

# Modal Logic S4 (K+T+4)

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## 1 Height preserving admissibility of weakening on the left

- Case(s) rule  $\rightarrow_R$

$$\frac{h_1 : F_4, \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{h_1 : \Delta_2, F_4, F_W \vdash \Delta_3, F_5} \text{IH}}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \rightarrow F_5} \text{ax} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash F_4, \Delta_3 \quad h_1 : \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{h_1 : \Delta_2, F_W \vdash \Delta_3, F_4} \text{IH} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{h_1 : \Delta_2, F_W \vdash \Delta_3, F_5} \text{IH}}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \wedge F_5} \text{IH} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash F_4, F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{h_1 : \Delta_2, F_W \vdash \Delta_3, F_4, F_5} \text{IH}}{\bullet h_1 : \Delta_2, F_W \vdash \Delta_3, F_4 \vee F_5} \text{ax} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{h_1 : \Delta_2, F_W \vdash \Delta_3} \text{IH}}{\bullet h_1 : \Delta_2, F_W \vdash \perp, \Delta_3} \text{ax} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \rightarrow \frac{}{\bullet h_1 : \Delta_2, F_W \vdash \top, \Delta_3} \top_R$$

- Case(s) rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_5}{\bullet h_1 : \Box \Gamma_2, \Delta_3 \vdash \Delta_4, [\Box] F_5} A4 \rightarrow \frac{\frac{h_1 : \Box \Gamma_2 \vdash F_5}{h_1 : \Delta_3, F_W, \Box \Gamma_2 \vdash \Delta_4, [\Box] F_5} \text{ax}}{\bullet h_1 : \Delta_3, F_W, \Box \Gamma_2 \vdash \Delta_4, [\Box] F_5} A4$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_1 : \Delta_2 \vdash F_3, \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{h_1 : \Delta_2, F_W \vdash \Delta_5, F_3} \text{IH} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{h_1 : \Delta_2, F_4, F_W \vdash \Delta_5} \text{IH}}{\bullet h_1 : \Delta_2, F_W, F_3 \rightarrow F_4 \vdash \Delta_5} \text{IH} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_1 : F_3, F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{h_1 : \Delta_2, F_3, F_4, F_W \vdash \Delta_5} \text{IH}}{\bullet h_1 : \Delta_2, F_W, F_3 \wedge F_4 \vdash \Delta_5} \text{ax} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_1 : F_3, \Delta_2 \vdash \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{h_1 : \Delta_2, F_3, F_W \vdash \Delta_5} \text{IH} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{h_1 : \Delta_2, F_4, F_W \vdash \Delta_5} \text{IH}}{\bullet h_1 : \Delta_2, F_W, F_3 \vee F_4 \vdash \Delta_5} \text{ax} \vee_L$$

- Case(s) rule  $AT$

$$\frac{h_1 : F_3, \Delta_2, []F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_2, []F_3 \vdash \Delta_4} AT \quad \rightarrow \quad \frac{\frac{h_1 : \Delta_2, F_3, []F_3 \vdash \Delta_4}{h_1 : \Delta_2, F_3, F_W, []F_3 \vdash \Delta_4} IH}{\bullet h_1 : \Delta_2, F_W, []F_3 \vdash \Delta_4} AT$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \quad \rightarrow \quad \frac{}{\bullet h_1 : \perp, \Delta_2, F_W \vdash \Delta_3} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \quad \rightarrow \quad \frac{}{\bullet h_1 : \Delta_2, F_W, p_3 \vdash \Delta_4, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \quad \rightarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{h_1 : \Delta_2, F_W \vdash \Delta_3} IH}{\bullet h_1 : \top, \Delta_2, F_W \vdash \Delta_3} \top_L$$

## 2 Height preserving admissibility of weakening on the right

- Case(s) rule  $\rightarrow_R$

$$\frac{h_1 : F_4, \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \text{ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \rightarrow F_5} \text{IH}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \rightarrow F_5} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash F_4, \Delta_3 \quad h_1 : \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4, F_W} \text{ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_5, F_W} \text{ax}}{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \wedge F_5} \text{IH} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \wedge F_5} \text{IH}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \wedge F_5} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash F_4, F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5, F_W} \text{ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5, F_W}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \vee F_5} \text{IH}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W, F_4 \vee F_5} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_W} \text{ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_W}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3, F_W} \text{IH}}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3, F_W} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \rightarrow \frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3, F_W} \top_R$$

- Case(s) rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_5}{\bullet h_1 : \Box \Gamma_2, \Delta_3 \vdash \Delta_4, [\Box F_5]} A4 \rightarrow \frac{\frac{h_1 : \Box \Gamma_2 \vdash F_5}{\bullet h_1 : \Delta_3, \Box \Gamma_2 \vdash \Delta_4, F_W, [\Box F_5]} \text{ax}}{\bullet h_1 : \Delta_3, \Box \Gamma_2 \vdash \Delta_4, F_W, [\Box F_5]} A4$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_1 : \Delta_2 \vdash F_3, \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3, F_W} \text{ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5, F_W} \text{ax}}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5, F_W} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_1 : F_3, F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5, F_W} \text{ax} \quad \frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5, F_W}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5, F_W} \text{IH}}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5, F_W} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_1 : F_3, \Delta_2 \vdash \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vdash \Delta_5, F_W} \text{ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5, F_W} \text{ax}}{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5, F_W}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5, F_W} \text{IH} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5, F_W}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5, F_W} \text{IH}}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5, F_W} \vee_L$$

- Case(s) rule  $AT$

$$\frac{h_1 : F_3, \Delta_2, [\Box F_3] \vdash \Delta_4}{\bullet h_1 : \Delta_2, [\Box F_3] \vdash \Delta_4} AT \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, [\Box F_3] \vdash \Delta_4}{\bullet h_1 : \Delta_2, F_3, [\Box F_3] \vdash \Delta_4, F_W} \text{ax} \quad \frac{h_1 : \Delta_2, F_3, [\Box F_3] \vdash \Delta_4, F_W}{\bullet h_1 : \Delta_2, [\Box F_3] \vdash \Delta_4, F_W} \text{IH}}{\bullet h_1 : \Delta_2, [\Box F_3] \vdash \Delta_4, F_W} AT$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \quad \rightarrow \quad \frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3, F_W} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \quad \rightarrow \quad \frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, F_W, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \quad \rightarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{ax} \quad \frac{}{h_1 : \Delta_2 \vdash \Delta_3, F_W} \text{IH}}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3, F_W} \top_L$$

### 3 Measure of derivations

- Case(s) rule  $\rightarrow_R$

$$\frac{h_1 : F_4, \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5} \text{ax}}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \text{IH} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash F_4, \Delta_3 \quad h_1 : \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4} \text{ax} \quad \frac{h_1 : \Delta_2 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_5} \text{ax}}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \text{IH} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash F_4, F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5} \text{ax}}{\bullet \bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \text{IH} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{ax}}{\bullet \bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \text{IH} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \rightarrow \frac{}{\bullet \bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R$$

- Case(s) rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_5}{\bullet h_1 : \Box \Gamma_2, \Delta_3 \vdash \Delta_4, [\Box F_5]} A4 \rightarrow \frac{\frac{h_1 : \Box \Gamma_2 \vdash F_5}{\bullet h_1 : \Box \Gamma_2 \vdash F_5} \text{ax}}{\bullet \bullet h_1 : \Delta_3, \Box \Gamma_2 \vdash \Delta_4, [\Box F_5]} \text{IH} A4$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_1 : \Delta_2 \vdash F_3, \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{ax}}{\bullet \bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \text{IH} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_1 : F_3, F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{ax}}{\bullet \bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \text{IH} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_1 : F_3, \Delta_2 \vdash \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vdash \Delta_5} \text{ax} \quad \frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{ax}}{\bullet \bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \text{IH} \vee_L$$

- Case(s) rule  $AT$

$$\frac{h_1 : F_3, \Delta_2, [\Box F_3] \vdash \Delta_4}{\bullet h_1 : \Delta_2, [\Box F_3] \vdash \Delta_4} AT \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, [\Box F_3] \vdash \Delta_4}{\bullet h_1 : \Delta_2, F_3, [\Box F_3] \vdash \Delta_4} \text{ax}}{\bullet \bullet h_1 : \Delta_2, [\Box F_3] \vdash \Delta_4} \text{IH} AT$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \quad \rightarrow \quad \frac{}{\bullet \bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I \quad \rightarrow \quad \frac{}{\bullet \bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I$$

- Case(s) rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \quad \rightarrow \quad \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{IH}}{\bullet \bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L$$

## 4 Invertibility of Rules

### 4.1 Status of $\rightarrow_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : F_5, \Delta_4 \vdash F_6, \Delta_7, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \rightarrow F_2), F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\frac{h_3 : \Delta_4, F_1, F_5 \vdash \Delta_7, F_2, F_6}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \rightarrow F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \rightarrow F_6} \rightarrow_R$$

$$\frac{h_1 : F_4, \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5} \text{ax}}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_3, F_5} \text{H}$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, \Delta_7, F_1 \rightarrow F_2 \quad h_3 : \Delta_4 \vdash F_6, \Delta_7, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \rightarrow F_2), F_5 \wedge F_6} \wedge_R \rightarrow \frac{\frac{h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \wedge F_6} \text{ax/ind} \quad \frac{h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_6}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \wedge F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \wedge F_6} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, F_6, \Delta_7, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \rightarrow F_2), F_5 \vee F_6} \vee_R \rightarrow \frac{\frac{h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5, F_6}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \vee F_6} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \Delta_7, F_2, F_5 \vee F_6} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \rightarrow F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \rightarrow F_2} \perp_R \rightarrow \frac{\frac{h_3 : \Delta_4, F_1 \vdash \Delta_5, F_2}{\bullet h_3 : \Delta_4, F_1 \vdash \perp, \Delta_5, F_2} \text{ax/ind}}{\bullet h_3 : \Delta_4, F_1 \vdash \perp, \Delta_5, F_2} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \rightarrow F_2} \top_R \rightarrow \frac{}{\bullet h_3 : \Delta_4, F_1 \vdash \top, \Delta_5, F_2} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_5 \vdash (\Delta_7, F_1 \rightarrow F_2), \Box F_6} A4 \rightarrow \frac{\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Delta_5, F_1, \Box \Gamma_4 \vdash \Delta_7, F_2, \Box F_6} \text{ax}}{\bullet h_3 : \Delta_5, F_1, \Box \Gamma_4 \vdash \Delta_7, F_2, \Box F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash F_6, \Delta_1, F_2 \rightarrow F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \rightarrow F_3} \rightarrow_L \rightarrow \frac{\frac{h_4 : \Delta_5, F_2 \vdash \Delta_1, F_3, F_6}{\bullet h_4 : \Delta_5, F_2, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \text{ax/ind} \quad \frac{h_4 : \Delta_5, F_2, F_7 \vdash \Delta_1, F_3}{\bullet h_4 : \Delta_5, F_2, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_2, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : F_6, F_7, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \rightarrow F_3} \wedge_L \rightarrow \frac{\frac{h_4 : \Delta_5, F_2, F_6, F_7 \vdash \Delta_1, F_3}{\bullet h_4 : \Delta_5, F_2, F_6 \wedge F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\bullet h_4 : \Delta_5, F_2, F_6 \wedge F_7 \vdash \Delta_1, F_3} \wedge_L$$



- Case rule  $\vee_L$

$$\frac{\frac{h_4 : F_6, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \rightarrow F_3} \vee_L}{\frac{\frac{h_4 : \Delta_5, F_2, F_6 \vdash \Delta_1, F_3}{\bullet h_4 : \Delta_5, F_2, F_6 \vee F_7 \vdash \Delta_1, F_3} \text{ax/ind} \quad \frac{h_4 : \Delta_5, F_2, F_7 \vdash \Delta_1, F_3}{\bullet h_4 : \Delta_5, F_2, F_6 \vee F_7 \vdash \Delta_1, F_3} \text{ax/ind}}{\vee_L} \rightarrow$$

- Case rule  $AT$

$$\frac{h_4 : F_6, \Delta_5, []F_6 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \Delta_5, []F_6 \vdash \Delta_1, F_2 \rightarrow F_3} AT \rightarrow \frac{\frac{h_4 : \Delta_5, F_2, F_6, []F_6 \vdash \Delta_1, F_3}{\bullet h_4 : \Delta_5, F_2, []F_6 \vdash \Delta_1, F_3} \text{ax/ind}}{AT}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3} \perp_L \rightarrow \frac{}{\bullet h_4 : \perp, \Delta_5, F_2 \vdash \Delta_1, F_3} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \rightarrow F_2} I \rightarrow \frac{}{\bullet h_3 : \Delta_4, F_1, p_5 \vdash \Delta_6, F_2, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \rightarrow F_3} \top_L \rightarrow \frac{\frac{h_4 : \Delta_5, F_2 \vdash \Delta_1, F_3}{\bullet h_4 : \top, \Delta_5, F_2 \vdash \Delta_1, F_3} \text{ax/ind}}{\top_L}$$

## 4.2 Status of $\wedge_R$ : (Left Premise): Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : F_5, \Delta_4 \vdash F_6, \Delta_7, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\frac{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_1, F_6}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \rightarrow F_6} \text{ax/ind}}{\rightarrow_R}$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, \Delta_7, F_1 \wedge F_2 \quad h_3 : \Delta_4 \vdash F_6, \Delta_7, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \wedge F_6} \wedge_R \rightarrow \frac{\frac{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \wedge F_6} \text{ax/ind} \quad \frac{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_6}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \wedge F_6} \text{ax/ind}}{\wedge_R}$$

$$\frac{h_1 : \Delta_2 \vdash F_4, \Delta_3 \quad h_1 : \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4} \text{ax}}{H}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, F_6, \Delta_7, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \vee F_6} \vee_R \rightarrow \frac{\frac{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5, F_6}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_5 \vee F_6} \text{ax/ind}}{\vee_R}$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \wedge F_2} \perp_R \rightarrow \frac{\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1} \text{ax/ind}}{\perp_R}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \wedge F_2} \top_R \rightarrow \frac{}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_5 \vdash (\Delta_7, F_1 \wedge F_2), [] F_6} A4 \rightarrow \frac{\frac{}{h_3 : \Box \Gamma_4 \vdash F_6} ax}{\bullet h_3 : \Delta_5, \Box \Gamma_4 \vdash \Delta_7, F_1, [] F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash F_6, \Delta_1, F_2 \wedge F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \wedge F_3} \rightarrow_L \rightarrow \frac{\frac{}{h_4 : \Delta_5 \vdash \Delta_1, F_2, F_6} ax/ind \quad \frac{}{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2} ax/ind}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : F_6, F_7, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \wedge F_3} \wedge_L \rightarrow \frac{\frac{}{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2} ax/ind}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : F_6, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \wedge F_3} \vee_L \rightarrow \frac{\frac{}{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2} ax/ind \quad \frac{}{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2} ax/ind}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2} \vee_L$$

- Case rule  $AT$

$$\frac{h_4 : F_6, \Delta_5, [] F_6 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, [] F_6 \vdash \Delta_1, F_2 \wedge F_3} AT \rightarrow \frac{\frac{}{h_4 : \Delta_5, F_6, [] F_6 \vdash \Delta_1, F_2} ax/ind}{\bullet h_4 : \Delta_5, [] F_6 \vdash \Delta_1, F_2} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \perp_L \rightarrow \frac{}{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \wedge F_2} I \rightarrow \frac{}{\bullet h_3 : \Delta_4, p_5 \vdash \Delta_6, F_1, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \top_L \rightarrow \frac{\frac{}{h_4 : \Delta_5 \vdash \Delta_1, F_2} ax/ind}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2} \top_L$$

### 4.3 Status of $\wedge_R$ (Right Premise): : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : F_5, \Delta_4 \vdash F_6, \Delta_7, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\overline{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_2, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \rightarrow F_6} \text{ax/ind} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, \Delta_7, F_1 \wedge F_2 \quad h_3 : \Delta_4 \vdash F_6, \Delta_7, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \wedge F_6} \wedge_R \rightarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \wedge F_6} \text{ax/ind} \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_2, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \wedge F_6} \text{ax/ind} \wedge_R$$

$$\frac{h_1 : \Delta_2 \vdash F_4, \Delta_3 \quad h_1 : \Delta_2 \vdash F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_3, F_5}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_5} \text{ax} \text{H}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, F_6, \Delta_7, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \wedge F_2), F_5 \vee F_6} \vee_R \rightarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_2, F_5 \vee F_6} \text{ax/ind} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \wedge F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \wedge F_2} \perp_R \rightarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_5, F_2}}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_2} \text{ax/ind} \perp_R$$

- Case rule  $\top_R$

$$\frac{\overline{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \wedge F_2}}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_2} \top_R \rightarrow \frac{\overline{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_2}}{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_2} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_5 \vdash (\Delta_7, F_1 \wedge F_2), [\Box F_6]} A4 \rightarrow \frac{\overline{h_3 : \Box \Gamma_4 \vdash F_6}}{\bullet h_3 : \Delta_5, \Box \Gamma_4 \vdash \Delta_7, F_2, [\Box F_6]} \text{ax} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash F_6, \Delta_1, F_2 \wedge F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \wedge F_3} \rightarrow_L \rightarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_3, F_6}}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \text{ax/ind} \frac{\overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_3}}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_3} \text{ax/ind} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : F_6, F_7, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \wedge F_3} \wedge_L \rightarrow \frac{\overline{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_3}}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_3} \text{ax/ind} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : F_6, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \wedge F_3} \vee_L \rightarrow \frac{\overline{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_3}}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_3} \text{ax/ind} \frac{\overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_3}}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_3} \text{ax/ind} \vee_L$$

- Case rule  $AT$

$$\frac{h_4 : F_6, \Delta_5, []F_6 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \Delta_5, []F_6 \vdash \Delta_1, F_2 \wedge F_3} AT \rightarrow \frac{\overline{h_4 : \Delta_5, F_6, []F_6 \vdash \Delta_1, F_3}}{\bullet h_4 : \Delta_5, []F_6 \vdash \Delta_1, F_3} \frac{ax/ind}{AT}$$

- Case rule  $\perp_L$

$$\overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \perp_L \rightarrow \overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_3} \perp_L$$

- Case rule  $I$

$$\overline{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \wedge F_2} I \rightarrow \overline{\bullet h_3 : \Delta_4, p_5 \vdash \Delta_6, F_2, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \wedge F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \wedge F_3} \top_L \rightarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_3}}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_3} \frac{ax/ind}{\top_L}$$

#### 4.4 Status of $\vee_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : F_5, \Delta_4 \vdash F_6, \Delta_7, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\overline{h_3 : \Delta_4, F_5 \vdash \Delta_7, F_1, F_2, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \rightarrow F_6} \frac{ax/ind}{\rightarrow_R}$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, \Delta_7, F_1 \vee F_2 \quad h_3 : \Delta_4 \vdash F_6, \Delta_7, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \wedge F_6} \wedge_R \rightarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5} \quad \overline{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \wedge F_6} \frac{ax/ind}{\wedge_R}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_4 \vdash F_5, F_6, \Delta_7, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash (\Delta_7, F_1 \vee F_2), F_5 \vee F_6} \vee_R \rightarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5, F_6}}{\bullet h_3 : \Delta_4 \vdash \Delta_7, F_1, F_2, F_5 \vee F_6} \frac{ax/ind}{\vee_R}$$

$$\frac{h_1 : \Delta_2 \vdash F_4, F_5, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5}}{\bullet h_1 : \Delta_2 \vdash \Delta_3, F_4, F_5} \frac{ax}{H}$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_4 \vdash \Delta_5, F_1 \vee F_2}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1 \vee F_2} \perp_R \rightarrow \frac{\overline{h_3 : \Delta_4 \vdash \Delta_5, F_1, F_2}}{\bullet h_3 : \Delta_4 \vdash \perp, \Delta_5, F_1, F_2} \frac{ax/ind}{\perp_R}$$

- Case rule  $\top_R$

$$\overline{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1 \vee F_2} \top_R \rightarrow \overline{\bullet h_3 : \Delta_4 \vdash \top, \Delta_5, F_1, F_2} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_5 \vdash (\Delta_7, F_1 \vee F_2), []F_6} A4 \rightarrow \frac{\overline{h_3 : \Box \Gamma_4 \vdash F_6}^{ax}}{\bullet h_3 : \Delta_5, \Box \Gamma_4 \vdash \Delta_7, F_1, F_2, []F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Delta_5 \vdash F_6, \Delta_1, F_2 \vee F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2 \vee F_3} \rightarrow_L \rightarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_2, F_3, F_6}^{ax/ind} \quad \overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2, F_3}^{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \rightarrow F_7 \vdash \Delta_1, F_2, F_3} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_4 : F_6, F_7, \Delta_5 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2 \vee F_3} \wedge_L \rightarrow \frac{\overline{h_4 : \Delta_5, F_6, F_7 \vdash \Delta_1, F_2, F_3}^{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \wedge F_7 \vdash \Delta_1, F_2, F_3} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_4 : F_6, \Delta_5 \vdash \Delta_1, F_2 \vee F_3 \quad h_4 : F_7, \Delta_5 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2 \vee F_3} \vee_L \rightarrow \frac{\overline{h_4 : \Delta_5, F_6 \vdash \Delta_1, F_2, F_3}^{ax/ind} \quad \overline{h_4 : \Delta_5, F_7 \vdash \Delta_1, F_2, F_3}^{ax/ind}}{\bullet h_4 : \Delta_5, F_6 \vee F_7 \vdash \Delta_1, F_2, F_3} \vee_L$$

- Case rule  $AT$

$$\frac{h_4 : F_6, \Delta_5, []F_6 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \Delta_5, []F_6 \vdash \Delta_1, F_2 \vee F_3} AT \rightarrow \frac{\overline{h_4 : \Delta_5, F_6, []F_6 \vdash \Delta_1, F_2, F_3}^{ax/ind}}{\bullet h_4 : \Delta_5, []F_6 \vdash \Delta_1, F_2, F_3} AT$$

- Case rule  $\perp_L$

$$\overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2 \vee F_3} \perp_L \rightarrow \overline{\bullet h_4 : \perp, \Delta_5 \vdash \Delta_1, F_2, F_3} \perp_L$$

- Case rule  $I$

$$\overline{\bullet h_3 : p_5, \Delta_4 \vdash p_5, \Delta_6, F_1 \vee F_2} I \rightarrow \overline{\bullet h_3 : \Delta_4, p_5 \vdash \Delta_6, F_1, F_2, p_5} I$$

- Case rule  $\top_L$

$$\frac{h_4 : \Delta_5 \vdash \Delta_1, F_2 \vee F_3}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2 \vee F_3} \top_L \rightarrow \frac{\overline{h_4 : \Delta_5 \vdash \Delta_1, F_2, F_3}^{ax/ind}}{\bullet h_4 : \top, \Delta_5 \vdash \Delta_1, F_2, F_3} \top_L$$

## 4.5 Status of $\perp_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_1 : F_3, \Delta_2 \vdash \perp, F_4, \Delta_5}{\bullet h_1 : \Delta_2 \vdash (\perp, \Delta_5), F_3 \rightarrow F_4} \rightarrow_R \rightarrow \frac{\overline{h_1 : \Delta_2, F_3 \vdash \Delta_5, F_4}^{ax/ind}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \rightarrow F_4} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{\frac{h_1 : \Delta_2 \vdash \perp, F_3, \Delta_5 \quad h_1 : \Delta_2 \vdash \perp, F_4, \Delta_5}{\bullet h_1 : \Delta_2 \vdash (\perp, \Delta_5), F_3 \wedge F_4}}{\wedge_R} \rightarrow \frac{\frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{ ax/ind} \quad \frac{h_1 : \Delta_2 \vdash \Delta_5, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_4} \text{ ax/ind}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \wedge F_4} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \perp, F_3, F_4, \Delta_5}{\bullet h_1 : \Delta_2 \vdash (\perp, \Delta_5), F_3 \vee F_4} \vee_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_5, F_3, F_4}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3 \vee F_4} \text{ ax/ind}}{\vee_R}$$

- Case rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \Delta_3} \perp_R \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{ ax}}{\text{H}}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \perp, \Delta_3} \top_R \rightarrow \frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R$$

- Case rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_4}{\bullet h_1 : \Box \Gamma_2, \Delta_3 \vdash (\perp, \Delta_5), [\Box F_4]} A4 \rightarrow \frac{\frac{h_1 : \Box \Gamma_2 \vdash F_4}{\bullet h_1 : \Delta_3, \Box \Gamma_2 \vdash \Delta_5, [\Box F_4]} \text{ ax}}{A4}$$

- Case rule  $\rightarrow_L$

$$\frac{h_2 : \Delta_3 \vdash \perp, F_4, \Delta_1 \quad h_2 : F_5, \Delta_3 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \perp, \Delta_1} \rightarrow_L \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, F_4}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \Delta_1} \text{ ax/ind} \quad \frac{h_2 : \Delta_3, F_5 \vdash \Delta_1}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \Delta_1} \text{ ax/ind}}{\rightarrow_L}$$

- Case rule  $\wedge_L$

$$\frac{h_2 : F_4, F_5, \Delta_3 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \wedge F_5 \vdash \perp, \Delta_1} \wedge_L \rightarrow \frac{\frac{h_2 : \Delta_3, F_4, F_5 \vdash \Delta_1}{\bullet h_2 : \Delta_3, F_4 \wedge F_5 \vdash \Delta_1} \text{ ax/ind}}{\wedge_L}$$

- Case rule  $\vee_L$

$$\frac{h_2 : F_4, \Delta_3 \vdash \perp, \Delta_1 \quad h_2 : F_5, \Delta_3 \vdash \perp, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \perp, \Delta_1} \vee_L \rightarrow \frac{\frac{h_2 : \Delta_3, F_4 \vdash \Delta_1}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \Delta_1} \text{ ax/ind} \quad \frac{h_2 : \Delta_3, F_5 \vdash \Delta_1}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \Delta_1} \text{ ax/ind}}{\vee_L}$$

- Case rule  $AT$

$$\frac{h_2 : F_4, \Delta_3, [\Box F_4 \vdash \perp, \Delta_1]}{\bullet h_2 : \Delta_3, [\Box F_4 \vdash \perp, \Delta_1]} AT \rightarrow \frac{\frac{h_2 : \Delta_3, F_4, [\Box F_4 \vdash \Delta_1]}{\bullet h_2 : \Delta_3, [\Box F_4 \vdash \Delta_1]} \text{ ax/ind}}{AT}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_2 : \perp, \Delta_3 \vdash \perp, \Delta_1} \perp_L \rightarrow \frac{}{\bullet h_2 : \perp, \Delta_3 \vdash \Delta_1} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_3, \Delta_2 \vdash p_3, \perp, \Delta_4} I \rightarrow \frac{}{\bullet h_1 : \Delta_2, p_3 \vdash \Delta_4, p_3} I$$

- Case rule  $\top_L$

$$\frac{h_2 : \Delta_3 \vdash \perp, \Delta_1}{\bullet h_2 : \top, \Delta_3 \vdash \perp, \Delta_1} \top_L \rightarrow \frac{\frac{}{h_2 : \Delta_3 \vdash \Delta_1} \text{ax/ind}}{\bullet h_2 : \top, \Delta_3 \vdash \Delta_1} \top_L$$

## 4.6 Status of $\top_R$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_1 : F_3, \Delta_2 \vdash \top, F_4, \Delta_5}{\bullet h_1 : \Delta_2 \vdash (\top, \Delta_5), F_3 \rightarrow F_4} \rightarrow_R \rightarrow \text{trivial}$$

- Case rule  $\wedge_R$

$$\frac{h_1 : \Delta_2 \vdash \top, F_3, \Delta_5 \quad h_1 : \Delta_2 \vdash \top, F_4, \Delta_5}{\bullet h_1 : \Delta_2 \vdash (\top, \Delta_5), F_3 \wedge F_4} \wedge_R \rightarrow \text{trivial}$$

- Case rule  $\vee_R$

$$\frac{h_1 : \Delta_2 \vdash \top, F_3, F_4, \Delta_5}{\bullet h_1 : \Delta_2 \vdash (\top, \Delta_5), F_3 \vee F_4} \vee_R \rightarrow \text{trivial}$$

- Case rule  $\perp_R$

$$\frac{h_1 : \Delta_2 \vdash \top, \Delta_3}{\bullet h_1 : \Delta_2 \vdash \perp, \top, \Delta_3} \perp_R \rightarrow \text{trivial}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_1 : \Delta_2 \vdash \top, \Delta_3} \top_R \rightarrow \text{trivial}$$

- Case rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_4}{\bullet h_1 : \Box \Gamma_2, \Delta_3 \vdash (\top, \Delta_5), [\Box F_4]} A4 \rightarrow \text{trivial}$$

- Case rule  $\rightarrow_L$

$$\frac{h_2 : \Delta_3 \vdash \top, F_4, \Delta_1 \quad h_2 : F_5, \Delta_3 \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \rightarrow F_5 \vdash \top, \Delta_1} \rightarrow_L \rightarrow \text{trivial}$$

- Case rule  $\wedge_L$

$$\frac{h_2 : F_4, F_5, \Delta_3 \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \wedge F_5 \vdash \top, \Delta_1} \wedge_L \rightarrow \text{trivial}$$

- Case rule  $\vee_L$

$$\frac{h_2 : F_4, \Delta_3 \vdash \top, \Delta_1 \quad h_2 : F_5, \Delta_3 \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, F_4 \vee F_5 \vdash \top, \Delta_1} \vee_L \rightarrow \text{trivial}$$

- Case rule  $AT$

$$\frac{h_2 : F_4, \Delta_3, [\Box F_4] \vdash \top, \Delta_1}{\bullet h_2 : \Delta_3, [\Box F_4] \vdash \top, \Delta_1} AT \rightarrow \text{trivial}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_2 : \perp, \Delta_3 \vdash \top, \Delta_1} \perp_L \rightarrow \text{trivial}$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_3, \Delta_2 \vdash p_3, \top, \Delta_4} I \rightarrow \text{trivial}$$

- Case rule  $\top_L$

$$\frac{h_2 : \Delta_3 \vdash \top, \Delta_1}{\bullet h_2 : \top, \Delta_3 \vdash \top, \Delta_1} \top_L \rightarrow \text{trivial}$$

#### 4.7 Status of A4: : Non invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : \Box \Gamma_1, F_5, \Delta_2 \vdash F_6, \Delta_7, [\Box F_3]}{\bullet h_4 : \Box \Gamma_1, \Delta_2 \vdash (\Delta_7, [\Box F_3]), F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind}}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Box \Gamma_1, \Delta_2 \vdash F_5, \Delta_7, [\Box F_3] \quad h_4 : \Box \Gamma_1, \Delta_2 \vdash F_6, \Delta_7, [\Box F_3]}{\bullet h_4 : \Box \Gamma_1, \Delta_2 \vdash (\Delta_7, [\Box F_3]), F_5 \wedge F_6} \wedge_R \rightarrow \frac{\frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind}}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Box \Gamma_1, \Delta_2 \vdash F_5, F_6, \Delta_7, [\Box F_3]}{\bullet h_4 : \Box \Gamma_1, \Delta_2 \vdash (\Delta_7, [\Box F_3]), F_5 \vee F_6} \vee_R \rightarrow \frac{\frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind}}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Box \Gamma_1, \Delta_2 \vdash \Delta_5, [\Box F_3]}{\bullet h_4 : \Box \Gamma_1, \Delta_2 \vdash \perp, \Delta_5, [\Box F_3]} \perp_R \rightarrow \frac{\frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind}}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$



- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Box \Gamma_1, \Delta_2 \vdash \top, \Delta_5, [] F_3} \top_R \rightarrow \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} \text{fail}$$

- Case rule  $A4$

$$\frac{h_2 : \Box \Gamma_5, \Box \Gamma_6 \vdash F_3}{\bullet h_2 : (\Box \Gamma_5, \Box \Gamma_6), \Box \Gamma_7, \Delta_8 \vdash (\Delta_4, [] F_1), [] F_3} A4 \rightarrow \frac{}{\bullet h_2 : \Box \Gamma_5, \Box \Gamma_7 \vdash F_1} \text{fail}$$

$$\frac{h_1 : \Box \Gamma_4, \Box \Gamma_5 \vdash F_3}{\bullet h_1 : (\Box \Gamma_4, \Box \Gamma_5), \Box \Gamma_6, \Delta_7 \vdash \Delta_2, [] F_3} A4 \rightarrow \frac{}{\bullet h_1 : \Box \Gamma_4, \Box \Gamma_6 \vdash F_3} \text{fail}$$

- Case rule  $\rightarrow_L$

$$\frac{h_4 : \Box \Gamma_1, \Delta_7 \vdash F_5, \Delta_2, [] F_3 \quad h_4 : \Box \Gamma_1, F_6, \Delta_7 \vdash \Delta_2, [] F_3}{\bullet h_4 : (\Box \Gamma_1, \Delta_7), F_5 \rightarrow F_6 \vdash \Delta_2, [] F_3} \rightarrow_L \rightarrow \frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind} \quad \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $\wedge_L$

$$\frac{h_4 : \Box \Gamma_1, F_5, F_6, \Delta_7 \vdash \Delta_2, [] F_3}{\bullet h_4 : (\Box \Gamma_1, \Delta_7), F_5 \wedge F_6 \vdash \Delta_2, [] F_3} \wedge_L \rightarrow \frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind} \quad \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $\vee_L$

$$\frac{h_4 : \Box \Gamma_1, F_5, \Delta_7 \vdash \Delta_2, [] F_3 \quad h_4 : \Box \Gamma_1, F_6, \Delta_7 \vdash \Delta_2, [] F_3}{\bullet h_4 : (\Box \Gamma_1, \Delta_7), F_5 \vee F_6 \vdash \Delta_2, [] F_3} \vee_L \rightarrow \frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind} \quad \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $AT$

$$\frac{h_4 : \Box \Gamma_6, F_5, \Delta_1, [] F_5 \vdash \Delta_2, [] F_3}{\bullet h_4 : (\Box \Gamma_6, \Delta_1), [] F_5 \vdash \Delta_2, [] F_3} AT \rightarrow \frac{}{h_4 : \Box \Gamma_6, [] F_5 \vdash F_3} \text{ax/ind} \quad \frac{}{\bullet h_4 : \Box \Gamma_6, [] F_5 \vdash F_3} H$$

$$\frac{h_4 : \Box \Gamma_1, F_5, \Delta_6, [] F_5 \vdash \Delta_2, [] F_3}{\bullet h_4 : (\Box \Gamma_1, \Delta_6), [] F_5 \vdash \Delta_2, [] F_3} AT \rightarrow \frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind} \quad \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_4 : \perp, \Box \Gamma_1, \Delta_5 \vdash \Delta_2, [] F_3} \perp_L \rightarrow \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} \text{fail}$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Box \Gamma_1, \Delta_6 \vdash p_4, \Delta_5, [] F_2} I \rightarrow \frac{}{\bullet h_3 : \Box \Gamma_1 \vdash F_2} \text{fail}$$

- Case rule  $\top_L$

$$\frac{h_4 : \Box \Gamma_1, \Delta_5 \vdash \Delta_2, [] F_3}{\bullet h_4 : \top, \Box \Gamma_1, \Delta_5 \vdash \Delta_2, [] F_3} \top_L \rightarrow \frac{}{h_4 : \Box \Gamma_1 \vdash F_3} \text{ax/ind} \quad \frac{}{\bullet h_4 : \Box \Gamma_1 \vdash F_3} H$$

## 4.8 Status of $\rightarrow_L$ : (Left Premise): Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : F_6, \Delta_1, F_2 \rightarrow F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightarrow \frac{\overline{h_4 : \Delta_1, F_6 \vdash \Delta_5, F_2, F_7}}{\bullet h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6 \rightarrow F_7} \text{ax/ind} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash F_6, \Delta_5 \quad h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \rightarrow \frac{\overline{h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6} \text{ax/ind} \quad \overline{h_4 : \Delta_1 \vdash \Delta_5, F_2, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash F_6, F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \rightarrow \frac{\overline{h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \Delta_5, F_2, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \perp, \Delta_5} \perp_R \rightarrow \frac{\overline{h_4 : \Delta_1 \vdash \Delta_5, F_2} \text{ax/ind}}{\bullet h_4 : \Delta_1 \vdash \perp, \Delta_5, F_2} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \top, \Delta_5} \top_R \rightarrow \frac{}{\bullet h_4 : \Delta_1 \vdash \top, \Delta_5, F_2} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_5, \Box F_6} A4 \rightarrow \frac{\overline{h_3 : \Box \Gamma_4 \vdash F_6} \text{ax}}{\bullet h_3 : \Delta_7, \Box \Gamma_4 \vdash \Delta_5, F_1, \Box F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash F_4, \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightarrow \frac{\overline{h_3 : \Delta_7 \vdash \Delta_6, F_1, F_4} \text{ax/ind} \quad \overline{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \rightarrow F_5 \vdash \Delta_6, F_1} \rightarrow_L$$

$$\frac{h_1 : \Delta_2 \vdash F_3, \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\overline{h_1 : \Delta_2 \vdash \Delta_5, F_3} \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_5, F_3} H$$

- Case rule  $\wedge_L$

$$\frac{h_3 : F_4, F_5, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightarrow \frac{\overline{h_3 : \Delta_7, F_4, F_5 \vdash \Delta_6, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \wedge F_5 \vdash \Delta_6, F_1} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : F_4, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightarrow \frac{\overline{h_3 : \Delta_7, F_4 \vdash \Delta_6, F_1} \text{ax/ind} \quad \overline{h_3 : \Delta_7, F_5 \vdash \Delta_6, F_1} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_4 \vee F_5 \vdash \Delta_6, F_1} \vee_L$$

- Case rule  $AT$

$$\frac{\frac{h_3 : F_4, \Delta_6, []F_4, F_1 \rightarrow F_2 \vdash \Delta_5}{\bullet h_3 : (\Delta_6, F_1 \rightarrow F_2), []F_4 \vdash \Delta_5} AT}{\bullet h_3 : \Delta_6, []F_4 \vdash \Delta_5, F_1} \xrightarrow{ax/ind} \frac{\frac{h_3 : \Delta_6, F_4, []F_4 \vdash \Delta_5, F_1}{\bullet h_3 : \Delta_6, []F_4 \vdash \Delta_5, F_1} AT}{\bullet h_3 : \Delta_6, []F_4 \vdash \Delta_5, F_1} \xrightarrow{ax/ind}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \perp_L \rightarrow \frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_4, F_1} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \rightarrow F_2 \vdash p_4, \Delta_5} I \rightarrow \frac{}{\bullet h_3 : \Delta_6, p_4 \vdash \Delta_5, F_1, p_4} I$$

- Case rule  $\top_L$

$$\frac{\frac{h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \top_L}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_4, F_1} \xrightarrow{ax/ind} \frac{\frac{h_3 : \Delta_5 \vdash \Delta_4, F_1}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_4, F_1} \top_L}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_4, F_1} \xrightarrow{ax/ind}$$

## 4.9 Status of $\rightarrow_L$ (Right Premise): : Invertible

- Case rule  $\rightarrow_R$

$$\frac{\frac{h_4 : F_6, \Delta_1, F_2 \rightarrow F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \xrightarrow{ax/ind} \frac{\frac{h_4 : \Delta_1, F_3, F_6 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \xrightarrow{ax/ind}$$

- Case rule  $\wedge_R$

$$\frac{\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash F_6, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \wedge F_7} \xrightarrow{ax/ind} \frac{\frac{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \wedge F_7} \xrightarrow{ax/ind}$$

- Case rule  $\vee_R$

$$\frac{\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash F_6, F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5, F_6 \vee F_7} \xrightarrow{ax/ind} \frac{\frac{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \vee F_7} \xrightarrow{ax/ind}$$

- Case rule  $\perp_R$

$$\frac{\frac{h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \perp, \Delta_5} \perp_R}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \perp, \Delta_5} \xrightarrow{ax/ind} \frac{\frac{h_4 : \Delta_1, F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_3 \vdash \perp, \Delta_5} \perp_R}{\bullet h_4 : \Delta_1, F_3 \vdash \perp, \Delta_5} \xrightarrow{ax/ind}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \rightarrow F_3 \vdash \top, \Delta_5} \top_R \rightarrow \frac{}{\bullet h_4 : \Delta_1, F_3 \vdash \top, \Delta_5} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_5, [\Box F_6]} A4 \rightarrow \frac{\overline{h_3 : \Box \Gamma_4 \vdash F_6}^{ax}}{\bullet h_3 : \Delta_7, F_2, \Box \Gamma_4 \vdash \Delta_5, [\Box F_6]} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \rightarrow F_2 \vdash F_4, \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightarrow \frac{\overline{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4}^{ax/ind} \quad \overline{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6}^{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

$$\frac{h_1 : \Delta_2 \vdash F_3, \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\overline{h_1 : \Delta_2, F_4 \vdash \Delta_5}^{ax}}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} H$$

- Case rule  $\wedge_L$

$$\frac{h_3 : F_4, F_5, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightarrow \frac{\overline{h_3 : \Delta_7, F_2, F_4, F_5 \vdash \Delta_6}^{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : F_4, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \rightarrow F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \rightarrow F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightarrow \frac{\overline{h_3 : \Delta_7, F_2, F_4 \vdash \Delta_6}^{ax/ind} \quad \overline{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6}^{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

- Case rule  $AT$

$$\frac{h_3 : F_4, \Delta_6, [\Box F_4, F_1 \rightarrow F_2 \vdash \Delta_5]}{\bullet h_3 : (\Delta_6, F_1 \rightarrow F_2), [\Box F_4 \vdash \Delta_5]} AT \rightarrow \frac{\overline{h_3 : \Delta_6, F_2, F_4, [\Box F_4 \vdash \Delta_5]}^{ax/ind}}{\bullet h_3 : \Delta_6, F_2, [\Box F_4 \vdash \Delta_5]} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \perp_L \rightarrow \frac{}{\bullet h_3 : \perp, \Delta_5, F_2 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \rightarrow F_2 \vdash p_4, \Delta_5} I \rightarrow \frac{}{\bullet h_3 : \Delta_6, F_2, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \rightarrow F_2 \vdash \Delta_4} \top_L \rightarrow \frac{\overline{h_3 : \Delta_5, F_2 \vdash \Delta_4}^{ax/ind}}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \top_L$$

#### 4.10 Status of $\wedge_L$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : F_6, \Delta_1, F_2 \wedge F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_2, F_3, F_6 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \wedge F_3 \vdash F_6, \Delta_5 \quad h_4 : \Delta_1, F_2 \wedge F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6} \text{ax/ind} \quad \frac{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \wedge F_3 \vdash F_6, F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \vee F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \wedge F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \perp, \Delta_5} \perp_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_2, F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \perp, \Delta_5} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \perp, \Delta_5} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \wedge F_3 \vdash \top, \Delta_5} \top_R \rightarrow \frac{}{\bullet h_4 : \Delta_1, F_2, F_3 \vdash \top, \Delta_5} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_7, F_1 \wedge F_2 \vdash \Delta_5, \Box F_6} A4 \rightarrow \frac{\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Delta_7, F_1, F_2, \Box \Gamma_4 \vdash \Delta_5, \Box F_6} \text{ax}}{\bullet h_3 : \Delta_7, F_1, F_2, \Box \Gamma_4 \vdash \Delta_5, \Box F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \wedge F_2 \vdash F_4, \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \wedge F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \wedge F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_1, F_2 \vdash \Delta_6, F_4}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ax/ind} \quad \frac{h_3 : \Delta_7, F_1, F_2, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_3 : F_4, F_5, \Delta_7, F_1 \wedge F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \wedge F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_1, F_2, F_4, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \wedge F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

$$\frac{h_1 : F_3, F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{ax}}{\bullet h_1 : \Delta_2, F_3, F_4 \vdash \Delta_5} \text{H}$$

- Case rule  $\vee_L$

$$\frac{h_3 : F_4, \Delta_7, F_1 \wedge F_2 \vdash \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \wedge F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \wedge F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_1, F_2, F_4 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ax/ind} \quad \frac{h_3 : \Delta_7, F_1, F_2, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_1, F_2, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

- Case rule  $AT$

$$\frac{h_3 : F_4, \Delta_6, []F_4, F_1 \wedge F_2 \vdash \Delta_5}{\bullet h_3 : (\Delta_6, F_1 \wedge F_2), []F_4 \vdash \Delta_5} AT \quad \rightarrow \quad \frac{\overline{h_3 : \Delta_6, F_1, F_2, F_4, []F_4 \vdash \Delta_5} \text{ ax/ind}}{\bullet h_3 : \Delta_6, F_1, F_2, []F_4 \vdash \Delta_5} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \wedge F_2 \vdash \Delta_4} \perp_L \quad \rightarrow \quad \frac{}{\bullet h_3 : \perp, \Delta_5, F_1, F_2 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \wedge F_2 \vdash p_4, \Delta_5} I \quad \rightarrow \quad \frac{}{\bullet h_3 : \Delta_6, F_1, F_2, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \wedge F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \wedge F_2 \vdash \Delta_4} \top_L \quad \rightarrow \quad \frac{\overline{h_3 : \Delta_5, F_1, F_2 \vdash \Delta_4} \text{ ax/ind}}{\bullet h_3 : \top, \Delta_5, F_1, F_2 \vdash \Delta_4} \top_L$$

#### 4.11 Status of $\vee_L$ : (Left Premise): Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : F_6, \Delta_1, F_2 \vee F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \quad \rightarrow \quad \frac{\overline{h_4 : \Delta_1, F_2, F_6 \vdash \Delta_5, F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash F_6, \Delta_5 \quad h_4 : \Delta_1, F_2 \vee F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \quad \rightarrow \quad \frac{\overline{h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6} \text{ ax/ind} \quad \overline{h_4 : \Delta_1, F_2 \vdash \Delta_5, F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash F_6, F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \quad \rightarrow \quad \frac{\overline{h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6, F_7} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_2 \vdash \Delta_5, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \perp, \Delta_5} \perp_R \quad \rightarrow \quad \frac{\overline{h_4 : \Delta_1, F_2 \vdash \Delta_5} \text{ ax/ind}}{\bullet h_4 : \Delta_1, F_2 \vdash \perp, \Delta_5} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \top, \Delta_5} \top_R \quad \rightarrow \quad \frac{}{\bullet h_4 : \Delta_1, F_2 \vdash \top, \Delta_5} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_7, F_1 \vee F_2 \vdash \Delta_5, [] F_6} A4 \rightarrow \frac{\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Delta_7, F_1, \Box \Gamma_4 \vdash \Delta_5, [] F_6} ax}{\bullet h_3 : \Delta_7, F_1, \Box \Gamma_4 \vdash \Delta_5, [] F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \vee F_2 \vdash F_4, \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_1 \vdash \Delta_6, F_4}{\bullet h_3 : \Delta_7, F_1, F_4 \rightarrow F_5 \vdash \Delta_6} ax/ind \quad \frac{h_3 : \Delta_7, F_1, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_4 \rightarrow F_5 \vdash \Delta_6} ax/ind}{\bullet h_3 : \Delta_7, F_1, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_3 : F_4, F_5, \Delta_7, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_1, F_4, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_4 \wedge F_5 \vdash \Delta_6} ax/ind}{\bullet h_3 : \Delta_7, F_1, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : F_4, \Delta_7, F_1 \vee F_2 \vdash \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_1, F_4 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_4 \vee F_5 \vdash \Delta_6} ax/ind \quad \frac{h_3 : \Delta_7, F_1, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_1, F_4 \vee F_5 \vdash \Delta_6} ax/ind}{\bullet h_3 : \Delta_7, F_1, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

$$\frac{h_1 : F_3, \Delta_2 \vdash \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_3 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} ax}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} H$$

- Case rule  $AT$

$$\frac{h_3 : F_4, \Delta_6, [] F_4, F_1 \vee F_2 \vdash \Delta_5}{\bullet h_3 : (\Delta_6, F_1 \vee F_2), [] F_4 \vdash \Delta_5} AT \rightarrow \frac{\frac{h_3 : \Delta_6, F_1, F_4, [] F_4 \vdash \Delta_5}{\bullet h_3 : \Delta_6, F_1, [] F_4 \vdash \Delta_5} ax/ind}{\bullet h_3 : \Delta_6, F_1, [] F_4 \vdash \Delta_5} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \perp_L \rightarrow \frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \vee F_2 \vdash p_4, \Delta_5} I \rightarrow \frac{}{\bullet h_3 : \Delta_6, F_1, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \top_L \rightarrow \frac{\frac{h_3 : \Delta_5, F_1 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vdash \Delta_4} ax/ind}{\bullet h_3 : \top, \Delta_5, F_1 \vdash \Delta_4} \top_L$$

## 4.12 Status of $\vee_L$ (Right Premise): : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_4 : F_6, \Delta_1, F_2 \vee F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_3, F_6 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \rightarrow F_7} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash F_6, \Delta_5 \quad h_4 : \Delta_1, F_2 \vee F_3 \vdash F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6} \text{ax/ind} \quad \frac{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_7}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \wedge F_7} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash F_6, F_7, \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6, F_7}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \vee F_7} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \Delta_5, F_6 \vee F_7} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_4 : \Delta_1, F_2 \vee F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \perp, \Delta_5} \perp_R \rightarrow \frac{\frac{h_4 : \Delta_1, F_3 \vdash \Delta_5}{\bullet h_4 : \Delta_1, F_3 \vdash \perp, \Delta_5} \text{ax/ind}}{\bullet h_4 : \Delta_1, F_3 \vdash \perp, \Delta_5} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_4 : \Delta_1, F_2 \vee F_3 \vdash \top, \Delta_5} \top_R \rightarrow \frac{}{\bullet h_4 : \Delta_1, F_3 \vdash \top, \Delta_5} \top_R$$

- Case rule  $A4$

$$\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4, \Delta_7, F_1 \vee F_2 \vdash \Delta_5, \Box F_6} A4 \rightarrow \frac{\frac{h_3 : \Box \Gamma_4 \vdash F_6}{\bullet h_3 : \Box \Gamma_4 \vdash F_6} \text{ax}}{\bullet h_3 : \Delta_7, F_2, \Box \Gamma_4 \vdash \Delta_5, \Box F_6} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_7, F_1 \vee F_2 \vdash F_4, \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_2 \vdash \Delta_6, F_4}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ax/ind} \quad \frac{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \rightarrow F_5 \vdash \Delta_6} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_3 : F_4, F_5, \Delta_7, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \wedge F_5 \vdash \Delta_6} \wedge_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_2, F_4, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \wedge F_5 \vdash \Delta_6} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_3 : F_4, \Delta_7, F_1 \vee F_2 \vdash \Delta_6 \quad h_3 : F_5, \Delta_7, F_1 \vee F_2 \vdash \Delta_6}{\bullet h_3 : (\Delta_7, F_1 \vee F_2), F_4 \vee F_5 \vdash \Delta_6} \vee_L \rightarrow \frac{\frac{h_3 : \Delta_7, F_2, F_4 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ax/ind} \quad \frac{h_3 : \Delta_7, F_2, F_5 \vdash \Delta_6}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \text{ax/ind}}{\bullet h_3 : \Delta_7, F_2, F_4 \vee F_5 \vdash \Delta_6} \vee_L$$

$$\frac{h_1 : F_3, \Delta_2 \vdash \Delta_5 \quad h_1 : F_4, \Delta_2 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_1 : \Delta_2, F_4 \vdash \Delta_5}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} \text{ax}}{\bullet h_1 : \Delta_2, F_4 \vdash \Delta_5} H$$



- Case rule  $AT$

$$\frac{h_3 : F_4, \Delta_6, []F_4, F_1 \vee F_2 \vdash \Delta_5}{\bullet h_3 : (\Delta_6, F_1 \vee F_2), []F_4 \vdash \Delta_5} AT \rightarrow \frac{\frac{h_3 : \Delta_6, F_2, F_4, []F_4 \vdash \Delta_5}{\bullet h_3 : \Delta_6, F_2, []F_4 \vdash \Delta_5} ax/ind}{\bullet h_3 : \Delta_6, F_2, []F_4 \vdash \Delta_5} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \perp_L \rightarrow \frac{}{\bullet h_3 : \perp, \Delta_5, F_2 \vdash \Delta_4} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_3 : p_4, \Delta_6, F_1 \vee F_2 \vdash p_4, \Delta_5} I \rightarrow \frac{}{\bullet h_3 : \Delta_6, F_2, p_4 \vdash \Delta_5, p_4} I$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_5, F_1 \vee F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_1 \vee F_2 \vdash \Delta_4} \top_L \rightarrow \frac{\frac{h_3 : \Delta_5, F_2 \vdash \Delta_4}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} ax/ind}{\bullet h_3 : \top, \Delta_5, F_2 \vdash \Delta_4} \top_L$$

### 4.13 Status of $AT$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : F_5, \Delta_1, []F_2 \vdash F_6, \Delta_4}{\bullet h_3 : \Delta_1, []F_2 \vdash \Delta_4, F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\frac{h_3 : \Delta_1, F_2, F_5, []F_2 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \rightarrow F_6} ax/ind}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \rightarrow F_6} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_1, []F_2 \vdash F_5, \Delta_4 \quad h_3 : \Delta_1, []F_2 \vdash F_6, \Delta_4}{\bullet h_3 : \Delta_1, []F_2 \vdash \Delta_4, F_5 \wedge F_6} \wedge_R \rightarrow \frac{\frac{h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \wedge F_6} ax/ind \quad \frac{h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \wedge F_6} ax/ind}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \wedge F_6} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_1, []F_2 \vdash F_5, F_6, \Delta_4}{\bullet h_3 : \Delta_1, []F_2 \vdash \Delta_4, F_5 \vee F_6} \vee_R \rightarrow \frac{\frac{h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5, F_6}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \vee F_6} ax/ind}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4, F_5 \vee F_6} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_1, []F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, []F_2 \vdash \perp, \Delta_4} \perp_R \rightarrow \frac{\frac{h_3 : \Delta_1, F_2, []F_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \perp, \Delta_4} ax/ind}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \perp, \Delta_4} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_1, []F_2 \vdash \top, \Delta_4} \top_R \rightarrow \frac{}{\bullet h_3 : \Delta_1, F_2, []F_2 \vdash \top, \Delta_4} \top_R$$

- Case rule  $A4$

$$\frac{\frac{h_2 : \Box \Gamma_6, []F_1 \vdash F_5}{\bullet h_2 : (\Box \Gamma_6, []F_1), \Delta_3 \vdash \Delta_4, []F_5} A4}{\bullet h_2 : \Delta_3, F_1, \Box \Gamma_6, []F_1 \vdash \Delta_4, []F_5} A4 \rightarrow \frac{\frac{h_2 : \Box \Gamma_6, []F_1 \vdash F_5}{\bullet h_2 : \Delta_3, F_1, \Box \Gamma_6, []F_1 \vdash \Delta_4, []F_5} ax}{\bullet h_2 : \Delta_3, F_1, \Box \Gamma_6, []F_1 \vdash \Delta_4, []F_5} A4$$

$$\frac{\frac{h_2 : \Box \Gamma_3 \vdash F_5}{\bullet h_2 : \Box \Gamma_3, \Delta_6, []F_1 \vdash \Delta_4, []F_5} A4}{\bullet h_2 : \Delta_6, F_1, \Box \Gamma_3, []F_1 \vdash \Delta_4, []F_5} A4 \rightarrow \frac{\frac{h_2 : \Box \Gamma_3 \vdash F_5}{\bullet h_2 : \Delta_6, F_1, \Box \Gamma_3, []F_1 \vdash \Delta_4, []F_5} ax}{\bullet h_2 : \Delta_6, F_1, \Box \Gamma_3, []F_1 \vdash \Delta_4, []F_5} A4$$

- Case rule  $\rightarrow_L$

$$\frac{\frac{h_2 : \Delta_6, []F_1 \vdash F_3, \Delta_5 \quad h_2 : F_4, \Delta_6, []F_1 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, []F_1), F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\frac{h_2 : \Delta_6, F_1, []F_1 \vdash \Delta_5, F_3}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \rightarrow F_4 \vdash \Delta_5} ax/ind \quad \frac{h_2 : \Delta_6, F_1, F_4, []F_1 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_4, []F_1 \vdash \Delta_5} ax/ind}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{\frac{h_2 : F_3, F_4, \Delta_6, []F_1 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, []F_1), F_3 \wedge F_4 \vdash \Delta_5} \wedge_L}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_2 : \Delta_6, F_1, F_3, F_4, []F_1 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \wedge F_4 \vdash \Delta_5} ax/ind}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{\frac{h_2 : F_3, \Delta_6, []F_1 \vdash \Delta_5 \quad h_2 : F_4, \Delta_6, []F_1 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, []F_1), F_3 \vee F_4 \vdash \Delta_5} \vee_L}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_2 : \Delta_6, F_1, F_3, []F_1 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \vee F_4 \vdash \Delta_5} ax/ind \quad \frac{h_2 : \Delta_6, F_1, F_4, []F_1 \vdash \Delta_5}{\bullet h_2 : \Delta_6, F_1, F_4, []F_1 \vdash \Delta_5} ax/ind}{\bullet h_2 : \Delta_6, F_1, []F_1, F_3 \vee F_4 \vdash \Delta_5} \vee_L$$

- Case rule  $AT$

$$\frac{\frac{h_2 : F_3, \Delta_5, []F_1, []F_3 \vdash \Delta_4}{\bullet h_2 : (\Delta_5, []F_1), []F_3 \vdash \Delta_4} AT}{\bullet h_2 : \Delta_5, F_1, []F_1, []F_3 \vdash \Delta_4} AT \rightarrow \frac{\frac{h_2 : \Delta_5, F_1, F_3, []F_1, []F_3 \vdash \Delta_4}{\bullet h_2 : \Delta_5, F_1, []F_1, []F_3 \vdash \Delta_4} ax/ind}{\bullet h_2 : \Delta_5, F_1, []F_1, []F_3 \vdash \Delta_4} AT$$

$$\frac{\frac{h_1 : F_3, \Delta_2, []F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_2, []F_3 \vdash \Delta_4} AT}{\bullet h_1 : \Delta_2, F_3, []F_3 \vdash \Delta_4} AT \rightarrow \frac{\frac{h_1 : \Delta_2, F_3, F_3, []F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_2, F_3, []F_3 \vdash \Delta_4} ax/ind}{\bullet h_1 : \Delta_2, F_3, []F_3 \vdash \Delta_4} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_2 : \perp, \Delta_4, []F_1 \vdash \Delta_3} \perp_L \rightarrow \frac{}{\bullet h_2 : \perp, \Delta_4, F_1, []F_1 \vdash \Delta_3} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_2 : p_3, \Delta_5, []F_1 \vdash p_3, \Delta_4} I \rightarrow \frac{}{\bullet h_2 : \Delta_5, F_1, p_3, []F_1 \vdash \Delta_4, p_3} I$$

- Case rule  $\top_L$

$$\frac{\frac{h_2 : \Delta_4, []F_1 \vdash \Delta_3}{\bullet h_2 : \top, \Delta_4, []F_1 \vdash \Delta_3} \top_L}{\bullet h_2 : \top, \Delta_4, F_1, []F_1 \vdash \Delta_3} \top_L \rightarrow \frac{\frac{h_2 : \Delta_4, F_1, []F_1 \vdash \Delta_3}{\bullet h_2 : \top, \Delta_4, F_1, []F_1 \vdash \Delta_3} ax/ind}{\bullet h_2 : \top, \Delta_4, F_1, []F_1 \vdash \Delta_3} \top_L$$

#### 4.14 Status of $\perp_L$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_2 : \perp, F_4, \Delta_1 \vdash F_5, \Delta_3}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \text{trivial}$$

- Case rule  $\wedge_R$

$$\frac{h_2 : \perp, \Delta_1 \vdash F_4, \Delta_3 \quad h_2 : \perp, \Delta_1 \vdash F_5, \Delta_3}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \text{trivial}$$

- Case rule  $\vee_R$

$$\frac{h_2 : \perp, \Delta_1 \vdash F_4, F_5, \Delta_3}{\bullet h_2 : \perp, \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightarrow \text{trivial}$$

- Case rule  $\perp_R$

$$\frac{h_2 : \perp, \Delta_1 \vdash \Delta_3}{\bullet h_2 : \perp, \Delta_1 \vdash \perp, \Delta_3} \perp_R \rightarrow \text{trivial}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_2 : \perp, \Delta_1 \vdash \top, \Delta_3} \top_R \rightarrow \text{trivial}$$

- Case rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_4}{\bullet h_1 : \Box \Gamma_2, \perp, \Delta_5 \vdash \Delta_3, [\Box F_4]} A4 \rightarrow \text{trivial}$$

- Case rule  $\rightarrow_L$

$$\frac{h_1 : \perp, \Delta_5 \vdash F_2, \Delta_4 \quad h_1 : \perp, F_3, \Delta_5 \vdash \Delta_4}{\bullet h_1 : (\perp, \Delta_5), F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L \rightarrow \text{trivial}$$

- Case rule  $\wedge_L$

$$\frac{h_1 : \perp, F_2, F_3, \Delta_5 \vdash \Delta_4}{\bullet h_1 : (\perp, \Delta_5), F_2 \wedge F_3 \vdash \Delta_4} \wedge_L \rightarrow \text{trivial}$$

- Case rule  $\vee_L$

$$\frac{h_1 : \perp, F_2, \Delta_5 \vdash \Delta_4 \quad h_1 : \perp, F_3, \Delta_5 \vdash \Delta_4}{\bullet h_1 : (\perp, \Delta_5), F_2 \vee F_3 \vdash \Delta_4} \vee_L \rightarrow \text{trivial}$$

- Case rule  $AT$

$$\frac{h_1 : \perp, F_2, \Delta_4, [\Box F_2 \vdash \Delta_3]}{\bullet h_1 : (\perp, \Delta_4), [\Box F_2 \vdash \Delta_3]} AT \rightarrow \text{trivial}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \Delta_2 \vdash \Delta_3} \perp_L \rightarrow \text{trivial}$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_2, \perp, \Delta_4 \vdash p_2, \Delta_3} I \rightarrow \text{trivial}$$

- Case rule  $\top_L$

$$\frac{h_1 : \perp, \Delta_3 \vdash \Delta_2}{\bullet h_1 : \top, \perp, \Delta_3 \vdash \Delta_2} \top_L \rightarrow \text{trivial}$$

#### 4.15 Status of $I$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_3 : F_4, \Delta_1, p_2 \vdash F_5, \Delta_6, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash (\Delta_6, p_2), F_4 \rightarrow F_5} \rightarrow_R \rightarrow \text{trivial}$$

- Case rule  $\wedge_R$

$$\frac{h_3 : \Delta_1, p_2 \vdash F_4, \Delta_6, p_2 \quad h_3 : \Delta_1, p_2 \vdash F_5, \Delta_6, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash (\Delta_6, p_2), F_4 \wedge F_5} \wedge_R \rightarrow \text{trivial}$$

- Case rule  $\vee_R$

$$\frac{h_3 : \Delta_1, p_2 \vdash F_4, F_5, \Delta_6, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash (\Delta_6, p_2), F_4 \vee F_5} \vee_R \rightarrow \text{trivial}$$

- Case rule  $\perp_R$

$$\frac{h_3 : \Delta_1, p_2 \vdash \Delta_4, p_2}{\bullet h_3 : \Delta_1, p_2 \vdash \perp, \Delta_4, p_2} \perp_R \rightarrow \text{trivial}$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_1, p_2 \vdash \top, \Delta_4, p_2} \top_R \rightarrow \text{trivial}$$

- Case rule  $A4$

$$\frac{h_2 : \Box \Gamma_3 \vdash F_4}{\bullet h_2 : \Box \Gamma_3, \Delta_6, p_1 \vdash (\Delta_5, p_1), \Box F_4} A4 \rightarrow \text{trivial}$$

- Case rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_6, p_1 \vdash F_4, \Delta_2, p_1 \quad h_3 : F_5, \Delta_6, p_1 \vdash \Delta_2, p_1}{\bullet h_3 : (\Delta_6, p_1), F_4 \rightarrow F_5 \vdash \Delta_2, p_1} \rightarrow_L \rightarrow \text{trivial}$$

- Case rule  $\wedge_L$

$$\frac{h_3 : F_4, F_5, \Delta_6, P_1 \vdash \Delta_2, P_1}{\bullet h_3 : (\Delta_6, P_1), F_4 \wedge F_5 \vdash \Delta_2, P_1} \wedge_L \rightarrow \text{trivial}$$

- Case rule  $\vee_L$

$$\frac{h_3 : F_4, \Delta_6, P_1 \vdash \Delta_2, P_1 \quad h_3 : F_5, \Delta_6, P_1 \vdash \Delta_2, P_1}{\bullet h_3 : (\Delta_6, P_1), F_4 \vee F_5 \vdash \Delta_2, P_1} \vee_L \rightarrow \text{trivial}$$

- Case rule  $AT$

$$\frac{h_3 : F_4, \Delta_5, P_1, []F_4 \vdash \Delta_2, P_1}{\bullet h_3 : (\Delta_5, P_1), []F_4 \vdash \Delta_2, P_1} AT \rightarrow \text{trivial}$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_4, P_1 \vdash \Delta_2, P_1} \perp_L \rightarrow \text{trivial}$$

- Case rule  $I$

$$\frac{}{\bullet h_2 : P_3, \Delta_5, P_1 \vdash P_3, \Delta_4, P_1} I \rightarrow \text{trivial}$$

$$\frac{}{\bullet h_1 : P_3, \Delta_2 \vdash P_3, \Delta_4} I \rightarrow \text{trivial}$$

- Case rule  $\top_L$

$$\frac{h_3 : \Delta_4, P_1 \vdash \Delta_2, P_1}{\bullet h_3 : \top, \Delta_4, P_1 \vdash \Delta_2, P_1} \top_L \rightarrow \text{trivial}$$

#### 4.16 Status of $\top_L$ : : Invertible

- Case rule  $\rightarrow_R$

$$\frac{h_2 : \top, F_4, \Delta_1 \vdash F_5, \Delta_3}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \frac{\frac{}{h_2 : \Delta_1, F_4 \vdash \Delta_3, F_5} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \rightarrow F_5} \rightarrow_R$$

- Case rule  $\wedge_R$

$$\frac{h_2 : \top, \Delta_1 \vdash F_4, \Delta_3 \quad h_2 : \top, \Delta_1 \vdash F_5, \Delta_3}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\frac{}{h_2 : \Delta_1 \vdash \Delta_3, F_4} \text{ax/ind} \quad \frac{}{h_2 : \Delta_1 \vdash \Delta_3, F_5} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \wedge F_5} \wedge_R$$

- Case rule  $\vee_R$

$$\frac{h_2 : \top, \Delta_1 \vdash F_4, F_5, \Delta_3}{\bullet h_2 : \top, \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \vee_R \rightarrow \frac{\frac{}{h_2 : \Delta_1 \vdash \Delta_3, F_4, F_5} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \Delta_3, F_4 \vee F_5} \vee_R$$

- Case rule  $\perp_R$

$$\frac{h_2 : \top, \Delta_1 \vdash \Delta_3}{\bullet h_2 : \top, \Delta_1 \vdash \perp, \Delta_3} \perp_R \rightarrow \frac{\frac{h_2 : \Delta_1 \vdash \Delta_3}{\bullet h_2 : \Delta_1 \vdash \perp, \Delta_3} \text{ax/ind}}{\bullet h_2 : \Delta_1 \vdash \perp, \Delta_3} \perp_R$$

- Case rule  $\top_R$

$$\frac{}{\bullet h_2 : \top, \Delta_1 \vdash \top, \Delta_3} \top_R \rightarrow \frac{}{\bullet h_2 : \Delta_1 \vdash \top, \Delta_3} \top_R$$

- Case rule  $A4$

$$\frac{h_1 : \Box \Gamma_2 \vdash F_4}{\bullet h_1 : \Box \Gamma_2, \top, \Delta_5 \vdash \Delta_3, [] F_4} A4 \rightarrow \frac{\frac{h_1 : \Box \Gamma_2 \vdash F_4}{\bullet h_1 : \Delta_5, \Box \Gamma_2 \vdash \Delta_3, [] F_4} \text{ax}}{\bullet h_1 : \Delta_5, \Box \Gamma_2 \vdash \Delta_3, [] F_4} A4$$

- Case rule  $\rightarrow_L$

$$\frac{h_1 : \top, \Delta_5 \vdash F_2, \Delta_4 \quad h_1 : \top, F_3, \Delta_5 \vdash \Delta_4}{\bullet h_1 : (\top, \Delta_5), F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L \rightarrow \frac{\frac{h_1 : \Delta_5 \vdash \Delta_4, F_2}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \text{ax/ind} \quad \frac{h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \text{ax/ind}}{\bullet h_1 : \Delta_5, F_2 \rightarrow F_3 \vdash \Delta_4} \rightarrow_L$$

- Case rule  $\wedge_L$

$$\frac{h_1 : \top, F_2, F_3, \Delta_5 \vdash \Delta_4}{\bullet h_1 : (\top, \Delta_5), F_2 \wedge F_3 \vdash \Delta_4} \wedge_L \rightarrow \frac{\frac{h_1 : \Delta_5, F_2, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \text{ax/ind}}{\bullet h_1 : \Delta_5, F_2 \wedge F_3 \vdash \Delta_4} \wedge_L$$

- Case rule  $\vee_L$

$$\frac{h_1 : \top, F_2, \Delta_5 \vdash \Delta_4 \quad h_1 : \top, F_3, \Delta_5 \vdash \Delta_4}{\bullet h_1 : (\top, \Delta_5), F_2 \vee F_3 \vdash \Delta_4} \vee_L \rightarrow \frac{\frac{h_1 : \Delta_5, F_2 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \text{ax/ind} \quad \frac{h_1 : \Delta_5, F_3 \vdash \Delta_4}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \text{ax/ind}}{\bullet h_1 : \Delta_5, F_2 \vee F_3 \vdash \Delta_4} \vee_L$$

- Case rule  $AT$

$$\frac{h_1 : \top, F_2, \Delta_4, [] F_2 \vdash \Delta_3}{\bullet h_1 : (\top, \Delta_4), [] F_2 \vdash \Delta_3} AT \rightarrow \frac{\frac{h_1 : \Delta_4, F_2, [] F_2 \vdash \Delta_3}{\bullet h_1 : \Delta_4, [] F_2 \vdash \Delta_3} \text{ax/ind}}{\bullet h_1 : \Delta_4, [] F_2 \vdash \Delta_3} AT$$

- Case rule  $\perp_L$

$$\frac{}{\bullet h_1 : \perp, \top, \Delta_3 \vdash \Delta_2} \perp_L \rightarrow \frac{}{\bullet h_1 : \perp, \Delta_3 \vdash \Delta_2} \perp_L$$

- Case rule  $I$

$$\frac{}{\bullet h_1 : p_2, \top, \Delta_4 \vdash p_2, \Delta_3} I \rightarrow \frac{}{\bullet h_1 : \Delta_4, p_2 \vdash \Delta_3, p_2} I$$

- Case rule  $\top_L$

$$\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \top, \Delta_2 \vdash \Delta_3} \top_L \rightarrow \frac{\frac{h_1 : \Delta_2 \vdash \Delta_3}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{ax}}{\bullet h_1 : \Delta_2 \vdash \Delta_3} \text{H}$$

## 5 Height preserving admissibility of contraction on the left

- Case(s) rule  $\rightarrow_R$

$$\frac{h_3 : F_5, \Delta_1, \Delta_2, \Delta_2 \vdash F_6, \Delta_4}{\bullet h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_5 \rightarrow F_6} \rightarrow_R \rightarrow \frac{\frac{h_3 : \Delta_1, \Delta_2, \Delta_2, F_5 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, \Delta_2, F_5 \vdash \Delta_4, F_6} \text{IH}}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_5 \rightarrow F_6} \text{ax} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash F_5, \Delta_4 \quad h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash F_6, \Delta_4}{\bullet h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_5 \wedge F_6} \wedge_R \rightarrow \frac{\frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_5} \text{IH} \quad \frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_6} \text{IH}}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_5 \wedge F_6} \text{ax} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash F_5, F_6, \Delta_4}{\bullet h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_5 \vee F_6} \vee_R \rightarrow \frac{\frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_5}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_5} \text{IH} \quad \frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4, F_6}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_6} \text{IH}}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4, F_5 \vee F_6} \text{ax} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \perp, \Delta_4} \perp_R \rightarrow \frac{\frac{h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \Delta_4}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \Delta_4} \text{IH}}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \perp, \Delta_4} \text{ax} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_3 : \Delta_1, \Delta_2, \Delta_2 \vdash \top, \Delta_4} \top_R \rightarrow \frac{}{\bullet h_3 : \Delta_1, \Delta_2 \vdash \top, \Delta_4} \top_R$$

- Case(s) rule  $A4$

$$\frac{h_1 : \Box r_4, \Box r_5, \Box r_5, \Box r_6 \vdash F_3}{\bullet h_1 : (\Box r_4, \Delta_7), (\Box r_5, \Box r_6, \Delta_8), \Box r_5, \Box r_6, \Delta_8 \vdash \Delta_2, [\Box F_3]} A4 \rightarrow \frac{\frac{h_1 : \Box r_4, \Box r_5, \Box r_5, \Box r_6 \vdash F_3}{\bullet h_1 : \Delta_7, \Delta_8, \Box r_4, \Box r_5, \Box r_6 \vdash \Delta_2, [\Box F_3]} \text{IH}}{\bullet h_1 : \Delta_7, \Delta_8, \Box r_4, \Box r_5, \Box r_6 \vdash \Delta_2, [\Box F_3]} \text{ax} A4$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_2 : \Delta_1, \Delta_6, \Delta_6, F_3 \rightarrow F_4 \vdash F_3, \Delta_5 \quad h_2 : F_4, \Delta_1, \Delta_6, \Delta_6, F_3 \rightarrow F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, (\Delta_6, F_3 \rightarrow F_4), \Delta_6, F_3 \rightarrow F_4 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_6, \Delta_6 \vdash \Delta_5, F_3, F_3}{\bullet h_2 : \Delta_1, \Delta_6 \vdash \Delta_5, F_3, F_3} \text{IH} \quad \frac{h_2 : \Delta_1, \Delta_6, \Delta_6, F_4, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, \Delta_6, F_4 \vdash \Delta_5} \text{IH}}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \rightarrow F_4 \vdash \Delta_5} \text{inv-th/ax} \rightarrow_L$$

$$\frac{h_2 : \Delta_1, \Delta_1, \Delta_6 \vdash F_3, \Delta_5 \quad h_2 : F_4, \Delta_1, \Delta_1, \Delta_6 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, F_3 \rightarrow F_4), \Delta_1, \Delta_1 \vdash \Delta_5} \rightarrow_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_1, \Delta_6 \vdash \Delta_5, F_3}{\bullet h_2 : \Delta_1, \Delta_6 \vdash \Delta_5, F_3} \text{IH} \quad \frac{h_2 : \Delta_1, \Delta_1, \Delta_6, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, \Delta_6, F_4 \vdash \Delta_5} \text{IH}}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \rightarrow F_4 \vdash \Delta_5} \text{ax} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_2 : F_3, F_4, \Delta_1, \Delta_6, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, (\Delta_6, F_3 \wedge F_4), \Delta_6, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_6, \Delta_6, F_3, F_3, F_4, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, \Delta_6, F_3, F_4 \vdash \Delta_5} \text{IH} \quad \frac{h_2 : \Delta_1, \Delta_6, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5} \text{inv-th/ax}}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L$$

$$\frac{h_2 : F_3, F_4, \Delta_1, \Delta_1, \Delta_6 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, F_3 \wedge F_4), \Delta_1, \Delta_1 \vdash \Delta_5} \wedge_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_1, \Delta_6, F_3, F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, \Delta_6, F_3, F_4 \vdash \Delta_5} \text{IH} \quad \frac{h_2 : \Delta_1, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5} \text{ax}}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \wedge F_4 \vdash \Delta_5} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\begin{array}{c}
\frac{h_2 : F_3, \Delta_1, \Delta_6, \Delta_6, F_3 \vee F_4 \vdash \Delta_5 \quad h_2 : F_4, \Delta_1, \Delta_6, \Delta_6, F_3 \vee F_4 \vdash \Delta_5}{\bullet h_2 : \Delta_1, (\Delta_6, F_3 \vee F_4), \Delta_6, F_3 \vee F_4 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_6, \Delta_6, F_3, F_3 \vdash \Delta_5}{h_2 : \Delta_1, \Delta_6, F_3 \vdash \Delta_5} \text{inv-th/ax} \quad \frac{h_2 : \Delta_1, \Delta_6, \Delta_6, F_4, F_4 \vdash \Delta_5}{h_2 : \Delta_1, \Delta_6, F_4 \vdash \Delta_5} \text{inv-th/ax}}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \vee F_4 \vdash \Delta_5} \text{IH} \vee_L
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_3, \Delta_1, \Delta_1, \Delta_6 \vdash \Delta_5 \quad h_2 : F_4, \Delta_1, \Delta_1, \Delta_6 \vdash \Delta_5}{\bullet h_2 : (\Delta_6, F_3 \vee F_4), \Delta_1, \Delta_1 \vdash \Delta_5} \vee_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_1, \Delta_6, F_3 \vdash \Delta_5}{h_2 : \Delta_1, \Delta_6, F_3 \vdash \Delta_5} \text{ax} \quad \frac{h_2 : \Delta_1, \Delta_1, \Delta_6, F_4 \vdash \Delta_5}{h_2 : \Delta_1, \Delta_6, F_4 \vdash \Delta_5} \text{ax}}{\bullet h_2 : \Delta_1, \Delta_6, F_3 \vee F_4 \vdash \Delta_5} \text{IH} \vee_L
\end{array}$$

- Case(s) rule  $AT$

$$\begin{array}{c}
\frac{h_2 : F_3, \Delta_1, \Delta_5, \Delta_5, []F_3, []F_3 \vdash \Delta_4}{\bullet h_2 : \Delta_1, (\Delta_5, []F_3), \Delta_5, []F_3 \vdash \Delta_4} AT \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_5, \Delta_5, F_3, []F_3, []F_3 \vdash \Delta_4}{h_2 : \Delta_1, \Delta_5, F_3, []F_3 \vdash \Delta_4} \text{ax} \quad \frac{h_2 : \Delta_1, \Delta_5, F_3, []F_3 \vdash \Delta_4}{\bullet h_2 : \Delta_1, \Delta_5, []F_3 \vdash \Delta_4} \text{IH}}{\bullet h_2 : \Delta_1, \Delta_5, []F_3 \vdash \Delta_4} AT
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_3, \Delta_1, \Delta_1, \Delta_5, []F_3 \vdash \Delta_4}{\bullet h_2 : (\Delta_5, []F_3), \Delta_1, \Delta_1 \vdash \Delta_4} AT \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_1, \Delta_5, F_3, []F_3 \vdash \Delta_4}{h_2 : \Delta_1, \Delta_5, F_3, []F_3 \vdash \Delta_4} \text{ax} \quad \frac{h_2 : \Delta_1, \Delta_5, F_3, []F_3 \vdash \Delta_4}{\bullet h_2 : \Delta_1, \Delta_5, []F_3 \vdash \Delta_4} \text{IH}}{\bullet h_2 : \Delta_1, \Delta_5, []F_3 \vdash \Delta_4} AT
\end{array}$$

- Case(s) rule  $\perp_L$

$$\begin{array}{c}
\frac{}{\bullet h_2 : (\perp, \Delta_4), \Delta_1, \Delta_1 \vdash \Delta_3} \perp_L \rightarrow \frac{}{\bullet h_2 : \perp, \Delta_1, \Delta_4 \vdash \Delta_3} \perp_L
\end{array}$$

$$\begin{array}{c}
\frac{}{\bullet h_2 : \Delta_1, (\perp, \Delta_4), \perp, \Delta_4 \vdash \Delta_3} \perp_L \rightarrow \frac{}{\bullet h_2 : \perp, \Delta_1, \Delta_4 \vdash \Delta_3} \perp_L
\end{array}$$

- Case(s) rule  $I$

$$\begin{array}{c}
\frac{}{\bullet h_2 : (\Delta_5, p_3), \Delta_1, \Delta_1 \vdash \Delta_4, p_3} I \rightarrow \frac{}{\bullet h_2 : \Delta_1, \Delta_5, p_3 \vdash \Delta_4, p_3} I
\end{array}$$

$$\begin{array}{c}
\frac{}{\bullet h_2 : \Delta_1, (\Delta_5, p_3), \Delta_5, p_3 \vdash \Delta_4, p_3} I \rightarrow \frac{}{\bullet h_2 : \Delta_1, \Delta_5, p_3 \vdash \Delta_4, p_3} I
\end{array}$$

- Case(s) rule  $\top_L$

$$\begin{array}{c}
\frac{h_2 : \Delta_1, \Delta_1, \Delta_4 \vdash \Delta_3}{\bullet h_2 : (\top, \Delta_4), \Delta_1, \Delta_1 \vdash \Delta_3} \top_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_1, \Delta_4 \vdash \Delta_3}{h_2 : \Delta_1, \Delta_4 \vdash \Delta_3} \text{ax} \quad \frac{h_2 : \Delta_1, \Delta_4 \vdash \Delta_3}{\bullet h_2 : \top, \Delta_1, \Delta_4 \vdash \Delta_3} \text{IH}}{\bullet h_2 : \top, \Delta_1, \Delta_4 \vdash \Delta_3} \top_L
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \top, \Delta_1, \Delta_4, \Delta_4 \vdash \Delta_3}{\bullet h_2 : \Delta_1, (\top, \Delta_4), \top, \Delta_4 \vdash \Delta_3} \top_L \rightarrow \frac{\frac{h_2 : \Delta_1, \Delta_4, \Delta_4 \vdash \Delta_3}{h_2 : \Delta_1, \Delta_4 \vdash \Delta_3} \text{inv-th/ax} \quad \frac{h_2 : \Delta_1, \Delta_4 \vdash \Delta_3}{\bullet h_2 : \top, \Delta_1, \Delta_4 \vdash \Delta_3} \text{IH}}{\bullet h_2 : \top, \Delta_1, \Delta_4 \vdash \Delta_3} \top_L
\end{array}$$



## 6 Height preserving admissibility of contraction on the Right

- Case(s) rule  $\rightarrow_R$

$$\frac{h_2 : F_4, \Delta_3 \vdash F_5, \Delta_1, \Delta_6, \Delta_6, F_4 \rightarrow F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, (\Delta_6, F_4 \rightarrow F_5), \Delta_6, F_4 \rightarrow F_5} \rightarrow_R \rightarrow \frac{\frac{h_2 : \Delta_3, F_4, F_4 \vdash \Delta_1, \Delta_6, \Delta_6, F_5, F_5}{h_2 : \Delta_3, F_4 \vdash \Delta_1, \Delta_6, \Delta_6, F_5, F_5} \text{IH-Mutual} \quad \frac{h_2 : \Delta_3, F_4 \vdash \Delta_1, \Delta_6, F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4 \rightarrow F_5} \text{IH}}{h_2 : \Delta_3, F_4 \vdash \Delta_1, \Delta_1, \Delta_6} \text{ax} \rightarrow_R$$

- Case(s) rule  $\wedge_R$

$$\frac{h_2 : \Delta_3 \vdash F_4, \Delta_1, \Delta_6, \Delta_6, F_4 \wedge F_5 \quad h_2 : \Delta_3 \vdash F_5, \Delta_1, \Delta_6, \Delta_6, F_4 \wedge F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, (\Delta_6, F_4 \wedge F_5), \Delta_6, F_4 \wedge F_5} \wedge_R \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, \Delta_6, F_4, F_4}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, \Delta_6, F_5, F_5}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4 \wedge F_5} \wedge_R$$

$$\frac{h_2 : \Delta_3 \vdash F_4, \Delta_1, \Delta_1, \Delta_6 \quad h_2 : \Delta_3 \vdash F_5, \Delta_1, \Delta_1, \Delta_6}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \wedge F_5), \Delta_1, \Delta_1} \wedge_R \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_6, F_4}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_6, F_5}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4 \wedge F_5} \wedge_R$$

- Case(s) rule  $\vee_R$

$$\frac{h_2 : \Delta_3 \vdash F_4, F_5, \Delta_1, \Delta_6, \Delta_6, F_4 \vee F_5}{\bullet h_2 : \Delta_3 \vdash \Delta_1, (\Delta_6, F_4 \vee F_5), \Delta_6, F_4 \vee F_5} \vee_R \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, \Delta_6, F_4, F_4, F_5, F_5}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4, F_5} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, \Delta_6, F_5, F_5}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4 \vee F_5} \vee_R$$

$$\frac{h_2 : \Delta_3 \vdash F_4, F_5, \Delta_1, \Delta_1, \Delta_6}{\bullet h_2 : \Delta_3 \vdash (\Delta_6, F_4 \vee F_5), \Delta_1, \Delta_1} \vee_R \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_6, F_4, F_5}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4, F_5} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_6, F_5}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_5} \text{IH}}{\bullet h_2 : \Delta_3 \vdash \Delta_1, \Delta_6, F_4 \vee F_5} \vee_R$$

- Case(s) rule  $\perp_R$

$$\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_4}{\bullet h_2 : \Delta_3 \vdash (\perp, \Delta_4), \Delta_1, \Delta_1} \perp_R \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_4}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_4} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_4}{\bullet h_2 : \Delta_3 \vdash \perp, \Delta_1, \Delta_4} \text{IH}}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_1, \Delta_4} \text{ax} \perp_R$$

$$\frac{h_2 : \Delta_3 \vdash \perp, \Delta_1, \Delta_4, \Delta_4}{\bullet h_2 : \Delta_3 \vdash \Delta_1, (\perp, \Delta_4), \perp, \Delta_4} \perp_R \rightarrow \frac{\frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_4, \Delta_4}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_4} \text{IH} \quad \frac{h_2 : \Delta_3 \vdash \Delta_1, \Delta_4, \Delta_4}{\bullet h_2 : \Delta_3 \vdash \perp, \Delta_1, \Delta_4} \text{IH}}{h_2 : \Delta_3 \vdash \Delta_1, \Delta_4, \Delta_4} \text{inv-th/ax} \perp_R$$

- Case(s) rule  $\top_R$

$$\frac{}{\bullet h_2 : \Delta_3 \vdash (\top, \Delta_4), \Delta_1, \Delta_1} \top_R \rightarrow \frac{}{\bullet h_2 : \Delta_3 \vdash \top, \Delta_1, \Delta_4} \top_R$$

$$\frac{}{\bullet h_2 : \Delta_3 \vdash \Delta_1, (\top, \Delta_4), \top, \Delta_4} \top_R \rightarrow \frac{}{\bullet h_2 : \Delta_3 \vdash \top, \Delta_1, \Delta_4} \top_R$$

- Case(s) rule  $A4$

$$\frac{h_2 : \Box \Gamma_3 \vdash F_5}{\bullet h_2 : \Box \Gamma_3, \Delta_4 \vdash \Delta_1, (\Delta_6, [\Box F_5]), \Delta_6, [\Box F_5]} A4 \rightarrow \frac{h_2 : \Box \Gamma_3 \vdash F_5}{\bullet h_2 : \Delta_4, \Box \Gamma_3 \vdash \Delta_1, \Delta_6, [\Box F_5]} \text{ax} A4$$

$$\frac{h_2 : \Box \Gamma_3 \vdash F_5}{\bullet h_2 : \Box \Gamma_3, \Delta_4 \vdash (\Delta_6, [\Box F_5]), \Delta_1, \Delta_1} A4 \rightarrow \frac{h_2 : \Box \Gamma_3 \vdash F_5}{\bullet h_2 : \Delta_4, \Box \Gamma_3 \vdash \Delta_1, \Delta_6, [\Box F_5]} \text{ax} A4$$

- Case(s) rule  $\rightarrow_L$

$$\frac{h_3 : \Delta_4 \vdash F_5, \Delta_1, \Delta_2, \Delta_2 \quad h_3 : F_6, \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, \Delta_2, \Delta_2} \rightarrow_L \quad \rightarrow \quad \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2, F_5}{h_3 : \Delta_4 \vdash \Delta_1, \Delta_2, F_5} \text{ ax} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, \Delta_2, \Delta_2}{h_3 : \Delta_4, F_6 \vdash \Delta_1, \Delta_2} \text{ IH}}{\bullet h_3 : \Delta_4, F_5 \rightarrow F_6 \vdash \Delta_1, \Delta_2} \rightarrow_L$$

- Case(s) rule  $\wedge_L$

$$\frac{h_3 : F_5, F_6, \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, \Delta_2, \Delta_2} \wedge_L \quad \rightarrow \quad \frac{\frac{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, \Delta_2, \Delta_2}{h_3 : \Delta_4, F_5, F_6 \vdash \Delta_1, \Delta_2} \text{ ax}}{\bullet h_3 : \Delta_4, F_5 \wedge F_6 \vdash \Delta_1, \Delta_2} \wedge_L$$

- Case(s) rule  $\vee_L$

$$\frac{h_3 : F_5, \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2 \quad h_3 : F_6, \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, \Delta_2, \Delta_2} \vee_L \quad \rightarrow \quad \frac{\frac{h_3 : \Delta_4, F_5 \vdash \Delta_1, \Delta_2, \Delta_2}{h_3 : \Delta_4, F_5 \vdash \Delta_1, \Delta_2} \text{ ax} \quad \frac{h_3 : \Delta_4, F_6 \vdash \Delta_1, \Delta_2, \Delta_2}{h_3 : \Delta_4, F_6 \vdash \Delta_1, \Delta_2} \text{ IH}}{\bullet h_3 : \Delta_4, F_5 \vee F_6 \vdash \Delta_1, \Delta_2} \vee_L$$

- Case(s) rule  $AT$

$$\frac{h_3 : F_5, \Delta_4, []F_5 \vdash \Delta_1, \Delta_2, \Delta_2}{\bullet h_3 : \Delta_4, []F_5 \vdash \Delta_1, \Delta_2, \Delta_2} AT \quad \rightarrow \quad \frac{\frac{h_3 : \Delta_4, F_5, []F_5 \vdash \Delta_1, \Delta_2, \Delta_2}{h_3 : \Delta_4, F_5, []F_5 \vdash \Delta_1, \Delta_2} \text{ ax}}{\bullet h_3 : \Delta_4, []F_5 \vdash \Delta_1, \Delta_2} AT$$

- Case(s) rule  $\perp_L$

$$\frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2} \perp_L \quad \rightarrow \quad \frac{}{\bullet h_3 : \perp, \Delta_4 \vdash \Delta_1, \Delta_2} \perp_L$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_1, (\Delta_5, p_4), \Delta_5, p_4} I \quad \rightarrow \quad \frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_1, \Delta_5, p_4} I$$

$$\frac{}{\bullet h_2 : \Delta_3, p_4 \vdash (\Delta_5, p_4), \Delta_1, \Delta_1} I \quad \rightarrow \quad \frac{}{\bullet h_2 : \Delta_3, p_4 \vdash \Delta_1, \Delta_5, p_4} I$$

- Case(s) rule  $\top_L$

$$\frac{h_3 : \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2} \top_L \quad \rightarrow \quad \frac{\frac{h_3 : \Delta_4 \vdash \Delta_1, \Delta_2, \Delta_2}{h_3 : \Delta_4 \vdash \Delta_1, \Delta_2} \text{ ax}}{\bullet h_3 : \top, \Delta_4 \vdash \Delta_1, \Delta_2} \top_L$$

## 7 Identity-Expansion

$$\frac{\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0, \boxed{\vdash} F_0 \vdash F_0} W}{- : \boxed{\vdash} F_0 \vdash F_0} AT}{- : \boxed{\vdash} F_0 \vdash \boxed{\vdash} F_0} A4$$

$$\frac{\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0 \vdash F_0, F_1} W}{- : F_0 \vee F_1 \vdash F_0, F_1} \vee_R}{\frac{\frac{\overline{- : F_1 \vdash F_1} \text{ IH}}{- : F_1 \vdash F_0, F_1} W}{- : F_0 \vee F_1 \vdash F_0 \vee F_1} \vee_L}$$

$$\frac{\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0, F_1 \vdash F_0} W}{- : F_0, F_1 \vdash F_0 \wedge F_1} \wedge_L}{\frac{\frac{\overline{- : F_1 \vdash F_1} \text{ IH}}{- : F_0, F_1 \vdash F_1} W}{- : F_0 \wedge F_1 \vdash F_0 \wedge F_1} \wedge_R}$$

$$\frac{\frac{\frac{\overline{- : F_0 \vdash F_0} \text{ IH}}{- : F_0 \vdash F_0, F_1} W}{- : F_0, F_0 \rightarrow F_1 \vdash F_1} \rightarrow_R}{\frac{\frac{\overline{- : F_1 \vdash F_1} \text{ IH}}{- : F_0, F_1 \vdash F_1} W}{- : F_0 \rightarrow F_1 \vdash F_0 \rightarrow F_1} \rightarrow_L}$$

$$\frac{}{- : \top \vdash \top} \top_R$$

$$\frac{}{- : \perp \vdash \perp} \perp_L$$

## 8 Cut-Elimination

### 8.1 Status of $\rightarrow_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{\frac{h_1 : F_7, \Delta_6 \vdash F_8, \Delta_{10}, F_{11} \rightarrow F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : F_{11}, \Delta_6, F_7 \rightarrow F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R}{\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}}{\rightarrow} \text{Cut}} \\
\frac{\frac{\frac{h_1 : \Delta_6, F_{11}, F_7 \vdash \Delta_{10}, F_{12}, F_8}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \rightarrow F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_{11}, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{12}}{\text{ax/W}}}{\frac{- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}}{\rightarrow_R} \text{hCut}} \\
\frac{\frac{h_2 : F_9, \Delta_8 \vdash F_7, F_{10}, \Delta_{14}, F_{12} \rightarrow F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : F_7, F_{12}, \Delta_8 \vdash F_{13}, \Delta_{14}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}} \rightarrow_R}{\frac{- : \Delta_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \rightarrow F_{10}}{\rightarrow} \text{Cut}} \\
\frac{\frac{\frac{h_2 : \Delta_8, F_{12}, F_9 \vdash \Delta_{14}, F_{10}, F_{13}, F_7}{\bullet h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_7, F_9 \rightarrow F_{10}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_8, F_{12}, F_7 \vdash \Delta_{14}, F_{13}, F_9 \rightarrow F_{10}}{\text{ax/W}}}{\frac{- : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \rightarrow F_{10}}{\rightarrow_R} \text{hCut}} \\
\frac{\frac{h_2 : F_{11}, \Delta_8 \vdash F_7, F_{12}, \Delta_{10}}{\bullet h_2 : \Delta_8 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7} \rightarrow_R \quad \frac{h_9 : F_7, F_{11}, \Delta_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R}{\frac{- : \Delta_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}}{\rightarrow} \text{Cut}} \\
\frac{\frac{\frac{h_2 : \Delta_8, F_{11} \vdash \Delta_{10}, F_{12}, F_7}{\text{ax/W}} \quad \frac{h_9 : \Delta_8, F_{11}, F_7 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : \Delta_8, F_{11}, F_7 \vdash \Delta_{10}, F_{12}} \text{ax/W}}{\frac{- : \Delta_8, F_{11} \vdash \Delta_{10}, F_{12}}{\rightarrow_R} \text{H}} \\
\frac{- : \Delta_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}}{\rightarrow_R} \text{hCut}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{\frac{h_1 : F_7, \Delta_6 \vdash F_8, \Delta_{10}, F_{11} \wedge F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash F_{11}, \Delta_{10} \quad h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R}{\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}}{\rightarrow} \text{Cut}} \\
\frac{\frac{\frac{h_1 : \Delta_6, F_7 \vdash \Delta_{10}, F_{11}, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_7 \rightarrow F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11}}{\text{ax/W}}}{\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11}}{\rightarrow_R} \text{hCut}} \quad \frac{\frac{h_1 : \Delta_6, F_7 \vdash \Delta_{10}, F_{12}, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{12}, F_7 \rightarrow F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{12}}{\text{ax/W}}}{\frac{- : \Delta_6 \vdash \Delta_{10}, F_{12}}{\rightarrow_R} \text{hCut}} \\
\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}}{\wedge_R} \\
\frac{\frac{h_2 : F_9, \Delta_8 \vdash F_7, F_{10}, \Delta_{14}, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \wedge F_{13}), F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash F_{12}, \Delta_{14}, F_9 \rightarrow F_{10} \quad h_{11} : F_7, \Delta_8 \vdash F_{13}, \Delta_{14}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \rightarrow F_{10}} \wedge_R}{\frac{- : \Delta_8 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \rightarrow F_{10}}{\rightarrow} \text{Cut}} \\
\frac{\frac{h_2 : \Delta_8, F_9 \vdash \Delta_{14}, F_{10}, F_7, F_{12} \wedge F_{13}}{\text{ax/W}} \quad \frac{h_{11} : \Delta_8, F_7, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \quad h_{11} : \Delta_8, F_7, F_9 \vdash \Delta_{14}, F_{10}, F_{13}}{\bullet h_{11} : \Delta_8, F_7, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \wedge F_{13}} \text{inv-th/ax}}{\frac{- : \Delta_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \wedge F_{13}}{\rightarrow_R} \text{hCut}} \\
\frac{- : \Delta_8 \vdash \Delta_{14}, F_9 \rightarrow F_{10}, F_{12} \wedge F_{13}}{\rightarrow_R}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_1 : F_7, \Delta_6 \vdash F_8, \Delta_{10}, F_{11} \vee F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash F_{11}, F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \\
\rightarrow \\
\frac{h_1 : \Delta_6, F_7 \vdash \Delta_{10}, F_{11}, F_{12}, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7 \rightarrow F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}, F_{11}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12} \quad \vee_R \\
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \text{hCut} \\
\rightarrow \\
\frac{h_2 : F_9, \Delta_8 \vdash F_7, F_{10}, \Delta_{14}, F_{12} \vee F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \vee F_{13}), F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash F_{12}, F_{13}, \Delta_{14}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \rightarrow F_{10}} \vee_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \rightarrow F_{10} \\
\rightarrow \\
\frac{h_2 : \Delta_8, F_9 \vdash \Delta_{14}, F_{10}, F_7, F_{12} \vee F_{13}}{\bullet h_2 : \Delta_8, F_9 \vdash \Delta_{14}, F_{10}, F_7, F_{12} \vee F_{13}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_7, F_9 \vdash \Delta_{14}, F_{10}, F_{12}, F_{13}}{\bullet h_{11} : \Delta_8, F_7, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \vee F_{13}} \text{inv-th/ax} \\
\hline
- : \Delta_8, F_9 \vdash \Delta_{14}, F_{10}, F_{12} \vee F_{13} \quad \vee_R \\
- : \Delta_8 \vdash \Delta_{14}, F_9 \rightarrow F_{10}, F_{12} \vee F_{13} \quad \text{hCut} \\
\rightarrow_R
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_1 : F_7, \Delta_6 \vdash F_8, \perp, \Delta_{10}}{\bullet h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \perp, \Delta_{10}} \perp_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \\
\rightarrow \\
\frac{\bullet h_1 : \Delta_6 \vdash \perp, \Delta_{10}, F_7 \rightarrow F_8}{- : \Delta_6 \vdash \perp, \Delta_{10}} \text{ax/W} \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \perp, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \perp, \Delta_{10}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \text{hCut} \\
\rightarrow \\
\frac{h_2 : F_9, \Delta_8 \vdash F_7, F_{10}, \perp, \Delta_{12}}{\bullet h_2 : \Delta_8 \vdash ((\perp, \Delta_{12}), F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash \Delta_{12}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\perp, \Delta_{12}), F_9 \rightarrow F_{10}} \perp_R \\
\hline
- : \Delta_8 \vdash (\perp, \Delta_{12}), F_9 \rightarrow F_{10} \\
\rightarrow \\
\frac{\bullet h_2 : \Delta_8 \vdash \perp, \Delta_{12}, F_7, F_9 \rightarrow F_{10}}{- : \Delta_8 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_7 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10}} \text{ax/W} \\
\hline
- : \Delta_8 \vdash \perp, \Delta_{12}, F_9 \rightarrow F_{10} \quad \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_1 : F_7, \Delta_6 \vdash F_8, \top, \Delta_{10}}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \rightarrow F_8} \rightarrow_R \quad \frac{h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \top, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \rightarrow F_8 \vdash \top, \Delta_{10}} \top_R \\
\hline
- : \Delta_6 \vdash \top, \Delta_{10} \\
\rightarrow \\
- : \Delta_6 \vdash \top, \Delta_{10} \quad \top_R \\
\rightarrow \\
\frac{h_2 : F_9, \Delta_8 \vdash F_7, F_{10}, \top, \Delta_{12}}{\bullet h_2 : \Delta_8 \vdash ((\top, \Delta_{12}), F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : \Delta_8, F_7 \vdash (\top, \Delta_{12}), F_9 \rightarrow F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\top, \Delta_{12}), F_9 \rightarrow F_{10}} \top_R \\
\hline
- : \Delta_8 \vdash (\top, \Delta_{12}), F_9 \rightarrow F_{10} \\
\rightarrow \\
- : \Delta_8 \vdash \top, \Delta_{12}, F_9 \rightarrow F_{10} \quad \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_1 : F_6, \Box \Gamma_9, \Delta_{12} \vdash F_7, \Delta_{10}, \Box F_{11}}{\bullet h_1 : \Box \Gamma_9, \Delta_{12} \vdash (\Delta_{10}, \Box F_{11}), F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Box \Gamma_9 \vdash F_{11}}{\bullet h_8 : (\Box \Gamma_9, \Delta_{12}), F_6 \rightarrow F_7 \vdash \Delta_{10}, \Box F_{11}} A4 \\
\hline
- : \Box \Gamma_9, \Delta_{12} \vdash \Delta_{10}, \Box F_{11} \\
\rightarrow \\
- : \Box \Gamma_9 \vdash F_{11} \quad \text{ax/W} \\
\hline
- : \Delta_{12}, \Box \Gamma_9 \vdash \Delta_{10}, \Box F_{11} \quad A4 \\
\hline
- : \Delta_{12}, \Box \Gamma_9 \vdash \Delta_{10}, \Box F_{11} \quad \text{Cut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_8, \Box \Gamma_{14}, \Delta_{11} \vdash \Box F_7, F_9, \Delta_{13}, \Box F_{12}}{\bullet h_2 : \Box \Gamma_{14}, \Delta_{11} \vdash ((\Delta_{13}, \Box F_{12}), F_8 \rightarrow F_9), \Box F_7} \rightarrow_R \quad \frac{h_{10} : \Box \Gamma_{14}, \Box F_7 \vdash F_{12}}{\bullet h_{10} : (\Box \Gamma_{14}, \Delta_{11}), \Box F_7 \vdash (\Delta_{13}, \Box F_{12}), F_8 \rightarrow F_9} \begin{array}{l} A4 \\ \text{Cut} \end{array} \\
\hline
- : \Box \Gamma_{14}, \Delta_{11} \vdash (\Delta_{13}, \Box F_{12}), F_8 \rightarrow F_9 \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11}, F_8, \Box \Gamma_{14} \vdash \Box F_7, \Delta_{13}, F_9, \Box F_{12}}{- : \Delta_{11}, F_8, \Box \Gamma_{14} \vdash \Delta_{13}, F_9, \Box F_{12}} \text{ ax/W} \quad \frac{h_{10} : \Box F_7, \Box \Gamma_{14} \vdash F_{12}}{\bullet h_{10} : \Box F_7, \Delta_{11}, F_8, \Box \Gamma_{14} \vdash \Delta_{13}, F_9, \Box F_{12}} \begin{array}{l} \text{ax/W} \\ A4 \end{array}}{\frac{- : \Delta_{11}, F_8, \Box \Gamma_{14} \vdash \Delta_{13}, F_9, \Box F_{12}}{- : \Delta_{11}, \Box \Gamma_{14} \vdash \Delta_{13}, \Box F_{12}, F_8 \rightarrow F_9} \rightarrow_R} \text{ hCut} \\
\hline
\frac{h_2 : F_8, \Box \Gamma_{11}, \Delta_{14} \vdash F_7, F_9, \Delta_{13}, \Box F_{12}}{\bullet h_2 : \Box \Gamma_{11}, \Delta_{14} \vdash ((\Delta_{13}, \Box F_{12}), F_8 \rightarrow F_9), F_7} \rightarrow_R \quad \frac{h_{10} : \Box \Gamma_{11} \vdash F_{12}}{\bullet h_{10} : (\Box \Gamma_{11}, \Delta_{14}), F_7 \vdash (\Delta_{13}, \Box F_{12}), F_8 \rightarrow F_9} \begin{array}{l} A4 \\ \text{Cut} \end{array} \\
\hline
- : \Box \Gamma_{11}, \Delta_{14} \vdash (\Delta_{13}, \Box F_{12}), F_8 \rightarrow F_9 \\
\rightarrow \\
\frac{- : \Box \Gamma_{11} \vdash F_{12}}{- : \Delta_{14}, \Box \Gamma_{11} \vdash \Delta_{13}, \Box F_{12}, F_8 \rightarrow F_9} \begin{array}{l} \text{ax/W} \\ A4 \end{array}
\end{array}$$

• Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_1 : F_6, \Delta_{12}, F_9 \rightarrow F_{10} \vdash F_7, \Delta_{11}}{\bullet h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Delta_{12}, F_6 \rightarrow F_7 \vdash F_9, \Delta_{11} \quad h_8 : F_{10}, \Delta_{12}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \rightarrow F_{10}), F_6 \rightarrow F_7 \vdash \Delta_{11}} \begin{array}{l} \rightarrow_L \\ \text{Cut} \end{array} \\
\hline
- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{12}, F_6 \vdash \Delta_{11}, F_7, F_9}}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6 \rightarrow F_7} \text{ inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_6 \rightarrow F_7 \vdash \Delta_{11}, F_9}}{- : \Delta_{12} \vdash \Delta_{11}, F_9} \text{ ax/W} \quad \frac{\frac{h_1 : \Delta_{12}, F_{10}, F_6 \vdash \Delta_{11}, F_7}}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \rightarrow F_7} \text{ inv-th/ax} \quad \frac{h_8 : \Delta_{12}, F_{10}, F_6 \rightarrow F_7 \vdash \Delta_{11}}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}} \begin{array}{l} \text{ax/W} \\ \text{hCut} \end{array}}{\frac{- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}}{- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}} \rightarrow_L}
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : F_8, \Delta_7 \vdash F_9, \Delta_{10}}{\bullet h_1 : \Delta_7 \vdash \Delta_{10}, F_8 \rightarrow F_9} \rightarrow_R \quad \frac{h_6 : \Delta_7 \vdash F_8, \Delta_{10} \quad h_6 : F_9, \Delta_7 \vdash \Delta_{10}}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{10}} \begin{array}{l} \rightarrow_L \\ \text{Cut} \end{array} \\
\hline
- : \Delta_7 \vdash \Delta_{10} \\
\rightarrow \\
\frac{- : \Delta_7, F_8 \vdash \Delta_{10}, F_9}{- : \Delta_7 \vdash \Delta_{10}, F_8} \text{ ax/W} \quad \frac{- : \Delta_7, F_8 \vdash \Delta_{10}, F_9}{- : \Delta_7, F_8 \vdash \Delta_{10}} \text{ ax/W} \quad \frac{- : \Delta_7, F_8, F_9 \vdash \Delta_{10}}{- : \Delta_7, F_8 \vdash \Delta_{10}} \text{ sCut}}{\frac{- : \Delta_7 \vdash \Delta_{10}}{- : \Delta_7 \vdash \Delta_{10}} \text{ sCut}}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_8, \Delta_{11} \vdash F_{12} \rightarrow F_{13}, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), F_{12} \rightarrow F_{13}} \rightarrow_R \quad \frac{h_{10} : \Delta_{11} \vdash F_{12}, \Delta_7, F_8 \rightarrow F_9 \quad h_{10} : F_{13}, \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8 \rightarrow F_9} \begin{array}{l} \rightarrow_L \\ \text{Cut} \end{array} \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9 \\
\rightarrow \\
\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, F_{12} \rightarrow F_{13}}{- : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \text{ ax/W} \quad \frac{h_{10} : \Delta_{11}, F_8 \vdash \Delta_7, F_{12}, F_9}{\bullet h_{10} : \Delta_{11}, F_8, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_9} \text{ inv-th/ax} \quad \frac{h_{10} : \Delta_{11}, F_{13}, F_8 \vdash \Delta_7, F_9}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \begin{array}{l} \text{inv-th/ax} \\ \text{hCut} \end{array}}{\frac{- : \Delta_{11}, F_8 \vdash \Delta_7, F_9}{- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \rightarrow_R}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_9, \Delta_{14}, F_{12} \rightarrow F_{13} \vdash F_7, F_{10}, \Delta_8}{\bullet h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : F_7, \Delta_{14} \vdash F_{12}, \Delta_8, F_9 \rightarrow F_{10} \quad h_{11} : F_7, F_{13}, \Delta_{14} \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \rightarrow F_{13}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \begin{array}{l} \rightarrow_L \\ \text{Cut} \end{array} \\
\hline
- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10} \\
\rightarrow \\
\frac{h_2 : \Delta_{14}, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_7}{- : \Delta_{14}, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}} \text{ ax/W} \quad \frac{h_{11} : \Delta_{14}, F_7, F_9 \vdash \Delta_8, F_{10}, F_{12}}{\bullet h_{11} : \Delta_{14}, F_7, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}} \text{ inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7, F_9 \vdash \Delta_8, F_{10}}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}} \begin{array}{l} \text{inv-th/ax} \\ \text{hCut} \end{array}}{\frac{- : \Delta_{14}, F_9, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R}
\end{array}$$

• Case rule  $\wedge_L$

- Case rule  $\forall_L$

- Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{h_1 : F_6, \Delta_{11}, [F_9 \vdash F_7, \Delta_{10}]}{\bullet h_1 : \Delta_{11}, [F_9 \vdash \Delta_{10}, F_6 \rightarrow F_7]} \rightarrow_R \quad \frac{h_8 : F_9, \Delta_{11}, [F_9, F_6 \rightarrow F_7 \vdash \Delta_{10}]}{\bullet h_8 : (\Delta_{11}, [F_9]), F_6 \rightarrow F_7 \vdash \Delta_{10}} \begin{array}{l} AT \\ Cut \end{array}}{\neg : \Delta_{11}, [F_9 \vdash \Delta_{10}]} \rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_{11}, F_9, [F_9 \vdash \Delta_{10}, F_6 \rightarrow F_7]}{\neg : \Delta_{11}, F_9, [F_9 \vdash \Delta_{10}]} \text{ax/W} \quad \frac{h_8 : \Delta_{11}, F_9, [F_9, F_6 \rightarrow F_7 \vdash \Delta_{10}]}{\neg : \Delta_{11}, [F_9 \vdash \Delta_{10}]} \begin{array}{l} \text{ax/W} \\ hCut \end{array}}{\neg : \Delta_{11}, [F_9 \vdash \Delta_{10}]} \begin{array}{l} AT \\ \end{array} \\
\\
\frac{\frac{h_2 : F_8, \Delta_{11} \vdash [F_{12}, F_9, \Delta_7]}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), [F_{12}]} \rightarrow_R \quad \frac{h_{10} : F_{12}, \Delta_{11}, [F_{12} \vdash \Delta_7, F_8 \rightarrow F_9]}{\bullet h_{10} : \Delta_{11}, [F_{12} \vdash \Delta_7, F_8 \rightarrow F_9]} \begin{array}{l} AT \\ Cut \end{array}}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \rightarrow \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \Delta_7, F_9, [F_{12}]}{\neg : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_{12}, F_8, [F_{12} \vdash \Delta_7, F_9]}{\bullet h_{10} : \Delta_{11}, F_8, [F_{12} \vdash \Delta_7, F_9]} \begin{array}{l} \text{inv-th/ax} \\ AT \end{array}}{\neg : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \begin{array}{l} hCut \\ \end{array}}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \rightarrow_R \\
\\
\frac{\frac{h_2 : F_9, \Delta_{13}, [F_{12} \vdash F_7, F_{10}, \Delta_8]}{\bullet h_2 : \Delta_{13}, [F_{12} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7]} \rightarrow_R \quad \frac{h_{11} : F_7, F_{12}, \Delta_{13}, [F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}]}{\bullet h_{11} : (\Delta_{13}, [F_{12}]), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \begin{array}{l} AT \\ Cut \end{array}}{\neg : \Delta_{13}, [F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}]} \rightarrow \\
\frac{\frac{\bullet h_2 : \Delta_{13}, F_{12}, [F_{12} \vdash \Delta_8, F_7, F_9 \rightarrow F_{10}]}{\neg : \Delta_{13}, F_{12}, [F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}]} \text{ax/W} \quad \frac{h_{11} : \Delta_{13}, F_{12}, F_7, [F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}]}{\neg : \Delta_{13}, [F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}]} \begin{array}{l} \text{ax/W} \\ hCut \end{array}}{\neg : \Delta_{13}, [F_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}]} \begin{array}{l} AT \\ \end{array}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_1 : F_6, \perp, \Delta_{10} \vdash F_7, \Delta_9}{\bullet h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : (\perp, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9}{\bullet h_8 : (\perp, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9}}{\neg : \perp, \Delta_{10} \vdash \Delta_9} \begin{array}{l} \perp_L \\ Cut \end{array} \rightarrow \\
\frac{\neg : \perp, \Delta_{10} \vdash \Delta_9}{\neg : \perp, \Delta_{10} \vdash \Delta_9} \perp_L \\
\\
\frac{\frac{h_2 : F_8, \Delta_{11} \vdash \perp, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), \perp} \rightarrow_R \quad \frac{h_{10} : \Delta_{11}, \perp \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, \perp \vdash \Delta_7, F_8 \rightarrow F_9}}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \begin{array}{l} \perp_L \\ Cut \end{array} \rightarrow \\
\frac{\frac{h_2 : \Delta_{11}, F_8 \vdash \perp, \Delta_7, F_9}{\neg : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \text{ax/W} \quad \frac{h_{10} : \perp, \Delta_{11}, F_8 \vdash \Delta_7, F_9}{\neg : \Delta_{11}, F_8 \vdash \Delta_7, F_9} \begin{array}{l} \perp_L \\ hCut \end{array}}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9} \rightarrow_R \\
\\
\frac{\frac{h_2 : F_9, \perp, \Delta_{12} \vdash F_7, F_{10}, \Delta_8}{\bullet h_2 : \perp, \Delta_{12} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : (\perp, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\perp, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \begin{array}{l} \perp_L \\ Cut \end{array}}{\neg : \perp, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow \\
\frac{\neg : \perp, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}}{\neg : \perp, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_1 : F_6, \Delta_{11}, p_9 \vdash F_7, \Delta_{10}, p_9}{\bullet h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : (\Delta_{11}, p_9), F_6 \rightarrow F_7 \vdash \Delta_{10}, p_9}{\bullet h_8 : (\Delta_{11}, p_9), F_6 \rightarrow F_7 \vdash \Delta_{10}, p_9}}{\neg : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9} \begin{array}{l} I \\ Cut \end{array} \rightarrow \\
\frac{\neg : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9}{\neg : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9} I \\
\\
\frac{\frac{h_2 : F_7, \Delta_{10} \vdash p_{11}, F_8, \Delta_{12}, p_{11}}{\bullet h_2 : \Delta_{10} \vdash ((\Delta_{12}, p_{11}), F_7 \rightarrow F_8), p_{11}} \rightarrow_R \quad \frac{h_9 : \Delta_{10}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7 \rightarrow F_8}{\bullet h_9 : \Delta_{10}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7 \rightarrow F_8}}{\neg : \Delta_{10} \vdash (\Delta_{12}, p_{11}), F_7 \rightarrow F_8} \begin{array}{l} I \\ Cut \end{array} \rightarrow \\
\frac{\frac{h_2 : \Delta_{10}, F_7 \vdash \Delta_{12}, F_8, p_{11}, p_{11}}{\neg : \Delta_{10}, F_7 \vdash \Delta_{12}, F_8, p_{11}} \text{ax/W} \quad \frac{h_9 : \Delta_{10}, F_7, p_{11} \vdash \Delta_{12}, F_8, p_{11}}{\neg : \Delta_{10}, F_7 \vdash \Delta_{12}, F_8, p_{11}} \begin{array}{l} I \\ hCut \end{array}}{\neg : \Delta_{10} \vdash \Delta_{12}, p_{11}, F_7 \rightarrow F_8} \rightarrow_R
\end{array}$$



$$\begin{array}{c}
\frac{h_2 : F_8, \Delta_{13}, p_{11} \vdash F_7, F_9, \Delta_{12}, p_{11}}{\bullet h_2 : \Delta_{13}, p_{11} \vdash ((\Delta_{12}, p_{11}), F_8 \rightarrow F_9), F_7} \rightarrow_R \quad \frac{h_{10} : (\Delta_{13}, p_{11}), F_7 \vdash (\Delta_{12}, p_{11}), F_8 \rightarrow F_9}{\bullet h_{10} : (\Delta_{13}, p_{11}), F_7 \vdash (\Delta_{12}, p_{11}), F_8 \rightarrow F_9} I \\
\hline
- : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_8 \rightarrow F_9 \\
\hline
\rightarrow \\
\hline
- : \Delta_{13}, p_{11} \vdash \Delta_{12}, p_{11}, F_8 \rightarrow F_9 \quad I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_1 : F_6, \top, \Delta_{10} \vdash F_7, \Delta_9}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7} \rightarrow_R \quad \frac{h_8 : \Delta_{10}, F_6 \rightarrow F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9} \top_L \\
\hline
- : \top, \Delta_{10} \vdash \Delta_9 \\
\hline
\rightarrow \\
\hline
\frac{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \rightarrow F_7}{- : \top, \Delta_{10} \vdash \Delta_9} \text{ax/W} \quad \frac{h_8 : \top, \Delta_{10}, F_6 \rightarrow F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \rightarrow F_7 \vdash \Delta_9} \text{ax/W} \\
\hline
- : \top, \Delta_{10} \vdash \Delta_9 \quad \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_8, \Delta_{11} \vdash \top, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \rightarrow F_9), \top} \rightarrow_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9}{\bullet h_{10} : \Delta_{11}, \top \vdash \Delta_7, F_8 \rightarrow F_9} \top_L \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9 \\
\hline
\rightarrow \\
\hline
- : \Delta_{11} \vdash \Delta_7, F_8 \rightarrow F_9 \quad \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : F_9, \top, \Delta_{12} \vdash F_7, F_{10}, \Delta_8}{\bullet h_2 : \top, \Delta_{12} \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_7} \rightarrow_R \quad \frac{h_{11} : F_7, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\top, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \top_L \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \\
\hline
\rightarrow \\
\hline
\frac{\bullet h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9 \rightarrow F_{10}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/W} \quad \frac{h_{11} : \top, \Delta_{12}, F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_{11} : (\top, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/W} \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \rightarrow F_{10} \quad \text{hCut}
\end{array}$$

## 8.2 Status of $\wedge_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, \Delta_{10}, F_{11} \rightarrow F_{12} \quad h_1 : \Delta_6 \vdash F_8, \Delta_{10}, F_{11} \rightarrow F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : F_{11}, \Delta_6, F_7 \wedge F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12} \\
\hline
\rightarrow \\
\hline
\frac{h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \wedge F_8} \text{inv-th/ax} \quad \frac{h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_8}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \wedge F_8} \text{inv-th/ax} \\
\hline
- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12} \quad \wedge_R \\
\hline
\frac{h_9 : \Delta_6, F_{11}, F_7 \wedge F_8 \vdash \Delta_{10}, F_{12}}{- : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12} \quad \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_8 \vdash F_7, F_9, \Delta_{14}, F_{12} \rightarrow F_{13} \quad h_2 : \Delta_8 \vdash F_7, F_{10}, \Delta_{14}, F_{12} \rightarrow F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : F_7, F_{12}, \Delta_8 \vdash F_{13}, \Delta_{14}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10}} \rightarrow_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \wedge F_{10} \\
\hline
\rightarrow \\
\hline
\frac{h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_7, F_9}{\bullet h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_7, F_9 \wedge F_{10}} \text{inv-th/ax} \quad \frac{h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{10}, F_{13}, F_7}{\bullet h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{10}, F_{13}, F_7} \text{inv-th/ax} \\
\hline
- : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10} \quad \wedge_R \\
\hline
\frac{h_{11} : \Delta_8, F_{12}, F_7 \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10}}{- : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \wedge F_{10}} \text{ax/W} \\
\hline
- : \Delta_8 \vdash \Delta_{14}, F_{12} \rightarrow F_{13}, F_9 \wedge F_{10} \quad \text{hCut}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, \Delta_{10}, F_{11} \wedge F_{12} \quad h_1 : \Delta_6 \vdash F_8, \Delta_{10}, F_{11} \wedge F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash F_{11}, \Delta_{10} \quad h_9 : \Delta_6, F_7 \wedge F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12} \\
\hline
\rightarrow \\
\hline
\frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash F_{11}, \Delta_{10}}{- : \Delta_6, F_7, F_8 \vdash \Delta_{10}, F_{11}} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash F_{12}, \Delta_{10}}{- : \Delta_6, F_7, F_8 \vdash \Delta_{10}, F_{12}} \text{inv-th/ax} \\
\hline
- : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12} \quad \wedge_R \\
\hline
\frac{- : \Delta_6 \vdash \Delta_{10}, F_7, F_{11} \wedge F_{12}}{- : \Delta_6 \vdash \Delta_{10}, F_7, F_{11} \wedge F_{12}} \text{ax/W} \quad \frac{- : \Delta_6, F_7 \vdash \Delta_{10}, F_{11} \wedge F_{12}}{- : \Delta_6, F_7 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \text{sCut} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_8 \vdash F_7, F_9, \Delta_{14}, F_{12} \wedge F_{13} \quad h_2 : \Delta_8 \vdash F_7, F_{10}, \Delta_{14}, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \wedge F_{13}), F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash F_{12}, \Delta_{14}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \wedge F_{10}} \wedge_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \wedge F_{10} \quad \rightarrow \\
\frac{\frac{h_2 : \Delta_8 \vdash \Delta_{14}, F_{12}, F_7, F_9 \quad \text{inv-th/ax} \quad h_2 : \Delta_8 \vdash \Delta_{14}, F_{10}, F_{12}, F_7 \quad \text{inv-th/ax}}{\bullet h_2 : \Delta_8 \vdash \Delta_{14}, F_{12}, F_7, F_9 \wedge F_{10}} \wedge_R \quad \frac{h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{12}, F_9 \wedge F_{10} \quad \text{ax/W}}{\bullet h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{12}, F_9 \wedge F_{10}} \text{hCut}}{\frac{- : \Delta_8 \vdash \Delta_{14}, F_{12}, F_9 \wedge F_{10}}{- : \Delta_8 \vdash \Delta_{14}, F_{12} \wedge F_{13}, F_9 \wedge F_{10}}} \wedge_R \\
\hline
\frac{h_2 : \Delta_8 \vdash F_7, F_{11}, \Delta_{10} \quad h_2 : \Delta_8 \vdash F_7, F_{12}, \Delta_{10}}{\bullet h_2 : \Delta_8 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7} \wedge_R \quad \frac{h_9 : F_7, \Delta_8 \vdash F_{11}, \Delta_{10} \quad h_9 : F_7, \Delta_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R \\
\hline
- : \Delta_8 \vdash \Delta_{10}, F_{11} \wedge F_{12} \quad \rightarrow \\
\frac{\frac{h_2 : \Delta_8 \vdash \Delta_{10}, F_{11}, F_7 \quad \text{ax/W} \quad h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11} \quad \text{ax/W}}{\bullet h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11}} \text{hCut} \quad \frac{h_2 : \Delta_8 \vdash \Delta_{10}, F_{12}, F_7 \quad \text{ax/W} \quad h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{12} \quad \text{ax/W}}{\bullet h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{12}} \text{hCut}}{\frac{- : \Delta_8 \vdash \Delta_{10}, F_{11} \quad - : \Delta_8 \vdash \Delta_{10}, F_{12}}{- : \Delta_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}}} \wedge_R
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, \Delta_{10}, F_{11} \vee F_{12} \quad h_1 : \Delta_6 \vdash F_8, \Delta_{10}, F_{11} \vee F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash F_{11}, F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \rightarrow \\
\frac{h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7 \quad \text{inv-th/ax} \quad h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_8 \quad \text{inv-th/ax}}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}, F_{11}, F_{12}} \text{ax/W} \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12} \quad \vee_R \\
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \\
\hline
\frac{h_2 : \Delta_8 \vdash F_7, F_9, \Delta_{14}, F_{12} \vee F_{13} \quad h_2 : \Delta_8 \vdash F_7, F_{10}, \Delta_{14}, F_{12} \vee F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \vee F_{13}), F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash F_{12}, F_{13}, \Delta_{14}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \wedge F_{10}} \vee_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \wedge F_{10} \quad \rightarrow \\
\frac{h_2 : \Delta_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_7, F_9 \quad \text{inv-th/ax} \quad h_2 : \Delta_8 \vdash \Delta_{14}, F_{10}, F_{12}, F_{13}, F_7 \quad \text{inv-th/ax}}{\bullet h_2 : \Delta_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_7, F_9 \wedge F_{10}} \wedge_R \quad \frac{h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \wedge F_{10}} \text{ax/W} \\
\hline
- : \Delta_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \wedge F_{10} \quad \vee_R \\
- : \Delta_8 \vdash \Delta_{14}, F_9 \wedge F_{10}, F_{12} \vee F_{13}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, \perp, \Delta_{10} \quad h_1 : \Delta_6 \vdash F_8, \perp, \Delta_{10}}{\bullet h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \wedge F_8} \wedge_R \quad \frac{h_9 : \Delta_6, F_7 \wedge F_8 \vdash \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \perp, \Delta_{10}} \perp_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10} \quad \rightarrow \\
\frac{\bullet h_1 : \Delta_6 \vdash \perp, \Delta_{10}, F_7 \wedge F_8 \quad \text{ax/W} \quad h_9 : \Delta_6, F_7 \wedge F_8 \vdash \perp, \Delta_{10} \quad \text{ax/W}}{\bullet h_1 : \Delta_6 \vdash \perp, \Delta_{10}} \text{hCut} \\
\hline
\frac{h_2 : \Delta_8 \vdash F_7, F_9, \perp, \Delta_{12} \quad h_2 : \Delta_8 \vdash F_7, F_{10}, \perp, \Delta_{12}}{\bullet h_2 : \Delta_8 \vdash ((\perp, \Delta_{12}), F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash \Delta_{12}, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\perp, \Delta_{12}), F_9 \wedge F_{10}} \perp_R \\
\hline
- : \Delta_8 \vdash (\perp, \Delta_{12}), F_9 \wedge F_{10} \quad \rightarrow \\
\frac{\bullet h_2 : \Delta_8 \vdash \perp, \Delta_{12}, F_7, F_9 \wedge F_{10} \quad \text{ax/W} \quad h_{11} : \Delta_8, F_7 \vdash \perp, \Delta_{12}, F_9 \wedge F_{10} \quad \text{ax/W}}{\bullet h_2 : \Delta_8 \vdash \perp, \Delta_{12}, F_9 \wedge F_{10}} \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, \top, \Delta_{10} \quad h_1 : \Delta_6 \vdash F_8, \top, \Delta_{10}}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \wedge F_8} \wedge_R \quad \frac{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \top, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \wedge F_8 \vdash \top, \Delta_{10}} \top_R \\
\hline
- : \Delta_6 \vdash \top, \Delta_{10} \quad \rightarrow \\
- : \Delta_6 \vdash \top, \Delta_{10} \quad \top_R
\end{array}$$



- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash F_6, \Delta_{11} \quad h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash F_7, \Delta_{11}}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \frac{h_8 : F_9, F_{10}, \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \wedge_L}{\frac{}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \text{Cut}} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \wedge F_7} \text{inv-th/ax} \quad \frac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_7}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_7} \text{inv-th/ax}}{\frac{}{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}} \wedge_R} \wedge_L \\
\frac{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : \Delta_{12}, F_{10}, F_9, F_6 \wedge F_7 \vdash \Delta_{11}} \text{ax/W} \\
\frac{}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \text{hCut} \\
\rightarrow \\
\frac{h_1 : \Delta_7 \vdash F_8, \Delta_{10} \quad h_1 : \Delta_7 \vdash F_9, \Delta_{10}}{\bullet h_1 : \Delta_7 \vdash \Delta_{10}, F_8 \wedge F_9} \wedge_R \quad \frac{h_6 : F_8, F_9, \Delta_7 \vdash \Delta_{10}}{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{10}} \wedge_L}{\frac{}{- : \Delta_7 \vdash \Delta_{10}} \text{Cut}} \\
\rightarrow \\
\frac{}{- : \Delta_7 \vdash \Delta_{10}, F_8} \text{ax/W} \quad \frac{}{- : \Delta_7, F_8 \vdash \Delta_{10}, F_9} \text{ax/W} \quad \frac{}{- : \Delta_7, F_8, F_9 \vdash \Delta_{10}} \text{ax/W} \\
\frac{}{- : \Delta_7 \vdash \Delta_{10}} \text{sCut} \\
\rightarrow \\
\frac{h_2 : \Delta_{11} \vdash F_{12} \wedge F_{13}, F_8, \Delta_7 \quad h_2 : \Delta_{11} \vdash F_{12} \wedge F_{13}, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), F_{12} \wedge F_{13}} \wedge_R \quad \frac{h_{10} : F_{12}, F_{13}, \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8 \wedge F_9} \wedge_L}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{Cut}} \\
\rightarrow \\
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_{12} \wedge F_{13} \vdash \Delta_7, F_8} \text{ax/W} \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8} \text{inv-th/ax}}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8} \wedge_L} \wedge_L \\
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_9, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_{11} \vdash \Delta_7, F_9, F_{12} \wedge F_{13} \vdash \Delta_7, F_9} \text{ax/W} \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_9} \text{inv-th/ax}}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_9} \wedge_L} \wedge_L \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \wedge_R \\
\rightarrow \\
\frac{h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash F_7, F_9, \Delta_8 \quad h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash F_7, F_{10}, \Delta_8}{\bullet h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash (\Delta_8, F_9 \wedge F_{10}), F_7} \wedge_R \quad \frac{h_{11} : F_7, F_{12}, F_{13}, \Delta_{14} \vdash \Delta_8, F_9 \wedge F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \wedge F_{13}), F_7 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_L}{\frac{}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut}} \\
\rightarrow \\
\frac{h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_7, F_9}{\bullet h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_7, F_9 \wedge F_{10}} \text{inv-th/ax} \quad \frac{h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_{10}, F_7}{\bullet h_2 : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_{10}, F_7} \text{inv-th/ax}}{\frac{}{- : \Delta_{14}, F_{12}, F_{13} \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R} \wedge_L \\
\frac{h_{11} : \Delta_{14}, F_{12}, F_{13}, F_7 \vdash \Delta_8, F_9 \wedge F_{10}}{\bullet h_{11} : \Delta_{14}, F_{12}, F_{13}, F_7 \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W} \\
\frac{}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \wedge F_{10}} \text{hCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash F_6, \Delta_{11} \quad h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash F_7, \Delta_{11}}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \wedge F_7} \wedge_R \quad \frac{h_8 : F_9, \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11} \quad h_8 : F_{10}, \Delta_{12}, F_6 \wedge F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \wedge F_7 \vdash \Delta_{11}} \vee_L}{\frac{}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \text{Cut}} \\
\rightarrow \\
\frac{h_2 : \Delta_{12}, F_6, F_7, F_9 \vdash \Delta_{11}}{\bullet h_2 : \Delta_{12}, F_6, F_7, F_9 \vdash \Delta_{11}} \text{inv-th/ax} \quad \frac{h_2 : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}}{\bullet h_2 : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}} \text{inv-th/ax}}{\frac{}{- : \Delta_{12}, F_6, F_7, F_9 \vdash \Delta_{11}} \vee_L} \vee_L \\
\frac{}{- : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}} \text{ax/W} \quad \frac{}{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}, F_7} \text{ax/W} \quad \frac{}{- : \Delta_{12}, F_6, F_7, F_9 \vee F_{10} \vdash \Delta_{11}} \text{ax/W} \\
\frac{}{- : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6} \text{sCut} \quad \frac{}{- : \Delta_{12}, F_6, F_9 \vee F_{10} \vdash \Delta_{11}} \text{sCut} \\
\rightarrow \\
\frac{h_2 : \Delta_{11} \vdash F_{12} \vee F_{13}, F_8, \Delta_7 \quad h_2 : \Delta_{11} \vdash F_{12} \vee F_{13}, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \wedge F_9), F_{12} \vee F_{13}} \wedge_R \quad \frac{h_{10} : F_{12}, \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9 \quad h_{10} : F_{13}, \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \vee F_{13} \vdash \Delta_7, F_8 \wedge F_9} \vee_L}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \text{Cut}} \\
\rightarrow \\
\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8}{\bullet h_2 : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \text{inv-th/ax} \quad \frac{h_2 : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_9}{\bullet h_2 : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_9} \text{inv-th/ax}}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \wedge_R} \wedge_R \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_{13}, F_8 \wedge F_9} \text{ax/W} \quad \frac{}{- : \Delta_{11}, F_{13} \vdash \Delta_7, F_{12}, F_8 \wedge F_9} \text{ax/W} \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_{12}, F_8 \wedge F_9} \text{sCut} \quad \frac{}{- : \Delta_{11}, F_{12} \vdash \Delta_7, F_8 \wedge F_9} \text{sCut} \\
\rightarrow \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \wedge F_9} \wedge_L
\end{array}$$



- Case rule  $\top_L$

### 8.3 Status of $V_R$ : OK

- $$\begin{array}{c}
\frac{\frac{h_1 : \Delta_6 \vdash F_7, F_8, \Delta_{10}, F_{11} \rightarrow F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : F_{11}, \Delta_6, F_7 \vee F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R}{\neg : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \text{Cut} \\
\frac{\frac{h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7, F_8}{\bullet h_1 : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}, F_7 \vee F_8} \text{inv-th/ax} \quad \frac{}{h_9 : \Delta_6, F_{11}, F_7 \vee F_8 \vdash \Delta_{10}, F_{12}} \text{ax/W}}{\neg : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}} \vee_R \quad \text{hCut} \\
\frac{\neg : \Delta_6, F_{11} \vdash \Delta_{10}, F_{12}}{\neg : \Delta_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_8 \vdash F_7, F_9, F_{10}, \Delta_{14}, F_{12} \rightarrow F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, F_{12}, \Delta_8 \vdash F_{13}, \Delta_{14}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vee F_{10}} \rightarrow_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{10}, F_{13}, F_7, F_9}{\bullet h_2 : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_7, F_9 \vee F_{10}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_8, F_{12}, F_7 \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}}{h_{11} : \Delta_8, F_{12}, F_7 \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}} \text{ax/W}}{\frac{- : \Delta_8, F_{12} \vdash \Delta_{14}, F_{13}, F_9 \vee F_{10}}{- : \Delta_8 \vdash \Delta_{14}, F_{12} \rightarrow F_{13}, F_9 \vee F_{10}} \vee_R} \text{hCut} \\
\rightarrow_R
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, F_8, \Delta_{10}, F_{11} \wedge F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash F_{11}, \Delta_{10} \quad h_9 : \Delta_6, F_7 \vee F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_7, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_7 \vee F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}}{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}} \text{ax/W}}{\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11}}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \vee_R} \text{hCut} \\
\wedge_R \\
\hline
\frac{h_2 : \Delta_8 \vdash F_7, F_9, F_{10}, \Delta_{14}, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash F_{12}, \Delta_{14}, F_9 \vee F_{10} \quad h_{11} : F_7, \Delta_8 \vdash F_{13}, \Delta_{14}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vee F_{10}} \wedge_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_8 \vdash \Delta_{14}, F_{10}, F_7, F_9, F_{12} \wedge F_{13}}{\bullet h_2 : \Delta_8 \vdash \Delta_{14}, F_{10}, F_7, F_9, F_{12} \wedge F_{13}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{10}, F_{12}, F_9}{\bullet h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{10}, F_9, F_{12} \wedge F_{13}} \text{inv-th/ax}}{\frac{- : \Delta_8 \vdash \Delta_{14}, F_{10}, F_9, F_{12} \wedge F_{13}}{- : \Delta_8 \vdash \Delta_{14}, F_{12} \wedge F_{13}, F_9 \vee F_{10}} \vee_R} \text{hCut} \\
\wedge_R
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, F_8, \Delta_{10}, F_{11} \vee F_{12}}{\bullet h_1 : \Delta_6 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash F_{11}, F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7, F_8}{\bullet h_1 : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}, F_7 \vee F_8} \text{inv-th/ax} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}, F_{11}, F_{12}} \text{ax/W}}{\frac{- : \Delta_6 \vdash \Delta_{10}, F_{11}, F_{12}}{- : \Delta_6 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R} \text{hCut} \\
\vee_R \\
\hline
\frac{h_2 : \Delta_8 \vdash F_7, F_9, F_{10}, \Delta_{14}, F_{12} \vee F_{13}}{\bullet h_2 : \Delta_8 \vdash ((\Delta_{14}, F_{12} \vee F_{13}), F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash F_{12}, F_{13}, \Delta_{14}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \vee F_{10}} \vee_R \\
\hline
- : \Delta_8 \vdash (\Delta_{14}, F_{12} \vee F_{13}), F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_8 \vdash \Delta_{14}, F_{10}, F_{12}, F_{13}, F_7, F_9}{\bullet h_2 : \Delta_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_7, F_9 \vee F_{10}} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10}}{h_{11} : \Delta_8, F_7 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10}} \text{ax/W}}{\frac{- : \Delta_8 \vdash \Delta_{14}, F_{12}, F_{13}, F_9 \vee F_{10}}{- : \Delta_8 \vdash \Delta_{14}, F_{12} \vee F_{13}, F_9 \vee F_{10}} \vee_R} \text{hCut} \\
\vee_R \\
\hline
\frac{h_2 : \Delta_8 \vdash F_7, F_{11}, F_{12}, \Delta_{10}}{\bullet h_2 : \Delta_8 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_7} \vee_R \quad \frac{h_9 : F_7, \Delta_8 \vdash F_{11}, F_{12}, \Delta_{10}}{\bullet h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R \\
\hline
- : \Delta_8 \vdash \Delta_{10}, F_{11} \vee F_{12} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_8 \vdash \Delta_{10}, F_{11}, F_{12}, F_7}{\bullet h_2 : \Delta_8 \vdash \Delta_{10}, F_{11}, F_{12}, F_7} \text{ax/W} \quad \frac{h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : \Delta_8, F_7 \vdash \Delta_{10}, F_{11}, F_{12}} \text{ax/W}}{\frac{- : \Delta_8 \vdash \Delta_{10}, F_{11}, F_{12}}{- : \Delta_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R} \text{hCut} \\
\vee_R
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, F_8, \perp, \Delta_{10}}{\bullet h_1 : \Delta_6 \vdash (\perp, \Delta_{10}), F_7 \vee F_8} \vee_R \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \Delta_{10}}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \perp, \Delta_{10}} \perp_R \\
\hline
\frac{}{- : \Delta_6 \vdash \perp, \Delta_{10}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{\bullet h_1 : \Delta_6 \vdash \perp, \Delta_{10}, F_7 \vee F_8}{- : \Delta_6 \vdash \perp, \Delta_{10}} \text{ax/W} \quad \frac{h_9 : \Delta_6, F_7 \vee F_8 \vdash \perp, \Delta_{10}}{h_9 : \Delta_6, F_7 \vee F_8 \vdash \perp, \Delta_{10}} \text{ax/W} \\
\hline
\frac{}{- : \Delta_6 \vdash \perp, \Delta_{10}} \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_8 \vdash F_7, F_9, F_{10}, \perp, \Delta_{12}}{\bullet h_2 : \Delta_8 \vdash ((\perp, \Delta_{12}), F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, \Delta_8 \vdash \Delta_{12}, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_8, F_7 \vdash (\perp, \Delta_{12}), F_9 \vee F_{10}} \perp_R \\
\hline
\frac{}{- : \Delta_8 \vdash (\perp, \Delta_{12}), F_9 \vee F_{10}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{\bullet h_2 : \Delta_8 \vdash \perp, \Delta_{12}, F_7, F_9 \vee F_{10}}{- : \Delta_8 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_7 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}}{h_{11} : \Delta_8, F_7 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}} \text{ax/W} \\
\hline
\frac{}{- : \Delta_8 \vdash \perp, \Delta_{12}, F_9 \vee F_{10}} \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_6 \vdash F_7, F_8, \top, \Delta_{10}}{\bullet h_1 : \Delta_6 \vdash (\top, \Delta_{10}), F_7 \vee F_8} \vee_R \quad \frac{}{\bullet h_9 : \Delta_6, F_7 \vee F_8 \vdash \top, \Delta_{10}} \top_R \\
\hline
\frac{}{- : \Delta_6 \vdash \top, \Delta_{10}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_6 \vdash \top, \Delta_{10}} \top_R
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_8 \vdash F_7, F_9, F_{10}, \top, \Delta_{12}}{\bullet h_2 : \Delta_8 \vdash ((\top, \Delta_{12}), F_9 \vee F_{10}), F_7} \vee_R \quad \frac{}{\bullet h_{11} : \Delta_8, F_7 \vdash (\top, \Delta_{12}), F_9 \vee F_{10}} \top_R \\
\hline
\frac{}{- : \Delta_8 \vdash (\top, \Delta_{12}), F_9 \vee F_{10}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_8 \vdash \top, \Delta_{12}, F_9 \vee F_{10}} \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_9, \Delta_{12} \vdash F_6, F_7, \Delta_{10}, \Box F_{11}}{\bullet h_1 : \Box \Gamma_9, \Delta_{12} \vdash (\Delta_{10}, \Box F_{11}), F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Box \Gamma_9 \vdash F_{11}}{\bullet h_8 : (\Box \Gamma_9, \Delta_{12}), F_6 \vee F_7 \vdash \Delta_{10}, \Box F_{11}} A4 \\
\hline
\frac{}{- : \Box \Gamma_9, \Delta_{12} \vdash \Delta_{10}, \Box F_{11}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Box \Gamma_9 \vdash F_{11}} \text{ax/W} \\
\hline
\frac{}{- : \Delta_{12}, \Box \Gamma_9 \vdash \Delta_{10}, \Box F_{11}} A4
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_{14}, \Delta_{11} \vdash \Box F_7, F_8, F_9, \Delta_{13}, \Box F_{12}}{\bullet h_2 : \Box \Gamma_{14}, \Delta_{11} \vdash ((\Delta_{13}, \Box F_{12}), F_8 \vee F_9), \Box F_7} \vee_R \quad \frac{h_{10} : \Box \Gamma_{14}, \Box F_7 \vdash F_{12}}{\bullet h_{10} : (\Box \Gamma_{14}, \Delta_{11}), \Box F_7 \vdash (\Delta_{13}, \Box F_{12}), F_8 \vee F_9} A4 \\
\hline
\frac{}{- : \Box \Gamma_{14}, \Delta_{11} \vdash (\Delta_{13}, \Box F_{12}), F_8 \vee F_9} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Box \Gamma_{14}, \Delta_{11} \vdash \Delta_{13}, \Box F_{12}, F_8 \vee F_9} \text{ax/W} \\
\hline
\frac{h_2 : \Delta_{11}, \Box \Gamma_{14} \vdash \Box F_7, \Delta_{13}, F_8, F_9, \Box F_{12}}{- : \Delta_{11}, \Box \Gamma_{14} \vdash \Delta_{13}, F_8, F_9, \Box F_{12}} \text{ax/W} \quad \frac{h_{10} : \Box F_7, \Box \Gamma_{14} \vdash F_{12}}{\bullet h_{10} : \Box F_7, \Delta_{11}, \Box \Gamma_{14} \vdash \Delta_{13}, F_8, F_9, \Box F_{12}} A4 \\
\hline
\frac{}{- : \Delta_{11}, \Box \Gamma_{14} \vdash \Delta_{13}, F_8, F_9, \Box F_{12}} \text{hCut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_{11}, \Box \Gamma_{14} \vdash \Delta_{13}, \Box F_{12}, F_8 \vee F_9} \vee_R
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_{11}, \Delta_{14} \vdash F_7, F_8, F_9, \Delta_{13}, \Box F_{12}}{\bullet h_2 : \Box \Gamma_{11}, \Delta_{14} \vdash ((\Delta_{13}, \Box F_{12}), F_8 \vee F_9), F_7} \vee_R \quad \frac{h_{10} : \Box \Gamma_{11} \vdash F_{12}}{\bullet h_{10} : (\Box \Gamma_{11}, \Delta_{14}), F_7 \vdash (\Delta_{13}, \Box F_{12}), F_8 \vee F_9} A4 \\
\hline
\frac{}{- : \Box \Gamma_{11}, \Delta_{14} \vdash (\Delta_{13}, \Box F_{12}), F_8 \vee F_9} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Box \Gamma_{11} \vdash F_{12}} \text{ax/W} \\
\hline
\frac{}{- : \Delta_{14}, \Box \Gamma_{11} \vdash \Delta_{13}, \Box F_{12}, F_8 \vee F_9} A4
\end{array}$$

- Case rule  $\rightarrow_L$



$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash F_6, F_7, \Delta_{11}}{\bullet h_1 : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash F_9, \Delta_{11} \quad h_8 : F_{10}, \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \rightarrow F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \rightarrow_L}{- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{12} \vdash \Delta_{11}, F_6, F_7, F_9}{\bullet h_1 : \Delta_{12} \vdash \Delta_{11}, F_9, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}, F_9}{- : \Delta_{12} \vdash \Delta_{11}, F_9} \text{inv-th/ax}}{- : \Delta_{12} \vdash \Delta_{11}, F_9} \vee_R \quad \frac{h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{12}, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}}{- : \Delta_{12}, F_{10} \vdash \Delta_{11}} \text{ax/w}}{- : \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}} \text{hCut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash F_{12} \rightarrow F_{13}, F_8, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), F_{12} \rightarrow F_{13}} \vee_R \quad \frac{h_{10} : \Delta_{11} \vdash F_{12}, \Delta_7, F_8 \vee F_9 \quad h_{10} : F_{13}, \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8 \vee F_9} \rightarrow_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, F_{12} \rightarrow F_{13}}{- : \Delta_{11} \vdash \Delta_7, F_8, F_9} \text{ax/w} \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_{12}, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \rightarrow F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax}}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \vee_R \quad \frac{h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8, F_9}{- : \Delta_{11}, F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax}}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{hCut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash F_7, F_9, F_{10}, \Delta_8}{\bullet h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, \Delta_{14} \vdash F_{12}, \Delta_8, F_9 \vee F_{10} \quad h_{11} : F_7, F_{13}, \Delta_{14} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \rightarrow F_{13}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \rightarrow_L}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_7, F_9}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_7, F_9} \text{ax/w} \quad \frac{h_{11} : \Delta_{14}, F_7 \vdash \Delta_8, F_{10}, F_{12}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax}}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_{10}, F_9} \vee_R \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_{10}, F_9}{- : \Delta_{14}, F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax}}{- : \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash F_6, F_7, \Delta_{11}}{\bullet h_1 : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : F_9, F_{10}, \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \wedge F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \wedge_L}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{12}, F_{10}, F_9, F_6 \vee F_7 \vdash \Delta_{11}}{- : \Delta_{12}, F_{10}, F_9 \vdash \Delta_{11}} \text{inv-th/ax}}{- : \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}} \wedge_L \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash F_{12} \wedge F_{13}, F_8, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), F_{12} \wedge F_{13}} \vee_R \quad \frac{h_{10} : F_{12}, F_{13}, \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8 \vee F_9} \wedge_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, F_{12} \wedge F_{13}}{- : \Delta_{11} \vdash \Delta_7, F_8, F_9} \text{ax/w} \quad \frac{h_{10} : \Delta_{11}, F_{12}, F_{13} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax}}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \vee_R \quad \frac{h_{10} : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8, F_9}{- : \Delta_{11}, F_{12} \wedge F_{13} \vdash \Delta_7, F_8, F_9} \wedge_L}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{hCut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash F_7, F_9, F_{10}, \Delta_8}{\bullet h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, F_{12}, F_{13}, \Delta_{14} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \wedge F_{13}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \wedge_L}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_7, F_9}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_7, F_9} \text{ax/w} \quad \frac{h_{11} : \Delta_{14}, F_{12}, F_{13}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax}}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9} \vee_R \quad \frac{h_{11} : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_{10}, F_9} \wedge_L}{- : \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash F_6, F_7, \Delta_{11}}{\bullet h_1 : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : F_9, \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11} \quad h_8 : F_{10}, \Delta_{12}, F_6 \vee F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Delta_{12}, F_9 \vee F_{10}), F_6 \vee F_7 \vdash \Delta_{11}} \vee_L}{\neg : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6, F_7}{\bullet h_1 : \Delta_{12}, F_9 \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{\text{inv-th/ax}}{h_8 : \Delta_{12}, F_9, F_6 \vee F_7 \vdash \Delta_{11}} \quad \frac{\text{ax/W}}{\bullet h_1 : \Delta_{12}, F_{10} \vdash \Delta_{11}, F_6 \vee F_7} \vee_R \quad \frac{\text{inv-th/ax}}{h_8 : \Delta_{12}, F_{10}, F_6 \vee F_7 \vdash \Delta_{11}} \vee_L}{\neg : \Delta_{12}, F_9 \vdash \Delta_{11}} \text{hCut} \\
\rightarrow \\
\frac{\neg : \Delta_{12}, F_9 \vdash \Delta_{11}}{\neg : \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}} \vee_L \\
\\
\frac{\frac{h_1 : \Delta_7 \vdash F_8, F_9, \Delta_{10}}{\bullet h_1 : \Delta_7 \vdash \Delta_{10}, F_8 \vee F_9} \vee_R \quad \frac{h_6 : F_8, \Delta_7 \vdash \Delta_{10} \quad h_6 : F_9, \Delta_7 \vdash \Delta_{10}}{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{10}} \vee_L}{\neg : \Delta_7 \vdash \Delta_{10}} \text{Cut} \\
\rightarrow \\
\frac{\neg : \Delta_7 \vdash \Delta_{10}, F_8, F_9}{\neg : \Delta_7 \vdash \Delta_{10}, F_8} \text{ax/W} \quad \frac{\neg : \Delta_7, F_9 \vdash \Delta_{10}, F_8}{\neg : \Delta_7 \vdash \Delta_{10}, F_8} \text{sCut} \quad \frac{\neg : \Delta_7, F_8 \vdash \Delta_{10}}{\neg : \Delta_7 \vdash \Delta_{10}} \text{sCut} \\
\rightarrow \\
\frac{\neg : \Delta_7 \vdash \Delta_{10}}{\neg : \Delta_7 \vdash \Delta_{10}} \\
\\
\frac{\frac{h_2 : \Delta_{11} \vdash F_{12} \vee F_{13}, F_8, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), F_{12} \vee F_{13}} \vee_R \quad \frac{h_{10} : F_{12}, \Delta_{11} \vdash \Delta_7, F_8 \vee F_9 \quad h_{10} : F_{13}, \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \vee F_{13} \vdash \Delta_7, F_8 \vee F_9} \vee_L}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, F_{12} \vee F_{13}}{\bullet h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, F_{12} \vee F_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_{12} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \vee F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{11}, F_{13} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, F_{12} \vee F_{13} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax}}{\neg : \Delta_{11} \vdash \Delta_7, F_8, F_9} \vee_L}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{hCut} \\
\rightarrow \\
\frac{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \\
\\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash F_7, F_9, F_{10}, \Delta_8}{\bullet h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, F_{12}, \Delta_{14} \vdash \Delta_8, F_9 \vee F_{10} \quad h_{11} : F_7, F_{13}, \Delta_{14} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{14}, F_{12} \vee F_{13}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \vee_L}{\neg : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_7, F_9}{\bullet h_2 : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_7, F_9} \text{ax/W} \quad \frac{\frac{h_{11} : \Delta_{14}, F_{12}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_7, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_{14}, F_{13}, F_7 \vdash \Delta_8, F_{10}, F_9}{\bullet h_{11} : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9} \text{inv-th/ax}}{\neg : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_{10}, F_9} \vee_L}{\neg : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut} \\
\rightarrow \\
\frac{\neg : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \vee F_{10}}{\neg : \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_8, F_9 \vee F_{10}}
\end{array}$$

• Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_{11}, []F_9 \vdash F_6, F_7, \Delta_{10}}{\bullet h_1 : \Delta_{11}, []F_9 \vdash \Delta_{10}, F_6 \vee F_7} \vee_R \quad \frac{h_8 : F_9, \Delta_{11}, []F_9, F_6 \vee F_7 \vdash \Delta_{10}}{\bullet h_8 : (\Delta_{11}, []F_9), F_6 \vee F_7 \vdash \Delta_{10}} AT}{\neg : \Delta_{11}, []F_9 \vdash \Delta_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{11}, F_9, []F_9 \vdash \Delta_{10}, F_6 \vee F_7}{\bullet h_1 : \Delta_{11}, F_9, []F_9 \vdash \Delta_{10}, F_6 \vee F_7} \text{ax/W} \quad \frac{h_8 : \Delta_{11}, F_9, []F_9, F_6 \vee F_7 \vdash \Delta_{10}}{\bullet h_8 : \Delta_{11}, F_9, []F_9 \vdash \Delta_{10}, F_6 \vee F_7} \text{hCut}}{\neg : \Delta_{11}, F_9, []F_9 \vdash \Delta_{10}} AT \\
\rightarrow \\
\frac{\neg : \Delta_{11}, F_9, []F_9 \vdash \Delta_{10}}{\neg : \Delta_{11}, F_9, []F_9 \vdash \Delta_{10}} \\
\\
\frac{\frac{h_2 : \Delta_{11} \vdash []F_{12}, F_8, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), []F_{12}} \vee_R \quad \frac{h_{10} : F_{12}, \Delta_{11}, []F_{12} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, []F_{12} \vdash \Delta_7, F_8 \vee F_9} AT}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, []F_{12}}{\bullet h_2 : \Delta_{11} \vdash \Delta_7, F_8, F_9, []F_{12}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{11}, F_{12}, []F_{12} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, []F_{12} \vdash \Delta_7, F_8, F_9} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{11}, F_{12}, []F_{12} \vdash \Delta_7, F_8, F_9}{\bullet h_{10} : \Delta_{11}, []F_{12} \vdash \Delta_7, F_8, F_9} AT}}{\neg : \Delta_{11} \vdash \Delta_7, F_8, F_9} \vee_R \\
\rightarrow \\
\frac{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\neg : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \\
\\
\frac{\frac{h_2 : \Delta_{13}, []F_{12} \vdash F_7, F_9, F_{10}, \Delta_8}{\bullet h_2 : \Delta_{13}, []F_{12} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, F_{12}, \Delta_{13}, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\Delta_{13}, []F_{12}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} AT}{\neg : \Delta_{13}, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 : \Delta_{13}, F_{12}, []F_{12} \vdash \Delta_8, F_7, F_9 \vee F_{10}}{\bullet h_2 : \Delta_{13}, F_{12}, []F_{12} \vdash \Delta_8, F_7, F_9 \vee F_{10}} \text{ax/W} \quad \frac{h_{11} : \Delta_{13}, F_{12}, F_7, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : \Delta_{13}, F_{12}, F_7, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut}}{\neg : \Delta_{13}, F_{12}, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}} AT \\
\rightarrow \\
\frac{\neg : \Delta_{13}, F_{12}, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}}{\neg : \Delta_{13}, F_{12}, []F_{12} \vdash \Delta_8, F_9 \vee F_{10}}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \perp, \Delta_{10} \vdash F_6, F_7, \Delta_9}{\bullet h_1 : \perp, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7} \vee_R \quad \frac{}{\bullet h_8 : (\perp, \Delta_{10}), F_6 \vee F_7 \vdash \Delta_9} \perp_L}{\frac{}{- : \perp, \Delta_{10} \vdash \Delta_9} \text{Cut}} \xrightarrow{\quad} \\
\frac{}{- : \perp, \Delta_{10} \vdash \Delta_9} \perp_L \\
\\
\frac{\frac{h_2 : \Delta_{11} \vdash \perp, F_8, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), \perp} \vee_R \quad \frac{}{\bullet h_{10} : \Delta_{11}, \perp \vdash \Delta_7, F_8 \vee F_9} \perp_L}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{Cut}} \xrightarrow{\quad} \\
\frac{\frac{h_2 : \Delta_{11} \vdash \perp, \Delta_7, F_8, F_9}{\bullet h_2 : \Delta_{11} \vdash \perp, \Delta_7, F_8, F_9} \text{ax/W} \quad \frac{}{\bullet h_{10} : \perp, \Delta_{11} \vdash \Delta_7, F_8, F_9} \perp_L}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8, F_9} \text{hCut}} \vee_R \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \vee_R \\
\\
\frac{\frac{h_2 : \perp, \Delta_{12} \vdash F_7, F_9, F_{10}, \Delta_8}{\bullet h_2 : \perp, \Delta_{12} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{}{\bullet h_{11} : (\perp, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \perp_L}{\frac{}{- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut}} \xrightarrow{\quad} \\
\frac{}{- : \perp, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10}} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_1 : \Delta_{11}, p_9 \vdash F_6, F_7, \Delta_{10}, p_9}{\bullet h_1 : \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), F_6 \vee F_7} \vee_R \quad \frac{}{\bullet h_8 : (\Delta_{11}, p_9), F_6 \vee F_7 \vdash \Delta_{10}, p_9} I}{\frac{}{- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9} \text{Cut}} \xrightarrow{\quad} \\
\frac{}{- : \Delta_{11}, p_9 \vdash \Delta_{10}, p_9} I \\
\\
\frac{\frac{h_2 : \Delta_{10} \vdash p_{11}, F_7, F_8, \Delta_{12}, p_{11}}{\bullet h_2 : \Delta_{10} \vdash ((\Delta_{12}, p_{11}), F_7 \vee F_8), p_{11}} \vee_R \quad \frac{}{\bullet h_9 : \Delta_{10}, p_{11} \vdash (\Delta_{12}, p_{11}), F_7 \vee F_8} I}{\frac{}{- : \Delta_{10} \vdash (\Delta_{12}, p_{11}), F_7 \vee F_8} \text{Cut}} \xrightarrow{\quad} \\
\frac{\frac{h_2 : \Delta_{10} \vdash \Delta_{12}, F_7, F_8, p_{11}, p_{11}}{\bullet h_2 : \Delta_{10} \vdash \Delta_{12}, F_7, F_8, p_{11}, p_{11}} \text{ax/W} \quad \frac{}{\bullet h_9 : \Delta_{10}, p_{11} \vdash \Delta_{12}, F_7, F_8, p_{11}} I}{\frac{}{- : \Delta_{10} \vdash \Delta_{12}, F_7, F_8, p_{11}} \text{hCut}} \vee_R \\
\frac{}{- : \Delta_{10} \vdash \Delta_{12}, p_{11}, F_7 \vee F_8} \vee_R \\
\\
\frac{\frac{h_2 : \Delta_{13}, p_{11} \vdash F_7, F_8, F_9, \Delta_{12}, p_{11}}{\bullet h_2 : \Delta_{13}, p_{11} \vdash ((\Delta_{12}, p_{11}), F_8 \vee F_9), F_7} \vee_R \quad \frac{}{\bullet h_{10} : (\Delta_{13}, p_{11}), F_7 \vdash (\Delta_{12}, p_{11}), F_8 \vee F_9} I}{\frac{}{- : \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), F_8 \vee F_9} \text{Cut}} \xrightarrow{\quad} \\
\frac{}{- : \Delta_{13}, p_{11} \vdash \Delta_{12}, p_{11}, F_8 \vee F_9} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{h_1 : \top, \Delta_{10} \vdash F_6, F_7, \Delta_9}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7} \vee_R \quad \frac{h_8 : \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9}{\bullet h_8 : (\top, \Delta_{10}), F_6 \vee F_7 \vdash \Delta_9} \top_L}{\frac{}{- : \top, \Delta_{10} \vdash \Delta_9} \text{Cut}} \xrightarrow{\quad} \\
\frac{\frac{h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7}{\bullet h_1 : \top, \Delta_{10} \vdash \Delta_9, F_6 \vee F_7} \text{ax/W} \quad \frac{h_8 : \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9}{\bullet h_8 : \top, \Delta_{10}, F_6 \vee F_7 \vdash \Delta_9} \text{ax/W}}{\frac{}{- : \top, \Delta_{10} \vdash \Delta_9} \text{hCut}} \\
\\
\frac{\frac{h_2 : \Delta_{11} \vdash \top, F_8, F_9, \Delta_7}{\bullet h_2 : \Delta_{11} \vdash (\Delta_7, F_8 \vee F_9), \top} \vee_R \quad \frac{h_{10} : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9}{\bullet h_{10} : \Delta_{11}, \top \vdash \Delta_7, F_8 \vee F_9} \top_L}{\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{Cut}} \xrightarrow{\quad} \\
\frac{}{- : \Delta_{11} \vdash \Delta_7, F_8 \vee F_9} \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \top, \Delta_{12} \vdash F_7, F_9, F_{10}, \Delta_8}{\bullet h_2 : \top, \Delta_{12} \vdash (\Delta_8, F_9 \vee F_{10}), F_7} \vee_R \quad \frac{h_{11} : F_7, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10}}{\bullet h_{11} : (\top, \Delta_{12}), F_7 \vdash \Delta_8, F_9 \vee F_{10}} \top_L \\
\hline
- : \top, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\bullet h_2 : \top, \Delta_{12} \vdash \Delta_8, F_7, F_9 \vee F_{10} \quad \text{ax/W} \quad \frac{h_{11} : \top, \Delta_{12}, F_7 \vdash \Delta_8, F_9 \vee F_{10} \quad \text{ax/W}}{h_{11} : \top, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10}} \text{hCut}}{- : \top, \Delta_{12} \vdash \Delta_8, F_9 \vee F_{10}}
\end{array}$$

#### 8.4 Status of $\perp_R$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \rightarrow F_8), \perp} \perp_R \quad \frac{h_5 : \perp, F_7, \Delta_4 \vdash F_8, \Delta_6}{\bullet h_5 : \Delta_4, \perp \vdash \Delta_6, F_7 \rightarrow F_8} \rightarrow_R \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8 \quad \text{ax/W}}{- : \Delta_4 \vdash \Delta_6, F_7 \rightarrow F_8} \\
\hline
\frac{h_2 : \Delta_6 \vdash F_5, \Delta_{10}, F_8 \rightarrow F_9}{\bullet h_2 : \Delta_6 \vdash (\perp, \Delta_{10}, F_8 \rightarrow F_9), F_5} \perp_R \quad \frac{h_7 : F_5, F_8, \Delta_6 \vdash \perp, F_9, \Delta_{10}}{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9} \rightarrow_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_2 : \Delta_6 \vdash \perp, \Delta_{10}, F_5, F_8 \rightarrow F_9 \quad \text{ax/W} \quad \frac{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9 \quad \text{ax/W}}{h_{11} : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9} \text{hCut}}{- : \Delta_6 \vdash \perp, \Delta_{10}, F_8 \rightarrow F_9}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \wedge F_8), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash F_7, \Delta_6 \quad h_5 : \perp, \Delta_4 \vdash F_8, \Delta_6}{\bullet h_5 : \Delta_4, \perp \vdash \Delta_6, F_7 \wedge F_8} \wedge_R \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8 \quad \text{ax/W}}{- : \Delta_4 \vdash \Delta_6, F_7 \wedge F_8} \\
\hline
\frac{h_2 : \Delta_6 \vdash F_5, \Delta_{10}, F_8 \wedge F_9}{\bullet h_2 : \Delta_6 \vdash (\perp, \Delta_{10}, F_8 \wedge F_9), F_5} \perp_R \quad \frac{h_7 : F_5, \Delta_6 \vdash \perp, F_8, \Delta_{10} \quad h_7 : F_5, \Delta_6 \vdash \perp, F_9, \Delta_{10}}{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \wedge F_9} \wedge_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10}, F_8 \wedge F_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_2 : \Delta_6 \vdash \perp, \Delta_{10}, F_5, F_8 \wedge F_9 \quad \text{ax/W} \quad \frac{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \wedge F_9 \quad \text{ax/W}}{h_{11} : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \wedge F_9} \text{hCut}}{- : \Delta_6 \vdash \perp, \Delta_{10}, F_8 \wedge F_9}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \Delta_6, F_7 \vee F_8}{\bullet h_1 : \Delta_4 \vdash (\Delta_6, F_7 \vee F_8), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash F_7, F_8, \Delta_6}{\bullet h_5 : \Delta_4, \perp \vdash \Delta_6, F_7 \vee F_8} \vee_R \\
\hline
- : \Delta_4 \vdash \Delta_6, F_7 \vee F_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_4 \vdash \Delta_6, F_7 \vee F_8 \quad \text{ax/W}}{- : \Delta_4 \vdash \Delta_6, F_7 \vee F_8} \\
\hline
\frac{h_2 : \Delta_6 \vdash F_5, \Delta_{10}, F_8 \vee F_9}{\bullet h_2 : \Delta_6 \vdash (\perp, \Delta_{10}, F_8 \vee F_9), F_5} \perp_R \quad \frac{h_7 : F_5, \Delta_6 \vdash \perp, F_8, F_9, \Delta_{10}}{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \vee_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_{10}, F_8 \vee F_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_2 : \Delta_6 \vdash \perp, \Delta_{10}, F_5, F_8 \vee F_9 \quad \text{ax/W} \quad \frac{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \vee F_9 \quad \text{ax/W}}{h_{11} : \Delta_6, F_5 \vdash \perp, \Delta_{10}, F_8 \vee F_9} \text{hCut}}{- : \Delta_6 \vdash \perp, \Delta_{10}, F_8 \vee F_9}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \perp, \Delta_6}{\bullet h_1 : \Delta_4 \vdash (\perp, \Delta_6), \perp} \perp_R \quad \frac{h_5 : \perp, \Delta_4 \vdash \Delta_6}{\bullet h_5 : \Delta_4, \perp \vdash \perp, \Delta_6} \perp_R \\
\hline
- : \Delta_4 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_4 \vdash \perp, \Delta_6 \quad \text{ax/W}}{- : \Delta_4 \vdash \perp, \Delta_6}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_6 \vdash F_5, \Delta_8}{\bullet h_2 : \Delta_6 \vdash (\perp, \Delta_8), F_5} \perp_R \quad \frac{h_7 : F_5, \Delta_6 \vdash \Delta_8}{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_8} \perp_R \\
\hline
- : \Delta_6 \vdash \perp, \Delta_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_2 : \Delta_6 \vdash \perp, \Delta_8, F_5}{- : \Delta_6 \vdash \perp, \Delta_8} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \Delta_8}{- : \Delta_6 \vdash \perp, \Delta_8} \text{ax/W} \\
\hline
\text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_1 : \Delta_4 \vdash \top, \Delta_6}{\bullet h_1 : \Delta_4 \vdash (\top, \Delta_6), \perp} \perp_R \quad \frac{}{\bullet h_5 : \Delta_4, \perp \vdash \top, \Delta_6} \top_R \\
\hline
- : \Delta_4 \vdash \top, \Delta_6 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{}{- : \Delta_4 \vdash \top, \Delta_6} \top_R \\
\hline
\frac{h_2 : \Delta_6 \vdash F_5, \top, \Delta_8}{\bullet h_2 : \Delta_6 \vdash (\perp, \top, \Delta_8), F_5} \perp_R \quad \frac{}{\bullet h_7 : \Delta_6, F_5 \vdash \perp, \top, \Delta_8} \top_R \\
\hline
- : \Delta_6 \vdash \perp, \top, \Delta_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{}{- : \Delta_6 \vdash \perp, \top, \Delta_8} \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_5, \Delta_8 \vdash \Delta_6, [\Box F_7]}{\bullet h_1 : \Box \Gamma_5, \Delta_8 \vdash (\Delta_6, [\Box F_7]), \perp} \perp_R \quad \frac{h_4 : \Box \Gamma_5 \vdash F_7}{\bullet h_4 : (\Box \Gamma_5, \Delta_8), \perp \vdash \Delta_6, [\Box F_7]} A4 \\
\hline
- : \Box \Gamma_5, \Delta_8 \vdash \Delta_6, [\Box F_7] \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{}{- : \Delta_8, \Box \Gamma_5 \vdash \Delta_6, [\Box F_7]} \text{ax/W} \\
\hline
\frac{h_2 : \Box \Gamma_{10}, \Delta_7 \vdash \Box F_5, \Delta_9, [\Box F_8]}{\bullet h_2 : \Box \Gamma_{10}, \Delta_7 \vdash (\perp, \Delta_9, [\Box F_8]), \Box F_5} \perp_R \quad \frac{h_6 : \Box \Gamma_{10}, \Box F_5 \vdash F_8}{\bullet h_6 : (\Box \Gamma_{10}, \Delta_7), \Box F_5 \vdash \perp, \Delta_9, [\Box F_8]} A4 \\
\hline
- : \Box \Gamma_{10}, \Delta_7 \vdash \perp, \Delta_9, [\Box F_8] \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_2 : \Delta_7, \Box \Gamma_{10} \vdash \perp, \Box F_5, \Delta_9, [\Box F_8]}{- : \Delta_7, \Box \Gamma_{10} \vdash \perp, \Delta_9, [\Box F_8]} \text{ax/W} \quad \frac{\bullet h_6 : \Box F_5, \Delta_7, \Box \Gamma_{10} \vdash \perp, \Delta_9, [\Box F_8]}{- : \Delta_7, \Box \Gamma_{10} \vdash \perp, \Delta_9, [\Box F_8]} \text{ax/W} \\
\hline
\text{hCut} \\
\hline
\frac{h_2 : \Box \Gamma_7, \Delta_{10} \vdash F_5, \Delta_9, [\Box F_8]}{\bullet h_2 : \Box \Gamma_7, \Delta_{10} \vdash (\perp, \Delta_9, [\Box F_8]), F_5} \perp_R \quad \frac{h_6 : \Box \Gamma_7 \vdash F_8}{\bullet h_6 : (\Box \Gamma_7, \Delta_{10}), F_5 \vdash \perp, \Delta_9, [\Box F_8]} A4 \\
\hline
- : \Box \Gamma_7, \Delta_{10} \vdash \perp, \Delta_9, [\Box F_8] \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{}{- : \Box \Gamma_7 \vdash F_8} \text{ax/W} \\
\hline
\frac{}{- : \Delta_{10}, \Box \Gamma_7 \vdash \perp, \Delta_9, [\Box F_8]} A4
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_1 : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7, \perp} \perp_R \quad \frac{h_4 : \perp, \Delta_8 \vdash F_5, \Delta_7 \quad h_4 : \perp, F_6, \Delta_8 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \rightarrow F_6), \perp \vdash \Delta_7} \rightarrow_L \\
\hline
- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \text{ax/W} \\
\hline
\frac{h_2 : \Delta_7 \vdash F_8 \rightarrow F_9, \Delta_5}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), F_8 \rightarrow F_9} \perp_R \quad \frac{h_6 : \Delta_7 \vdash \perp, F_8, \Delta_5 \quad h_6 : F_9, \Delta_7 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \perp, \Delta_5} \rightarrow_L \\
\hline
- : \Delta_7 \vdash \perp, \Delta_5 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \rightarrow F_9}{- : \Delta_7 \vdash \perp, \Delta_5} \text{ax/W} \quad \frac{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \perp, \Delta_5}{- : \Delta_7 \vdash \perp, \Delta_5} \text{ax/W} \\
\hline
\text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash F_5, \Delta_6}{\bullet h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : F_5, \Delta_{10} \vdash \perp, F_8, \Delta_6 \quad h_7 : F_5, F_9, \Delta_{10} \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \rightarrow F_9), F_5 \vdash \perp, \Delta_6} \rightarrow_L \\
\hline
- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6, F_5}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_{10}, F_5, F_8 \rightarrow F_9 \vdash \perp, \Delta_6}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6} \text{ax/W}}{- : \Delta_{10}, F_8 \rightarrow F_9 \vdash \perp, \Delta_6} \text{hCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7, \perp} \perp_R \quad \frac{h_4 : \perp, F_5, F_6, \Delta_8 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \wedge F_6), \perp \vdash \Delta_7} \wedge_L \\
\hline
- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7}{- : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \text{ax/W} \\
\hline
\frac{h_2 : \Delta_7 \vdash F_8 \wedge F_9, \Delta_5}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), F_8 \wedge F_9} \perp_R \quad \frac{h_6 : F_8, F_9, \Delta_7 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \perp, \Delta_5} \wedge_L \\
\hline
- : \Delta_7 \vdash \perp, \Delta_5 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \wedge F_9}{- : \Delta_7 \vdash \perp, \Delta_5} \text{ax/W} \quad \frac{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \perp, \Delta_5}{- : \Delta_7 \vdash \perp, \Delta_5} \text{ax/W}}{- : \Delta_7 \vdash \perp, \Delta_5} \text{hCut} \\
\hline
\frac{h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash F_5, \Delta_6}{\bullet h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : F_5, F_8, F_9, \Delta_{10} \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \wedge F_9), F_5 \vdash \perp, \Delta_6} \wedge_L \\
\hline
- : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6, F_5}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_{10}, F_5, F_8 \wedge F_9 \vdash \perp, \Delta_6}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6} \text{ax/W}}{- : \Delta_{10}, F_8 \wedge F_9 \vdash \perp, \Delta_6} \text{hCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{\bullet h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7, \perp} \perp_R \quad \frac{h_4 : \perp, F_5, \Delta_8 \vdash \Delta_7 \quad h_4 : \perp, F_6, \Delta_8 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \vee F_6), \perp \vdash \Delta_7} \vee_L \\
\hline
- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7}{- : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \text{ax/W} \\
\hline
\frac{h_2 : \Delta_7 \vdash F_8 \vee F_9, \Delta_5}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), F_8 \vee F_9} \perp_R \quad \frac{h_6 : F_8, \Delta_7 \vdash \perp, \Delta_5 \quad h_6 : F_9, \Delta_7 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \perp, \Delta_5} \vee_L \\
\hline
- : \Delta_7 \vdash \perp, \Delta_5 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, F_8 \vee F_9}{- : \Delta_7 \vdash \perp, \Delta_5} \text{ax/W} \quad \frac{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \perp, \Delta_5}{- : \Delta_7 \vdash \perp, \Delta_5} \text{ax/W}}{- : \Delta_7 \vdash \perp, \Delta_5} \text{hCut} \\
\hline
\frac{h_2 : \Delta_{10}, F_8 \vee F_9 \vdash F_5, \Delta_6}{\bullet h_2 : \Delta_{10}, F_8 \vee F_9 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : F_5, F_8, \Delta_{10} \vdash \perp, \Delta_6 \quad h_7 : F_5, F_9, \Delta_{10} \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \vee F_9), F_5 \vdash \perp, \Delta_6} \vee_L \\
\hline
- : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\frac{h_2 : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6, F_5}{- : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6} \text{ax/W} \quad \frac{\bullet h_7 : \Delta_{10}, F_5, F_8 \vee F_9 \vdash \perp, \Delta_6}{- : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6} \text{ax/W}}{- : \Delta_{10}, F_8 \vee F_9 \vdash \perp, \Delta_6} \text{hCut}
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{h_1 : \Delta_7, \boxed{F_5} \vdash \Delta_6}{\bullet h_1 : \Delta_7, \boxed{F_5} \vdash \Delta_6, \perp} \perp_R \quad \frac{h_4 : \perp, F_5, \Delta_7, \boxed{F_5} \vdash \Delta_6}{\bullet h_4 : (\Delta_7, \boxed{F_5}), \perp \vdash \Delta_6} AT \\
\hline
- : \Delta_7, \boxed{F_5} \vdash \Delta_6 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{- : \Delta_7, \boxed{F_5} \vdash \Delta_6}{- : \Delta_7, \boxed{F_5} \vdash \Delta_6} \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash \Box F_8, \Delta_5}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), \Box F_8} \perp_R \quad \frac{h_6 : F_8, \Delta_7, \Box F_8 \vdash \perp, \Delta_5}{\bullet h_6 : \Delta_7, \Box F_8 \vdash \perp, \Delta_5} AT \\
\hline
- : \Delta_7 \vdash \perp, \Delta_5 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5, \Box F_8}{- : \Delta_7 \vdash \perp, \Delta_5} ax/W \quad \frac{\bullet h_6 : \Delta_7, \Box F_8 \vdash \perp, \Delta_5}{- : \Delta_7 \vdash \perp, \Delta_5} hCut
\end{array}$$
  

$$\begin{array}{c}
\frac{h_2 : \Delta_9, \Box F_8 \vdash F_5, \Delta_6}{\bullet h_2 : \Delta_9, \Box F_8 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{h_7 : F_5, F_8, \Delta_9, \Box F_8 \vdash \perp, \Delta_6}{\bullet h_7 : (\Delta_9, \Box F_8), F_5 \vdash \perp, \Delta_6} AT \\
\hline
- : \Delta_9, \Box F_8 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{h_2 : \Delta_9, \Box F_8 \vdash \perp, \Delta_6, F_5}{- : \Delta_9, \Box F_8 \vdash \perp, \Delta_6} ax/W \quad \frac{\bullet h_7 : \Delta_9, F_5, \Box F_8 \vdash \perp, \Delta_6}{- : \Delta_9, \Box F_8 \vdash \perp, \Delta_6} hCut
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{h_1 : \Delta_5 \vdash \Delta_6}{\bullet h_1 : \Delta_5 \vdash \Delta_6, \perp} \perp_R \quad \frac{}{\bullet h_4 : \Delta_5, \perp \vdash \Delta_6} \perp_L \\
\hline
- : \Delta_5 \vdash \Delta_6 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_5 \vdash \Delta_6} ax/W
\end{array}$$
  

$$\begin{array}{c}
\frac{h_2 : \Delta_7 \vdash \perp, \Delta_5}{\bullet h_2 : \Delta_7 \vdash (\perp, \Delta_5), \perp} \perp_R \quad \frac{}{\bullet h_6 : \Delta_7, \perp \vdash \perp, \Delta_5} \perp_L \\
\hline
- : \Delta_7 \vdash \perp, \Delta_5 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_7 \vdash \perp, \Delta_5} ax/W
\end{array}$$
  

$$\begin{array}{c}
\frac{h_2 : \perp, \Delta_8 \vdash F_5, \Delta_6}{\bullet h_2 : \perp, \Delta_8 \vdash (\perp, \Delta_6), F_5} \perp_R \quad \frac{}{\bullet h_7 : (\perp, \Delta_8), F_5 \vdash \perp, \Delta_6} \perp_L \\
\hline
- : \perp, \Delta_8 \vdash \perp, \Delta_6 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \perp, \Delta_8 \vdash \perp, \Delta_6} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_1 : \Delta_7, p_5 \vdash \Delta_6, p_5}{\bullet h_1 : \Delta_7, p_5 \vdash (\Delta_6, p_5), \perp} \perp_R \quad \frac{}{\bullet h_4 : (\Delta_7, p_5), \perp \vdash \Delta_6, p_5} I \\
\hline
- : \Delta_7, p_5 \vdash \Delta_6, p_5 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} I
\end{array}$$
  

$$\begin{array}{c}
\frac{h_2 : \Delta_6 \vdash p_7, \Delta_8, p_7}{\bullet h_2 : \Delta_6 \vdash (\perp, \Delta_8, p_7), p_7} \perp_R \quad \frac{}{\bullet h_5 : \Delta_6, p_7 \vdash \perp, \Delta_8, p_7} I \\
\hline
- : \Delta_6 \vdash \perp, \Delta_8, p_7 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{h_2 : \Delta_6 \vdash \perp, \Delta_8, p_7, p_7}{- : \Delta_6 \vdash \perp, \Delta_8, p_7} ax/W \quad \frac{\bullet h_5 : \Delta_6, p_7 \vdash \perp, \Delta_8, p_7}{- : \Delta_6 \vdash \perp, \Delta_8, p_7} hCut
\end{array}$$
  

$$\begin{array}{c}
\frac{h_2 : \Delta_9, p_7 \vdash F_5, \Delta_8, p_7}{\bullet h_2 : \Delta_9, p_7 \vdash (\perp, \Delta_8, p_7), F_5} \perp_R \quad \frac{}{\bullet h_6 : (\Delta_9, p_7), F_5 \vdash \perp, \Delta_8, p_7} I \\
\hline
- : \Delta_9, p_7 \vdash \perp, \Delta_8, p_7 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \Delta_9, p_7 \vdash \perp, \Delta_8, p_7} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_1 : \top, \Delta_6 \vdash \Delta_5}{\bullet h_1 : \top, \Delta_6 \vdash \Delta_5, \perp} \perp_R \quad \frac{h_4 : \perp, \Delta_6 \vdash \Delta_5}{\bullet h_4 : (\top, \Delta_6), \perp \vdash \Delta_5} \top_L \\
\hline
- : \top, \Delta_6 \vdash \Delta_5 \quad \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\frac{}{- : \top, \Delta_6 \vdash \Delta_5} ax/W
\end{array}$$





- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Delta_4 \vdash (\perp, \Delta_6), \top} \top_R \quad \frac{h_5 : \top, \Delta_4 \vdash \Delta_6}{\bullet h_5 : \Delta_4, \top \vdash \perp, \Delta_6} \perp_R}{- : \Delta_4 \vdash \perp, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_4 \vdash \perp, \top, \Delta_6}{- : \Delta_4 \vdash \perp, \Delta_6} \text{ax/W} \quad \frac{h_5 : \top, \Delta_4 \vdash \perp, \Delta_6}{\bullet h_5 : \Delta_4, \top \vdash \perp, \Delta_6} \text{hCut}}{- : \Delta_4 \vdash \perp, \Delta_6} \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \Delta_6 \vdash (\top, \perp, \Delta_8), F_5} \top_R \quad \frac{h_7 : F_5, \Delta_6 \vdash \top, \Delta_8}{\bullet h_7 : \Delta_6, F_5 \vdash \top, \perp, \Delta_8} \perp_R}{- : \Delta_6 \vdash \top, \perp, \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_6 \vdash \perp, \top, \Delta_8} \top_R
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Delta_4 \vdash (\top, \Delta_6), \top} \top_R \quad \frac{h_5 : \top, \Delta_4 \vdash \top, \Delta_6}{\bullet h_5 : \Delta_4, \top \vdash \top, \Delta_6} \top_R}{- : \Delta_4 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_4 \vdash \top, \Delta_6} \top_R \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \Delta_6 \vdash (\top, \Delta_8), F_5} \top_R \quad \frac{h_7 : \Delta_6, F_5 \vdash \top, \Delta_8}{\bullet h_7 : \Delta_6, F_5 \vdash \top, \Delta_8} \top_R}{- : \Delta_6 \vdash \top, \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_6 \vdash \top, \Delta_8} \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Box \Gamma_5, \Delta_8 \vdash (\Delta_6, \llbracket F_7 \rrbracket), \top} \top_R \quad \frac{h_4 : \Box \Gamma_5 \vdash F_7}{\bullet h_4 : (\Box \Gamma_5, \Delta_8), \top \vdash \Delta_6, \llbracket F_7 \rrbracket} A4}{- : \Box \Gamma_5, \Delta_8 \vdash \Delta_6, \llbracket F_7 \rrbracket} \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_5 \vdash F_7} \text{ax/W}}{- : \Delta_8, \Box \Gamma_5 \vdash \Delta_6, \llbracket F_7 \rrbracket} A4 \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \Box \Gamma_{10}, \Delta_7 \vdash (\top, \Delta_9, \llbracket F_8 \rrbracket), \Box F_5} \top_R \quad \frac{h_6 : \Box \Gamma_{10}, \Box F_5 \vdash F_8}{\bullet h_6 : (\Box \Gamma_{10}, \Delta_7), \Box F_5 \vdash \top, \Delta_9, \llbracket F_8 \rrbracket} A4}{- : \Box \Gamma_{10}, \Delta_7 \vdash \top, \Delta_9, \llbracket F_8 \rrbracket} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_7, \Box \Gamma_{10} \vdash \top, \Delta_9, \llbracket F_8 \rrbracket} \top_R \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \Box \Gamma_7, \Delta_{10} \vdash (\top, \Delta_9, \llbracket F_8 \rrbracket), F_5} \top_R \quad \frac{h_6 : \Box \Gamma_7 \vdash F_8}{\bullet h_6 : (\Box \Gamma_7, \Delta_{10}), F_5 \vdash \top, \Delta_9, \llbracket F_8 \rrbracket} A4}{- : \Box \Gamma_7, \Delta_{10} \vdash \top, \Delta_9, \llbracket F_8 \rrbracket} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_{10}, \Box \Gamma_7 \vdash \top, \Delta_9, \llbracket F_8 \rrbracket} \top_R
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7, \top} \top_R \quad \frac{h_4 : \top, \Delta_8 \vdash F_5, \Delta_7 \quad h_4 : \top, F_6, \Delta_8 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \rightarrow F_6), \top \vdash \Delta_7} \rightarrow_L}{- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7} \text{Cut} \\
\rightarrow \\
\frac{\frac{\frac{}{\bullet h_1 : \Delta_8 \vdash \top, \Delta_7, F_5} \top_R \quad \frac{h_4 : \top, \Delta_8 \vdash \Delta_7, F_5}{\bullet h_4 : \Delta_8, F_6 \vdash \top, \Delta_7} \text{ax/W}}{- : \Delta_8 \vdash \Delta_7, F_5} \text{hCut} \quad \frac{\frac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : \Delta_8, F_6 \vdash \top, \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\bullet h_4 : \Delta_8, F_6 \vdash \top, \Delta_7} \text{ax/W}}{- : \Delta_8, F_6 \vdash \Delta_7} \text{hCut} \\
\rightarrow \\
- : \Delta_8, F_5 \rightarrow F_6 \vdash \Delta_7 \rightarrow_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), F_8 \rightarrow F_9}{\vdash : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{h_6 : \Delta_7 \vdash \top, F_8, \Delta_5 \quad h_6 : F_9, \Delta_7 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \rightarrow F_9 \vdash \top, \Delta_5} \rightarrow_L}{\vdash : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_7 \vdash \top, \Delta_5}{\vdash : \Delta_7 \vdash \top, \Delta_5} \top_R \\
\\
\frac{\frac{\bullet h_2 : \Delta_{10}, F_8 \rightarrow F_9 \vdash (\top, \Delta_6), F_5}{\vdash : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \top_R \quad \frac{h_7 : F_5, \Delta_{10} \vdash \top, F_8, \Delta_6 \quad h_7 : F_5, F_9, \Delta_{10} \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \rightarrow F_9), F_5 \vdash \top, \Delta_6} \rightarrow_L}{\vdash : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6}{\vdash : \Delta_{10}, F_8 \rightarrow F_9 \vdash \top, \Delta_6} \top_R
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7, \top}{\vdash : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, F_5, F_6, \Delta_8 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \wedge F_6), \top \vdash \Delta_7} \wedge_L}{\vdash : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_8, F_5, F_6 \vdash \top, \Delta_7}{\vdash : \Delta_8, F_5, F_6 \vdash \top, \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_5, F_6 \vdash \Delta_7}{\vdash : \Delta_8, F_5, F_6 \vdash \Delta_7} \text{ax/W}}{\vdash : \Delta_8, F_5 \wedge F_6 \vdash \Delta_7} \text{hCut} \\
\wedge_L \\
\\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), F_8 \wedge F_9}{\vdash : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{h_6 : F_8, F_9, \Delta_7 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \wedge F_9 \vdash \top, \Delta_5} \wedge_L}{\vdash : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_7 \vdash \top, \Delta_5}{\vdash : \Delta_7 \vdash \top, \Delta_5} \top_R \\
\\
\frac{\frac{\bullet h_2 : \Delta_{10}, F_8 \wedge F_9 \vdash (\top, \Delta_6), F_5}{\vdash : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \top_R \quad \frac{h_7 : F_5, F_8, F_9, \Delta_{10} \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \wedge F_9), F_5 \vdash \top, \Delta_6} \wedge_L}{\vdash : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6}{\vdash : \Delta_{10}, F_8 \wedge F_9 \vdash \top, \Delta_6} \top_R
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \vee F_6 \vdash \Delta_7, \top}{\vdash : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \top_R \quad \frac{h_4 : \top, F_5, \Delta_8 \vdash \Delta_7 \quad h_4 : \top, F_6, \Delta_8 \vdash \Delta_7}{\bullet h_4 : (\Delta_8, F_5 \vee F_6), \top \vdash \Delta_7} \vee_L}{\vdash : \Delta_8, F_5 \vee F_6 \vdash \Delta_7} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_8, F_5 \vdash \top, \Delta_7}{\vdash : \Delta_8, F_5 \vdash \top, \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_5 \vdash \Delta_7}{\vdash : \Delta_8, F_5 \vdash \Delta_7} \text{ax/W}}{\vdash : \Delta_8, F_5 \vdash \top, \Delta_7} \text{hCut} \\
\frac{\frac{\bullet h_1 : \Delta_8, F_6 \vdash \top, \Delta_7}{\vdash : \Delta_8, F_6 \vdash \top, \Delta_7} \top_R \quad \frac{h_4 : \top, \Delta_8, F_6 \vdash \Delta_7}{\vdash : \Delta_8, F_6 \vdash \Delta_7} \text{ax/W}}{\vdash : \Delta_8, F_6 \vdash \top, \Delta_7} \text{hCut} \\
\vee_L \\
\vdash : \Delta_8, F_5 \vee F_6 \vdash \Delta_7 \\
\\
\frac{\frac{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), F_8 \vee F_9}{\vdash : \Delta_7 \vdash \top, \Delta_5} \top_R \quad \frac{h_6 : F_8, \Delta_7 \vdash \top, \Delta_5 \quad h_6 : F_9, \Delta_7 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, F_8 \vee F_9 \vdash \top, \Delta_5} \vee_L}{\vdash : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_7 \vdash \top, \Delta_5}{\vdash : \Delta_7 \vdash \top, \Delta_5} \top_R \\
\\
\frac{\frac{\bullet h_2 : \Delta_{10}, F_8 \vee F_9 \vdash (\top, \Delta_6), F_5}{\vdash : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \top_R \quad \frac{h_7 : F_5, F_8, \Delta_{10} \vdash \top, \Delta_6 \quad h_7 : F_5, F_9, \Delta_{10} \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_{10}, F_8 \vee F_9), F_5 \vdash \top, \Delta_6} \vee_L}{\vdash : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6}{\vdash : \Delta_{10}, F_8 \vee F_9 \vdash \top, \Delta_6} \top_R
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_1 : \Delta_7, []F_5 \vdash \Delta_6, \top} \top_R \quad \frac{h_4 : \top, F_5, \Delta_7, []F_5 \vdash \Delta_6}{\bullet h_4 : (\Delta_7, []F_5), \top \vdash \Delta_6} AT}{- : \Delta_7, []F_5 \vdash \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{\frac{}{\bullet h_1 : \Delta_7, F_5, []F_5 \vdash \top, \Delta_6} \text{ax/W} \quad \frac{h_4 : \top, \Delta_7, F_5, []F_5 \vdash \Delta_6}{\bullet h_4 : (\Delta_7, []F_5), \top \vdash \Delta_6} \text{ax/W}}{- : \Delta_7, F_5, []F_5 \vdash \Delta_6} \text{hCut} \\
\frac{}{- : \Delta_7, []F_5 \vdash \Delta_6} AT
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), []F_8} \top_R \quad \frac{h_6 : F_8, \Delta_7, []F_8 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, []F_8 \vdash \top, \Delta_5} AT}{- : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \top_R
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_2 : \Delta_9, []F_8 \vdash (\top, \Delta_6), F_5} \top_R \quad \frac{h_7 : F_5, F_8, \Delta_9, []F_8 \vdash \top, \Delta_6}{\bullet h_7 : (\Delta_9, []F_8), F_5 \vdash \top, \Delta_6} AT}{- : \Delta_9, []F_8 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_9, []F_8 \vdash \top, \Delta_6} \top_R
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_1 : \perp, \Delta_6 \vdash \Delta_5, \top} \top_R \quad \frac{}{\bullet h_4 : (\perp, \Delta_6), \top \vdash \Delta_5} \perp_L}{- : \perp, \Delta_6 \vdash \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{}{- : \perp, \Delta_6 \vdash \Delta_5} \perp_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), \perp} \top_R \quad \frac{}{\bullet h_6 : \Delta_7, \perp \vdash \top, \Delta_5} \perp_L}{- : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \top_R
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_2 : \perp, \Delta_8 \vdash (\top, \Delta_6), F_5} \top_R \quad \frac{}{\bullet h_7 : (\perp, \Delta_8), F_5 \vdash \top, \Delta_6} \perp_L}{- : \perp, \Delta_8 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{}{- : \perp, \Delta_8 \vdash \top, \Delta_6} \top_R
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_1 : \Delta_7, p_5 \vdash (\Delta_6, p_5), \top} \top_R \quad \frac{}{\bullet h_4 : (\Delta_7, p_5), \top \vdash \Delta_6, p_5} I}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_7, p_5 \vdash \Delta_6, p_5} I
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_2 : \Delta_6 \vdash (\top, \Delta_8, p_7), p_7} \top_R \quad \frac{}{\bullet h_5 : \Delta_6, p_7 \vdash \top, \Delta_8, p_7} I}{- : \Delta_6 \vdash \top, \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_6 \vdash \top, \Delta_8, p_7} \top_R
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_2 : \Delta_9, p_7 \vdash (\top, \Delta_8, p_7), F_5} \top_R \quad \frac{}{\bullet h_6 : (\Delta_9, p_7), F_5 \vdash \top, \Delta_8, p_7} I}{- : \Delta_9, p_7 \vdash \top, \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_9, p_7 \vdash \top, \Delta_8, p_7} \top_R
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_1 : \Delta_5 \vdash \Delta_6, \top} \top_R \quad \frac{h_4 : \Delta_5 \vdash \Delta_6}{\bullet h_4 : \Delta_5, \top \vdash \Delta_6} \top_L}{- : \Delta_5 \vdash \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_5 \vdash \Delta_6} \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_2 : \Delta_7 \vdash (\top, \Delta_5), \top} \top_R \quad \frac{h_6 : \Delta_7 \vdash \top, \Delta_5}{\bullet h_6 : \Delta_7, \top \vdash \top, \Delta_5} \top_L}{- : \Delta_7 \vdash \top, \Delta_5} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Delta_7 \vdash \top, \Delta_5} \top_R \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \top, \Delta_8 \vdash (\top, \Delta_6), F_5} \top_R \quad \frac{h_7 : F_5, \Delta_8 \vdash \top, \Delta_6}{\bullet h_7 : (\top, \Delta_8), F_5 \vdash \top, \Delta_6} \top_L}{- : \top, \Delta_8 \vdash \top, \Delta_6} \text{Cut} \\
\rightarrow \\
\frac{}{- : \top, \Delta_8 \vdash \top, \Delta_6} \top_R
\end{array}$$

## 8.6 Status of A4: OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Box \Gamma_6, \Delta_7 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), \Box F_8} A4 \quad \frac{h_9 : \Box \Gamma_6, F_{11}, \Delta_7, \Box F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : (\Box \Gamma_6, \Delta_7), \Box F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R}{- : \Box \Gamma_6, \Delta_7 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \text{Cut} \\
\rightarrow \\
\frac{\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Delta_7, F_{11}, \Box \Gamma_6 \vdash \Delta_{10}, F_{12}, \Box F_8} \text{ax/W} \quad \frac{h_9 : \Delta_7, F_{11}, \Box \Gamma_6, \Box F_8 \vdash \Delta_{10}, F_{12}}{\bullet h_9 : (\Delta_7, F_{11}), \Box \Gamma_6 \vdash \Delta_{10}, F_{12}} A4}{- : \Delta_7, F_{11}, \Box \Gamma_6 \vdash \Delta_{10}, F_{12}} \text{hCut} \\
\rightarrow_R \\
\frac{}{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R \\
\\
\frac{\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_9 \vdash ((\Delta_{14}, F_{12} \rightarrow F_{13}), \Box F_{10}), F_8} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, F_{12}, \Delta_9 \vdash F_{13}, \Delta_{14}, \Box F_{10}}{\bullet h_{11} : (\Box \Gamma_7, \Delta_9), F_8 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), \Box F_{10}} \rightarrow_R}{- : \Box \Gamma_7, \Delta_9 \vdash (\Delta_{14}, F_{12} \rightarrow F_{13}), \Box F_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_7 \vdash F_{10}} \text{ax/W}}{- : \Delta_9, \Box \Gamma_7 \vdash \Delta_{14}, \Box F_{10}, F_{12} \rightarrow F_{13}} A4
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Box \Gamma_6, \Delta_7 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), \Box F_8} A4 \quad \frac{h_9 : \Box \Gamma_6, \Delta_7, \Box F_8 \vdash F_{11}, \Delta_{10} \quad h_9 : \Box \Gamma_6, \Delta_7, \Box F_8 \vdash F_{12}, \Delta_{10}}{\bullet h_9 : (\Box \Gamma_6, \Delta_7), \Box F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R}{- : \Box \Gamma_6, \Delta_7 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \text{Cut} \\
\rightarrow \\
\frac{\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11}, \Box F_8} \text{ax/W} \quad \frac{h_9 : \Delta_7, \Box \Gamma_6, \Box F_8 \vdash \Delta_{10}, F_{11}}{\bullet h_9 : (\Delta_7, \Box \Gamma_6), \Box F_8 \vdash \Delta_{10}, F_{11}} A4}{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11}} \text{hCut} \\
\wedge_R \\
\frac{\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11}, \Box F_8} \text{ax/W} \quad \frac{h_9 : \Delta_7, \Box \Gamma_6, \Box F_8 \vdash \Delta_{10}, F_{11}}{\bullet h_9 : (\Delta_7, \Box \Gamma_6), \Box F_8 \vdash \Delta_{10}, F_{11}} A4}{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11}} \text{hCut} \\
\wedge_R \\
\frac{}{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R \\
\\
\frac{\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_9 \vdash ((\Delta_{14}, F_{12} \wedge F_{13}), \Box F_{10}), F_8} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, \Delta_9 \vdash F_{12}, \Delta_{14}, \Box F_{10} \quad h_{11} : \Box \Gamma_7, F_8, \Delta_9 \vdash F_{13}, \Delta_{14}, \Box F_{10}}{\bullet h_{11} : (\Box \Gamma_7, \Delta_9), F_8 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), \Box F_{10}} \wedge_R}{- : \Box \Gamma_7, \Delta_9 \vdash (\Delta_{14}, F_{12} \wedge F_{13}), \Box F_{10}} \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_7 \vdash F_{10}} \text{ax/W}}{- : \Delta_9, \Box \Gamma_7 \vdash \Delta_{14}, \Box F_{10}, F_{12} \wedge F_{13}} A4
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Box \Gamma_6, \Delta_7 \vdash (\Delta_{10}, F_{11} \vee F_{12}), \Box F_8} A4 \quad \frac{h_9 : \Box \Gamma_6, \Delta_7, \Box F_8 \vdash F_{11}, F_{12}, \Delta_{10}}{\bullet h_9 : (\Box \Gamma_6, \Delta_7), \Box F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R}{- : \Box \Gamma_6, \Delta_7 \vdash \Delta_{10}, F_{11} \vee F_{12}} \text{Cut} \\
\rightarrow \\
\frac{\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11}, F_{12}, \Box F_8} \text{ax/W} \quad \frac{h_9 : \Delta_7, \Box \Gamma_6, \Box F_8 \vdash \Delta_{10}, F_{11}, F_{12}}{\bullet h_9 : (\Delta_7, \Box \Gamma_6), \Box F_8 \vdash \Delta_{10}, F_{11}, F_{12}} A4}{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11}, F_{12}} \text{hCut} \\
\vee_R \\
\frac{}{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_9 \vdash ((\Delta_{14}, F_{12} \vee F_{13}), [\Box F_{10}], F_8)} \quad A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, \Delta_9 \vdash F_{12}, F_{13}, \Delta_{14}, [\Box F_{10}]}{\bullet h_{11} : (\Box \Gamma_7, \Delta_9), F_8 \vdash (\Delta_{14}, F_{12} \vee F_{13}), [\Box F_{10}]} \quad \vee_R \\
\hline
- : \Box \Gamma_7, \Delta_9 \vdash (\Delta_{14}, F_{12} \vee F_{13}), [\Box F_{10}] \quad \text{Cut} \\
\rightarrow \\
\frac{- : \Box \Gamma_7 \vdash F_{10} \quad \text{ax/W}}{- : \Delta_9, \Box \Gamma_7 \vdash \Delta_{14}, [\Box F_{10}, F_{12} \vee F_{13}]} \quad A4
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Box \Gamma_6, \Delta_7 \vdash (\perp, \Delta_{10}), [\Box F_8]} \quad A4 \quad \frac{h_9 : \Box \Gamma_6, \Delta_7, [\Box F_8] \vdash \Delta_{10}}{\bullet h_9 : (\Box \Gamma_6, \Delta_7), [\Box F_8] \vdash \perp, \Delta_{10}} \quad \perp_R \\
\hline
- : \Box \Gamma_6, \Delta_7 \vdash \perp, \Delta_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\bullet h_1 : \Delta_7, \Box \Gamma_6 \vdash \perp, \Delta_{10}, [\Box F_8] \quad \text{ax/W}}{- : \Delta_7, \Box \Gamma_6 \vdash \perp, \Delta_{10}} \quad \text{hCut} \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_9 \vdash ((\perp, \Delta_{12}), [\Box F_{10}], F_8)} \quad A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, \Delta_9 \vdash \Delta_{12}, [\Box F_{10}]}{\bullet h_{11} : (\Box \Gamma_7, \Delta_9), F_8 \vdash (\perp, \Delta_{12}), [\Box F_{10}]} \quad \perp_R \\
\hline
- : \Box \Gamma_7, \Delta_9 \vdash (\perp, \Delta_{12}), [\Box F_{10}] \quad \text{Cut} \\
\rightarrow \\
\frac{- : \Box \Gamma_7 \vdash F_{10} \quad \text{ax/W}}{- : \Delta_9, \Box \Gamma_7 \vdash \perp, \Delta_{12}, [\Box F_{10}]} \quad A4
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_6 \vdash F_8}{\bullet h_1 : \Box \Gamma_6, \Delta_7 \vdash (\top, \Delta_{10}), [\Box F_8]} \quad A4 \quad \frac{h_9 : (\Box \Gamma_6, \Delta_7), [\Box F_8] \vdash \top, \Delta_{10}}{\bullet h_9 : (\Box \Gamma_6, \Delta_7), [\Box F_8] \vdash \top, \Delta_{10}} \quad \top_R \\
\hline
- : \Box \Gamma_6, \Delta_7 \vdash \top, \Delta_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{- : \Delta_7, \Box \Gamma_6 \vdash \top, \Delta_{10}}{- : \Delta_7, \Box \Gamma_6 \vdash \top, \Delta_{10}} \quad \top_R \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_9 \vdash ((\top, \Delta_{12}), [\Box F_{10}], F_8)} \quad A4 \quad \frac{h_{11} : (\Box \Gamma_7, \Delta_9), F_8 \vdash (\top, \Delta_{12}), [\Box F_{10}]}{\bullet h_{11} : (\Box \Gamma_7, \Delta_9), F_8 \vdash (\top, \Delta_{12}), [\Box F_{10}]} \quad \top_R \\
\hline
- : \Box \Gamma_7, \Delta_9 \vdash (\top, \Delta_{12}), [\Box F_{10}] \quad \text{Cut} \\
\rightarrow \\
\frac{- : \Delta_9, \Box \Gamma_7 \vdash \top, \Delta_{12}, [\Box F_{10}]}{- : \Delta_9, \Box \Gamma_7 \vdash \top, \Delta_{12}, [\Box F_{10}]} \quad \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_{10}, \Box \Gamma_{12} \vdash F_6}{\bullet h_1 : (\Box \Gamma_{10}, \Box \Gamma_{12}), \Box \Gamma_{11}, \Delta_{13} \vdash (\Delta_8, [\Box F_9]), [\Box F_6]} \quad A4 \quad \frac{h_7 : \Box \Gamma_{10}, \Box \Gamma_{11}, [\Box F_6] \vdash F_9}{\bullet h_7 : ((\Box \Gamma_{10}, \Box \Gamma_{12}), \Box \Gamma_{11}, \Delta_{13}), [\Box F_6] \vdash \Delta_8, [\Box F_9]} \quad A4 \\
\hline
- : (\Box \Gamma_{10}, \Box \Gamma_{12}), \Box \Gamma_{11}, \Delta_{13} \vdash \Delta_8, [\Box F_9] \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Box \Gamma_{10}, \Box \Gamma_{12} \vdash F_6 \quad \text{ax/W}}{\bullet h_1 : \Box \Gamma_{10}, \Box \Gamma_{11}, \Box \Gamma_{12} \vdash F_9, [\Box F_6]} \quad A4 \quad \frac{h_7 : \Box \Gamma_{10}, \Box \Gamma_{11}, \Box \Gamma_{12}, [\Box F_6] \vdash F_9 \quad \text{ax/W}}{- : \Box \Gamma_{10}, \Box \Gamma_{11}, \Box \Gamma_{12} \vdash F_9} \quad \text{hCut}}{- : \Delta_{13}, \Box \Gamma_{10}, \Box \Gamma_{11}, \Box \Gamma_{12} \vdash \Delta_8, [\Box F_9]} \quad A4 \\
\frac{h_1 : \Box \Gamma_{10}, \Box \Gamma_{12} \vdash F_6}{\bullet h_1 : (\Box \Gamma_{10}, \Box \Gamma_{12}), \Box \Gamma_{11}, \Delta_{13} \vdash (\Delta_8, [\Box F_9]), [\Box F_6]} \quad A4 \quad \frac{h_7 : \Box \Gamma_{10}, \Box \Gamma_{11} \vdash F_9}{\bullet h_7 : ((\Box \Gamma_{10}, \Box \Gamma_{12}), \Box \Gamma_{11}, \Delta_{13}), [\Box F_6] \vdash \Delta_8, [\Box F_9]} \quad A4 \\
\hline
- : (\Box \Gamma_{10}, \Box \Gamma_{12}), \Box \Gamma_{11}, \Delta_{13} \vdash \Delta_8, [\Box F_9] \quad \text{Cut} \\
\rightarrow \\
\frac{- : \Box \Gamma_{10}, \Box \Gamma_{11} \vdash F_9 \quad \text{ax/W}}{- : \Delta_{13}, \Box \Gamma_{10}, \Box \Gamma_{11}, \Box \Gamma_{12} \vdash \Delta_8, [\Box F_9]} \quad A4 \\
\frac{h_2 : \Box \Gamma_{12}, \Box \Gamma_{14} \vdash F_8}{\bullet h_2 : (\Box \Gamma_{12}, \Box \Gamma_{14}), \Box \Gamma_{13}, \Delta_{15} \vdash ((\Delta_{11}, [\Box F_{10}]), [\Box F_8]), \Box F_7} \quad A4 \quad \frac{h_9 : \Box \Gamma_{12}, \Box \Gamma_{13}, \Box F_7 \vdash F_{10}}{\bullet h_9 : ((\Box \Gamma_{12}, \Box \Gamma_{14}), \Box \Gamma_{13}, \Delta_{15}), \Box F_7 \vdash (\Delta_{11}, [\Box F_{10}]), [\Box F_8]} \quad A4 \\
\hline
- : (\Box \Gamma_{12}, \Box \Gamma_{14}), \Box \Gamma_{13}, \Delta_{15} \vdash (\Delta_{11}, [\Box F_{10}]), [\Box F_8] \quad \text{Cut} \\
\rightarrow \\
\frac{- : \Box \Gamma_{12}, \Box \Gamma_{14} \vdash F_8 \quad \text{ax/W}}{- : \Delta_{15}, \Box \Gamma_{12}, \Box \Gamma_{13}, \Box \Gamma_{14} \vdash \Delta_{11}, [\Box F_{10}], [\Box F_8]} \quad A4
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box\Gamma_{11}, \Box\Gamma_{13} \vdash F_{10}}{\bullet h_2 : (\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14} \vdash (\Delta_9, \Box F_{10}), \Box F_7} A4 \quad \frac{h_8 : \Box\Gamma_{11}, \Box\Gamma_{12}, \Box F_7 \vdash F_{10}}{\bullet h_8 : ((\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14}), \Box F_7 \vdash \Delta_9, \Box F_{10}} A4 \\
\hline
- : (\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14} \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{- : \Box\Gamma_{11}, \Box\Gamma_{13} \vdash F_{10} \quad \text{ax/W}}{- : \Delta_{14}, \Box\Gamma_{11}, \Box\Gamma_{12}, \Box\Gamma_{13} \vdash \Delta_9, \Box F_{10}} A4}{- : (\Box\Gamma_{12}, \Box\Gamma_{14}), \Box\Gamma_{13}, \Delta_{15} \vdash (\Delta_{11}, \Box F_{10}), \Box F_8} A4 \\
\frac{h_2 : \Box\Gamma_{12}, \Box\Gamma_{14} \vdash F_8}{\bullet h_2 : (\Box\Gamma_{12}, \Box\Gamma_{14}), \Box\Gamma_{13}, \Delta_{15} \vdash ((\Delta_{11}, \Box F_{10}), \Box F_8), F_7} A4 \quad \frac{h_9 : \Box\Gamma_{12}, \Box\Gamma_{13} \vdash F_{10}}{\bullet h_9 : ((\Box\Gamma_{12}, \Box\Gamma_{14}), \Box\Gamma_{13}, \Delta_{15}), F_7 \vdash (\Delta_{11}, \Box F_{10}), \Box F_8} A4 \\
\hline
- : (\Box\Gamma_{12}, \Box\Gamma_{14}), \Box\Gamma_{13}, \Delta_{15} \vdash (\Delta_{11}, \Box F_{10}), \Box F_8 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{- : \Box\Gamma_{12}, \Box\Gamma_{13} \vdash F_{10} \quad \text{ax/W}}{- : \Delta_{15}, \Box\Gamma_{12}, \Box\Gamma_{13}, \Box\Gamma_{14} \vdash \Delta_{11}, \Box F_{10}, \Box F_8} A4}{- : (\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14} \vdash \Delta_9, \Box F_{10}} A4 \\
\frac{h_2 : \Box\Gamma_{11}, \Box\Gamma_{13} \vdash F_{10}}{\bullet h_2 : (\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14} \vdash (\Delta_9, \Box F_{10}), F_7} A4 \quad \frac{h_8 : \Box\Gamma_{11}, \Box\Gamma_{12} \vdash F_{10}}{\bullet h_8 : ((\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14}), F_7 \vdash \Delta_9, \Box F_{10}} A4 \\
\hline
- : (\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14} \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{- : \Box\Gamma_{11}, \Box\Gamma_{12} \vdash F_{10} \quad \text{ax/W}}{- : \Delta_{14}, \Box\Gamma_{11}, \Box\Gamma_{12}, \Box\Gamma_{13} \vdash \Delta_9, \Box F_{10}} A4}{- : (\Box\Gamma_{11}, \Box\Gamma_{13}), \Box\Gamma_{12}, \Delta_{14} \vdash \Delta_9, \Box F_{10}} A4
\end{array}$$

• Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_1 : \Box\Gamma_6 \vdash F_7}{\bullet h_1 : \Box\Gamma_6, \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11}, \Box F_7} A4 \quad \frac{h_8 : \Box\Gamma_6, \Delta_{12}, \Box F_7 \vdash F_9, \Delta_{11} \quad h_8 : \Box\Gamma_6, F_{10}, \Delta_{12}, \Box F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Box\Gamma_6, \Delta_{12}, F_9 \rightarrow F_{10}), \Box F_7 \vdash \Delta_{11}} \rightarrow_L \\
\hline
- : \Box\Gamma_6, \Delta_{12}, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Box\Gamma_6 \vdash F_7 \quad \text{ax/W}}{\bullet h_1 : \Delta_{12}, \Box\Gamma_6 \vdash \Delta_{11}, F_9, \Box F_7} A4 \quad \frac{h_8 : \Delta_{12}, \Box\Gamma_6, \Box F_7 \vdash \Delta_{11}, F_9}{\bullet h_8 : (\Box\Gamma_6, \Delta_{12}, F_9 \rightarrow F_{10}), \Box F_7 \vdash \Delta_{11}} \text{ax/W}}{- : \Delta_{12}, \Box\Gamma_6 \vdash \Delta_{11}, F_9} \text{hCut} \\
\frac{\frac{\frac{h_1 : \Box\Gamma_6 \vdash F_7 \quad \text{ax/W}}{\bullet h_1 : \Delta_{12}, F_{10}, \Box\Gamma_6 \vdash \Delta_{11}, \Box F_7} A4 \quad \frac{h_8 : \Delta_{12}, F_{10}, \Box\Gamma_6, \Box F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Box\Gamma_6, \Delta_{12}, F_9 \rightarrow F_{10}), \Box F_7 \vdash \Delta_{11}} \text{ax/W}}{- : \Delta_{12}, F_{10}, \Box\Gamma_6 \vdash \Delta_{11}} \text{hCut} \\
\hline
- : \Delta_{12}, \Box\Gamma_6, F_9 \rightarrow F_{10} \vdash \Delta_{11} \quad \rightarrow_L
\end{array}$$
  

$$\begin{array}{c}
\frac{h_2 : \Box\Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box\Gamma_7, \Delta_8 \vdash (\Delta_9, \Box F_{10}), F_{12} \rightarrow F_{13}} A4 \quad \frac{h_{11} : \Box\Gamma_7, \Delta_8 \vdash F_{12}, \Delta_9, \Box F_{10} \quad h_{11} : \Box\Gamma_7, F_{13}, \Delta_8 \vdash \Delta_9, \Box F_{10}}{\bullet h_{11} : (\Box\Gamma_7, \Delta_8), F_{12} \rightarrow F_{13} \vdash \Delta_9, \Box F_{10}} \rightarrow_L \\
\hline
- : \Box\Gamma_7, \Delta_8 \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{- : \Box\Gamma_7 \vdash F_{10} \quad \text{ax/W}}{- : \Delta_8, \Box\Gamma_7 \vdash \Delta_9, \Box F_{10}} A4}{- : \Box\Gamma_7, \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_9, \Box F_{10}} A4 \\
\frac{h_2 : \Box\Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box\Gamma_7, \Delta_{14}, F_{12} \rightarrow F_{13} \vdash (\Delta_9, \Box F_{10}), F_8} A4 \quad \frac{h_{11} : \Box\Gamma_7, F_8, \Delta_{14} \vdash F_{12}, \Delta_9, \Box F_{10} \quad h_{11} : \Box\Gamma_7, F_8, F_{13}, \Delta_{14} \vdash \Delta_9, \Box F_{10}}{\bullet h_{11} : (\Box\Gamma_7, \Delta_{14}, F_{12} \rightarrow F_{13}), F_8 \vdash \Delta_9, \Box F_{10}} \rightarrow_L \\
\hline
- : \Box\Gamma_7, \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{- : \Box\Gamma_7 \vdash F_{10} \quad \text{ax/W}}{- : \Delta_{14}, \Box\Gamma_7, F_{12} \rightarrow F_{13} \vdash \Delta_9, \Box F_{10}} A4}{- : \Box\Gamma_7, \Delta_{14}, F_{12} \rightarrow F_{13} \vdash \Delta_9, \Box F_{10}} A4
\end{array}$$

• Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_1 : \Box\Gamma_6 \vdash F_7}{\bullet h_1 : \Box\Gamma_6, \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11}, \Box F_7} A4 \quad \frac{h_8 : \Box\Gamma_6, F_9, F_{10}, \Delta_{12}, \Box F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Box\Gamma_6, \Delta_{12}, F_9 \wedge F_{10}), \Box F_7 \vdash \Delta_{11}} \wedge_L \\
\hline
- : \Box\Gamma_6, \Delta_{12}, F_9 \wedge F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Box\Gamma_6 \vdash F_7 \quad \text{ax/W}}{\bullet h_1 : \Delta_{12}, F_{10}, F_9, \Box\Gamma_6 \vdash \Delta_{11}, \Box F_7} A4 \quad \frac{h_8 : \Delta_{12}, F_{10}, F_9, \Box\Gamma_6, \Box F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Box\Gamma_6, \Delta_{12}, F_9 \wedge F_{10}), \Box F_7 \vdash \Delta_{11}} \text{ax/W}}{- : \Delta_{12}, F_{10}, F_9, \Box\Gamma_6 \vdash \Delta_{11}} \text{hCut} \\
\hline
- : \Delta_{12}, \Box\Gamma_6, F_9 \wedge F_{10} \vdash \Delta_{11} \quad \wedge_L
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_8 \vdash (\Delta_9, \Box F_{10}), F_{12} \wedge F_{13}} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_{12}, F_{13}, \Delta_8 \vdash \Delta_9, \Box F_{10}}{\bullet h_{11} : (\Box \Gamma_7, \Delta_8), F_{12} \wedge F_{13} \vdash \Delta_9, \Box F_{10}} \wedge_L \\
\hline
- : \Box \Gamma_7, \Delta_8 \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_7 \vdash F_{10}}{ax/W}}{- : \Delta_8, \Box \Gamma_7 \vdash \Delta_9, \Box F_{10}} A4 \\
\hline
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_{14}, F_{12} \wedge F_{13} \vdash (\Delta_9, \Box F_{10}), F_8} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, F_{12}, F_{13}, \Delta_{14} \vdash \Delta_9, \Box F_{10}}{\bullet h_{11} : (\Box \Gamma_7, \Delta_{14}, F_{12} \wedge F_{13}), F_8 \vdash \Delta_9, \Box F_{10}} \wedge_L \\
\hline
- : \Box \Gamma_7, \Delta_{14}, F_{12} \wedge F_{13} \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_7 \vdash F_{10}}{ax/W}}{- : \Delta_{14}, \Box \Gamma_7, F_{12} \wedge F_{13} \vdash \Delta_9, \Box F_{10}} A4
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Box \Gamma_6, \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11}, \Box F_7} A4 \quad \frac{h_8 : \Box \Gamma_6, F_9, \Delta_{12}, \Box F_7 \vdash \Delta_{11} \quad h_8 : \Box \Gamma_6, F_{10}, \Delta_{12}, \Box F_7 \vdash \Delta_{11}}{\bullet h_8 : (\Box \Gamma_6, \Delta_{12}, F_9 \vee F_{10}), \Box F_7 \vdash \Delta_{11}} \vee_L \\
\hline
- : \Box \Gamma_6, \Delta_{12}, F_9 \vee F_{10} \vdash \Delta_{11} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Delta_{12}, F_9, \Box \Gamma_6 \vdash \Delta_{11}, \Box F_7} ax/W \quad \frac{h_8 : \Delta_{12}, F_9, \Box \Gamma_6, \Box F_7 \vdash \Delta_{11}}{hCut}}{- : \Delta_{12}, F_9, \Box \Gamma_6 \vdash \Delta_{11}} A4 \quad \frac{\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Delta_{12}, F_{10}, \Box \Gamma_6 \vdash \Delta_{11}, \Box F_7} ax/W \quad \frac{h_8 : \Delta_{12}, F_{10}, \Box \Gamma_6, \Box F_7 \vdash \Delta_{11}}{hCut}}{- : \Delta_{12}, F_{10}, \Box \Gamma_6 \vdash \Delta_{11}} A4 \\
\hline
- : \Delta_{12}, \Box \Gamma_6, F_9 \vee F_{10} \vdash \Delta_{11} \quad \vee_L \\
\rightarrow \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_8 \vdash (\Delta_9, \Box F_{10}), F_{12} \vee F_{13}} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_{12}, \Delta_8 \vdash \Delta_9, \Box F_{10} \quad h_{11} : \Box \Gamma_7, F_{13}, \Delta_8 \vdash \Delta_9, \Box F_{10}}{\bullet h_{11} : (\Box \Gamma_7, \Delta_8), F_{12} \vee F_{13} \vdash \Delta_9, \Box F_{10}} \vee_L \\
\hline
- : \Box \Gamma_7, \Delta_8 \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_7 \vdash F_{10}}{ax/W}}{- : \Delta_8, \Box \Gamma_7 \vdash \Delta_9, \Box F_{10}} A4 \\
\hline
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_{14}, F_{12} \vee F_{13} \vdash (\Delta_9, \Box F_{10}), F_8} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, F_{12}, \Delta_{14} \vdash \Delta_9, \Box F_{10} \quad h_{11} : \Box \Gamma_7, F_8, F_{13}, \Delta_{14} \vdash \Delta_9, \Box F_{10}}{\bullet h_{11} : (\Box \Gamma_7, \Delta_{14}, F_{12} \vee F_{13}), F_8 \vdash \Delta_9, \Box F_{10}} \vee_L \\
\hline
- : \Box \Gamma_7, \Delta_{14}, F_{12} \vee F_{13} \vdash \Delta_9, \Box F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{}{- : \Box \Gamma_7 \vdash F_{10}}{ax/W}}{- : \Delta_{14}, \Box \Gamma_7, F_{12} \vee F_{13} \vdash \Delta_9, \Box F_{10}} A4
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_{11}, \Box F_9 \vdash F_7}{\bullet h_1 : (\Box \Gamma_{11}, \Box F_9), \Delta_6 \vdash \Delta_{10}, \Box F_7} A4 \quad \frac{h_8 : \Box \Gamma_{11}, F_9, \Delta_6, \Box F_7 \vdash \Delta_{10}}{\bullet h_8 : ((\Box \Gamma_{11}, \Box F_9), \Delta_6), \Box F_7 \vdash \Delta_{10}} AT \\
\hline
- : (\Box \Gamma_{11}, \Box F_9), \Delta_6 \vdash \Delta_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_6, F_9, \Box \Gamma_{11}, \Box F_9 \vdash \Delta_{10}, \Box F_7}{- : \Delta_6, F_9, \Box \Gamma_{11}, \Box F_9 \vdash \Delta_{10}} ax/W \quad \frac{h_8 : \Delta_6, F_9, \Box \Gamma_{11}, \Box F_7 \vdash \Delta_{10}}{hCut}}{- : \Delta_6, \Box \Gamma_{11}, \Box F_9 \vdash \Delta_{10}} AT \\
\hline
\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Box \Gamma_6, \Delta_{11}, \Box F_9 \vdash \Delta_{10}, \Box F_7} A4 \quad \frac{h_8 : \Box \Gamma_6, F_9, \Delta_{11}, \Box F_7 \vdash \Delta_{10}}{\bullet h_8 : (\Box \Gamma_6, \Delta_{11}, \Box F_9), \Box F_7 \vdash \Delta_{10}} AT \\
\hline
- : \Box \Gamma_6, \Delta_{11}, \Box F_9 \vdash \Delta_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 : \Delta_{11}, F_9, \Box \Gamma_6, \Box F_9 \vdash \Delta_{10}, \Box F_7}{- : \Delta_{11}, F_9, \Box \Gamma_6, \Box F_9 \vdash \Delta_{10}} ax/W \quad \frac{h_8 : \Delta_{11}, F_9, \Box \Gamma_6, \Box F_7 \vdash \Delta_{10}}{hCut}}{- : \Delta_{11}, \Box \Gamma_6, \Box F_9 \vdash \Delta_{10}} AT
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_9}{\bullet h_1 : \Box \Gamma_6, \Delta_7 \vdash \Delta_{10}, [\Box F_9]} A4 \quad \frac{h_8 : \Box \Gamma_6, F_9, \Delta_7, [\Box F_9 \vdash \Delta_{10}] AT}{\bullet h_8 : (\Box \Gamma_6, \Delta_7), [\Box F_9 \vdash \Delta_{10}] AT} AT}{\frac{- : \Box \Gamma_6, \Delta_7 \vdash \Delta_{10}}{\rightarrow} Cut} \\
\frac{\frac{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}, F_9}{ax/W} \quad \frac{\frac{\bullet h_1 : \Delta_7, F_9, \Box \Gamma_6 \vdash \Delta_{10}, [\Box F_9]}{ax/W} \quad \frac{h_8 : \Delta_7, F_9, \Box \Gamma_6, [\Box F_9 \vdash \Delta_{10}] ax/W}{\rightarrow} hCut}{\frac{- : \Delta_7, F_9, \Box \Gamma_6 \vdash \Delta_{10}}{sCut}} \\
\frac{- : \Delta_7, \Box \Gamma_6 \vdash \Delta_{10}}{\rightarrow} \\
\frac{\frac{h_2 : \Box \Gamma_{13}, [\Box F_{12} \vdash F_{10}]}{\bullet h_2 : (\Box \Gamma_{13}, [\Box F_{12}]), \Delta_8 \vdash (\Delta_9, [\Box F_{10}]), F_7} A4 \quad \frac{h_{11} : \Box \Gamma_{13}, F_7, F_{12}, \Delta_8, [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]] AT}{\bullet h_{11} : ((\Box \Gamma_{13}, [\Box F_{12}]), \Delta_8), F_7 \vdash \Delta_9, [\Box F_{10}]] AT} AT}{\frac{- : (\Box \Gamma_{13}, [\Box F_{12}]), \Delta_8 \vdash \Delta_9, [\Box F_{10}]]}{\rightarrow} Cut} \\
\frac{- : \Box \Gamma_{13}, [\Box F_{12} \vdash F_{10}]}{ax/W} \\
\frac{- : \Delta_8, \Box \Gamma_{13}, [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]]}{A4} \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_8 \vdash (\Delta_9, [\Box F_{10}]), [\Box F_{12}]} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_{12}, \Delta_8, [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]] AT}{\bullet h_{11} : (\Box \Gamma_7, \Delta_8), [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]] AT} AT}{\frac{- : \Box \Gamma_7, \Delta_8 \vdash \Delta_9, [\Box F_{10}]]}{\rightarrow} Cut} \\
\frac{- : \Box \Gamma_7 \vdash F_{10}}{ax/W} \\
\frac{- : \Delta_8, \Box \Gamma_7 \vdash \Delta_9, [\Box F_{10}]]}{A4} \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_{13}, [\Box F_{12} \vdash (\Delta_9, [\Box F_{10}]), F_8]} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, F_{12}, \Delta_{13}, [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]] AT}{\bullet h_{11} : (\Box \Gamma_7, \Delta_{13}, [\Box F_{12}]), F_8 \vdash \Delta_9, [\Box F_{10}]] AT} AT}{\frac{- : \Box \Gamma_7, \Delta_{13}, [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]]}{\rightarrow} Cut} \\
\frac{- : \Box \Gamma_7 \vdash F_{10}}{ax/W} \\
\frac{- : \Delta_{13}, \Box \Gamma_7, [\Box F_{12} \vdash \Delta_9, [\Box F_{10}]]}{A4}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Box \Gamma_6, \perp, \Delta_{10} \vdash \Delta_9, [\Box F_7]} A4 \quad \frac{\bullet h_8 : (\Box \Gamma_6, \perp, \Delta_{10}), [\Box F_7 \vdash \Delta_9]}{\rightarrow} \perp_L}{\frac{- : \Box \Gamma_6, \perp, \Delta_{10} \vdash \Delta_9}{\rightarrow} Cut} \\
\frac{- : \perp, \Delta_{10}, \Box \Gamma_6 \vdash \Delta_9}{\perp_L} \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_8 \vdash (\Delta_9, [\Box F_{10}]), \perp} A4 \quad \frac{\bullet h_{11} : (\Box \Gamma_7, \Delta_8), \perp \vdash \Delta_9, [\Box F_{10}]] \perp_L}{\rightarrow} \perp_L}{\frac{- : \Box \Gamma_7, \Delta_8 \vdash \Delta_9, [\Box F_{10}]]}{\rightarrow} Cut} \\
\frac{- : \Box \Gamma_7 \vdash F_{10}}{ax/W} \\
\frac{- : \Delta_8, \Box \Gamma_7 \vdash \Delta_9, [\Box F_{10}]]}{A4} \\
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \perp, \Delta_{12} \vdash (\Delta_9, [\Box F_{10}]), F_8} A4 \quad \frac{\bullet h_{11} : (\Box \Gamma_7, \perp, \Delta_{12}), F_8 \vdash \Delta_9, [\Box F_{10}]] \perp_L}{\rightarrow} \perp_L}{\frac{- : \Box \Gamma_7, \perp, \Delta_{12} \vdash \Delta_9, [\Box F_{10}]]}{\rightarrow} Cut} \\
\frac{- : \perp, \Delta_{12}, \Box \Gamma_7 \vdash \Delta_9, [\Box F_{10}]]}{\perp_L}
\end{array}$$

- Case rule  $I$

$$\frac{\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Box \Gamma_6, \Delta_{11}, p_9 \vdash (\Delta_{10}, p_9), [\Box F_7]} A4 \quad \frac{\bullet h_8 : (\Box \Gamma_6, \Delta_{11}, p_9), [\Box F_7 \vdash \Delta_{10}, p_9]}{\rightarrow} I}{\frac{- : \Box \Gamma_6, \Delta_{11}, p_9 \vdash \Delta_{10}, p_9}{\rightarrow} Cut} \\
\frac{- : \Delta_{11}, \Box \Gamma_6, p_9 \vdash \Delta_{10}, p_9}{I}$$



$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_7 \vdash F_9}{\bullet h_2 : \Box \Gamma_7, \Delta_8 \vdash ((\Delta_{12}, p_{11}), [F_9], p_{11})} A4 \quad \frac{\bullet h_{10} : (\Box \Gamma_7, \Delta_8), p_{11} \vdash (\Delta_{12}, p_{11}), [F_9]}{\vdash : \Box \Gamma_7, \Delta_8 \vdash (\Delta_{12}, p_{11}), [F_9]} I \\
\text{Cut} \\
\rightarrow \\
\frac{\vdash : \Box \Gamma_7 \vdash F_9}{\vdash : \Delta_8, \Box \Gamma_7 \vdash \Delta_{12}, p_{11}, [F_9]} ax/W \\
A4 \\
\vdash : \Delta_8, \Box \Gamma_7 \vdash \Delta_{12}, p_{11}, [F_9]
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_7 \vdash F_9}{\bullet h_2 : \Box \Gamma_7, \Delta_{13}, p_{11} \vdash ((\Delta_{12}, p_{11}), [F_9], F_8)} A4 \quad \frac{\bullet h_{10} : (\Box \Gamma_7, \Delta_{13}, p_{11}), F_8 \vdash (\Delta_{12}, p_{11}), [F_9]}{\vdash : \Box \Gamma_7, \Delta_{13}, p_{11} \vdash (\Delta_{12}, p_{11}), [F_9]} I \\
\text{Cut} \\
\rightarrow \\
\frac{\vdash : \Delta_{13}, \Box \Gamma_7, p_{11} \vdash \Delta_{12}, p_{11}, [F_9]}{\vdash : \Delta_{13}, \Box \Gamma_7, p_{11} \vdash \Delta_{12}, p_{11}, [F_9]} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_1 : \Box \Gamma_6 \vdash F_7}{\bullet h_1 : \Box \Gamma_6, \top, \Delta_{10} \vdash \Delta_9, [F_7]} A4 \quad \frac{h_8 : \Box \Gamma_6, \Delta_{10}, [F_7] \vdash \Delta_9}{\bullet h_8 : (\Box \Gamma_6, \top, \Delta_{10}), [F_7] \vdash \Delta_9} \top_L \\
\text{Cut} \\
\vdash : \Box \Gamma_6, \top, \Delta_{10} \vdash \Delta_9 \\
\rightarrow \\
\frac{\bullet h_1 : \top, \Delta_{10}, \Box \Gamma_6 \vdash \Delta_9, [F_7]}{\vdash : \top, \Delta_{10}, \Box \Gamma_6 \vdash \Delta_9} ax/W \quad \frac{h_8 : \top, \Delta_{10}, \Box \Gamma_6, [F_7] \vdash \Delta_9}{\vdash : \top, \Delta_{10}, \Box \Gamma_6 \vdash \Delta_9} hCut
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \Delta_8 \vdash (\Delta_9, [F_{10}]), \top} A4 \quad \frac{h_{11} : \Box \Gamma_7, \Delta_8 \vdash \Delta_9, [F_{10}]}{\bullet h_{11} : (\Box \Gamma_7, \Delta_8), \top \vdash \Delta_9, [F_{10}]} \top_L \\
\text{Cut} \\
\vdash : \Box \Gamma_7, \Delta_8 \vdash \Delta_9, [F_{10}] \\
\rightarrow \\
\frac{\vdash : \Delta_8, \Box \Gamma_7 \vdash \Delta_9, [F_{10}]}{\vdash : \Delta_8, \Box \Gamma_7 \vdash \Delta_9, [F_{10}]} ax/W
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \Box \Gamma_7 \vdash F_{10}}{\bullet h_2 : \Box \Gamma_7, \top, \Delta_{12} \vdash (\Delta_9, [F_{10}]), F_8} A4 \quad \frac{h_{11} : \Box \Gamma_7, F_8, \Delta_{12} \vdash \Delta_9, [F_{10}]}{\bullet h_{11} : (\Box \Gamma_7, \top, \Delta_{12}), F_8 \vdash \Delta_9, [F_{10}]} \top_L \\
\text{Cut} \\
\vdash : \Box \Gamma_7, \top, \Delta_{12} \vdash \Delta_9, [F_{10}] \\
\rightarrow \\
\frac{\vdash : \Box \Gamma_7 \vdash F_{10}}{\vdash : \top, \Delta_{12}, \Box \Gamma_7 \vdash \Delta_9, [F_{10}]} ax/W \\
A4 \\
\vdash : \top, \Delta_{12}, \Box \Gamma_7 \vdash \Delta_9, [F_{10}]
\end{array}$$

## 8.7 Status of $\rightarrow_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_8 \vdash F_7, F_9, \Delta_{12}, F_{13} \rightarrow F_{14} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \rightarrow F_{14}}{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_7} \rightarrow_L \quad \frac{h_{11} : F_7, F_{13}, \Delta_8, F_9 \rightarrow F_{10} \vdash F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \rightarrow F_{10}), F_7 \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \rightarrow_R \\
\text{Cut} \\
\vdash : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14} \\
\rightarrow \\
\frac{h_3 : \Delta_8, F_{13} \vdash \Delta_{12}, F_{14}, F_7, F_9}{\bullet h_3 : \Delta_8, F_{13}, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{14}, F_7} inv\text{-th}/ax \quad \frac{h_3 : \Delta_8, F_{10}, F_{13} \vdash \Delta_{12}, F_{14}, F_7}{\vdash : \Delta_8, F_{13}, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{14}} inv\text{-th}/ax \\
\rightarrow_L \\
\frac{\vdash : \Delta_8, F_{13}, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{14}}{\vdash : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} ax/W \\
hCut \\
\rightarrow_R \\
\vdash : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_8 \vdash F_7, F_9, \Delta_{12}, F_{13} \wedge F_{14} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \wedge F_{14}}{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_7} \rightarrow_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \rightarrow F_{10} \vdash F_{13}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \rightarrow F_{10}), F_7 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \rightarrow_R \\
\text{Cut} \\
\vdash : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14} \\
\rightarrow \\
\frac{h_3 : \Delta_8 \vdash \Delta_{12}, F_{13}, F_7, F_9}{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13}, F_7} inv\text{-th}/ax \quad \frac{h_3 : \Delta_8, F_{10} \vdash \Delta_{12}, F_{13}, F_7}{\vdash : \Delta_8, F_7, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13}} inv\text{-th}/ax \\
\rightarrow_L \\
\frac{\vdash : \Delta_8, F_7, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13}}{\vdash : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14}} ax/W \\
hCut \\
\rightarrow_R \\
\vdash : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_8 \vdash F_7, F_9, \Delta_{12}, F_{13} \vee F_{14} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \vee F_{14} \rightarrow_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \rightarrow F_{10} \vdash F_{13}, F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \rightarrow F_{10}), F_7 \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R}{\frac{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_7}{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}} \rightarrow} \text{Cut} \\
\\
\frac{\frac{h_3 : \Delta_8 \vdash \Delta_{12}, F_{13}, F_{14}, F_7, F_9}{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_8, F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7}{\bullet h_{11} : (\Delta_8, F_7, F_9 \rightarrow F_{10}) \vdash \Delta_{12}, F_{13}, F_{14}} \text{inv-th/ax}}{\frac{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R} \text{ax/W} \\
\text{hCut}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_8 \vdash F_7, F_9, \perp, \Delta_{12} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \perp, \Delta_{12} \rightarrow_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \rightarrow F_{10} \vdash \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \rightarrow F_{10}), F_7 \vdash \perp, \Delta_{12}} \perp_R}{\frac{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash (\perp, \Delta_{12}), F_7}{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \perp, \Delta_{12}} \rightarrow} \text{Cut} \\
\\
\frac{\frac{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash \perp, \Delta_{12}, F_7}{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \perp, \Delta_{12}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_7, F_9 \rightarrow F_{10} \vdash \perp, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_7, F_9 \rightarrow F_{10}) \vdash \perp, \Delta_{12}} \text{ax/W}}{\text{hCut}}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_8 \vdash F_7, F_9, \top, \Delta_{12} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \top, \Delta_{12} \rightarrow_L \quad \frac{h_{11} : (\Delta_8, F_9 \rightarrow F_{10}), F_7 \vdash \top, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \rightarrow F_{10}), F_7 \vdash \top, \Delta_{12}} \top_R}{\frac{\bullet h_3 : \Delta_8, F_9 \rightarrow F_{10} \vdash (\top, \Delta_{12}), F_7}{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \top, \Delta_{12}} \rightarrow} \text{Cut} \\
\\
\frac{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \top, \Delta_{12}}{- : \Delta_8, F_9 \rightarrow F_{10} \vdash \top, \Delta_{12}} \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_3 : \Box \Gamma_{13}, \Delta_{14} \vdash \Box F_7, F_8, \Delta_{11}, \Box F_{12} \quad h_3 : F_9, \Box \Gamma_{13}, \Delta_{14} \vdash \Box F_7, \Delta_{11}, \Box F_{12} \rightarrow_L \quad \frac{h_{10} : \Box \Gamma_{13}, \Box F_7 \vdash F_{12}}{\bullet h_{10} : ((\Box \Gamma_{13}, \Delta_{14}), F_8 \rightarrow F_9) \vdash \Box F_7 \vdash \Delta_{11}, \Box F_{12}} \rightarrow}{\frac{\bullet h_3 : ((\Box \Gamma_{13}, \Delta_{14}), F_8 \rightarrow F_9) \vdash (\Delta_{11}, \Box F_{12}), \Box F_7}{- : ((\Box \Gamma_{13}, \Delta_{14}), F_8 \rightarrow F_9) \vdash \Delta_{11}, \Box F_{12}} \rightarrow} \text{Cut} \\
\\
\frac{\frac{h_3 : \Delta_{14}, \Box \Gamma_{13} \vdash \Box F_7, \Delta_{11}, F_8, \Box F_{12}}{- : \Delta_{14}, \Box \Gamma_{13} \vdash \Delta_{11}, F_8, \Box F_{12}} \text{ax/W} \quad \frac{\frac{h_{10} : \Box F_7, \Box \Gamma_{13} \vdash F_{12}}{\bullet h_{10} : \Box F_7, \Delta_{14}, \Box \Gamma_{13} \vdash \Delta_{11}, F_8, \Box F_{12}} \text{ax/W}}{- : \Delta_{14}, \Box \Gamma_{13} \vdash \Delta_{11}, F_8, \Box F_{12}} \text{A4}}{\frac{- : \Delta_{14}, \Box \Gamma_{13}, F_8 \rightarrow F_9 \vdash \Delta_{11}, \Box F_{12}}{- : \Delta_{14}, \Box \Gamma_{13}, F_8 \rightarrow F_9 \vdash \Delta_{11}, \Box F_{12}} \text{hCut}} \text{ax/W} \\
\\
\frac{h_3 : \Box \Gamma_{11}, \Delta_{14} \vdash F_7, F_8, \Delta_{12}, \Box F_{13} \quad h_3 : F_9, \Box \Gamma_{11}, \Delta_{14} \vdash F_7, \Delta_{12}, \Box F_{13} \rightarrow_L \quad \frac{h_{10} : \Box \Gamma_{11} \vdash F_{13}}{\bullet h_{10} : ((\Box \Gamma_{11}, \Delta_{14}), F_8 \rightarrow F_9) \vdash F_7 \vdash \Delta_{12}, \Box F_{13}} \text{A4}}{\frac{\bullet h_3 : ((\Box \Gamma_{11}, \Delta_{14}), F_8 \rightarrow F_9) \vdash (\Delta_{12}, \Box F_{13}), F_7}{- : ((\Box \Gamma_{11}, \Delta_{14}), F_8 \rightarrow F_9) \vdash \Delta_{12}, \Box F_{13}} \rightarrow} \text{Cut} \\
\\
\frac{- : \Box \Gamma_{11} \vdash F_{13}}{- : \Delta_{14}, \Box \Gamma_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}, \Box F_{13}} \text{ax/W} \text{A4}
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_7 \vdash F_{11} \rightarrow F_{12}, F_8, \Delta_{13} \quad h_3 : F_9, \Delta_7 \vdash F_{11} \rightarrow F_{12}, \Delta_{13} \rightarrow_L \quad \frac{h_{10} : \Delta_7, F_8 \rightarrow F_9 \vdash F_{11}, \Delta_{13} \quad h_{10} : F_{12}, \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \rightarrow_L}{\frac{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash F_{11} \rightarrow F_{12}}{- : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \rightarrow} \text{Cut} \\
\\
\frac{\frac{- : \Delta_7, F_{11} \vdash \Delta_{13}, F_{12}, F_8}{- : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{12}} \text{inv-th/ax} \quad \frac{- : \Delta_7, F_{11}, F_9 \vdash \Delta_{13}, F_{12}}{- : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{12}} \text{inv-th/ax}}{\frac{- : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{12}}{- : \Delta_7, F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{13}} \rightarrow_L} \text{ax/W} \\
\\
\frac{- : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}, F_{11}}{- : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{13}} \text{ax/W} \text{sCut}
\end{array}$$



$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7 \vdash [\Box F_{11}, F_8, \Delta_{12} \quad h_3 : F_9, \Delta_7 \vdash [\Box F_{11}, \Delta_{12}]}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}, [\Box F_{11}]} \rightarrow_L \quad \frac{h_{10} : F_{11}, \Delta_7, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}]}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), [\Box F_{11} \vdash \Delta_{12}]} \begin{array}{l} AT \\ Cut \end{array}}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7 \vdash \Delta_{12}, F_8, [\Box F_{11}]}{\neg : \Delta_7 \vdash \Delta_{12}, F_8} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, [\Box F_{11} \vdash \Delta_{12}, F_8]}{\bullet h_{10} : \Delta_7, [\Box F_{11} \vdash \Delta_{12}, F_8]} \text{inv-th/ax} \quad \frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, [\Box F_{11}]}{\neg : \Delta_7, F_9 \vdash \Delta_{12}} \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_{11}, F_9, [\Box F_{11} \vdash \Delta_{12}]}{\bullet h_{10} : \Delta_7, F_9, [\Box F_{11} \vdash \Delta_{12}]} \text{inv-th/ax}}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}} \begin{array}{l} AT \\ hCut \end{array} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{13}, [\Box F_{11} \vdash F_7, F_8, \Delta_{12} \quad h_3 : F_9, \Delta_{13}, [\Box F_{11} \vdash F_7, \Delta_{12}]}{\bullet h_3 : (\Delta_{13}, [\Box F_{11}]), F_8 \rightarrow F_9 \vdash \Delta_{12}, F_7} \rightarrow_L \quad \frac{h_{10} : F_7, F_{11}, \Delta_{13}, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}]}{\bullet h_{10} : ((\Delta_{13}, [\Box F_{11}]), F_8 \rightarrow F_9), F_7 \vdash \Delta_{12}} \begin{array}{l} AT \\ Cut \end{array}}{\neg : (\Delta_{13}, [\Box F_{11}]), F_8 \rightarrow F_9 \vdash \Delta_{12}} \\
\rightarrow \\
\frac{\frac{\bullet h_3 : \Delta_{13}, F_{11}, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}, F_7]}{\neg : \Delta_{13}, F_{11}, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}]} \text{ax/W} \quad \frac{h_{10} : \Delta_{13}, F_{11}, F_7, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}]}{\neg : \Delta_{13}, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}]} \text{ax/W}}{\neg : \Delta_{13}, [\Box F_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}]} \begin{array}{l} hCut \\ AT \end{array}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7 \vdash \perp, F_8, \Delta_{11} \quad h_3 : F_9, \Delta_7 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}, \perp} \rightarrow_L \quad \frac{h_{10} : (\Delta_7, F_8 \rightarrow F_9), \perp \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), \perp \vdash \Delta_{11}} \begin{array}{l} \perp_L \\ Cut \end{array}}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7 \vdash \perp, \Delta_{11}, F_8]}{\neg : \Delta_7 \vdash \Delta_{11}, F_8} \text{ax/W} \quad \frac{\frac{\bullet h_{10} : \perp, \Delta_7 \vdash \Delta_{11}, F_8}}{\neg : \Delta_7, F_9 \vdash \Delta_{11}} \perp_L \quad \frac{h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}}{\neg : \Delta_7, F_9 \vdash \Delta_{11}} \text{ax/W} \quad \frac{\bullet h_{10} : \perp, \Delta_7, F_9 \vdash \Delta_{11}}{\neg : \Delta_7, F_9 \vdash \Delta_{11}} \perp_L}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \begin{array}{l} hCut \\ \rightarrow_L \end{array} \\
\rightarrow \\
\frac{\frac{h_3 : \perp, \Delta_{12} \vdash F_7, F_8, \Delta_{11} \quad h_3 : F_9, \perp, \Delta_{12} \vdash F_7, \Delta_{11}}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}, F_7} \rightarrow_L \quad \frac{h_{10} : ((\perp, \Delta_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{11}}{\bullet h_{10} : ((\perp, \Delta_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{11}} \begin{array}{l} \perp_L \\ Cut \end{array}}{\neg : (\perp, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\rightarrow \\
\frac{\neg : \perp, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\neg : \perp, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7 \vdash p_{11}, F_8, \Delta_{12}, p_{11} \quad h_3 : F_9, \Delta_7 \vdash p_{11}, \Delta_{12}, p_{11}}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash (\Delta_{12}, p_{11}), p_{11}} \rightarrow_L \quad \frac{h_{10} : (\Delta_7, F_8 \rightarrow F_9), p_{11} \vdash \Delta_{12}, p_{11}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), p_{11} \vdash \Delta_{12}, p_{11}} \begin{array}{l} I \\ Cut \end{array}}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7 \vdash \Delta_{12}, F_8, p_{11}, p_{11}}{\neg : \Delta_7 \vdash \Delta_{12}, F_8, p_{11}} \text{ax/W} \quad \frac{\frac{\bullet h_{10} : \Delta_7, p_{11} \vdash \Delta_{12}, F_8, p_{11}}{\neg : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}, p_{11}} I \quad \frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}, p_{11}}{\neg : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}} \text{ax/W} \quad \frac{\bullet h_{10} : \Delta_7, F_9, p_{11} \vdash \Delta_{12}, p_{11}}{\neg : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}} I}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} \begin{array}{l} hCut \\ \rightarrow_L \end{array} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{13}, p_{11} \vdash F_7, F_8, \Delta_{12}, p_{11} \quad h_3 : F_9, \Delta_{13}, p_{11} \vdash F_7, \Delta_{12}, p_{11}}{\bullet h_3 : (\Delta_{13}, p_{11}), F_8 \rightarrow F_9 \vdash (\Delta_{12}, p_{11}), F_7} \rightarrow_L \quad \frac{h_{10} : ((\Delta_{13}, p_{11}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{12}, p_{11}}{\bullet h_{10} : ((\Delta_{13}, p_{11}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{12}, p_{11}} \begin{array}{l} I \\ Cut \end{array}}{\neg : (\Delta_{13}, p_{11}), F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} \\
\rightarrow \\
\frac{\neg : \Delta_{13}, p_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}}{\neg : \Delta_{13}, p_{11}, F_8 \rightarrow F_9 \vdash \Delta_{12}, p_{11}} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_7 \vdash \top, F_8, \Delta_{11} \quad h_3 : F_9, \Delta_7 \vdash \top, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}, \top} \rightarrow_L \quad \frac{h_{10} : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \rightarrow F_9), \top \vdash \Delta_{11}} \begin{array}{l} \top_L \\ Cut \end{array}}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \\
\rightarrow \\
\frac{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\neg : \Delta_7, F_8 \rightarrow F_9 \vdash \Delta_{11}} \text{ax/W}
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : \top, \Delta_{12} \vdash F_7, F_8, \Delta_{11} \quad h_3 : F_9, \top, \Delta_{12} \vdash F_7, \Delta_{11}}{\bullet h_3 : (\top, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11}, F_7} \rightarrow_L \quad \frac{h_{10} : F_7, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \rightarrow F_9), F_7 \vdash \Delta_{11}} \top_L \\
\hline
- : (\top, \Delta_{12}), F_8 \rightarrow F_9 \vdash \Delta_{11} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\bullet h_3 : \top, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}, F_7 \quad \text{ax/W}}{- : \top, \Delta_{12}, F_8 \rightarrow F_9 \vdash \Delta_{11}} \quad \frac{h_{10} : \top, \Delta_{12}, F_7, F_8 \rightarrow F_9 \vdash \Delta_{11}}{\text{hCut}} \text{ax/W}
\end{array}$$

## 8.8 Status of $\wedge_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : F_9, F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \rightarrow F_{14}}{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_7} \wedge_L \quad \frac{h_{11} : F_7, F_{13}, \Delta_8, F_9 \wedge F_{10} \vdash F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \wedge F_{10}), F_7 \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \rightarrow_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \Delta_8, F_{10}, F_{13}, F_9 \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_{13}, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}, F_7} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_8, F_{13}, F_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14}}{\text{hCut}} \text{ax/W} \\
\hline
- : \Delta_8, F_{13}, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{14} \quad \rightarrow_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : F_9, F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \wedge F_{14}}{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_7} \wedge_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \wedge F_{10} \vdash F_{13}, \Delta_{12} \quad h_{11} : F_7, \Delta_8, F_9 \wedge F_{10} \vdash F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \wedge F_{10}), F_7 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \wedge_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \Delta_8, F_{10}, F_9 \vdash \Delta_{12}, F_7, F_{13} \wedge F_{14}}{\bullet h_3 : \Delta_8, F_{10}, F_9 \vdash \Delta_{12}, F_7, F_{13} \wedge F_{14}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_{10}, F_7, F_9 \vdash \Delta_{12}, F_{13}}{\bullet h_{11} : \Delta_8, F_{10}, F_7, F_9 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \text{inv-th/ax} \\
\hline
- : \Delta_8, F_{10}, F_9 \vdash \Delta_{12}, F_7, F_{13} \wedge F_{14} \quad \wedge_L \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14} \quad \text{hCut}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : F_9, F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \vee F_{14}}{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_7} \wedge_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \wedge F_{10} \vdash F_{13}, F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \wedge F_{10}), F_7 \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \Delta_8, F_{10}, F_9 \vdash \Delta_{12}, F_{13}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7} \text{inv-th/ax} \quad \frac{h_{11} : \Delta_8, F_7, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{\text{hCut}} \text{ax/W} \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13}, F_{14} \quad \vee_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : F_9, F_{10}, \Delta_8 \vdash F_7, \perp, \Delta_{12}}{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash (\perp, \Delta_{12}), F_7} \wedge_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \wedge F_{10}), F_7 \vdash \perp, \Delta_{12}} \perp_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \perp, \Delta_{12} \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}, F_7}{\text{ax/W}} \quad \frac{h_{11} : \Delta_8, F_7, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}}{\text{hCut}} \text{ax/W} \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \perp, \Delta_{12}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : F_9, F_{10}, \Delta_8 \vdash F_7, \top, \Delta_{12}}{\bullet h_3 : \Delta_8, F_9 \wedge F_{10} \vdash (\top, \Delta_{12}), F_7} \wedge_L \quad \frac{h_{11} : (\Delta_8, F_9 \wedge F_{10}), F_7 \vdash \top, \Delta_{12}}{\text{hCut}} \top_R \\
\hline
- : \Delta_8, F_9 \wedge F_{10} \vdash \top, \Delta_{12} \quad \text{Cut} \\
\hline
\rightarrow \\
- : \Delta_8, F_9 \wedge F_{10} \vdash \top, \Delta_{12} \quad \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Box \Gamma_{13}, \Delta_{14} \vdash \Box F_7, \Delta_{11}, [] F_{12}}{\bullet h_3 : (\Box \Gamma_{13}, \Delta_{14}), F_8 \wedge F_9 \vdash (\Delta_{11}, [] F_{12}), \Box F_7} \wedge_L \quad \frac{h_{10} : \Box \Gamma_{13}, \Box F_7 \vdash F_{12}}{\bullet h_{10} : ((\Box \Gamma_{13}, \Delta_{14}), F_8 \wedge F_9), \Box F_7 \vdash \Delta_{11}, [] F_{12}} A4 \\
\frac{}{- : (\Box \Gamma_{13}, \Delta_{14}), F_8 \wedge F_9 \vdash \Delta_{11}, [] F_{12}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{14}, F_8, F_9, \Box \Gamma_{13} \vdash \Box F_7, \Delta_{11}, [] F_{12}}{- : \Delta_{14}, F_8, F_9, \Box \Gamma_{13} \vdash \Delta_{11}, [] F_{12}} \text{ax/W} \quad \frac{h_{10} : \Box F_7, \Box \Gamma_{13} \vdash F_{12}}{\bullet h_{10} : \Box F_7, \Delta_{14}, F_8, F_9, \Box \Gamma_{13} \vdash \Delta_{11}, [] F_{12}} \text{ax/W}}{- : \Delta_{14}, F_8, F_9, \Box \Gamma_{13} \vdash \Delta_{11}, [] F_{12}} \wedge_L \\
\frac{}{- : \Delta_{14}, \Box \Gamma_{13}, F_8 \wedge F_9 \vdash \Delta_{11}, [] F_{12}} \text{hCut} \\
\frac{h_3 : F_8, F_9, \Box \Gamma_{11}, \Delta_{14} \vdash F_7, \Delta_{12}, [] F_{13}}{\bullet h_3 : (\Box \Gamma_{11}, \Delta_{14}), F_8 \wedge F_9 \vdash (\Delta_{12}, [] F_{13}), F_7} \wedge_L \quad \frac{h_{10} : \Box \Gamma_{11} \vdash F_{13}}{\bullet h_{10} : ((\Box \Gamma_{11}, \Delta_{14}), F_8 \wedge F_9), F_7 \vdash \Delta_{12}, [] F_{13}} A4 \\
\frac{}{- : (\Box \Gamma_{11}, \Delta_{14}), F_8 \wedge F_9 \vdash \Delta_{12}, [] F_{13}} \text{Cut} \\
\rightarrow \\
\frac{}{- : \Box \Gamma_{11} \vdash F_{13}} \text{ax/W} \\
\frac{}{- : \Delta_{14}, \Box \Gamma_{11}, F_8 \wedge F_9 \vdash \Delta_{12}, [] F_{13}} A4
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash F_{11} \rightarrow F_{12}, \Delta_{13}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}} \wedge_L \quad \frac{h_{10} : \Delta_7, F_8 \wedge F_9 \vdash F_{11}, \Delta_{13} \quad h_{10} : F_{12}, \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \rightarrow_L \\
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}}{- : \Delta_7, F_8, F_9 \vdash \Delta_{13}} \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11}}{\bullet h_{10} : \Delta_7, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \text{inv-th/ax}}{- : \Delta_7, F_8, F_9 \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}} \text{hCut} \\
\frac{h_3 : F_8, F_9, \Delta_{14}, F_{11} \rightarrow F_{12} \vdash F_7, \Delta_{13}}{\bullet h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \wedge_L \quad \frac{h_{10} : F_7, \Delta_{14}, F_8 \wedge F_9 \vdash F_{11}, \Delta_{13} \quad h_{10} : F_7, F_{12}, \Delta_{14}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{13}} \rightarrow_L \\
\frac{}{- : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{14}, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}, F_7}{- : \Delta_{14}, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \text{ax/W} \quad \frac{h_{10} : \Delta_{14}, F_7, F_8, F_9 \vdash \Delta_{13}, F_{11}}{\bullet h_{10} : \Delta_{14}, F_7, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \text{inv-th/ax}}{- : \Delta_{14}, F_8, F_9, F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : \Delta_{14}, F_{11} \rightarrow F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}} \text{hCut} \\
\rightarrow_L
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash F_{11} \wedge F_{12}, \Delta_{13}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}} \wedge_L \quad \frac{h_{10} : F_{11}, F_{12}, \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{- : \Delta_7, F_8, F_9 \vdash \Delta_{13}} \text{ax/W} \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_8, F_9, F_{11} \wedge F_{12} \vdash \Delta_{13}} \text{inv-th/ax}}{- : \Delta_7, F_8, F_9 \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}} \text{hCut} \\
\frac{h_3 : F_8, F_9, \Delta_{14}, F_{11} \wedge F_{12} \vdash F_7, \Delta_{13}}{\bullet h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \wedge_L \quad \frac{h_{10} : F_7, F_{11}, F_{12}, \Delta_{14}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{14}, F_{11}, F_{12}, F_8, F_9 \vdash \Delta_{13}, F_7}{- : \Delta_{14}, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \text{ax/W}}{- : \Delta_{14}, F_{11}, F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}} \wedge_L \\
\frac{}{- : \Delta_{14}, F_{11} \wedge F_{12}, F_8 \wedge F_9 \vdash \Delta_{13}} \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_{10}, F_{11}, \Delta_8 \vdash F_7, \Delta_{12}}{\bullet h_3 : \Delta_8, F_{10} \wedge F_{11} \vdash \Delta_{12}, F_7} \wedge_L \quad \frac{h_9 : F_7, F_{10}, F_{11}, \Delta_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_8, F_{10} \wedge F_{11}), F_7 \vdash \Delta_{12}} \wedge_L \\
\hline
- : \Delta_8, F_{10} \wedge F_{11} \vdash \Delta_{12} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_8, F_{10}, F_{11} \vdash \Delta_{12}, F_7}{- : \Delta_8, F_{10}, F_{11} \vdash \Delta_{12}} \text{ax/W} \quad \frac{\frac{h_9 : \Delta_8, F_{10}, F_{11}, F_7 \vdash \Delta_{12}}{\bullet h_9 : \Delta_8, F_{10}, F_{11}, F_7 \vdash \Delta_{12}} \text{ax/W} \quad \frac{}{H} \text{H}}{\bullet h_{10} : (\Delta_8, F_{10} \wedge F_{11}), F_7 \vdash \Delta_{12}} \text{hCut} \\
\hline
- : \Delta_8, F_{10} \wedge F_{11} \vdash \Delta_{12} \quad \wedge_L
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash F_{11} \vee F_{12}, \Delta_{13}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}} \wedge_L \quad \frac{h_{10} : F_{11}, \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \quad h_{10} : F_{12}, \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \vee_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}}{- : \Delta_7, F_8, F_9 \vdash \Delta_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_7, F_{12}, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{inv-th/ax}}{\bullet h_{10} : \Delta_7, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{hCut} \\
\hline
- : \Delta_7, F_8, F_9 \vdash \Delta_{13} \quad \wedge_L \\
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{13} \\
\frac{h_3 : F_8, F_9, \Delta_{14}, F_{11} \vee F_{12} \vdash F_7, \Delta_{13}}{\bullet h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \wedge F_9 \vdash \Delta_{13}, F_7} \wedge_L \quad \frac{h_{10} : F_7, F_{11}, \Delta_{14}, F_8 \wedge F_9 \vdash \Delta_{13} \quad h_{10} : F_7, F_{12}, \Delta_{14}, F_8 \wedge F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \vee F_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{13}} \vee_L \\
\hline
- : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \wedge F_9 \vdash \Delta_{13} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{14}, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}, F_7}{- : \Delta_{14}, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_{14}, F_{11}, F_7, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_{14}, F_7, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{inv-th/ax} \quad \frac{h_{10} : \Delta_{14}, F_{12}, F_7, F_8, F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_{14}, F_7, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{inv-th/ax}}{\bullet h_{10} : \Delta_{14}, F_7, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13}} \text{hCut} \\
\hline
- : \Delta_{14}, F_8, F_9, F_{11} \vee F_{12} \vdash \Delta_{13} \quad \wedge_L \\
- : \Delta_{14}, F_8 \wedge F_9, F_{11} \vee F_{12} \vdash \Delta_{13}
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash \boxed{F_{11}}, \Delta_{12}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12}, \boxed{F_{11}}} \wedge_L \quad \frac{h_{10} : F_{11}, \Delta_7, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), \boxed{F_{11}} \vdash \Delta_{12}} AT \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{12}, \boxed{F_{11}}}{- : \Delta_7, F_8, F_9 \vdash \Delta_{12}} \text{ax/W} \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, F_8, F_9, \boxed{F_{11}} \vdash \Delta_{12}}{\bullet h_{10} : \Delta_7, F_8, F_9, \boxed{F_{11}} \vdash \Delta_{12}} \text{inv-th/ax} \quad \frac{}{AT} AT}}{\bullet h_{10} : \Delta_7, F_8, F_9, \boxed{F_{11}} \vdash \Delta_{12}} \text{hCut} \\
\hline
- : \Delta_7, F_8, F_9 \vdash \Delta_{12} \quad \wedge_L \\
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12} \\
\frac{h_3 : F_8, F_9, \Delta_{13}, \boxed{F_{11}} \vdash F_7, \Delta_{12}}{\bullet h_3 : (\Delta_{13}, \boxed{F_{11}}), F_8 \wedge F_9 \vdash \Delta_{12}, F_7} \wedge_L \quad \frac{h_{10} : F_7, F_{11}, \Delta_{13}, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}}{\bullet h_{10} : ((\Delta_{13}, \boxed{F_{11}}), F_8 \wedge F_9), F_7 \vdash \Delta_{12}} AT \\
\hline
- : (\Delta_{13}, \boxed{F_{11}}), F_8 \wedge F_9 \vdash \Delta_{12} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{13}, F_{11}, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}, F_7}{- : \Delta_{13}, F_{11}, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}} \text{ax/W} \quad \frac{h_{10} : \Delta_{13}, F_{11}, F_7, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}}{\bullet h_{10} : \Delta_{13}, F_{11}, F_7, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}} \text{ax/W}}{\bullet h_{10} : \Delta_{13}, F_{11}, F_7, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}} \text{hCut} \\
\hline
- : \Delta_{13}, F_{11}, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12} \quad AT \\
- : \Delta_{13}, \boxed{F_{11}}, F_8 \wedge F_9 \vdash \Delta_{12}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}, \perp} \wedge_L \quad \frac{h_{10} : (\Delta_7, F_8 \wedge F_9), \perp \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), \perp \vdash \Delta_{11}} \perp_L \\
\hline
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7, F_8, F_9 \vdash \perp, \Delta_{11}}{- : \Delta_7, F_8, F_9 \vdash \perp} \text{ax/W} \quad \frac{h_{10} : \perp, \Delta_7, F_8, F_9 \vdash \Delta_{11}}{\bullet h_{10} : \perp, \Delta_7, F_8, F_9 \vdash \Delta_{11}} \perp_L}{\bullet h_{10} : \perp, \Delta_7, F_8, F_9 \vdash \Delta_{11}} \text{hCut} \\
\hline
- : \Delta_7, F_8, F_9 \vdash \perp \quad \wedge_L \\
- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \perp, \Delta_{12} \vdash F_7, \Delta_{11}}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}, F_7} \wedge_L \quad \frac{}{\bullet h_{10} : ((\perp, \Delta_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{11}} \perp_L \\
\hline
\frac{}{- : (\perp, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{}{- : \perp, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11}} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash p_{11}, \Delta_{12}, p_{11}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash (\Delta_{12}, p_{11}), p_{11}} \wedge_L \quad \frac{}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), p_{11} \vdash \Delta_{12}, p_{11}} I \\
\hline
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12}, p_{11}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{12}, p_{11}, p_{11}}{\bullet h_3 : \Delta_7, F_8, F_9 \vdash \Delta_{12}, p_{11}} \text{ax/w} \quad \frac{}{\bullet h_{10} : \Delta_7, F_8, F_9, p_{11} \vdash \Delta_{12}, p_{11}} I \\
\hline
\frac{}{- : \Delta_7, F_8, F_9 \vdash \Delta_{12}, p_{11}} \text{hCut} \\
\hline
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{12}, p_{11}} \wedge_L
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_{13}, p_{11} \vdash F_7, \Delta_{12}, p_{11}}{\bullet h_3 : (\Delta_{13}, p_{11}), F_8 \wedge F_9 \vdash (\Delta_{12}, p_{11}), F_7} \wedge_L \quad \frac{}{\bullet h_{10} : ((\Delta_{13}, p_{11}), F_8 \wedge F_9), F_7 \vdash \Delta_{12}, p_{11}} I \\
\hline
\frac{}{- : (\Delta_{13}, p_{11}), F_8 \wedge F_9 \vdash \Delta_{12}, p_{11}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{}{- : \Delta_{13}, p_{11}, F_8 \wedge F_9 \vdash \Delta_{12}, p_{11}} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \Delta_7 \vdash \top, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}, \top} \wedge_L \quad \frac{h_{10} : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \wedge F_9), \top \vdash \Delta_{11}} \top_L \\
\hline
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{}{- : \Delta_7, F_8 \wedge F_9 \vdash \Delta_{11}} \text{ax/w}
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_8, F_9, \top, \Delta_{12} \vdash F_7, \Delta_{11}}{\bullet h_3 : (\top, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}, F_7} \wedge_L \quad \frac{h_{10} : F_7, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \wedge F_9), F_7 \vdash \Delta_{11}} \top_L \\
\hline
\frac{}{- : (\top, \Delta_{12}), F_8 \wedge F_9 \vdash \Delta_{11}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{\bullet h_3 : \top, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11}, F_7}{\bullet h_3 : \top, \Delta_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{11}} \text{ax/w} \quad \frac{h_{10} : \top, \Delta_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{11}}{\bullet h_{10} : \top, \Delta_{12}, F_7, F_8 \wedge F_9 \vdash \Delta_{11}} \text{ax/w} \\
\hline
\frac{}{- : \top, \Delta_{12}, F_8 \wedge F_9 \vdash \Delta_{11}} \text{hCut}
\end{array}$$

## 8.9 Status of $\vee_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : F_9, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \rightarrow F_{14} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \rightarrow F_{14}}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash (\Delta_{12}, F_{13} \rightarrow F_{14}), F_7} \vee_L \quad \frac{h_{11} : F_7, F_{13}, \Delta_8, F_9 \vee F_{10} \vdash F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \vee F_{10}), F_7 \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \rightarrow_R \\
\hline
\frac{}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{h_3 : \Delta_8, F_{13}, F_9 \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_{13}, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_8, F_{10}, F_{13} \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_{10}, F_{13} \vdash \Delta_{12}, F_{14}, F_7} \text{inv-th/ax} \\
\hline
\frac{}{- : \Delta_8, F_{13}, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}} \vee_L \quad \frac{h_{11} : \Delta_8, F_{13}, F_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}}{\bullet h_{11} : \Delta_8, F_{13}, F_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}} \text{ax/w} \\
\hline
\frac{}{- : \Delta_8, F_{13}, F_9 \vee F_{10} \vdash \Delta_{12}, F_{14}} \rightarrow_R \quad \frac{}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \rightarrow F_{14}} \text{hCut}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : F_9, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \wedge F_{14} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \wedge F_{14}}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash (\Delta_{12}, F_{13} \wedge F_{14}), F_7} \vee_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \vee F_{10} \vdash F_{13}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \vee F_{10}), F_7 \vdash \Delta_{12}, F_{13} \wedge F_{14}} \wedge_R \\
\hline
\frac{}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14}} \text{Cut} \\
\hline
\frac{}{\rightarrow} \\
\hline
\frac{h_3 : \Delta_8, F_9 \vdash \Delta_{12}, F_{13}, F_7}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_8, F_{10} \vdash \Delta_{12}, F_{13}, F_7}{\bullet h_3 : \Delta_8, F_{10} \vdash \Delta_{12}, F_{13}, F_7} \text{inv-th/ax} \\
\hline
\frac{}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_7} \vee_L \quad \frac{h_{11} : \Delta_8, F_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}}{\bullet h_{11} : \Delta_8, F_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}} \text{ax/w} \\
\hline
\frac{}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}} \text{hCut} \quad \frac{h_3 : \Delta_8, F_9 \vdash \Delta_{12}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_9 \vdash \Delta_{12}, F_{14}, F_7} \text{inv-th/ax} \\
\hline
\frac{}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \wedge F_{14}} \wedge_R
\end{array}$$



- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : F_9, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \vee F_{14} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12}, F_{13} \vee F_{14}}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash (\Delta_{12}, F_{13} \vee F_{14}), F_7} \vee_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \vee F_{10} \vdash F_{13}, F_{14}, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \vee F_{10}), F_7 \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_8, F_9 \vdash \Delta_{12}, F_{13}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7} \text{inv-th/ax} \quad \frac{h_3 : \Delta_8, F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7} \text{inv-th/ax}}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}, F_7} \vee_L \quad \frac{h_{11} : \Delta_8, F_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}}{h_{11} : \Delta_8, F_7, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}} \text{ax/W} \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13}, F_{14}} \text{hCut} \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}, F_{13} \vee F_{14}} \vee_R
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : F_9, \Delta_8 \vdash F_7, \perp, \Delta_{12} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \perp, \Delta_{12}}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash (\perp, \Delta_{12}), F_7} \vee_L \quad \frac{h_{11} : F_7, \Delta_8, F_9 \vee F_{10} \vdash \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \vee F_{10}), F_7 \vdash \perp, \Delta_{12}} \perp_R \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{Cut} \\
\rightarrow \\
\frac{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash \perp, \Delta_{12}, F_7}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{ax/W} \quad \frac{h_{11} : \Delta_8, F_7, F_9 \vee F_{10} \vdash \perp, \Delta_{12}}{h_{11} : \Delta_8, F_7, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{ax/W} \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \perp, \Delta_{12}} \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : F_9, \Delta_8 \vdash F_7, \top, \Delta_{12} \quad h_3 : F_{10}, \Delta_8 \vdash F_7, \top, \Delta_{12}}{\bullet h_3 : \Delta_8, F_9 \vee F_{10} \vdash (\top, \Delta_{12}), F_7} \vee_L \quad \frac{h_{11} : (\Delta_8, F_9 \vee F_{10}), F_7 \vdash \top, \Delta_{12}}{\bullet h_{11} : (\Delta_8, F_9 \vee F_{10}), F_7 \vdash \top, \Delta_{12}} \top_R \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \top, \Delta_{12}} \text{Cut} \\
\rightarrow \\
\frac{\quad}{- : \Delta_8, F_9 \vee F_{10} \vdash \top, \Delta_{12}} \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_3 : F_8, \Box \Gamma_{13}, \Delta_{14} \vdash \Box F_7, \Delta_{11}, \Box F_{12} \quad h_3 : F_9, \Box \Gamma_{13}, \Delta_{14} \vdash \Box F_7, \Delta_{11}, \Box F_{12}}{\bullet h_3 : (\Box \Gamma_{13}, \Delta_{14}), F_8 \vee F_9 \vdash (\Delta_{11}, \Box F_{12}), \Box F_7} \vee_L \quad \frac{h_{10} : \Box \Gamma_{13}, \Box F_7 \vdash F_{12}}{\bullet h_{10} : ((\Box \Gamma_{13}, \Delta_{14}), F_8 \vee F_9), \Box F_7 \vdash \Delta_{11}, \Box F_{12}} \text{Cut} \\
\frac{\quad}{- : (\Box \Gamma_{13}, \Delta_{14}), F_8 \vee F_9 \vdash \Delta_{11}, \Box F_{12}} \rightarrow \\
\frac{\frac{h_3 : \Delta_{14}, F_8, \Box \Gamma_{13} \vdash \Box F_7, \Delta_{11}, \Box F_{12}}{\bullet h_3 : \Delta_{14}, F_8, \Box \Gamma_{13} \vdash \Delta_{11}, \Box F_{12}} \text{ax/W} \quad \frac{h_{10} : \Box F_7, \Box \Gamma_{13} \vdash F_{12}}{\bullet h_{10} : \Box F_7, \Delta_{14}, F_8, \Box \Gamma_{13} \vdash \Delta_{11}, \Box F_{12}} \text{ax/W}}{\bullet h_{10} : \Box F_7, \Delta_{14}, F_8, \Box \Gamma_{13} \vdash \Delta_{11}, \Box F_{12}} \text{hCut} \quad \frac{h_{10} : \Box F_7, \Delta_{14}, F_8, \Box \Gamma_{13} \vdash \Delta_{11}, \Box F_{12}}{\bullet h_{10} : \Box F_7, \Delta_{14}, F_8, \Box \Gamma_{13} \vdash \Delta_{11}, \Box F_{12}} \text{ax/W} \\
\frac{\quad}{- : \Delta_{14}, \Box \Gamma_{13}, F_8 \vee F_9 \vdash \Delta_{11}, \Box F_{12}} \vee \\
\frac{\quad}{- : \Delta_{14}, \Box \Gamma_{13}, F_8 \vee F_9 \vdash \Delta_{11}, \Box F_{12}} A4 \\
\frac{h_3 : F_8, \Box \Gamma_{11}, \Delta_{14} \vdash F_7, \Delta_{12}, \Box F_{13} \quad h_3 : F_9, \Box \Gamma_{11}, \Delta_{14} \vdash F_7, \Delta_{12}, \Box F_{13}}{\bullet h_3 : (\Box \Gamma_{11}, \Delta_{14}), F_8 \vee F_9 \vdash (\Delta_{12}, \Box F_{13}), F_7} \vee_L \quad \frac{h_{10} : \Box \Gamma_{11} \vdash F_{13}}{\bullet h_{10} : ((\Box \Gamma_{11}, \Delta_{14}), F_8 \vee F_9), F_7 \vdash \Delta_{12}, \Box F_{13}} A4 \\
\frac{\quad}{- : (\Box \Gamma_{11}, \Delta_{14}), F_8 \vee F_9 \vdash \Delta_{12}, \Box F_{13}} \text{Cut} \\
\rightarrow \\
\frac{\quad}{- : \Box \Gamma_{11} \vdash F_{13}} \text{ax/W} \\
\frac{\quad}{- : \Delta_{14}, \Box \Gamma_{11}, F_8 \vee F_9 \vdash \Delta_{12}, \Box F_{13}} A4
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7 \vdash F_{11} \rightarrow F_{12}, \Delta_{13} \quad h_3 : F_9, \Delta_7 \vdash F_{11} \rightarrow F_{12}, \Delta_{13}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \rightarrow F_{12}} \vee_L \quad \frac{h_{10} : \Delta_7, F_8 \vee F_9 \vdash F_{11}, \Delta_{13} \quad h_{10} : F_{12}, \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \rightarrow F_{12} \vdash \Delta_{13}} \rightarrow_L \\
\frac{\quad}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}} \text{Cut} \\
\rightarrow \\
\frac{\frac{\quad}{- : \Delta_7, F_{11}, F_8 \vdash \Delta_{13}, F_{12}} \text{inv-th/ax} \quad \frac{\quad}{- : \Delta_7, F_{11}, F_9 \vdash \Delta_{13}, F_{12}} \text{inv-th/ax}}{\bullet h_{10} : (\Delta_7, F_{11}, F_8 \vee F_9) \vdash \Delta_{13}, F_{12}} \vee_L \quad \frac{\quad}{- : \Delta_7, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}} \text{ax/W} \\
\frac{\quad}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}} \text{ax/W} \quad \frac{\quad}{- : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}} \text{sCut} \\
\frac{\quad}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_{14}, F_{11} \rightarrow F_{12} \vdash F_7, \Delta_{13} \quad h_3 : F_9, \Delta_{14}, F_{11} \rightarrow F_{12} \vdash F_7, \Delta_{13}}{\bullet h_3 : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L \quad \frac{h_{10} : F_7, \Delta_{14}, F_8 \vee F_9 \vdash F_{11}, \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9) \vdash F_{11}, \Delta_{13}} \\
\hline
\frac{}{- : (\Delta_{14}, F_{11} \rightarrow F_{12}), F_8 \vee F_9 \vdash \Delta_{13}}{\rightarrow} \\
\frac{\frac{h_3 : \Delta_{14}, F_8 \vdash \Delta_{13}, F_{11}, F_7}{\bullet h_3 : \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_7} \text{ inv-th/ax} \quad \frac{h_3 : \Delta_{14}, F_9 \vdash \Delta_{13}, F_{11}, F_7}{\vee_L} \text{ inv-th/ax}}{\frac{}{- : \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}}} \vee_L \quad \frac{h_{10} : \Delta_{14}, F_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}}{\text{ax/W}} \quad \frac{h_3 : \Delta_{14}, F_{12}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{ inv-th/ax}}{\frac{}{- : \Delta_{14}, F_{11} \rightarrow F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}} \text{ hCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7 \vdash F_{11} \wedge F_{12}, \Delta_{13} \quad h_3 : F_9, \Delta_7 \vdash F_{11} \wedge F_{12}, \Delta_{13}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}} \vee_L \quad \frac{h_{10} : F_{11}, F_{12}, \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \wedge F_{12} \vdash \Delta_{13}} \wedge_L \\
\hline
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\rightarrow} \text{ Cut} \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{\text{ax/W}} \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_8 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_8, F_{11} \wedge F_{12} \vdash \Delta_{13}} \text{ inv-th/ax}}{\frac{}{- : \Delta_7, F_8 \vdash \Delta_{13}}} \wedge_L \quad \frac{h_3 : \Delta_7, F_9 \vdash \Delta_{13}, F_{11} \wedge F_{12}}{\text{ax/W}} \quad \frac{h_{10} : \Delta_7, F_{11}, F_{12}, F_9 \vdash \Delta_{13}}{\bullet h_{10} : \Delta_7, F_9, F_{11} \wedge F_{12} \vdash \Delta_{13}} \text{ inv-th/ax}}{\frac{}{- : \Delta_7, F_9 \vdash \Delta_{13}}} \wedge_L \\
\hline
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\vee_L} \\
\frac{h_3 : F_8, \Delta_{14}, F_{11} \wedge F_{12} \vdash F_7, \Delta_{13} \quad h_3 : F_9, \Delta_{14}, F_{11} \wedge F_{12} \vdash F_7, \Delta_{13}}{\bullet h_3 : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L \quad \frac{h_{10} : F_7, F_{11}, F_{12}, \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{13}} \wedge_L \\
\hline
\frac{}{- : (\Delta_{14}, F_{11} \wedge F_{12}), F_8 \vee F_9 \vdash \Delta_{13}}{\rightarrow} \text{ Cut} \\
\frac{\frac{h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{ inv-th/ax} \quad \frac{h_3 : \Delta_{14}, F_{11}, F_{12}, F_9 \vdash \Delta_{13}, F_7}{\vee_L} \text{ inv-th/ax}}{\frac{}{- : \Delta_{14}, F_{11}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}} \wedge_L \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_{12}, F_7, F_8 \vee F_9 \vdash \Delta_{13}}{\text{ax/W}} \\
\hline
\frac{}{- : \Delta_{14}, F_{11} \wedge F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}{\wedge_L}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7 \vdash F_{11} \vee F_{12}, \Delta_{13} \quad h_3 : F_9, \Delta_7 \vdash F_{11} \vee F_{12}, \Delta_{13}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11} \vee F_{12}} \vee_L \quad \frac{h_{10} : F_{11}, \Delta_7, F_8 \vee F_9 \vdash \Delta_{13} \quad h_{10} : F_{12}, \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), F_{11} \vee F_{12} \vdash \Delta_{13}} \vee_L \\
\hline
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\rightarrow} \text{ Cut} \\
\frac{\frac{}{- : \Delta_7, F_8 \vdash \Delta_{13}, F_{11}, F_{12}}{\text{inv-th/ax}} \quad \frac{}{- : \Delta_7, F_9 \vdash \Delta_{13}, F_{11}, F_{12}}{\text{inv-th/ax}}}{\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}, F_{12}}} \vee_L \quad \frac{}{- : \Delta_7, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_{11}}{\text{ax/W}} \quad \frac{}{- : \Delta_7, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}}{\text{sCut}} \\
\hline
\frac{}{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{13}}{\vee_L} \\
\frac{h_3 : F_8, \Delta_{14}, F_{11} \vee F_{12} \vdash F_7, \Delta_{13} \quad h_3 : F_9, \Delta_{14}, F_{11} \vee F_{12} \vdash F_7, \Delta_{13}}{\bullet h_3 : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \vee F_9 \vdash \Delta_{13}, F_7} \vee_L \quad \frac{h_{10} : F_7, F_{11}, \Delta_{14}, F_8 \vee F_9 \vdash \Delta_{13}}{\bullet h_{10} : ((\Delta_{14}, F_{11} \vee F_{12}), F_8 \vee F_9) \vdash \Delta_{13}} \\
\hline
\frac{}{- : (\Delta_{14}, F_{11} \vee F_{12}), F_8 \vee F_9 \vdash \Delta_{13}}{\rightarrow} \\
\frac{\frac{h_3 : \Delta_{14}, F_{11}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{ inv-th/ax} \quad \frac{h_3 : \Delta_{14}, F_{11}, F_9 \vdash \Delta_{13}, F_7}{\vee_L} \text{ inv-th/ax}}{\frac{}{- : \Delta_{14}, F_{11}, F_8 \vee F_9 \vdash \Delta_{13}}} \vee_L \quad \frac{h_{10} : \Delta_{14}, F_{11}, F_7, F_8 \vee F_9 \vdash \Delta_{13}}{\text{ax/W}} \quad \frac{h_3 : \Delta_{14}, F_{12}, F_8 \vdash \Delta_{13}, F_7}{\bullet h_3 : \Delta_{14}, F_{12}, F_8 \vee F_9 \vdash \Delta_{13}, F_7} \text{ inv-th/ax}}{\frac{}{- : \Delta_{14}, F_{11} \vee F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}} \text{ hCut} \\
\hline
\frac{}{- : \Delta_{14}, F_{11} \vee F_{12}, F_8 \vee F_9 \vdash \Delta_{13}}{\vee_L} \\
\frac{h_3 : F_{10}, \Delta_8 \vdash F_7, \Delta_{12} \quad h_3 : F_{11}, \Delta_8 \vdash F_7, \Delta_{12}}{\bullet h_3 : \Delta_8, F_{10} \vee F_{11} \vdash \Delta_{12}, F_7} \vee_L \quad \frac{h_9 : F_7, F_{10}, \Delta_8 \vdash \Delta_{12} \quad h_9 : F_7, F_{11}, \Delta_8 \vdash \Delta_{12}}{\bullet h_9 : (\Delta_8, F_{10} \vee F_{11}), F_7 \vdash \Delta_{12}} \vee_L \\
\hline
\frac{}{- : \Delta_8, F_{10} \vee F_{11} \vdash \Delta_{12}}{\rightarrow} \text{ Cut} \\
\frac{\frac{h_3 : \Delta_8, F_{10} \vdash \Delta_{12}, F_7}{\text{ax/W}} \quad \frac{h_9 : \Delta_8, F_{10}, F_7 \vdash \Delta_{12}}{\bullet h_9 : \Delta_8, F_{10}, F_7 \vdash \Delta_{12}} \text{ H}}{\frac{}{- : \Delta_8, F_{10} \vdash \Delta_{12}}} \text{ hCut} \quad \frac{h_3 : \Delta_8, F_{11} \vdash \Delta_{12}, F_7}{\text{ax/W}} \quad \frac{h_9 : \Delta_8, F_{11}, F_7 \vdash \Delta_{12}}{\bullet h_9 : \Delta_8, F_{11}, F_7 \vdash \Delta_{12}} \text{ H}}{\frac{}{- : \Delta_8, F_{11} \vdash \Delta_{12}}} \text{ hCut} \\
\hline
\frac{}{- : \Delta_8, F_{10} \vee F_{11} \vdash \Delta_{12}}{\vee_L}
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \Delta_7 \vdash \boxed{F_{11}}, \Delta_{12} \quad h_3 : F_9, \Delta_7 \vdash \boxed{F_{11}}, \Delta_{12}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, \boxed{F_{11}}} \vee_L \quad \frac{h_{10} : F_{11}, \Delta_7, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), \boxed{F_{11}} \vdash \Delta_{12}} AT}{\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}}{\rightarrow} \text{Cut}} \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{12}, \boxed{F_{11}}}{- : \Delta_7, F_8 \vdash \Delta_{12}} ax/W \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, F_8, \boxed{F_{11}} \vdash \Delta_{12}}{\bullet h_{10} : \Delta_7, F_8, \boxed{F_{11}} \vdash \Delta_{12}} inv\text{-th}/ax \quad \frac{h_{10} : \Delta_7, F_{11}, F_8, \boxed{F_{11}} \vdash \Delta_{12}}{AT}}{\frac{- : \Delta_7, F_8 \vdash \Delta_{12}}{hCut}} \quad \frac{\frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, \boxed{F_{11}}}{- : \Delta_7, F_9 \vdash \Delta_{12}} ax/W \quad \frac{\frac{h_{10} : \Delta_7, F_{11}, F_9, \boxed{F_{11}} \vdash \Delta_{12}}{\bullet h_{10} : \Delta_7, F_9, \boxed{F_{11}} \vdash \Delta_{12}} inv\text{-th}/ax \quad \frac{h_{10} : \Delta_7, F_{11}, F_9, \boxed{F_{11}} \vdash \Delta_{12}}{AT}}{\frac{- : \Delta_7, F_9 \vdash \Delta_{12}}{hCut}} \vee_L \\
\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}}{\rightarrow}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \Delta_{13}, \boxed{F_{11}} \vdash F_7, \Delta_{12} \quad h_3 : F_9, \Delta_{13}, \boxed{F_{11}} \vdash F_7, \Delta_{12}}{\bullet h_3 : (\Delta_{13}, \boxed{F_{11}}), F_8 \vee F_9 \vdash \Delta_{12}, F_7} \vee_L \quad \frac{h_{10} : F_7, F_{11}, \Delta_{13}, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}}{\bullet h_{10} : ((\Delta_{13}, \boxed{F_{11}}), F_8 \vee F_9), F_7 \vdash \Delta_{12}} AT}{\frac{- : (\Delta_{13}, \boxed{F_{11}}), F_8 \vee F_9 \vdash \Delta_{12}}{\rightarrow} \text{Cut}} \\
\frac{\frac{\bullet h_3 : \Delta_{13}, F_{11}, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}, F_7}{- : \Delta_{13}, F_{11}, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}} ax/W \quad \frac{h_{10} : \Delta_{13}, F_{11}, F_7, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}}{ax/W}}{\frac{- : \Delta_{13}, F_{11}, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}}{hCut}} AT \\
\frac{- : \Delta_{13}, \boxed{F_{11}}, F_8 \vee F_9 \vdash \Delta_{12}}{\rightarrow}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \Delta_7 \vdash \perp, \Delta_{11} \quad h_3 : F_9, \Delta_7 \vdash \perp, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}, \perp} \vee_L \quad \frac{h_{10} : (\Delta_7, F_8 \vee F_9), \perp \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), \perp \vdash \Delta_{11}} \perp_L}{\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{\rightarrow} \text{Cut}} \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \perp, \Delta_{11}}{- : \Delta_7, F_8 \vdash \Delta_{11}} ax/W \quad \frac{\bullet h_{10} : \perp, \Delta_7, F_8 \vdash \Delta_{11}}{\perp_L} \quad \frac{h_3 : \Delta_7, F_9 \vdash \perp, \Delta_{11}}{- : \Delta_7, F_9 \vdash \Delta_{11}} ax/W \quad \frac{\bullet h_{10} : \perp, \Delta_7, F_9 \vdash \Delta_{11}}{\perp_L}}{\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{hCut} \vee_L} \\
\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{\rightarrow}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \perp, \Delta_{12} \vdash F_7, \Delta_{11} \quad h_3 : F_9, \perp, \Delta_{12} \vdash F_7, \Delta_{11}}{\bullet h_3 : (\perp, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}, F_7} \vee_L \quad \frac{h_{10} : ((\perp, \Delta_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{11}}{\bullet h_{10} : ((\perp, \Delta_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{11}} \perp_L}{\frac{- : (\perp, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}}{\rightarrow} \text{Cut}} \\
\frac{- : \perp, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}}{\rightarrow} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \Delta_7 \vdash p_{11}, \Delta_{12}, p_{11} \quad h_3 : F_9, \Delta_7 \vdash p_{11}, \Delta_{12}, p_{11}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash (\Delta_{12}, p_{11}), p_{11}} \vee_L \quad \frac{h_{10} : (\Delta_7, F_8 \vee F_9), p_{11} \vdash \Delta_{12}, p_{11}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), p_{11} \vdash \Delta_{12}, p_{11}} I}{\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}}{\rightarrow} \text{Cut}} \\
\frac{\frac{h_3 : \Delta_7, F_8 \vdash \Delta_{12}, p_{11}, p_{11}}{- : \Delta_7, F_8 \vdash \Delta_{12}, p_{11}} ax/W \quad \frac{\bullet h_{10} : \Delta_7, F_8, p_{11} \vdash \Delta_{12}, p_{11}}{I} \quad \frac{h_3 : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}, p_{11}}{- : \Delta_7, F_9 \vdash \Delta_{12}, p_{11}} ax/W \quad \frac{\bullet h_{10} : \Delta_7, F_9, p_{11} \vdash \Delta_{12}, p_{11}}{I}}{\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}}{hCut} \vee_L} \\
\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}}{\rightarrow}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \Delta_{13}, p_{11} \vdash F_7, \Delta_{12}, p_{11} \quad h_3 : F_9, \Delta_{13}, p_{11} \vdash F_7, \Delta_{12}, p_{11}}{\bullet h_3 : (\Delta_{13}, p_{11}), F_8 \vee F_9 \vdash (\Delta_{12}, p_{11}), F_7} \vee_L \quad \frac{h_{10} : ((\Delta_{13}, p_{11}), F_8 \vee F_9), F_7 \vdash \Delta_{12}, p_{11}}{\bullet h_{10} : ((\Delta_{13}, p_{11}), F_8 \vee F_9), F_7 \vdash \Delta_{12}, p_{11}} I}{\frac{- : (\Delta_{13}, p_{11}), F_8 \vee F_9 \vdash \Delta_{12}, p_{11}}{\rightarrow} \text{Cut}} \\
\frac{- : \Delta_{13}, p_{11}, F_8 \vee F_9 \vdash \Delta_{12}, p_{11}}{\rightarrow} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{h_3 : F_8, \Delta_7 \vdash \top, \Delta_{11} \quad h_3 : F_9, \Delta_7 \vdash \top, \Delta_{11}}{\bullet h_3 : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}, \top} \vee_L \quad \frac{h_{10} : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{\bullet h_{10} : (\Delta_7, F_8 \vee F_9), \top \vdash \Delta_{11}} \top_L}{\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{\rightarrow} \text{Cut}} \\
\frac{- : \Delta_7, F_8 \vee F_9 \vdash \Delta_{11}}{\rightarrow} ax/W
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_8, \top, \Delta_{12} \vdash F_7, \Delta_{11} \quad h_3 : F_9, \top, \Delta_{12} \vdash F_7, \Delta_{11}}{\bullet h_3 : (\top, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11}, F_7} \vee_L \quad \frac{h_{10} : F_7, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}}{\bullet h_{10} : ((\top, \Delta_{12}), F_8 \vee F_9), F_7 \vdash \Delta_{11}} \top_L \\
\hline
- : (\top, \Delta_{12}), F_8 \vee F_9 \vdash \Delta_{11} \\
\hline
\frac{\bullet h_3 : \top, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}, F_7 \quad \text{ax/W} \quad \frac{h_{10} : \top, \Delta_{12}, F_7, F_8 \vee F_9 \vdash \Delta_{11}}{\text{hCut}} \quad \text{ax/W}}{- : \top, \Delta_{12}, F_8 \vee F_9 \vdash \Delta_{11}} \text{hCut}
\end{array}$$

## 8.10 Status of AT: OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7, [F_8 \vdash F_6, \Delta_{10}, F_{11} \rightarrow F_{12}] \quad \text{AT} \quad \frac{h_9 : F_6, F_{11}, \Delta_7, [F_8 \vdash F_{12}, \Delta_{10}]}{\bullet h_9 : (\Delta_7, [F_8]), F_6 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}} \rightarrow_R}{\bullet h_3 : \Delta_7, [F_8 \vdash (\Delta_{10}, F_{11} \rightarrow F_{12}), F_6]} \text{Cut} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}] \\
\hline
\frac{\bullet h_3 : \Delta_7, F_8, [F_8 \vdash \Delta_{10}, F_6, F_{11} \rightarrow F_{12}] \quad \text{ax/W} \quad \frac{\bullet h_9 : \Delta_7, F_6, F_8, [F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}]}{\text{hCut}} \quad \text{ax/W}}{- : \Delta_7, F_8, [F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}]} \text{hCut} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}] \quad \text{AT} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \rightarrow F_{12}]
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7, [F_8 \vdash F_6, \Delta_{10}, F_{11} \wedge F_{12}] \quad \text{AT} \quad \frac{h_9 : F_6, \Delta_7, [F_8 \vdash F_{11}, \Delta_{10}]}{\bullet h_9 : (\Delta_7, [F_8]), F_6 \vdash \Delta_{10}, F_{11} \wedge F_{12}} \wedge_R}{\bullet h_3 : \Delta_7, [F_8 \vdash (\Delta_{10}, F_{11} \wedge F_{12}), F_6]} \text{Cut} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}] \\
\hline
\frac{\bullet h_3 : \Delta_7, F_8, [F_8 \vdash \Delta_{10}, F_6, F_{11} \wedge F_{12}] \quad \text{ax/W} \quad \frac{\bullet h_9 : \Delta_7, F_6, F_8, [F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}]}{\text{hCut}} \quad \text{ax/W}}{- : \Delta_7, F_8, [F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}]} \text{hCut} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}] \quad \text{AT} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \wedge F_{12}]
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7, [F_8 \vdash F_6, \Delta_{10}, F_{11} \vee F_{12}] \quad \text{AT} \quad \frac{h_9 : F_6, \Delta_7, [F_8 \vdash F_{11}, F_{12}, \Delta_{10}]}{\bullet h_9 : (\Delta_7, [F_8]), F_6 \vdash \Delta_{10}, F_{11} \vee F_{12}} \vee_R}{\bullet h_3 : \Delta_7, [F_8 \vdash (\Delta_{10}, F_{11} \vee F_{12}), F_6]} \text{Cut} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}] \\
\hline
\frac{\bullet h_3 : \Delta_7, F_8, [F_8 \vdash \Delta_{10}, F_6, F_{11} \vee F_{12}] \quad \text{ax/W} \quad \frac{\bullet h_9 : \Delta_7, F_6, F_8, [F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}]}{\text{hCut}} \quad \text{ax/W}}{- : \Delta_7, F_8, [F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}]} \text{hCut} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}] \quad \text{AT} \\
\hline
- : \Delta_7, [F_8 \vdash \Delta_{10}, F_{11} \vee F_{12}]
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7, [F_8 \vdash F_6, \perp, \Delta_{10}] \quad \text{AT} \quad \frac{h_9 : F_6, \Delta_7, [F_8 \vdash \Delta_{10}]}{\bullet h_9 : (\Delta_7, [F_8]), F_6 \vdash \perp, \Delta_{10}} \perp_R}{\bullet h_3 : \Delta_7, [F_8 \vdash (\perp, \Delta_{10}), F_6]} \text{Cut} \\
\hline
- : \Delta_7, [F_8 \vdash \perp, \Delta_{10}] \\
\hline
\frac{\bullet h_3 : \Delta_7, [F_8 \vdash \perp, \Delta_{10}, F_6] \quad \text{ax/W} \quad \frac{h_9 : \Delta_7, F_6, [F_8 \vdash \perp, \Delta_{10}]}{\text{hCut}} \quad \text{ax/W}}{- : \Delta_7, [F_8 \vdash \perp, \Delta_{10}]} \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : F_8, \Delta_7, [F_8 \vdash F_6, \top, \Delta_{10}] \quad \text{AT} \quad \frac{h_9 : (\Delta_7, [F_8]), F_6 \vdash \top, \Delta_{10}}{\bullet h_9 : (\Delta_7, [F_8]), F_6 \vdash \top, \Delta_{10}} \top_R}{\bullet h_3 : \Delta_7, [F_8 \vdash (\top, \Delta_{10}), F_6]} \text{Cut} \\
\hline
- : \Delta_7, [F_8 \vdash \top, \Delta_{10}] \\
\hline
\frac{- : \Delta_7, [F_8 \vdash \top, \Delta_{10}]}{- : \Delta_7, [F_8 \vdash \top, \Delta_{10}]} \top_R
\end{array}$$

- Case rule A4

$$\begin{array}{c}
\frac{h_3 : F_7, (\Box \Gamma_{12}, \Delta_9), [\Box F_6, \Delta_{10}, \Box F_{11}] \quad \bullet h_3 : ((\Box \Gamma_{12}, \Delta_9), [\Box F_6, \Delta_{10}, \Box F_{11}])}{- : (\Box \Gamma_{12}, \Delta_9), [\Box F_7 \vdash \Delta_{10}, \Box F_{11}]} \text{AT} \quad \frac{h_8 : \Box \Gamma_{12}, \Box F_6, [\Box F_7 \vdash F_{11}]}{\bullet h_8 : ((\Box \Gamma_{12}, \Delta_9), [\Box F_7], \Box F_6 \vdash \Delta_{10}, \Box F_{11})} \text{A4} \\
\hline
- : (\Box \Gamma_{12}, \Delta_9), [\Box F_7 \vdash \Delta_{10}, \Box F_{11}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_9, F_7, \Box \Gamma_{12}, [\Box F_6, \Delta_{10}, \Box F_{11}]}{- : \Delta_9, F_7, \Box \Gamma_{12}, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}]} \text{ax/W} \quad \frac{\bullet h_8 : \Box F_6, \Delta_9, F_7, \Box \Gamma_{12}, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}]}{- : \Delta_9, \Box \Gamma_{12}, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}]} \text{ax/W} \\
\hline
- : \Delta_9, F_7, \Box \Gamma_{12}, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}] \quad \text{hCut} \\
\hline
- : \Delta_9, \Box \Gamma_{12}, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}] \quad \text{AT}
\end{array} \\
\\
\frac{h_3 : F_7, (\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash F_6, \Delta_9, \Box F_{10}] \quad \bullet h_3 : ((\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash F_6, \Delta_9, \Box F_{10}])}{- : (\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash \Delta_9, \Box F_{10}]} \text{AT} \quad \frac{h_8 : \Box \Gamma_{11}, [\Box F_7 \vdash F_{10}]}{\bullet h_8 : ((\Box \Gamma_{11}, \Delta_{12}), [\Box F_7], F_6 \vdash \Delta_9, \Box F_{10})} \text{A4} \\
\hline
- : (\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash \Delta_9, \Box F_{10}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{- : \Box \Gamma_{11}, [\Box F_7 \vdash F_{10}]}{- : \Delta_{12}, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}]} \text{ax/W} \\
\hline
- : \Delta_{12}, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}] \quad \text{A4}
\end{array} \\
\\
\frac{h_3 : F_7, (\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash \Box F_6, \Delta_9, \Box F_{10}] \quad \bullet h_3 : ((\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash (\Delta_9, \Box F_{10}), \Box F_6])}{- : (\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash \Delta_9, \Box F_{10}]} \text{AT} \quad \frac{h_8 : \Box \Gamma_{11}, \Box F_6 \vdash F_{10}}{\bullet h_8 : ((\Box \Gamma_{11}, \Delta_{12}), [\Box F_7], \Box F_6 \vdash \Delta_9, \Box F_{10})} \text{A4} \\
\hline
- : (\Box \Gamma_{11}, \Delta_{12}), [\Box F_7 \vdash \Delta_9, \Box F_{10}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_{12}, F_7, \Box \Gamma_{11}, [\Box F_7 \vdash \Box F_6, \Delta_9, \Box F_{10}]}{- : \Delta_{12}, F_7, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}]} \text{ax/W} \quad \frac{\bullet h_8 : \Box F_6, \Delta_{12}, F_7, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}]}{- : \Delta_{12}, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}]} \text{ax/W} \\
\hline
- : \Delta_{12}, F_7, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}] \quad \text{hCut} \\
\hline
- : \Delta_{12}, \Box \Gamma_{11}, [\Box F_7 \vdash \Delta_9, \Box F_{10}] \quad \text{AT}
\end{array} \\
\\
\frac{h_3 : F_7, (\Box \Gamma_9, \Delta_{12}), [\Box F_7 \vdash F_6, \Delta_{10}, \Box F_{11}] \quad \bullet h_3 : ((\Box \Gamma_9, \Delta_{12}), [\Box F_7 \vdash (\Delta_{10}, \Box F_{11}), F_6])}{- : (\Box \Gamma_9, \Delta_{12}), [\Box F_7 \vdash \Delta_{10}, \Box F_{11}]} \text{AT} \quad \frac{h_8 : \Box \Gamma_9 \vdash F_{11}}{\bullet h_8 : ((\Box \Gamma_9, \Delta_{12}), [\Box F_7], F_6 \vdash \Delta_{10}, \Box F_{11})} \text{A4} \\
\hline
- : (\Box \Gamma_9, \Delta_{12}), [\Box F_7 \vdash \Delta_{10}, \Box F_{11}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{- : \Box \Gamma_9 \vdash F_{11}}{- : \Delta_{12}, \Box \Gamma_9, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}]} \text{ax/W} \\
\hline
- : \Delta_{12}, \Box \Gamma_9, [\Box F_7 \vdash \Delta_{10}, \Box F_{11}] \quad \text{A4}
\end{array}
\end{array}$$

• Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : F_7, \Delta_6, [\Box F_7 \vdash F_9 \rightarrow F_{10}, \Delta_{11}] \quad \bullet h_3 : (\Delta_6, [\Box F_7 \vdash \Delta_{11}, F_9 \rightarrow F_{10}])}{- : \Delta_6, [\Box F_7 \vdash \Delta_{11}]} \text{AT} \quad \frac{h_8 : \Delta_6, [\Box F_7 \vdash F_9, \Delta_{11}] \quad h_8 : F_{10}, \Delta_6, [\Box F_7 \vdash \Delta_{11}]}{\bullet h_8 : ((\Delta_6, [\Box F_7]), F_9 \rightarrow F_{10} \vdash \Delta_{11})} \text{A4} \\
\hline
- : \Delta_6, [\Box F_7 \vdash \Delta_{11}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}, F_9 \rightarrow F_{10}]}{- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}]} \text{ax/W} \quad \frac{\bullet h_8 : \Delta_6, F_7, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}]}{- : \Delta_6, [\Box F_7 \vdash \Delta_{11}]} \text{ax/W} \\
\hline
- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}] \quad \text{hCut} \\
\hline
- : \Delta_6, [\Box F_7 \vdash \Delta_{11}] \quad \text{AT}
\end{array} \\
\\
\frac{h_3 : F_7, (\Delta_{12}, F_9 \rightarrow F_{10}), [\Box F_7 \vdash F_6, \Delta_{11}] \quad \bullet h_3 : ((\Delta_{12}, F_9 \rightarrow F_{10}), [\Box F_7 \vdash \Delta_{11}, F_6])}{- : (\Delta_{12}, F_9 \rightarrow F_{10}), [\Box F_7 \vdash \Delta_{11}]} \text{AT} \quad \frac{h_8 : F_6, \Delta_{12}, [\Box F_7 \vdash F_9, \Delta_{11}] \quad h_8 : F_6, F_{10}, \Delta_{12}, [\Box F_7 \vdash \Delta_{11}]}{\bullet h_8 : ((\Delta_{12}, F_9 \rightarrow F_{10}), [\Box F_7], F_6 \vdash \Delta_{11})} \text{A4} \\
\hline
- : (\Delta_{12}, F_9 \rightarrow F_{10}), [\Box F_7 \vdash \Delta_{11}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_{12}, F_7, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}, F_6]}{- : \Delta_{12}, F_7, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}]} \text{ax/W} \quad \frac{\bullet h_8 : \Delta_{12}, F_6, F_7, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}]}{- : \Delta_{12}, F_7, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}]} \text{ax/W} \\
\hline
- : \Delta_{12}, F_7, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}] \quad \text{hCut} \\
\hline
- : \Delta_{12}, [\Box F_7, F_9 \rightarrow F_{10} \vdash \Delta_{11}] \quad \text{AT}
\end{array}
\end{array}$$

• Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : F_7, \Delta_6, [\Box F_7 \vdash F_9 \wedge F_{10}, \Delta_{11}] \quad \bullet h_3 : (\Delta_6, [\Box F_7 \vdash \Delta_{11}, F_9 \wedge F_{10}])}{- : \Delta_6, [\Box F_7 \vdash \Delta_{11}]} \text{AT} \quad \frac{h_8 : F_9, F_{10}, \Delta_6, [\Box F_7 \vdash \Delta_{11}]}{\bullet h_8 : ((\Delta_6, [\Box F_7]), F_9 \wedge F_{10} \vdash \Delta_{11})} \wedge_L \\
\hline
- : \Delta_6, [\Box F_7 \vdash \Delta_{11}] \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{h_3 : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}, F_9 \wedge F_{10}]}{- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}]} \text{ax/W} \quad \frac{\bullet h_8 : \Delta_6, F_7, [\Box F_7, F_9 \wedge F_{10} \vdash \Delta_{11}]}{- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}]} \text{ax/W} \\
\hline
- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{11}] \quad \text{hCut} \\
\hline
- : \Delta_6, [\Box F_7 \vdash \Delta_{11}] \quad \text{AT}
\end{array}
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_7, (\Delta_{12}, F_9 \wedge F_{10}), [F_7 \vdash F_6, \Delta_{11}] \quad AT \quad \frac{h_8 : F_6, F_9, F_{10}, \Delta_{12}, [F_7 \vdash \Delta_{11}] \quad \wedge_L}{\bullet h_8 : ((\Delta_{12}, F_9 \wedge F_{10}), [F_7], F_6 \vdash \Delta_{11})} \text{Cut}}{\vdash : (\Delta_{12}, F_9 \wedge F_{10}), [F_7 \vdash \Delta_{11}]} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{12}, F_7, [F_7, F_9 \wedge F_{10} \vdash \Delta_{11}, F_6] \quad ax/W \quad \bullet h_8 : \Delta_{12}, F_6, F_7, [F_7, F_9 \wedge F_{10} \vdash \Delta_{11}] \quad ax/W}{\vdash : \Delta_{12}, F_7, [F_7, F_9 \wedge F_{10} \vdash \Delta_{11}] \quad hCut} \quad AT}{\vdash : \Delta_{12}, [F_7, F_9 \wedge F_{10} \vdash \Delta_{11}]}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{h_3 : F_7, \Delta_6, [F_7 \vdash F_9 \vee F_{10}, \Delta_{11}] \quad AT \quad \frac{h_8 : F_9, \Delta_6, [F_7 \vdash \Delta_{11}] \quad h_8 : F_{10}, \Delta_6, [F_7 \vdash \Delta_{11}] \quad \vee_L}{\bullet h_8 : (\Delta_6, [F_7], F_9 \vee F_{10} \vdash \Delta_{11})} \text{Cut}}{\vdash : \Delta_6, [F_7 \vdash \Delta_{11}]} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_6, F_7, [F_7 \vdash \Delta_{11}, F_9 \vee F_{10}] \quad ax/W \quad \bullet h_8 : \Delta_6, F_7, [F_7, F_9 \vee F_{10} \vdash \Delta_{11}] \quad ax/W}{\vdash : \Delta_6, F_7, [F_7 \vdash \Delta_{11}] \quad hCut} \quad AT}{\vdash : \Delta_6, [F_7 \vdash \Delta_{11}]} \\
\\
\frac{h_3 : F_7, (\Delta_{12}, F_9 \vee F_{10}), [F_7 \vdash F_6, \Delta_{11}] \quad AT \quad \frac{h_8 : F_6, F_9, \Delta_{12}, [F_7 \vdash \Delta_{11}] \quad h_8 : F_6, F_{10}, \Delta_{12}, [F_7 \vdash \Delta_{11}] \quad \vee_L}{\bullet h_8 : ((\Delta_{12}, F_9 \vee F_{10}), [F_7], F_6 \vdash \Delta_{11})} \text{Cut}}{\vdash : (\Delta_{12}, F_9 \vee F_{10}), [F_7 \vdash \Delta_{11}]} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{12}, F_7, [F_7, F_9 \vee F_{10} \vdash \Delta_{11}, F_6] \quad ax/W \quad \bullet h_8 : \Delta_{12}, F_6, F_7, [F_7, F_9 \vee F_{10} \vdash \Delta_{11}] \quad ax/W}{\vdash : \Delta_{12}, F_7, [F_7, F_9 \vee F_{10} \vdash \Delta_{11}] \quad hCut} \quad AT}{\vdash : \Delta_{12}, [F_7, F_9 \vee F_{10} \vdash \Delta_{11}]}
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{h_3 : F_7, \Delta_6, [F_7 \vdash [F_9, \Delta_{10}]] \quad AT \quad \frac{h_8 : F_9, \Delta_6, [F_7, [F_9 \vdash \Delta_{10}]] \quad AT}{\bullet h_8 : (\Delta_6, [F_7], [F_9 \vdash \Delta_{10}])} \text{Cut}}{\vdash : \Delta_6, [F_7 \vdash \Delta_{10}]} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_6, F_7, [F_7 \vdash \Delta_{10}, [F_9]] \quad ax/W \quad \bullet h_8 : \Delta_6, F_7, [F_7, [F_9 \vdash \Delta_{10}]] \quad ax/W}{\vdash : \Delta_6, F_7, [F_7 \vdash \Delta_{10}] \quad hCut} \quad AT}{\vdash : \Delta_6, [F_7 \vdash \Delta_{10}]} \\
\\
\frac{h_3 : F_7, (\Delta_{11}, [F_9]), [F_7 \vdash F_6, \Delta_{10}] \quad AT \quad \frac{h_8 : F_6, F_9, \Delta_{11}, [F_7, [F_9 \vdash \Delta_{10}]] \quad AT}{\bullet h_8 : ((\Delta_{11}, [F_9]), [F_7], F_6 \vdash \Delta_{10})} \text{Cut}}{\vdash : (\Delta_{11}, [F_9]), [F_7 \vdash \Delta_{10}]} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_{11}, F_7, [F_7, [F_9 \vdash \Delta_{10}, F_6]] \quad ax/W \quad \bullet h_8 : \Delta_{11}, F_6, F_7, [F_7, [F_9 \vdash \Delta_{10}]] \quad ax/W}{\vdash : \Delta_{11}, F_7, [F_7, [F_9 \vdash \Delta_{10}]] \quad hCut} \quad AT}{\vdash : \Delta_{11}, [F_7, [F_9 \vdash \Delta_{10}]} \\
\\
\frac{h_3 : F_9, \Delta_7, [F_9 \vdash F_6, \Delta_{10}] \quad AT \quad \frac{h_8 : F_6, F_9, \Delta_7, [F_9 \vdash \Delta_{10}] \quad AT}{\bullet h_8 : (\Delta_7, [F_9], F_6 \vdash \Delta_{10})} \text{Cut}}{\vdash : \Delta_7, [F_9 \vdash \Delta_{10}]} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_7, F_9, [F_9 \vdash \Delta_{10}, F_6]] \quad ax/W \quad \bullet h_8 : \Delta_7, F_6, F_9, [F_9 \vdash \Delta_{10}] \quad ax/W}{\vdash : \Delta_7, F_9, [F_9 \vdash \Delta_{10}] \quad hCut} \quad AT}{\vdash : \Delta_7, [F_9 \vdash \Delta_{10}]}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : F_7, \Delta_6, [\Box F_7 \vdash \perp, \Delta_9]}{\bullet h_3 : \Delta_6, [\Box F_7 \vdash \Delta_9, \perp]} AT \quad \frac{}{\bullet h_8 : (\Delta_6, [\Box F_7], \perp \vdash \Delta_9)} \perp_L}{\frac{}{- : \Delta_6, [\Box F_7 \vdash \Delta_9]} \text{Cut}} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_6, F_7, [\Box F_7 \vdash \perp, \Delta_9]}{\bullet h_3 : \perp, \Delta_6, F_7, [\Box F_7 \vdash \Delta_9]} ax/W \quad \frac{}{\bullet h_8 : \perp, \Delta_6, F_7, [\Box F_7 \vdash \Delta_9]} \perp_L}{\frac{}{- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_9]} hCut} \\
\frac{}{- : \Delta_6, [\Box F_7 \vdash \Delta_9]} AT
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_7, (\perp, \Delta_{10}), [\Box F_7 \vdash F_6, \Delta_9]}{\bullet h_3 : (\perp, \Delta_{10}), [\Box F_7 \vdash \Delta_9, F_6]} AT \quad \frac{}{\bullet h_8 : ((\perp, \Delta_{10}), [\Box F_7], F_6 \vdash \Delta_9)} \perp_L}{\frac{}{- : (\perp, \Delta_{10}), [\Box F_7 \vdash \Delta_9]} \text{Cut}} \\
\rightarrow \\
\frac{}{- : \perp, \Delta_{10}, [\Box F_7 \vdash \Delta_9]} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{h_3 : F_7, \Delta_6, [\Box F_7 \vdash p_9, \Delta_{10}, p_9]}{\bullet h_3 : \Delta_6, [\Box F_7 \vdash (\Delta_{10}, p_9), p_9]} AT \quad \frac{}{\bullet h_8 : (\Delta_6, [\Box F_7], p_9 \vdash \Delta_{10}, p_9)} I}{\frac{}{- : \Delta_6, [\Box F_7 \vdash \Delta_{10}, p_9]} \text{Cut}} \\
\rightarrow \\
\frac{\frac{h_3 : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{10}, p_9, p_9]}{\bullet h_3 : \Delta_6, F_7, p_9, [\Box F_7 \vdash \Delta_{10}, p_9]} ax/W \quad \frac{}{\bullet h_8 : \Delta_6, F_7, p_9, [\Box F_7 \vdash \Delta_{10}, p_9]} I}{\frac{}{- : \Delta_6, F_7, [\Box F_7 \vdash \Delta_{10}, p_9]} hCut} \\
\frac{}{- : \Delta_6, [\Box F_7 \vdash \Delta_{10}, p_9]} AT
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_7, (\Delta_{11}, p_9), [\Box F_7 \vdash F_6, \Delta_{10}, p_9]}{\bullet h_3 : (\Delta_{11}, p_9), [\Box F_7 \vdash (\Delta_{10}, p_9), F_6]} AT \quad \frac{}{\bullet h_8 : ((\Delta_{11}, p_9), [\Box F_7], F_6 \vdash \Delta_{10}, p_9)} I}{\frac{}{- : (\Delta_{11}, p_9), [\Box F_7 \vdash \Delta_{10}, p_9]} \text{Cut}} \\
\rightarrow \\
\frac{}{- : \Delta_{11}, p_9, [\Box F_7 \vdash \Delta_{10}, p_9]} I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{h_3 : F_7, \Delta_6, [\Box F_7 \vdash \top, \Delta_9]}{\bullet h_3 : \Delta_6, [\Box F_7 \vdash \Delta_9, \top]} AT \quad \frac{h_8 : \Delta_6, [\Box F_7 \vdash \Delta_9]}{\bullet h_8 : (\Delta_6, [\Box F_7], \top \vdash \Delta_9)} \top_L}{\frac{}{- : \Delta_6, [\Box F_7 \vdash \Delta_9]} \text{Cut}} \\
\rightarrow \\
\frac{}{- : \Delta_6, [\Box F_7 \vdash \Delta_9]} ax/W
\end{array}$$

$$\begin{array}{c}
\frac{h_3 : F_7, (\top, \Delta_{10}), [\Box F_7 \vdash F_6, \Delta_9]}{\bullet h_3 : (\top, \Delta_{10}), [\Box F_7 \vdash \Delta_9, F_6]} AT \quad \frac{h_8 : F_6, \Delta_{10}, [\Box F_7 \vdash \Delta_9]}{\bullet h_8 : ((\top, \Delta_{10}), [\Box F_7], F_6 \vdash \Delta_9)} \top_L}{\frac{}{- : (\top, \Delta_{10}), [\Box F_7 \vdash \Delta_9]} \text{Cut}} \\
\rightarrow \\
\frac{\frac{}{\bullet h_3 : \top, \Delta_{10}, [\Box F_7 \vdash \Delta_9, F_6]} ax/W \quad \frac{}{\bullet h_8 : \top, \Delta_{10}, F_6, [\Box F_7 \vdash \Delta_9]} ax/W}{\frac{}{- : \top, \Delta_{10}, [\Box F_7 \vdash \Delta_9]} hCut}
\end{array}$$

## 8.11 Status of $\perp_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{}{\bullet h_3 : \perp, \Delta_6 \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_5} \perp_L \quad \frac{h_7 : \perp, F_5, F_9, \Delta_6 \vdash F_{10}, \Delta_8}{\bullet h_7 : (\perp, \Delta_6), F_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R}{\frac{}{- : \perp, \Delta_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut}} \\
\rightarrow \\
\frac{}{- : \perp, \Delta_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \perp_L
\end{array}$$

- Case rule  $\wedge_R$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_6 \vdash (\Delta_8, F_9 \wedge F_{10}), F_5}}{\perp_L} \quad \frac{\frac{h_7 : \perp, F_5, \Delta_6 \vdash F_9, \Delta_8 \quad h_7 : \perp, F_5, \Delta_6 \vdash F_{10}, \Delta_8}{\bullet h_7 : (\perp, \Delta_6), F_5 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R}{\frac{}{- : \perp, \Delta_6 \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut}} \rightarrow \frac{}{- : \perp, \Delta_6 \vdash \Delta_8, F_9 \wedge F_{10}} \perp_L$$

- Case rule  $\vee_R$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_6 \vdash (\Delta_8, F_9 \vee F_{10}), F_5}}{\perp_L} \quad \frac{\frac{h_7 : \perp, F_5, \Delta_6 \vdash F_9, F_{10}, \Delta_8}{\bullet h_7 : (\perp, \Delta_6), F_5 \vdash \Delta_8, F_9 \vee F_{10}} \vee_R}{\frac{}{- : \perp, \Delta_6 \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut}} \rightarrow \frac{}{- : \perp, \Delta_6 \vdash \Delta_8, F_9 \vee F_{10}} \perp_L$$

- Case rule  $\perp_R$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_6 \vdash (\perp, \Delta_8), F_5}}{\perp_L} \quad \frac{\frac{h_7 : \perp, F_5, \Delta_6 \vdash \Delta_8}{\bullet h_7 : (\perp, \Delta_6), F_5 \vdash \perp, \Delta_8}}{\perp_R}}{\frac{}{- : \perp, \Delta_6 \vdash \perp, \Delta_8}} \text{Cut} \rightarrow \frac{}{- : \perp, \Delta_6 \vdash \perp, \Delta_8} \perp_L$$

- Case rule  $\top_R$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_6 \vdash (\top, \Delta_8), F_5}}{\perp_L} \quad \frac{\frac{h_7 : (\perp, \Delta_6), F_5 \vdash \top, \Delta_8}{\bullet h_7 : (\perp, \Delta_6), F_5 \vdash \top, \Delta_8}}{\top_R}}{\frac{}{- : \perp, \Delta_6 \vdash \top, \Delta_8}} \text{Cut} \rightarrow \frac{}{- : \perp, \Delta_6 \vdash \top, \Delta_8} \top_R$$

- Case rule  $A4$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Box \Gamma_9, \Delta_{10} \vdash (\Delta_7, \Box F_8), \Box F_5}}{\perp_L} \quad \frac{\frac{h_6 : \Box \Gamma_9, \Box F_5 \vdash F_8}{\bullet h_6 : (\perp, \Box \Gamma_9, \Delta_{10}), \Box F_5 \vdash \Delta_7, \Box F_8}}{A4}}{\frac{}{- : \perp, \Box \Gamma_9, \Delta_{10} \vdash \Delta_7, \Box F_8}} \text{Cut} \rightarrow \frac{}{- : \perp, \Delta_{10}, \Box \Gamma_9 \vdash \Delta_7, \Box F_8} \perp_L$$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Box \Gamma_7, \Delta_{10} \vdash (\Delta_8, \Box F_9), F_5}}{\perp_L} \quad \frac{\frac{h_6 : \Box \Gamma_7 \vdash F_9}{\bullet h_6 : (\perp, \Box \Gamma_7, \Delta_{10}), F_5 \vdash \Delta_8, \Box F_9}}{A4}}{\frac{}{- : \perp, \Box \Gamma_7, \Delta_{10} \vdash \Delta_8, \Box F_9}} \text{Cut} \rightarrow \frac{}{- : \perp, \Delta_{10}, \Box \Gamma_7 \vdash \Delta_8, \Box F_9} \perp_L$$

- Case rule  $\rightarrow_L$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8}}{\perp_L} \quad \frac{\frac{h_6 : \perp, \Delta_5 \vdash F_7, \Delta_9 \quad h_6 : \perp, F_8, \Delta_5 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_5), F_7 \rightarrow F_8 \vdash \Delta_9}}{\rightarrow_L}}{\frac{}{- : \perp, \Delta_5 \vdash \Delta_9}} \text{Cut} \rightarrow \frac{}{- : \perp, \Delta_5 \vdash \Delta_9} \perp_L$$

$$\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5}}{\perp_L} \quad \frac{\frac{h_6 : \perp, F_5, \Delta_{10} \vdash F_7, \Delta_9 \quad h_6 : \perp, F_5, F_8, \Delta_{10} \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_{10}, F_7 \rightarrow F_8), F_5 \vdash \Delta_9}}{\rightarrow_L}}{\frac{}{- : \perp, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9}} \text{Cut} \rightarrow \frac{}{- : \perp, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \perp_L$$

- Case rule  $\wedge_L$



$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_9, F_7 \wedge F_8}{\vdash : \perp, \Delta_5 \vdash \Delta_9} \perp_L \quad \frac{\frac{h_6 : \perp, F_7, F_8, \Delta_5 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_5), F_7 \wedge F_8 \vdash \Delta_9} \wedge_L}{\vdash : \perp, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_5 \vdash \Delta_9} \perp_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5}{\vdash : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \perp_L \quad \frac{\frac{h_6 : \perp, F_5, F_7, F_8, \Delta_{10} \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_{10}, F_7 \wedge F_8), F_5 \vdash \Delta_9} \wedge_L}{\vdash : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \perp_L
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_9, F_7 \vee F_8}{\vdash : \perp, \Delta_5 \vdash \Delta_9} \perp_L \quad \frac{\frac{h_6 : \perp, F_7, \Delta_5 \vdash \Delta_9 \quad h_6 : \perp, F_8, \Delta_5 \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_5), F_7 \vee F_8 \vdash \Delta_9} \vee_L}{\vdash : \perp, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_5 \vdash \Delta_9} \perp_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5}{\vdash : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \perp_L \quad \frac{\frac{h_6 : \perp, F_5, F_7, \Delta_{10} \vdash \Delta_9 \quad h_6 : \perp, F_5, F_8, \Delta_{10} \vdash \Delta_9}{\bullet h_6 : (\perp, \Delta_{10}, F_7 \vee F_8), F_5 \vdash \Delta_9} \vee_L}{\vdash : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \perp_L
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_8, [\Box F_7]}{\vdash : \perp, \Delta_5 \vdash \Delta_8} \perp_L \quad \frac{\frac{h_6 : \perp, F_7, \Delta_5, [\Box F_7] \vdash \Delta_8}{\bullet h_6 : (\perp, \Delta_5), [\Box F_7] \vdash \Delta_8} AT}{\vdash : \perp, \Delta_5 \vdash \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_5 \vdash \Delta_8} \perp_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_9, [\Box F_7] \vdash \Delta_8, F_5}{\vdash : \perp, \Delta_9, [\Box F_7] \vdash \Delta_8} \perp_L \quad \frac{\frac{h_6 : \perp, F_5, F_7, \Delta_9, [\Box F_7] \vdash \Delta_8}{\bullet h_6 : (\perp, \Delta_9, [\Box F_7]), F_5 \vdash \Delta_8} AT}{\vdash : \perp, \Delta_9, [\Box F_7] \vdash \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_9, [\Box F_7] \vdash \Delta_8} \perp_L
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_6 \vdash \Delta_8, F_5}{\vdash : \perp, \Delta_6 \vdash \Delta_8} \perp_L \quad \frac{\frac{h_7 : (\perp, \Delta_6), F_5 \vdash \Delta_8}{\bullet h_7 : (\perp, \Delta_6), F_5 \vdash \Delta_8} \perp_L}{\vdash : \perp, \Delta_6 \vdash \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_6 \vdash \Delta_8} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_5 \vdash (\Delta_8, p_7), p_7}{\vdash : \perp, \Delta_5 \vdash \Delta_8, p_7} \perp_L \quad \frac{\frac{h_6 : (\perp, \Delta_5), p_7 \vdash \Delta_8, p_7}{\bullet h_6 : (\perp, \Delta_5), p_7 \vdash \Delta_8, p_7} I}{\vdash : \perp, \Delta_5 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_5 \vdash \Delta_8, p_7} \perp_L
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\bullet h_3 : \perp, \Delta_9, p_7 \vdash (\Delta_8, p_7), F_5}{\vdash : \perp, \Delta_9, p_7 \vdash \Delta_8, p_7} \perp_L \quad \frac{\frac{h_6 : (\perp, \Delta_9, p_7), F_5 \vdash \Delta_8, p_7}{\bullet h_6 : (\perp, \Delta_9, p_7), F_5 \vdash \Delta_8, p_7} I}{\vdash : \perp, \Delta_9, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{}{\vdash : \perp, \Delta_9, p_7 \vdash \Delta_8, p_7} \perp_L
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_3 : \perp, \Delta_5 \vdash \Delta_7, \top} \perp_L \quad \frac{h_6 : \perp, \Delta_5 \vdash \Delta_7}{\bullet h_6 : (\perp, \Delta_5), \top \vdash \Delta_7} \top_L}{\frac{}{- : \perp, \Delta_5 \vdash \Delta_7} \text{Cut}} \rightarrow \\
\frac{}{- : \perp, \Delta_5 \vdash \Delta_7} \perp_L \\
\\
\frac{\frac{\frac{}{\bullet h_3 : \perp, \top, \Delta_8 \vdash \Delta_7, F_5} \perp_L \quad \frac{h_6 : \perp, F_5, \Delta_8 \vdash \Delta_7}{\bullet h_6 : (\perp, \top, \Delta_8), F_5 \vdash \Delta_7} \top_L}{\frac{}{- : \perp, \top, \Delta_8 \vdash \Delta_7} \text{Cut}} \rightarrow \\
\frac{}{- : \perp, \top, \Delta_8 \vdash \Delta_7} \perp_L
\end{array}$$

## 8.12 Status of $I$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Delta_5, p_6 \vdash (\Delta_8, F_9 \rightarrow F_{10}), p_6} I \quad \frac{h_7 : F_9, \Delta_5, p_6, p_6 \vdash F_{10}, \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R}{\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{Cut}} \rightarrow \\
\frac{\frac{\frac{}{\bullet h_1 : \Delta_5, F_9, p_6 \vdash \Delta_8, F_{10}, p_6} I \quad \frac{h_7 : \Delta_5, F_9, p_6, p_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\Delta_5, F_9, p_6), p_6 \vdash \Delta_8, F_{10}} \text{ax/w}}{\frac{}{- : \Delta_5, F_9, p_6 \vdash \Delta_8, F_{10}} \text{hCut}} \rightarrow_R \\
\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \Delta_7, p_8 \vdash ((\Delta_{12}, F_{10} \rightarrow F_{11}), p_8), F_6} I \quad \frac{h_9 : F_6, F_{10}, \Delta_7, p_8 \vdash F_{11}, \Delta_{12}, p_8}{\bullet h_9 : (\Delta_7, p_8), F_6 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), p_8} \rightarrow_R}{\frac{}{- : \Delta_7, p_8 \vdash (\Delta_{12}, F_{10} \rightarrow F_{11}), p_8} \text{Cut}} \rightarrow \\
\frac{}{- : \Delta_7, p_8 \vdash \Delta_{12}, p_8, F_{10} \rightarrow F_{11}} I
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Delta_5, p_6 \vdash (\Delta_8, F_9 \wedge F_{10}), p_6} I \quad \frac{\frac{h_7 : \Delta_5, p_6, p_6 \vdash F_9, \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R}{\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \wedge F_{10}} \text{Cut}} \rightarrow \\
\frac{\frac{\frac{}{\bullet h_1 : \Delta_5, p_6 \vdash \Delta_8, F_9, p_6} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_9}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9} \text{ax/w}}{\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9} \text{hCut}} \frac{\frac{\frac{}{\bullet h_1 : \Delta_5, p_6 \vdash \Delta_8, F_{10}, p_6} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_{10}}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_{10}} \text{ax/w}}{\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_{10}} \text{hCut}} \wedge_R \\
\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \wedge F_{10}} \\
\\
\frac{\frac{\frac{}{\bullet h_2 : \Delta_7, p_8 \vdash ((\Delta_{12}, F_{10} \wedge F_{11}), p_8), F_6} I \quad \frac{\frac{h_9 : F_6, \Delta_7, p_8 \vdash F_{10}, \Delta_{12}, p_8}{\bullet h_9 : (\Delta_7, p_8), F_6 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), p_8} \wedge_R}{\frac{}{- : \Delta_7, p_8 \vdash (\Delta_{12}, F_{10} \wedge F_{11}), p_8} \text{Cut}} \rightarrow \\
\frac{}{- : \Delta_7, p_8 \vdash \Delta_{12}, p_8, F_{10} \wedge F_{11}} I
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{\frac{\frac{}{\bullet h_1 : \Delta_5, p_6 \vdash (\Delta_8, F_9 \vee F_{10}), p_6} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash F_9, F_{10}, \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_9 \vee F_{10}} \vee_R}{\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \vee F_{10}} \text{Cut}} \rightarrow \\
\frac{\frac{\frac{}{\bullet h_1 : \Delta_5, p_6 \vdash \Delta_8, F_{10}, F_9, p_6} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8, F_{10}, F_9}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \Delta_8, F_{10}, F_9} \text{ax/w}}{\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_{10}, F_9} \text{hCut}} \vee_R \\
\frac{}{- : \Delta_5, p_6 \vdash \Delta_8, F_9 \vee F_{10}}
\end{array}$$

$$\begin{array}{c}
\frac{\bullet h_2 : \Delta_7, p_8 \vdash ((\Delta_{12}, F_{10} \vee F_{11}), p_8), F_6}{- : \Delta_7, p_8 \vdash (\Delta_{12}, F_{10} \vee F_{11}), p_8} I \quad \frac{h_9 : F_6, \Delta_7, p_8 \vdash F_{10}, F_{11}, \Delta_{12}, p_8}{\bullet h_9 : (\Delta_7, p_8), F_6 \vdash (\Delta_{12}, F_{10} \vee F_{11}), p_8} \vee_R \\
\text{Cut} \\
\frac{}{- : \Delta_7, p_8 \vdash \Delta_{12}, p_8, F_{10} \vee F_{11}} \rightarrow I
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\perp, \Delta_8), p_6}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} I \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \Delta_8}{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \perp, \Delta_8} \perp_R \\
\text{Cut} \\
\frac{}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} \rightarrow \\
\frac{\bullet h_1 : \Delta_5, p_6 \vdash \perp, \Delta_8, p_6}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} \text{ax/W} \quad \frac{h_7 : \Delta_5, p_6, p_6 \vdash \perp, \Delta_8}{h_7 : \Delta_5, p_6, p_6 \vdash \perp, \Delta_8} \text{hCut} \\
\frac{}{- : \Delta_5, p_6 \vdash \perp, \Delta_8} \rightarrow \\
\frac{\bullet h_2 : \Delta_7, p_8 \vdash ((\perp, \Delta_{10}), p_8), F_6}{- : \Delta_7, p_8 \vdash (\perp, \Delta_{10}), p_8} I \quad \frac{h_9 : F_6, \Delta_7, p_8 \vdash \Delta_{10}, p_8}{\bullet h_9 : (\Delta_7, p_8), F_6 \vdash (\perp, \Delta_{10}), p_8} \perp_R \\
\text{Cut} \\
\frac{}{- : \Delta_7, p_8 \vdash \perp, \Delta_{10}, p_8} \rightarrow I
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{\bullet h_1 : \Delta_5, p_6 \vdash (\top, \Delta_8), p_6}{- : \Delta_5, p_6 \vdash \top, \Delta_8} I \quad \frac{\bullet h_7 : (\Delta_5, p_6), p_6 \vdash \top, \Delta_8}{- : \Delta_5, p_6 \vdash \top, \Delta_8} \top_R \\
\text{Cut} \\
\frac{}{- : \Delta_5, p_6 \vdash \top, \Delta_8} \rightarrow \top_R \\
\frac{\bullet h_2 : \Delta_7, p_8 \vdash ((\top, \Delta_{10}), p_8), F_6}{- : \Delta_7, p_8 \vdash (\top, \Delta_{10}), p_8} I \quad \frac{\bullet h_9 : (\Delta_7, p_8), F_6 \vdash (\top, \Delta_{10}), p_8}{- : \Delta_7, p_8 \vdash (\top, \Delta_{10}), p_8} \top_R \\
\text{Cut} \\
\frac{}{- : \Delta_7, p_8 \vdash \top, \Delta_{10}, p_8} \rightarrow \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{\bullet h_1 : (\Box \Gamma_7, \Delta_{10}), p_5 \vdash (\Delta_8, \llbracket F_9 \rrbracket), p_5}{- : (\Box \Gamma_7, \Delta_{10}), p_5 \vdash \Delta_8, \llbracket F_9 \rrbracket} I \quad \frac{h_6 : \Box \Gamma_7 \vdash F_9}{\bullet h_6 : ((\Box \Gamma_7, \Delta_{10}), p_5), p_5 \vdash \Delta_8, \llbracket F_9 \rrbracket} A4 \\
\text{Cut} \\
\frac{}{- : (\Box \Gamma_7, \Delta_{10}), p_5 \vdash \Delta_8, \llbracket F_9 \rrbracket} \rightarrow \\
\frac{}{- : \Box \Gamma_7 \vdash F_9} \text{ax/W} \\
\frac{}{- : \Delta_{10}, \Box \Gamma_7, p_5 \vdash \Delta_8, \llbracket F_9 \rrbracket} A4 \\
\frac{\bullet h_2 : (\Box \Gamma_{11}, \Delta_{12}), p_7 \vdash ((\Delta_{10}, \llbracket F_9 \rrbracket), p_7), \Box F_6}{- : (\Box \Gamma_{11}, \Delta_{12}), p_7 \vdash (\Delta_{10}, \llbracket F_9 \rrbracket), p_7} I \quad \frac{h_8 : \Box \Gamma_{11}, \Box F_6 \vdash F_9}{\bullet h_8 : ((\Box \Gamma_{11}, \Delta_{12}), p_7), \Box F_6 \vdash (\Delta_{10}, \llbracket F_9 \rrbracket), p_7} A4 \\
\text{Cut} \\
\frac{}{- : (\Box \Gamma_{11}, \Delta_{12}), p_7 \vdash (\Delta_{10}, \llbracket F_9 \rrbracket), p_7} \rightarrow \\
\frac{}{- : \Delta_{12}, \Box \Gamma_{11}, p_7 \vdash \Delta_{10}, p_7, \llbracket F_9 \rrbracket} I \\
\frac{\bullet h_2 : (\Box \Gamma_9, \Delta_{12}), p_7 \vdash ((\Delta_{11}, \llbracket F_{10} \rrbracket), p_7), F_6}{- : (\Box \Gamma_9, \Delta_{12}), p_7 \vdash (\Delta_{11}, \llbracket F_{10} \rrbracket), p_7} I \quad \frac{h_8 : \Box \Gamma_9 \vdash F_{10}}{\bullet h_8 : ((\Box \Gamma_9, \Delta_{12}), p_7), F_6 \vdash (\Delta_{11}, \llbracket F_{10} \rrbracket), p_7} A4 \\
\text{Cut} \\
\frac{}{- : (\Box \Gamma_9, \Delta_{12}), p_7 \vdash (\Delta_{11}, \llbracket F_{10} \rrbracket), p_7} \rightarrow \\
\frac{}{- : \Delta_{12}, \Box \Gamma_9, p_7 \vdash \Delta_{11}, p_7, \llbracket F_{10} \rrbracket} I
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\Delta_{10}, F_7 \rightarrow F_8), p_5 \vdash \Delta_9, p_5}{- : (\Delta_{10}, F_7 \rightarrow F_8), p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, p_5, p_5 \vdash F_7, \Delta_9 \quad h_6 : F_8, \Delta_{10}, p_5, p_5 \vdash \Delta_9}{\bullet h_6 : ((\Delta_{10}, F_7 \rightarrow F_8), p_5), p_5 \vdash \Delta_9} \rightarrow_L}{- : (\Delta_{10}, F_7 \rightarrow F_8), p_5 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_{10}, p_5 \vdash \Delta_9, F_7, p_5}{- : \Delta_{10}, p_5 \vdash \Delta_9, F_7} I \quad \frac{h_6 : \Delta_{10}, p_5, p_5 \vdash \Delta_9, F_7}{h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_1 : \Delta_{10}, F_8, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_8, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9}{- : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/W}}{- : \Delta_{10}, p_5, F_7 \rightarrow F_8 \vdash \Delta_9} \text{hCut} \\
\rightarrow_L \\
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \rightarrow F_{11}}{- : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \rightarrow F_{11}} I \quad \frac{h_9 : \Delta_6, p_7 \vdash F_{10}, \Delta_8, p_7 \quad h_9 : F_{11}, \Delta_6, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), F_{10} \rightarrow F_{11} \vdash \Delta_8, p_7} \rightarrow_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_2 : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash (\Delta_8, p_7), F_6} I \quad \frac{h_9 : F_6, \Delta_{12}, p_7 \vdash F_{10}, \Delta_8, p_7 \quad h_9 : F_6, F_{11}, \Delta_{12}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{12}, F_{10} \rightarrow F_{11}), p_7), F_6 \vdash \Delta_8, p_7} \rightarrow_L}{- : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{- : (\Delta_{12}, F_{10} \rightarrow F_{11}), p_7 \vdash \Delta_8, p_7}{- : \Delta_{12}, p_7, F_{10} \rightarrow F_{11} \vdash \Delta_8, p_7} I
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\Delta_{10}, F_7 \wedge F_8), p_5 \vdash \Delta_9, p_5}{- : (\Delta_{10}, F_7 \wedge F_8), p_5 \vdash \Delta_9} I \quad \frac{h_6 : F_7, F_8, \Delta_{10}, p_5, p_5 \vdash \Delta_9}{\bullet h_6 : ((\Delta_{10}, F_7 \wedge F_8), p_5), p_5 \vdash \Delta_9} \wedge_L}{- : (\Delta_{10}, F_7 \wedge F_8), p_5 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_{10}, F_7, F_8, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_7, F_8, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_7, F_8, p_5, p_5 \vdash \Delta_9}{h_6 : \Delta_{10}, F_7, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/W}}{- : \Delta_{10}, p_5, F_7 \wedge F_8 \vdash \Delta_9} \text{hCut} \\
\wedge_L \\
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \wedge F_{11}}{- : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \wedge F_{11}} I \quad \frac{h_9 : F_{10}, F_{11}, \Delta_6, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), F_{10} \wedge F_{11} \vdash \Delta_8, p_7} \wedge_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_2 : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash (\Delta_8, p_7), F_6} I \quad \frac{h_9 : F_6, F_{10}, F_{11}, \Delta_{12}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{12}, F_{10} \wedge F_{11}), p_7), F_6 \vdash \Delta_8, p_7} \wedge_L}{- : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{- : (\Delta_{12}, F_{10} \wedge F_{11}), p_7 \vdash \Delta_8, p_7}{- : \Delta_{12}, p_7, F_{10} \wedge F_{11} \vdash \Delta_8, p_7} I
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\Delta_{10}, F_7 \vee F_8), p_5 \vdash \Delta_9, p_5}{- : (\Delta_{10}, F_7 \vee F_8), p_5 \vdash \Delta_9} I \quad \frac{h_6 : F_7, \Delta_{10}, p_5, p_5 \vdash \Delta_9 \quad h_6 : F_8, \Delta_{10}, p_5, p_5 \vdash \Delta_9}{\bullet h_6 : ((\Delta_{10}, F_7 \vee F_8), p_5), p_5 \vdash \Delta_9} \vee_L}{- : (\Delta_{10}, F_7 \vee F_8), p_5 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_{10}, F_7, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_7, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_7, p_5, p_5 \vdash \Delta_9}{h_6 : \Delta_{10}, F_7, p_5, p_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_1 : \Delta_{10}, F_8, p_5 \vdash \Delta_9, p_5}{- : \Delta_{10}, F_8, p_5 \vdash \Delta_9} I \quad \frac{h_6 : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9}{- : \Delta_{10}, F_8, p_5, p_5 \vdash \Delta_9} \text{ax/W}}{- : \Delta_{10}, p_5, F_7 \vee F_8 \vdash \Delta_9} \text{hCut} \\
\vee_L \\
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \vee F_{11}}{- : \Delta_6, p_7 \vdash (\Delta_8, p_7), F_{10} \vee F_{11}} I \quad \frac{h_9 : F_{10}, \Delta_6, p_7 \vdash \Delta_8, p_7 \quad h_9 : F_{11}, \Delta_6, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), F_{10} \vee F_{11} \vdash \Delta_8, p_7} \vee_L}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \\
\frac{\frac{\bullet h_2 : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash (\Delta_8, p_7), F_6} I \quad \frac{h_9 : F_6, F_{10}, \Delta_{12}, p_7 \vdash \Delta_8, p_7 \quad h_9 : F_6, F_{11}, \Delta_{12}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{12}, F_{10} \vee F_{11}), p_7), F_6 \vdash \Delta_8, p_7} \vee_L}{- : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{- : (\Delta_{12}, F_{10} \vee F_{11}), p_7 \vdash \Delta_8, p_7}{- : \Delta_{12}, p_7, F_{10} \vee F_{11} \vdash \Delta_8, p_7} I
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\Delta_9, \llbracket F_7 \rrbracket, p_5 \vdash \Delta_8, p_5}{- : (\Delta_9, \llbracket F_7 \rrbracket, p_5 \vdash \Delta_8} I \quad \frac{\frac{h_6 : F_7, \Delta_9, p_5, p_5, \llbracket F_7 \rrbracket \vdash \Delta_8}{\bullet h_6 : ((\Delta_9, \llbracket F_7 \rrbracket, p_5), p_5 \vdash \Delta_8} AT}{\text{Cut}} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \Delta_9, F_7, p_5, \llbracket F_7 \rrbracket \vdash \Delta_8, p_5}{- : \Delta_9, F_7, p_5, \llbracket F_7 \rrbracket \vdash \Delta_8} I \quad \frac{h_6 : \Delta_9, F_7, p_5, p_5, \llbracket F_7 \rrbracket \vdash \Delta_8}{\bullet h_6 : ((\Delta_9, \llbracket F_7 \rrbracket, p_5), p_5 \vdash \Delta_8} ax/W}{\text{hCut}} \\
\frac{- : \Delta_9, F_7, p_5, \llbracket F_7 \rrbracket \vdash \Delta_8}{- : \Delta_9, p_5, \llbracket F_7 \rrbracket \vdash \Delta_8} AT
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), \llbracket F_{10} \rrbracket}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{\frac{h_9 : F_{10}, \Delta_6, p_7, \llbracket F_{10} \rrbracket \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), \llbracket F_{10} \rrbracket \vdash \Delta_8, p_7} AT}{\text{Cut}} \\
\rightarrow \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : (\Delta_{11}, \llbracket F_{10} \rrbracket, p_7 \vdash (\Delta_8, p_7), F_6}{- : (\Delta_{11}, \llbracket F_{10} \rrbracket, p_7 \vdash \Delta_8, p_7} I \quad \frac{\frac{h_9 : F_6, F_{10}, \Delta_{11}, p_7, \llbracket F_{10} \rrbracket \vdash \Delta_8, p_7}{\bullet h_9 : ((\Delta_{11}, \llbracket F_{10} \rrbracket, p_7), F_6 \vdash \Delta_8, p_7} AT}{\text{Cut}} \\
\rightarrow \\
\frac{- : (\Delta_{11}, \llbracket F_{10} \rrbracket, p_7 \vdash \Delta_8, p_7}{- : \Delta_{11}, p_7, \llbracket F_{10} \rrbracket \vdash \Delta_8, p_7} I
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\perp, \Delta_8), p_5 \vdash \Delta_7, p_5}{- : (\perp, \Delta_8), p_5 \vdash \Delta_7} I \quad \frac{\frac{\bullet h_6 : ((\perp, \Delta_8), p_5), p_5 \vdash \Delta_7}{- : (\perp, \Delta_8), p_5 \vdash \Delta_7} \perp_L}{\text{Cut}} \\
\rightarrow \\
\frac{- : \perp, \Delta_8, p_5 \vdash \Delta_7}{- : \perp, \Delta_8, p_5 \vdash \Delta_7} \perp_L
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), \perp}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{\frac{\bullet h_9 : (\Delta_6, p_7), \perp \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} \perp_L}{\text{Cut}} \\
\rightarrow \\
\frac{- : \Delta_6, p_7 \vdash \Delta_8, p_7}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : (\perp, \Delta_{10}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\perp, \Delta_{10}), p_7 \vdash \Delta_8, p_7} I \quad \frac{\frac{\bullet h_9 : ((\perp, \Delta_{10}), p_7), F_6 \vdash \Delta_8, p_7}{- : (\perp, \Delta_{10}), p_7 \vdash \Delta_8, p_7} \perp_L}{\text{Cut}} \\
\rightarrow \\
\frac{- : \perp, \Delta_{10}, p_7 \vdash \Delta_8, p_7}{- : \perp, \Delta_{10}, p_7 \vdash \Delta_8, p_7} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : \Delta_5, p_7 \vdash (\Delta_8, p_7), p_7}{- : \Delta_5, p_7 \vdash \Delta_8, p_7} I \quad \frac{\frac{\bullet h_6 : (\Delta_5, p_7), p_7 \vdash \Delta_8, p_7}{- : \Delta_5, p_7 \vdash \Delta_8, p_7} I}{\text{Cut}} \\
\rightarrow \\
\frac{- : \Delta_5, p_7 \vdash \Delta_8, p_7}{- : \Delta_5, p_7 \vdash \Delta_8, p_7} I
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_1 : (\Delta_9, p_7), p_5 \vdash (\Delta_8, p_7), p_5}{- : (\Delta_9, p_7), p_5 \vdash \Delta_8, p_7} I \quad \frac{\frac{\bullet h_6 : ((\Delta_9, p_7), p_5), p_5 \vdash \Delta_8, p_7}{- : (\Delta_9, p_7), p_5 \vdash \Delta_8, p_7} I}{\text{Cut}} \\
\rightarrow \\
\frac{- : \Delta_9, p_5, p_7 \vdash \Delta_8, p_7}{- : \Delta_9, p_5, p_7 \vdash \Delta_8, p_7} I
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : \Delta_6, p_7 \vdash ((\Delta_{10}, p_9), p_7), p_9}{- : \Delta_6, p_7 \vdash (\Delta_{10}, p_9), p_7} I \quad \frac{\frac{\bullet h_8 : (\Delta_6, p_7), p_9 \vdash (\Delta_{10}, p_9), p_7}{- : \Delta_6, p_7 \vdash (\Delta_{10}, p_9), p_7} I}{\text{Cut}} \\
\rightarrow \\
\frac{- : \Delta_6, p_7 \vdash \Delta_{10}, p_7, p_9}{- : \Delta_6, p_7 \vdash \Delta_{10}, p_7, p_9} I
\end{array}$$
  

$$\begin{array}{c}
\frac{\frac{\bullet h_2 : (\Delta_{11}, p_9), p_7 \vdash ((\Delta_{10}, p_9), p_7), F_6}{- : (\Delta_{11}, p_9), p_7 \vdash (\Delta_{10}, p_9), p_7} I \quad \frac{\frac{\bullet h_8 : ((\Delta_{11}, p_9), p_7), F_6 \vdash (\Delta_{10}, p_9), p_7}{- : (\Delta_{11}, p_9), p_7 \vdash (\Delta_{10}, p_9), p_7} I}{\text{Cut}} \\
\rightarrow \\
\frac{- : \Delta_{11}, p_7, p_9 \vdash \Delta_{10}, p_7, p_9}{- : \Delta_{11}, p_7, p_9 \vdash \Delta_{10}, p_7, p_9} I
\end{array}$$

$$\begin{array}{c}
\frac{\bullet h_2 : \Delta_7, p_9 \vdash (\Delta_{10}, p_9), F_6}{- : \Delta_7, p_9 \vdash \Delta_{10}, p_9} I \quad \frac{\bullet h_8 : (\Delta_7, p_9), F_6 \vdash \Delta_{10}, p_9}{- : \Delta_7, p_9 \vdash \Delta_{10}, p_9} I \\
\text{Cut} \\
\rightarrow \\
- : \Delta_7, p_9 \vdash \Delta_{10}, p_9 \quad I
\end{array}$$

- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\bullet h_1 : (\top, \Delta_8), p_5 \vdash \Delta_7, p_5}{- : (\top, \Delta_8), p_5 \vdash \Delta_7} I \quad \frac{h_6 : \Delta_8, p_5, p_5 \vdash \Delta_7}{\bullet h_6 : ((\top, \Delta_8), p_5), p_5 \vdash \Delta_7} \top_L \\
\text{Cut} \\
\rightarrow \\
\frac{\bullet h_1 : \top, \Delta_8, p_5 \vdash \Delta_7, p_5}{- : \top, \Delta_8, p_5 \vdash \Delta_7} I \quad \frac{h_6 : \top, \Delta_8, p_5, p_5 \vdash \Delta_7}{- : \top, \Delta_8, p_5 \vdash \Delta_7} \text{ax/W} \\
\text{hCut} \\
\\
\frac{\bullet h_2 : \Delta_6, p_7 \vdash (\Delta_8, p_7), \top}{- : \Delta_6, p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : \Delta_6, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : (\Delta_6, p_7), \top \vdash \Delta_8, p_7} \top_L \\
\text{Cut} \\
\rightarrow \\
- : \Delta_6, p_7 \vdash \Delta_8, p_7 \quad I \\
\\
\frac{\bullet h_2 : (\top, \Delta_{10}), p_7 \vdash (\Delta_8, p_7), F_6}{- : (\top, \Delta_{10}), p_7 \vdash \Delta_8, p_7} I \quad \frac{h_9 : F_6, \Delta_{10}, p_7 \vdash \Delta_8, p_7}{\bullet h_9 : ((\top, \Delta_{10}), p_7), F_6 \vdash \Delta_8, p_7} \top_L \\
\text{Cut} \\
\rightarrow \\
- : \top, \Delta_{10}, p_7 \vdash \Delta_8, p_7 \quad I
\end{array}$$

### 8.13 Status of $\top_L$ : OK

- Case rule  $\rightarrow_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_6 \vdash F_5, \Delta_8, F_9 \rightarrow F_{10}}{\bullet h_3 : \top, \Delta_6 \vdash (\Delta_8, F_9 \rightarrow F_{10}), F_5} \top_L \quad \frac{h_7 : \top, F_5, F_9, \Delta_6 \vdash F_{10}, \Delta_8}{\bullet h_7 : (\top, \Delta_6), F_5 \vdash \Delta_8, F_9 \rightarrow F_{10}} \rightarrow_R \\
\text{Cut} \\
- : \top, \Delta_6 \vdash \Delta_8, F_9 \rightarrow F_{10} \\
\rightarrow \\
\frac{h_3 : \top, \Delta_6 \vdash \Delta_8, F_5, F_9 \rightarrow F_{10}}{- : \top, \Delta_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_6, F_5 \vdash \Delta_8, F_9 \rightarrow F_{10}}{- : \top, \Delta_6 \vdash \Delta_8, F_9 \rightarrow F_{10}} \text{ax/W} \\
\text{hCut}
\end{array}$$

- Case rule  $\wedge_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_6 \vdash F_5, \Delta_8, F_9 \wedge F_{10}}{\bullet h_3 : \top, \Delta_6 \vdash (\Delta_8, F_9 \wedge F_{10}), F_5} \top_L \quad \frac{h_7 : \top, F_5, \Delta_6 \vdash F_9, \Delta_8 \quad h_7 : \top, F_5, \Delta_6 \vdash F_{10}, \Delta_8}{\bullet h_7 : (\top, \Delta_6), F_5 \vdash \Delta_8, F_9 \wedge F_{10}} \wedge_R \\
\text{Cut} \\
- : \top, \Delta_6 \vdash \Delta_8, F_9 \wedge F_{10} \\
\rightarrow \\
\frac{h_3 : \top, \Delta_6 \vdash \Delta_8, F_5, F_9 \wedge F_{10}}{- : \top, \Delta_6 \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_6, F_5 \vdash \Delta_8, F_9 \wedge F_{10}}{- : \top, \Delta_6 \vdash \Delta_8, F_9 \wedge F_{10}} \text{ax/W} \\
\text{hCut}
\end{array}$$

- Case rule  $\vee_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_6 \vdash F_5, \Delta_8, F_9 \vee F_{10}}{\bullet h_3 : \top, \Delta_6 \vdash (\Delta_8, F_9 \vee F_{10}), F_5} \top_L \quad \frac{h_7 : \top, F_5, \Delta_6 \vdash F_9, F_{10}, \Delta_8}{\bullet h_7 : (\top, \Delta_6), F_5 \vdash \Delta_8, F_9 \vee F_{10}} \vee_R \\
\text{Cut} \\
- : \top, \Delta_6 \vdash \Delta_8, F_9 \vee F_{10} \\
\rightarrow \\
\frac{h_3 : \top, \Delta_6 \vdash \Delta_8, F_5, F_9 \vee F_{10}}{- : \top, \Delta_6 \vdash \Delta_8, F_9 \vee F_{10}} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_6, F_5 \vdash \Delta_8, F_9 \vee F_{10}}{- : \top, \Delta_6 \vdash \Delta_8, F_9 \vee F_{10}} \text{ax/W} \\
\text{hCut}
\end{array}$$

- Case rule  $\perp_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_6 \vdash F_5, \perp, \Delta_8}{\bullet h_3 : \top, \Delta_6 \vdash (\perp, \Delta_8), F_5} \top_L \quad \frac{h_7 : \top, F_5, \Delta_6 \vdash \Delta_8}{\bullet h_7 : (\top, \Delta_6), F_5 \vdash \perp, \Delta_8} \perp_R \\
\hline
- : \top, \Delta_6 \vdash \perp, \Delta_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \top, \Delta_6 \vdash \perp, \Delta_8, F_5}{- : \top, \Delta_6 \vdash \perp, \Delta_8} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_6, F_5 \vdash \perp, \Delta_8}{- : \top, \Delta_6 \vdash \perp, \Delta_8} \text{ax/W} \\
\hline
- : \top, \Delta_6 \vdash \perp, \Delta_8 \quad \text{hCut}
\end{array}$$

- Case rule  $\top_R$

$$\begin{array}{c}
\frac{h_3 : \Delta_6 \vdash F_5, \top, \Delta_8}{\bullet h_3 : \top, \Delta_6 \vdash (\top, \Delta_8), F_5} \top_L \quad \frac{}{\bullet h_7 : (\top, \Delta_6), F_5 \vdash \top, \Delta_8} \top_R \\
\hline
- : \top, \Delta_6 \vdash \top, \Delta_8 \quad \text{Cut} \\
\hline
\rightarrow \\
- : \top, \Delta_6 \vdash \top, \Delta_8 \quad \top_R
\end{array}$$

- Case rule  $A4$

$$\begin{array}{c}
\frac{h_3 : \Box \Gamma_9, \Delta_{10} \vdash \Box F_5, \Delta_7, []F_8}{\bullet h_3 : \top, \Box \Gamma_9, \Delta_{10} \vdash (\Delta_7, []F_8), \Box F_5} \top_L \quad \frac{h_6 : \Box \Gamma_9, \Box F_5 \vdash F_8}{\bullet h_6 : (\top, \Box \Gamma_9, \Delta_{10}), \Box F_5 \vdash \Delta_7, []F_8} A4 \\
\hline
- : \top, \Box \Gamma_9, \Delta_{10} \vdash \Delta_7, []F_8 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \top, \Delta_{10}, \Box \Gamma_9 \vdash \Box F_5, \Delta_7, []F_8}{- : \top, \Delta_{10}, \Box \Gamma_9 \vdash \Delta_7, []F_8} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Box F_5, \Delta_{10}, \Box \Gamma_9 \vdash \Delta_7, []F_8}{- : \top, \Delta_{10}, \Box \Gamma_9 \vdash \Delta_7, []F_8} \text{ax/W} \\
\hline
- : \top, \Delta_{10}, \Box \Gamma_9 \vdash \Delta_7, []F_8 \quad \text{hCut} \\
\hline
\frac{h_3 : \Box \Gamma_7, \Delta_{10} \vdash F_5, \Delta_8, []F_9}{\bullet h_3 : \top, \Box \Gamma_7, \Delta_{10} \vdash (\Delta_8, []F_9), F_5} \top_L \quad \frac{h_6 : \Box \Gamma_7 \vdash F_9}{\bullet h_6 : (\top, \Box \Gamma_7, \Delta_{10}), F_5 \vdash \Delta_8, []F_9} A4 \\
\hline
- : \top, \Box \Gamma_7, \Delta_{10} \vdash \Delta_8, []F_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{}{- : \Box \Gamma_7 \vdash F_9} \text{ax/W} \\
\hline
- : \top, \Delta_{10}, \Box \Gamma_7 \vdash \Delta_8, []F_9 \quad A4
\end{array}$$

- Case rule  $\rightarrow_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_5 \vdash F_7 \rightarrow F_8, \Delta_9}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8} \top_L \quad \frac{h_6 : \top, \Delta_5 \vdash F_7, \Delta_9 \quad h_6 : \top, F_8, \Delta_5 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_5), F_7 \rightarrow F_8 \vdash \Delta_9} \rightarrow_L \\
\hline
- : \top, \Delta_5 \vdash \Delta_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \rightarrow F_8}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, F_7 \rightarrow F_8 \vdash \Delta_9}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \\
\hline
- : \top, \Delta_5 \vdash \Delta_9 \quad \text{hCut} \\
\hline
\frac{h_3 : \Delta_{10}, F_7 \rightarrow F_8 \vdash F_5, \Delta_9}{\bullet h_3 : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5} \top_L \quad \frac{h_6 : \top, F_5, \Delta_{10} \vdash F_7, \Delta_9 \quad h_6 : \top, F_5, F_8, \Delta_{10} \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_{10}, F_7 \rightarrow F_8), F_5 \vdash \Delta_9} \rightarrow_L \\
\hline
- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9, F_5}{- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \rightarrow F_8 \vdash \Delta_9}{- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9} \text{ax/W} \\
\hline
- : \top, \Delta_{10}, F_7 \rightarrow F_8 \vdash \Delta_9 \quad \text{hCut}
\end{array}$$

- Case rule  $\wedge_L$

$$\begin{array}{c}
\frac{h_3 : \Delta_5 \vdash F_7 \wedge F_8, \Delta_9}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \wedge F_8} \top_L \quad \frac{h_6 : \top, F_7, F_8, \Delta_5 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_5), F_7 \wedge F_8 \vdash \Delta_9} \wedge_L \\
\hline
- : \top, \Delta_5 \vdash \Delta_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \wedge F_8}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, F_7 \wedge F_8 \vdash \Delta_9}{- : \top, \Delta_5 \vdash \Delta_9} \text{ax/W} \\
\hline
- : \top, \Delta_5 \vdash \Delta_9 \quad \text{hCut} \\
\hline
\frac{h_3 : \Delta_{10}, F_7 \wedge F_8 \vdash F_5, \Delta_9}{\bullet h_3 : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5} \top_L \quad \frac{h_6 : \top, F_5, F_7, F_8, \Delta_{10} \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_{10}, F_7 \wedge F_8), F_5 \vdash \Delta_9} \wedge_L \\
\hline
- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9 \quad \text{Cut} \\
\hline
\rightarrow \\
\frac{h_3 : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9, F_5}{- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \wedge F_8 \vdash \Delta_9}{- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9} \text{ax/W} \\
\hline
- : \top, \Delta_{10}, F_7 \wedge F_8 \vdash \Delta_9 \quad \text{hCut}
\end{array}$$

- Case rule  $\vee_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_5 \vdash F_7 \vee F_8, \Delta_9}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \vee F_8} \top_L \quad \frac{\frac{h_6 : \top, F_7, \Delta_5 \vdash \Delta_9 \quad h_6 : \top, F_8, \Delta_5 \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_5), F_7 \vee F_8 \vdash \Delta_9} \vee_L}{- : \top, \Delta_5 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \vee F_8}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_9, F_7 \vee F_8} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, F_7 \vee F_8 \vdash \Delta_9}{\bullet h_6 : \top, \Delta_5, F_7 \vee F_8 \vdash \Delta_9} \text{hCut}}{- : \top, \Delta_5 \vdash \Delta_9}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_{10}, F_7 \vee F_8 \vdash F_5, \Delta_9}{\bullet h_3 : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5} \top_L \quad \frac{\frac{h_6 : \top, F_5, F_7, \Delta_{10} \vdash \Delta_9 \quad h_6 : \top, F_5, F_8, \Delta_{10} \vdash \Delta_9}{\bullet h_6 : (\top, \Delta_{10}, F_7 \vee F_8), F_5 \vdash \Delta_9} \vee_L}{- : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5}{\bullet h_3 : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9, F_5} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \vee F_8 \vdash \Delta_9}{\bullet h_6 : \top, \Delta_{10}, F_5, F_7 \vee F_8 \vdash \Delta_9} \text{hCut}}{- : \top, \Delta_{10}, F_7 \vee F_8 \vdash \Delta_9}
\end{array}$$

- Case rule  $AT$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_5 \vdash []F_7, \Delta_8}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_8, []F_7} \top_L \quad \frac{\frac{h_6 : \top, F_7, \Delta_5, []F_7 \vdash \Delta_8}{\bullet h_6 : (\top, \Delta_5), []F_7 \vdash \Delta_8} AT}{- : \top, \Delta_5 \vdash \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, []F_7}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_8, []F_7} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, []F_7 \vdash \Delta_8}{\bullet h_6 : \top, \Delta_5, []F_7 \vdash \Delta_8} \text{hCut}}{- : \top, \Delta_5 \vdash \Delta_8}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_9, []F_7 \vdash F_5, \Delta_8}{\bullet h_3 : \top, \Delta_9, []F_7 \vdash \Delta_8, F_5} \top_L \quad \frac{\frac{h_6 : \top, F_5, F_7, \Delta_9, []F_7 \vdash \Delta_8}{\bullet h_6 : (\top, \Delta_9, []F_7), F_5 \vdash \Delta_8} AT}{- : \top, \Delta_9, []F_7 \vdash \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_9, []F_7 \vdash \Delta_8, F_5}{\bullet h_3 : \top, \Delta_9, []F_7 \vdash \Delta_8, F_5} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_9, F_5, []F_7 \vdash \Delta_8}{\bullet h_6 : \top, \Delta_9, F_5, []F_7 \vdash \Delta_8} \text{hCut}}{- : \top, \Delta_9, []F_7 \vdash \Delta_8}
\end{array}$$

- Case rule  $\perp_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_5 \vdash \perp, \Delta_7}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_7, \perp} \top_L \quad \frac{\frac{}{\bullet h_6 : (\top, \Delta_5), \perp \vdash \Delta_7} \perp_L}{- : \top, \Delta_5 \vdash \Delta_7} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \perp, \Delta_7}{\bullet h_3 : \top, \Delta_5 \vdash \perp, \Delta_7} \text{ax/W} \quad \frac{\bullet h_6 : \perp, \top, \Delta_5 \vdash \Delta_7}{\bullet h_6 : \perp, \top, \Delta_5 \vdash \Delta_7} \perp_L}{- : \top, \Delta_5 \vdash \Delta_7} \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : \perp, \Delta_8 \vdash F_5, \Delta_7}{\bullet h_3 : \top, \perp, \Delta_8 \vdash \Delta_7, F_5} \top_L \quad \frac{\frac{}{\bullet h_6 : (\top, \perp, \Delta_8), F_5 \vdash \Delta_7} \perp_L}{- : \top, \perp, \Delta_8 \vdash \Delta_7} \text{Cut} \\
\rightarrow \\
\frac{}{- : \perp, \top, \Delta_8 \vdash \Delta_7} \perp_L
\end{array}$$

- Case rule  $I$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_5 \vdash p_7, \Delta_8, p_7}{\bullet h_3 : \top, \Delta_5 \vdash (\Delta_8, p_7), p_7} \top_L \quad \frac{\frac{}{\bullet h_6 : (\top, \Delta_5), p_7 \vdash \Delta_8, p_7} I}{- : \top, \Delta_5 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_5 \vdash \Delta_8, p_7, p_7}{\bullet h_3 : \top, \Delta_5 \vdash \Delta_8, p_7, p_7} \text{ax/W} \quad \frac{\bullet h_6 : \top, \Delta_5, p_7 \vdash \Delta_8, p_7}{\bullet h_6 : \top, \Delta_5, p_7 \vdash \Delta_8, p_7} I}{- : \top, \Delta_5 \vdash \Delta_8, p_7} \text{hCut}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_9, p_7 \vdash F_5, \Delta_8, p_7}{\bullet h_3 : \top, \Delta_9, p_7 \vdash (\Delta_8, p_7), F_5} \top_L \quad \frac{\frac{}{\bullet h_6 : (\top, \Delta_9, p_7), F_5 \vdash \Delta_8, p_7} I}{- : \top, \Delta_9, p_7 \vdash \Delta_8, p_7} \text{Cut} \\
\rightarrow \\
\frac{}{- : \top, \Delta_9, p_7 \vdash \Delta_8, p_7} I
\end{array}$$



- Case rule  $\top_L$

$$\begin{array}{c}
\frac{\frac{h_3 : \Delta_6 \vdash F_5, \Delta_8}{\bullet h_3 : \top, \Delta_6 \vdash \Delta_8, F_5} \top_L \quad \frac{h_7 : F_5, \Delta_6 \vdash \Delta_8}{\bullet h_7 : (\top, \Delta_6), F_5 \vdash \Delta_8} \top_L}{- : \top, \Delta_6 \vdash \Delta_8} \text{Cut} \\
\rightarrow \\
\frac{\frac{h_3 : \top, \Delta_6 \vdash \Delta_8, F_5}{- : \top, \Delta_6 \vdash \Delta_8} \text{ax/W} \quad \frac{\bullet h_7 : \top, \Delta_6, F_5 \vdash \Delta_8}{- : \top, \Delta_6 \vdash \Delta_8} \text{ax/W}}{- : \top, \Delta_6 \vdash \Delta_8} \text{hCut}
\end{array}$$