee F_8} \text{Cut} \\
\sim \\
\frac{\frac{\frac{\bullet h_1 : \Delta_4 \vdash \top, \Delta_6, F_7, F_8}{\vdash : \Delta_4 \vdash \Delta_6, F_7, F_8} \top_R \quad \frac{h_5 : \top, \Delta_4 \vdash \Delta_6, F_7, F_8}{\vdash : \Delta_4 \vdash \Delta_6, F_7, F_8} \text{ax/W}}{\vdash : \Delta_4 \vdash \Delta_6, F_7 \vee F_8} \text{hCut} \\
\vee_R \\
\vdash : \Delta_4 \vdash \Delta_6, F_7 \vee F_8 \\
\sim \\
\frac{\frac{\frac{\bullet h_2 : \Delta_5 \vdash (\top, \Delta_{10}, F_8 \vee F_9), F_6}{\vdash : \Delta_5 \vdash \top, \Delta_{10}, F_8 \vee F_9} \top_R \quad \frac{\frac{h_7 : \Delta_5, F_6 \vdash \top, \Delta_{10}, F_8, F_9}{\bullet h_7 : \Delta_5, F_6 \vdash \top, \Delta_{10}, F_8 \vee F_9} \vee_R}{\vdash : \Delta_5 \vdash \top, \Delta_{10}, F_8 \vee F_9} \text{Cut} \\
\sim \\
\vdash : \Delta_5 \vdash \top, \Delta_{10}, F_8 \vee F_9 \top_R
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_4 \vdash (\perp, \Delta_6), \top}{\vdash : \Delta_4 \vdash \perp, \Delta_6} \top_R \quad \frac{\frac{h_5 : \top, \Delta_4 \vdash \Delta_6}{\bullet h_5 : \Delta_4, \top \vdash \perp, \Delta_6} \perp_R}{\vdash : \Delta_4 \vdash \perp, \Delta_6} \text{Cut} \\
\sim \\
\frac{\frac{\frac{\bullet h_1 : \Delta_4 \vdash \perp, \top, \Delta_6}{\vdash : \Delta_4 \vdash \perp, \Delta_6} \text{ax/W} \quad \frac{h_5 : \top, \Delta_4 \vdash \perp, \Delta_6}{\vdash : \Delta_4 \vdash \perp, \Delta_6} \text{ax/W}}{\vdash : \Delta_4 \vdash \perp, \Delta_6} \text{hCut} \\
\sim \\
\frac{\frac{\frac{\bullet h_2 : \Delta_5 \vdash (\top, \perp, \Delta_8), F_6}{\vdash : \Delta_5 \vdash \top, \perp, \Delta_8} \top_R \quad \frac{\frac{h_7 : \Delta_5, F_6 \vdash \top, \Delta_8}{\bullet h_7 : \Delta_5, F_6 \vdash \top, \perp, \Delta_8} \perp_R}{\vdash : \Delta_5 \vdash \top, \perp, \Delta_8} \text{Cut} \\
\sim \\
\vdash : \Delta_5 \vdash \top, \perp, \Delta_8 \top_R
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_4 \vdash (\top, \Delta_6), \top}{\vdash : \Delta_4 \vdash \top, \Delta_6} \top_R \quad \frac{\frac{h_5 : \Delta_4, \top \vdash \top, \Delta_6}{\bullet h_5 : \Delta_4, \top \vdash \top, \Delta_6} \top_R}{\vdash : \Delta_4 \vdash \top, \Delta_6} \text{Cut} \\
\sim \\
\vdash : \Delta_4 \vdash \top, \Delta_6 \top_R \\
\sim \\
\frac{\frac{\frac{\bullet h_2 : \Delta_5 \vdash (\top, \Delta_8), F_6}{\vdash : \Delta_5 \vdash \top, \Delta_8} \top_R \quad \frac{h_7 : \Delta_5, F_6 \vdash \top, \Delta_8}{\bullet h_7 : \Delta_5, F_6 \vdash \top, \Delta_8} \top_R}{\vdash : \Delta_5 \vdash \top, \Delta_8} \text{Cut} \\
\sim \\
\vdash : \Delta_5 \vdash \top, \Delta_8 \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7, \top}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5 \quad h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \rightarrow F_6), \top \vdash \Delta_7} \rightarrow_L}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \text{Cut} \\
\\
\frac{\frac{\bullet h_1 : \Delta_8 \vdash \top, \Delta_7, F_5}{- : \Delta_8 \vdash \Delta_7, F_5} \top_R \quad \frac{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5}{- : \Delta_8 \vdash \Delta_7, F_5} \text{ax/W} \quad \frac{\bullet h_1 : \Delta_8, F_6 \vdash \top, \Delta_7}{- : \Delta_8, F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{- : \Delta_8, F_6 \vdash \Delta_7} \text{ax/W}}{\frac{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \text{hCut}} \rightarrow_L \\
\\
\frac{\frac{\bullet h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash (\top, \Delta_6), F_5}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \top_R \quad \frac{h_7 : \Delta_{10}, F_5 \vdash \top, \Delta_6, F_8 \quad h_7 : \Delta_{10}, F_5, F_9 \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \rightarrow F_9), F_5 \vdash \top, \Delta_6} \rightarrow_L}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \text{Cut} \\
\\
\frac{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \text{ax/W} \quad \frac{\top_R}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \top_R \\
\\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), F_8 \rightarrow F_9}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{h_6 : \Delta_7 \vdash \top, \Delta_5, F_8 \quad h_6 : \Delta_7, F_9 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \top, \Delta_5} \rightarrow_L}{- : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\\
\frac{- : \Delta_7 \vdash \top, \Delta_5}{- : \Delta_7 \vdash \top, \Delta_5} \text{ax/W} \quad \frac{\top_R}{- : \Delta_7 \vdash \top, \Delta_5} \top_R
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7, \top}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_5, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \wedge F_6), \top \vdash \Delta_7} \wedge_L}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \text{Cut} \\
\\
\frac{\frac{\bullet h_1 : \Delta_8, F_5, F_6 \vdash \top, \Delta_7}{- : \Delta_8, F_5, F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_5, F_6 \vdash \Delta_7}{- : \Delta_8, F_5, F_6 \vdash \Delta_7} \text{ax/W}}{\frac{- : \Delta_8, F_5, F_6 \vdash \Delta_7}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \wedge_L} \text{hCut} \\
\\
\frac{\frac{\bullet h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash (\top, \Delta_6), F_5}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \top_R \quad \frac{h_7 : \Delta_{10}, F_5, F_8, F_9 \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \wedge F_9), F_5 \vdash \top, \Delta_6} \wedge_L}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \text{Cut} \\
\\
\frac{- : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \text{ax/W} \quad \frac{\top_R}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \top_R \\
\\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), F_8 \wedge F_9}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{h_6 : \Delta_7, F_8, F_9 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \top, \Delta_5} \wedge_L}{- : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\\
\frac{- : \Delta_7 \vdash \top, \Delta_5}{- : \Delta_7 \vdash \top, \Delta_5} \text{ax/W} \quad \frac{\top_R}{- : \Delta_7 \vdash \top, \Delta_5} \top_R
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7, \top}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_5 \vdash \Delta_7 \quad h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \vee F_6), \top \vdash \Delta_7} \vee_L}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \text{Cut} \\
\\
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \vdash \top, \Delta_7}{- : \Delta_8, F_5 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_5 \vdash \Delta_7}{- : \Delta_8, F_5 \vdash \Delta_7} \text{ax/W} \quad \frac{\bullet h_1 : \Delta_8, F_6 \vdash \top, \Delta_7}{- : \Delta_8, F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{- : \Delta_8, F_6 \vdash \Delta_7} \text{ax/W}}{\frac{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \vee_L} \text{hCut} \\
\\
\frac{\frac{\bullet h_2 : \Delta_{10}, F_8 \vee F_9 \vdash (\top, \Delta_6), F_5}{- : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \top_R \quad \frac{h_7 : \Delta_{10}, F_5, F_8 \vdash \top, \Delta_6 \quad h_7 : \Delta_{10}, F_5, F_9 \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \vee F_9), F_5 \vdash \top, \Delta_6} \vee_L}{- : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \text{Cut} \\
\\
\frac{- : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6}{- : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \text{ax/W} \quad \frac{\top_R}{- : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \top_R \\
\\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), F_8 \vee F_9}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{h_6 : \Delta_7, F_8 \vdash \top, \Delta_5 \quad h_6 : \Delta_7, F_9 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \top, \Delta_5} \vee_L}{- : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\\
\frac{- : \Delta_7 \vdash \top, \Delta_5}{- : \Delta_7 \vdash \top, \Delta_5} \text{ax/W} \quad \frac{\top_R}{- : \Delta_7 \vdash \top, \Delta_5} \top_R
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \perp, \Delta_6 \vdash \Delta_5, \top}{- : \perp, \Delta_6 \vdash \Delta_5} \top_R \quad \frac{\bullet h_4 : (\perp, \Delta_6), \top \vdash \Delta_5}{\text{Cut}} \perp_L}{\frac{}{- : \perp, \Delta_6 \vdash \Delta_5} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \perp, \Delta_6 \vdash \Delta_5} \perp_L \\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), \perp}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{\bullet h_6 : \Delta_7, \perp \vdash \top, \Delta_5}{\text{Cut}} \perp_L}{\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \\
\frac{\frac{\bullet h_2 : \perp, \Delta_8 \vdash (\top, \Delta_6), F_5}{- : \perp, \Delta_8 \vdash \top, \Delta_6} \top_R \quad \frac{\bullet h_7 : (\perp, \Delta_8), F_5 \vdash \top, \Delta_6}{\text{Cut}} \perp_L}{\frac{}{- : \perp, \Delta_8 \vdash \top, \Delta_6} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \perp, \Delta_8 \vdash \top, \Delta_6} \top_R
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_7, p_5 \vdash (\Delta_6, p_5), \top}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} \top_R \quad \frac{\bullet h_4 : (\Delta_7, p_5), \top \vdash \Delta_6, p_5}{\text{Cut}} I}{\frac{}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} I \\
\frac{\frac{\bullet h_2 : \Delta_6 \vdash (\top, \Delta_8, p_7), p_7}{- : \Delta_6 \vdash \top, \Delta_8, p_7} \top_R \quad \frac{\bullet h_5 : \Delta_6, p_7 \vdash \top, \Delta_8, p_7}{\text{Cut}} I}{\frac{}{- : \Delta_6 \vdash \top, \Delta_8, p_7} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \Delta_6 \vdash \top, \Delta_8, p_7} \top_R \\
\frac{\frac{\bullet h_2 : \Delta_9, p_7 \vdash (\top, \Delta_8, p_7), F_5}{- : \Delta_9, p_7 \vdash \top, \Delta_8, p_7} \top_R \quad \frac{\bullet h_6 : (\Delta_9, p_7), F_5 \vdash \top, \Delta_8, p_7}{\text{Cut}} I}{\frac{}{- : \Delta_9, p_7 \vdash \top, \Delta_8, p_7} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \Delta_9, p_7 \vdash \top, \Delta_8, p_7} \top_R
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_5 \vdash \Delta_6, \top}{- : \Delta_5 \vdash \Delta_6} \top_R \quad \frac{\frac{h_4 : \Delta_5 \vdash \Delta_6}{\bullet h_4 : \Delta_5, \top \vdash \Delta_6} \top_L}{\text{Cut}} \top_L}{\frac{}{- : \Delta_5 \vdash \Delta_6} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \Delta_5 \vdash \Delta_6} \text{ax/w} \\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), \top}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{\frac{h_6 : \Delta_7 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, \top \vdash \top, \Delta_5} \top_L}{\text{Cut}} \top_L}{\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \\
\frac{\frac{\bullet h_2 : \top, \Delta_8 \vdash (\top, \Delta_6), F_5}{- : \top, \Delta_8 \vdash \top, \Delta_6} \top_R \quad \frac{\frac{h_7 : \Delta_8, F_5 \vdash \top, \Delta_6}{\bullet h_7 : (\top, \Delta_8), F_5 \vdash \top, \Delta_6} \top_L}{\text{Cut}} \top_L}{\frac{}{- : \top, \Delta_8 \vdash \top, \Delta_6} \rightsquigarrow} \text{Cut} \\
\frac{}{- : \top, \Delta_8 \vdash \top, \Delta_6} \top_R
\end{array}$$

## 8.6 Status of $\rightarrow_L$ : OK

- Case rule  $\rightarrow_R$





$$\begin{array}{c}
\frac{h_3 : \Delta_7 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}, F_8 \quad h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12} \rightarrow_L \quad \frac{h_{10} : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{11} \quad h_{10} : \Delta_7, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13} \rightarrow_L}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \text{Cut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \\
\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13} \\
\frac{\frac{\vdash : \Delta_7, F_{11} \vdash \Delta_{13}, F_{12}, F_8 \quad \text{inv-th/ax} \quad \vdash : \Delta_7, F_{11}, F_9 \vdash \Delta_{13}, F_{12} \rightarrow_L \quad \text{inv-th/ax}}{\vdash : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{12}} \rightarrow_L \quad \frac{\vdash : \Delta_7, F_{11}, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\vdash : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}} \text{sCut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \text{ax/W} \\
\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13} \\
\frac{h_3 : \Delta_7 \vdash \Delta_{12}, F_8, F_{10} \quad h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8 \rightarrow_L \quad \frac{h_9 : \Delta_7, F_8 \vdash \Delta_{12}, F_{10} \quad h_9 : \Delta_7, F_8, F_{11} \vdash \Delta_{12} \rightarrow_L}{\bullet h_9 : (\Delta_7, F_{10} \rightarrow F_{11}), F_8 \vdash \Delta_{12}} \text{Cut}}{\vdash : \Delta_7, F_{10} \rightarrow F_{11} \vdash \Delta_{12}} \\
\vdash : \Delta_7, F_{10} \rightarrow F_{11} \vdash \Delta_{12} \\
\frac{\frac{h_3 : \Delta_7 \vdash \Delta_{12}, F_{10}, F_8 \quad \text{ax/W} \quad \frac{h_9 : \Delta_7, F_8 \vdash \Delta_{12}, F_{10} \quad \text{ax/W}}{\bullet h_9 : \Delta_7, F_8 \vdash \Delta_{12}, F_{10}} \text{height}}{\vdash : \Delta_7 \vdash \Delta_{12}, F_{10}} \text{hCut} \quad \frac{\frac{h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8 \quad \text{ax/W} \quad \frac{h_9 : \Delta_7, F_{11}, F_8 \vdash \Delta_{12} \quad \text{ax/W}}{\bullet h_9 : \Delta_7, F_{11}, F_8 \vdash \Delta_{12}} \text{height}}{\vdash : \Delta_7, F_{11} \vdash \Delta_{12}} \text{hCut}}{\vdash : \Delta_7, F_{10} \rightarrow F_{11} \vdash \Delta_{12}} \rightarrow_L
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_{14}, F_{11} \wedge F_{12} \vdash \Delta_{13}, F_7, F_8 \quad h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_9 \vdash \Delta_{13}, F_7 \rightarrow_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_{11}, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{13}} \text{Cut}}{\vdash : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \rightarrow F_9 \vdash \Delta_{13}} \\
\vdash : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \rightarrow F_9 \vdash \Delta_{13} \\
\frac{\frac{h_3 : \Delta_{14}, F_{11}, F_{12} \vdash \Delta_{13}, F_7, F_8 \quad \text{inv-th/ax} \quad h_3 : \Delta_{14}, F_{11}, F_{12}, F_9 \vdash \Delta_{13}, F_7 \rightarrow_L \quad \text{inv-th/ax}}{\bullet h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_7} \rightarrow_L \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_{12}, F_7, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\vdash : \Delta_{14}, F_{11}, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}} \text{ax/W}}{\vdash : \Delta_{14}, F_8 \rightarrow F_9, F_{11} \wedge F_{12} \vdash \Delta_{13}} \text{hCut} \\
\vdash : \Delta_{14}, F_8 \rightarrow F_9, F_{11} \wedge F_{12} \vdash \Delta_{13} \\
\frac{h_3 : \Delta_7 \vdash \Delta_{13}, F_{11} \wedge F_{12}, F_8 \quad h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12} \rightarrow_L \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), F_{11} \wedge F_{12} \vdash \Delta_{13}} \text{Cut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \\
\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13} \\
\frac{\frac{h_3 : \Delta_7 \vdash \Delta_{13}, F_8, F_{11} \wedge F_{12} \quad \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12} \vdash \Delta_{13}, F_8 \quad \text{inv-th/ax}}{\bullet h_{10} : \Delta_7, F_{11} \wedge F_{12} \vdash \Delta_{13}, F_8} \wedge_L}{\vdash : \Delta_7 \vdash \Delta_{13}, F_8} \text{hCut} \quad \frac{h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12} \quad \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_9 \vdash \Delta_{13} \quad \text{inv-t.}}{\bullet h_{10} : \Delta_7, F_9, F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L}{\vdash : \Delta_7, F_9 \vdash \Delta_{13}} \text{hCut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \rightarrow_L
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_{14}, F_{11} \vee F_{12} \vdash \Delta_{13}, F_7, F_8 \quad h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_9 \vdash \Delta_{13}, F_7 \rightarrow_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \vee F_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{13}} \text{Cut}}{\vdash : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \rightarrow F_9 \vdash \Delta_{13}} \\
\vdash : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \rightarrow F_9 \vdash \Delta_{13} \\
\frac{\frac{h_3 : \Delta_{14}, F_{11} \vee F_{12} \vdash \Delta_{13}, F_7, F_8 \quad \text{ax/W} \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_7 \vdash \Delta_{13}, F_8 \quad \text{inv-th/ax}}{\bullet h_{10} : \Delta_{14}, F_7, F_{11} \vee F_{12} \vdash \Delta_{13}, F_8} \wedge_L}{\vdash : \Delta_{14}, F_{11} \vee F_{12} \vdash \Delta_{13}, F_8} \text{hCut} \quad \frac{h_{10} : \Delta_{14}, F_{12}, F_7 \vdash \Delta_{13}, F_8 \quad \text{inv-th/ax}}{\vdash : \Delta_{14}, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \vee_L}{\vdash : \Delta_{14}, F_8 \rightarrow F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \\
\vdash : \Delta_{14}, F_8 \rightarrow F_9, F_{11} \vee F_{12} \vdash \Delta_{13} \\
\frac{h_3 : \Delta_7 \vdash \Delta_{13}, F_{11} \vee F_{12}, F_8 \quad h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \vee F_{12} \rightarrow_L \quad \frac{h_{10} : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13} \quad h_{10} : \Delta_7, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \text{Cut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \\
\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13} \\
\frac{\frac{\vdash : \Delta_7 \vdash \Delta_{13}, F_{11}, F_{12}, F_8 \quad \text{inv-th/ax} \quad \vdash : \Delta_7, F_9 \vdash \Delta_{13}, F_{11}, F_{12} \rightarrow_L \quad \text{inv-th/ax}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{11}, F_{12}} \rightarrow_L \quad \frac{\vdash : \Delta_7, F_{12}, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{11} \quad \text{ax/W}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{11}} \text{sCut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \rightarrow_L
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7 \vdash \Delta_{11}, \perp, F_8 \quad h_3 : \Delta_7, F_9 \vdash \Delta_{11}, \perp}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}, \perp} \rightarrow_L \quad \frac{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), \perp \vdash \Delta_{11}}{\vdash_L} \text{Cut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\sim \\
\frac{\frac{h_3 : \Delta_7 \vdash \perp, \Delta_{11}, F_8}{\vdash : \Delta_7 \vdash \Delta_{11}, F_8} \text{ax/W} \quad \frac{\bullet h_{10} : \perp, \Delta_7 \vdash \Delta_{11}, F_8}{\vdash : \Delta_7, F_9 \vdash \Delta_{11}} \text{hCut} \quad \frac{\frac{h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}}{\vdash : \Delta_7, F_9 \vdash \Delta_{11}} \text{ax/W} \quad \frac{\bullet h_{10} : \perp, \Delta_7, F_9 \vdash \Delta_{11}}{\vdash_L} \text{hCut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \rightarrow_L \\
\sim \\
\frac{h_3 : \perp, \Delta_{12} \vdash \Delta_{11}, F_7, F_8 \quad h_3 : (\perp, \Delta_{12}), F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}, F_7} \rightarrow_L \quad \frac{\bullet h_{10} : ((\perp, \Delta_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{11}}{\vdash_L} \text{Cut}}{\vdash : (\perp, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\sim \\
\vdash : \perp, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11} \quad \vdash_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_3 : \Delta_7 \vdash (\Delta_{12}, p_{11}), p_{11}, F_8 \quad h_3 : \Delta_7, F_9 \vdash (\Delta_{12}, p_{11}), p_{11}}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash (\Delta_{12}, p_{11}), p_{11}} \rightarrow_L \quad \frac{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), p_{11} \vdash \Delta_{12}, p_{11}}{\vdash_L} \text{Cut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} \\
\sim \\
\frac{\frac{h_3 : \Delta_7 \vdash \Delta_{12}, F_8, p_{11}, p_{11}}{\vdash : \Delta_7 \vdash \Delta_{12}, F_8, p_{11}} \text{ax/W} \quad \frac{\bullet h_{10} : \Delta_7, p_{11} \vdash \Delta_{12}, F_8, p_{11}}{\vdash : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}} I \text{ hCut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} \rightarrow_L \\
\sim \\
\frac{h_3 : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7, F_8 \quad h_3 : (\Delta_{13}, p_{11}), F_9 \vdash (\Delta_{12}, p_{11}), F_7}{\bullet h_3 : (\Delta_{13}, p_{11}), F_8 \rightarrow F_9 \vdash (\Delta_{12}, p_{11}), F_7} \rightarrow_L \quad \frac{\bullet h_{10} : ((\Delta_{13}, p_{11}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{12}, p_{11}}{\vdash_L} \text{Cut}}{\vdash : (\Delta_{13}, p_{11}), F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} \\
\sim \\
\vdash : \Delta_{13}, p_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11} \quad I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_7 \vdash \Delta_{11}, \top, F_8 \quad h_3 : \Delta_7, F_9 \vdash \Delta_{11}, \top}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}, \top} \rightarrow_L \quad \frac{h_{10} : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), \top \vdash \Delta_{11}} \top_L \text{Cut}}{\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\sim \\
\vdash : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11} \quad \text{ax/W} \\
\sim \\
\frac{h_3 : \top, \Delta_{12} \vdash \Delta_{11}, F_7, F_8 \quad h_3 : (\top, \Delta_{12}), F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : (\top, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}, F_7} \rightarrow_L \quad \frac{h_{10} : \Delta_{12}, F_7, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{11}} \top_L \text{Cut}}{\vdash : (\top, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\sim \\
\frac{\bullet h_3 : \top, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}, F_7}{\vdash : \top, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}} \text{ax/W} \quad \frac{h_{10} : \top, \Delta_{12}, F_7, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\text{hCut}} \text{ax/W}
\end{array}$$

## 8.7 Status of $\wedge_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9, F_{10} \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_8}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_8} \wedge_L \quad \frac{h_{11} : \Delta_7, F_8, F_{13}, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \wedge F_{10}), F_8 \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \rightarrow_R \text{Cut}}{\vdash : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_{10}, F_{13}, F_9 \vdash \Delta_{12}, F_{14}, F_8}{\bullet h_3 : \Delta_7, F_{13}, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}, F_8} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_7, F_{13}, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}}{\vdash : \Delta_7, F_{13}, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}} \text{ax/W} \text{hCut}}{\vdash : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \rightarrow_R
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9, F_{10} \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_8}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_8} \wedge_L \quad \frac{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \quad h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \wedge F_{10}), F_8 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \wedge_R \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_3 : \Delta_7, F_{10}, F_9 \vdash \Delta_{12}, F_8, F_{13} \wedge F_{14}}{h_3 : \Delta_7, F_{10}, F_9 \vdash \Delta_{12}, F_8, F_{13} \wedge F_{14}} \text{ax/W} \quad \frac{\frac{h_{11} : \Delta_7, F_{10}, F_8, F_9 \vdash \Delta_{12}, F_{13}}{h_{11} : \Delta_7, F_{10}, F_8, F_9 \vdash \Delta_{12}, F_{13}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_7, F_{10}, F_8, F_9 \vdash \Delta_{12}, F_{14}}{h_{11} : \Delta_7, F_{10}, F_8, F_9 \vdash \Delta_{12}, F_{14}} \text{inv-th/ax}}{\bullet h_{11} : \Delta_7, F_{10}, F_8, F_9 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \wedge_R \\
\hline
- : \Delta_7, F_{10}, F_9 \vdash \Delta_{12}, F_{13} \wedge F_{14} \quad \text{hCut} \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14} \quad \wedge_L
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9, F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_8}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_8} \wedge_L \quad \frac{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \wedge F_{10}), F_8 \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_3 : \Delta_7, F_{10}, F_9 \vdash \Delta_{12}, F_{13}, F_{14}, F_8}{\bullet h_3 : \Delta_7, F_{10}, F_9 \vdash \Delta_{12}, F_{13}, F_{14}, F_8} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14}} \text{ax/W}}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_8} \wedge_L \quad \text{hCut} \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14} \quad \vee_R \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9, F_{10} \vdash (\perp, \Delta_{12}), F_8}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash (\perp, \Delta_{12}), F_8} \wedge_L \quad \frac{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \Delta_{12}}{\bullet h_{11} : (\Delta_7, F_9 \wedge F_{10}), F_8 \vdash \perp, \Delta_{12}} \perp_R \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \perp, \Delta_{12} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_3 : \Delta_7, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}, F_8} \text{ax/W} \quad \frac{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}}{h_{11} : \Delta_7, F_8, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}} \text{ax/W}}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}} \wedge_L \quad \text{hCut} \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9, F_{10} \vdash (\top, \Delta_{12}), F_8}{\bullet h_3 : \Delta_7, F_9 \wedge F_{10} \vdash (\top, \Delta_{12}), F_8} \wedge_L \quad \frac{h_{11} : (\Delta_7, F_9 \wedge F_{10}), F_8 \vdash \top, \Delta_{12}}{\bullet h_{11} : (\Delta_7, F_9 \wedge F_{10}), F_8 \vdash \top, \Delta_{12}} \top_R \\
\hline
- : \Delta_7, F_9 \wedge F_{10} \vdash \top, \Delta_{12} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
- : \Delta_7, F_9 \wedge F_{10} \vdash \top, \Delta_{12} \quad \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8, F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \wedge_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \quad h_{10} : \Delta_{14}, F_7, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{13}} \rightarrow_L \\
\hline
- : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_3 : \Delta_{14}, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}, F_7}{h_3 : \Delta_{14}, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}, F_7} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{14}, F_7, F_8, F_9 \vdash \Delta_{13}, F_{11}}{h_{10} : \Delta_{14}, F_7, F_8, F_9 \vdash \Delta_{13}, F_{11}} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{14}, F_{12}, F_7, F_8, F_9 \vdash \Delta_{13}}{h_{10} : \Delta_{14}, F_{12}, F_7, F_8, F_9 \vdash \Delta_{13}} \text{inv-th/ax}}{\bullet h_{10} : \Delta_{14}, F_7, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \wedge_L \quad \text{hCut} \\
\hline
- : \Delta_{14}, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13} \quad \wedge_L \\
\hline
- : \Delta_{14}, F_{11} \rightarrow F_{12}, F_8 \wedge F_9 \vdash \Delta_{13} \\
\hline
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}} \wedge_L \quad \frac{h_{10} : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \quad h_{10} : \Delta_7, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \rightarrow_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{Cut} \\
\hline
\rightsquigarrow \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}}{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11}}{h_{10} : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11}} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_7, F_{12}, F_8, F_9 \vdash \Delta_{13}}{h_{10} : \Delta_7, F_{12}, F_8, F_9 \vdash \Delta_{13}} \text{inv-th/ax}}{\bullet h_{10} : \Delta_7, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \wedge_L \quad \text{hCut} \\
\hline
- : \Delta_7, F_8, F_9 \vdash \Delta_{13} \quad \wedge_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_8, F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \wedge_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{13}} \wedge_L \\
\hline
- : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_{14}, F_{11}, F_{12}, F_8, F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \text{inv-th/ax} \wedge_L \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_7), F_8 \wedge F_9 \vdash \Delta_{13}} \text{ax/W} \\
\hline
- : \Delta_{14}, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{hCut} \\
\hline
- : \Delta_{14}, F_{11} \wedge F_{12}, F_8 \wedge F_9 \vdash \Delta_{13} \quad \wedge_L
\end{array} \\
\hline
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}} \wedge_L \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}} \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8, F_9), F_{11} \wedge F_{12} \vdash \Delta_{13}} \text{inv-th/ax} \\
\hline
- : \Delta_7, F_8, F_9 \vdash \Delta_{13} \quad \text{hCut} \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \quad \wedge_L
\end{array} \\
\hline
\frac{h_3 : \Delta_7, F_{10}, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_8} \wedge_L \quad \frac{h_9 : \Delta_7, F_8, F_{10}, F_{11} \vdash \Delta_{12}}{\bullet h_9 : (\Delta_7, F_{10} \wedge F_{11}), F_8 \vdash \Delta_{12}} \wedge_L \\
\hline
- : \Delta_7, F_{10} \wedge F_{11} \vdash \Delta_{12} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_7, F_{10}, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{10}, F_{11} \vdash \Delta_{12}, F_8} \text{ax/W} \quad \frac{h_9 : \Delta_7, F_{10}, F_{11}, F_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_7, F_{10}, F_{11}), F_8 \vdash \Delta_{12}} \text{height} \\
\hline
- : \Delta_7, F_{10}, F_{11} \vdash \Delta_{12} \quad \text{hCut} \\
\hline
- : \Delta_7, F_{10} \wedge F_{11} \vdash \Delta_{12} \quad \wedge_L
\end{array}
\end{array}$$

• Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_8, F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \wedge_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_{11}, F_8 \wedge F_9 \vdash \Delta_{13} \quad h_{10} : \Delta_{14}, F_7, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \vee F_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{13}} \vee_L \\
\hline
- : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_{14}, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}, F_7} \text{ax/W} \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_7, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_{14}, F_7, F_8, F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \text{inv-th/ax} \\
\hline
- : \Delta_{14}, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13} \quad \text{hCut} \\
\hline
- : \Delta_{14}, F_8 \wedge F_9, F_{11} \vee F_{12} \vdash \Delta_{13} \quad \wedge_L
\end{array} \\
\hline
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}} \wedge_L \quad \frac{h_{10} : \Delta_7, F_{11}, F_8 \wedge F_9 \vdash \Delta_{13} \quad h_{10} : \Delta_7, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \vee_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}}{\bullet h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}} \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_{11}, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8, F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \text{inv-th/ax} \\
\hline
- : \Delta_7, F_8, F_9 \vdash \Delta_{13} \quad \text{hCut} \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \quad \wedge_L
\end{array}
\end{array}$$

• Case rule  $\perp_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{11}, \perp}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}, \perp} \wedge_L \quad \frac{h_{10} : (\Delta_7, F_8 \wedge F_9), \perp \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), \perp \vdash \Delta_{11}} \perp_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8, F_9 \vdash \perp, \Delta_{11}} \text{ax/W} \quad \frac{h_{10} : \perp, \Delta_7, F_8, F_9 \vdash \Delta_{11}}{\bullet h_{10} : \perp, \Delta_7, F_8, F_9 \vdash \Delta_{11}} \perp_L \\
\hline
- : \Delta_7, F_8, F_9 \vdash \Delta_{11} \quad \text{hCut} \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11} \quad \wedge_L
\end{array} \\
\hline
\frac{h_3 : (\perp, \Delta_{12}), F_8, F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}, F_7} \wedge_L \quad \frac{h_{10} : ((\perp, \Delta_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{11}}{\bullet h_{10} : ((\perp, \Delta_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{11}} \perp_L \\
\hline
- : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : (\perp, \Delta_{12}), F_8, F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}, F_7} \text{ax/W} \quad \frac{h_{10} : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}} \perp_L \\
\hline
- : \perp, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11} \quad \text{hCut} \\
\hline
- : \perp, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11} \quad \perp_L
\end{array}
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_8, F_9 \vdash (\Delta_{12}, P_{11}), P_{11} \quad \bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash (\Delta_{12}, P_{11}), P_{11} \quad \wedge_L \quad \frac{}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), P_{11} \vdash \Delta_{12}, P_{11}} I}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12}, P_{11}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{12}, P_{11}, P_{11}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12}, P_{11}} \text{ax/W} \quad \frac{\bullet h_{10} : \Delta_7, F_8, F_9, P_{11} \vdash \Delta_{12}, P_{11}}{\bullet h_{10} : ((\Delta_{13}, P_{11}), F_8 \wedge F_9), F_7 \vdash \Delta_{12}, P_{11}} I}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12}, P_{11}} \wedge_L \text{hCut} \\
\sim \\
\frac{h_3 : (\Delta_{13}, P_{11}), F_8, F_9 \vdash (\Delta_{12}, P_{11}), F_7 \quad \bullet h_3 : (\Delta_{13}, P_{11}), F_8 \wedge F_9 \vdash (\Delta_{12}, P_{11}), F_7 \quad \wedge_L \quad \frac{}{\bullet h_{10} : ((\Delta_{13}, P_{11}), F_8 \wedge F_9), F_7 \vdash \Delta_{12}, P_{11}} I}{- : (\Delta_{13}, P_{11}), F_8 \wedge F_9 \vdash \Delta_{12}, P_{11}} \text{Cut} \\
\sim \\
- : \Delta_{13}, P_{11}, F_8 \wedge F_9 \vdash \Delta_{12}, P_{11} \quad I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{11}, \top \quad \bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}, \top \quad \wedge_L \quad \frac{h_{10} : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), \top \vdash \Delta_{11}} \top_L}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}} \text{Cut} \\
\sim \\
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11} \quad \text{ax/W} \\
\sim \\
\frac{h_3 : (\top, \Delta_{12}), F_8, F_9 \vdash \Delta_{11}, F_7 \quad \bullet h_3 : (\top, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}, F_7 \quad \wedge_L \quad \frac{h_{10} : \Delta_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{11}} \top_L}{- : (\top, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}} \text{Cut} \\
\sim \\
\frac{\bullet h_3 : \top, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11}, F_7 \quad \text{ax/W} \quad \frac{h_{10} : \top, \Delta_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{11}} \text{ax/W}}{- : \top, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11}} \text{hCut}
\end{array}$$

## 8.8 Status of $\vee_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9 \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_8 \quad \bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_8 \quad \wedge_L \quad \frac{h_{11} : \Delta_7, F_8, F_{13}, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \rightarrow_R}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_{13}, F_9 \vdash \Delta_{12}, F_{14}, F_8}{\bullet h_3 : \Delta_7, F_{13}, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}, F_8} \text{inv-th/ax} \quad \frac{h_3 : \Delta_7, F_{10}, F_{13} \vdash \Delta_{12}, F_{14}, F_8}{\bullet h_3 : \Delta_7, F_{10}, F_{13} \vee F_{10} \vdash \Delta_{12}, F_{14}, F_8} \text{inv-th/ax}}{- : \Delta_7, F_{13}, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}} \wedge_L \text{hCut} \\
\sim \\
- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14} \quad \rightarrow_R
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9 \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_8 \quad \bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_8 \quad \wedge_L \quad \frac{h_{11} : \Delta_7, F_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \wedge_R}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, F_{13}, F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_8} \text{inv-th/ax} \quad \frac{h_3 : \Delta_7, F_{10} \vdash \Delta_{12}, F_{13}, F_8}{\bullet h_3 : \Delta_7, F_{10} \vee F_{10} \vdash \Delta_{12}, F_{13}, F_8} \text{inv-th/ax}}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}} \wedge_L \text{hCut} \\
\sim \\
\frac{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \quad \frac{h_{11} : \Delta_7, F_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \text{ax/W}}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14}} \text{hCut}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9 \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_8 \quad h_3 : \Delta_7, F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_8} \vee_L \quad \frac{h_{11} : \Delta_7, F_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R \\
\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}} \text{Cut} \\
\frac{}{\sim} \\
\frac{\frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, F_{13}, F_{14}, F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_8} \text{inv-th/ax} \quad \frac{h_3 : \Delta_7, F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_8} \text{inv-th/ax}}{\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}} \vee_L} \vee_L \\
\frac{h_{11} : \Delta_7, F_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \Delta_{12}, F_{13}, F_{14}} \text{ax/W} \\
\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}} \text{hCut} \\
\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9 \vdash (\perp, \Delta_{12}), F_8 \quad h_3 : \Delta_7, F_{10} \vdash (\perp, \Delta_{12}), F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash (\perp, \Delta_{12}), F_8} \vee_L \quad \frac{h_{11} : \Delta_7, F_8, F_9 \vee F_{10} \vdash \Delta_{12}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \perp, \Delta_{12}} \perp_R \\
\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{Cut} \\
\frac{}{\sim} \\
\frac{\frac{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash \perp, \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{ax/W} \quad \frac{h_{11} : \Delta_7, F_8, F_9 \vee F_{10} \vdash \perp, \Delta_{12}}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \perp, \Delta_{12}} \text{ax/W}}{\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{hCut}}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_7, F_9 \vdash (\top, \Delta_{12}), F_8 \quad h_3 : \Delta_7, F_{10} \vdash (\top, \Delta_{12}), F_8}{\bullet h_3 : \Delta_7, F_9 \vee F_{10} \vdash (\top, \Delta_{12}), F_8} \vee_L \quad \frac{}{\bullet h_{11} : (\Delta_7, F_9 \vee F_{10}), F_8 \vdash \top, \Delta_{12}} \top_R \\
\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \top, \Delta_{12}} \text{Cut} \\
\frac{}{\sim} \\
\frac{}{- : \Delta_7, F_9 \vee F_{10} \vdash \top, \Delta_{12}} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vdash \Delta_{13}, F_7 \quad h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9) \vdash \Delta_{13}} \rightarrow_R \\
\frac{}{- : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\frac{}{\sim} \\
\frac{\frac{h_3 : \Delta_{14}, F_8 \vdash \Delta_{13}, F_{11}, F_7}{\bullet h_3 : \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_{14}, F_9 \vdash \Delta_{13}, F_{11}, F_7}{\bullet h_3 : \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_7} \text{inv-th/ax}}{\frac{}{- : \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}} \vee_L} \vee_L \\
\frac{h_{10} : \Delta_{14}, F_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9) \vdash \Delta_{13}, F_{11}} \text{ax/W} \\
\frac{}{- : \Delta_{14}, F_{11} \rightarrow F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \text{hCut} \\
\frac{}{- : \Delta_{14}, F_{11} \rightarrow F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \rightarrow_L \\
\frac{}{\sim} \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{13}, F_{11} \rightarrow F_{12} \quad h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}} \vee_L \quad \frac{h_{10} : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \quad h_{10} : \Delta_7, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \rightarrow_R \\
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\frac{}{\sim} \\
\frac{\frac{}{- : \Delta_7, F_{11}, F_8 \vdash \Delta_{13}, F_{12}} \text{inv-th/ax} \quad \frac{}{- : \Delta_7, F_{11}, F_9 \vdash \Delta_{13}, F_{12}} \text{inv-th/ax}}{\frac{}{- : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}, F_{12}} \vee_L} \vee_L \\
\frac{}{- : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}, F_{12}} \text{ax/W} \\
\frac{}{- : \Delta_7, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \text{sCut} \\
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vdash \Delta_{13}, F_7 \quad h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\frac{}{\sim} \\
\frac{\frac{h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_{14}, F_{11}, F_{12}, F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{inv-th/ax}}{\frac{}{- : \Delta_{14}, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L} \vee_L \\
\frac{h_{10} : \Delta_{14}, F_{11}, F_{12}, F_7, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{13}} \text{ax/W} \\
\frac{}{- : \Delta_{14}, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \text{hCut} \\
\frac{}{- : \Delta_{14}, F_{11} \wedge F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \wedge_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{13}, F_{11} \wedge F_{12} \quad h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}} \vee_L \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{- : \Delta_7, F_8 \vdash \Delta_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_8, F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L \quad \frac{h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{- : \Delta_7, F_9 \vdash \Delta_{13}} \text{ax/W}}{\bullet h_{10} : \Delta_7, F_9, F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{hCut} \quad \frac{- : \Delta_7, F_9 \vdash \Delta_{13}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \vee_L
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \vdash \Delta_{13}, F_7 \quad h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_9 \vdash \Delta_{13}, F_7}{\bullet h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \vee F_{12}), F_8 \vee F_9) \vdash \Delta_{13}} \wedge_L}{- : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_{14}, F_{11}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_{14}, F_{11}, F_9 \vdash \Delta_{13}, F_7}{- : \Delta_{14}, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}} \text{inv-th/ax}}{- : \Delta_{14}, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}} \vee_L \quad \frac{\frac{h_{10} : \Delta_{14}, F_{11}, F_7, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_{14}, F_{11}, F_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{ax/W} \quad \frac{h_3 : \Delta_{14}, F_{12}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{inv-th/ax}}{- : \Delta_{14}, F_{11} \vee F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \text{hCut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{13}, F_{11} \vee F_{12} \quad h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}} \vee_L \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \wedge_L \quad \frac{h_{10} : \Delta_7, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \wedge_L}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\sim \\
\frac{\frac{- : \Delta_7, F_8 \vdash \Delta_{13}, F_{11}, F_{12}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_{12}} \text{inv-th/ax} \quad \frac{- : \Delta_7, F_9 \vdash \Delta_{13}, F_{11}, F_{12}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_{12}} \text{inv-th/ax}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_{12}} \vee_L \quad \frac{\frac{- : \Delta_7, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}}{\bullet h_{10} : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}} \text{ax/W} \quad \frac{- : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}} \text{ax/W}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{sCut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_{10} \vdash \Delta_{12}, F_8 \quad h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{10} \vee F_{11} \vdash \Delta_{12}, F_8} \vee_L \quad \frac{\frac{h_9 : \Delta_7, F_8, F_{10} \vdash \Delta_{12}}{\bullet h_9 : (\Delta_7, F_{10} \vee F_{11}), F_8 \vdash \Delta_{12}} \wedge_L \quad \frac{h_9 : \Delta_7, F_8, F_{11} \vdash \Delta_{12}}{\bullet h_9 : (\Delta_7, F_{10} \vee F_{11}), F_8 \vdash \Delta_{12}} \wedge_L}{- : \Delta_7, F_{10} \vee F_{11} \vdash \Delta_{12}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_{10} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{10} \vdash \Delta_{12}, F_8} \text{ax/W} \quad \frac{\frac{h_9 : \Delta_7, F_{10}, F_8 \vdash \Delta_{12}}{\bullet h_9 : \Delta_7, F_{10}, F_8 \vdash \Delta_{12}} \text{ax/W} \quad \frac{h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8} \text{height}}{- : \Delta_7, F_{10} \vdash \Delta_{12}} \text{hCut} \quad \frac{\frac{h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8} \text{ax/W} \quad \frac{\frac{h_9 : \Delta_7, F_{11}, F_8 \vdash \Delta_{12}}{\bullet h_9 : \Delta_7, F_{11}, F_8 \vdash \Delta_{12}} \text{ax/W} \quad \frac{h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8}{\bullet h_3 : \Delta_7, F_{11} \vdash \Delta_{12}, F_8} \text{height}}{- : \Delta_7, F_{11} \vdash \Delta_{12}} \text{hCut} \\
\sim \\
- : \Delta_7, F_{10} \vee F_{11} \vdash \Delta_{12}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{11}, \perp \quad h_3 : \Delta_7, F_9 \vdash \Delta_{11}, \perp}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}, \perp} \vee_L \quad \frac{h_{10} : (\Delta_7, F_8 \vee F_9), \perp \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), \perp \vdash \Delta_{11}} \perp_L}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}} \text{Cut} \\
\sim \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \vdash \perp, \Delta_{11}} \text{ax/W} \quad \frac{\frac{h_{10} : \perp, \Delta_7, F_8 \vdash \Delta_{11}}{\bullet h_{10} : \perp, \Delta_7, F_8 \vdash \Delta_{11}} \perp_L \quad \frac{h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}} \text{ax/W}}{- : \Delta_7, F_8 \vdash \Delta_{11}} \text{hCut} \quad \frac{\frac{h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}} \text{ax/W} \quad \frac{\frac{h_{10} : \perp, \Delta_7, F_9 \vdash \Delta_{11}}{\bullet h_{10} : \perp, \Delta_7, F_9 \vdash \Delta_{11}} \perp_L}{- : \Delta_7, F_9 \vdash \Delta_{11}} \text{hCut}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}} \vee_L \\
\sim \\
\frac{h_3 : (\perp, \Delta_{12}), F_8 \vdash \Delta_{11}, F_7 \quad h_3 : (\perp, \Delta_{12}), F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}, F_7} \vee_L \quad \frac{h_{10} : ((\perp, \Delta_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{11}}{\bullet h_{10} : ((\perp, \Delta_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{11}} \perp_L}{- : (\perp, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}} \text{Cut} \\
\sim \\
- : \perp, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11} \quad \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7, F_8 \vdash (\Delta_{12}, p_{11}), p_{11} \quad h_3 : \Delta_7, F_9 \vdash (\Delta_{12}, p_{11}), p_{11}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash (\Delta_{12}, p_{11}), p_{11}} \vee_L \quad \frac{}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), p_{11} \vdash \Delta_{12}, p_{11}} I}{\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}} \text{Cut}} \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{12}, p_{11}, p_{11}}{- : \Delta_7, F_8 \vdash \Delta_{12}, p_{11}} \text{ax/w} \quad \frac{\bullet h_{10} : \Delta_7, F_8, p_{11} \vdash \Delta_{12}, p_{11}}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}} I \quad \frac{\frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}, p_{11}}{- : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}} \text{ax/w} \quad \frac{\bullet h_{10} : \Delta_7, F_9, p_{11} \vdash \Delta_{12}, p_{11}}{- : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}} I}{\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}} \text{hCut}} \\
\frac{\frac{h_3 : (\Delta_{13}, p_{11}), F_8 \vdash (\Delta_{12}, p_{11}), F_7 \quad h_3 : (\Delta_{13}, p_{11}), F_9 \vdash (\Delta_{12}, p_{11}), F_7}{\bullet h_3 : (\Delta_{13}, p_{11}), F_8 \vee F_9 \vdash (\Delta_{12}, p_{11}), F_7} \vee_L \quad \frac{}{\bullet h_{10} : ((\Delta_{13}, p_{11}), F_8 \vee F_9), F_7 \vdash \Delta_{12}, p_{11}} I}{\frac{}{- : (\Delta_{13}, p_{11}), F_8 \vee F_9 \vdash \Delta_{12}, p_{11}} \text{Cut}} \\
\frac{}{- : \Delta_{13}, p_{11}, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{11}, \top \quad h_3 : \Delta_7, F_9 \vdash \Delta_{11}, \top}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}, \top} \vee_L \quad \frac{h_{10} : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), \top \vdash \Delta_{11}} \top_L}{\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}} \text{Cut}} \\
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}} \text{ax/w} \\
\frac{\frac{h_3 : (\top, \Delta_{12}), F_8 \vdash \Delta_{11}, F_7 \quad h_3 : (\top, \Delta_{12}), F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : (\top, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}, F_7} \vee_L \quad \frac{h_{10} : \Delta_{12}, F_7, F_8 \vee F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{11}} \top_L}{\frac{}{- : (\top, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}} \text{Cut}} \\
\frac{\frac{\bullet h_3 : \top, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}, F_7}{- : \top, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}} \text{ax/w} \quad \frac{h_{10} : \top, \Delta_{12}, F_7, F_8 \vee F_9 \vdash \Delta_{11}}{- : \top, \Delta_{12}, F_7, F_8 \vee F_9 \vdash \Delta_{11}} \text{ax/w}}{- : \top, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}} \text{hCut}
\end{array}$$

## 8.9 Status of $\perp_L$ : OK

- Case rule  $\rightarrow_R$

$$\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_6}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \perp_L \quad \frac{h_7 : \perp, \Delta_5, F_6, F_9 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\perp, \Delta_5), F_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R}{\frac{}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut}} \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \perp_L$$

- Case rule  $\wedge_R$

$$\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash (\Delta_8, F_9 \wedge F_{10}), F_6}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \perp_L \quad \frac{h_7 : \perp, \Delta_5, F_6 \vdash \Delta_8, F_9 \quad h_7 : \perp, \Delta_5, F_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\perp, \Delta_5), F_6 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R}{\frac{}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut}} \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \perp_L$$

- Case rule  $\vee_R$

$$\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash (\Delta_8, F_9 \vee F_{10}), F_6}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \perp_L \quad \frac{h_7 : \perp, \Delta_5, F_6 \vdash \Delta_8, F_9, F_{10}}{\bullet h_7 : (\perp, \Delta_5), F_6 \vdash \Delta_8, F_9 \vee F_{10}} \vee_R}{\frac{}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut}} \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \perp_L$$

- Case rule  $\perp_R$



$$\begin{array}{c}
\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash (\perp, \Delta_8), F_6} \perp_L \quad \frac{h_7 : \perp, \Delta_5, F_6 \vdash \Delta_8}{\bullet h_7 : (\perp, \Delta_5), F_6 \vdash \perp, \Delta_8} \perp_R}{- : \perp, \Delta_5 \vdash \perp, \Delta_8} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_5 \vdash \perp, \Delta_8} \perp_L
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash (\top, \Delta_8), F_6} \perp_L \quad \frac{}{\bullet h_7 : (\perp, \Delta_5), F_6 \vdash \top, \Delta_8} \top_R}{- : \perp, \Delta_5 \vdash \top, \Delta_8} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_5 \vdash \top, \Delta_8} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_3 : \perp, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5} \perp_L \quad \frac{h_6 : \perp, \Delta_{10}, F_5 \vdash \Delta_9, F_7 \quad h_6 : \perp, \Delta_{10}, F_5, F_8 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_{10}, F_7 \rightarrow F_8), F_5 \vdash \Delta_9} \rightarrow_L}{- : \perp, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \perp_L \\
\\
\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8} \perp_L \quad \frac{h_6 : \perp, \Delta_5 \vdash \Delta_9, F_7 \quad h_6 : \perp, \Delta_5, F_8 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_5), F_7 \rightarrow F_8 \vdash \Delta_9} \rightarrow_L}{- : \perp, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_9} \perp_L
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_3 : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5} \perp_L \quad \frac{h_6 : \perp, \Delta_{10}, F_5, F_7, F_8 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_{10}, F_7 \wedge F_8), F_5 \vdash \Delta_9} \wedge_L}{- : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \perp_L \\
\\
\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_9, F_7 \wedge F_8} \perp_L \quad \frac{h_6 : \perp, \Delta_5, F_7, F_8 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_5), F_7 \wedge F_8 \vdash \Delta_9} \wedge_L}{- : \perp, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_9} \perp_L
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_3 : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5} \perp_L \quad \frac{h_6 : \perp, \Delta_{10}, F_5, F_7 \vdash \Delta_9 \quad h_6 : \perp, \Delta_{10}, F_5, F_8 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_{10}, F_7 \vee F_8), F_5 \vdash \Delta_9} \vee_L}{- : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \perp_L \\
\\
\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_9, F_7 \vee F_8} \perp_L \quad \frac{h_6 : \perp, \Delta_5, F_7 \vdash \Delta_9 \quad h_6 : \perp, \Delta_5, F_8 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_5), F_7 \vee F_8 \vdash \Delta_9} \vee_L}{- : \perp, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_9} \perp_L
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_8, F_6} \perp_L \quad \frac{}{\bullet h_7 : (\perp, \Delta_5), F_6 \vdash \Delta_8} \perp_L}{- : \perp, \Delta_5 \vdash \Delta_8} \text{Cut} \\
\sim\!\!\sim \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_8} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash (\Delta_8, p_7), p_7}{- : \perp, \Delta_5 \vdash \Delta_8, p_7} \perp_L \quad \frac{\bullet h_6 : (\perp, \Delta_5), p_7 \vdash \Delta_8, p_7}{\text{Cut}} I}{\frac{- : \perp, \Delta_5 \vdash \Delta_8, p_7}{\sim} \perp_L} \\
\frac{\bullet h_3 : \perp, \Delta_9, p_7 \vdash (\Delta_8, p_7), F_5}{- : \perp, \Delta_9, p_7 \vdash \Delta_8, p_7} \perp_L \quad \frac{\bullet h_6 : (\perp, \Delta_9, p_7), F_5 \vdash \Delta_8, p_7}{\text{Cut}} I}{\frac{- : \perp, \Delta_9, p_7 \vdash \Delta_8, p_7}{\sim} \perp_L}
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_7, \top}{- : \perp, \Delta_5 \vdash \Delta_7} \perp_L \quad \frac{h_6 : \perp, \Delta_5 \vdash \Delta_7}{\bullet h_6 : (\perp, \Delta_5), \top \vdash \Delta_7} \top_L}{\frac{- : \perp, \Delta_5 \vdash \Delta_7}{\sim} \perp_L} \text{Cut} \\
\frac{\frac{\bullet h_3 : \perp, \top, \Delta_8 \vdash \Delta_7, F_5}{- : \perp, \top, \Delta_8 \vdash \Delta_7} \perp_L \quad \frac{h_6 : \perp, \Delta_8, F_5 \vdash \Delta_7}{\bullet h_6 : (\perp, \top, \Delta_8), F_5 \vdash \Delta_7} \top_L}{\frac{- : \perp, \top, \Delta_8 \vdash \Delta_7}{\sim} \perp_L} \text{Cut}
\end{array}$$

## 8.10 Status of $I$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_8 \vdash ((\Delta_{12}, F_{10} \rightarrow F_{11}), p_8), F_7}{- : \Delta_6, p_8 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), p_8} I \quad \frac{h_9 : \Delta_6, F_7, F_{10}, p_8 \vdash \Delta_{12}, F_{11}, p_8}{\bullet h_9 : (\Delta_6, p_8), F_7 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), p_8} \rightarrow_R}{\frac{- : \Delta_6, p_8 \vdash \Delta_{12}, p_8, F_{10} \rightarrow F_{11}}{\sim} I} \text{Cut} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\Delta_8, F_9 \rightarrow F_{10}), p_6}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} I \quad \frac{h_7 : \Delta_5, F_9, p_6, p_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R}{\frac{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}}{\sim} \text{Cut}} \\
\frac{\frac{\bullet h_1 : \Delta_5, F_9, p_6 \vdash \Delta_8, F_{10}, p_6}{- : \Delta_5, F_9, p_6 \vdash \Delta_8, F_{10}} I \quad \frac{h_7 : \Delta_5, F_9, p_6, p_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/w}}{\frac{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}}{\sim} \rightarrow_R} \text{hCut}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_8 \vdash ((\Delta_{12}, F_{10} \wedge F_{11}), p_8), F_7}{- : \Delta_6, p_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), p_8} I \quad \frac{h_9 : \Delta_6, F_7, p_8 \vdash \Delta_{12}, F_{10}, p_8 \quad h_9 : \Delta_6, F_7, p_8 \vdash \Delta_{12}, F_{11}, p_8}{\bullet h_9 : (\Delta_6, p_8), F_7 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), p_8} \wedge_R}{\frac{- : \Delta_6, p_8 \vdash \Delta_{12}, p_8, F_{10} \wedge F_{11}}{\sim} I} \text{Cut} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\Delta_8, F_9 \wedge F_{10}), p_6}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \wedge F_{10}} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_9 \quad h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R}{\frac{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \wedge F_{10}}{\sim} \text{Cut}} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash \Delta_8, F_9, p_6}{- : \Delta_5, p_6 \vdash \Delta_8, F_9} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_9}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/w}}{\frac{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \wedge F_{10}}{\sim} \rightarrow_R} \text{hCut}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_8 \vdash ((\Delta_{12}, F_{10} \vee F_{11}), p_8), F_7}{- : \Delta_6, p_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), p_8} I \quad \frac{h_9 : \Delta_6, F_7, p_8 \vdash \Delta_{12}, F_{10}, F_{11}, p_8}{\bullet h_9 : (\Delta_6, p_8), F_7 \vdash (\Delta_{12}, F_{10} \vee F_{11}), p_8} \vee_R}{- : \Delta_6, p_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), p_8} \text{Cut} \\
\frac{}{- : \Delta_6, p_8 \vdash \Delta_{12}, p_8, F_{10} \vee F_{11}} \text{I} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\Delta_8, F_9 \vee F_{10}), p_6}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \vee F_{10}} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_9, F_{10}}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \vee F_{10}} \vee_R}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash \Delta_8, F_{10}, F_9, p_6}{- : \Delta_5, p_6 \vdash \Delta_8, F_{10}, F_9} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_{10}, F_9}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_{10}, F_9} \text{ax/W}}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut} \vee_R
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_8 \vdash ((\perp, \Delta_{10}), p_8), F_7}{- : \Delta_6, p_8 \vdash (\perp, \Delta_{10}), p_8} I \quad \frac{h_9 : \Delta_6, F_7, p_8 \vdash \Delta_{10}, p_8}{\bullet h_9 : (\Delta_6, p_8), F_7 \vdash (\perp, \Delta_{10}), p_8} \perp_R}{- : \Delta_6, p_8 \vdash (\perp, \Delta_{10}), p_8} \text{Cut} \\
\frac{}{- : \Delta_6, p_8 \vdash \perp, \Delta_{10}, p_8} \text{I} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\perp, \Delta_8), p_6}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \perp, \Delta_8} \perp_R}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} \text{Cut} \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash \perp, \Delta_8, p_6}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \perp, \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \perp, \Delta_8} \text{ax/W}}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_8 \vdash ((\top, \Delta_{10}), p_8), F_7}{- : \Delta_6, p_8 \vdash (\top, \Delta_{10}), p_8} I \quad \frac{h_9 : (\Delta_6, p_8), F_7 \vdash (\top, \Delta_{10}), p_8}{\bullet h_9 : (\Delta_6, p_8), F_7 \vdash (\top, \Delta_{10}), p_8} \top_R}{- : \Delta_6, p_8 \vdash (\top, \Delta_{10}), p_8} \text{Cut} \\
\frac{}{- : \Delta_6, p_8 \vdash \top, \Delta_{10}, p_8} \top_R \\
\frac{\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\top, \Delta_8), p_6}{- : \Delta_5, p_6 \vdash \top, \Delta_8} I \quad \frac{h_7 : (\Delta_5, p_6), p_6 \vdash \top, \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \top, \Delta_8} \top_R}{- : \Delta_5, p_6 \vdash \top, \Delta_8} \text{Cut} \\
\frac{}{- : \Delta_5, p_6 \vdash \top, \Delta_8} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_{12}, F_6, p_7 \vdash \Delta_8, F_{10}, p_7 \quad h_9 : \Delta_{12}, F_6, F_{11}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{12}, F_{10} \rightarrow F_{11}), p_7), F_6 \vdash \Delta_8, p_7} \rightarrow_L}{- : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\frac{}{- : \Delta_{12}, p_7, F_{10} \rightarrow F_{11} \vdash \Delta_8, p_7} \text{I} \\
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \rightarrow F_{11}}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_6, p_7 \vdash \Delta_8, F_{10}, p_7 \quad h_9 : \Delta_6, F_{11}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), F_{10} \rightarrow F_{11} \vdash \Delta_8, p_7} \rightarrow_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\frac{}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{I}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\Delta_{10}, F_7 \rightarrow F_8), p_5 \vdash \Delta_9, p_5}{- : (\Delta_{10}, F_7 \rightarrow F_8), p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, p_5, p_5 \vdash \Delta_9, F_7 \quad h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9}{\bullet h_6 : ((\Delta_{10}, F_7 \rightarrow F_8), p_5), p_5 \vdash \Delta_9} \rightarrow_L}{- : (\Delta_{10}, F_7 \rightarrow F_8), p_5 \vdash \Delta_9} \text{Cut} \\
\sim \\
\frac{\frac{\bullet h_1 : \Delta_{10}, p_5 \vdash \Delta_9, F_7, p_5}{- : \Delta_{10}, p_5 \vdash \Delta_9, F_7} I \quad \frac{h_6 : \Delta_{10}, p_5, p_5 \vdash \Delta_9, F_7}{- : \Delta_{10}, p_5, p_5 \vdash \Delta_9, F_7} \text{ax/w} \quad \frac{\bullet h_1 : \Delta_{10}, F_8, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_8, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9}{- : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/w}}{- : \Delta_{10}, p_5, F_7 \rightarrow F_8 \vdash \Delta_9} \text{hCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_{12}, F_6, F_{10}, F_{11}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{12}, F_{10} \wedge F_{11}), p_7), F_6 \vdash \Delta_8, p_7} \wedge_L}{- : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\sim \\
\frac{- : \Delta_{12}, p_7, F_{10} \wedge F_{11} \vdash \Delta_8, p_7}{- : \Delta_{12}, p_7, F_{10} \wedge F_{11} \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \wedge F_{11}}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_6, F_{10}, F_{11}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), F_{10} \wedge F_{11} \vdash \Delta_8, p_7} \wedge_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\sim \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_1 : (\Delta_{10}, F_7 \wedge F_8), p_5 \vdash \Delta_9, p_5}{- : (\Delta_{10}, F_7 \wedge F_8), p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_7, F_8, p_5, p_5 \vdash \Delta_9}{\bullet h_6 : ((\Delta_{10}, F_7 \wedge F_8), p_5), p_5 \vdash \Delta_9} \wedge_L}{- : (\Delta_{10}, F_7 \wedge F_8), p_5 \vdash \Delta_9} \text{Cut} \\
\sim \\
\frac{\frac{\bullet h_1 : \Delta_{10}, F_7, F_8, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_7, F_8, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_7, F_8, p_5, p_5 \vdash \Delta_9}{- : \Delta_{10}, F_7, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/w}}{- : \Delta_{10}, p_5, F_7 \wedge F_8 \vdash \Delta_9} \text{hCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_{12}, F_6, F_{10}, p_7 \vdash \Delta_8, p_7 \quad h_9 : \Delta_{12}, F_6, F_{11}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{12}, F_{10} \vee F_{11}), p_7), F_6 \vdash \Delta_8, p_7} \vee_L}{- : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\sim \\
\frac{- : \Delta_{12}, p_7, F_{10} \vee F_{11} \vdash \Delta_8, p_7}{- : \Delta_{12}, p_7, F_{10} \vee F_{11} \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \vee F_{11}}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_6, F_{10}, p_7 \vdash \Delta_8, p_7 \quad h_9 : \Delta_6, F_{11}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), F_{10} \vee F_{11} \vdash \Delta_8, p_7} \vee_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\sim \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_1 : (\Delta_{10}, F_7 \vee F_8), p_5 \vdash \Delta_9, p_5}{- : (\Delta_{10}, F_7 \vee F_8), p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_7, p_5, p_5 \vdash \Delta_9 \quad h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9}{\bullet h_6 : ((\Delta_{10}, F_7 \vee F_8), p_5), p_5 \vdash \Delta_9} \vee_L}{- : (\Delta_{10}, F_7 \vee F_8), p_5 \vdash \Delta_9} \text{Cut} \\
\sim \\
\frac{\frac{\bullet h_1 : \Delta_{10}, F_7, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_7, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_7, p_5, p_5 \vdash \Delta_9}{- : \Delta_{10}, F_7, p_5, p_5 \vdash \Delta_9} \text{ax/w} \quad \frac{\bullet h_1 : \Delta_{10}, F_8, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_8, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9}{- : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/w}}{- : \Delta_{10}, p_5, F_7 \vee F_8 \vdash \Delta_9} \text{hCut}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), \perp}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{\bullet h_9 : (\Delta_6, p_7), \perp \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \perp_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\sim \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I
\end{array}$$

$$\begin{array}{c}
\frac{\bullet h_2 : (\perp, \Delta_{10}), p_7 \vdash (\Delta_8, p_7), F_6 \quad I \quad \bullet h_9 : ((\perp, \Delta_{10}), p_7), F_6 \vdash \Delta_8, p_7}{\vdash : (\perp, \Delta_{10}), p_7 \vdash \Delta_8, p_7} \text{Cut} \quad \perp_L \\
\frac{\vdash : (\perp, \Delta_{10}), p_7 \vdash \Delta_8, p_7}{\vdash : \perp, \Delta_{10}, p_7 \vdash \Delta_8, p_7} \text{I} \\
\frac{\bullet h_1 : (\perp, \Delta_8), p_5 \vdash \Delta_7, p_5 \quad I \quad \bullet h_6 : ((\perp, \Delta_8), p_5), p_5 \vdash \Delta_7}{\vdash : (\perp, \Delta_8), p_5 \vdash \Delta_7} \text{Cut} \quad \perp_L \\
\frac{\vdash : (\perp, \Delta_8), p_5 \vdash \Delta_7}{\vdash : \perp, \Delta_8, p_5 \vdash \Delta_7} \text{I}
\end{array}$$

• Case rule  $I$

$$\begin{array}{c}
\frac{\bullet h_2 : \Delta_6, p_7 \vdash ((\Delta_{10}, p_9), p_7), p_9 \quad I \quad \bullet h_8 : (\Delta_6, p_7), p_9 \vdash (\Delta_{10}, p_9), p_7}{\vdash : \Delta_6, p_7 \vdash (\Delta_{10}, p_9), p_7} \text{Cut} \quad I \\
\frac{\vdash : \Delta_6, p_7 \vdash (\Delta_{10}, p_9), p_7}{\vdash : \Delta_6, p_7 \vdash \Delta_{10}, p_7, p_9} \text{I} \\
\frac{\bullet h_2 : (\Delta_{11}, p_9), p_7 \vdash ((\Delta_{10}, p_9), p_7), F_6 \quad I \quad \bullet h_8 : ((\Delta_{11}, p_9), p_7), F_6 \vdash (\Delta_{10}, p_9), p_7}{\vdash : (\Delta_{11}, p_9), p_7 \vdash (\Delta_{10}, p_9), p_7} \text{Cut} \quad I \\
\frac{\vdash : (\Delta_{11}, p_9), p_7 \vdash (\Delta_{10}, p_9), p_7}{\vdash : \Delta_{11}, p_7, p_9 \vdash \Delta_{10}, p_7, p_9} \text{I} \\
\frac{\bullet h_2 : \Delta_6, p_9 \vdash (\Delta_{10}, p_9), F_7 \quad I \quad \bullet h_8 : (\Delta_6, p_9), F_7 \vdash \Delta_{10}, p_9}{\vdash : \Delta_6, p_9 \vdash \Delta_{10}, p_9} \text{Cut} \quad I \\
\frac{\vdash : \Delta_6, p_9 \vdash \Delta_{10}, p_9}{\vdash : \Delta_6, p_9 \vdash \Delta_{10}, p_9} \text{I} \\
\frac{\bullet h_1 : \Delta_5, p_7 \vdash (\Delta_8, p_7), p_7 \quad I \quad \bullet h_6 : (\Delta_5, p_7), p_7 \vdash \Delta_8, p_7}{\vdash : \Delta_5, p_7 \vdash \Delta_8, p_7} \text{Cut} \quad I \\
\frac{\vdash : \Delta_5, p_7 \vdash \Delta_8, p_7}{\vdash : \Delta_5, p_7 \vdash \Delta_8, p_7} \text{I} \\
\frac{\bullet h_1 : (\Delta_9, p_7), p_5 \vdash (\Delta_8, p_7), p_5 \quad I \quad \bullet h_6 : ((\Delta_9, p_7), p_5), p_5 \vdash \Delta_8, p_7}{\vdash : (\Delta_9, p_7), p_5 \vdash \Delta_8, p_7} \text{Cut} \quad I \\
\frac{\vdash : (\Delta_9, p_7), p_5 \vdash \Delta_8, p_7}{\vdash : \Delta_9, p_5, p_7 \vdash \Delta_8, p_7} \text{I}
\end{array}$$

• Case rule  $\top_L$

$$\begin{array}{c}
\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), \top \quad I \quad \frac{h_9 : \Delta_6, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), \top \vdash \Delta_8, p_7} \top_L}{\vdash : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \quad \top_L \\
\frac{\vdash : \Delta_6, p_7 \vdash \Delta_8, p_7}{\vdash : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{I} \\
\frac{\bullet h_2 : (\top, \Delta_{10}), p_7 \vdash (\Delta_8, p_7), F_6 \quad I \quad \frac{h_9 : \Delta_{10}, F_6, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\top, \Delta_{10}), p_7), F_6 \vdash \Delta_8, p_7} \top_L}{\vdash : (\top, \Delta_{10}), p_7 \vdash \Delta_8, p_7} \text{Cut} \quad \top_L \\
\frac{\vdash : (\top, \Delta_{10}), p_7 \vdash \Delta_8, p_7}{\vdash : \top, \Delta_{10}, p_7 \vdash \Delta_8, p_7} \text{I} \\
\frac{\bullet h_1 : (\top, \Delta_8), p_5 \vdash \Delta_7, p_5 \quad I \quad \frac{h_6 : \Delta_8, p_5, p_5 \vdash \Delta_7}{\bullet h_6 : ((\top, \Delta_8), p_5), p_5 \vdash \Delta_7} \top_L}{\vdash : (\top, \Delta_8), p_5 \vdash \Delta_7} \text{Cut} \quad \top_L \\
\frac{\vdash : (\top, \Delta_8), p_5 \vdash \Delta_7}{\vdash : \top, \Delta_8, p_5 \vdash \Delta_7} \text{I} \\
\frac{\bullet h_1 : \top, \Delta_8, p_5 \vdash \Delta_7, p_5 \quad I \quad \frac{h_6 : \top, \Delta_8, p_5, p_5 \vdash \Delta_7}{\bullet h_6 : \top, \Delta_8, p_5, p_5 \vdash \Delta_7} \text{ax/W}}{\vdash : \top, \Delta_8, p_5 \vdash \Delta_7} \text{hCut}
\end{array}$$

## 8.11 Status of $\top_L$ : OK

- Case rule  $\rightarrow_R$

$$\frac{\frac{h_3 : \Delta_5 \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_6}{\bullet h_3 : \top, \Delta_5 \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_6} \top_L \quad \frac{h_7 : \top, \Delta_5, F_6, F_9 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\top, \Delta_5), F_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut}$$

$$\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, F_6, F_9 \rightarrow F_{10}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_5, F_6 \vdash \Delta_8, F_9 \rightarrow F_{10}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{hCut}$$

- Case rule  $\wedge_R$

$$\frac{\frac{h_3 : \Delta_5 \vdash (\Delta_8, F_9 \wedge F_{10}), F_6}{\bullet h_3 : \top, \Delta_5 \vdash (\Delta_8, F_9 \wedge F_{10}), F_6} \top_L \quad \frac{h_7 : \top, \Delta_5, F_6 \vdash \Delta_8, F_9 \quad h_7 : \top, \Delta_5, F_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\top, \Delta_5), F_6 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut}$$

$$\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, F_6, F_9 \wedge F_{10}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_5, F_6 \vdash \Delta_8, F_9 \wedge F_{10}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \wedge F_{10}} \text{hCut}$$

- Case rule  $\vee_R$

$$\frac{\frac{h_3 : \Delta_5 \vdash (\Delta_8, F_9 \vee F_{10}), F_6}{\bullet h_3 : \top, \Delta_5 \vdash (\Delta_8, F_9 \vee F_{10}), F_6} \top_L \quad \frac{h_7 : \top, \Delta_5, F_6 \vdash \Delta_8, F_9, F_{10}}{\bullet h_7 : (\top, \Delta_5), F_6 \vdash \Delta_8, F_9 \vee F_{10}} \vee_R}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut}$$

$$\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, F_6, F_9 \vee F_{10}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_5, F_6 \vdash \Delta_8, F_9 \vee F_{10}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut}$$

- Case rule  $\perp_R$

$$\frac{\frac{h_3 : \Delta_5 \vdash (\perp, \Delta_8), F_6}{\bullet h_3 : \top, \Delta_5 \vdash (\perp, \Delta_8), F_6} \top_L \quad \frac{h_7 : \top, \Delta_5, F_6 \vdash \Delta_8}{\bullet h_7 : (\top, \Delta_5), F_6 \vdash \perp, \Delta_8} \perp_R}{- : \top, \Delta_5 \vdash \perp, \Delta_8} \text{Cut}$$

$$\frac{\frac{h_3 : \top, \Delta_5 \vdash \perp, \Delta_8, F_6}{- : \top, \Delta_5 \vdash \perp, \Delta_8} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_5, F_6 \vdash \perp, \Delta_8}{- : \top, \Delta_5 \vdash \perp, \Delta_8} \text{ax/W}}{- : \top, \Delta_5 \vdash \perp, \Delta_8} \text{hCut}$$

- Case rule  $\top_R$

$$\frac{\frac{h_3 : \Delta_5 \vdash (\top, \Delta_8), F_6}{\bullet h_3 : \top, \Delta_5 \vdash (\top, \Delta_8), F_6} \top_L \quad \frac{\bullet h_7 : (\top, \Delta_5), F_6 \vdash \top, \Delta_8}{- : \top, \Delta_5 \vdash \top, \Delta_8} \top_R}{- : \top, \Delta_5 \vdash \top, \Delta_8} \text{Cut}$$

$$\frac{- : \top, \Delta_5 \vdash \top, \Delta_8}{- : \top, \Delta_5 \vdash \top, \Delta_8} \top_R$$

- Case rule  $\rightarrow_L$

$$\frac{\frac{h_3 : \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5}{\bullet h_3 : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5} \top_L \quad \frac{h_6 : \top, \Delta_{10}, F_5 \vdash \Delta_9, F_7 \quad h_6 : \top, \Delta_{10}, F_5, F_8 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_{10}, F_7 \rightarrow F_8), F_5 \vdash \Delta_9} \rightarrow_L}{- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{Cut}$$

$$\frac{\frac{h_3 : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5}{- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \rightarrow F_8 \vdash \Delta_9}{- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{hCut}$$

$$\frac{\frac{h_3 : \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8} \top_L \quad \frac{h_6 : \top, \Delta_5 \vdash \Delta_9, F_7 \quad h_6 : \top, \Delta_5, F_8 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_5), F_7 \rightarrow F_8 \vdash \Delta_9} \rightarrow_L}{- : \top, \Delta_5 \vdash \Delta_9} \text{Cut}$$

$$\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, F_7 \rightarrow F_8 \vdash \Delta_9}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_9} \text{hCut}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5}{\bullet h_3 : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5} \top_L \quad \frac{h_6 : \top, \Delta_{10}, F_5, F_7, F_8 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_{10}, F_7 \wedge F_8), F_5 \vdash \Delta_9} \wedge_L}{- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_3 : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5}{- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \wedge F_8 \vdash \Delta_9}{- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{hCut} \\
\\
\frac{\frac{h_3 : \Delta_5 \vdash \Delta_9, F_7 \wedge F_8}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \wedge F_8} \top_L \quad \frac{h_6 : \top, \Delta_5, F_7, F_8 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_5), F_7 \wedge F_8 \vdash \Delta_9} \wedge_L}{- : \top, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \wedge F_8}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, F_7 \wedge F_8 \vdash \Delta_9}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_9} \text{hCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5}{\bullet h_3 : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5} \top_L \quad \frac{\frac{h_6 : \top, \Delta_{10}, F_5, F_7 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_{10}, F_7 \vee F_8), F_5 \vdash \Delta_9} \vee_L \quad h_6 : \top, \Delta_{10}, F_5, F_8 \vdash \Delta_9}{- : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_3 : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5}{- : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \vee F_8 \vdash \Delta_9}{- : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{hCut} \\
\\
\frac{\frac{h_3 : \Delta_5 \vdash \Delta_9, F_7 \vee F_8}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \vee F_8} \top_L \quad \frac{\frac{h_6 : \top, \Delta_5, F_7 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_5), F_7 \vee F_8 \vdash \Delta_9} \vee_L \quad h_6 : \top, \Delta_5, F_8 \vdash \Delta_9}{- : \top, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \vee F_8}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, F_7 \vee F_8 \vdash \Delta_9}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_9} \text{hCut}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_5 \vdash \Delta_7, \perp}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_7, \perp} \top_L \quad \frac{h_6 : (\top, \Delta_5), \perp \vdash \Delta_7}{\bullet h_6 : (\top, \Delta_5), \perp \vdash \Delta_7} \perp_L}{- : \top, \Delta_5 \vdash \Delta_7} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_7, \perp}{- : \top, \Delta_5 \vdash \Delta_7} \text{ax/W} \quad \frac{\bullet h_6 : \perp, \top, \Delta_5 \vdash \Delta_7}{- : \top, \Delta_5 \vdash \Delta_7} \perp_L}{- : \top, \Delta_5 \vdash \Delta_7} \text{hCut} \\
\\
\frac{\frac{h_3 : \perp, \Delta_8 \vdash \Delta_7, F_5}{\bullet h_3 : \top, \perp, \Delta_8 \vdash \Delta_7, F_5} \top_L \quad \frac{h_6 : (\top, \perp, \Delta_8), F_5 \vdash \Delta_7}{\bullet h_6 : (\top, \perp, \Delta_8), F_5 \vdash \Delta_7} \perp_L}{- : \top, \perp, \Delta_8 \vdash \Delta_7} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \perp, \top, \Delta_8 \vdash \Delta_7} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_5 \vdash (\Delta_8, p_7), p_7}{\bullet h_3 : \top, \Delta_5 \vdash (\Delta_8, p_7), p_7} \top_L \quad \frac{h_6 : (\top, \Delta_5), p_7 \vdash \Delta_8, p_7}{\bullet h_6 : (\top, \Delta_5), p_7 \vdash \Delta_8, p_7} I}{- : \top, \Delta_5 \vdash \Delta_8, p_7} \text{Cut} \\
\rightsquigarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, p_7, p_7}{- : \top, \Delta_5 \vdash \Delta_8, p_7} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, p_7 \vdash \Delta_8, p_7}{- : \top, \Delta_5 \vdash \Delta_8, p_7} I}{- : \top, \Delta_5 \vdash \Delta_8, p_7} \text{hCut} \\
\\
\frac{\frac{h_3 : \Delta_9, p_7 \vdash (\Delta_8, p_7), F_5}{\bullet h_3 : \top, \Delta_9, p_7 \vdash (\Delta_8, p_7), F_5} \top_L \quad \frac{h_6 : (\top, \Delta_9, p_7), F_5 \vdash \Delta_8, p_7}{\bullet h_6 : (\top, \Delta_9, p_7), F_5 \vdash \Delta_8, p_7} I}{- : \top, \Delta_9, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightsquigarrow \\
\frac{}{- : \top, \Delta_9, p_7 \vdash \Delta_8, p_7} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
 \frac{\frac{h_3 : \Delta_5 \vdash \Delta_8, F_6}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_8, F_6} \top_L \quad \frac{\frac{h_7 : \Delta_5, F_6 \vdash \Delta_8}{\bullet h_7 : (\top, \Delta_5), F_6 \vdash \Delta_8} \top_L}{- : \top, \Delta_5 \vdash \Delta_8} \text{Cut} \\
 \sim\!\!\!\rightarrow \\
 \frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, F_6}{- : \top, \Delta_5 \vdash \Delta_8} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_5, F_6 \vdash \Delta_8}{- : \top, \Delta_5 \vdash \Delta_8} \text{ax/W}}{- : \top, \Delta_5 \vdash \Delta_8} \text{hCut}
 \end{array}$$



## 9 Admissibility of $A \rightarrow A \wedge A$

- Case(s) rule  $\rightarrow_R$

$$\frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_5, F_1 \wedge F_1}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \rightarrow F_5), F_1 \wedge F_1} \rightarrow_R \rightsquigarrow \frac{\frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_5, F_1 \wedge F_1}{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \rightarrow F_5} \text{ax} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_2 : \Delta_3 \vdash \Delta_6, F_4, F_1 \wedge F_1 \quad h_2 : \Delta_3 \vdash \Delta_6, F_5, F_1 \wedge F_1}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \wedge F_5), F_1 \wedge F_1} \wedge_R \rightsquigarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_6, F_4, F_1 \wedge F_1}{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_6, F_5, F_1 \wedge F_1}{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \wedge F_5} \wedge_R \text{ax}$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_4} \wedge_R \rightsquigarrow \frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4} \text{height}$$

- Case(s) rule  $\vee_R$

$$\frac{h_2 : \Delta_3 \vdash \Delta_6, F_4, F_5, F_1 \wedge F_1}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \vee F_5), F_1 \wedge F_1} \vee_R \rightsquigarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_6, F_4, F_5, F_1 \wedge F_1}{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \vee F_5} \text{ax} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_2 : \Delta_3 \vdash \Delta_4, F_1 \wedge F_1}{\bullet h_2 : \Delta_3 \vdash (\perp, \Delta_4), F_1 \wedge F_1} \perp_R \rightsquigarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_4, F_1 \wedge F_1}{h_2 : \Delta_3 \vdash \Delta_4, F_1} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \perp, \Delta_4, F_1} \perp_R \text{ax}$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_2 : \Delta_3 \vdash (\top, \Delta_4), F_1 \wedge F_1} \top_R \rightsquigarrow \frac{}{\bullet h_2 : \Delta_3 \vdash \top, \Delta_4, F_1} \top_R$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_4 \vdash \Delta_1, F_5, F_2 \wedge F_2 \quad h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2 \wedge F_2}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, F_2 \wedge F_2} \rightarrow_L \rightsquigarrow \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_5, F_2 \wedge F_2}{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_5} \text{IH} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2 \wedge F_2}{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2} \text{IH}}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, F_2} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2 \wedge F_2}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2 \wedge F_2} \wedge_L \rightsquigarrow \frac{\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2 \wedge F_2}{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2} \text{IH}}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2} \wedge_L \text{ax}$$

- Case(s) rule  $\vee_L$

$$\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2 \wedge F_2 \quad h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2 \wedge F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2 \wedge F_2} \vee_L \rightsquigarrow \frac{\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2 \wedge F_2}{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2} \text{IH} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2 \wedge F_2}{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2} \text{IH}}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2} \vee_L \text{ax}$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, F_2 \wedge F_2} \perp_L \rightsquigarrow \frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, F_2} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_2 : \Delta_3, p_4 \vdash (\Delta_5, p_4), F_1 \wedge F_1} I \rightsquigarrow \frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_5, F_1, p_4} I$$

- Case(s) rule  $\top_L$

$$\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2 \wedge F_2}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2 \wedge F_2} \top_L \quad \rightsquigarrow \quad \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2 \wedge F_2}{h_3 : \Delta_4 \vdash \Delta_1, F_2} \text{IH}}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2} \text{ax} \top_L$$

## 10 Admissibility of $A \wedge A \rightarrow A$

- Case(s) rule  $\rightarrow_R$

$$\frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_5}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \rightarrow F_5), F_1} \rightarrow_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3, F_4 \vdash \Delta_6, F_1, F_5}{\bullet h_2 : \Delta_3, F_4 \vdash \Delta_6, F_5, F_1 \wedge F_1} \text{IH}}{\bullet \bullet h_2 : \Delta_3 \vdash \Delta_6, F_4 \rightarrow F_5, F_1 \wedge F_1} \text{ax} \rightarrow_R$$

$$\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, (F_4 \rightarrow F_5) \wedge (F_4 \rightarrow F_5)} \wedge_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4 \quad h_2 : \Delta_3 \vdash \Delta_6, F_1, F_5}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \wedge F_5), F_1} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_4, F_1 \wedge F_1} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_5, F_1 \wedge F_1} \text{IH}}{\bullet \bullet h_2 : \Delta_3 \vdash \Delta_6, F_1 \wedge F_1, F_4 \wedge F_5} \wedge_R$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4 \quad h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \text{ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, (F_4 \wedge F_5) \wedge (F_4 \wedge F_5)} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4, F_5}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \vee F_5), F_1} \vee_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_6, F_1, F_4, F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_6, F_4, F_5, F_1 \wedge F_1} \text{IH}}{\bullet \bullet h_2 : \Delta_3 \vdash \Delta_6, F_1 \wedge F_1, F_4 \vee F_5} \text{ax} \vee_R$$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, (F_4 \vee F_5) \wedge (F_4 \vee F_5)} \wedge_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, \perp} \perp_R \quad \rightsquigarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \quad \frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, \perp \wedge \perp} \wedge_R$$

$$\frac{h_2 : \Delta_3 \vdash \Delta_4, F_1}{\bullet h_2 : \Delta_3 \vdash (\perp, \Delta_4), F_1} \perp_R \quad \rightsquigarrow \quad \frac{\frac{h_2 : \Delta_3 \vdash \Delta_4, F_1}{\bullet h_2 : \Delta_3 \vdash \Delta_4, F_1 \wedge F_1} \text{IH}}{\bullet \bullet h_2 : \Delta_3 \vdash \perp, \Delta_4, F_1 \wedge F_1} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \Delta_3, \top} \top_R \quad \rightsquigarrow \quad \frac{\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \quad \frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, \top \wedge \top} \wedge_R$$

$$\frac{}{\bullet h_2 : \Delta_3 \vdash (\top, \Delta_4), F_1} \top_R \quad \rightsquigarrow \quad \frac{}{\bullet \bullet h_2 : \Delta_3 \vdash \top, \Delta_4, F_1 \wedge F_1} \top_R$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_5 \quad h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, F_2} \rightarrow_L \quad \rightsquigarrow \quad \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2, F_5}{\bullet h_3 : \Delta_4 \vdash \Delta_1, F_5, F_2 \wedge F_2} \text{IH} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2 \wedge F_2} \text{IH}}{\bullet \bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, F_2 \wedge F_2} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2} \wedge_L \rightsquigarrow \frac{\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, F_2 \wedge F_2} \text{IH}}{\bullet \bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, F_2 \wedge F_2} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2 \quad h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2} \vee_L \rightsquigarrow \frac{\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_5 \vdash \Delta_1, F_2 \wedge F_2} \text{IH} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4, F_6 \vdash \Delta_1, F_2 \wedge F_2} \text{IH}}{\bullet \bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, F_2 \wedge F_2} \vee_L$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, F_2} \perp_L \rightsquigarrow \frac{}{\bullet \bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, F_2 \wedge F_2} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_2 : \Delta_3, p_4 \vdash (\Delta_5, p_4), F_1} I \rightsquigarrow \frac{}{\bullet \bullet h_2 : \Delta_3, p_4 \vdash \Delta_5, p_4, F_1 \wedge F_1} I$$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \rightsquigarrow \frac{\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \quad \frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I}{\bullet \bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3 \wedge p_3} \wedge_R$$

- Case(s) rule  $\top_L$

$$\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2} \top_L \rightsquigarrow \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, F_2}{\bullet h_3 : \Delta_4 \vdash \Delta_1, F_2 \wedge F_2} \text{IH}}{\bullet \bullet h_3 : \top, \Delta_4 \vdash \Delta_1, F_2 \wedge F_2} \top_L$$

**Note:** This rule does not preserve the height of the derivation but it produces proofs of at most  $s(n)$  steps. See the specification in `prop-and-A.maude`.