# System G3i

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## 1 Height preserving admissibility of weakening

• Case(s) rule  $\top_R$ 

$$\frac{}{\bullet \mathbf{h}_1 : \Delta_2 \vdash \top} \ ^{\top}R \qquad \rightarrow \qquad \frac{}{\bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_0 \vdash \top} \ ^{\top}R$$

• Case(s) rule  $\rightarrow_R$ 

• Case(s) rule  $\wedge_R$ 

$$\frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3 \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \quad \wedge_R \qquad \rightarrow \qquad \frac{\frac{\overline{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3}}{\mathbf{h}_1:\Delta_2,\mathbf{F}_0 \vdash \mathbf{F}_3} \quad \text{ax}}{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_0 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \quad \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\mathbf{h}_1:\Delta_2,\mathbf{F}_0 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \quad \text{if} \quad \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\mathbf{h}_1:\Delta_2,\mathbf{F}_0 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \quad \wedge_R$$

• Case(s) rule  $\vee_1$ 

• Case(s) rule  $\vee_2$ 

• Case(s) rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}\rightarrow_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}\quad \mathbf{ax}}{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_0,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}}\quad \mathbf{IH} \quad \frac{\overline{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}\quad \mathbf{ax}}{\mathbf{h}_1:\Delta_2,\mathbf{F}_0,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}\quad \mathbf{IH}} \rightarrow_L \quad \rightarrow \quad \mathbf{h}_1:\Delta_2,\mathbf{F}_0,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}$$

• Case(s) rule  $\wedge_L$ 

$$\begin{array}{c} \mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5} \\ \bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3}\land\mathbf{f}_{4}\vdash\mathbf{f}_{5} \end{array} \land_{L} \qquad \rightarrow \qquad \frac{\overbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5}}^{\mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5}}^{\mathbf{ax}} \ \mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5}} \ \mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5}} \ \mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5}} \ \mathbf{h}_{1}:\Delta_{2},\mathbf{f}_{3},\mathbf{f}_{4}\vdash\mathbf{f}_{5}} \ \mathbf{h}_{2}$$

• Case(s) rule  $\vee_L$ 

$$\frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash\mathbf{F}_{5}\quad\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{5}}{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\lor\mathbf{F}_{4}\vdash\mathbf{F}_{5}}\;\vee_{L}\qquad\rightarrow\qquad\frac{\frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash\mathbf{F}_{5}}{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{0},\mathbf{F}_{3}\vdash\mathbf{F}_{5}}}_{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{0},\mathbf{F}_{3}\lor\mathbf{F}_{5}\vdash\mathbf{F}_{5}}^{\bullet\mathbf{xx}}\frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{5}}{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{0},\mathbf{F}_{4}\vdash\mathbf{F}_{5}}}^{\bullet\mathbf{xx}}\underset{\vdash}{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{0}\vdash\mathbf{F}_{5}}^{\bullet\mathbf{xx}}$$

• Case(s) rule  $\perp_L$ 

 $\bullet$  Case(s) rule I

• Case(s) rule  $\top_L$ 

#### 2 Measure of derivations

• Case(s) rule  $\top_R$ 

• Case(s) rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_4}{\bullet\mathbf{h}_1:\Delta_2\vdash\mathbf{F}_3\to\mathbf{F}_4}\to_R \qquad \to \qquad \frac{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_4}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_4} \quad ^{\mathrm{ax}}}{\bullet \quad \bullet \quad \mathbf{h}_1:\Delta_2\vdash\mathbf{F}_3\to\mathbf{F}_4} \to_R$$

• Case(s) rule  $\wedge_R$ 

$$\frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3 \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \quad \wedge_R \qquad \rightarrow \qquad \frac{\overbrace{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3}^{} \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_3} \quad \overset{\mathbf{ax}}{\mathbf{H}} \quad \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4} \quad \overset{\mathbf{ax}}{\mathbf{H}} \quad \overset{\mathbf{ax}}{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4} \quad \overset{\mathbf{ax}}{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4} \quad \overset{\mathbf{ax}}{\mathbf{H}} \quad \overset{\mathbf{ax}}{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4} \quad \overset{\mathbf{ax}}{\mathbf{h}_1:\Delta_2 \vdash \mathbf{H}_4} \quad \overset{\mathbf{ax}}{\mathbf{h}_1:\Delta_2$$

• Case(s) rule  $\vee_1$ 

• Case(s) rule  $\vee_2$ 

$$\frac{\begin{smallmatrix} \mathbf{h}_1 : \Delta_2 \vdash \mathtt{F}_4 \\ \bullet \mathbf{h}_1 : \Delta_2 \vdash \mathtt{F}_3 \lor \mathtt{F}_4 \end{smallmatrix}}{\bullet \mathbf{h}_1 : \Delta_2 \vdash \mathtt{F}_3 \lor \mathtt{F}_4} \ \lor_2 \qquad \rightarrow \qquad \frac{\begin{smallmatrix} \overline{\mathbf{h}_1 : \Delta_2 \vdash \mathtt{F}_4} \\ \bullet \mathbf{h}_1 : \Delta_2 \vdash \mathtt{F}_4 \end{smallmatrix}}{\bullet \bullet \mathbf{h}_1 : \Delta_2 \vdash \mathtt{F}_3 \lor \mathtt{F}_4} \ \lor_2$$

• Case(s) rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}\rightarrow_L \qquad \rightarrow \qquad \underbrace{\frac{\overline{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}}_{\bullet\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5} \stackrel{\mathrm{ax}}{=} \underbrace{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}}_{\bullet\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5} \stackrel{\mathrm{ax}}{=} \underbrace{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}}_{\bullet\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5} \stackrel{\mathrm{ax}}{=} \underbrace{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}}_{\bullet\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}$$

• Case(s) rule  $\wedge_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\land\mathbf{F}_4\vdash\mathbf{F}_5} & \wedge_L & \rightarrow & \frac{\overline{\mathbf{h}_1:\Delta_2,\mathbf{F}_3,\mathbf{F}_4\vdash\mathbf{F}_5}}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3,\mathbf{F}_4\vdash\mathbf{F}_5} \overset{\mathrm{ax}}{\to} \\ \bullet & \wedge_1:\Delta_2,\mathbf{F}_3\land\mathbf{F}_4\vdash\mathbf{F}_5 & \wedge_L \end{array}$$

• Case(s) rule  $\vee_L$ 

$$\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_5\quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_5} \quad \vee_L \qquad \rightarrow \qquad \frac{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_5}\quad \mathbf{H}}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_5}\quad \mathbf{H} \quad \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}\quad \mathbf{H} \quad \vee_L$$

• Case(s) rule  $\perp_L$ 

 $\bullet$  Case(s) rule I

• Case(s) rule  $\top_L$ 

## 3 Invertibility of Rules

## 3.1 Status of $\top_R$ : Invertible

• Case rule  $\top_R$ 

$$\frac{}{\bullet \mathbf{h}_1 : \Delta_2 \vdash \top} \ ^{\top}R \qquad \rightarrow \qquad \mathbf{trivial}$$

- Case rule  $\rightarrow_R$
- Case rule  $\wedge_R$
- Case rule  $\vee_1$
- Case rule  $\vee_2$
- Case rule  $\rightarrow_L$

$$\begin{array}{ccc} \mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3 & \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\top\\ & \bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\top\end{array} \to_L \qquad \to \qquad \mathsf{trivial}$$

• Case rule  $\wedge_L$ 

$$\begin{array}{ccc} \mathbf{h}_1:\Delta_2,\mathbf{F}_3,\mathbf{F}_4 \vdash \top \\ \bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3 \wedge \mathbf{F}_4 \vdash \top \end{array} \ \wedge_L \qquad \rightarrow \qquad \mathsf{trivial}$$

$$\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\top\quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\top}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\top}\ \vee_L \qquad \rightarrow \qquad \mathsf{trivial}$$

- $\bullet\,$  Case rule I
- Case rule  $\top_L$

$$\frac{\mathbf{h}_1: \Delta_2 \vdash \top}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \top} \ \top_L \qquad \rightarrow \qquad \mathsf{trivial}$$

## 3.2 Status of $\rightarrow_R$ : Invertible

- Case rule  $\top_R$
- Case rule  $\rightarrow_R$

$$\frac{\mathsf{h}_1:\Delta_2,\mathsf{F}_3\vdash\mathsf{F}_4}{\bullet\mathsf{h}_1:\Delta_2\vdash\mathsf{F}_3\to\mathsf{F}_4}\to_R \qquad\to\qquad \frac{\overline{\mathsf{h}_1:\Delta_2,\mathsf{F}_3\vdash\mathsf{F}_4}}{\bullet\mathsf{h}_1:\Delta_2,\mathsf{F}_3\vdash\mathsf{F}_4} \overset{\mathsf{ax}}{\to}$$

- Case rule  $\wedge_R$
- Case rule  $\vee_1$
- Case rule  $\vee_2$
- Case rule  $\rightarrow_L$

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\rightarrow\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_1\rightarrow\mathbf{F}_2}\rightarrow_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5}\quad_W^{\text{ax}}}{\underbrace{\mathbf{h}_3:\Delta_4,\mathbf{F}_1,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5}\quad_W^{\text{ax}}}\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_1,\mathbf{F}_6\vdash\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_1,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_2}\qquad_\Delta^{\text{ax/ind}}\rightarrow_L$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{f}_5,\mathbf{f}_6\vdash\mathbf{f}_1\to\mathbf{f}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{f}_5\land\mathbf{f}_6\vdash\mathbf{f}_1\to\mathbf{f}_2} \ \land_L \qquad \to \qquad \frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{f}_1,\mathbf{f}_5,\mathbf{f}_6\vdash\mathbf{f}_2}}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{f}_1,\mathbf{f}_5\land\mathbf{f}_6\vdash\mathbf{f}_2} \ \stackrel{\mathsf{ax/ind}}{\wedge}_L$$

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1\to\mathbf{F}_2\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\to\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vee\mathbf{F}_6\vdash\mathbf{F}_1\to\mathbf{F}_2}\quad\forall_L\qquad\rightarrow\qquad\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_1,\mathbf{F}_5\vdash\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_1,\mathbf{F}_5\vee\mathbf{F}_6\vdash\mathbf{F}_2}\quad\overset{\mathrm{ax/ind}}{\bullet}\quad\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_1,\mathbf{F}_6\vdash\mathbf{F}_2}{\vee_L}$$

- $\bullet\,$  Case rule I
- Case rule  $\top_L$

## 3.3 Status of $\wedge_R$ : (Left Premise): Invertible

- Case rule  $\top_R$
- Case rule  $\rightarrow_R$
- Case rule  $\wedge_R$

$$\frac{\mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_3 \quad \mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_4}{\bullet \mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_3 \land \mathtt{F}_4} \quad \wedge_R \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_3}}{\bullet \mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_3} \quad \overset{\mathtt{ax}}{\mathtt{H}}$$

- Case rule  $\vee_1$
- Case rule  $\vee_2$
- Case rule  $\rightarrow_L$

$$\frac{\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_5\quad\mathtt{h}_3:\Delta_4,\mathtt{F}_6\vdash\mathtt{F}_1\land\mathtt{F}_2}{\bullet\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_1\land\mathtt{F}_2}\ \to L \qquad \to \qquad \frac{\overline{\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_5}\quad \text{ax}\quad \overline{\mathtt{h}_3:\Delta_4,\mathtt{F}_6\vdash\mathtt{F}_1}\quad \text{ax/ind}\quad \to L}{\bullet\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_1}$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1\land\mathbf{F}_2\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\land\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\lor\mathbf{F}_6\vdash\mathbf{F}_1\land\mathbf{F}_2}\quad\vee_L\qquad\rightarrow\qquad\frac{\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\lor\mathbf{F}_6\vdash\mathbf{F}_1}\quad\frac{\mathbf{ax/ind}}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1}\quad\frac{\mathbf{ax/ind}}{\lor_L}\quad\vee_L$$

- ullet Case rule I
- Case rule  $\top_L$

## 3.4 Status of $\wedge_R$ (Right Premise): : Invertible

- Case rule  $\top_R$
- Case rule  $\rightarrow_R$
- Case rule  $\wedge_R$

- Case rule  $\vee_1$
- Case rule  $\vee_2$
- Case rule  $\rightarrow_L$

$$\frac{\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_5\quad\mathtt{h}_3:\Delta_4,\mathtt{F}_6\vdash\mathtt{F}_1\land\mathtt{F}_2}{\bullet\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_1\land\mathtt{F}_2}\ \to L \qquad \to \qquad \frac{\overline{\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_5}\quad \text{ax}\quad \overline{\mathtt{h}_3:\Delta_4,\mathtt{F}_6\vdash\mathtt{F}_2}\quad \text{ax/ind}\quad \to L}{\bullet\mathtt{h}_3:\Delta_4,\mathtt{F}_5\to\mathtt{F}_6\vdash\mathtt{F}_2}$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1\land\mathbf{F}_2\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\land\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\lor\mathbf{F}_6\vdash\mathbf{F}_1\land\mathbf{F}_2}\quad\vee_L\qquad\rightarrow\qquad\frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_2}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_2}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_2}\quad\vee_L$$

- $\bullet$  Case rule I
- Case rule  $\top_L$

$$\frac{\mathbf{h}_3: \Delta_4 \vdash \mathbf{F}_1 \land \mathbf{F}_2}{\bullet \mathbf{h}_3: \top, \Delta_4 \vdash \mathbf{F}_1 \land \mathbf{F}_2} \ \top_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3: \Delta_4 \vdash \mathbf{F}_2}}{\bullet \mathbf{h}_3: \top, \Delta_4 \vdash \mathbf{F}_2} \ \top_L$$

## 3.5 Status of $\vee_1$ : Non invertible

- Case rule  $\top_R$
- Case rule  $\rightarrow_R$
- Case rule  $\wedge_R$
- Case rule  $\vee_1$

• Case rule  $\vee_2$ 

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2} \ \rightarrow L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5}\quad \text{ax} \quad \overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1}\quad \frac{\mathbf{ax}/\mathbf{ind}}{\rightarrow L} \rightarrow L \qquad \rightarrow L$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1\vee\mathbf{F}_2\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vee\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2}\quad\vee_L\qquad\rightarrow\qquad\frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1}\quad\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1}\quad\vee_L}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vee\mathbf{F}_6\vdash\mathbf{F}_1}\quad\vee_L$$

- ullet Case rule I
- Case rule  $\top_L$

$$\frac{\mathbf{h}_3:\Delta_4 \vdash \mathbf{F}_1 \vee \mathbf{F}_2}{\bullet \mathbf{h}_3:\top,\Delta_4 \vdash \mathbf{F}_1 \vee \mathbf{F}_2} \ \top_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_4 \vdash \mathbf{F}_1}}{\bullet \mathbf{h}_3:\top,\Delta_4 \vdash \mathbf{F}_1} \ ^{\mathrm{ax/ind}} \top_L$$

## 3.6 Status of $\vee_2$ : Non invertible

- Case rule  $\top_R$
- Case rule  $\rightarrow_R$
- Case rule  $\wedge_R$
- Case rule  $\vee_1$

$$\frac{\mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_3}{\bullet \mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_3 \vee \mathtt{F}_4} \ \lor_1 \qquad \rightarrow \qquad \overline{\bullet \mathtt{h}_1:\Delta_2 \vdash \mathtt{F}_4} \ \mathtt{fail}$$

• Case rule  $\vee_2$ 

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2} \ \rightarrow L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5}\quad \text{ax} \quad \overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_2}\quad \frac{\mathbf{ax}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_2} \ \rightarrow L \qquad \rightarrow L \qquad \frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5}\quad \text{ax} \quad \overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_2}\quad \frac{\mathbf{ax}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_2} \ \rightarrow L \qquad \rightarrow L \qquad \frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\rightarrow\mathbf{F}_6\vdash\mathbf{F}_5}\quad \overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_2}\quad \overline{\mathbf{h}_3:\Delta_4,$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_1\vee\mathbf{F}_2\quad\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2}{\bullet\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vee\mathbf{F}_6\vdash\mathbf{F}_1\vee\mathbf{F}_2}\quad\vee_L\qquad\rightarrow\qquad\frac{\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_5\vdash\mathbf{F}_2}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_4,\mathbf{F}_6\vdash\mathbf{F}_2}\quad\frac{\mathbf{ax/ind}}{\vee_L}\quad\frac{\mathbf{ax/ind}}{\vee_L}$$

$$\frac{}{\bullet \mathsf{h}_3:\bot,\Delta_4 \vdash \mathsf{F}_1 \vee \mathsf{F}_2} \ ^{\bot}L \qquad \rightarrow \qquad \frac{}{\bullet \mathsf{h}_3:\bot,\Delta_4 \vdash \mathsf{F}_2} \ ^{\bot}L$$

- ullet Case rule I
- Case rule  $\top_L$

$$\frac{\mathbf{h}_3:\Delta_4 \vdash \mathbf{F}_1 \vee \mathbf{F}_2}{\bullet \mathbf{h}_3:\top,\Delta_4 \vdash \mathbf{F}_1 \vee \mathbf{F}_2} \ \top_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_4 \vdash \mathbf{F}_2}}{\bullet \mathbf{h}_3:\top,\Delta_4 \vdash \mathbf{F}_2} \ ^{\mathrm{ax/ind}}$$

#### 3.7 Status of $\rightarrow_L$ : (Left Premise): Non invertible

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_5,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6} \end{array} \to_R \qquad \to \qquad \begin{array}{c} \bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_2 \end{array} \ \, \text{fail}$$

• Case rule  $\wedge_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_5\quad\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_5\land\mathbf{F}_6}\quad\wedge_R\qquad\rightarrow\qquad\frac{\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_2}}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_2}\stackrel{\mathrm{ax/ind}}{\to}$$

• Case rule  $\vee_1$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_5}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_1\qquad\rightarrow\qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_2}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_2}\ _{\mathsf{H}}^{\mathsf{ax/ind}}$$

• Case rule  $\vee_2$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_6}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_2\qquad\rightarrow\qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_2}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_2}\ ^{\mathtt{ax/ind}}_{\mathtt{H}}$$

• Case rule  $\rightarrow_L$ 

$$\begin{array}{c} \underline{\mathbf{h}_3:\Delta_7, \mathbf{F}_1 \rightarrow \mathbf{F}_2, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_4 \quad \mathbf{h}_3:\Delta_7, \mathbf{F}_5, \mathbf{F}_1 \rightarrow \mathbf{F}_2 \vdash \mathbf{F}_6} \\ \bullet \underline{\mathbf{h}_3:(\Delta_7, \mathbf{F}_1 \rightarrow \mathbf{F}_2), \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_6} \end{array} \rightarrow_L \\ \end{array} \rightarrow L \begin{array}{c} \overline{\mathbf{h}_3:\Delta_7, \mathbf{F}_1 \rightarrow \mathbf{F}_2, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_1} \\ \bullet \underline{\mathbf{h}_3:\Delta_7, \mathbf{F}_1 \rightarrow \mathbf{F}_2, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_1} \end{array} \xrightarrow{\mathbf{ax/ind}} \\ \underline{\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3 \quad \mathbf{h}_1:\Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_5} \\ \bullet \underline{\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3} \end{array} \xrightarrow{\mathbf{ax/ind}} \\ \underline{\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3} \xrightarrow{\mathbf{ax/ind}} \\ \underline{\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3}} \xrightarrow{\mathbf{ax/ind}} \\ \underline{\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3} \xrightarrow{\mathbf{h}_1:\Delta_2, \mathbf{h}_2 \vdash \mathbf{h}_2} \xrightarrow{\mathbf{h}_1:\Delta_2, \mathbf{h}_2 \vdash \mathbf{h}_3} \xrightarrow{\mathbf{h}_1:\Delta_2, \mathbf{h}_2 \vdash \mathbf{h}_$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_5,\mathbf{F}_1\to\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\to\mathbf{F}_2),\mathbf{F}_4\land\mathbf{F}_5\vdash\mathbf{F}_6}~\land L~~\rightarrow~~\frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_5,\mathbf{F}_1\to\mathbf{F}_2\vdash\mathbf{F}_1}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1\to\mathbf{F}_2,\mathbf{F}_4\land\mathbf{F}_5\vdash\mathbf{F}_1}~\overset{\mathsf{ax/ind}}{\land} L$$

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_6\quad\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\rightarrow\mathbf{F}_2),\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad\vee_L\qquad\rightarrow\qquad\frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_1}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_1}\quad\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_1}\quad\vee_L}\circ\mathbf{h}_3:\Delta_7,\mathbf{F}_1\rightarrow\mathbf{F}_2,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_1}$$

 $\bullet$  Case rule I

$$\frac{}{\bullet \mathsf{h}_3: \mathsf{p}_4, \Delta_5, \mathsf{F}_1 \to \mathsf{F}_2 \vdash \mathsf{p}_4} \quad I \qquad \to \qquad \frac{}{\bullet \mathsf{h}_3: \Delta_5, \mathsf{p}_4, \mathsf{F}_1 \to \mathsf{F}_2 \vdash \mathsf{F}_1} \quad \mathsf{fail}$$

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_3:\Delta_5,\mathbf{F}_1\to\mathbf{F}_2\vdash\mathbf{F}_4}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1\to\mathbf{F}_2\vdash\mathbf{F}_4}\ \top_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_5,\mathbf{F}_1\to\mathbf{F}_2\vdash\mathbf{F}_1}\ ^{\mathrm{ax/ind}}}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1\to\mathbf{F}_2\vdash\mathbf{F}_1}\ ^{\mathrm{T}}_L$$

#### 3.8 Status of $\rightarrow_L$ (Right Premise): : Invertible

• Case rule  $\top_R$ 

$$\frac{}{\bullet \mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\top} \ ^\top R \qquad \to \qquad \frac{}{\bullet \mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\top} \ ^\top R$$

• Case rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_5,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6}\to_R \qquad\to\qquad \frac{\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_3,\mathbf{F}_5\vdash\mathbf{F}_6}}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6}\to_R$$

• Case rule  $\wedge_R$ 

• Case rule  $\vee_1$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_5}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_1 \qquad\to\qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_5}\ ^{ax/ind}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_1$$

• Case rule  $\vee_2$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_6}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\to\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_2\qquad\rightarrow\qquad\frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_6}\ \ \text{ax/ind}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\vee_2$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_1\rightarrow\mathbf{F}_2,\mathbf{F}_4\rightarrow\mathbf{F}_5\vdash\mathbf{F}_4\quad\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\rightarrow\mathbf{F}_2),\mathbf{F}_4\rightarrow\mathbf{F}_5\vdash\mathbf{F}_6}\rightarrow_L\qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\rightarrow\mathbf{F}_5\vdash\mathbf{F}_4}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_5\vdash\mathbf{F}_6}\quad\text{ax/ind}\quad\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_5\vdash\mathbf{F}_6}\rightarrow_L$$

$$\frac{\mathtt{h}_1:\Delta_2,\mathtt{F}_3\to\mathtt{F}_4\vdash\mathtt{F}_3\quad\mathtt{h}_1:\Delta_2,\mathtt{F}_4\vdash\mathtt{F}_5}{\bullet\mathtt{h}_1:\Delta_2,\mathtt{F}_3\to\mathtt{F}_4\vdash\mathtt{F}_5}\to_L \qquad \to \qquad \frac{\overline{\mathtt{h}_1:\Delta_2,\mathtt{F}_4\vdash\mathtt{F}_5}}{\bullet\mathtt{h}_1:\Delta_2,\mathtt{F}_4\vdash\mathtt{F}_5} \overset{\mathsf{ax}}{\to} \\$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\rightarrow\mathbf{F}_2),\mathbf{F}_4\wedge\mathbf{F}_5\vdash\mathbf{F}_6}} \ \land_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4,\mathbf{F}_5\vdash\mathbf{F}_6}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\wedge\mathbf{F}_5\vdash\mathbf{F}_6}} \ \stackrel{\mathrm{ax/ind}}{\wedge}L$$

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_6\quad\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\rightarrow\mathbf{F}_2),\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad\vee_L\qquad\rightarrow\qquad\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}\quad\frac{\mathbf{ax/ind}}{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}\quad\vee_L$$

• Case rule  $\perp_L$ 

ullet Case rule I

$$\frac{}{\bullet \mathsf{h}_3 : \mathsf{p}_4, \Delta_5, \mathsf{F}_1 \to \mathsf{F}_2 \vdash \mathsf{p}_4} \quad I \qquad \to \qquad \frac{}{\bullet \mathsf{h}_3 : \Delta_5, \mathsf{F}_2, \mathsf{p}_4 \vdash \mathsf{p}_4} \quad I$$

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_3:\Delta_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_4}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1\rightarrow\mathbf{F}_2\vdash\mathbf{F}_4}\ \top_L \qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_5,\mathbf{F}_2\vdash\mathbf{F}_4}}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_2\vdash\mathbf{F}_4}\ \top_L$$

#### 3.9 Status of $\wedge_L$ : Invertible

• Case rule  $\top_R$ 

$$\frac{}{\bullet \mathbf{h}_4 : \Delta_1, \mathbf{F}_2 \wedge \mathbf{F}_3 \vdash \top} \ \top_R \qquad \rightarrow \qquad \frac{}{\bullet \mathbf{h}_4 : \Delta_1, \mathbf{F}_2, \mathbf{F}_3 \vdash \top} \ \top_R$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \underbrace{\begin{array}{c} \mathbf{h}_4:\Delta_1,\mathbf{F}_5,\mathbf{F}_2\wedge\mathbf{F}_3\vdash\mathbf{F}_6 \\ \bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\wedge\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6 \end{array}}_{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2,\mathbf{F}_3,\mathbf{F}_5\vdash\mathbf{F}_6} \xrightarrow{\mathtt{ax/ind}} \xrightarrow{\bullet}_R \\ \end{array}} \xrightarrow{\bullet}_R$$

• Case rule  $\wedge_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\wedge\mathbf{F}_3\vdash\mathbf{F}_5\quad\mathbf{h}_4:\Delta_1,\mathbf{F}_2\wedge\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\wedge\mathbf{F}_3\vdash\mathbf{F}_6}\quad\wedge_R\qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_2,\mathbf{F}_3\vdash\mathbf{F}_5}\quad\text{ax/ind}\quad\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_2,\mathbf{F}_3\vdash\mathbf{F}_6}\quad\frac{\mathbf{ax/ind}}{\wedge_R}\quad\wedge_R$$

• Case rule  $\vee_1$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\wedge\mathtt{F}_3\vdash\mathtt{F}_5}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\wedge\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6} \ \vee_1 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_2,\mathtt{F}_3\vdash\mathtt{F}_5} \ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2,\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6} \ \vee_1$$

• Case rule  $\vee_2$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\wedge\mathtt{F}_3\vdash\mathtt{F}_6}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\wedge\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_2\qquad\rightarrow\qquad\frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_2,\mathtt{F}_3\vdash\mathtt{F}_6}\ ^{\mathsf{ax/ind}}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2,\mathtt{F}_3\vdash\mathtt{F}_6\vee\mathtt{F}_6}\ \vee_2$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4\to\mathbf{F}_5,\mathbf{F}_1\land\mathbf{F}_2\vdash\mathbf{F}_4\quad\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\land\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\land\mathbf{F}_2),\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6}}\to_L \qquad \to \qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_4}\quad \mathbf{ax/ind} \quad \overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_5\vdash\mathbf{F}_6}\quad \mathbf{ax/ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6}\quad \to L} \to L$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\mathbf{a}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_1\wedge\mathbf{F}_2\vdash\mathbf{F}_6\quad\mathbf{a}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\wedge\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\wedge\mathbf{F}_2),\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\ \vee_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_6}\quad \mathbf{a}^{\times}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad \mathbf{a}^{\times}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad \vee_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_6}\quad \mathbf{a}^{\times}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad \mathbf{a}^{\times}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad \nabla_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_6}\quad \mathbf{a}^{\times}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}}\quad \nabla_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_6}\quad \mathbf{a}^{\times}/\mathbf{ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}\quad \mathbf{a}^{\times}/\mathbf{ind}}$$

• Case rule  $\perp_L$ 

ullet Case rule I

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_3:\Delta_5,\mathbf{F}_1\wedge\mathbf{F}_2\vdash\mathbf{F}_4}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1\wedge\mathbf{F}_2\vdash\mathbf{F}_4}\ \top_L \qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_5,\mathbf{F}_1,\mathbf{F}_2\vdash\mathbf{F}_4}\ ^{\mathrm{ax/ind}}}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1,\mathbf{F}_2\vdash\mathbf{F}_4}\ ^{\mathrm{ax/ind}}$$

#### 3.10 Status of $\vee_L$ : (Left Premise): Invertible

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_5,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6}\ \to_R \qquad \to \qquad \frac{\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_2,\mathbf{F}_5\vdash\mathbf{F}_6}}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_5\to\mathbf{F}_6} \overset{\mathsf{ax/ind}}{\to_R}$$

• Case rule  $\wedge_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_5\quad\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_6}\quad\wedge_R\qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_5}\quad\text{ax/ind}\quad\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_6}\quad\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_6}{\wedge_R}\quad\wedge_R$$

• Case rule  $\vee_1$ 

$$\frac{{}^{\text{h}}_4:\Delta_1,{}^{\text{f}}_2\vee{}^{\text{f}}_3\vdash{}^{\text{f}}_5}{\bullet{}^{\text{h}}_4:\Delta_1,{}^{\text{f}}_2\vee{}^{\text{f}}_3\vdash{}^{\text{f}}_5\vee{}^{\text{f}}_6}}{\bullet{}^{\text{h}}_4:\Delta_1,{}^{\text{f}}_2\vdash{}^{\text{f}}_5\vee{}^{\text{f}}_6}}\vee_1 \qquad \rightarrow \qquad \frac{\overline{{}^{\text{h}}_4:\Delta_1,{}^{\text{f}}_2\vdash{}^{\text{f}}_5}}{\bullet{}^{\text{h}}_4:\Delta_1,{}^{\text{f}}_2\vdash{}^{\text{f}}_5\vee{}^{\text{f}}_6}}\vee_1$$

• Case rule  $\vee_2$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vee\mathtt{F}_3\vdash\mathtt{F}_6}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vee\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_2 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vdash\mathtt{F}_6}\ \mathtt{ax/ind}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vdash\mathtt{F}_5\vee\mathtt{F}_6}\ \vee_2$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{f}_4\to\mathbf{f}_5,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_4\quad\mathbf{h}_3:\Delta_7,\mathbf{f}_5,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{f}_1\vee\mathbf{f}_2),\mathbf{f}_4\to\mathbf{f}_5\vdash\mathbf{f}_6} \ \to_L \qquad \to \qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_1,\mathbf{f}_4\to\mathbf{f}_5\vdash\mathbf{f}_4}\quad \frac{\mathbf{ax/ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{f}_1,\mathbf{f}_4\to\mathbf{f}_5\vdash\mathbf{f}_6} \ \frac{\mathbf{ax/ind}}{\to_L} \ \to_L$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_6\quad\mathbf{h}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\vee\mathbf{F}_2)\cdot\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}} \quad \vee_L \qquad \rightarrow \qquad \frac{\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}} \quad \overset{\mathrm{ax/ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}} \quad \vee_L \qquad \rightarrow \qquad \frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_1,\mathbf{F}_4\vee\mathbf{F}_5\vdash\mathbf{F}_6}} \quad \overset{\mathrm{ax/ind}}{\vee_L} \quad \overset{\mathrm{ax/ind}}{\vee_L} \quad \vee_L \quad \overset{\mathrm{ax/ind}}{\vee_L} \quad \overset{\mathrm{ax/$$

$$\frac{\mathtt{h}_1:\Delta_2,\mathtt{F}_3\vdash\mathtt{F}_5\quad\mathtt{h}_1:\Delta_2,\mathtt{F}_4\vdash\mathtt{F}_5}{\bullet\mathtt{h}_1:\Delta_2,\mathtt{F}_3\vee\mathtt{F}_4\vdash\mathtt{F}_5}\ \vee_L \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_1:\Delta_2,\mathtt{F}_3\vdash\mathtt{F}_5}}{\bullet\mathtt{h}_1:\Delta_2,\mathtt{F}_3\vdash\mathtt{F}_5}\ ^{\mathtt{ax}}_{\mathtt{H}}$$

• Case rule  $\perp_L$ 

$$\frac{}{\bullet \mathbf{h}_3:\bot,\Delta_5,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_4} \ ^\perp L \qquad \rightarrow \qquad \frac{}{\bullet \mathbf{h}_3:\bot,\Delta_5,\mathbf{f}_1\vdash\mathbf{f}_4} \ ^\perp L$$

ullet Case rule I

$$\frac{}{\bullet \mathbf{h}_3: \mathbf{p}_4, \Delta_5, \mathbf{f}_1 \vee \mathbf{f}_2 \vdash \mathbf{p}_4} \quad I \qquad \rightarrow \qquad \frac{}{\bullet \mathbf{h}_3: \Delta_5, \mathbf{f}_1, \mathbf{p}_4 \vdash \mathbf{p}_4} \quad I$$

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_3:\Delta_5,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_4}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_4}\ \top_L \qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_5,\mathbf{F}_1\vdash\mathbf{F}_4}\ ^{\mathrm{ax/ind}}}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{F}_1\vdash\mathbf{F}_4}\ ^{\mathrm{T}}_L$$

#### 3.11 Status of $\vee_L$ (Right Premise): : Invertible

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_5,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6}\ \to_R \qquad \to \qquad \frac{\overline{\mathbf{h}_4:\Delta_1,\mathbf{F}_3,\mathbf{F}_5\vdash\mathbf{F}_6}}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_5\to\mathbf{F}_6} \overset{\mathrm{ax/ind}}{\to_R}$$

• Case rule  $\wedge_R$ 

$$\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_5\quad\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_6}\quad\wedge_R\qquad\rightarrow\qquad\frac{\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_5}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_5}\quad\text{ax/ind}\quad\frac{\mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_5\wedge\mathbf{F}_6}\quad\wedge_R$$

• Case rule  $\vee_1$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vee\mathtt{F}_3\vdash\mathtt{F}_5}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vee\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6} \ \vee_1 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_5} \ ^{\mathrm{ax/ind}}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6} \ ^{\mathrm{v}_1}$$

• Case rule  $\vee_2$ 

$$\frac{\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vee\mathtt{F}_3\vdash\mathtt{F}_6}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_2\vee\mathtt{F}_3\vdash\mathtt{F}_5\vee\mathtt{F}_6} \ \vee_2 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_6} \ \mathtt{ax/ind}}{\bullet\mathtt{h}_4:\Delta_1,\mathtt{F}_3\vdash\mathtt{F}_6\vee\mathtt{F}_6} \vee_2$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{a}_3:\Delta_7,\mathbf{F}_4\to\mathbf{F}_5,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_4\quad\mathbf{a}_3:\Delta_7,\mathbf{F}_5,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\vee\mathbf{F}_2),\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6} \to_L \longrightarrow \frac{\overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_4}\quad \overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6}\quad \overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6}\quad \overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6}\quad \overline{\mathbf{a}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\to\mathbf{F}_5\vdash\mathbf{F}_6}$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{F}_4,\mathbf{F}_5,\mathbf{F}_1\vee\mathbf{F}_2\vdash\mathbf{F}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{F}_1\vee\mathbf{F}_2),\mathbf{F}_4\wedge\mathbf{F}_5\vdash\mathbf{F}_6}\wedge_L\qquad\rightarrow\qquad\frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4,\mathbf{F}_5\vdash\mathbf{F}_6}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{F}_2,\mathbf{F}_4\wedge\mathbf{F}_5\vdash\mathbf{F}_6}\wedge_L$$

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_3:\Delta_7,\mathbf{f}_4,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_6\quad\mathbf{h}_3:\Delta_7,\mathbf{f}_5,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_6}{\bullet\mathbf{h}_3:(\Delta_7,\mathbf{f}_1\vee\mathbf{f}_2),\mathbf{f}_4\vee\mathbf{f}_5\vdash\mathbf{f}_6}}\;\vee_L\qquad\rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vdash\mathbf{f}_6}\quad \frac{\mathbf{ax/ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vee\mathbf{f}_5\vdash\mathbf{f}_6}} \\{\bullet}\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vee\mathbf{f}_5\vdash\mathbf{f}_6} \\\\\vee_L\qquad \rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vdash\mathbf{f}_6}\quad \frac{\mathbf{ax/ind}}{\bullet\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vee\mathbf{f}_5\vdash\mathbf{f}_6}} \\\vee_L\qquad \rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vdash\mathbf{f}_6}\quad \overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vee\mathbf{f}_5\vdash\mathbf{f}_6}} \\\vee_L\qquad \rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_2,\mathbf{f}_4\vee\mathbf{f}_5\vdash\mathbf{f}_6}} \\\vee_L\qquad \rightarrow\qquad \frac{\overline{\mathbf{h}_3:\Delta_7,\mathbf{f}_5\vdash\mathbf{f}_6}} \\\vee_L\qquad$$

$$\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_5\quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_5}\ \vee_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_5}\ ^{\mathrm{ax}}$$

ullet Case rule I

$$\frac{}{\bullet \mathsf{h}_3: \mathsf{p}_4, \Delta_5, \mathsf{F}_1 \vee \mathsf{F}_2 \vdash \mathsf{p}_4} \quad I \qquad \rightarrow \qquad \frac{}{\bullet \mathsf{h}_3: \Delta_5, \mathsf{F}_2, \mathsf{p}_4 \vdash \mathsf{p}_4} \quad I$$

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_3:\Delta_5,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_4}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{f}_1\vee\mathbf{f}_2\vdash\mathbf{f}_4}\ \top_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_3:\Delta_5,\mathbf{f}_2\vdash\mathbf{f}_4}}{\bullet\mathbf{h}_3:\top,\Delta_5,\mathbf{f}_2\vdash\mathbf{f}_4}\ \top_L$$

#### 3.12 Status of $\perp_L$ : Invertible

• Case rule  $\top_R$ 

$$\frac{}{\bullet^{\mathbf{h}_2}:\bot,\Delta_1\vdash\top} \ ^{\top}R \qquad \rightarrow \qquad \mathsf{trivial}$$

• Case rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_2:\bot,\Delta_1,\mathbf{F}_3\vdash\mathbf{F}_4}{\bullet\mathbf{h}_2:\bot,\Delta_1\vdash\mathbf{F}_3\to\mathbf{F}_4}\ \to R\qquad \to\qquad \mathsf{trivial}$$

• Case rule  $\wedge_R$ 

$$\frac{\mathbf{h}_2:\bot,\Delta_1 \vdash \mathbf{F}_3 \quad \mathbf{h}_2:\bot,\Delta_1 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_2:\bot,\Delta_1 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \ \land_R \qquad \rightarrow \qquad \mathtt{trivial}$$

• Case rule  $\vee_1$ 

$$\frac{\mathtt{h}_2:\bot,\Delta_1\vdash\mathtt{F}_3}{\bullet\mathtt{h}_2:\bot,\Delta_1\vdash\mathtt{F}_3\vee\mathtt{F}_4}\ \vee_1 \qquad \to \qquad \mathtt{trivial}$$

• Case rule  $\vee_2$ 

$$\frac{ \mbox{${\rm h}$}_2 : \bot, \Delta_1 \vdash \mbox{${\rm F}$}_4 }{ \mbox{${\rm eh}$}_2 : \bot, \Delta_1 \vdash \mbox{${\rm F}$}_3 \vee \mbox{${\rm F}$}_4 } \ \ \, \vee_2 \qquad \rightarrow \qquad \mbox{trivial}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_1:\bot,\Delta_5,\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_2\quad\mathbf{h}_1:\bot,\Delta_5,\mathbf{F}_3\vdash\mathbf{F}_4}{\bullet\mathbf{h}_1:(\bot,\Delta_5),\mathbf{F}_2\to\mathbf{F}_3\vdash\mathbf{F}_4}\ \to_L \qquad \to \qquad \mathsf{trivial}$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_1:\bot,\Delta_5,\mathbf{F}_2,\mathbf{F}_3\vdash\mathbf{F}_4}{\bullet\mathbf{h}_1:(\bot,\Delta_5),\mathbf{F}_2\land\mathbf{F}_3\vdash\mathbf{F}_4}\ \land_L \qquad \rightarrow \qquad \mathsf{trivial}$$

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_1:\bot,\Delta_5,\mathbf{F}_2\vdash\mathbf{F}_4\quad\mathbf{h}_1:\bot,\Delta_5,\mathbf{F}_3\vdash\mathbf{F}_4}{\bullet\mathbf{h}_1:(\bot,\Delta_5),\mathbf{F}_2\vee\mathbf{F}_3\vdash\mathbf{F}_4}\ \vee_L \qquad \rightarrow \qquad \text{trivial}$$

• Case rule  $\perp_L$ 

ullet Case rule I

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_1:\bot,\Delta_3\vdash\mathbf{F}_2}{\bullet\mathbf{h}_1:\top,\bot,\Delta_3\vdash\mathbf{F}_2}\ \top_L \qquad \rightarrow \qquad \mathsf{trivial}$$

#### 3.13 Status of I: Invertible

- Case rule  $\top_R$
- Case rule  $\rightarrow_R$
- Case rule  $\wedge_R$
- Case rule  $\vee_1$
- Case rule  $\vee_2$
- Case rule  $\rightarrow_L$

$$\frac{\mathbf{h}_2:\Delta_5,\mathbf{p}_1,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_2:\Delta_5,\mathbf{F}_4,\mathbf{p}_1\vdash\mathbf{p}_1}{\bullet\mathbf{h}_2:(\Delta_5,\mathbf{p}_1),\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{p}_1}\ \to_L \qquad \to \qquad \mathtt{trivial}$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_2:\Delta_5,\mathbf{F}_3,\mathbf{F}_4,\mathbf{p}_1\vdash\mathbf{p}_1}{\bullet\mathbf{h}_2:(\Delta_5,\mathbf{p}_1),\mathbf{F}_3\wedge\mathbf{F}_4\vdash\mathbf{p}_1}\ \wedge_L \qquad \rightarrow \qquad \mathtt{trivial}$$

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_2:\Delta_5,\mathbf{F}_3,\mathbf{p}_1\vdash\mathbf{p}_1\quad\mathbf{h}_2:\Delta_5,\mathbf{F}_4,\mathbf{p}_1\vdash\mathbf{p}_1}{\bullet\mathbf{h}_2:(\Delta_5,\mathbf{p}_1),\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{p}_1}\ \vee_L \qquad \rightarrow \qquad \mathsf{trivial}$$

ullet Case rule I

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_2:\Delta_3,\mathbf{p}_1\vdash\mathbf{p}_1}{\bullet\mathbf{h}_2:\top,\Delta_3,\mathbf{p}_1\vdash\mathbf{p}_1}\ \top_L \qquad \rightarrow \qquad \mathtt{trivial}$$

## 3.14 Status of $\top_L$ : Invertible

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_2: \top, \Delta_1, \mathbf{F}_3 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_2: \top, \Delta_1 \vdash \mathbf{F}_3 \to \mathbf{F}_4} \ \to_R \qquad \to \qquad \frac{\overline{\mathbf{h}_2: \Delta_1, \mathbf{F}_3 \vdash \mathbf{F}_4}}{\bullet \mathbf{h}_2: \Delta_1 \vdash \mathbf{F}_3 \to \mathbf{F}_4} \overset{\mathrm{ax/ind}}{\to}_R$$

• Case rule  $\wedge_R$ 

$$\frac{\mathbf{h}_2: \top, \Delta_1 \vdash \mathbf{F}_3 \quad \mathbf{h}_2: \top, \Delta_1 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_2: \top, \Delta_1 \vdash \mathbf{F}_3 \land \mathbf{F}_4} \quad \wedge_R \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_2: \Delta_1 \vdash \mathbf{F}_3} \quad \text{ax/ind} \quad \overline{\mathbf{h}_2: \Delta_1 \vdash \mathbf{F}_4} \quad \overline{\mathbf{h}_2: \Delta_1 \vdash \mathbf{F}_4} \quad \wedge_R \quad \wedge_R$$

• Case rule  $\vee_1$ 

$$\frac{\mathtt{h}_2: \top, \Delta_1 \vdash \mathtt{F}_3}{\bullet \mathtt{h}_2: \top, \Delta_1 \vdash \mathtt{F}_3 \vee \mathtt{F}_4} \ \vee_1 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_2: \Delta_1 \vdash \mathtt{F}_3} \ \text{ax/ind}}{\bullet \mathtt{h}_2: \Delta_1 \vdash \mathtt{F}_3 \vee \mathtt{F}_4} \vee_1$$

• Case rule  $\vee_2$ 

$$\frac{\mathtt{h}_2: \top, \Delta_1 \vdash \mathtt{F}_4}{\bullet \mathtt{h}_2: \top, \Delta_1 \vdash \mathtt{F}_3 \vee \mathtt{F}_4} \ \vee_2 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_2: \Delta_1 \vdash \mathtt{F}_4}}{\bullet \mathtt{h}_2: \Delta_1 \vdash \mathtt{F}_3 \vee \mathtt{F}_4} \vee_2$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_1: \top, \Delta_5, \mathbf{F}_2 \rightarrow \mathbf{F}_3 \vdash \mathbf{F}_2 \quad \mathbf{h}_1: \top, \Delta_5, \mathbf{F}_3 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1: (\top, \Delta_5), \mathbf{F}_2 \rightarrow \mathbf{F}_3 \vdash \mathbf{F}_4} \quad \rightarrow L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_1: \Delta_5, \mathbf{F}_2 \rightarrow \mathbf{F}_3 \vdash \mathbf{F}_2} \quad \overset{\mathrm{ax/ind}}{\bullet \mathbf{h}_1: \Delta_5, \mathbf{F}_2 \rightarrow \mathbf{F}_3 \vdash \mathbf{F}_4} \quad \xrightarrow{\mathbf{h}_1: \Delta_5, \mathbf{h}_2 \rightarrow \mathbf{h}_1: \Delta_5, \mathbf{h}_2 \rightarrow \mathbf{h}_2: \Delta_5, \mathbf{h}_2: \Delta_$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_1: \top, \Delta_5, \mathbf{F}_2, \mathbf{F}_3 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1: (\top, \Delta_5), \mathbf{F}_2 \land \mathbf{F}_3 \vdash \mathbf{F}_4} \ \land_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_1: \Delta_5, \mathbf{F}_2, \mathbf{F}_3 \vdash \mathbf{F}_4}}{\bullet \mathbf{h}_1: \Delta_5, \mathbf{F}_2 \land \mathbf{F}_3 \vdash \mathbf{F}_4} \ \stackrel{\mathsf{ax/ind}}{\wedge} L$$

$$\frac{\mathbf{h}_1: \top, \Delta_5, \mathbf{F}_2 \vdash \mathbf{F}_4 \quad \mathbf{h}_1: \top, \Delta_5, \mathbf{F}_3 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1: (\top, \Delta_5), \mathbf{F}_3 \lor \mathbf{F}_4} \quad \vee_L \qquad \rightarrow \qquad \frac{\overline{\mathbf{h}_1: \Delta_5, \mathbf{F}_2 \vdash \mathbf{F}_4} \quad \text{ax/ind}}{\bullet \mathbf{h}_1: \Delta_5, \mathbf{F}_2 \lor \mathbf{F}_3 \vdash \mathbf{F}_4} \quad \frac{\mathbf{ax/ind}}{\vee_L} \quad \vee_L \quad \vee_L$$

• Case rule  $\perp_L$ 

 $\bullet\,$  Case rule I

• Case rule  $\top_L$ 

## 4 Height preserving admissibility of contraction

• Case(s) rule  $\top_R$ 

$$\frac{}{\bullet \mathbf{h}_3 : \Delta_1, \mathbf{F}_2, \mathbf{F}_2 \vdash \top} \ ^{\top}R \qquad \rightarrow \qquad \frac{}{\bullet \mathbf{h}_3 : \Delta_1, \mathbf{F}_2 \vdash \top} \ ^{\top}R$$

• Case(s) rule  $\rightarrow_R$ 

$$\frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5}\to_R \qquad \to \qquad \frac{\frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2\vdash\mathbf{F}_4\vdash\mathbf{F}_5}{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_5}}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5} \qquad \overset{\mathrm{ax}}{\to} \\ \frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5} \qquad \to \\ \frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5} \qquad \to \\ \frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_4\vdash\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5} \qquad \to \\ \frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\to\mathbf{F}_5} \qquad \to \\ \frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_3\to\mathbf{F}$$

• Case(s) rule  $\wedge_R$ 

$$\frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2\vdash\mathbf{F}_4\quad\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2\vdash\mathbf{F}_5}{\bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2\vdash\mathbf{F}_4\land\mathbf{F}_5}\quad\wedge_R\qquad\rightarrow\qquad \frac{\frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2,\mathbf{F}_2\vdash\mathbf{F}_4}{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4}\quad\underset{\mathbf{H}}{\text{ax}}\quad\frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_5}{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_5}\\\bullet\\ \bullet\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\land\mathbf{F}_5} \\ & \qquad \bullet\\ \mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4\land\mathbf{F}_5 \\ \end{pmatrix} \xrightarrow{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_4} \frac{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_5}{\mathbf{h}_3:\Delta_1,\mathbf{F}_2\vdash\mathbf{F}_5} \\ \wedge_R$$

• Case(s) rule  $\vee_1$ 

$$\frac{ \begin{smallmatrix} \mathbf{h}_3 : \Delta_1, \mathbf{F}_2, \mathbf{F}_2 \vdash \mathbf{F}_4 \\ \bullet \mathbf{h}_3 : \Delta_1, \mathbf{F}_2, \mathbf{F}_2 \vdash \mathbf{F}_4 \lor \mathbf{F}_5 \end{smallmatrix}}{ \bullet \mathbf{h}_3 : \Delta_1, \mathbf{F}_2, \mathbf{F}_2 \vdash \mathbf{F}_4 }} \ \ \mathbf{\overset{ax}{h}}_{\mathsf{1H}} \\ \frac{ \begin{smallmatrix} \mathbf{h}_3 : \Delta_1, \mathbf{F}_2 \vdash \mathbf{F}_4 \\ \bullet \mathbf{h}_3 : \Delta_1, \mathbf{F}_2 \vdash \mathbf{F}_4 \lor \mathbf{F}_5 \end{smallmatrix}}{ \bullet \mathbf{h}_3 : \Delta_1, \mathbf{F}_2 \vdash \mathbf{F}_4 \lor \mathbf{F}_5}} \ \ \mathbf{\overset{ax}{h}}_{\mathsf{1H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{2H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{2H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{2H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{2H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{2H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \lor \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \vdash \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} \\ \mathbf{\overset{\bullet}{h}}_{\mathsf{3H}} : \Delta_1, \mathbf{\overset{\bullet}{h}}_{\mathsf{3H$$

• Case(s) rule  $\vee_2$ 

$$\frac{\mathtt{h}_3:\Delta_1,\mathtt{F}_2,\mathtt{F}_2 \vdash \mathtt{F}_5}{\bullet \mathtt{h}_3:\Delta_1,\mathtt{F}_2,\mathtt{F}_2 \vdash \mathtt{F}_4 \lor \mathtt{F}_5} \ \lor_2 \qquad \rightarrow \qquad \frac{\overline{\mathtt{h}_3:\Delta_1,\mathtt{F}_2,\mathtt{F}_2 \vdash \mathtt{F}_5}}{\bullet \mathtt{h}_3:\Delta_1,\mathtt{F}_2 \vdash \mathtt{F}_5} \overset{\mathsf{ax}}{\underset{\mathsf{I}_3}{\sqcup}} \underbrace{\mathtt{h}_3:\Delta_1,\mathtt{F}_2 \vdash \mathtt{F}_5}_{} \lor_2$$

• Case(s) rule  $\rightarrow_L$ 

$$\begin{array}{c} \underline{\mathbf{h}_2:\Delta_1,\mathbf{F}_3\rightarrow\mathbf{F}_4,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}_{\bullet\mathbf{h}_2:\Delta_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3},\underline{\mathbf{h}_2:\Delta_1,\mathbf{F}_4\rightarrow\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3} \end{array} \rightarrow_L \qquad \rightarrow \qquad \begin{array}{c} \underline{\mathbf{h}_2:\Delta_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}_{\bullet\mathbf{h}_2:\Delta_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3},\underline{\mathbf{h}_2:\Delta_1,\mathbf{F}_4\rightarrow\mathbf{F}_5}_{\mathrm{IH}} \xrightarrow{\mathbf{h}_2:\Delta_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3} \\ \underline{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_5} \end{array} \rightarrow_L \\ \underline{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_5} \xrightarrow{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3} \xrightarrow{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3} \xrightarrow{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3} \xrightarrow{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3} \xrightarrow{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5} \xrightarrow{\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_5}_{\bullet\mathbf{h}_2:\Delta_6,\mathbf{F}_1,\mathbf{F$$

• Case(s) rule  $\wedge_L$ 

$$\frac{ \begin{array}{l} \frac{h_2:\Delta_1,F_3,F_4,F_3 \wedge F_4 \vdash F_5}{\bullet h_2:\Delta_1,F_3,F_3,F_4 \vdash F_5} \end{array}{}_{IH}}{ \begin{array}{l} \frac{h_2:\Delta_1,F_3,F_3,F_4,F_4 \vdash F_5}{\bullet h_2:\Delta_1,F_3,F_3,F_4 \vdash F_5} \end{array}{}_{IH}}{ \begin{array}{l} \frac{h_2:\Delta_1,F_3,F_3,F_4 \vdash F_5}{\bullet h_2:\Delta_1,F_3,F_4 \vdash F_5} \end{array}{}_{IH}}{ \begin{array}{l} \frac{h_2:\Delta_1,F_3,F_3,F_4 \vdash F_5}{\bullet h_2:\Delta_1,F_3,F_4 \vdash F_5} \end{array}{}_{IH}}{ \begin{array}{l} \frac{h_2:\Delta_1,F_3,F_3,F_4 \vdash F_5}{\bullet h_2:\Delta_1,F_3,F_4 \vdash F_5} \end{array}{}_{IH}} \end{array}} \\ \frac{h_2:\Delta_1,F_3,F_4 \vdash F_5}{\bullet h_2:\Delta_1,F_3,F_4 \vdash F_5} \end{array}{}_{IH}$$

• Case(s) rule  $\vee_L$ 

$$\frac{\mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5} \quad \mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{4}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}{\bullet \mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}} \vee_{L} \rightarrow \underbrace{\frac{\mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3}, \mathbf{F}_{3} \vdash \mathbf{F}_{5}}{\mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}} \vee_{L} \rightarrow \underbrace{\frac{\mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}} \vee_{L} \rightarrow \underbrace{\frac{\mathbf{h}_{2} : \Delta_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vdash \mathbf{F}_{5}} \underbrace{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vdash \mathbf{F}_{5}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}} \underbrace{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}} \underbrace{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}} \underbrace{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}} \underbrace{\mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}_{\bullet \mathbf{h}_{2} : \Delta_{6}, \mathbf{F}_{1}, \mathbf{F}_{3} \vee \mathbf{F}_{4} \vdash \mathbf{F}_{5}}}$$

• Case(s) rule  $\perp_L$ 

$$\frac{}{\bullet \mathtt{h}_2: (\bot, \Delta_4), \mathtt{F}_1, \mathtt{F}_1 \vdash \mathtt{F}_3} \ ^\bot L \qquad \rightarrow \qquad \frac{}{\bullet \mathtt{h}_2: \bot, \Delta_4, \mathtt{F}_1 \vdash \mathtt{F}_3} \ ^\bot L$$

• Case(s) rule I

• Case(s) rule  $\top_L$ 

$$\frac{\mathbf{h}_2:\Delta_4,\mathbf{F}_1,\mathbf{F}_1\vdash\mathbf{F}_3}{\bullet\mathbf{h}_2:(\top,\Delta_4),\mathbf{F}_1,\mathbf{F}_1\vdash\mathbf{F}_3}\ \top_L \qquad \rightarrow \qquad \frac{\frac{\mathbf{h}_2:\Delta_4,\mathbf{F}_1,\mathbf{F}_1\vdash\mathbf{F}_3}{\bullet\mathbf{h}_2:\Delta_4,\mathbf{F}_1\vdash\mathbf{F}_3}}{\bullet\mathbf{h}_2:\top,\Delta_4,\mathbf{F}_1\vdash\mathbf{F}_3} \ \stackrel{\mathrm{ax}}{\top_L}$$

## 5 Identity-Expansion

$$\begin{array}{c|c} \hline -: F_0 \vdash F_0 & \text{IH} & \hline -: F_1 \vdash F_1 & \text{IH} \\ \hline -: F_0 \vdash F_0 \lor F_1 & \lor_1 & \hline -: F_1 \vdash F_0 \lor F_1 & \lor_2 \\ \hline -: F_0 \vdash F_0 \lor F_1 \vdash F_0 \lor F_1 & \lor_L \\ \hline \hline \hline -: F_0 \vdash F_0 & W & \hline -: F_0, F_1 \vdash F_1 & W \\ \hline \hline -: F_0, F_1 \vdash F_0 & W & \hline -: F_0, F_1 \vdash F_1 & W \\ \hline \hline -: F_0, F_1 \vdash F_0 \land F_1 & \land_L & \\ \hline \hline \hline -: F_0 \vdash F_0 & \text{IH} & \hline -: F_0, F_1 \vdash F_1 & W \\ \hline \hline -: F_0, F_0 \to F_1 \vdash F_0 & W & \hline -: F_0, F_1 \vdash F_1 & W \\ \hline \hline -: F_0, F_0 \to F_1 \vdash F_0 \to F_1 & \to_R \\ \hline \hline -: F_0 \to F_1 \vdash F_0 \to F_1 & \to_R \\ \hline \hline -: T \vdash T & \top_R \\ \hline \hline \hline -: \bot \vdash \bot & \bot_L \\ \hline \hline \end{array}$$

### 6 Cut-Elimination

#### 6.1 Status of $\top_R$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c|c} \hline { \bullet \mathbf{h}_1 : \Delta_3 \vdash \top} & \top_R & \hline { \bullet \mathbf{h}_4 : \Delta_3, \top \vdash \top} & \top_R \\ \hline { - : \Delta_3 \vdash \top} & \hline { \leftarrow} \\ \hline { - : \Delta_3 \vdash \top} & \top_R \\ \hline \hline { - : \Delta_3 \vdash \top} & \top_R \\ \hline \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c|c} \frac{\bullet \mathbf{h}_1 : \Delta_3 \vdash \top}{\bullet \mathbf{h}_1 : \Delta_3 \vdash \top} & \frac{\mathbf{h}_4 : \top, \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_4 : \Delta_3, \top \vdash \mathbf{F}_5 \to \mathbf{F}_6} & \mathsf{Cut} \\ \hline -: \Delta_3 \vdash \mathbf{F}_5 \to \mathbf{F}_6 & \to \\ \hline \frac{\bullet \mathbf{h}_1 : \Delta_3, \mathbf{F}_5 \vdash \top}{\bullet \mathbf{h}_4 : \top, \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6} & \frac{\mathsf{ax/W}}{\mathsf{hCut}} \\ \hline \frac{-: \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6}{-: \Delta_3 \vdash \mathbf{F}_5 \to \mathbf{F}_6} & \to_R \end{array}$$

• Case rule  $\wedge_R$ 

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \bullet_{\mathbf{h}_1:\,\Delta_3 \,\vdash\, \top} \\ \bullet_{\mathbf{h}_1:\,\Delta_3 \,\vdash\, \top} \end{array} \begin{array}{c} \top_R & \frac{\mathbf{h}_4:\,\top,\,\Delta_3 \,\vdash\, \mathbf{F}_5}{\bullet_{\mathbf{h}_4:\,\Delta_3,\,\top \,\vdash\, \mathbf{F}_5 \,\vee\, \mathbf{F}_6} \\ \\ -:\,\Delta_3 \,\vdash\, \mathbf{F}_5 \,\vee\, \mathbf{F}_6 \\ \end{array} \begin{array}{c} \to \\ \bullet_{\mathbf{h}_1:\,\Delta_3 \,\vdash\, \top} \end{array} \begin{array}{c} \to \\ & \uparrow_R \\ \hline \bullet_{\mathbf{h}_4:\,\top,\,\Delta_3 \,\vdash\, \mathbf{F}_5} \\ \hline -:\,\Delta_3 \,\vdash\, \mathbf{F}_5 \\ \hline -:\,\Delta_3 \,\vdash\, \mathbf{F}_5 \,\vee\, \mathbf{F}_6 \end{array} \begin{array}{c} \vee_1 \\ \bullet \mathsf{Cut} \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c|c} \underline{\bullet \mathbf{h}_1 : \Delta_3 \vdash \top} & \top_R & \frac{\mathbf{h}_4 : \top, \Delta_3 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_4 : \Delta_3, \top \vdash \mathbf{F}_5 \vee \mathbf{F}_6} \\ \hline -: \Delta_3 \vdash \mathbf{F}_5 \vee \mathbf{F}_6 & \rightarrow \\ \underline{\bullet \mathbf{h}_1 : \Delta_3 \vdash \top} & \top_R & \frac{\rightarrow}{\mathbf{h}_4 : \top, \Delta_3 \vdash \mathbf{F}_6} \\ \underline{-: \Delta_3 \vdash \mathbf{F}_6} & \vee_2 \\ \hline -: \Delta_3 \vdash \mathbf{F}_5 \vee \mathbf{F}_6 & \vee_2 \end{array} \quad \text{ax/W} \\ \underline{-: \Delta_3 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} \\ \underline{-: \Delta_3 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} & \vee_2 \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \top}_{\bullet \mathbf{h}_1} \ \top_R \ \frac{\mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_4 \quad \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_3 : (\Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5), \top \vdash \mathbf{F}_6} \ \mathbf{Cut}} \\ \frac{-: \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_6} \ \mathbf{h}_{\mathsf{Cut}}}{\bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{T}} \ \frac{\mathsf{ax/W}}{\mathsf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6}} \\ \frac{-: \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_2 : \Delta_7, \mathbf{F}_4 \rightarrow \mathbf{F}_5 \vdash \mathbf{F}_6} \ \rightarrow_L$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c} \frac{\bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_4 \wedge \mathbf{F}_5 \vdash \top}{\bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_4 \wedge \mathbf{F}_5 \vdash \top} & \frac{\mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_3 : (\Delta_7, \mathbf{F}_4 \wedge \mathbf{F}_5), \top \vdash \mathbf{F}_6} & \wedge L \\ \hline & -: \Delta_7, \mathbf{F}_4 \wedge \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \hline & \bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \top & \top_R & \frac{\rightarrow}{\mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \mathbf{F}_6} \\ \hline & \frac{-: \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \mathbf{F}_6}{-: \Delta_7, \mathbf{F}_4 \wedge \mathbf{F}_5 \vdash \mathbf{F}_6} & \wedge_L & \text{ax/W} \\ \hline \end{array}$$

• Case rule  $\vee_L$ 

$$\frac{\underbrace{\begin{array}{l} \bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_4 \vee \mathbf{F}_5 \vdash \top}_{\bullet} \quad \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_4 \vdash \mathbf{F}_6 \quad \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_3 : (\Delta_7, \mathbf{F}_4 \vee \mathbf{F}_5), \top \vdash \mathbf{F}_6 \end{array}}_{\bullet} \quad \mathbf{Cut}}{-: \Delta_7, \mathbf{F}_4 \vdash \mathbf{F}_6} \quad \underbrace{\begin{array}{l} \bullet \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_3 : (\Delta_7, \mathbf{F}_4 \vee \mathbf{F}_5), \top \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_4 : \Delta_7, \mathbf{F}_4 \vdash \mathbf{F}_6 \end{array}}_{\bullet} \quad \underbrace{\begin{array}{l} \bullet \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1 : \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_7 \vdash \mathbf{h}_7 \\ \bullet \mathbf{h}_2 : \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \end{array}}_{\bullet} \quad \underbrace{\begin{array}{l} \bullet \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_3 : \top, \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_4 : \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_5 : \Delta_7, \mathbf{F}_5 \vdash \mathbf{F}_6 \end{array}}_{\bullet} \quad \mathbf{h}_{\bullet} \mathbf$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \hline \bullet_{\mathbf{h}_1}: \bot, \Delta_5 \vdash \top & \top_R & \hline \bullet_{\mathbf{h}_3}: (\bot, \Delta_5), \top \vdash \mathbf{F}_4 \\ \hline -: \bot, \Delta_5 \vdash \mathbf{F}_4 \\ \hline \rightarrow \\ \hline -: \bot, \Delta_5 \vdash \mathbf{F}_4 & \bot_L \end{array} \quad \mathbf{Cut}$$

ullet Case rule I

$$\cfrac{\cfrac{\bullet \mathbf{h}_1:\Delta_5,\mathbf{p}_4 \vdash \top}{} \ \ \ \cfrac{\top_R}{\bullet \mathbf{h}_3:(\Delta_5,\mathbf{p}_4),\top \vdash \mathbf{p}_4} \ \ \cfrac{I}{ \ \ \ } \\ \cfrac{-:\Delta_5,\mathbf{p}_4 \vdash \mathbf{p}_4}{-:\Delta_5,\mathbf{p}_4 \vdash \mathbf{p}_4} \ \ I }$$

• Case rule  $\top_L$ 

$$\frac{ \underbrace{ \begin{array}{c} \mathbf{h}_1 : \Delta_4 \vdash \top}_{} \quad \top_R \quad \frac{\mathbf{h}_3 : \Delta_4 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_3 : \Delta_4, \top \vdash \mathbf{F}_5} \\ - : \Delta_4 \vdash \mathbf{F}_5 \\ \hline - : \Delta_4 \vdash \mathbf{F}_5 \end{array} }_{} \quad \begin{array}{c} \top_L \\ \text{Cut} \end{array}$$

#### 6.2 Status of $\rightarrow_R$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_5, \mathbf{F}_6 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \to_R & \\ \frac{\bullet \mathbf{h}_3:\Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \vdash \top}{-:\Delta_5 \vdash \top} & \\ \frac{-:\Delta_5 \vdash \top}{-:\Delta_5 \vdash \top} & \\ \top_R & \\ \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_5, \mathbf{F}_6 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \to_R & \frac{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} & \to_R \\ \hline -:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} & \to \\ \bullet \mathbf{h}_1:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_6 \to \mathbf{F}_7 & \bullet \mathbf{xx/W} & \frac{-}{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}} & \bullet \mathbf{xx/W} \\ \hline & \frac{-:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{-:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \to_R \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\underbrace{\frac{\mathbf{h}_1: \Delta_5, \mathbf{F}_6 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7}}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \to_R \underbrace{\frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \quad \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10}}}_{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10}}}_{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9}} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10}}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_5, \mathbf{F}_6 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_{R} \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \begin{array}{c} \vee_1 \\ \text{Cut} \\ \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \begin{array}{c} -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \hline \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \\ \\ \hline -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \hline -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_5, \mathbf{F}_6 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_R \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array}}{ \begin{array}{c} -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \begin{array}{c} -: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_5 \vdash \mathbf{F}_{10} \\ -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array}} \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\frac{h_1:(\Delta_{11},F_8\to F_9),F_5\vdash F_6}{\bullet h_1:\Delta_{11},F_8\to F_9\vdash F_5\to F_6}}_{\bullet h_1:\Delta_{11},F_8\to F_9\vdash F_5\to F_6}\to R} \xrightarrow{h_7:\Delta_{11},F_5\to F_6,F_8\to F_9\vdash F_8} \underbrace{h_7:\Delta_{11},F_9,F_5\to F_6\vdash F_{10}}_{\bullet h_7:(\Delta_{11},F_8\to F_9),F_5\to F_6\vdash F_{10}} Cut} \to L \xrightarrow{-:\Delta_{11},F_8\to F_9\vdash F_{10}} \underbrace{-:\Delta_{11},F_8\to F_9\vdash F_8}_{h_7:\Delta_{11},F_9\to F_6} \xrightarrow{inv-th/ax} \underbrace{-:\Delta_{11},F_8\to F_9\vdash F_8}_{h_7:\Delta_{11},F_9\to F_9\vdash F_{10}} \to L \xrightarrow{-:\Delta_{11},F_9\to F_9\vdash F_{10}} \to L \xrightarrow{-:\Delta_{11},F_9\to F_9\vdash F_9} Cut} \xrightarrow{-:\Delta_{11},F_9\to F_8} \underbrace{-:\Delta_6\vdash F_9}_{\bullet h_1:\Delta_6\vdash F_7\to F_8} \xrightarrow{ax/W} \underbrace{-:\Delta_6,F_7\to F_8\vdash F_9}_{\bullet h_2:\Delta_6,F_7\to F_8\vdash F_9} \underbrace{-:\Delta_6,F_7\to F_8\vdash F_9}_{\bullet L} \xrightarrow{-:\Delta_6,F_7\vdash F_8} \underbrace{-:\Delta_6,F_7\to F_8\vdash F_9}_{\bullet L} \xrightarrow{-:\Delta_6,F_7\vdash F_8} \underbrace{-:\Delta_6,F_7\to F_8\vdash F_9}_{\bullet L} \xrightarrow{-:\Delta_6,F_7\vdash F_9} \underbrace{-:\Delta_6,F_7\vdash F_9}_{\bullet L} \xrightarrow{-:\Delta_6,F_7\vdash F_9} \underbrace{-:\Delta_6,F_7\vdash F_9}_{\bullet L} \xrightarrow{-:\Delta_6,F_7\vdash F_9}}_{\bullet L} \xrightarrow{-:\Delta_6,F_7\vdash F_9}_{\bullet L} \xrightarrow{$$

• Case rule  $\wedge_L$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1 : (\Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9), \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_5 \to \mathbf{F}_6 \end{array} \rightarrow_R \quad \frac{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_7 : (\Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9), \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10}} \quad \begin{array}{c} \wedge_L \\ \bullet \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_5, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_6 \end{array} \rightarrow_R \quad \frac{\bullet}{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10}} \\ \bullet \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_5 \to \mathbf{F}_6 \end{array} \rightarrow_R \quad \frac{\bullet}{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10}} \\ \bullet \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_{10}} \\ \hline - : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_{10} \end{array} \rightarrow_L \\ \end{array}$$

$$\frac{\underbrace{\frac{h_1:(\Delta_{11},F_8\vee F_9),F_5\vdash F_6}{\bullet h_1:\Delta_{11},F_8\vee F_9\vdash F_5\to F_6}}_{\bullet h_1:\Delta_{11},F_8\vee F_9\vdash F_5\to F_6}\to_R \underbrace{\frac{h_7:\Delta_{11},F_8,F_5\to F_6\vdash F_{10}}{\bullet h_7:(\Delta_{11},F_8\vee F_9),F_5\to F_6\vdash F_{10}}}_{-:\Delta_{11},F_8\vdash F_5\to F_6}\underbrace{Cut}$$

$$\frac{-:\Delta_{11},F_8\vee F_9\vdash F_{10}}{h_1:\Delta_{11},F_5,F_8\vdash F_6}\underbrace{\frac{inv-th/ax}{h_7:\Delta_{11},F_8,F_5\to F_6\vdash F_{10}}}_{h_7:\Delta_{11},F_8,F_5\to F_6\vdash F_{10}}\underbrace{\frac{ax/W}{\bullet h_1:\Delta_{11},F_9\vdash F_5\to F_6}}_{\bullet h_1:\Delta_{11},F_9\vdash F_5\to F_6}\underbrace{\frac{inv-th/ax}{h_7:\Delta_{11},F_9,F_5\to F_6\vdash F_{10}}}_{-:\Delta_{11},F_8\vee F_9\vdash F_{10}}\underbrace{\frac{ax/W}{\bullet h_7:\Delta_{11},F_9\vdash F_5\to F_6\vdash F_{10}}}_{-:\Delta_{11},F_9\vdash F_{10}}\vee_L$$

• Case rule  $\perp_L$ 

$$\begin{array}{c} \frac{h_1: (\bot, \Delta_9), F_5 \vdash F_6}{\bullet h_1: \bot, \Delta_9 \vdash F_5 \to F_6} \to_{\mathit{R}} & \\ \frac{\bullet h_7: (\bot, \Delta_9), F_5 \to F_6 \vdash F_8}{-: \bot, \Delta_9 \vdash F_8} & \bot_{\mathit{L}} & \\ & \xrightarrow{-: \bot, \Delta_9 \vdash F_8} & \bot_{\mathit{L}} & \\ \end{array}$$

ullet Case rule I

$$\frac{ \frac{\mathbf{h}_1: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_9, \mathbf{p}_8 \vdash \mathbf{F}_5 \rightarrow \mathbf{F}_6} \rightarrow_R \frac{}{\bullet \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \rightarrow \mathbf{F}_6 \vdash \mathbf{p}_8} }{-: \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8} \frac{I}{-: \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8} I$$

• Case rule  $\top_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: (\top, \Delta_9), \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \to \mathbf{F}_6} \to_R & \frac{\mathbf{h}_7: \Delta_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_8} & \top_L \\ \hline & -: \top, \Delta_9 \vdash \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \to \mathbf{F}_6 & \mathbf{ax/W} \\ \hline & -: \top, \Delta_9 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline & -: \top, \Delta_9 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline & \mathbf{h}_7: \top, \Delta_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline & \mathbf{h}_7: \top, \Delta_9 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \end{array}$$

#### 6.3 Status of $\wedge_R$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \quad \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \wedge_R & \frac{\bullet}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \top} & \top_R \\ \hline & -:\Delta_5 \vdash \top \\ \hline & \frac{-:\Delta_5 \vdash \top}{-:\Delta_5 \vdash \top} & \top_R \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \quad \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \wedge_R \quad \frac{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \quad \xrightarrow{-:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \quad \xrightarrow{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10}} \quad \overset{\mathbf{ax/W}}{\vdash} \\ \frac{-:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{-:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \quad \to_R \end{array}$$

• Case rule  $\wedge_R$ 

• Case rule  $\vee_1$ 

$$\frac{\mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \quad \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_7}{\underbrace{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \land \mathbf{F}_7}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \land \mathbf{F}_7}} \land_R \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \\ -: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \\ \underbrace{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \land \mathbf{F}_7}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \underset{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9} \quad \underset{\mathbf{h}_{Cut}}{\operatorname{ax/W}} \\ \underbrace{-: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}}_{-: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \lor_1$$

• Case rule  $\vee_2$ 

$$\frac{\begin{array}{c} \mathbf{h}_1:\Delta_5\vdash F_6\quad \mathbf{h}_1:\Delta_5\vdash F_7}{\bullet \mathbf{h}_1:\Delta_5\vdash F_6\land F_7} \ \land_R \ \frac{\begin{array}{c} \mathbf{h}_8:\Delta_5,F_6\land F_7\vdash F_{10} \\ \bullet \mathbf{h}_8:\Delta_5,F_6\land F_7\vdash F_9\lor F_{10} \\ \hline \\ -:\Delta_5\vdash F_9\lor F_{10} \\ \hline \\ \bullet \mathbf{h}_1:\Delta_5\vdash F_6\land F_7 \end{array} \xrightarrow[\mathbf{h}_8:\Delta_5,F_6\land F_7\vdash F_{10} \\ \bullet \mathbf{h}_1:\Delta_5\vdash F_6\land F_7 \end{array}} \begin{array}{c} \mathbf{cut} \\ \bullet \mathbf{h}_1:\Delta_5\vdash F_6\land F_7 \end{array} \xrightarrow[\mathbf{h}_8:\Delta_5,F_6\land F_7\vdash F_{10} \\ \bullet \mathbf{h}_2:\Delta_5\vdash F_9\lor F_{10} \\ \hline \\ -:\Delta_5\vdash F_9\lor F_{10} \end{array} \bigvee_{\mathbf{h}} \mathbf{cut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\frac{h_1:\Delta_{11},F_8\to F_9\vdash F_5}{\bullet h_1:\Delta_{11},F_8\to F_9\vdash F_6}}_{\bullet h_1:\Delta_{11},F_8\to F_9\vdash F_5\wedge F_6} \land \underbrace{\frac{h_7:\Delta_{11},F_8\to F_9,F_5\wedge F_6\vdash F_8}{\bullet h_7:(\Delta_{11},F_8\to F_9),F_5\wedge F_6\vdash F_{10}}}_{\bullet h_7:(\Delta_{11},F_8\to F_9),F_5\wedge F_6\vdash F_{10}} \underbrace{\frac{-:\Delta_{11},F_8\to F_9\vdash F_8}{\bullet h_7:\Delta_{11},F_8\to F_9\vdash F_8}}_{\bullet h_2:\Delta_{11},F_8\to F_9\vdash F_8} \underbrace{\frac{h_7:\Delta_{11},F_8\to F_9\vdash F_6}{\bullet h_1:\Delta_{11},F_9\vdash F_5}}_{\bullet h_2:\Delta_{11},F_9\vdash F_5\wedge F_6} \underbrace{\frac{inv-th/ax}{h_1:\Delta_{11},F_9\vdash F_6}}_{h_7:\Delta_{11},F_9\vdash F_{10}} \underbrace{\frac{-:\Delta_{11},F_9\vdash F_8}{\bullet h_2:\Delta_{11},F_9\vdash F_{10}}}_{-:\Delta_{11},F_9\vdash F_{10}} \underbrace{-:\Delta_{11},F_9\vdash F_{10}}_{\bullet h_2:\Delta_{11},F_9\vdash F_{10}} \underbrace{-:\Delta_{11},F_9\vdash F_{10}}_{\bullet h_2:\Delta_{11},F_9\vdash F_{10}}$$

• Case rule  $\wedge_L$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{5}\quad\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{5}\wedge\mathbf{F}_{6}}} \wedge_{R} \quad \frac{\frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{9},\mathbf{F}_{5}\wedge\mathbf{F}_{6}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\wedge\mathbf{F}_{9}),\mathbf{F}_{5}\wedge\mathbf{F}_{6}\vdash\mathbf{F}_{10}}} \quad \wedge_{L} \quad \text{Cut}}{\frac{-:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{6}}} \quad \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{9},\mathbf{F}_{5}\wedge\mathbf{F}_{6}\vdash\mathbf{F}_{10}}}{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{9},\mathbf{F}_{5}\wedge\mathbf{F}_{6}\vdash\mathbf{F}_{10}}} \quad \frac{\mathbf{Ax}/\mathbf{W}}{\mathbf{hCut}}$$

$$\frac{\begin{array}{l} \frac{\mathbf{h}_1:\Delta_6 \vdash \mathbf{F}_7 \quad \mathbf{h}_1:\Delta_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_1:\Delta_6 \vdash \mathbf{F}_7 \land \mathbf{F}_8} \quad \wedge_R \quad \frac{\mathbf{h}_5:\Delta_6,\mathbf{F}_7,\mathbf{F}_8 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_5:\Delta_6,\mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{F}_9} \quad \wedge_L \\ \\ -:\Delta_6 \vdash \mathbf{F}_9 \quad \\ \hline -:\Delta_6,\mathbf{F}_7 \vdash \mathbf{F}_8 \quad \text{ax/W} \quad \frac{}{-:\Delta_6,\mathbf{F}_7,\mathbf{F}_8 \vdash \mathbf{F}_9} \quad \text{ax/W} \\ \hline -:\Delta_6,\mathbf{F}_7 \vdash \mathbf{F}_9 \quad \text{sCut} \\ \\ -:\Delta_6 \vdash \mathbf{F}_9 \end{array}$$

• Case rule  $\vee_L$ 

$$\frac{\underbrace{\frac{h_1 : \Delta_{11}, F_8 \vee F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_8 \vee F_9 \vdash F_6}}_{\bullet h_1 : \Delta_{11}, F_8 \vee F_9 \vdash F_5} \quad h_1 : \Delta_{11}, F_8 \vee F_9 \vdash F_6}{\bullet h_1 : \Delta_{11}, F_8 \vee F_9 \vdash F_5 \wedge F_6} \quad \wedge_R \quad \underbrace{\frac{h_7 : \Delta_{11}, F_8, F_5 \wedge F_6 \vdash F_{10}}{\bullet h_7 : (\Delta_{11}, F_8 \vee F_9), F_5 \wedge F_6 \vdash F_{10}}}_{\bullet h_7 : (\Delta_{11}, F_8 \vee F_9), F_5 \wedge F_6 \vdash F_{10}} \quad \wedge_L \quad \\ \underbrace{\frac{-: \Delta_{11}, F_8 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_8 \vdash F_5 \wedge F_6}}_{\bullet h_7 : \Delta_{11}, F_8 \vdash F_5 \wedge F_6 \vdash F_{10}} \quad \underbrace{\frac{-: \Delta_{11}, F_8 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_8 \vdash F_5}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_8 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5 \wedge F_6}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5 \wedge F_6}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5 \wedge F_6}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5 \wedge F_6}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1 : \Delta_{11}, F_9 \vdash F_5 \wedge F_6}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F_9 \vdash F_5}{\bullet h_1}}_{\bullet h_{10}} \underbrace{\frac{-: \Delta_{11}, F$$

• Case rule  $\perp_L$ 

$$\begin{array}{c} \mathbf{h}_1: \bot, \Delta_9 \vdash \mathbf{F}_5 \quad \mathbf{h}_1: \bot, \Delta_9 \vdash \mathbf{F}_6 \\ \\ \underline{\bullet \mathbf{h}_1: \bot, \Delta_9 \vdash \mathbf{F}_5 \land \mathbf{F}_6} \\ \hline \\ -: \bot, \Delta_9 \vdash \mathbf{F}_8 \\ \hline \\ -: \bot, \Delta_9 \vdash \mathbf{F}_8 \\ \hline \\ -: \bot, \Delta_9 \vdash \mathbf{F}_8 \\ \hline \\ \bot_L \end{array} \quad \begin{array}{c} \bot_L \\ \text{Cut} \\ \hline \end{array}$$

 $\bullet$  Case rule I

$$\frac{\mathbf{h}_1:\Delta_9,\mathbf{p}_8\vdash \mathbf{F}_5\quad \mathbf{h}_1:\Delta_9,\mathbf{p}_8\vdash \mathbf{F}_6}{\underbrace{\begin{array}{c}\bullet\mathbf{h}_1:\Delta_9,\mathbf{p}_8\vdash \mathbf{F}_6\\\\ -:\Delta_9,\mathbf{p}_8\vdash \mathbf{p}_8\\\\ -:\Delta_9,\mathbf{p}_8\vdash \mathbf{p}_8\end{array}}_{} \stackrel{\bullet}{\longrightarrow} I} \xrightarrow{\mathbf{Cut}}$$

$$\frac{\mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \quad \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_6}{\underbrace{\bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \land \mathbf{F}_6}_{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}_{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}} \underbrace{\begin{array}{c} \top_L \\ \bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \land \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \land \mathbf{F}_6 \\ \hline -: \top, \Delta_9 \vdash \mathbf{F}_8 \\ \hline -: \top, \Delta_9 \vdash \mathbf{F}_8 \end{array}}_{\bullet \mathbf{h}_7: \top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8} \underbrace{\begin{array}{c} \mathbf{ax/W} \\ \mathsf{hCut} \\ \bullet \mathsf{hCut} \\ \end{array}}_{\bullet \mathsf{hCut}}$$

#### 6.4 Status of $\vee_1$ : OK

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7} \bigvee_{1} & \frac{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} & \rightarrow_R \\ \hline -:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} & \rightarrow \\ \hline \bullet \mathbf{h}_1:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 & \text{ax/W} & \rightarrow \\ \hline \frac{-:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{-:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \to_R & \rightarrow_R \\ \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\underbrace{\begin{array}{l} \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array}}_{\bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7} \lor_1 \quad \underbrace{\begin{array}{l} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 & \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \hline \\ -: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array}}_{\bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_6} \underbrace{\begin{array}{l} \mathbf{ax} \lor \mathbf{w} \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array}}_{\bullet \mathbf{h}_3: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}} \underbrace{\begin{array}{l} \mathbf{ax} \lor \mathbf{w} \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_4: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_4: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_5: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_7: \Delta_5 \vdash \mathbf{h}_7: \Delta_$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \vee_1 & \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \hline \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} & \begin{array}{c} \vee_1 \\ \hline \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} & \begin{array}{c} \mathbf{cut} \\ \hline \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} & \mathbf{ax/W} \\ \hline \\ \hline \begin{array}{c} \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} & \mathbf{ax/W} \\ \hline \\ \hline \begin{array}{c} -: \Delta_5 \vdash \mathbf{F}_9 \\ \hline -: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} & \vee_1 \end{array} & \mathbf{hCut} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \vee_1 \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \quad \mathbf{Cut} \\ \\ -: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \quad \mathbf{ax/W} \quad \frac{}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}} \quad \mathbf{ax/W} \\ \\ -: \Delta_5 \vdash \mathbf{F}_{10} \quad \vee_2 \\ \hline -: \Delta_5 \vdash \mathbf{F}_{9} \lor \mathbf{F}_{10} \quad \vee_2 \\ \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}}} \vee_{1} \frac{\frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9},\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{8}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}),\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{10}}} \operatorname{Cut} \\ -:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \to \\ \frac{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9},\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{8}} \operatorname{ax/W} \\ -:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} \\ \to \\ -:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} \to_{L} \end{aligned} \qquad \mathbf{ax/W}$$

• Case rule  $\wedge_L$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_5 \\ \bullet \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_5 \vee \mathbf{F}_6 \end{array} \vee_1 \quad \begin{array}{c} \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_7 : (\Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9), \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ & - : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ & \rightarrow \\ \hline \\ \bullet \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_5 \end{array} \quad \begin{array}{c} \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ & \rightarrow \\ \hline \\ \bullet \mathbf{h}_1 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_5 \vee \mathbf{F}_6 \end{array} \quad \vee_1 \qquad \begin{array}{c} \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ & \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ & - : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ & - : \Delta_{11}, \mathbf{F}_8 \wedge \mathbf{F}_9 \vdash \mathbf{F}_{10} \end{array} \quad \wedge_L \end{array} \quad \mathbf{h}_7 : \mathbf$$

• Case rule  $\vee_L$ 

$$\frac{\begin{array}{c} h_1: \Delta_{11}, F_8 \vee F_9 \vdash F_5 \\ \hline \bullet h_1: \Delta_{11}, F_8 \vee F_9 \vdash F_5 \vee F_6 \end{array}}{\bullet h_1: \Delta_{11}, F_8 \vee F_9 \vdash F_5 \vee F_6} \vee_1 & \frac{h_7: \Delta_{11}, F_8, F_5 \vee F_6 \vdash F_{10}}{\bullet h_7: (\Delta_{11}, F_8 \vee F_9), F_5 \vee F_6 \vdash F_{10}} \\ \hline -: \Delta_{11}, F_8 \vee F_9 \vdash F_{10} \\ \hline \\ \hline -: \Delta_{11}, F_8 \vee F_9 \vdash F_{10} \\ \hline \hline \bullet h_1: \Delta_{11}, F_8 \vdash F_5 \end{array}} & \frac{1}{h_7: \Delta_{11}, F_8, F_5 \vee F_6 \vdash F_{10}} & \frac{1}{h_7: \Delta_{11}, F_8, F_5 \vee F_6 \vdash F_{10}} \\ \hline \\ \bullet h_1: \Delta_{11}, F_8 \vdash F_5 \vee F_6 \end{array} & \frac{1}{h_7: \Delta_{11}, F_8, F_5 \vee F_6 \vdash F_{10}} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline \\ \hline -: \Delta_{11}, F_8 \vdash F_{10} & \frac{1}{h_1: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_8 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_8 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_8 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_8 \vdash F_9 & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_8 \vdash F_9 & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5 \vee F_6} \vee_1 \\ \hline -: \Delta_{11}, F_9 \vdash F_{10} & \frac{1}{h_7: \Delta_{11}, F_9 \vdash F_5} \vee_1 \\$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \frac{h_1:\bot,\Delta_9\vdash F_5}{\bullet h_1:\bot,\Delta_9\vdash F_5\vee F_6} & \vee_1 & \hline\\ -:\bot,\Delta_9\vdash F_8 & \to \\ \hline\\ -:\bot,\Delta_9\vdash F_8 & \bot_L \\ \hline\\ -:\bot,\Delta_9\vdash F_8 & \bot_L \end{array}$$
 Cut

 $\bullet\,$  Case rule I

$$\frac{ \begin{array}{c|c} \mathbf{h}_1: \Delta_9, \mathbf{p}_8 \vdash \mathbf{F}_5 \\ \hline \bullet \mathbf{h}_1: \Delta_9, \mathbf{p}_8 \vdash \mathbf{F}_5 \vee \mathbf{F}_6 \end{array} \vee_1 & \frac{}{\bullet} \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{p}_8 \\ \hline \begin{array}{c} -: \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \\ \hline -: \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \end{array} & I \end{array} }{\mathbf{Cut}}$$

• Case rule  $\top_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} & \vee_1 & \frac{\mathbf{h}_7: \Delta_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8} & \tau_L \\ \hline & -: \top, \Delta_9 \vdash \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_1: \top, \Delta_9 \vdash \mathbf{F}_5 \lor \mathbf{F}_6 & \text{ax/W} & \frac{}{h_7: \top, \Delta_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8} \\ \hline & -: \top, \Delta_9 \vdash \mathbf{F}_8 & \text{hCut} \end{array}$$

#### 6.5 Status of $\vee_2$ : OK

• Case rule  $\top_R$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1 : \Delta_5 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1 : \Delta_5 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_2 \quad \frac{}{\bullet \mathbf{h}_8 : \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \top} \quad \begin{array}{c} \top_R \\ \hline -: \Delta_5 \vdash \top \\ \hline \\ \hline -: \Delta_5 \vdash \top \end{array} \quad \top_R \end{array} }$$
 Cut

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7} \bigvee_2 \begin{array}{c} \mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline -:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_1:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \begin{array}{c} \rightarrow \\ \mathbf{ax/W} \\ \hline -:\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline -:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline -:\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \end{array} \begin{array}{c} \rightarrow_R \\ \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\underbrace{\begin{array}{l} \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array}}_{\bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9} \underbrace{\begin{array}{l} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 & \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \hline \\ -: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_9 \\ \hline \\ -: \Delta_5 \vdash \mathbf{F}_9 \\ \hline \\ -: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \hline \end{array}}_{\bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \hline \\ -: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \hline \end{array}}_{\bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10} \\ \hline \\ -: \Delta_5 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \\ \hline \end{array}}_{\bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{F}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{h}_6 \lor \mathbf{h}_7 \vdash \mathbf{h}_{10} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{h}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{h}_6 \lor \mathbf{h}_7 \vdash \mathbf{h}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10}} \underbrace{\begin{array}{l} \mathbf{a}_8: \Delta_5, \mathbf{h}_6 \lor \mathbf{h}_7 \vdash \mathbf{h}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \hline \end{array}_{\bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10} \\ \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{h}_{10}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_2 & \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \\ & -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ & \xrightarrow{\bullet} \\ \hline \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \hline \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \hline -: \Delta_5 \vdash \mathbf{F}_9 \\ \hline -: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \begin{array}{c} \vee_1 \\ \mathsf{cut} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \vee_2 \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \quad \mathbf{Cut} \\ \\ -: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_1: \Delta_5 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \quad \mathbf{ax/W} \quad \frac{}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}} \\ \hline -: \Delta_5 \vdash \mathbf{F}_{10} \quad \vee_2 \\ \hline -: \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} \quad \mathbf{hCut}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}}} \vee_{2} \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9},\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{8}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}),\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{10}}} \operatorname{Cut} \\ -:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} \to \frac{\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{8}} \frac{\mathbf{inv}-\mathbf{th}/\mathbf{ax}}{\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{9}\vdash\mathbf{F}_{6}} \frac{\mathbf{inv}-\mathbf{th}/\mathbf{ax}}{\mathbf{h}_{1}:\Delta_{11},\mathbf{F}_{9}\vdash\mathbf{F}_{5}\vee\mathbf{F}_{6}} \vee_{2} \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9},\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{10}}{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9}\to\mathbf{F}_{5}\vee\mathbf{F}_{6}\vdash\mathbf{F}_{10}} \to L$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

• Case rule  $\perp_L$ 

$$\begin{array}{c} \frac{h_1:\bot,\Delta_9\vdash F_6}{\bullet h_1:\bot,\Delta_9\vdash F_5\vee F_6} \;\; \vee_2 \quad \\ \frac{\bullet h_7:(\bot,\Delta_9),F_5\vee F_6\vdash F_8}{-:\bot,\Delta_9\vdash F_8} \quad \\ \frac{-:\bot,\Delta_9\vdash F_8}{-:\bot,\Delta_9\vdash F_8} \;\; \bot_L \end{array} \quad \text{Cut}$$

ullet Case rule I

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_9,\mathbf{p}_8 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_9,\mathbf{p}_8 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} & \vee_2 & \\ \hline -:\Delta_9,\mathbf{p}_8 \vdash \mathbf{p}_8 & \rightarrow \\ \hline -:\Delta_9,\mathbf{p}_8 \vdash \mathbf{p}_8 & \rightarrow \\ \hline -:\Delta_9,\mathbf{p}_8 \vdash \mathbf{p}_8 & I \end{array} \quad \mathbf{Cut}$$

• Case rule  $\top_L$ 

$$\begin{array}{c} \mathbf{h}_1: \top, \Delta_9 \vdash F_6 \\ \hline \bullet \mathbf{h}_1: \top, \Delta_9 \vdash F_5 \vee F_6 \\ \hline \\ -: \top, \Delta_9 \vdash F_8 \\ \hline \\ -: \top, \Delta_9 \vdash F_8 \\ \hline \\ \bullet \mathbf{h}_1: \top, \Delta_9 \vdash F_5 \vee F_6 \\ \hline \\ \hline \bullet \mathbf{h}_1: \top, \Delta_9 \vdash F_5 \vee F_6 \\ \hline \\ -: \top, \Delta_9 \vdash F_8 \\ \hline \\ -: \top, \Delta_9 \vdash F_8 \\ \hline \\ -: \top, \Delta_9 \vdash F_8 \\ \hline \end{array} \begin{array}{c} \top_L \\ \mathsf{Cut} \\ \hline \\ \mathsf{h}_1: \top, \Delta_9 \vdash F_5 \vee F_6 \vdash F_8 \\ \hline \\ \mathsf{h}_1: \top, \Delta_9 \vdash F_8 \\ \hline \end{array}$$

### 6.6 Status of $\rightarrow_L$ : OK

• Case rule  $\top_R$ 

$$\frac{\mathbf{h}_1:\Delta_6,\mathbf{f}_8\to\mathbf{f}_9\vdash\mathbf{f}_8\quad\mathbf{h}_1:\Delta_6,\mathbf{f}_9\vdash\mathbf{f}_7}{\underbrace{\begin{array}{c}\bullet\mathbf{h}_1:\Delta_6,\mathbf{f}_8\to\mathbf{f}_9\vdash\mathbf{f}_7\\ \hline \bullet\mathbf{h}_1:\Delta_6,\mathbf{f}_8\to\mathbf{f}_9\vdash\mathbf{f}_7\\ \hline \\ -:\Delta_6,\mathbf{f}_8\to\mathbf{f}_9\vdash\top\\ \hline \\ -:\Delta_6,\mathbf{f}_8\to\mathbf{f}_9\vdash\top\\ \hline \end{array}}_{}^{}} \xrightarrow{\uparrow_R} \frac{\mathbf{h}_1:\Delta_6,\mathbf{f}_8\to\mathbf{f}_9\vdash\mathbf{f}_7\\ }_{}^{} \subset \mathbf{h}_1$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_6,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_8\quad\mathbf{h}_1:\Delta_6,\mathbf{F}_9\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_6,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_7}\to_L & \frac{\mathbf{h}_{10}:\Delta_6,\mathbf{F}_7,\mathbf{F}_{11},\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{10}:(\Delta_6,\mathbf{F}_8\to\mathbf{F}_9),\mathbf{F}_7\vdash\mathbf{F}_{11}\to\mathbf{F}_{12}} & \to_R \\ \hline & -:\Delta_6,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{11}\to\mathbf{F}_{12} & \to\\ & \frac{\bullet}{\mathbf{h}_1:\Delta_6,\mathbf{F}_{11},\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_7} & \mathbf{ax/W} & \frac{\to}{\mathbf{h}_{10}:\Delta_6,\mathbf{F}_{11},\mathbf{F}_7,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{12}} \\ \hline & -:\Delta_6,\mathbf{F}_{11},\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{12} \\ \hline & -:\Delta_6,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{11}\to\mathbf{F}_{12} & \to_R \\ \hline \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_8 \quad \mathbf{h}_1: \Delta_6, \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_7 \end{array} \rightarrow_L \quad \begin{array}{c} \mathbf{h}_{10}: \Delta_6, \mathbf{F}_7, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \quad \mathbf{h}_{10}: \Delta_6, \mathbf{F}_7, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{10}: (\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9), \mathbf{F}_7 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \rightarrow_{\mathbf{Cut}} \\ \hline -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \\ \bullet \mathbf{h}_{10}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \bullet \mathbf{h}_{21}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_7 \end{array} \xrightarrow{\mathbf{ax/W}} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_{10}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{21}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \end{array} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ -: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \wedge \mathbf{F}_{12} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \xrightarrow{\mathbf{h}_{22}: \Delta_6, \mathbf{F}_8 \rightarrow \mathbf{H}_9 \vdash \mathbf{H}_{12} \rightarrow \mathbf{H}_9 \rightarrow \mathbf{H}_9 \rightarrow \mathbf{H}_9 \rightarrow \mathbf{H}_9 \rightarrow \mathbf{H}_9 \rightarrow \mathbf{H}_9$$

• Case rule  $\vee_1$ 

$$\begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{8}\quad\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{9}\vdash\mathbf{F}_{7}}{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{7}} \to_{L} & \frac{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}}{\bullet\mathbf{h}_{10}:(\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}),\mathbf{F}_{7}\vdash\mathbf{F}_{11}\vee\mathbf{F}_{12}} & \mathbf{cut} \\ & \frac{-:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}\vee\mathbf{F}_{12}}{\bullet} & \frac{\to}{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}} & \mathbf{ax/W} \\ & \frac{-:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}}{-:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}} & \mathbf{v}_{1} & \mathbf{hCut} \\ & \frac{-:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}}{-:\Delta_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{11}\vee\mathbf{F}_{12}} & \mathbf{v}_{1} & \mathbf{hCut} \\ \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_8 \quad \mathbf{h}_1:\Delta_6, \mathbf{F}_9 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_7} \rightarrow_L \quad \frac{\mathbf{h}_{10}:\Delta_6, \mathbf{F}_7, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}}{\bullet \mathbf{h}_{10}:(\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9), \mathbf{F}_7 \vdash \mathbf{F}_{11} \vee \mathbf{F}_{12}} \quad \mathbf{Cut} \\ \\ \frac{-:\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{11} \vee \mathbf{F}_{12}}{\bullet \mathbf{h}_{10}:\Delta_6, \mathbf{F}_7, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \quad \mathbf{ax/W} \\ \frac{-:\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}}{-:\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_9 \vdash \mathbf{F}_{12}} \quad \forall_2 \\ \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\mathbf{h}_{1}: (\Delta_{13}, F_{10} \to F_{11}), F_{7} \to F_{8} \vdash F_{7} \quad \mathbf{h}_{1}: (\Delta_{13}, F_{10} \to F_{11}), F_{8} \vdash F_{6}}{\bullet \mathbf{h}_{1}: (\Delta_{13}, F_{10} \to F_{11}), F_{7} \to F_{8} \vdash F_{6}} \to_{L} \quad \frac{\mathbf{h}_{9}: \Delta_{13}, F_{6}, F_{7} \to F_{8}, F_{10} \to F_{11}), F_{7} \to F_{8}), F_{6} \vdash F_{12}}{\bullet \mathbf{h}_{9}: ((\Delta_{13}, F_{10} \to F_{11}), F_{7} \to F_{8}), F_{6} \vdash F_{12}} \quad \mathbf{Cut} \\ & - : (\Delta_{13}, F_{10} \to F_{11}), F_{7} \to F_{8} \vdash F_{12} \\ & - : (\Delta_{13}, F_{10} \to F_{11}), F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{13}, F_{10} \to F_{11}, F_{7} \to F_{8} \vdash F_{7} \\ & - : \Delta_{13}, F_{10} \to F_{11}, F_{7} \to F_{8} \vdash F_{7} \\ & - : \Delta_{13}, F_{10} \to F_{11}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{13}, F_{10} \to F_{11}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{13}, F_{10} \to F_{11}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{13}, F_{10} \to F_{11}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{6}, F_{7} \to F_{8} \vdash F_{7} & h_{1} : \Delta_{6}, F_{8} \vdash F_{10} \to F_{11} \\ & - : \Delta_{6}, F_{7} \to F_{8} \vdash F_{10} \to F_{11} \\ & - : \Delta_{6}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{6}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{6}, F_{7} \to F_{8} \vdash F_{12} \\ & - : \Delta_{6}, F_{8} \to F_{10} \to F_{11} \\ & - : \Delta_{6}, F_{8} \to F_{12} \\ & - : \Delta_{6}, F_{8} \to F_{12$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_{1}:(\Delta_{13},\mathbf{F}_{10}\wedge\mathbf{F}_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:(\Delta_{13},\mathbf{F}_{10}\wedge\mathbf{F}_{11}),\mathbf{F}_{8}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:(\Delta_{13},\mathbf{F}_{10}\wedge\mathbf{F}_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{6}}\rightarrow_{L}\frac{\mathbf{h}_{9}:\Delta_{13},\mathbf{F}_{6},\mathbf{F}_{10},\mathbf{F}_{11},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{9}:((\Delta_{13},\mathbf{F}_{10}\wedge\mathbf{F}_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}),\mathbf{F}_{6}\vdash\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{1}:\Delta_{13},\mathbf{F}_{10}\wedge\mathbf{F}_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{1}:\Delta_{13},\mathbf{F}_{8},\mathbf{F}_{10}\wedge\mathbf{F}_{11}\vdash\mathbf{F}_{6}}\mathbf{ax/W}\frac{\mathbf{h}_{9}:\Delta_{13},\mathbf{F}_{10},\mathbf{F}_{11},\mathbf{F}_{6},\mathbf{F}_{8}\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{1}:\Delta_{13},\mathbf{F}_{8},\mathbf{F}_{10}\wedge\mathbf{F}_{11}\vdash\mathbf{F}_{6}}\mathbf{ax/W}\frac{\mathbf{h}_{9}:\Delta_{13},\mathbf{F}_{10},\mathbf{F}_{11},\mathbf{F}_{6},\mathbf{F}_{8}\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{1}:\Delta_{13},\mathbf{F}_{8},\mathbf{F}_{10}\wedge\mathbf{F}_{11}\vdash\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{9}:\Delta_{13},\mathbf{F}_{10},\mathbf{F}_{11}\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{9}:\Delta_{13},\mathbf{F}_{10},\mathbf{F}_{11}\vdash\mathbf{F}_{12}}\to_{L}$$

$$\frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}}{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\mathbf{F}_{10}\wedge\mathbf{F}_{11}}\to_{L}\frac{\mathbf{h}_{9}:\Delta_{6},\mathbf{F}_{10},\mathbf{F}_{11},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{12}}{\bullet\mathbf{h}_{9}:(\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}),\mathbf{F}_{10}\wedge\mathbf{F}_{11}\vdash\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\to_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\to_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\to_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}\wedge_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{F}_{12}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{13}\to\mathbf{h}_{11}\to\mathbf{h}_{13}}\wedge_{L}\frac{\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{h}_{13}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{h}_{13}}\wedge_{L}\frac{\mathbf{h}_{13}:\Delta_{13},\mathbf{F}_{11}\to\mathbf{h}_{13}}{\bullet\mathbf{h}_{11}:\Delta_{13},\mathbf{F}_{13}\to\mathbf{h}_{13}}\wedge_{L}\frac{\mathbf{h}_{13}:\Delta_{13},\mathbf{F}_{13},\mathbf{F}_{13}\to\mathbf{h}_{13}}{\bullet\mathbf{h}_{13}:\Delta_{13},\mathbf{F}_{13},\mathbf{F}_{13}\to\mathbf{h}_{13}}\wedge_{L}\frac{\mathbf{h}_{13}:\Delta_{13},\mathbf{F}_$$

• Case rule  $\vee_L$ 

$$\frac{\mathbf{h}_{1}: (\Delta_{13}, F_{10} \vee F_{11}), F_{7} \rightarrow F_{8} \vdash F_{7} \quad \mathbf{h}_{1}: (\Delta_{13}, F_{10} \vee F_{11}), F_{8} \vdash F_{6}}{\bullet \mathbf{h}_{1}: (\Delta_{13}, F_{10} \vee F_{11}), F_{7} \rightarrow F_{8} \vdash F_{6}} \rightarrow_{L} \quad \frac{\mathbf{h}_{9}: \Delta_{13}, F_{6}, F_{10}, F_{7} \rightarrow F_{8} \vdash F_{12} \quad \mathbf{h}_{9}: \Delta_{13}, F_{6}, F_{11}, F_{7} \rightarrow F_{8} \vdash F_{12}}{\bullet \mathbf{h}_{9}: ((\Delta_{13}, F_{10} \vee F_{11}), F_{7} \rightarrow F_{8} \vdash F_{12}} \quad \mathbf{Cut}$$

$$-: (\Delta_{13}, F_{10} \vee F_{11}), F_{7} \rightarrow F_{8} \vdash F_{12} \quad \cdots \quad \frac{\mathbf{h}_{9}: \Delta_{13}, F_{10} \vee F_{11}), F_{7} \rightarrow F_{8} \vdash F_{12}}{\bullet \mathbf{h}_{9}: \Delta_{13}, F_{10}, F_{6}, F_{8} \vdash F_{12}} \quad \text{inv-th/ax} \quad \frac{\mathbf{h}_{9}: \Delta_{13}, F_{11}, F_{6}, F_{8} \vdash F_{12}}{\bullet \mathbf{h}_{9}: \Delta_{13}, F_{10}, F_{6}, F_{8} \vdash F_{12}} \quad \mathbf{h}_{0}$$

$$-: \Delta_{13}, F_{7} \rightarrow F_{8}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \frac{\mathbf{h}_{1}: \Delta_{13}, F_{8}, F_{10} \vee F_{11} \vdash F_{12}}{\bullet \mathbf{h}_{9}: \Delta_{13}, F_{10}, F_{7} \rightarrow F_{8} \vdash F_{12}} \quad \mathbf{h}_{0}$$

$$-: \Delta_{13}, F_{7} \rightarrow F_{8}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{11} \vdash F_{12} \quad \cdots \quad \mathbf{h}_{1} : \Delta_{13}, F_{10} \vee F_{$$

$$\frac{\mathbf{h}_1:\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8 \vdash \mathbf{F}_7 \quad \mathbf{h}_1:\Delta_6, \mathbf{F}_8 \vdash \bot}{\underbrace{\begin{array}{c} \bullet \mathbf{h}_1:\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8 \vdash \bot} \\ -:\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8 \vdash \mathbf{F}_{10} \\ \end{array}} \underbrace{\begin{array}{c} \bullet \mathbf{h}_2:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ -:\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8 \vdash \mathbf{F}_{10} \\ \hline \\ -:\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8 \vdash \mathbf{F}_7 \\ \end{array}} \underbrace{\begin{array}{c} \bot_L \\ \bullet \mathbf{h}_2:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_2:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_7 \rightarrow \mathbf{F}_8), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_{10}), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_{10}), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_{10}), \bot \vdash \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_3:(\Delta_6, \mathbf{F}_8 \rightarrow \mathbf{F}_{10}), \bot \vdash \mathbf{h}_3:(\Delta_6, \mathbf{F}_8 \rightarrow \mathbf$$

$$\frac{\mathbf{h}_{1}:(\bot,\Delta_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:(\bot,\Delta_{11}),\mathbf{F}_{8}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:(\bot,\Delta_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{6}}\to_{L}\quad\frac{\bullet\mathbf{h}_{9}:((\bot,\Delta_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}),\mathbf{F}_{6}\vdash\mathbf{F}_{10}}{-:(\bot,\Delta_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{10}}\xrightarrow{\bot_{L}}\quad\mathbf{Cut}$$

ullet Case rule I

$$\frac{\mathbf{h}_{1}:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{8}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{6}}\to_{L}\frac{\bullet\mathbf{h}_{9}:((\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\to\mathbf{F}_{8}),\mathbf{F}_{6}\vdash\mathbf{p}_{10}}{-:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{p}_{10}}\xrightarrow{\bullet}I$$

$$\frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\mathbf{p}_{10}}{\bullet}\to_{L}\frac{\bullet\mathbf{h}_{9}:(\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}),\mathbf{p}_{10}\vdash\mathbf{p}_{10}}{\bullet}I$$

$$\frac{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{p}_{10}}{-:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{p}_{10}}\to_{L}\frac{\bullet}{\bullet}$$

$$\frac{\bullet\mathbf{h}_{9}:(\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}),\mathbf{p}_{10}\vdash\mathbf{p}_{10}}{\bullet}Cut$$

$$\frac{-:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{p}_{10}}{\bullet}ax/\mathbf{W}$$

• Case rule  $\top_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\top}{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\top}\to L \quad \frac{\mathbf{h}_{9}:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{9}:(\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}),\top\vdash\mathbf{F}_{10}} \quad \top_{L} \\ & \xrightarrow{\bullet} \\ & \xrightarrow{-:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{10}} \\ & \xrightarrow{\bullet} \\ & \xrightarrow{-:\Delta_{6},\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{10}} \quad \mathbf{ax/W} \\ \\ \frac{\mathbf{h}_{1}:(\top,\Delta_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:(\top,\Delta_{11}),\mathbf{F}_{8}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:(\top,\Delta_{11}),\mathbf{F}_{7}\to\mathbf{F}_{8}\vdash\mathbf{F}_{10}} \quad \xrightarrow{\bullet} \\ & \xrightarrow{\bullet} \\ &$$

#### 6.7 Status of $\wedge_L$ : OK

• Case rule  $\top_R$ 

$$\frac{\begin{array}{l} \mathbf{h}_1: \Delta_6, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_6, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_7 \end{array} \land_L \quad \frac{}{\bullet \mathbf{h}_{10}: (\Delta_6, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_7 \vdash \top} \quad \begin{array}{l} \top_R \\ \mathsf{Cut} \\ & \xrightarrow{} \\ \hline -: \Delta_6, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \top \\ \hline -: \Delta_6, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \top \end{array} \mid_T_R \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_6, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_6, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_7} \ \land_L \ \frac{\mathbf{h}_{10}:\Delta_6, \mathbf{F}_7, \mathbf{F}_{11}, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{12}}{\bullet \mathbf{h}_{10}:(\Delta_6, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_7 \vdash \mathbf{F}_{11} \to \mathbf{F}_{12}} \\ \hline -:\Delta_6, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{11} \to \mathbf{F}_{12} \\ \hline \bullet \mathbf{h}_1:\Delta_6, \mathbf{F}_{11}, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_7} \ \mathbf{ax/W} \ \frac{\rightarrow}{\mathbf{h}_{10}:\Delta_6, \mathbf{F}_{11}, \mathbf{F}_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{12}} \\ \hline -:\Delta_6, \mathbf{F}_{11}, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{12} \\ \hline -:\Delta_6, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{11} \to \mathbf{F}_{12} \end{array} \rightarrow_R \end{array} \qquad \begin{array}{c} \rightarrow_R \\ \text{Cut} \\ \text{hCut} \\ \hline \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{7}}{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{7}}}{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{7}}} \wedge_{L} \xrightarrow{\frac{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{11}}{\bullet\mathbf{h}_{10}:(\Delta_{6},\mathbf{F}_{8}\wedge\mathbf{F}_{9}),\mathbf{F}_{7}\vdash\mathbf{F}_{11}\wedge\mathbf{F}_{12}}} \text{Cut} \\ -\frac{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{11}\wedge\mathbf{F}_{12}}{\bullet\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{11}} \xrightarrow{\mathbf{inv-th/ax}} \frac{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{11}}{\bullet\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{11}\wedge\mathbf{F}_{12}}} \wedge_{R}} \xrightarrow{\bullet\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{11}\wedge\mathbf{F}_{12}} \wedge_{L}} \bullet_{\mathbf{h}_{10}}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1 : \Delta_6, F_8, F_9 \vdash F_7 \\ \hline \bullet \mathbf{h}_1 : \Delta_6, F_8 \land F_9 \vdash F_7 \end{array} \wedge_L \quad \frac{\mathbf{h}_{10} : \Delta_6, F_7, F_8 \land F_9 \vdash F_{11} \\ \hline \bullet \mathbf{h}_{10} : (\Delta_6, F_8 \land F_9), F_7 \vdash F_{11} \lor F_{12} \\ \hline - : \Delta_6, F_8 \land F_9 \vdash F_{11} \lor F_{12} \\ \hline \bullet \mathbf{h}_1 : \Delta_6, F_8 \land F_9 \vdash F_7 \quad \text{ax/W} \quad \frac{\rightarrow}{\mathbf{h}_{10} : \Delta_6, F_7, F_8 \land F_9 \vdash F_{11}} \\ \hline - : \Delta_6, F_8 \land F_9 \vdash F_{11} \\ \hline - : \Delta_6, F_8 \land F_9 \vdash F_{11} \lor F_{12} \end{array} \vee_1 \quad \begin{array}{c} \mathsf{Ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_{1} : \Delta_{6}, \mathbf{F}_{8}, \mathbf{F}_{9} \vdash \mathbf{F}_{7} \\ \bullet \mathbf{h}_{1} : \Delta_{6}, \mathbf{F}_{8} \land \mathbf{F}_{9} \vdash \mathbf{F}_{7} \end{array} \wedge_{L} \quad \begin{array}{c} \mathbf{h}_{10} : \Delta_{6}, \mathbf{F}_{7}, \mathbf{F}_{8} \land \mathbf{F}_{9} \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{10} : (\Delta_{6}, \mathbf{F}_{8} \land \mathbf{F}_{9}), \mathbf{F}_{7} \vdash \mathbf{F}_{11} \lor \mathbf{F}_{12} \end{array} \quad \begin{array}{c} \vee_{2} \\ \mathsf{Cut} \\ \\ \bullet \mathbf{h}_{1} : \Delta_{6}, \mathbf{F}_{8} \land \mathbf{F}_{9} \vdash \mathbf{F}_{7} & \mathsf{ax/W} & \overset{\bullet}{\rightarrow} \\ \mathbf{h}_{10} : \Delta_{6}, \mathbf{F}_{7}, \mathbf{F}_{8} \land \mathbf{F}_{9} \vdash \mathbf{F}_{12} \\ \hline - : \Delta_{6}, \mathbf{F}_{8} \land \mathbf{F}_{9} \vdash \mathbf{F}_{12} & \vee_{2} \end{array} \quad \begin{array}{c} \mathsf{ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\frac{h_{1}:(\Delta_{13},F_{10}\to F_{11}),F_{7},F_{8}\vdash F_{6}}{\bullet h_{1}:(\Delta_{13},F_{10}\to F_{11}),F_{7}\land F_{8}\vdash F_{6}}}{\bullet h_{1}:(\Delta_{13},F_{10}\to F_{11}),F_{7}\land F_{8}\vdash F_{6}}} \wedge_{L} \frac{\frac{h_{9}:\Delta_{13},F_{6},F_{10}\to F_{11},F_{7}\land F_{8}\vdash F_{10}}{\bullet h_{9}:((\Delta_{13},F_{10}\to F_{11}),F_{7}\land F_{8}),F_{6}\vdash F_{12}}}{\bullet h_{9}:((\Delta_{13},F_{10}\to F_{11}),F_{7}\land F_{8}),F_{6}\vdash F_{12}}} \wedge_{L} \\ -:(\Delta_{13},F_{10}\to F_{11}),F_{7}\land F_{8}\vdash F_{12}} \\ \frac{-:(\Delta_{13},F_{10}\to F_{11}),F_{7}\land F_{8}\vdash F_{12}}{\bullet h_{9}:\Delta_{13},F_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{10}}} \text{ inv-th/ax } \frac{h_{9}:\Delta_{13},F_{11},F_{6},F_{7},F_{8}\vdash F_{12}}}{h_{9}:\Delta_{13},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{13},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}{-:\Delta_{13},F_{10}\to F_{11},F_{7}\land F_{8}\vdash F_{12}}} \wedge_{L} \\ \frac{h_{1}:\Delta_{6},F_{7},F_{8}\vdash F_{10}\to F_{11}}{\bullet h_{1}:\Delta_{6},F_{7}\land F_{8}\vdash F_{10}\to F_{11}}} \wedge_{L} \frac{h_{9}:\Delta_{6},F_{10}\to F_{11}\vdash F_{12}}{\bullet h_{9}:(\Delta_{6},F_{7}\land F_{8}),F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \\ \frac{h_{1}:\Delta_{6},F_{7},F_{8}\vdash F_{10}\to F_{11}}{\bullet h_{1}:\Delta_{6},F_{7}\land F_{8}\vdash F_{10}\to F_{11}}} \wedge_{L} \frac{h_{9}:\Delta_{6},F_{10}\to F_{11}\vdash F_{10}}}{\bullet h_{9}:(\Delta_{6},F_{7}\land F_{8}),F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{6},F_{7},F_{8}\vdash F_{12}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{10}}} \text{ inv-th/ax } \frac{h_{9}:\Delta_{6},F_{11},F_{7},F_{8}\vdash F_{12}}}{h_{9}:\Delta_{6},F_{7},F_{8}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{10}}} \wedge_{L} \frac{h_{9}:\Delta_{6},F_{11},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \frac{h_{9}:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \frac{h_{9}:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8},F_{10}\to F_{11}\vdash F_{12}}} \wedge_{L} \frac{h_{9}:\Delta_{6},F_{7},F_{8}\vdash F_{12}}}{\bullet h_{9}:\Delta_{6},F_{7},F_{8}\vdash F_{12}}} \wedge_{L} \\ \frac{-:\Delta_{6},F_{7}$$

• Case rule  $\wedge_L$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: (\Delta_{13}, F_{10} \wedge F_{11}), F_7, F_8 \vdash F_6 \\ \bullet \mathbf{h}_1: (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8 \vdash F_6 \end{array}}{ \bullet \mathbf{h}_1: (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8 \vdash F_6} \\ \begin{array}{c} -: (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8 \vdash F_{12} \\ \hline \\ -: (\Delta_{13}, F_{10} \wedge F_{11}), F_7 \wedge F_8 \vdash F_{12} \\ \hline \\ \bullet \mathbf{h}_1: \Delta_{13}, F_{10}, F_{11}, F_7, F_8 \vdash F_6 \end{array}} \\ \begin{array}{c} \mathbf{h}_1: \Delta_{13}, F_{10}, F_{11}, F_7, F_8 \vdash F_6 \\ \hline \\ \bullet \mathbf{h}_1: \Delta_{13}, F_{10}, F_{11}, F_7 \wedge F_8 \vdash F_6 \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10}, F_{11}, F_7 \wedge F_8 \vdash F_{12} \\ \hline \\ -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10}, F_{11}, F_7 \wedge F_8 \vdash F_{12} \\ \hline \\ -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \\ \hline \\ -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \\ \hline \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array} \\ \begin{array}{c} -: \Delta_{13}, F_{10} \wedge F_{11}, F_7 \wedge F_8 \vdash F_{12} \end{array} \\ \end{array}$$

$$\begin{array}{c} \frac{h_1:\Delta_6,F_7,F_8\vdash F_{10}\wedge F_{11}}{\bullet h_1:\Delta_6,F_7\wedge F_8\vdash F_{10}\wedge F_{11}} \ \wedge_L \ \frac{h_9:\Delta_6,F_{10},F_{11},F_7\wedge F_8\vdash F_{12}}{\bullet h_9:(\Delta_6,F_7\wedge F_8),F_{10}\wedge F_{11}\vdash F_{12}} \ \wedge_L \\ \hline \\ -:\Delta_6,F_7\wedge F_8\vdash F_{12} \\ \hline \\ \frac{h_9:\Delta_6,F_7\wedge F_8\vdash F_{12}}{\bullet h_9:\Delta_6,F_{10},F_{11},F_7,F_8\vdash F_{12}} \ \text{out} \\ \hline \\ \frac{h_1:\Delta_6,F_7,F_8\vdash F_{10}\wedge F_{11}}{\bullet h_9:\Delta_6,F_7,F_8\vdash F_{12}} \wedge_L \\ \hline \\ \frac{-:\Delta_6,F_7\wedge F_8\vdash F_{12}}{-:\Delta_6,F_7\wedge F_8\vdash F_{12}} \wedge_L \\ \hline \\ \frac{h_1:\Delta_6,F_9,F_{10}\vdash F_7}{\bullet h_1:\Delta_6,F_9\wedge F_{10}\vdash F_7} \wedge_L \ \frac{h_8:\Delta_6,F_7,F_9,F_{10}\vdash F_{11}}{\bullet h_8:(\Delta_6,F_9\wedge F_{10}),F_7\vdash F_{11}} \wedge_L \\ \hline \\ \frac{-:\Delta_6,F_9\wedge F_{10}\vdash F_7}{\bullet h_1:\Delta_6,F_{10},F_9\vdash F_7} \ \text{ax/W} \ \frac{h_8:\Delta_6,F_{10},F_7,F_9\vdash F_{11}}{\bullet h_8:\Delta_6,F_{10},F_7,F_9\vdash F_{11}} \ H_{hCut} \\ \hline \\ \frac{-:\Delta_6,F_9\wedge F_{10}\vdash F_{11}}{-:\Delta_6,F_9\wedge F_{10}\vdash F_{11}} \wedge_L \end{array}$$

• Case rule  $\vee_L$ 

$$\frac{\frac{h_1:(\Delta_{13},F_{10}\vee F_{11}),F_7,F_8\vdash F_6}{\bullet h_1:(\Delta_{13},F_{10}\vee F_{11}),F_7\wedge F_8\vdash F_6}}{\wedge L} \frac{h_9:\Delta_{13},F_6,F_{10},F_7\wedge F_8\vdash F_{12}}{\bullet h_9:((\Delta_{13},F_{10}\vee F_{11}),F_7\wedge F_8),F_6\vdash F_{12}} \vee L}{-:(\Delta_{13},F_{10}\vee F_{11}),F_7\wedge F_8\vdash F_{12}} \times L} \frac{(\Delta_{13},F_{10}\vee F_{11}),F_7\wedge F_8\vdash F_{12}}{\wedge h_9:\Delta_{13},F_{10}\vee F_{11}}} \vee L} \frac{(\Delta_{13},F_{10}\vee F_{11}),F_7\wedge F_8\vdash F_{12}}{\wedge h_9:\Delta_{13},F_{10}\vee F_{11}}} \times L} \frac{(\Delta_{13},F_{10}\vee F_{11}),F_7\wedge F_8\vdash F_{12}}{\wedge h_9:\Delta_{13},F_{10}\vee F_{11}\vdash F_{12}}} \wedge L} \frac{(\Delta_{13},F_{10}\vee F_{11}\vdash F_{12}})}{\wedge L} \times L}{\frac{-:\Delta_{13},F_7,F_8,F_{10}\vee F_{11}\vdash F_{12}}{\wedge L}}{\wedge h_9:\Delta_{13},F_7,F_8,F_{10}\vee F_{11}\vdash F_{12}}} \wedge L} \times L}{\frac{-:\Delta_{13},F_7,F_8,F_{10}\vee F_{11}\vdash F_{12}}{\wedge L}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{10}\vee F_{11}\vdash F_{12}}} \times L} \times L}{\frac{-:\Delta_6,F_7,F_8\vdash F_{10}\vee F_{11}}{\wedge L}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}} \times L} \times L} \times L}{\frac{-:\Delta_6,F_7\wedge F_8\vdash F_{10}\vee F_{11}\vdash F_{12}}{\wedge L}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}} \times L} \times L}{\frac{-:\Delta_6,F_7\wedge F_8\vdash F_{12}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}}}{\wedge h_9:\Delta_6,F_{11},F_7,F_8\vdash F_{12}}} \times L} \times L}{\frac{-:\Delta_6,F_7\wedge F_8\vdash F_{12}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}} \wedge L}}{\frac{-:\Delta_6,F_7\wedge F_8\vdash F_{12}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}}}{\wedge h_9:\Delta_6,F_7,F_8\vdash F_{12}}} \times L}$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \frac{h_1:\Delta_6,F_7,F_8\vdash\bot}{\bullet h_1:\Delta_6,F_7\land F_8\vdash\bot} & \land_L & \frac{\bullet h_9:(\Delta_6,F_7\land F_8),\bot\vdash F_{10}}{\bullet h_9:(\Delta_6,F_7\land F_8),\bot\vdash F_{10}} & \Box_L \\ \hline & -:\Delta_6,F_7\land F_8\vdash F_{10} & \\ \hline & \frac{\to}{h_1:\Delta_6,F_7,F_8\vdash\bot} & \text{ax/W} & \frac{\to}{\bullet h_9:\bot,\Delta_6,F_7,F_8\vdash F_{10}} & \bot_L \\ \hline & \frac{-:\Delta_6,F_7,F_8\vdash F_{10}}{-:\Delta_6,F_7\land F_8\vdash F_{10}} & \land_L \\ \hline & \frac{h_1:(\bot,\Delta_{11}),F_7,F_8\vdash F_6}{\bullet h_1:(\bot,\Delta_{11}),F_7\land F_8\vdash F_6} & \land_L & \\ \hline & \frac{\bullet h_9:((\bot,\Delta_{11}),F_7\land F_8\vdash F_{10}}{-:(\bot,\Delta_{11}),F_7\land F_8\vdash F_{10}} & \bot_L \\ \hline & -:(\bot,\Delta_{11}),F_7\land F_8\vdash F_{10} & \\ \hline & -:\bot,\Delta_{11},F_7\land F_8\vdash F_{10} & \bot_L \\ \hline \end{array}$$

ullet Case rule I

$$\frac{ \frac{\mathbf{h}_1 : (\Delta_{11}, \mathbf{p}_{10}), \mathbf{F}_7, \mathbf{F}_8 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1 : (\Delta_{11}, \mathbf{p}_{10}), \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{F}_6} \land_L \quad \frac{\bullet \mathbf{h}_9 : ((\Delta_{11}, \mathbf{p}_{10}), \mathbf{F}_7 \land \mathbf{F}_8), \mathbf{F}_6 \vdash \mathbf{p}_{10}}{-: (\Delta_{11}, \mathbf{p}_{10}), \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10}} \quad Cut \\ \frac{-: (\Delta_{11}, \mathbf{p}_{10}), \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10}}{-: \Delta_{11}, \mathbf{p}_{10}, \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10}} \quad I$$

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_6, \mathbf{F}_7, \mathbf{F}_8 \vdash \mathbf{p}_{10} \\ \bullet \mathbf{h}_1: \Delta_6, \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10} \end{array} \land_L \quad \frac{\bullet \mathbf{h}_9: (\Delta_6, \mathbf{F}_7 \land \mathbf{F}_8), \mathbf{p}_{10} \vdash \mathbf{p}_{10}}{-: \Delta_6, \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10}} \quad \mathbf{Cut} \\ \frac{-: \Delta_6, \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10}}{-: \Delta_6, \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{p}_{10}} \quad \mathbf{ax/W} \end{array}$$

• Case rule  $\top_L$ 

$$\begin{array}{c|c} \underline{\mathbf{h}_1:\Delta_6, \mathbf{F}_7, \mathbf{F}_8 \vdash \top} & \wedge_L & \underline{\mathbf{h}_9:\Delta_6, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} \\ \underline{-:\Delta_6, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \top} & \wedge_L & \underline{\bullet \mathbf{h}_9:(\Delta_6, \mathbf{F}_7 \wedge \mathbf{F}_8), \top \vdash \mathbf{F}_{10}} \\ \underline{-:\Delta_6, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \\ \underline{-:\Delta_6, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \\ \underline{-:\Delta_6, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \\ \underline{\bullet \mathbf{h}_1:(\top,\Delta_{11}), \mathbf{F}_7, \mathbf{F}_8 \vdash \mathbf{F}_6} & \wedge_L & \underline{\mathbf{h}_9:\Delta_{11}, \mathbf{F}_6, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} \\ \underline{\bullet \mathbf{h}_1:(\top,\Delta_{11}), \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_6} & \underline{\bullet \mathbf{h}_9:((\top,\Delta_{11}), \mathbf{F}_7 \wedge \mathbf{F}_8), \mathbf{F}_6 \vdash \mathbf{F}_{10}} \\ \underline{-:(\top,\Delta_{11}), \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\bullet \mathbf{x}/\mathbf{W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_6} & \underline{\mathsf{ax/W}} & \underline{\bullet \mathbf{x}/\mathbf{W}} \\ \underline{-:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11}, \mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\mathsf{ax/W}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\bullet \mathbf{h}_1:\top,\Delta_{11},\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_11,\mathbf{F}_7 \wedge \mathbf{F}_8 \vdash \mathbf{F}_{10}} & \underline{\bullet \mathbf{h}_1:\top,\Delta_11,\mathbf{h}_1 \wedge \mathbf{h}_1 \wedge \mathbf{h}_2} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_11,\mathbf{h}_1 \wedge \mathbf{h}_2} & \underline{\bullet \mathbf{h}_1:\top,\Delta_11,\mathbf{h}_2 \wedge \mathbf{h}_2} & \underline{\bullet \mathbf{h}_1 \wedge \mathbf{h}_2} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_11,\mathbf{h}_2 \wedge \mathbf{h}_2 \wedge \mathbf{h}_2} & \underline{\bullet \mathbf{h}_2 \wedge \mathbf{h}_2} \\ \underline{\bullet \mathbf{h}_1:\top,\Delta_11,\mathbf{h}_2 \wedge \mathbf{h}_2} & \underline{\bullet \mathbf{h}_2 \wedge \mathbf{h}_2} & \underline{\bullet$$

## 6.8 Status of $\vee_L$ : OK

• Case rule  $\top_R$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \vdash \mathbf{F}_7 & \mathbf{h}_1 : \Delta_6, \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \top \\ \hline & & \rightarrow \\ \hline & - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \top \end{array} } \begin{array}{c} \top_{\mathit{R}} \\ \text{Cut} \\ \hline \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\frac{ \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \vdash \mathbf{F}_7 \quad \mathbf{h}_1 : \Delta_6, \mathbf{F}_9 \vdash \mathbf{F}_7 }{ \underbrace{ \begin{array}{c} \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline \\ - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \to \mathbf{F}_{12} \\ \hline \\ \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_{11}, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline \\ \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_{11}, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline \\ \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_{11}, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{12} \\ \hline \\ - : \Delta_6, \mathbf{F}_{11}, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{12} \\ \hline \\ - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \to \mathbf{F}_{12} \\ \hline \\ - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \to \mathbf{F}_{12} \\ \hline \end{array} \begin{array}{c} \Delta_R \\ \bullet \mathbf{h} \\ \bullet$$

• Case rule  $\wedge_R$ 

$$\frac{\underbrace{\begin{array}{l} \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{8} \vdash \mathbf{F}_{7} \quad \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{9} \vdash \mathbf{F}_{7} \\ \bullet \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{7} \end{array}}_{\bullet \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{7}} \lor_{L} \underbrace{\begin{array}{l} \mathbf{h}_{10}: \Delta_{6}, \mathbf{F}_{7}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{11} \quad \mathbf{h}_{10}: \Delta_{6}, \mathbf{F}_{7}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{10}: (\Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9}), \mathbf{F}_{7} \vdash \mathbf{F}_{11} \land \mathbf{F}_{12} \\ \bullet \\ \bullet \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{7} \end{array}}_{\bullet \mathbf{h}_{10}: \Delta_{6}, \mathbf{F}_{7}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{11} \\ \bullet \mathbf{h}_{Cut} \underbrace{\begin{array}{l} \mathbf{a} \mathbf{x} / \mathbf{w} \\ \bullet \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{Cut} \\ \hline -: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{11} \\ \bullet \mathbf{h}_{Cut} \\ \hline -: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{11} \land \mathbf{F}_{12} \\ \end{array}}_{\bullet \mathbf{h}_{Cut}} \underbrace{\begin{array}{l} \mathbf{a} \mathbf{x} / \mathbf{w} \\ \bullet \mathbf{h}_{1}: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{12} \\ -: \Delta_{6}, \mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{12} \\ \bullet \mathbf{h}_{Cut} \\ \end{array}}_{\bullet \mathbf{h}_{Cut}}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \vdash \mathbf{F}_7 & \mathbf{h}_1 : \Delta_6, \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \lor \mathbf{F}_{12} \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_1 \lor \mathbf{F}_{12} \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_7 \\ \hline & \bullet \mathbf{h}_1 : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline & - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline & - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline & - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{F}_7, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline \mathbf{h}_{Cut} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline - : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline \mathbf{h}_{10} : \Delta_6, \mathbf{F}_8 \lor \mathbf{F}_9 \vdash \mathbf{F}_{11} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \end{pmatrix} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \end{pmatrix} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \end{pmatrix} \begin{array}{c} \mathbf{h}_{10} : \Delta_6, \mathbf{h}_{10} \lor \mathbf{h}_{10} \\ \hline \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{9}\vdash\mathbf{F}_{7}}{\underbrace{-\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{7}}_{}} \quad \vee_{L}\quad \frac{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{7},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{12}}{\underbrace{-\mathbf{h}_{10}:(\Delta_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}),\mathbf{F}_{7}\vdash\mathbf{F}_{11}\vee\mathbf{F}_{12}}_{}} \quad \overset{\vee_{2}}{\leftarrow} \underbrace{\frac{\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{11}\vee\mathbf{F}_{12}}{\underbrace{-\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{12}}_{}}}_{\mathbf{hCut}} \quad \overset{\mathbf{ax/W}}{\leftarrow} \underbrace{\frac{-\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{12}}{-\mathbf{h}_{10}:\Delta_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{12}}}_{} \quad \vee_{2}$$

• Case rule  $\rightarrow_L$ 

$$\frac{h_1: (\Delta_{13}, F_{10} \to F_{11}), F_7 \vdash F_6 \quad h_1: (\Delta_{13}, F_{10} \to F_{11}), F_8 \vdash F_6}{\bullet h_1: (\Delta_{13}, F_{10} \to F_{11}), F_7 \lor F_8 \vdash F_6} \quad \vee_L \quad \frac{h_9: \Delta_{13}, F_6, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \quad h_9: \Delta_{13}, F_6, F_{11}}{\bullet h_9: ((\Delta_{13}, F_{10} \to F_{11}), F_7 \lor F_8 \vdash F_{12}} \\ -: (\Delta_{13}, F_{10} \to F_{11}), F_7 \lor F_8 \vdash F_{12} \\ \hline \bullet h_1: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_6} \quad \text{ax/W} \quad \frac{\bullet h_1: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10}}{h_9: \Delta_{13}, F_6, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10}} \quad \text{ax/W} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11}, F_7 \lor F_8 \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13}, F_{10} \to F_{11} \vdash F_{10} \\ \hline -: \Delta_{13},$$

• Case rule  $\wedge_L$ 

$$\frac{\mathbf{h}_{1}: (\Delta_{13}, F_{10} \wedge F_{11}), F_{7} \vdash F_{6} \quad h_{1}: (\Delta_{13}, F_{10} \wedge F_{11}), F_{8} \vdash F_{6}}{\bullet \mathbf{h}_{1}: (\Delta_{13}, F_{10} \wedge F_{11}), F_{7} \vee F_{8} \vdash F_{6}} \quad \vee_{L} \quad \frac{\mathbf{h}_{9}: \Delta_{13}, F_{6}, F_{10}, F_{11}, F_{7} \vee F_{8} \vdash F_{12}}{\bullet \mathbf{h}_{9}: ((\Delta_{13}, F_{10} \wedge F_{11}), F_{7} \vee F_{8} \vdash F_{12})} \quad \wedge_{L} \quad \mathbf{Cut} \\ \\ -: (\Delta_{13}, F_{10} \wedge F_{11}), F_{7} \vee F_{8} \vdash F_{12} \\ \hline \frac{\mathbf{h}_{1}: \Delta_{13}, F_{10}, F_{11}, F_{7} \vdash F_{6}}{\bullet \mathbf{h}_{1}: \Delta_{13}, F_{10}, F_{11}, F_{8} \vdash F_{6}} \quad \mathbf{inv} - \mathbf{th} / \mathbf{ax} \\ & \qquad \qquad \rightarrow \\ h_{1}: \Delta_{13}, F_{10}, F_{11}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ -: \Delta_{13}, F_{10}, F_{11}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ -: \Delta_{13}, F_{10}, F_{11}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ -: \Delta_{13}, F_{10} \wedge F_{11}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ -: \Delta_{6}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ h_{1}: \Delta_{6}, F_{7} \vdash F_{10} \wedge F_{11} \quad h_{1}: \Delta_{6}, F_{8} \vdash F_{10} \wedge F_{11} \\ \hline \\ \bullet h_{1}: \Delta_{6}, F_{7} \vee F_{8} \vdash F_{10} \wedge F_{11} \\ \hline \\ -: \Delta_{6}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ h_{9}: \Delta_{6}, F_{10}, F_{11}, F_{7} \vdash F_{12} \\ \hline \\ \bullet h_{9}: \Delta_{6}, F_{10}, F_{11}, F_{8} \vdash F_{12} \\ \hline \\ -: \Delta_{6}, F_{7} \vdash F_{10} \wedge F_{11} \\ \hline \\ -: \Delta_{6}, F_{7} \vdash F_{12} \\ \hline \\ -: \Delta_{6}, F_{7} \vdash F_{12} \\ \hline \\ -: \Delta_{6}, F_{7} \vee F_{8} \vdash F_{12} \\ \hline \\ -: \Delta_{6},$$

• Case rule  $\vee_L$ 

$$\frac{h_1: (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vdash F_6 \quad h_1: (\Delta_{13}, F_{10} \vee F_{11}), F_8 \vdash F_6}{\bullet h_1: (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vee F_8 \vdash F_6} \quad \vee_L \frac{h_2: \Delta_{13}, F_6, F_{10}, F_7 \vee F_8 \vdash F_{12}}{\bullet h_2: ((\Delta_{13}, F_{10} \vee F_{11}), F_7 \vee F_8 \vdash F_8)} \\ -: (\Delta_{13}, F_{10} \vee F_{11}), F_7 \vee F_8 \vdash F_6} \\ \frac{h_1: \Delta_{13}, F_{10}, F_7 \vdash F_6}{\bullet h_1: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_6} \\ \frac{h_1: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_6}{\vee_L} \frac{h_2: \Delta_{13}, F_{10}, F_8 \vdash F_8}{\vee_L} \frac{h_2: \Delta_{13}, F_{10}, F_8 \vee F_8 \vdash F_{12}}{\bullet h_2: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12}} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10}, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_{13}, F_{10} \vee F_{11}, h_1: \Delta_6, F_8 \vdash F_{10} \vee F_{11} \\ -: \Delta_6, F_7 \vee F_8 \vdash F_{10} \vee F_{11} \\ -: \Delta_6, F_7 \vee F_8 \vdash F_{10} \vee F_{11} \\ -: \Delta_6, F_7 \vee F_8 \vdash F_{12} \\ -: \Delta_6, F_7 \vee F_8 \vdash F_{11} \\ -: \Delta_6, F_7 \vee F_8 \vdash F_{11} \\ -: \Delta_6, F_7 \vee F_8 \vdash F_{11} \\ -: \Delta_6, F_7 \vee F_9 \vdash F_{11} \\ -: \Delta_6, F_7 \vee F_9 \vdash F_{11} \\ -: \Delta_6, F_9 \vee F_{10} \vdash F_{11} \\ -: \Delta_6, F_9$$

• Case rule  $\perp_L$ 

$$\frac{\frac{h_1:\Delta_6,F_7\vdash\bot\ h_1:\Delta_6,F_8\vdash\bot}{\bullet h_1:\Delta_6,F_7\lor F_8\vdash\bot}\ \lor_L}{\frac{\bullet h_1:\Delta_6,F_7\lor F_8\vdash\bot}{-:\Delta_6,F_7\lor F_8\vdash F_{10}}} \overset{\bot_L}{\to} \underbrace{\frac{h_1:\Delta_6,F_7\vdash\bot\ h_1:\Delta_6,F_7\vdash F_{10}}{\bullet h_9:\bot,\Delta_6,F_7\vdash F_{10}}} \overset{\bot_L}{\to} \underbrace{\frac{h_1:\Delta_6,F_8\vdash\bot}{-:\Delta_6,F_8\vdash F_{10}}}_{-:\Delta_6,F_8\vdash F_{10}} \lor_L} \xrightarrow{\frac{\bot_L}{h\text{Cut}}} \underbrace{\frac{\bot_L}{h\text{Cut}}}_{\bullet h_9:\bot,\Delta_6,F_8\vdash F_{10}} \lor_L} \underbrace{\frac{\bot_L}{h\text{Cut}}}_{\bullet h_9:\bot,\Delta_6,F_8\vdash F_{10}} \lor_L} \underbrace{\frac{\bot_L}{h\text{Cut}}}_{\bullet h_9:\bot,\Delta_6,F_8\vdash F_{10}} \lor_L} \underbrace{\frac{\bot_L}{h\text{Cut}}}_{-:\bot,\Delta_{11}),F_7\lor F_8\vdash F_{10}}}_{-:\bot,\Delta_{11}),F_7\lor F_8\vdash F_{10}} \underbrace{\frac{\bot_L}{-:\bot,\Delta_{11},F_7\lor F_8\vdash F_{10}}}_{-:\bot,\Delta_{11},F_7\lor F_8\vdash F_{10}} \bot_L$$

ullet Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_{1}:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\vdash\mathbf{F}_{6}\quad\mathbf{h}_{1}:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{8}\vdash\mathbf{F}_{6}}{\bullet}\mathbf{h}_{1}:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{6}} \quad \vee_{L} \quad & \frac{\bullet}{\bullet}\mathbf{h}_{9}:((\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\vee\mathbf{F}_{8}),\mathbf{F}_{6}\vdash\mathbf{p}_{10}}{-:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{p}_{10}} \quad & Cut \\ & \frac{-:(\Delta_{11},\mathbf{p}_{10}),\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{p}_{10}}{-:\Delta_{11},\mathbf{p}_{10},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{p}_{10}} \quad I \\ \\ \frac{\bullet\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{p}_{10}\quad\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\mathbf{p}_{10}}{\bullet} \quad \vee_{L} \quad & \frac{\bullet}{\bullet}\mathbf{h}_{9}:(\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}),\mathbf{p}_{10}\vdash\mathbf{p}_{10}}{\bullet} \quad Cut \\ & \frac{-:\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{p}_{10}}{-:\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{p}_{10}} \quad ax/\mathsf{W} \end{array}$$

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{7}\vdash\top\ \mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\top\ }{\bullet \mathbf{h}_{1}:\Delta_{6},\mathbf{F}_{8}\vdash\top\ } \vee_{L} \quad \frac{\mathbf{h}_{9}:\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}}{\bullet \mathbf{h}_{9}:(\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}),\top\vdash\mathbf{F}_{10}} \quad \overset{\top}{\subset}_{L} \\ \frac{-:\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}}{-:\Delta_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}} \quad \mathbf{ax/W} \\ \\ \frac{\mathbf{h}_{1}:(\top,\Delta_{11}),\mathbf{F}_{7}\vdash\mathbf{F}_{6} \quad \mathbf{h}_{1}:(\top,\Delta_{11}),\mathbf{F}_{8}\vdash\mathbf{F}_{6}}{-:(\top,\Delta_{11}),\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}} \quad \overset{\mathbf{h}_{9}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}}{\bullet \mathbf{h}_{9}:((\top,\Delta_{11}),\mathbf{F}_{7}\vee\mathbf{F}_{8}),\mathbf{F}_{6}\vdash\mathbf{F}_{10}} \quad \overset{\top}{\subset}_{L} \\ \\ \frac{-:(\top,\Delta_{11}),\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}}{\bullet \mathbf{h}_{1}:\top,\Delta_{11},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{6}} \quad \mathbf{ax/W} \\ \\ \frac{\bullet}{\mathbf{h}_{1}:\top,\Delta_{11},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{6}} \quad \mathbf{ax/W} \\ \\ \frac{-:\top,\Delta_{11},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}}{\bullet \mathbf{h}_{1}:\top,\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{7}\vee\mathbf{F}_{8}\vdash\mathbf{F}_{10}} \quad \mathbf{ax/W} \\ \\ \mathbf{hCut} \\ \end{array}$$

# 6.9 Status of $\perp_L$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c|c} \overline{\bullet \mathbf{h}_1 : \bot, \Delta_4 \vdash \mathbf{F}_5} & \bot_L & \overline{\bullet \mathbf{h}_6 : (\bot, \Delta_4), \mathbf{F}_5 \vdash \top} & \top_R \\ \hline & - : \bot, \Delta_4 \vdash \top \\ & \xrightarrow{-} & \bot_R \\ \hline & - : \bot, \Delta_4 \vdash \top & \top_R \end{array}$$
 Cut

• Case rule  $\rightarrow_R$ 

$$\frac{\overbrace{\bullet \mathbf{h}_1: \bot, \Delta_4 \vdash \mathbf{F}_5}^{\bullet \mathbf{h}_1: \bot, \Delta_4 \vdash \mathbf{F}_5} \ \bot_L \ \ \frac{\mathbf{h}_6: (\bot, \Delta_4), \mathbf{F}_5, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6: (\bot, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \ \ \frac{-: \bot, \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8}{-: \bot, \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \ \ \bot_L}{-: \bot, \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8}$$

• Case rule  $\wedge_R$ 

$$\begin{array}{c|c} \underline{\bullet \mathbf{h}_1 : \bot, \Delta_4 \vdash \mathbf{F}_5} & \bot_L & \frac{\mathbf{h}_6 : \bot, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \quad \mathbf{h}_6 : \bot, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6 : (\bot, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8} & \mathbf{Cut} \\ \hline & - : \bot, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ & \xrightarrow{} \\ \hline & - : \bot, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 & \bot_L \end{array}$$

• Case rule  $\vee_1$ 

$$\frac{\underbrace{ \begin{array}{c} \mathbf{h}_{6}: \bot, \Delta_{4}, F_{5} \vdash F_{7} \\ \hline \bullet \mathbf{h}_{1}: \bot, \Delta_{4} \vdash F_{5} \end{array} \bot_{L} \quad \begin{array}{c} \mathbf{h}_{6}: \bot, \Delta_{4}, F_{5} \vdash F_{7} \\ \hline \bullet \mathbf{h}_{6}: (\bot, \Delta_{4}), F_{5} \vdash F_{7} \lor F_{8} \\ \hline -: \bot, \Delta_{4} \vdash F_{7} \lor F_{8} \end{array} }{-: \bot, \Delta_{4} \vdash F_{7} \lor F_{8}} \quad \bot_{L} \end{array}} \quad \begin{array}{c} \lor_{1} \\ \mathsf{Cut} \\ \hline \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_6: \bot, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_1: \bot, \Delta_4 \vdash \mathbf{F}_5} & \bot_L & \frac{\mathbf{h}_6: (\bot, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6: (\bot, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \lor \mathbf{F}_8} & \mathsf{Cut} \\ & \xrightarrow{-: \bot, \Delta_4 \vdash \mathbf{F}_7 \lor \mathbf{F}_8} & \bot_L \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{ \underbrace{ \begin{array}{c} \bullet_{h_1} : \bot, \Delta_9, F_6 \to F_7 \vdash F_6 \\ \bullet_{h_2} : \bot, \Delta_9, F_6 \to F_7 \vdash F_8 \end{array}}_{ \begin{array}{c} \bullet_{h_3} : \bot, \Delta_9, F_4, F_6 \to F_7 \vdash F_6 \\ \bullet_{h_5} : (\bot, \Delta_9, F_6 \to F_7), F_4 \vdash F_8 \\ \hline \\ - : \bot, \Delta_9, F_6 \to F_7 \vdash F_8 \\ \hline \\ \hline \\ - : \bot, \Delta_9, F_6 \to F_7 \vdash F_8 \\ \hline \\ \bullet_{h_2} : \bot, \Delta_4, F_6 \to F_7 \vdash F_6 \\ \hline \\ \bullet_{h_5} : \bot, \Delta_4, F_6 \to F_7 \vdash F_8 \\ \hline \\ \bullet_{h_5} : (\bot, \Delta_4), F_6 \to F_7 \vdash F_8 \\ \hline \\ - : \bot, \Delta_4 \vdash F_8 \\ \hline \\ - : \bot, \Delta_4 \vdash F_8 \\ \hline \\ - : \bot, \Delta_4 \vdash F_8 \\ \hline \end{array}_{D} Cut$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c|c} \bullet h_1: \bot, \Delta_9, F_6 \wedge F_7 \vdash F_4 \\ \hline \bullet h_2: \bot, \Delta_9, F_6 \wedge F_7), F_4 \vdash F_8 \\ \hline -: \bot, \Delta_9, F_6 \wedge F_7 \vdash F_8 \\ \hline -: \bot, \Delta_9, F_6 \wedge F_7 \vdash F_8 \\ \hline -: \bot, \Delta_9, F_6 \wedge F_7 \vdash F_8 \\ \hline \bot_L \\ \hline \bullet h_1: \bot, \Delta_4 \vdash F_6 \wedge F_7 \\ \hline \bot_L \\ \hline \bullet h_2: \bot, \Delta_4 \vdash F_8 \\ \hline -: \bot, \Delta_4 \vdash F_8 \\ \hline \bot_L \\ \hline \end{array} \begin{array}{c} \land L \\ \land L$$

• Case rule  $\vee_L$ 

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \underline{\bullet \mathbf{h}_1 : \bot, \Delta_4 \vdash \mathbf{F}_5} & \bot_L & \overline{\bullet \mathbf{h}_6 : (\bot, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7} \\ - : \bot, \Delta_4 \vdash \mathbf{F}_7 & \to \\ \hline - : \bot, \Delta_4 \vdash \mathbf{F}_7 & \bot_L \end{array}$$
 Cut

 $\bullet$  Case rule I

$$\begin{array}{c|c} & \frac{\bullet \mathbf{h}_1: \bot, \Delta_7, \mathbf{p}_6 \vdash \mathbf{F}_4}{\bullet} \stackrel{\bot_L}{\longrightarrow} \frac{\bullet \mathbf{h}_5: (\bot, \Delta_7, \mathbf{p}_6), \mathbf{F}_4 \vdash \mathbf{p}_6}{\bullet} \stackrel{I}{\subset} \mathbf{Cut}} \\ & \xrightarrow{-: \bot, \Delta_7, \mathbf{p}_6 \vdash \mathbf{p}_6} \xrightarrow{\bot_L} \\ & \xrightarrow{\bullet \mathbf{h}_1: \bot, \Delta_4 \vdash \mathbf{p}_6} \stackrel{\bot_L}{\longrightarrow} \frac{\bullet}{\bullet \mathbf{h}_5: (\bot, \Delta_4), \mathbf{p}_6 \vdash \mathbf{p}_6} \stackrel{I}{\subset} \mathbf{Cut}} \\ & \xrightarrow{-: \bot, \Delta_4 \vdash \mathbf{p}_6} \xrightarrow{\bot_L} \\ & \xrightarrow{-: \bot, \Delta_4 \vdash \mathbf{p}_6} \stackrel{\bot_L}{\longrightarrow} \\ & \xrightarrow{-: \bot, \Delta_4 \vdash \mathbf{p}_6} \stackrel{\bot_L}{\longrightarrow} \end{array}$$

$$\begin{array}{c|c} \hline \bullet \mathbf{h}_1 : \bot, \Delta_4 \vdash \top & \bot_L & \frac{\mathbf{h}_5 : \bot, \Delta_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_5 : (\bot, \Delta_4), \top \vdash \mathbf{F}_6} & \top_L \\ \hline & -: \bot, \Delta_4 \vdash \mathbf{F}_6 & \\ \hline & -: \bot, \Delta_4 \vdash \mathbf{F}_6 & \bot_L \\ \hline \\ \hline \bullet \mathbf{h}_1 : \bot, \top, \Delta_7 \vdash \mathbf{F}_4 & \bot_L & \frac{\mathbf{h}_5 : \bot, \Delta_7, \mathbf{F}_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_5 : (\bot, \top, \Delta_7), \mathbf{F}_4 \vdash \mathbf{F}_6} & \top_L \\ \hline & -: \bot, \top, \Delta_7 \vdash \mathbf{F}_6 & \\ \hline & -: \bot, \top, \Delta_7 \vdash \mathbf{F}_6 & \bot_L \\ \hline \end{array}$$

### 6.10 Status of I: OK

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\frac{ \begin{array}{c} \bullet_{h_1}: \Delta_4, p_5 \vdash p_5 \end{array} I \quad \begin{array}{c} h_6: \Delta_4, F_7, p_5, p_5 \vdash F_8 \\ \bullet_{h_6}: (\Delta_4, p_5), p_5 \vdash F_7 \rightarrow F_8 \end{array} \begin{array}{c} \rightarrow_R \\ \text{Cut} \\ \hline \\ \bullet_{h_1}: \Delta_4, F_7, p_5 \vdash p_5 \end{array} I \quad \begin{array}{c} \rightarrow \\ \hline \bullet_{h_6}: \Delta_4, F_7, p_5, p_5 \vdash F_8 \end{array} \begin{array}{c} \text{ax/W} \\ \bullet_{h_1}: \Delta_4, F_7, p_5 \vdash p_5 \end{array} I \begin{array}{c} \rightarrow \\ \hline \bullet_{h_1}: \Delta_4, F_7, p_5 \vdash F_8 \\ \hline -: \Delta_4, F_7, P_5 \vdash F_8 \\ \hline -: \Delta_4, p_5 \vdash F_7 \rightarrow F_8 \end{array} \rightarrow_R \end{array}$$

• Case rule  $\wedge_R$ 

• Case rule  $\vee_1$ 

$$\begin{array}{c|c} & \frac{\mathbf{h}_{6}:\Delta_{4},\mathbf{p}_{5},\mathbf{p}_{5} \vdash \mathbf{F}_{7}}{\bullet \mathbf{h}_{1}:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{p}_{5}} & I & \frac{\mathbf{h}_{6}:\Delta_{4},\mathbf{p}_{5},\mathbf{p}_{5} \vdash \mathbf{F}_{7} \lor \mathbf{F}_{8}}{\bullet \mathbf{h}_{6}:(\Delta_{4},\mathbf{p}_{5}),\mathbf{p}_{5} \vdash \mathbf{F}_{7} \lor \mathbf{F}_{8}} & \mathbf{Cut} \\ & -:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{7} \lor \mathbf{F}_{8} & + \\ & + & + \\ & \frac{\bullet \mathbf{h}_{1}:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{p}_{5}}{-:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{7}} & \mathbf{ax/W} & \mathbf{hCut} \\ & \frac{-:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{7}}{-:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{7} \lor \mathbf{F}_{8}} & \lor_{1} \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c|c} & \mathbf{h}_{6}:\Delta_{4},\mathbf{p}_{5},\mathbf{p}_{5} \vdash \mathbf{F}_{8} \\ \hline \bullet \mathbf{h}_{1}:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{p}_{5} & I & \bullet \mathbf{h}_{6}:(\Delta_{4},\mathbf{p}_{5}),\mathbf{p}_{5} \vdash \mathbf{F}_{7} \vee \mathbf{F}_{8} \\ \hline & -:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{7} \vee \mathbf{F}_{8} \\ \hline \bullet \mathbf{h}_{1}:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{p}_{5} & I & \to \\ \hline & \bullet \mathbf{h}_{1}:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{p}_{5} & I & \to \\ \hline & -:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{8} \\ \hline & -:\Delta_{4},\mathbf{p}_{5} \vdash \mathbf{F}_{8} & \vee_{2} \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\frac{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash p_{4}}{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash p_{4}}}_{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash F_{8}}}I \xrightarrow{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash F_{8}}_{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash F_{8}}}\underbrace{Cut} \xrightarrow{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash F_{8}}_{\bullet_{1}:(\Delta_{9},F_{6}\to F_{7}),p_{4}\vdash F_{8}}}\underbrace{I \xrightarrow{\bullet_{1}:(\Delta_{9},P_{4},P_{4},F_{6}\to F_{7}\vdash F_{6})}_{\bullet_{1}:(\Delta_{9},F_{7},p_{4}\vdash P_{4})}}\underbrace{I \xrightarrow{\bullet_{1}:(\Delta_{9},F_{7},p_{4}\vdash F_{8})}_{\bullet_{1}:(\Delta_{9},F_{7},p_{4}\vdash F_{8})}}_{\bullet_{1}:(\Delta_{9},F_{7},p_{4}\vdash F_{8})} \xrightarrow{\bullet_{1}:(\Delta_{9},F_{7},p_{4}\vdash F_{8})}_{\bullet_{1}:(\Delta_{9},F_{7},p_{4}\vdash F_{8})}}$$

• Case rule  $\wedge_L$ 

• Case rule  $\vee_L$ 

$$\frac{\underbrace{\bullet h_1 : (\Delta_9, F_6 \vee F_7), p_4 \vdash p_4}_{\bullet h_1 : (\Delta_9, F_6 \vee F_7), p_4 \vdash F_8} I \xrightarrow{\bullet h_5 : ((\Delta_9, F_6 \vee F_7), p_4), p_4 \vdash F_8}_{\bullet h_1 : \Delta_9, F_6, p_4 \vdash p_4} Cut} \lor_L$$

$$\frac{-: (\Delta_9, F_6 \vee F_7), p_4 \vdash F_8}{\bullet h_1 : \Delta_9, F_6, p_4 \vdash p_4} I \xrightarrow{h_5 : \Delta_9, F_6, p_4 \vdash F_8}_{\bullet h_2 : \Delta_9, F_6, p_4 \vdash F_8} I \xrightarrow{\bullet h_1 : \Delta_9, F_7, p_4 \vdash p_4}_{\bullet h_2 : \Delta_9, F_7, p_4 \vdash F_8} V_L$$

$$\frac{\bullet h_1 : \Delta_9, F_6, p_4 \vdash p_4}{-: \Delta_9, F_6, p_4 \vdash F_8} \lor_L$$

$$\frac{\bullet h_1 : \Delta_9, F_7, p_4 \vdash p_4}{-: \Delta_9, F_7, p_4 \vdash F_8} \lor_L$$

• Case rule  $\perp_L$ 

$$\frac{ \underbrace{ \bullet_{\mathbf{h}_1} : (\bot, \Delta_7), \mathbf{p}_4 \vdash \mathbf{p}_4}_{\bullet \mathbf{h}_1} \ I \quad \underbrace{ \bullet_{\mathbf{h}_5} : ((\bot, \Delta_7), \mathbf{p}_4), \mathbf{p}_4 \vdash \mathbf{F}_6}_{-: (\bot, \Delta_7), \mathbf{p}_4 \vdash \mathbf{F}_6} \ \underbrace{ \begin{matrix} \bot_L \\ \\ -: \bot, \Delta_7, \mathbf{p}_4 \vdash \mathbf{F}_6 \end{matrix}}_{-: \bot, \Delta_7, \mathbf{p}_4 \vdash \mathbf{F}_6} \ \bot_L$$

 $\bullet\,$  Case rule I

• Case rule  $\top_L$ 

$$\frac{\underbrace{\bullet \mathbf{h}_1: (\top, \Delta_7), \mathbf{p}_4 \vdash \mathbf{p}_4}_{\bullet \mathbf{h}_1: (\top, \Delta_7), \mathbf{p}_4 \vdash \mathbf{p}_4} I \quad \frac{\mathbf{h}_5: \Delta_7, \mathbf{p}_4, \mathbf{p}_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_5: ((\top, \Delta_7), \mathbf{p}_4), \mathbf{p}_4 \vdash \mathbf{F}_6} \quad \frac{\top_L}{\mathsf{Cut}}}{-: (\top, \Delta_7), \mathbf{p}_4 \vdash \mathbf{F}_6} \quad \frac{\rightarrow}{\mathbf{h}_5: \top, \Delta_7, \mathbf{p}_4, \mathbf{p}_4 \vdash \mathbf{F}_6}} \quad \frac{\mathsf{ax/W}}{\mathsf{hCut}}$$

## 6.11 Status of $\top_L$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_4 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1:\top,\Delta_4 \vdash \mathbf{F}_5} & \top_L & \hline{\bullet \mathbf{h}_6:(\top,\Delta_4),\mathbf{F}_5 \vdash \top} & \top_R \\ \hline & -:\top,\Delta_4 \vdash \top \\ \hline & \hline & -:\top,\Delta_4 \vdash \top \\ \hline & -:\top,\Delta_4 \vdash \top \end{array} \uparrow_R$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_4 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1:\top,\Delta_4 \vdash \mathbf{F}_5} & \top_L & \frac{\mathbf{h}_6:\top,\Delta_4,\mathbf{F}_5,\mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6:(\top,\Delta_4),\mathbf{F}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \\ \hline -:\top,\Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8 \\ \hline \frac{\mathbf{h}_1:\top,\Delta_4 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_6:\top,\Delta_4,\mathbf{F}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} & \mathbf{ax/W} \\ \hline -:\top,\Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h} \odot \mathbf{t} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_4 \vdash \mathbf{F}_5 \\ \bullet \mathbf{h}_1: \top, \Delta_4 \vdash \mathbf{F}_5 \end{array}}{ \begin{array}{c} \bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 & \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_6: (\top, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \end{array}} \begin{array}{c} \mathbf{Cut} \\ \bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \end{array}} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{hCut} \end{array}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{l} \mathbf{h}_1 : \Delta_4 \vdash \mathbf{F}_5 \\ \hline \bullet \mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_5 \end{array} \top_L \begin{array}{l} \frac{\mathbf{h}_6 : \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_6 : (\top, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \hline \\ \frac{\mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_6 : \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \end{array} \begin{array}{l} \mathbf{v}_1 \\ \mathbf{cut} \\ \hline \bullet \mathbf{h}_6 : \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \end{array} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1 : \Delta_4 \vdash \mathbf{F}_5 \\ \hline \bullet \mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_5 \end{array} \top_L \quad \frac{\mathbf{h}_6 : \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6 : (\top, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \hline \\ \hline \bullet_{\mathbf{h}_1} : \top, \Delta_4 \vdash \mathbf{F}_5 \end{array} \quad \begin{array}{c} \mathbf{h}_6 : \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_6 : \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_4 \\ \bullet \mathbf{h}_1: \top, \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_4 \end{array}}{ \bullet \mathbf{h}_1: \top, \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_4} \begin{array}{c} \top_L & \frac{\mathbf{h}_5: \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_6 \quad \mathbf{h}_5: \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline -: \top, \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array}} \begin{array}{c} \mathbf{Cut} \\ \hline \\ \frac{\mathbf{h}_1: \top, \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1: \top, \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_4} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array}} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array}} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_2: \top, \Delta_9, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_3: \top, \Delta_4 \vdash \mathbf{F}_6 \rightarrow \mathbf{F}_7 \end{array} \\ \hline \bullet \mathbf{h}_1: \Delta_4 \vdash \mathbf{F}_6 \rightarrow \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_5: (\top, \Delta_4), \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Cut} \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_8 \end{array} \end{array} \begin{array}{c} \mathbf{Cut} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \rightarrow \mathbf{F}_7 \vdash \mathbf{h}_8 \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet \mathbf{h}_5: \top, \Delta$$

### • Case rule $\wedge_L$

$$\begin{array}{c|c} \frac{\mathbf{h}_1: \Delta_9, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1: \top, \Delta_9, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_4} & \top_L & \frac{\mathbf{h}_5: \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5: (\top, \Delta_9, \mathbf{F}_6 \wedge \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8} & \wedge_L \\ \hline -: \top, \Delta_9, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \rightarrow \\ \hline \frac{\mathbf{h}_1: \top, \Delta_9, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_5: \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{ax/W} \\ \hline -: \top, \Delta_9, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \rightarrow \\ \hline \bullet \mathbf{h}_1: \Delta_4 \vdash \mathbf{F}_6 \wedge \mathbf{F}_7 & \top_L & \frac{\mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5: (\top, \Delta_4), \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8} & \wedge_L \\ \hline -: \top, \Delta_4 \vdash \mathbf{F}_8 & \rightarrow \\ \hline \bullet \mathbf{h}_1: \top, \Delta_4 \vdash \mathbf{F}_6 \wedge \mathbf{F}_7 & \mathbf{ax/W} & \rightarrow \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4, \mathbf{F}_6 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 & \rightarrow \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_5: \top, \Delta_4 \vdash \mathbf{h}_8 \wedge \mathbf{h}_7 \wedge \mathbf{h}_7 \wedge \mathbf{h}_8 & \mathbf{h}_7 \wedge \mathbf{h}_7$$

### • Case rule $\vee_L$

$$\begin{array}{c|c} \frac{\mathbf{h}_1 : \Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1 : \top, \Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_4} & \top_L & \frac{\mathbf{h}_5 : \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \vdash \mathbf{F}_8 \quad \mathbf{h}_5 : \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5 : (\top, \Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8} & \mathbf{Cut} \\ & & \xrightarrow{\bullet} \frac{\rightarrow}{\mathbf{h}_1 : \top, \Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_4} & \frac{\rightarrow}{\bullet \mathbf{h}_5 : \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \frac{\mathbf{ax/W}}{\bullet \mathbf{h}_5 : \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} \\ & \xrightarrow{\bullet} \frac{\mathbf{h}_1 : \Delta_4 \vdash \mathbf{F}_6 \vee \mathbf{F}_7} & \top_L & \frac{\mathbf{h}_5 : \top, \Delta_4, \mathbf{F}_6 \vdash \mathbf{F}_8 \quad \mathbf{h}_5 : \top, \Delta_4, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5 : (\top, \Delta_4), \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{Cut} \\ & \xrightarrow{\bullet} \frac{\mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_6 \vee \mathbf{F}_7} & \mathbf{ax/W} & \xrightarrow{\bullet} \frac{\rightarrow}{\bullet \mathbf{h}_5 : \top, \Delta_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{ax/W} \\ & \xrightarrow{\bullet} \frac{\mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_6 \vee \mathbf{F}_7} & \mathbf{ax/W} & \xrightarrow{\bullet} \frac{\rightarrow}{\bullet \mathbf{h}_5 : \top, \Delta_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{ax/W} \\ & \xrightarrow{\bullet} \frac{\rightarrow}{\bullet \mathbf{h}_5 : \top, \Delta_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{ax/W} \\ & \xrightarrow{\bullet} \frac{\rightarrow}{\bullet \mathbf{h}_5 : \top, \Delta_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{ax/W} \\ & \xrightarrow{\bullet} \mathbf{h}_5 : \top, \Delta_4 \vdash \mathbf{F}_8 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{h}_8 \vee \mathbf{F}_7 \vee \mathbf{F}_8 \vee \mathbf{F}_7 \vee \mathbf{F}_8 \vee \mathbf{F}_7 \vee \mathbf{F}_8 \vee \mathbf{F}_7 \vee \mathbf{F}_8$$

#### • Case rule $\perp_L$

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_4 \vdash \bot}{\bullet \mathbf{h}_1:\top,\Delta_4 \vdash \bot} \; \top_L \quad & \bullet \mathbf{h}_5:(\top,\Delta_4),\bot \vdash \mathbf{F}_6 \\ \hline -:\top,\Delta_4 \vdash \mathbf{F}_6 \\ \hline \\ \frac{}{\bullet \mathbf{h}_1:\top,\Delta_4 \vdash \bot} \quad & \mathbf{ax/W} \quad & \bullet \\ \hline -:\top,\Delta_4 \vdash \mathbf{F}_6 \\ \hline \\ \frac{\mathbf{h}_1:\top,\Delta_4 \vdash \bot}{} \quad & \mathbf{ax/W} \quad & \bullet \\ \hline -:\top,\Delta_4 \vdash \mathbf{F}_6 \\ \hline \\ \bullet \mathbf{h}_5:\bot,\top,\Delta_4 \vdash \mathbf{F}_6 \\ \hline \\ \bullet \mathbf{h}_1:\bot,\Delta_7 \vdash \mathbf{F}_4 \\ \hline \bullet \mathbf{h}_1:\top,\bot,\Delta_7 \vdash \mathbf{F}_4 \\ \hline \\ \bullet \mathbf{h}_1:\top,\bot,\Delta_7 \vdash \mathbf{F}_4 \\ \hline \\ -:\top,\bot,\Delta_7 \vdash \mathbf{F}_6 \\ \hline \\ -:\bot,\top,\Delta_7 \vdash \mathbf{F}_6 \\ \hline \\ \hline \\ -:\bot,\top,\Delta_7 \vdash \mathbf{F}_6 \\ \hline \\ \hline \\ -:\bot,\top,\Delta_7 \vdash \mathbf{F}_6 \\ \hline \\ \hline \end{array}$$

### ullet Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_7,\mathbf{p}_6 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\top,\Delta_7,\mathbf{p}_6 \vdash \mathbf{F}_4} \quad \top_L \quad \overbrace{\bullet \mathbf{h}_5:(\top,\Delta_7,\mathbf{p}_6),\mathbf{F}_4 \vdash \mathbf{p}_6}^{} \quad I \\ \hline -:\top,\Delta_7,\mathbf{p}_6 \vdash \mathbf{p}_6 \\ \qquad \qquad -:\top,\Delta_7,\mathbf{p}_6 \vdash \mathbf{p}_6 \end{array} \quad I \\ \\ \frac{\mathbf{h}_1:\Delta_4 \vdash \mathbf{p}_6}{\bullet \mathbf{h}_1:\top,\Delta_4 \vdash \mathbf{p}_6} \quad \top_L \quad \overbrace{\bullet \mathbf{h}_5:(\top,\Delta_4),\mathbf{p}_6 \vdash \mathbf{p}_6}^{} \quad I \\ \hline -:\top,\Delta_4 \vdash \mathbf{p}_6 \\ \qquad \qquad \qquad \longrightarrow \\ \hline -:\top,\Delta_4 \vdash \mathbf{p}_6 \end{array} \quad \mathbf{Cut}$$

## • Case rule $\top_L$

$$\frac{ \begin{array}{l} \mathbf{h}_1 : \Delta_4 \vdash \mathbf{F}_5 \\ \bullet \mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_5 \end{array}}{ - : \top, \Delta_4 \vdash \mathbf{F}_7} \begin{array}{l} \top_L \\ \bullet \mathbf{h}_6 : (\top, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \\ \hline \\ - : \top, \Delta_4 \vdash \mathbf{F}_7 \\ \hline \\ \bullet \mathbf{h}_1 : \top, \Delta_4 \vdash \mathbf{F}_5 \end{array}} \begin{array}{l} \top_L \\ \bullet \mathbf{h}_6 : (\top, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_6 : (\top, \Delta_4), \mathbf{F}_5 \vdash \mathbf{F}_7 \\ \hline \\ - : \top, \Delta_4 \vdash \mathbf{F}_7 \end{array}} \begin{array}{l} \mathbf{a}_{X/W} \\ \bullet \mathbf{b}_{Cut} \end{array}$$

# 7 Cut-Elimination

# 7.1 Status of $\top_R$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c|c} \hline \bullet_{\mathbf{h}_1 : \Delta_2 \vdash \top} & \top_R & \hline \bullet_{\mathbf{h}_4 : \Delta_3, \top \vdash \top} & \top_R \\ \hline - : \Delta_2, \Delta_3 \vdash \top & \\ \hline - : \Delta_2, \Delta_3 \vdash \top & \top_R \\ \hline \hline - : \Delta_2, \Delta_3 \vdash \top & \top_R \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_4: \top, \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_4: \Delta_3, \top \vdash \mathbf{F}_5 \to \mathbf{F}_6} \\ -: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \to \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1: * \vdash \top \end{array} \begin{array}{c} \top_R & \frac{\mathbf{h}_4: \top, \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_4: \Delta_3, \top \vdash \mathbf{F}_5 \to \mathbf{F}_6} \\ \\ \frac{\bullet \mathbf{h}_1: * \vdash \top}{\bullet \mathbf{h}_4: \top, \Delta_2, \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6} \\ \hline -: \Delta_2, \Delta_3, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \hline -: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \to \mathbf{F}_6} \end{array} \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\underbrace{\begin{array}{l} \bullet \mathbf{h}_1 : \Delta_2 \vdash \top}_{\bullet} \ \top_R \ \frac{\mathbf{h}_4 : \top, \Delta_3 \vdash \mathbf{F}_5 \quad \mathbf{h}_4 : \top, \Delta_3 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_4 : \Delta_3, \top \vdash \mathbf{F}_5 \land \mathbf{F}_6} \ \mathbf{Cut} \\ \hline -: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \land \mathbf{F}_6 \\ \hline \\ \bullet \mathbf{h}_1 : * \vdash \top \ T_R \ \frac{\mathbf{h}_4 : \top, \Delta_2, \Delta_3 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_4 : \top, \Delta_2, \Delta_3 \vdash \mathbf{F}_5} \ \frac{\mathbf{ax/W}}{\mathsf{hCut}} \\ \hline -: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \land \mathbf{F}_6 \\ \hline -: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \land \mathbf{F}_6 \\ \hline -: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \land \mathbf{F}_6 \end{array} \quad \begin{array}{l} \mathbf{ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\vee_1$ 

$$\begin{array}{c} \frac{\bullet \mathbf{h}_1 : \Delta_2 \vdash \top}{\bullet} \; \top_R \; \; \frac{\mathbf{h}_4 : \top, \Delta_3 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_4 : \Delta_3, \top \vdash \mathbf{F}_5 \lor \mathbf{F}_6} \; \underset{\mathsf{Cut}}{\leftarrow} \\ \frac{-: \Delta_2, \Delta_3 \vdash \mathbf{F}_5 \lor \mathbf{F}_6}{\to} \; \; \frac{\rightarrow}{\mathbf{h}_4 : \top, \Delta_2, \Delta_3 \vdash \mathbf{F}_5} \\ \frac{-: \Delta_2, \Delta_3 \vdash \mathbf{F}_5}{-: \Delta_2, \Delta_3 \vdash \mathbf{F}_5} \; \vee_1 \end{array} \begin{array}{c} \mathsf{ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{\bullet \mathbf{h}_1: \Delta_2 \vdash \top}{-: \Delta_2, \Delta_3 \vdash F_5 \lor F_6} \xrightarrow{\bullet \mathbf{h}_4: \Delta_3, \top \vdash F_5 \lor F_6} \underbrace{\begin{array}{c} \vee_2 \\ \bullet \mathbf{h}_4: \Delta_3, \top \vdash F_5 \lor F_6 \\ -: \Delta_2, \Delta_3 \vdash F_5 \lor F_6 \\ \hline \bullet \mathbf{h}_1: * \vdash \top \end{array}}_{\bullet \mathbf{h}_4: \top, \Delta_2, \Delta_3 \vdash F_6} \underbrace{\begin{array}{c} \mathsf{ax/W} \\ \mathsf{hCut} \\ -: \Delta_2, \Delta_3 \vdash F_6 \lor V_2 \\ \hline -: \Delta_2, \Delta_3 \vdash F_6 \lor V_2 \\ \end{array}}_{\bullet \mathsf{hCut}}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\begin{array}{l} \bullet_{h_1}: \Delta_2 \vdash \top} \quad \top_R \quad \frac{h_3: \top, \Delta_7, F_4 \to F_5 \vdash F_4 \quad h_3: \top, \Delta_7, F_5 \vdash F_6 \\ \bullet h_3: (\Delta_7, F_4 \to F_5), \top \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_4} \quad \underbrace{\begin{array}{l} \bullet_{h_3}: \top, \Delta_7, F_5 \vdash F_6 \\ \bullet h_3: (\Delta_7, F_4 \to F_5), \top \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_4} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \top \quad \top_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet h_1: * \vdash \top \quad T_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \top \quad \top_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet h_1: * \vdash \top \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \top \quad T_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet h_1: * \vdash \top \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \top \quad T_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet h_1: * \vdash \top \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \top \quad T_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet \quad h_1: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_4 \to F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \top \quad T_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet \quad h_1: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \vdash \quad T_R \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet \quad h_1: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet \quad h_1: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet \quad h_1: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \\ \bullet \quad h_1: * \vdash \vdash \quad h_3: \top, \Delta_2, \Delta_7, F_5 \vdash F_6 \end{array}}_{-: \Delta_2, \Delta_7, F_5 \vdash F_6} \quad \underbrace{\begin{array}{l} \bullet_{h_1}: * \vdash \vdash \quad h_3: \Delta_2, \Delta_7, F_5 \vdash \vdash \vdash \quad h_3: \Delta_7, \Delta_7, F_5 \vdash \vdash \vdash \quad h_3: \Delta_7, \Delta_7, A_7, A_7 \vdash \vdash \quad h_7 \vdash \vdash \quad h_$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c|c} & \frac{\mathbf{h}_3: \top, \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_3: (\Delta_7, \mathbf{F}_4 \land \mathbf{F}_5), \top \vdash \mathbf{F}_6} & \wedge_L \\ \hline & -: \Delta_2, \Delta_7, \mathbf{F}_4 \land \mathbf{F}_5 \vdash \mathbf{F}_6} & -: \Delta_2, \Delta_7, \mathbf{F}_4 \land \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1: * \vdash \top & \top_R & \xrightarrow{h_3: \top, \Delta_2, \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \mathbf{F}_6} \\ \hline & \frac{-: \Delta_2, \Delta_7, \mathbf{F}_4, \mathbf{F}_5 \vdash \mathbf{F}_6}{-: \Delta_2, \Delta_7, \mathbf{F}_4 \land \mathbf{F}_5 \vdash \mathbf{F}_6} & \wedge_L \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\vee_L$ 

$$\frac{\underbrace{\begin{array}{c} \bullet_{\mathbf{h}_1:\,\Delta_2\,\vdash\,\top} \\ \bullet_{\mathbf{h}_1:\,\Delta_2\,\vdash\,\top} \end{array}}_{\mathbf{h}_1:\,\Delta_2\,\vdash\,\top} \frac{\mathbf{h}_3:\,\top,\,\Delta_7,\,\mathbf{F}_4\,\vdash\,\mathbf{F}_6 \quad \mathbf{h}_3:\,\top,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6}{\bullet\,\mathbf{h}_3:\,(\Delta_7,\,\mathbf{F}_4\,\vee\,\mathbf{F}_5),\,\top\,\vdash\,\mathbf{F}_6} \underbrace{\mathbf{Cut}}_{\mathbf{c}} \\ \underbrace{\begin{array}{c} \bullet_{\mathbf{h}_1:\,\ast\,\vdash\,\top} \\ \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_4\,\vdash\,\mathbf{F}_6} \\ \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_4\,\vdash\,\mathbf{F}_6} \end{array}}_{\mathbf{h}_{\mathbf{C}}\mathbf{u}} \underbrace{\begin{array}{c} \bullet_{\mathbf{h}_1:\,\ast\,\vdash\,\top} \\ \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6} \\ \bullet_{\mathbf{h}_1:\,\ast\,\vdash\,\top} \end{array}}_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6} \underbrace{\begin{array}{c} \bullet_{\mathbf{x}}/\mathcal{W} \\ \bullet_{\mathbf{h}}/\mathcal{W} \\ \bullet_{\mathbf{h}}/\mathcal{W}} \end{array}}_{\mathbf{h}_{\mathbf{C}}\mathbf{u}} \underbrace{\begin{array}{c} \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6} \\ \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6} \end{array}}_{\mathbf{h}_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6}} \underbrace{\begin{array}{c} \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6} \\ \bullet_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6} \end{aligned}}_{\mathbf{h}_{\mathbf{h}_3:\,\top,\,\Delta_2,\,\Delta_7,\,\mathbf{F}_5\,\vdash\,\mathbf{F}_6}} \underbrace{\begin{array}{c} \bullet_{\mathbf{h}_3:$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \hline { \bullet \mathbf{h}_1 : \Delta_2 \vdash \top} & \top_R & \hline { \bullet \mathbf{h}_3 : (\bot, \Delta_5), \top \vdash \mathbf{F}_4 } \\ \hline {- : \Delta_2, \bot, \Delta_5 \vdash \mathbf{F}_4} \\ \hline { \rightarrow \\ \hline {- : \bot, \Delta_2, \Delta_5 \vdash \mathbf{F}_4 } } & \bot_L \end{array} \quad \mathbf{Cut}$$

ullet Case rule I

$$\begin{array}{c|c} \hline { \bullet \mathbf{h}_1 : \Delta_2 \vdash \top} & \top_R & \hline { \bullet \mathbf{h}_3 : (\Delta_5, \mathbf{p}_4), \top \vdash \mathbf{p}_4} & I \\ \hline { - : \Delta_2, \Delta_5, \mathbf{p}_4 \vdash \mathbf{p}_4} & \\ \hline { - : \Delta_2, \Delta_5, \mathbf{p}_4 \vdash \mathbf{p}_4} & I \\ \hline \end{array}$$

$$\frac{\underbrace{ \begin{array}{l} \bullet_{\mathbf{h}_1} : \Delta_2 \vdash \top}_{} \top_R & \frac{\mathbf{h}_3 : \Delta_4 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_3 : \Delta_4, \top \vdash \mathbf{F}_5} \\ - : \Delta_2, \Delta_4 \vdash \mathbf{F}_5 \\ \hline - : \Delta_2, \Delta_4 \vdash \mathbf{F}_5 \end{array}}_{- : \Delta_2, \Delta_4 \vdash \mathbf{F}_5} \text{ ax/W}} \top_L$$

## 7.2 Status of $\rightarrow_R$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_6\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_2\vdash\mathbf{F}_6\to\mathbf{F}_7} \to_R & \\ \hline \bullet\mathbf{h}_8:\Delta_5,\mathbf{F}_6\to\mathbf{F}_7\vdash\top \\ \hline -:\Delta_2,\Delta_5\vdash\top \\ \hline & \\ \hline -:\Delta_2,\Delta_5\vdash\top \\ \hline \end{array} \uparrow_R$$
 Cut

• Case rule  $\rightarrow_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_6 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_R \quad \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_R \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_9 \to \mathbf{F}_{10} \to \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_R \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_2: \Delta_2, \Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_5, \mathbf{F}_9 \to \mathbf{F}_{10} \end{array} \rightarrow_R \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\underbrace{\frac{\mathbf{h}_1: \Delta_2, \mathbf{F}_6 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \to_R \underbrace{\frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \quad \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{9} \land \mathbf{F}_{10}}}_{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{9} \land \mathbf{F}_{10}}}_{\bullet \mathbf{t}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{h}_9 \to \mathbf{h}_{10}}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_6 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_R \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h} \mathbf{Cut} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_6 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \rightarrow_{R} \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array}}{ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} } \begin{array}{c} \vee_2 \\ \text{Cut} \\ \hline \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_2: \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_{10} \vee_2 \end{array} \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_2: \Delta_3 \vdash \mathbf{h}_3 \vee_2 \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_5\vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2\vdash \mathbf{F}_5\to \mathbf{F}_6}}{\bullet \mathbf{h}_1:\Delta_2\vdash \mathbf{F}_5\to \mathbf{F}_6}} \to_R \frac{\mathbf{h}_7:\Delta_{11},\mathbf{F}_5\to \mathbf{F}_6,\mathbf{F}_8\to \mathbf{F}_9\vdash \mathbf{F}_8}{\bullet \mathbf{h}_7:(\Delta_{11},\mathbf{F}_8\to \mathbf{F}_9),\mathbf{F}_5\to \mathbf{F}_6\vdash \mathbf{F}_{10}}} \subset_{\mathbf{Cut}} \\ -:\Delta_2,\Delta_{11},\mathbf{F}_8\to \mathbf{F}_9\vdash \mathbf{F}_{10}} \subset_{\mathbf{Cut}} \\ \hline \bullet \mathbf{h}_1:\Delta_2\vdash \mathbf{F}_5\to \mathbf{F}_6} \xrightarrow{\mathbf{ax/W}} \frac{\mathbf{h}_7:\Delta_{11},\mathbf{F}_5\to \mathbf{F}_6,\mathbf{F}_8\to \mathbf{F}_9\vdash \mathbf{F}_8}{\bullet \mathbf{h}_7:\Delta_{11},\mathbf{F}_5\to \mathbf{F}_6,\mathbf{F}_8\to \mathbf{F}_9\vdash \mathbf{F}_{10}}} \xrightarrow{\mathbf{ax/W}} \frac{\mathbf{h}_7:\Delta_{11},\mathbf{F}_9,\mathbf{F}_5\to \mathbf{F}_6\vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_1:\Delta_2\vdash \mathbf{F}_5\to \mathbf{F}_6}} \xrightarrow{\mathbf{ax/W}} \frac{\mathbf{h}_7:\Delta_{11},\mathbf{F}_9,\mathbf{F}_5\to \mathbf{F}_6\vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_1:\Delta_2\vdash \mathbf{F}_5\to \mathbf{F}_6} \xrightarrow{\mathbf{ax/W}} \to_{\mathbf{T}}:\Delta_{11},\Delta_2,\mathbf{F}_9\vdash \mathbf{F}_{10}} \to_L \\ \hline \frac{\mathbf{h}_1:\Delta_2\vdash \mathbf{F}_7\to \mathbf{F}_8}{\bullet \mathbf{h}_1:\Delta_2\vdash \mathbf{F}_7\to \mathbf{F}_8} \xrightarrow{\mathbf{h}_8} \frac{\mathbf{h}_5:\Delta_6,\mathbf{F}_7\to \mathbf{F}_8\vdash \mathbf{F}_7}{\bullet \mathbf{h}_5:\Delta_6,\mathbf{F}_7\to \mathbf{F}_8\vdash \mathbf{F}_9}} \xrightarrow{\mathbf{Cut}} \xrightarrow{\mathbf{cut}} \to_L \\ \hline -:\Delta_2,\Delta_6\vdash \mathbf{F}_9} \xrightarrow{\mathbf{ax/W}} \xrightarrow{\mathbf{h}_5:\Delta_6,\mathbf{F}_7\to \mathbf{F}_8\vdash \mathbf{F}_7} \xrightarrow{\mathbf{ax/W}} \xrightarrow{\mathbf{ax/W}} \xrightarrow{\mathbf{cut}} \xrightarrow{\mathbf{c$$

• Case rule  $\wedge_L$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_5 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \to \mathbf{F}_6 \end{array} \rightarrow_R \begin{array}{c} \mathbf{h}_7: \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_7: (\Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10} \end{array} }{ \begin{array}{c} \wedge_L \\ \bullet \mathbf{h}_7: (\Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10} \end{array} } \begin{array}{c} \wedge_L \\ \mathsf{Cut} \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \to \mathbf{F}_6 \end{array} \begin{array}{c} \mathsf{ax/W} \\ \hline \bullet \mathbf{h}_7: \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_7: \Delta_{11}, \Delta_2, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_{11}, \Delta_2, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10} \end{array} } \begin{array}{c} \mathsf{AL} \\ \mathsf{hCut} \end{array}$$

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{5}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{2}\vdash\mathbf{F}_{5}\to\mathbf{F}_{6}}\to_{R}}{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{5}\to\mathbf{F}_{6}\vdash\mathbf{F}_{10}}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\vee\mathbf{F}_{9}),\mathbf{F}_{5}\to\mathbf{F}_{6}\vdash\mathbf{F}_{10}}}_{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\vee\mathbf{F}_{9}),\mathbf{F}_{5}\to\mathbf{F}_{6}\vdash\mathbf{F}_{10}}}}_{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10}}}_{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{5}\to\mathbf{F}_{6}\vdash\mathbf{F}_{10}}}_{\bullet\mathbf{h}_{Cut}}\to_{\mathbf{h}_{7}:\Delta_{11},\Delta_{2},\mathbf{F}_{8}\vdash\mathbf{F}_{10}}}_{\bullet\mathbf{h}_{11}:\Delta_{2}\vdash\mathbf{F}_{5}\to\mathbf{F}_{6}}}_{\bullet\mathbf{h}_{11}:\Delta_{2}\vdash\mathbf{F}_{5}\to\mathbf{F}_{6}}}_{\bullet\mathbf{h}_{11}:\Delta_{2}\vdash\mathbf{F}_{5}\to\mathbf{F}_{6}}^{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9},\mathbf{F}_{5}\to\mathbf{F}_{6}\vdash\mathbf{F}_{10}}}_{\bullet\mathbf{h}_{Cut}}$$

• Case rule  $\perp_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_5 \to \mathbf{F}_6} \to_R & \frac{}{\bullet \mathbf{h}_7:(\bot,\Delta_9), \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_8} \\ -:\Delta_2, \bot, \Delta_9 \vdash \mathbf{F}_8 & \xrightarrow{} \\ -:\bot, \Delta_2, \Delta_9 \vdash \mathbf{F}_8 & \bot_L \end{array} \quad \text{Cut}$$

 $\bullet$  Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \rightarrow \mathbf{F}_6} \xrightarrow{} \mathcal{A}_R & \frac{}{\bullet \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \rightarrow \mathbf{F}_6 \vdash \mathbf{p}_8} \\ & -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \\ & \xrightarrow{} -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \end{array} \xrightarrow{I} \mathbf{Cut}$$

• Case rule  $\top_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2, \mathbf{F}_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \to \mathbf{F}_6} \xrightarrow{} \rightarrow_R & \frac{\mathbf{h}_7: \Delta_9, \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \to \mathbf{F}_6 \vdash \mathbf{F}_8} & \overset{\top_L}{\cot} \\ \hline & -: \Delta_2, \top, \Delta_9 \vdash \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \to \mathbf{F}_6 & \text{ax/W} & \overset{\rightarrow}{\cot} \\ & -: \top, \Delta_2, \Delta_9 \vdash \mathbf{F}_8 & \text{ax/W} \\ \hline & -: \top, \Delta_2, \Delta_9 \vdash \mathbf{F}_8 & \text{hCut} \end{array}$$

# 7.3 Status of $\wedge_R$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \quad \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \wedge_R \quad \frac{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \top}{-: \Delta_2, \Delta_5 \vdash \top} \quad \overset{\top_R}{\subset} \text{Cut} \\ & \qquad \qquad \frac{\bullet}{-: \Delta_2, \Delta_5 \vdash \top} \quad \top_R \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \wedge_R \quad \frac{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \quad \xrightarrow{\bullet} \quad \text{Cut} \\ \\ \underbrace{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7}_{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \text{ax/W} \quad \frac{\rightarrow}{\mathbf{h}_8:\Delta_5, \mathbf{F}_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_2:\Delta_2,\Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10}} \quad \xrightarrow{\bullet}_R \quad \text{ax/W}} \quad \frac{-:\Delta_2,\Delta_5,\mathbf{F}_9 \vdash \mathbf{F}_{10}}{-:\Delta_2,\Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10}} \quad \to_R \end{array}}$$

• Case rule  $\wedge_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \quad \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7 \end{array} \wedge_R \quad \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \quad \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \end{array}}{ \begin{array}{c} \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \end{array}} \quad \mathbf{Cut} \\ \\ \underline{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \mathbf{ax/W} \quad \begin{array}{c} \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \land \mathbf{F}_{10} \end{array}} \quad \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7 \\ \bullet \mathbf{h}_2: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_3: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10} \end{array}} \quad \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7 \\ \bullet \mathbf{h}_2: \Delta_2, \Delta_5 \vdash \mathbf{F}_{10} \end{array}} \quad \mathbf{h}_1: \Delta_2 \vdash \mathbf{h}_2: \Delta_2, \Delta_5 \vdash \mathbf{h}_3: \Delta_3 \vdash \mathbf{h}_3: \Delta_4 \vdash \mathbf{h}_3: \Delta_5, \mathbf{h}_4: \Delta_5 \vdash \mathbf{h}_4: \Delta_4 \vdash \mathbf{h}_4:$$

• Case rule  $\vee_1$ 

$$\frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \quad \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7}{\underbrace{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7}} \land_R \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \xrightarrow{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \underbrace{\mathbf{ax/W}}_{h8: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9} \underbrace{\mathbf{ax/W}}_{h\text{Cut}} \\ \xrightarrow{-: \Delta_2, \Delta_5 \vdash \mathbf{F}_9} \lor_{\mathbf{f}_{10}} \lor_{\mathbf{1}}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \\ \wedge_R & \frac{\mathbf{h}_8:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \\ \\ -:\Delta_2,\Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ & \xrightarrow{\bullet} \\ \frac{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7}{\bullet \mathbf{x}/W} & \frac{\mathbf{a} \mathbf{x}/W}{\mathbf{h}_8:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_{10}} \\ & \frac{-:\Delta_2,\Delta_5 \vdash \mathbf{F}_{10}}{-:\Delta_2,\Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \\ & & & & \\ \end{array} \quad \begin{array}{c} \mathsf{a} \mathbf{x}/W \\ \mathsf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\frac{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \quad \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6}}_{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6}}}_{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \land \mathbf{F}_{6}} \land_{R} \quad \underbrace{\frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9},\mathbf{F}_{5} \land \mathbf{F}_{6} \vdash \mathbf{F}_{8} \quad \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9},\mathbf{F}_{5} \land \mathbf{F}_{6} \vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9}),\mathbf{F}_{5} \land \mathbf{F}_{6} \vdash \mathbf{F}_{10}} \underbrace{\mathbf{Cut}}_{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \land \mathbf{F}_{6}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9},\mathbf{F}_{5} \land \mathbf{F}_{6} \vdash \mathbf{F}_{8}}_{\bullet \mathbf{h}_{2}}}_{\bullet \mathbf{h}_{2}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9},\mathbf{F}_{5} \land \mathbf{F}_{6} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_{2}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{7}:\Delta_{11},\Delta_{2},\mathbf{F}_{9} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_{2}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{7}:\Delta_{11},\Delta_{2},\mathbf{F}_{9} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_{2}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{2}}}_{\bullet \mathbf{h}_{3}:\Delta_{2} \vdash \mathbf{F}_{5} \land \mathbf{F}_{6}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{7}:\Delta_{11},\Delta_{2},\mathbf{F}_{9} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_{2}:\Delta_{2},\mathbf{F}_{9} \vdash \mathbf{F}_{10}} \quad \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{2}:\Delta_{2}}}_{\bullet \mathbf{h}_{3}:\Delta_{2},\mathbf{F}_{9} \vdash \mathbf{F}_{10}}$$

• Case rule  $\wedge_L$ 

$$\frac{\frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_7 \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_7 \land \mathbf{F}_8} \quad \wedge_R \quad \frac{\mathbf{h}_5:\Delta_6, \mathbf{F}_7, \mathbf{F}_8 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_5:\Delta_6, \mathbf{F}_7 \land \mathbf{F}_8 \vdash \mathbf{F}_9} \quad \wedge_L}{-:\Delta_2,\Delta_6 \vdash \mathbf{F}_9} \quad \text{Cut}} \\ \frac{-:\Delta_2,\Delta_6 \vdash \mathbf{F}_9}{\bullet \mathbf{x}/\mathsf{W}} \quad \frac{\rightarrow}{-:\Delta_6,\Delta_6, \mathbf{F}_7, \mathbf{F}_8 \vdash \mathbf{F}_9} \quad \text{ax/W}}{-:\Delta_6,\Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9}} \\ \frac{-:\Delta_2 \vdash \mathbf{F}_7}{\bullet \mathbf{x}/\mathsf{W}} \quad \frac{-:\Delta_2,\Delta_6,\Delta_6,\mathbf{F}_7 \vdash \mathbf{F}_9}{\bullet \mathbf{x}/\mathsf{W}} \quad \text{sCut}}{-:\Delta_2,\Delta_6,\Delta_6,\mathbf{F}_7 \vdash \mathbf{F}_9} \quad \text{sCut}}$$

• Case rule  $\vee_L$ 

$$\frac{\underbrace{\frac{\mathbf{h}_{1}:\Delta_{2}\vdash F_{5}\quad \mathbf{h}_{1}:\Delta_{2}\vdash F_{6}}{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash F_{5}\wedge F_{6}}}_{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash F_{5}\wedge F_{6}} \wedge_{R} \underbrace{\frac{\mathbf{h}_{7}:\Delta_{11},F_{8},F_{5}\wedge F_{6}\vdash F_{10}\quad \mathbf{h}_{7}:\Delta_{11},F_{9},F_{5}\wedge F_{6}\vdash F_{10}}{\bullet \mathbf{h}_{7}:(\Delta_{11},F_{8}\vee F_{9}),F_{5}\wedge F_{6}\vdash F_{10}}}_{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash F_{5}\wedge F_{6}}} \underbrace{\mathbf{Cut}}_{-:\Delta_{2},\Delta_{11},F_{8},F_{5}\wedge F_{6}\vdash F_{10}} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{1}:\Delta_{2}\vdash F_{5}\wedge F_{6}}}_{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash F_{5}\wedge F_{6}} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_{7}:\Delta_{11},F_{9},F_{5}\wedge F_{6}\vdash F_{10}}}_{-:\Delta_{11},\Delta_{2},F_{9}\vdash F_{10}}}_{\bullet \mathbf{h}_{2}} \underbrace{\mathbf{ax/W}}_{\bullet \mathbf{h}_{2}}$$

• Case rule  $\perp_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_5 \quad \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_5 \land \mathbf{F}_6} \quad \wedge_R \quad \frac{\bullet \mathbf{h}_7:(\bot,\Delta_9), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet} \quad \frac{\bot_L}{\mathsf{Cut}} \\ & \xrightarrow{-:\Delta_2,\bot,\Delta_9 \vdash \mathbf{F}_8} \quad \bot_L \end{array}$$

 $\bullet$  Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \quad \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \land \mathbf{F}_6} \quad \bigwedge_R \quad \frac{\bullet \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{p}_8}{\bullet \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{p}_8} \quad \mathbf{Cut} \\ & \qquad \qquad -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \\ & \qquad \qquad -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \end{array} \quad I \end{array}$$

• Case rule  $\top_L$ 

$$\frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \quad \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6}{\underbrace{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \land \mathbf{F}_6}_{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}} \underbrace{\uparrow_L}_{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \land \mathbf{F}_6} \underbrace{\uparrow_L}_{\bullet \mathbf{h}_7: (\top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}}_{\bullet \mathbf{h}_7: (\top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8} \underbrace{\uparrow_L}_{\bullet \mathbf{h}_7: (\top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}}_{\bullet \mathbf{h}_7: (\top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8} \underbrace{\uparrow_L}_{\bullet \mathbf{h}_1: (\top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}}_{\bullet \mathbf{h}_1: (\top, \Delta_9, \mathbf{F}_5 \land \mathbf{F}_6 \vdash \mathbf{F}_8}$$

## 7.4 Status of $\vee_1$ : OK

• Case rule  $\top_R$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_1 \quad \frac{}{\bullet \mathbf{h}_8 : \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \top} \quad \begin{array}{c} \top_R \\ \hline - : \Delta_2, \Delta_5 \vdash \top \\ \hline \hline - : \Delta_2, \Delta_5 \vdash \top \end{array} \quad \top_R \end{array}}$$

• Case rule  $\rightarrow_R$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \lor_1 & \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_9, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} & \begin{array}{c} \mathbf{ax/W} \\ \hline \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} & \mathbf{ax/W} \\ \hline \\ \hline -: \Delta_2, \Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \end{array} & \rightarrow_R \end{array} & \mathbf{ax/W} \\ \hline \\ \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{2}\vdash \mathbf{F}_{6}}{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash \mathbf{F}_{6}\vee \mathbf{F}_{7}}}{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash \mathbf{F}_{6}\vee \mathbf{F}_{7}}} \bigvee_{1} \frac{\mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6}\vee \mathbf{F}_{7}\vdash \mathbf{F}_{9}}{\bullet \mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6}\vee \mathbf{F}_{7}\vdash \mathbf{F}_{10}}} \underbrace{\mathbf{Cut}} \\ \frac{-:\Delta_{2},\Delta_{5}\vdash \mathbf{F}_{9}\wedge \mathbf{F}_{10}}{\bullet \mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6}\vee \mathbf{F}_{7}\vdash \mathbf{F}_{9}}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}\mathbf{Cut}} \xrightarrow{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash \mathbf{F}_{6}\vee \mathbf{F}_{7}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6}\vee \mathbf{F}_{7}\vdash \mathbf{F}_{10}}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}\mathbf{Cut}} \xrightarrow{\bullet \mathbf{h}_{1}:\Delta_{2}\vdash \mathbf{F}_{6}\vee \mathbf{F}_{7}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6}\vee \mathbf{F}_{7}\vdash \mathbf{F}_{10}}} \bigwedge_{\mathbf{h}\mathbf{Cut}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}\mathbf{Cut}}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \vee_1 \quad \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \xrightarrow[\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} \vee_1 \\ \end{array} \xrightarrow[\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7]{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7} \xrightarrow[\mathbf{h}_2]{\mathbf{h}_2: \Delta_5 \vdash \mathbf{F}_9} \times_{\mathbf{h}_2: \Delta_5 \vdash \mathbf{h}_9} \times_{\mathbf{h}_3: \Delta_5 \vdash$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \vee_1 \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10}} \quad \begin{array}{c} \vee_2 \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \hline \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \lor \mathbf{F}_7 \end{array} \quad \underset{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet} \quad \underset{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \lor \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet} \quad \underset{\mathbf{h}_{Cut}}{\bullet} \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_{10} \quad \vee_2 \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} \quad \underset{\mathbf{h}_{Cut}}{\bullet} \\ \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} \lor_1}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} \lor_1 \underbrace{\frac{\mathbf{h}_7: \Delta_{11}, \mathbf{F}_8 \to \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_7: (\Delta_{11}, \mathbf{F}_8 \to \mathbf{F}_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_7: (\Delta_{11}, \mathbf{F}_8 \to \mathbf{F}_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}}} \underbrace{\mathbf{Cut}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_7: \Delta_{11}, \mathbf{F}_8 \to \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8}}_{-: \Delta_{11}, \Delta_2, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_8} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_7: \Delta_{11}, \Delta_2, \mathbf{F}_9 \vdash \mathbf{F}_{10}}}_{-: \Delta_{11}, \Delta_2, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_{10}} \to_L} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_7: \Delta_{11}, \Delta_2, \mathbf{F}_9 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} \underbrace{\frac{\mathbf{ax/W}}{\mathbf{h}_7: \Delta_{11}, \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_9 \vdash \mathbf{F}_{10}} \to_L}$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} & \vee_1 & \frac{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_7 : (\Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}} & \wedge_L \\ \hline & -: \Delta_2, \Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline & \bullet \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6 & \mathbf{ax/W} & \frac{\rightarrow}{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}} \\ \hline & \frac{-: \Delta_{11}, \Delta_2, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{-: \Delta_{11}, \Delta_2, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \wedge_L \end{array} \quad \begin{array}{c} \mathbf{Ax/W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\vee_L$ 

$$\frac{\frac{h_1: \Delta_2 \vdash F_5}{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6}}{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6} \lor_1 \quad \frac{\frac{h_7: \Delta_{11}, F_8, F_5 \lor F_6 \vdash F_{10}}{\bullet h_7: (\Delta_{11}, F_8 \lor F_9), F_5 \lor F_6 \vdash F_{10}}}{\bullet h_7: (\Delta_{11}, F_8 \lor F_9), F_5 \lor F_6 \vdash F_{10}} \quad \vee_L \\ \frac{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6}{\bullet h_7: \Delta_{11}, F_8, F_5 \lor F_6 \vdash F_{10}}}{h_7: \Delta_{11}, F_8, F_5 \lor F_6 \vdash F_{10}} \quad \frac{\text{ax/W}}{\text{hCut}} \quad \frac{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6}{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6} \quad \text{ax/W}} \quad \frac{h_7: \Delta_{11}, F_9, F_5 \lor F_6 \vdash F_{10}}{h_7: \Delta_{11}, F_9, F_5 \lor F_6 \vdash F_{10}} \quad \text{ax/W}}{h_{\text{Cut}}} \quad \frac{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6}{\bullet h_1: \Delta_2 \vdash F_5 \lor F_6} \quad \text{ax/W}} \quad \frac{h_7: \Delta_{11}, F_9, F_5 \lor F_6 \vdash F_{10}}{h_7: \Delta_{11}, F_9, F_5 \lor F_6 \vdash F_{10}} \quad \vee_L \\ \frac{h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{h_5: \Delta_6, F_7 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9, F_5 \lor F_6 \vdash F_{10}}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9, F_7 \lor F_8 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9, F_7 \lor F_8 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9, F_7 \lor F_8 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9, F_7 \lor F_8 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9 \vdash F_9}{\bullet h_5: \Delta_6, F_7 \lor F_8 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2 \vdash F_7 \lor F_8} \quad \bigvee_1 \quad \frac{\bullet h_7: \Delta_1, F_9 \vdash F_9}{\bullet h_1: \Delta_2, F_9 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2, F_9 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2, F_9 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2, F_9 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_1, F_9 \vdash F_9}{\bullet h_1: \Delta_2, F_9 \vdash F_9} \quad \text{Cut} \\ \frac{\bullet h_1: \Delta_2 \vdash F_7}{\bullet h_1: \Delta_2, F_9 \vdash F_9} \quad \text{Cut$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} & \vee_1 & \\ \hline \bullet \mathbf{h}_7: (\bot, \Delta_9), \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_8 \\ \hline -: \Delta_2, \bot, \Delta_9 \vdash \mathbf{F}_8 \\ \hline -: \bot, \Delta_2, \Delta_9 \vdash \mathbf{F}_8 & \bot_L \end{array} \quad \begin{array}{c} \bot_L \\ \text{Cut} \\ \hline \end{array}$$

 $\bullet$  Case rule I

$$\begin{array}{c|c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} & \vee_1 & \\ \hline \bullet \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{p}_8 \\ \hline & -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \\ \hline & -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \end{array} \begin{array}{c} I \\ \text{Cut} \end{array}$$

• Case rule  $\top_L$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6 \end{array} \lor_1 & \begin{array}{c} \mathbf{h}_7: \Delta_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8 \end{array} }{ \begin{array}{c} -: \Delta_2, \top, \Delta_9 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6 \end{array} } & \begin{array}{c} \Delta_x / \mathbf{w} \\ \bullet \mathbf{h}_7: \top, \Delta_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_7: \top, \Delta_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8 \end{array} } \begin{array}{c} \mathbf{ax} / \mathbf{w} \\ \mathbf{h} \mathsf{Cutt} \end{array}$$

### 7.5 Status of $\vee_2$ : OK

• Case rule  $\top_R$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_2 \begin{array}{c} \hline \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \top \\ \hline -: \Delta_2, \Delta_5 \vdash \top \\ \hline \hline -: \Delta_2, \Delta_5 \vdash \top \end{array} } \begin{array}{c} \top_R \\ \mathbf{Cut} \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_2 \quad \begin{array}{c} \mathbf{h}_8: \Delta_5, \mathbf{F}_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \\ \hline \\ \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \xrightarrow{\mathbf{ax/W}} \begin{array}{c} \rightarrow \\ \mathbf{h}_8: \Delta_5, \mathbf{F}_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline \\ \mathbf{h}_8: \Delta_5, \mathbf{F}_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \hline \\ -: \Delta_2, \Delta_5, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \end{array} \xrightarrow{\mathbf{Ax/W}} \begin{array}{c} \rightarrow \\ \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{7}}{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6} \lor \mathbf{F}_{7}}}{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6} \lor \mathbf{F}_{7}}} \bigvee_{2} \frac{\mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6} \lor \mathbf{F}_{7} \vdash \mathbf{F}_{9} \quad \mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6} \lor \mathbf{F}_{7} \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6} \lor \mathbf{F}_{7} \vdash \mathbf{F}_{9} \land \mathbf{F}_{10}}} \underbrace{\mathbf{Cut}} \\ -:\Delta_{2},\Delta_{5} \vdash \mathbf{F}_{9} \land \mathbf{F}_{10}} \\ \bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6} \lor \mathbf{F}_{7}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6} \lor \mathbf{F}_{7} \vdash \mathbf{F}_{9}}} \underbrace{\mathbf{ax}/\mathbb{W}}_{\mathbf{h}_{Cut}} \underbrace{-:\Delta_{2},\Delta_{5} \vdash \mathbf{F}_{9}} \underbrace{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6} \lor \mathbf{F}_{7}}_{-:\Delta_{2},\Delta_{5} \vdash \mathbf{F}_{10}}} \underbrace{\mathbf{h}_{8}:\Delta_{5},\mathbf{F}_{6} \lor \mathbf{F}_{7} \vdash \mathbf{F}_{10}}_{\mathbf{h}_{Cut}} \underbrace{\mathbf{h}_{Cut}}_{\mathbf{h}_{Cut}}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_2 \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10}} \quad \begin{array}{c} \vee_1 \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \hline \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \xrightarrow{\mathbf{ax/W}} \begin{array}{c} \rightarrow \\ \hline \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \\ \hline \bullet \mathbf{h}_3: \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \qquad \mathbf{ax/W} \\ \hline \begin{array}{c} -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \vee_1 \end{array} \qquad \mathbf{ax/W}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \vee_2 \quad \frac{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10}} \quad \begin{array}{c} \vee_2 \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \hline \\ \hline \bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 \end{array} \quad \frac{\mathbf{ax/W}}{\mathbf{h}_8: \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_{10}} \quad \frac{\mathbf{ax/W}}{\mathbf{h}_{\text{Cut}}} \\ \hline \\ -: \Delta_2, \Delta_5 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_5 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \end{array} \quad \vee_2 \\ \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{\underbrace{\frac{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6}}{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \lor \mathbf{F}_{6}}}_{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \lor \mathbf{F}_{6}} \lor_{2} \quad \underbrace{\frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9},\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{8}}{\bullet \mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9}),\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9} \vdash \mathbf{F}_{10})} \underbrace{\mathbf{Cut}}_{-:\Delta_{2},\Delta_{11},\mathbf{F}_{8} \to \mathbf{F}_{9},\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{8}}_{\bullet \mathbf{ax}/\mathbf{W}} \underbrace{\frac{\bullet}{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \lor \mathbf{F}_{6}}_{\bullet} \underbrace{\frac{\mathbf{ax}/\mathbf{W}}{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9},\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{10}}_{-:\Delta_{11},\Delta_{2},\mathbf{F}_{9} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{Lut}} \underbrace{\frac{\bullet}{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \lor \mathbf{F}_{6}}_{\bullet} \underbrace{\frac{\mathbf{ax}/\mathbf{W}}{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{9},\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{10}}}_{\bullet \mathbf{Lut}}}_{\bullet \mathbf{Lut}}$$

• Case rule  $\wedge_L$ 

$$\frac{ \begin{array}{c|c} \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_6 \\ \bullet \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6 \end{array}}{\bullet \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} \lor_2 & \frac{\mathbf{h}_7 : (\Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_7 : (\Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}} & \overset{\wedge_L}{\cot} \\ \hline & - : \Delta_2, \Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} \\ \hline & \frac{\bullet \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6}{\bullet} & \frac{\mathsf{ax/W}}{\mathsf{h}_7 : \Delta_{11}, \mathbf{F}_8, \mathbf{F}_9, \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_{10}} \\ \hline & \frac{- : \Delta_{11}, \Delta_2, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{- : \Delta_{11}, \Delta_2, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \overset{\wedge_L}{\land} \\ \hline \end{array} \quad \begin{array}{c} \mathsf{ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\vee_L$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6}}{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{6}}}{\bullet \mathbf{h}_{1}:\Delta_{2} \vdash \mathbf{F}_{5} \lor \mathbf{F}_{6}}} \lor_{2} \quad \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8} \lor \mathbf{F}_{9}),\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{10}}} \quad \mathbf{Cut} \\ \frac{-:\Delta_{2},\Delta_{11},\mathbf{F}_{8} \lor \mathbf{F}_{9} \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{8},\mathbf{F}_{5} \lor \mathbf{F}_{6} \vdash \mathbf{F}_{10}} \quad \mathbf{ax/W} \quad \mathbf{hCut}}{\bullet \mathbf{hCut}} \\ \frac{-:\Delta_{11},\Delta_{2},\mathbf{F}_{8} \vdash \mathbf{F}_{10}}{-:\Delta_{11},\Delta_{2},\mathbf{F}_{8} \vdash \mathbf{F}_{10}} \quad \vee_{L} \quad \mathbf{hCut}}$$

$$\begin{array}{c|c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} & \vee_2 & \frac{\mathbf{h}_5: \Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9 & \mathbf{h}_5: \Delta_6, \mathbf{F}_8 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_5: \Delta_6, \mathbf{F}_7 \vee \mathbf{F}_8 \vdash \mathbf{F}_9} & \mathrm{Cut} \\ \hline \\ -: \Delta_2, \Delta_6 \vdash \mathbf{F}_9 & \\ \hline -: \Delta_2 \vdash \mathbf{F}_8 & \mathrm{ax/W} & \\ \hline -: \Delta_6, \mathbf{F}_8 \vdash \mathbf{F}_9 & \mathrm{ax/W} \\ \hline -: \Delta_2, \Delta_6 \vdash \mathbf{F}_9 & \mathrm{sCut} \\ \end{array}$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} & \vee_2 & \frac{}{\bullet \mathbf{h}_7: (\bot, \Delta_9), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{F}_8} \\ \hline & -: \Delta_2, \bot, \Delta_9 \vdash \mathbf{F}_8 \\ \hline & -: \bot, \Delta_2, \Delta_9 \vdash \mathbf{F}_8 & \bot_L \end{array} \quad \text{Cut}$$

ullet Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \lor \mathbf{F}_6} & \vee_2 & \\ \hline \bullet \mathbf{h}_7: (\Delta_9, \mathbf{p}_8), \mathbf{F}_5 \lor \mathbf{F}_6 \vdash \mathbf{p}_8 \\ \hline & -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \\ \hline & -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \end{array} \quad \mathbf{Cut} \\ \hline \\ \hline -: \Delta_2, \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 \quad I \end{array}$$

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} \; \vee_2 \; \; \frac{\mathbf{h}_7: \Delta_9, \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_7: (\top, \Delta_9), \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_8} \; \frac{\top_L}{\mathsf{Cut}} \\ \\ \frac{-: \Delta_2, \top, \Delta_9 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \vee \mathbf{F}_6} \; \frac{\mathsf{ax/W}}{\bullet \mathbf{h}_7: \top, \Delta_9, \mathbf{F}_5 \vee \mathbf{F}_6 \vdash \mathbf{F}_8} \\ \\ \frac{\bullet \mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5 \vee \mathbf{F}_6}{-: \top, \Delta_2, \Delta_9 \vdash \mathbf{F}_8} \; \frac{\mathsf{ax/W}}{\mathsf{hCut}} \end{array}$$

# 7.6 Status of $\rightarrow_L$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_7} \to_L & \frac{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\top}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\top} & \top_R \\ \hline \\ \frac{-:(\Delta_2,\mathbf{F}_3\to\mathbf{F}_4),\Delta_6\vdash\top}{-:\Delta_2,\Delta_6,\mathbf{F}_3\to\mathbf{F}_4\vdash\top} & \top_R \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3}{\bullet} \quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7 \\ \hline \\ \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_7}{-:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4),\Delta_6\vdash\mathbf{F}_9\rightarrow\mathbf{F}_{10}} \\ \hline \\ \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet} \quad \frac{\rightarrow}{\mathbf{h}_8:\Delta_6,\mathbf{F}_7,\mathbf{F}_9\vdash\mathbf{F}_{10}} \\ \hline \\ \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet} \quad \frac{\rightarrow}{\mathbf{h}_8:\Delta_6,\mathbf{F}_7,\mathbf{F}_9\vdash\mathbf{F}_{10}} \\ \hline \\ \frac{-:\Delta_2,\Delta_6,\mathbf{F}_9,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_{10}}{-:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_9\rightarrow\mathbf{F}_{10}} \\ \hline \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \\ \hline \end{array}$$

• Case rule  $\wedge_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_7} \rightarrow_L & \frac{\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\quad\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}} & \mathbf{Cut} \\ & -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4),\Delta_6\vdash\mathbf{F}_9\wedge\mathbf{F}_{10} & \\ & \rightarrow & \\ & & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3 & \mathbf{ax/W} & \frac{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}} & \mathbf{ax/W} \\ & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3 & \mathbf{ax/W} & \bullet \\ & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3 & \mathbf{Ax/W} & \bullet \\ & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3 & \mathbf{Ax/W} & \bullet \\ \end{array}$$

• Case rule  $\vee_1$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3 \quad \mathbf{h}_1:\Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_7} \rightarrow_{L} & \frac{\mathbf{h}_8:\Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9}{\bullet \mathbf{h}_8:\Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10}} \\ \hline & -:(\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4), \Delta_6 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10} \\ \hline & \stackrel{\bullet}{\bullet} \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_7} & \xrightarrow{\mathbf{ax/W}} & \frac{\mathbf{ax/W}}{\mathbf{h}_8:\Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9} \\ \hline & \frac{-:\Delta_2,\Delta_6, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_9}{-:\Delta_2,\Delta_6, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_9 \vee \mathbf{F}_{10}} & \vee_1 \end{array} \quad \begin{array}{c} \mathsf{V_1} \\ \mathsf{Cut} \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_7}\rightarrow_L & \frac{\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\vee\mathbf{F}_{10}} & \vee_2\\ \hline & -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4),\Delta_6\vdash\mathbf{F}_9\vee\mathbf{F}_{10}\\ \hline & \rightarrow\\ \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_7}{\circ\mathbf{ax/W}} & \frac{\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_{10}}{\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_{10}} & \mathbf{ax/W}\\ \hline & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_{10}\\ \hline & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_{10}\\ \hline & -:\Delta_2,\Delta_6,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_{10}\\ \hline \end{array} & \vee_2 \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3}{\bullet} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet} \to L & \frac{\mathbf{h}_7:\Delta_{11},\mathbf{F}_6,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_8}{\bullet} \frac{\mathbf{h}_7:\Delta_{11},\mathbf{F}_6,\mathbf{F}_9\vdash\mathbf{F}_{10}}{\bullet} \\ \hline -:(\Delta_2,\mathbf{F}_3\to\mathbf{F}_4),\Delta_{11},\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{10} \\ \hline & \to \\ \hline & \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_3}{\bullet} \\ \hline -:\Delta_{11},\Delta_2,\mathbf{F}_3\to\mathbf{F}_4,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_3 \\ \hline & -:\Delta_{11},\Delta_2,\mathbf{F}_3\to\mathbf{F}_4,\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_3 \\ \hline & -:\Delta_{11},\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_8\to\mathbf{F}_9\vdash\mathbf{F}_{10} \\ \hline & \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet} \\ \hline & -:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_8\to\mathbf{F}_9 \\ \hline & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3 \\ \hline & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_1 \\ \hline & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_1 \\ \hline & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_1 \\ \hline & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_{10} \\ \hline & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_{10} \\ \hline \end{array}$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{3}\quad\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{6}} \to L & \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\land\mathbf{F}_{9}),\mathbf{F}_{6}\vdash\mathbf{F}_{10}} & \land L \\ \hline & -:(\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}),\Delta_{11},\mathbf{F}_{8}\land\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline & \bullet \\ \hline & \bullet \\ \hline & \bullet \\ \hline & -:\Delta_{11},\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{6} & \mathbf{ax/W} & \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline & -:\Delta_{11},\Delta_{2},\mathbf{F}_{8},\mathbf{F}_{9},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{10} \\ \hline & -:\Delta_{11},\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4},\mathbf{F}_{8}\land\mathbf{F}_{9}\vdash\mathbf{F}_{10} & \land L \\ \hline \end{array} \right.$$

$$\frac{\begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3 \quad \mathbf{h}_1: \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_9 \\ \hline \underline{\bullet} \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_8 \wedge \mathbf{F}_9 \\ \hline \\ -: (\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4), \Delta_7 \vdash \mathbf{F}_{10} \\ \hline \\ -: (\Delta_2, \mathbf{F}_3 \rightarrow \mathbf{F}_4), \Delta_7 \vdash \mathbf{F}_{10} \\ \hline \\ -: \Delta_2, \Delta_7, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3 \\ \hline \\ -: \Delta_2, \Delta_7, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3 \\ \hline \\ -: \Delta_2, \Delta_7, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_3 \\ \hline \end{array} \begin{array}{c} \mathbf{ax/W} \\ \hline \\ \mathbf{hCut} \\ \hline \\ -: \Delta_2, \Delta_7, \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline \\ -: \Delta_2, \Delta_7, \mathbf{F}_3 \rightarrow \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline \end{array} \begin{array}{c} \mathbf{ax/W} \\ \mathbf{hCut} \\ \hline \end{array}$$

• Case rule  $\vee_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{3}\quad\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{6}} \to L & \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\vdash\mathbf{F}_{10}\quad\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{9}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\vee\mathbf{F}_{9}),\mathbf{F}_{6}\vdash\mathbf{F}_{10}} & \mathbf{Cut} \\ & -:(\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}),\Delta_{11},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ & \to \\ & \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{6}\quad\mathbf{ax/W}}{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \mathbf{ax/W} \\ \hline -:\Delta_{11},\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{3}\quad\mathbf{ax/W} & -:\Delta_{11},\Delta_{2},\mathbf{F}_{4},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline -:\Delta_{11},\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{8}\vee\mathbf{F}_{9} & \to L & \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\vdash\mathbf{F}_{10}\quad\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline \bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{8}\vee\mathbf{F}_{9} & \to L & \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\vdash\mathbf{F}_{10}\quad\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline -:(\Delta_{2},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{8}\vee\mathbf{F}_{9}) & \bullet\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{3} & \mathbf{ax/W} & \bullet\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{3} & \mathbf{ax/W} & \bullet\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\vee\mathbf{F}_{9}\vdash\mathbf{F}_{10} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{10} & \bullet\mathbf{h}_{1}\to\mathbf{h}_{2}\to\mathbf{h}_{2} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{10} & \bullet\mathbf{h}_{1}\to\mathbf{h}_{2}\to\mathbf{h}_{2} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{10} & \bullet\mathbf{h}_{2}\to\mathbf{h}_{2} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{F}_{4}\vdash\mathbf{F}_{10} & \bullet\mathbf{h}_{2}\to\mathbf{h}_{2} \\ \hline -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\to\mathbf{h}_{4$$

• Case rule  $\perp_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3\quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\bot}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\bot} \to L & \frac{\bullet\mathbf{h}_6:\Delta_7,\bot\vdash\mathbf{F}_8}{\bullet\mathbf{h}_6:\Delta_7,\bot\vdash\mathbf{F}_8} & \frac{\bot_L}{\mathsf{Cut}} \\ & -:(\Delta_2,\mathbf{F}_3\to\mathbf{F}_4),\Delta_7\vdash\mathbf{F}_8 \\ & \to \\ & \frac{\bullet}{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\bot} & \mathsf{ax/W} & \frac{\bullet\mathbf{h}_6:\bot,\Delta_7\vdash\mathbf{F}_8}{\bullet\mathbf{h}_6:\bot,\Delta_7\vdash\mathbf{F}_8} & \frac{\bot_L}{\mathsf{hCut}} \\ \hline -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3 & \frac{\bullet}{\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_8} & \to_L \\ \hline & \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6} & \to_L & \frac{\bullet}{\mathbf{h}_7:(\bot,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8} & \bot_L \\ \hline & \frac{\bullet}{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_6} & \to_L & \frac{\bullet}{\mathbf{h}_7:(\bot,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8} & \bot_L \\ \hline & -:(\Delta_2,\mathbf{F}_3\to\mathbf{F}_4),\bot,\Delta_9\vdash\mathbf{F}_8 & \to \\ \hline & -:\bot,\Delta_2,\Delta_9,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_8 & \bot_L \\ \hline \end{array}$$

ullet Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3\quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_6}\to_L \quad \frac{\bullet\mathbf{h}_7:(\Delta_9,\mathbf{p}_8),\mathbf{F}_6\vdash\mathbf{p}_8}{\bullet\mathbf{h}_7:(\Delta_9,\mathbf{p}_8),\mathbf{F}_6\vdash\mathbf{p}_8} \quad \mathbf{Cut} \\ & -:(\Delta_2,\mathbf{F}_3\to\mathbf{F}_4),\Delta_9,\mathbf{p}_8\vdash\mathbf{p}_8 \\ & -:\Delta_2,\Delta_9,\mathbf{p}_8,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{p}_8 \quad I \\ \\ \hline \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{F}_3\quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{p}_8}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{p}_8} \to_L \quad \frac{\bullet\mathbf{h}_6:\Delta_7,\mathbf{p}_8\vdash\mathbf{p}_8}{\bullet\mathbf{h}_6:\Delta_7,\mathbf{p}_8\vdash\mathbf{p}_8} \quad \mathbf{Cut} \\ \hline & -:(\Delta_2,\mathbf{F}_3\to\mathbf{F}_4),\Delta_7\vdash\mathbf{p}_8 \\ & -:\Delta_2,\Delta_7,\mathbf{F}_3\to\mathbf{F}_4\vdash\mathbf{p}_8 \quad \mathbf{ax/W} \end{array}$$

• Case rule  $\top_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\top}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\top} \rightarrow_L \quad \frac{\mathbf{h}_6:\Delta_7\vdash\mathbf{F}_8}{\bullet\mathbf{h}_6:\Delta_7,\top\vdash\mathbf{F}_8} \quad \mathsf{T}_L \\ \hline -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4),\Delta_7\vdash\mathbf{F}_8 \\ \hline -:\Delta_2,\Delta_7,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_8 \quad \mathsf{ax/W} \\ \hline \\ \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_3\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_6} \rightarrow_L \quad \frac{\mathbf{h}_7:\Delta_9,\mathbf{F}_6\vdash\mathbf{F}_8}{\bullet\mathbf{h}_7:(\top,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8} \quad \mathsf{T}_L \\ \hline \\ \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet} \rightarrow_L \quad \frac{\bullet\mathbf{h}_7:\Delta_9,\mathbf{F}_6\vdash\mathbf{F}_8}{\bullet} \quad \mathsf{Cut} \\ \hline \\ -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4),\top,\Delta_9\vdash\mathbf{F}_8 \\ \hline \\ \bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_6 \quad \mathsf{ax/W} \\ \hline \\ -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_8) \quad \mathsf{ax/W} \\ \hline \\ -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_8) \quad \mathsf{ax/W} \\ \hline \\ -:(\Delta_2,\mathbf{F}_3\rightarrow\mathbf{F}_4\vdash\mathbf{F}_8) \quad \mathsf{ax/W} \\ \hline \\ \bullet\mathbf{h}_7:\top,\Delta_9,\mathbf{F}_6\vdash\mathbf{F}_8 \\ \mathsf{hCut} \\ \hline \end{array}$$

## 7.7 Status of $\wedge_L$ : OK

• Case rule  $\top_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \end{array} \land_L \quad \frac{}{\bullet \mathbf{h}_8: \Delta_6, \mathbf{F}_7 \vdash \top} \quad \begin{array}{c} \top_R \\ \mathsf{Cut} \\ \hline \\ -: (\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_6 \vdash \top \\ \hline \\ -: \Delta_2, \Delta_6, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \top \end{array} } \quad \top_R$$

• Case rule  $\rightarrow_R$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \end{array} \land_L \begin{array}{c} \mathbf{h}_8: \Delta_6, \mathbf{F}_7, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \end{array} }{ -: (\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_6 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} } \begin{array}{c} \rightarrow_R \\ \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_8: \Delta_6, \mathbf{F}_7, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{h}_8: \Delta_6, \mathbf{F}_7, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_6, \mathbf{F}_9, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_6, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_9 \to \mathbf{F}_{10} \end{array} \begin{array}{c} \mathbf{ax/W} \\ \bullet \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\wedge\mathbf{F}_4\vdash\mathbf{F}_7} \ \wedge_L \ \frac{\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9 \ \mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}} \ \\ \frac{-:(\Delta_2,\mathbf{F}_3\wedge\mathbf{F}_4),\Delta_6\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}} \ \\ \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}} \ \\ \frac{-:\Delta_2,\Delta_6,\mathbf{F}_3,\mathbf{F}_4\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}}{-:\Delta_2,\Delta_6,\mathbf{F}_3\wedge\mathbf{F}_4\vdash\mathbf{F}_9\wedge\mathbf{F}_{10}} \ \wedge_L \end{array} \begin{array}{c} \mathbf{Ax/W} \\ \bullet\mathbf{Cut} \end{array}$$

• Case rule  $\vee_1$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, F_3, F_4 \vdash F_7 \\ \bullet \mathbf{h}_1: \Delta_2, F_3 \land F_4 \vdash F_7 \end{array} \land_L \quad \begin{array}{c} \mathbf{h}_8: \Delta_6, F_7 \vdash F_9 \\ \bullet \mathbf{h}_8: \Delta_6, F_7 \vdash F_9 \lor F_{10} \end{array} \\ \hline -: (\Delta_2, F_3 \land F_4), \Delta_6 \vdash F_9 \lor F_{10} \\ \hline \bullet \mathbf{h}_1: \Delta_2, F_3 \land F_4 \vdash F_7 \\ \hline -: \Delta_2, \Delta_6, F_3 \land F_4 \vdash F_9 \\ \hline -: \Delta_2, \Delta_6, F_3 \land F_4 \vdash F_9 \\ \hline -: \Delta_2, \Delta_6, F_3 \land F_4 \vdash F_9 \lor F_{10} \end{array} } \begin{array}{c} \mathsf{Ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1: \Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_7 \\ \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \end{array} \land_L \quad \begin{array}{c} \mathbf{h}_8: \Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_{10} \\ \bullet \mathbf{h}_8: \Delta_6, \mathbf{F}_7 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \end{array} \\ \hline -: (\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_6 \vdash \mathbf{F}_9 \lor \mathbf{F}_{10} \\ \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \\ \hline \bullet \mathbf{h}_1: \Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_7 \\ \hline -: \Delta_2, \Delta_6, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_6, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline -: \Delta_2, \Delta_6, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline \end{array} \lor_2 \end{array} \quad \begin{array}{c} \mathsf{dx/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\wedge\mathbf{F}_{4}\vdash\mathbf{F}_{6}} \wedge_{L} & \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{8} & \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{9}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}),\mathbf{F}_{6}\vdash\mathbf{F}_{10}} & \mathbf{Cut} \\ & -:(\Delta_{2},\mathbf{F}_{3}\wedge\mathbf{F}_{4}),\Delta_{11},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10} & \\ & \frac{-}{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{6}} & \mathbf{ax}/W & \frac{-}{\bullet\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \mathbf{ax}/W \\ & \frac{-}{-:\Delta_{11},\Delta_{2},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \wedge_{L} \\ & \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{8}\to\mathbf{F}_{9}}{-:\Delta_{11},\Delta_{2},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \wedge_{L} \\ & \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{8}\to\mathbf{F}_{9}}{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\wedge\mathbf{F}_{4}\vdash\mathbf{F}_{8}\to\mathbf{F}_{9}} & \wedge_{L} & \frac{\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \mathbf{Cut} \\ & -:(\Delta_{2},\mathbf{F}_{3}\wedge\mathbf{F}_{4}),\Delta_{7}\vdash\mathbf{F}_{10} & \\ & \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{8}\to\mathbf{F}_{9}}{\bullet\mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\to\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \mathbf{ax}/W \\ & \frac{-:\Delta_{2},\Delta_{7},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{10}}{-:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\wedge\mathbf{F}_{4}\vdash\mathbf{F}_{10}} & \wedge_{L} \\ & & \mathbf{h}_{C}\mathbf{u}\mathbf{t} \\ & \frac{-:\Delta_{2},\Delta_{7},\mathbf{F}_{3},\mathbf{F}_{4}\vdash\mathbf{F}_{10}}{-:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\wedge\mathbf{F}_{4}\vdash\mathbf{F}_{10}} & \wedge_{L} \\ \end{array}$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_6} & \wedge_L & \frac{\mathbf{h}_7:\Delta_{11}, \mathbf{F}_6, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_7:(\Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9), \mathbf{F}_6 \vdash \mathbf{F}_{10}} & \wedge_L \\ \hline & -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_{11}, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \rightarrow \\ \hline \frac{\rightarrow}{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_6} & \mathbf{ax/W} & \bullet \\ \hline & -:\Delta_{11}, \Delta_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \wedge_L \\ \hline -:\Delta_{11}, \Delta_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \wedge_L \\ \hline \bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_8 \land \mathbf{F}_9} & \wedge_L & \frac{\mathbf{h}_6:\Delta_7, \mathbf{F}_8, \mathbf{F}_9 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_6:\Delta_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \wedge_L \\ \hline \bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_8 \land \mathbf{F}_9} & \wedge_L & \frac{\mathbf{h}_6:\Delta_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_6:\Delta_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} & \wedge_L \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_7 \vdash \mathbf{F}_{10} & \bullet_{\mathbf{h}_6:\Delta_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} \\ \hline \bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_8 \land \mathbf{F}_9} & \mathbf{ax/W} \\ \hline \bullet \mathbf{h}_6:\Delta_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10} & \bullet_{\mathbf{h}_6:\Delta_7, \mathbf{F}_8 \land \mathbf{F}_9 \vdash \mathbf{F}_{10}} \\ \hline -:\Delta_2,\Delta_7, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_{10} & \wedge_L \\ \hline -:\Delta_2,\Delta_7, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_{10} & \wedge_L \\ \hline \end{array}$$

• Case rule  $\vee_L$ 

$$\begin{array}{c} \frac{h_1:\Delta_2,F_3,F_4 \vdash F_6}{\bullet h_1:\Delta_2,F_3 \land F_4 \vdash F_6} \land_L & \frac{h_7:\Delta_{11},F_6,F_8 \vdash F_{10} \quad h_7:\Delta_{11},F_6,F_9 \vdash F_{10}}{\bullet h_7:(\Delta_{11},F_8 \lor F_9),F_6 \vdash F_{10}} & \text{Cut} \\ \hline \\ \frac{-:(\Delta_2,F_3 \land F_4) \land_{11},F_8 \lor F_9 \vdash F_{10}}{\bullet h_7:\Delta_{11},F_6,F_8 \lor F_9 \vdash F_{10}} & \frac{\text{ax/W}}{\bullet h_7:\Delta_{11},F_6,F_8 \lor F_9 \vdash F_{10}} \\ \hline \\ \frac{-:\Delta_{11},\Delta_2,F_3,F_4 \vdash F_6}{-:\Delta_{11},\Delta_2,F_3,F_4,F_8 \lor F_9 \vdash F_{10}} & \wedge_L \\ \hline \\ \frac{h_1:\Delta_2,F_3,F_4 \vdash F_8 \lor F_9}{-:\Delta_{11},\Delta_2,F_3 \land F_4,F_8 \lor F_9 \vdash F_{10}} & \wedge_L \\ \hline \\ \frac{\bullet h_1:\Delta_2,F_3,F_4 \vdash F_8 \lor F_9}{\bullet h_1:\Delta_2,F_3 \land F_4 \vdash F_8 \lor F_9} & \wedge_L & \frac{h_6:\Delta_7,F_8 \vdash F_{10} \quad h_6:\Delta_7,F_9 \vdash F_{10}}{\bullet h_6:\Delta_7,F_8 \lor F_9 \vdash F_{10}} & \text{Cut} \\ \hline \\ \frac{-:(\Delta_2,F_3,F_4 \vdash F_8 \lor F_9) \quad \text{ax/W}}{\bullet h_6:\Delta_7,F_8 \lor F_9 \vdash F_{10}} & \text{ax/W} \\ \hline \\ \frac{-:\Delta_2,\Delta_7,F_3,F_4 \vdash F_{10}}{-:\Delta_2,\Delta_7,F_3,F_4 \vdash F_{10}} & \wedge_L \\ \hline \end{array}$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \bot}{-\mathbf{e}\mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \bot} \land_L & \frac{\bullet}{\mathbf{h}_6:\Delta_7, \bot \vdash \mathbf{F}_8} & \overset{\bot_L}{\mathsf{Cut}} \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_7 \vdash \mathbf{F}_8 & \overset{}{\rightarrow} \\ \hline \frac{}{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \bot} & \mathsf{ax/W} & \frac{}{\bullet}\mathbf{h}_6:\bot, \Delta_7 \vdash \mathbf{F}_8 & \overset{\bot_L}{\mathsf{h}_{\mathsf{Cut}}} \\ \hline \frac{-:\Delta_2, \Delta_7, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_8}{-:\Delta_2, \Delta_7, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_8} & \wedge_L \\ \hline \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_6}{-:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_6} & \wedge_L & \frac{}{\bullet}\mathbf{h}_7:(\bot, \Delta_9), \mathbf{F}_6 \vdash \mathbf{F}_8} \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \bot, \Delta_9 \vdash \mathbf{F}_8 & & \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \bot, \Delta_9 \vdash \mathbf{F}_8 & & \\ \hline -:\bot, \Delta_2, \Delta_9, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_8 & & \bot_L \\ \hline \end{array}$$

ullet Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_6} \ \land L & \frac{\bullet \mathbf{h}_7:(\Delta_9, \mathbf{p}_8), \mathbf{F}_6 \vdash \mathbf{p}_8}{\bullet \mathbf{h}_7:(\Delta_9, \mathbf{p}_8), \mathbf{F}_6 \vdash \mathbf{p}_8} & \mathbf{Cut} \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_9, \mathbf{p}_8 \vdash \mathbf{p}_8 & \\ \hline -:\Delta_2, \Delta_9, \mathbf{p}_8, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{p}_8 & I \\ \hline \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{p}_8}{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{p}_8} & \wedge L & \frac{\bullet \mathbf{h}_6:\Delta_7, \mathbf{p}_8 \vdash \mathbf{p}_8}{\bullet \mathbf{h}_6:\Delta_7, \mathbf{p}_8 \vdash \mathbf{p}_8} & \mathbf{Cut} \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_7 \vdash \mathbf{p}_8 & \\ \hline \rightarrow & -:\Delta_2, \Delta_7, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{p}_8 & \mathbf{ax/W} \end{array}$$

• Case rule  $\top_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \top}{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \top} & \wedge_L & \frac{\mathbf{h}_6:\Delta_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6:\Delta_7, \top \vdash \mathbf{F}_8} & \top_L \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \Delta_7 \vdash \mathbf{F}_8 & \\ \hline -:\Delta_2, \Delta_7, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \\ \frac{\mathbf{h}_1:\Delta_2, \mathbf{F}_3, \mathbf{F}_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_6} & \wedge_L & \frac{\mathbf{h}_7:\Delta_9, \mathbf{F}_6 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_7:(\top,\Delta_9), \mathbf{F}_6 \vdash \mathbf{F}_8} & \nabla_L \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4), \top, \Delta_9 \vdash \mathbf{F}_8 & \\ \hline \rightarrow \mathbf{ax/W} & \\ \hline -:(\Delta_2, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_6) & \mathbf{ax/W} \\ \hline -:\top, \Delta_2, \Delta_9, \mathbf{F}_3 \land \mathbf{F}_4 \vdash \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \end{pmatrix} \\ \mathbf{hOut} \end{array}$$

## 7.8 Status of $\vee_L$ : OK

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_7\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7} \vee_L & \frac{\mathbf{h}_8:\Delta_6,\mathbf{F}_7,\mathbf{F}_9\vdash\mathbf{F}_{10}}{\bullet \mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\to\mathbf{F}_{10}} \\ \hline \\ \frac{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_7}{-:(\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4),\Delta_6\vdash\mathbf{F}_9\to\mathbf{F}_{10}} & \rightarrow_R \\ \hline \\ \frac{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet \mathbf{k}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_7} & \frac{\mathbf{ax}/\mathbf{W}}{\bullet \mathbf{h}_8:\Delta_6,\mathbf{F}_7,\mathbf{F}_9\vdash\mathbf{F}_{10}} \\ \hline \\ \frac{-:\Delta_2,\Delta_6,\mathbf{F}_9,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_{10}}{-:\Delta_2,\Delta_6,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_{10}} \to_R \\ \hline \end{array} \begin{array}{c} \mathbf{ax}/\mathbf{W} \\ \mathbf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash\mathbf{F}_{7}\quad\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{7}}{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{7}}}{\circ}\vee_{L} \quad \frac{\mathbf{h}_{8}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{F}_{9}\quad\mathbf{h}_{8}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{F}_{10}}{\bullet\mathbf{h}_{8}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{F}_{9}\wedge\mathbf{F}_{10}}}{\circ}\wedge_{R} \\ \frac{-:(\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{7}}{\bullet\mathbf{h}_{8}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{F}_{9}} \quad \frac{\mathbf{ax}/\mathbf{W}}{\mathbf{h}_{8}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{F}_{9}} \quad \frac{\mathbf{ax}/\mathbf{W}}{\bullet} \\ \frac{-:\Delta_{2},\Delta_{6},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{9}}{\bullet} \quad \frac{\mathbf{ax}/\mathbf{W}}{\bullet} \quad \frac{\bullet\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{7}} \quad \mathbf{ax}/\mathbf{W}}{\bullet} \quad \frac{\mathbf{h}_{8}:\Delta_{6},\mathbf{F}_{7}\vdash\mathbf{F}_{10}}{\bullet} \quad \wedge_{R}} \\ \bullet \wedge_{Cut} \quad \frac{-:\Delta_{2},\Delta_{6},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{9}}{\bullet} \quad \wedge_{R} \quad \bullet \wedge_{Cut} \quad \frac{\bullet}{\bullet} \quad \nabla_{\mathbf{F}_{1}} \quad \nabla_{\mathbf{F}_{1}} \quad \nabla_{\mathbf{F}_{2}} \quad \nabla_{\mathbf{F}_{3}} \quad \nabla_{\mathbf{F}_{4}} \quad \nabla_{\mathbf{F}$$

• Case rule  $\vee_1$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_7\quad\mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_7}{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_7} \vee_L & \frac{\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9}{\bullet\mathbf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9\vee\mathbf{F}_{10}} & \nabla_L\\ \hline \\ \frac{\bullet\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_7}{-:(\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_7} & \frac{\mathsf{ax/W}}{\bullet\mathbf{x}/\mathsf{W}} & \frac{\mathsf{ax/W}}{\mathsf{h}_8:\Delta_6,\mathbf{F}_7\vdash\mathbf{F}_9} & \frac{\mathsf{ax/W}}{\mathsf{hCut}}\\ \hline \\ \frac{-:\Delta_2,\Delta_6,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_9}{-:\Delta_2,\Delta_6,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_9\vee\mathbf{F}_{10}} & \vee_1 \end{array}$$

• Case rule  $\vee_2$ 

$$\frac{\begin{array}{l} \mathbf{h}_1:\Delta_2, \mathbf{F}_3\vdash \mathbf{F}_7 \quad \mathbf{h}_1:\Delta_2, \mathbf{F}_4\vdash \mathbf{F}_7 \\ \\ \underline{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_7} \\ -:(\Delta_2, \mathbf{F}_3\lor \mathbf{F}_4), \Delta_6\vdash \mathbf{F}_9\lor \mathbf{F}_{10} \\ \\ \underline{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_7} \\ \\ \underline{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_7} \\ \underline{\bullet \mathbf{h}_1:\Delta_2, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_7} \\ \underline{-:\Delta_2,\Delta_6, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_{10}} \\ -:\Delta_2,\Delta_6, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_{10} \\ \hline -:\Delta_2,\Delta_6, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_{10} \\ \hline -:\Delta_2,\Delta_6, \mathbf{F}_3\lor \mathbf{F}_4\vdash \mathbf{F}_{10} \\ \hline \end{array} \begin{array}{c} \mathbf{h}_8:\Delta_6, \mathbf{F}_7\vdash \mathbf{F}_{10} \\ \mathbf{h}_8:\Delta_6, \mathbf{F}_7\vdash \mathbf{F}_{10} \\ \mathbf{h}_{Cut} \\ \end{array} \begin{array}{c} \mathbf{ax/W} \\ \mathbf{h}_{Cut} \\ \end{array}$$

• Case rule  $\rightarrow_L$ 

$$\frac{ \begin{array}{c} \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \vdash \mathbf{F}_6 \quad \mathbf{h}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_6 \end{array} \vee_L \quad \frac{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_6, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_8 \quad \mathbf{h}_7 : \Delta_{11}, \mathbf{F}_6, \mathbf{F}_9 \vdash \mathbf{F}_{10} \\ \hline \bullet \mathbf{h}_7 : (\Delta_{11}, \mathbf{F}_8 \to \mathbf{F}_9), \mathbf{F}_6 \vdash \mathbf{F}_{10} \end{array} } \rightarrow_L \\ \hline \bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_6 \quad \mathbf{ax} / \mathbf{w} \quad \frac{\rightarrow}{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_6, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_8} \quad \mathbf{ax} / \mathbf{w} \\ \hline \bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_6 \quad \mathbf{ax} / \mathbf{w} \quad \frac{\rightarrow}{\mathbf{h}_7 : \Delta_{11}, \mathbf{F}_6, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_8} \quad \mathbf{ax} / \mathbf{w} \\ \hline - : \Delta_{11}, \Delta_2, \mathbf{F}_8 \to \mathbf{F}_9, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_8} \quad \mathbf{ax} / \mathbf{w} \quad \mathbf{h}_{11} : \Delta_2, \mathbf{F}_8 \to \mathbf{F}_9, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_{10}} \\ \hline - : \Delta_{11}, \Delta_2, \mathbf{F}_8 \to \mathbf{F}_9, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_8 \to \mathbf{F}_9} \quad \mathbf{v}_L \quad \mathbf{h}_6 : \Delta_7, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_8 \quad \mathbf{h}_6 : \Delta_7, \mathbf{F}_9 \vdash \mathbf{F}_{10}} \\ \hline \bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_8 \to \mathbf{F}_9} \quad \mathbf{v}_L \quad \mathbf{h}_6 : \Delta_7, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_8 \quad \mathbf{h}_6 : \Delta_7, \mathbf{F}_9 \vdash \mathbf{F}_{10}} \\ \hline \bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_8 \to \mathbf{F}_9} \quad \mathbf{v}_L \quad \mathbf{h}_6 : \Delta_7, \mathbf{F}_8 \to \mathbf{F}_9 \vdash \mathbf{F}_{10}} \quad \mathbf{cut} \\ \hline - : (\Delta_2, \mathbf{F}_3 \lor \mathbf{F}_4), \Delta_7 \vdash \mathbf{F}_{10}} \quad \mathbf{h}_{11} : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_8 \to \mathbf{F}_9} \quad \mathbf{ax} / \mathbf{w} \\ \hline \bullet \mathbf{h}_1 : \Delta_2, \mathbf{F}_3 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_8 \to \mathbf{F}_9} \quad \mathbf{ax} / \mathbf{w} \\ \hline - : \Delta_2, \Delta_7, \mathbf{F}_3 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_2, \Delta_7, \mathbf{F}_3 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_2, \Delta_7, \mathbf{F}_3 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_2, \Delta_7, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_2, \Delta_7, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{F}_4 \vdash \mathbf{F}_{10} \\ \hline - : \Delta_2, \Delta_7, \mathbf{F}_3 \lor \mathbf{F}_4 \vdash \mathbf{F}_{10} \quad \mathbf{u}_1 : \Delta_2, \mathbf{u}_1 \vdash \mathbf{u}_1 : \Delta_2, \mathbf{u}_2 \vdash \mathbf{u}_1 : \Delta_2, \mathbf{u}_2 \vdash \mathbf{u}_1 : \Delta_2, \mathbf{u}_1 \vdash \mathbf{u}_2 \vdash \mathbf{u}_1 : \Delta_2, \mathbf{u}_$$

• Case rule  $\wedge_L$ 

$$\frac{ \begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash\mathbf{F}_{6}\quad \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{6}}{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{6}} & \vee_{L} & \frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{10}}{\bullet \mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\wedge\mathbf{F}_{9}),\mathbf{F}_{6}\vdash\mathbf{F}_{10}} & \wedge_{L} \\ \hline & -:(\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}),\Delta_{11},\mathbf{F}_{8}\wedge\mathbf{F}_{9}\vdash\mathbf{F}_{10}} \\ \hline & \frac{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{6}}{\bullet \mathbf{m}_{2}} & \frac{\mathbf{m}_{2}}{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8},\mathbf{F}_{9}\vdash\mathbf{F}_{10}} & \mathbf{m}_{2}$$

### • Case rule $\vee_L$

$$\frac{\frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{6}\quad \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash \mathbf{F}_{6}}{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\lor \mathbf{F}_{4}\vdash \mathbf{F}_{6}}}{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\lor \mathbf{F}_{4}\vdash \mathbf{F}_{6}}} \vee_{L} \frac{\frac{\mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\vdash \mathbf{F}_{10}\quad \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{9}\vdash \mathbf{F}_{10}}{\bullet \mathbf{h}_{7}:(\Delta_{11},\mathbf{F}_{8}\lor \mathbf{F}_{9}),\mathbf{F}_{6}\vdash \mathbf{F}_{10}}} \underbrace{\mathbf{Cut}} \vee_{L} \\ \frac{-:(\Delta_{2},\mathbf{F}_{3}\lor \mathbf{F}_{6})}{\bullet \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}} \underbrace{\mathbf{ax}/\mathbf{W}}_{\bullet \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}}^{\bullet \mathbf{ax}/\mathbf{W}} \underbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash \mathbf{F}_{6}}_{\bullet \mathbf{L}u} \underbrace{\mathbf{ax}/\mathbf{W}}_{\bullet \mathbf{h}_{7}:\Delta_{11},\mathbf{F}_{6},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}}^{\bullet \mathbf{xx}/\mathbf{W}} \underbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash \mathbf{F}_{6}}_{\bullet \mathbf{L}u} \vee_{L} \underbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash \mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{L}u} \vee_{L} \underbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}\quad \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{9}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{9}\vdash \mathbf{F}_{10}} \vee_{L} \underbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}\quad \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{9}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}} \vee_{L} \underbrace{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}\quad \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}} \underbrace{\mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}} \underbrace{\mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{6}:\Delta_{7},\mathbf{F}_{8}\lor \mathbf{F}_{9}\vdash \mathbf{F}_{10}} \underbrace{\mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}} \underbrace{\mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}}_{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}}}_{\bullet \mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{F}_{10}} \underbrace{\mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}:\Delta_{2},\mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_{1}\vdash \mathbf{L}_$$

### • Case rule $\perp_L$

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\bot \quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\bot}{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\bot} \quad \vee_L \quad \frac{\bullet \mathbf{h}_6:\Delta_7,\bot\vdash\mathbf{F}_8}{\bullet \mathbf{h}_6:\Delta_7,\bot\vdash\mathbf{F}_8} \quad \frac{\bot_L}{\mathsf{Cut}} \\ \\ \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\bot}{\bullet \mathbf{h}_6:\bot,\Delta_7\vdash\mathbf{F}_8} \quad \frac{\mathsf{ax/W}}{\bullet \mathbf{h}_6:\bot,\Delta_7\vdash\mathbf{F}_8} \quad \frac{\mathsf{ax/W}}{\mathsf{h}_{\mathsf{Cut}}} \quad \frac{\mathsf{ax/W}}{\bullet \mathsf{h}_6:\bot,\Delta_7\vdash\mathbf{F}_8} \quad \frac{\bot_L}{\bullet \mathsf{h}_1:\Delta_2,\mathbf{F}_4\vdash\bot} \quad \frac{\mathsf{ax/W}}{\bullet \mathbf{h}_6:\bot,\Delta_7\vdash\mathbf{F}_8} \quad \frac{\bot_L}{\bullet \mathsf{h}_1:\Delta_2,\Delta_7,\mathbf{F}_3\vdash\mathbf{F}_8} \quad \vee_L \\ \\ \frac{-:\Delta_2,\Delta_7,\mathbf{F}_3\vdash\mathbf{F}_8}{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_6} \quad \mathsf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6} \quad \vee_L \quad \frac{\bullet_{\mathsf{h}_7:(\bot,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8}}{\bullet \mathsf{h}_7:(\bot,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8} \quad \frac{\bot_L}{\mathsf{Cut}} \\ \\ \frac{-:(\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet} \quad \vee_L \quad \frac{\bullet}{\bullet \mathsf{h}_7:(\bot,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8} \quad \bot_L}{\bullet \mathsf{h}_2,\Delta_2,\Delta_9,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_8} \quad \bot_L \\ \end{array}$$

## $\bullet\,$ Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash\mathbf{F}_{6}\quad\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{F}_{6}}{\bullet}\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{F}_{6}} \quad\forall_{L} \quad & \frac{\bullet}{\bullet}\mathbf{h}_{7}:(\Delta_{9},\mathbf{p}_{8}),\mathbf{F}_{6}\vdash\mathbf{p}_{8}}{\bullet} \quad & \mathbf{Cut} \\ \hline & -:(\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}),\Delta_{9},\mathbf{p}_{8}\vdash\mathbf{p}_{8}} \\ & -:(\Delta_{2},\Delta_{9},\mathbf{p}_{8},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{p}_{8}} \quad & I \\ \hline & \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vdash\mathbf{p}_{8}\quad\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{4}\vdash\mathbf{p}_{8}}{\bullet} \quad & \mathbf{h}_{6}:\Delta_{7},\mathbf{p}_{8}\vdash\mathbf{p}_{8}} \quad & I \\ \hline & \frac{\bullet}{\bullet}\mathbf{h}_{1}:\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{p}_{8}} \\ \hline & -:(\Delta_{2},\mathbf{F}_{3}\vee\mathbf{F}_{4}),\Delta_{7}\vdash\mathbf{p}_{8}} \quad & \mathbf{Cut} \\ \hline & -:\Delta_{2},\Delta_{7},\mathbf{F}_{3}\vee\mathbf{F}_{4}\vdash\mathbf{p}_{8}} \quad & \mathbf{ax/W} \\ \hline \end{array}$$

• Case rule  $\top_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\top \quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\top}{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\top} \vee_L & \frac{\mathbf{h}_6:\Delta_7\vdash\mathbf{F}_8}{\bullet \mathbf{h}_6:\Delta_7,\top\vdash\mathbf{F}_8} & \top_L \\ \hline -:(\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4),\Delta_7\vdash\mathbf{F}_8 & \\ \hline -:\Delta_2,\Delta_7,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_8 & \mathbf{ax/W} \\ \hline \\ \frac{\mathbf{h}_1:\Delta_2,\mathbf{F}_3\vdash\mathbf{F}_6 \quad \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_4\vdash\mathbf{F}_6} & \vee_L & \frac{\mathbf{h}_7:\Delta_9,\mathbf{F}_6\vdash\mathbf{F}_8}{\bullet \mathbf{h}_7:(\top,\Delta_9),\mathbf{F}_6\vdash\mathbf{F}_8} & \top_L \\ \hline \\ \frac{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_6}{-:(\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4),\top,\Delta_9\vdash\mathbf{F}_8} & \frac{\mathbf{T}_L}{\bullet} \\ \hline \\ \frac{\bullet \mathbf{h}_1:\Delta_2,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_6}{\bullet} & \frac{\mathbf{Ax/W}}{\bullet} & \frac{\mathbf{Ax/W}}{\bullet} \\ \hline \\ -:\top,\Delta_2,\Delta_9,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_8 & \frac{\mathbf{Ax/W}}{\bullet} \\ \hline \\ -:\top,\Delta_2,\Delta_9,\mathbf{F}_3\vee\mathbf{F}_4\vdash\mathbf{F}_8 & \frac{\mathbf{Ax/W}}{\bullet} \\ \hline \end{array}$$

# 7.9 Status of $\perp_L$ : OK

• Case rule  $\top_R$ 

$$\begin{array}{c|c} \hline \bullet_{\mathbf{h}_1}: \bot, \Delta_2 \vdash \mathsf{F}_5 & \bot_L & \hline \bullet_{\mathbf{h}_6}: \Delta_4, \mathsf{F}_5 \vdash \top \\ \hline -: (\bot, \Delta_2), \Delta_4 \vdash \top & \mathsf{Cut} \\ \hline & \hline -: \bot, \Delta_2, \Delta_4 \vdash \top & \top_R \end{array}$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c|c} \underline{\bullet \mathbf{h}_1: \bot, \Delta_2 \vdash \mathbf{F}_5} & \bot_L & \frac{\mathbf{h}_6: \Delta_4, \mathbf{F}_5, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} & \to_R \\ \hline -: (\bot, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8 & \to \\ \hline -: \bot, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8 & \bot_L \end{array}$$

• Case rule  $\wedge_R$ 

$$\begin{array}{c|c} \underline{\bullet \mathbf{h}_1: \bot, \Delta_2 \vdash \mathbf{F}_5} & \bot_L & \underline{\mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \quad \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8} \\ -: (\bot, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 & \\ \hline & -: (\bot, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 & \bot_L \end{array} \quad \begin{array}{c} \wedge_R \\ \\ \hline \\ -: \bot, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 & \bot_L \end{array}$$

• Case rule  $\vee_1$ 

$$\begin{array}{c|c} \frac{\bullet \mathbf{h}_1: \bot, \Delta_2 \vdash \mathbf{F}_5}{\bullet} & \bot_L & \frac{\mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \hline -: (\bot, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \hline -: \bot, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 & \bot_L \end{array} \quad \begin{array}{c} \vee_1 \\ \mathrm{Cut} \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c|c} \underline{\bullet \mathbf{h}_1 : \bot, \Delta_2 \vdash \mathbf{F}_5} & \bot_L & \frac{\mathbf{h}_6 : \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6 : \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} & \bigvee_2 \\ \hline - : (\bot, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ & \xrightarrow{} \\ \hline - : \bot, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 & \bot_L \end{array} \quad \mathbf{Cut}$$

• Case rule  $\rightarrow_L$ 

$$\begin{array}{c} \underbrace{ \begin{array}{c} \bullet \mathbf{h}_1 : \bot, \Delta_2 \vdash \mathbf{F}_4 \\ \bullet \mathbf{h}_5 : \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_6 & \mathbf{h}_5 : \Delta_9, \mathbf{F}_4, \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_5 : (\Delta_9, \mathbf{F}_6 \to \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8 \\ & - : (\bot, \Delta_2), \Delta_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ & - \vdots \bot, \Delta_2, \Delta_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array}} \begin{array}{c} \mathsf{Cut} \\ \bullet \mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \bot_L \\ \bullet \mathbf{h}_4 : \bot, \Delta_2 \vdash \mathbf{F}_6 \to \mathbf{F}_7 \\ \hline - : (\bot, \Delta_2), \Delta_5 \vdash \mathbf{F}_8 \\ & - \vdots \bot, \Delta_2, \Delta_5 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathsf{Cut} \\ \bullet \mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline - : (\bot, \Delta_2), \Delta_5 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \bot_L \\ \bullet \mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline - : \bot, \Delta_2, \Delta_5 \vdash \mathbf{F}_8 \end{array} \begin{array}{c} \mathsf{L}_L \end{array}$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c|c} \bullet_{\mathbf{h}_1}: \bot, \Delta_2 \vdash \mathbf{F}_4 & \bot_L & \frac{\mathbf{h}_5: \Delta_9, \mathbf{F}_4, \mathbf{F}_6, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5: (\Delta_9, \mathbf{F}_6 \land \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8} & \land_L \\ \hline -: (\bot, \Delta_2), \Delta_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8 & \to \\ \hline -: \bot, \Delta_2, \Delta_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8 & \bot_L \\ \hline \\ \bullet_{\mathbf{h}_1}: \bot, \Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7 & \bot_L & \frac{\mathbf{h}_4: \Delta_5, \mathbf{F}_6, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_4: \Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} & \land_L \\ \hline -: (\bot, \Delta_2), \Delta_5 \vdash \mathbf{F}_8 & \to \\ \hline -: \bot, \Delta_2, \Delta_5 \vdash \mathbf{F}_8 & \bot_L \\ \hline \end{array}$$

• Case rule  $\vee_L$ 

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \hline \bullet \mathbf{h}_1 : \bot, \Delta_2 \vdash \bot & \bullet \mathbf{h}_4 : \Delta_5, \bot \vdash \mathbf{F}_6 \\ \hline - : (\bot, \Delta_2), \Delta_5 \vdash \mathbf{F}_6 \\ \hline & - : \bot, \Delta_2, \Delta_5 \vdash \mathbf{F}_6 \\ \hline - : \bot, \Delta_2, \Delta_5 \vdash \mathbf{F}_6 \\ \hline \bullet \mathbf{h}_1 : \bot, \Delta_2 \vdash \mathbf{F}_4 & \bot_L & \bullet \mathbf{h}_5 : (\bot, \Delta_7), \mathbf{F}_4 \vdash \mathbf{F}_6 \\ \hline - : (\bot, \Delta_2), \bot, \Delta_7 \vdash \mathbf{F}_6 \\ \hline & - : \bot, \bot, \Delta_2, \Delta_7 \vdash \mathbf{F}_6 \\ \hline - : \bot, \bot, \Delta_2, \Delta_7 \vdash \mathbf{F}_6 \\ \hline \end{array}$$

 $\bullet\,$  Case rule I

$$\begin{array}{c|c} \hline { \bullet \mathbf{h}_1 : \bot, \Delta_2 \vdash \mathbf{F}_4} & \bot_L & \hline { \bullet \mathbf{h}_5 : (\Delta_7, \mathbf{p}_6), \mathbf{F}_4 \vdash \mathbf{p}_6} \\ \hline { - : (\bot, \Delta_2), \Delta_7, \mathbf{p}_6 \vdash \mathbf{p}_6} \\ \hline { - : \bot, \Delta_2, \Delta_7, \mathbf{p}_6 \vdash \mathbf{p}_6} & \bot_L \\ \hline \end{array}$$
 Cut

$$\begin{array}{c|c} \hline { \bullet \mathbf{h}_1 : \bot, \Delta_2 \vdash \mathbf{p}_6} \ \bot_L & \hline { \bullet \mathbf{h}_4 : \Delta_5, \mathbf{p}_6 \vdash \mathbf{p}_6 } \\ \hline { - : (\bot, \Delta_2), \Delta_5 \vdash \mathbf{p}_6} \\ \hline { \rightarrow \\ \hline { - : \bot, \Delta_2, \Delta_5 \vdash \mathbf{p}_6 } \ \bot_L } \end{array} \ \mathbf{Cut}$$

• Case rule  $\top_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_{1}: \bot, \Delta_{2} \vdash \top}{-: (\bot, \Delta_{2}), \Delta_{5} \vdash F_{6}} & \top_{L} \\ \hline -: (\bot, \Delta_{2}), \Delta_{5} \vdash F_{6} & \mathbf{cut} \\ \hline & -: (\bot, \Delta_{2}), \Delta_{5} \vdash F_{6} \\ \hline & -: \bot, \Delta_{2}, \Delta_{5} \vdash F_{6} & \bot_{L} \\ \hline \\ \frac{\bullet \mathbf{h}_{1}: \bot, \Delta_{2} \vdash F_{4}}{-: \bot, \Delta_{2} \vdash F_{4}} & \bot_{L} & \frac{\mathbf{h}_{5}: \Delta_{7}, F_{4} \vdash F_{6}}{\bullet \mathbf{h}_{5}: (\top, \Delta_{7}), F_{4} \vdash F_{6}} & \top_{L} \\ \hline & -: (\bot, \Delta_{2}), \top, \Delta_{7} \vdash F_{6} \\ \hline & -: \bot, \top, \Delta_{2}, \Delta_{7} \vdash F_{6} & \bot_{L} \\ \hline \end{array}$$

## 7.10 Status of I: OK

• Case rule  $\top_R$ 

• Case rule  $\rightarrow_R$ 

$$\frac{\overbrace{\bullet \mathbf{h}_1 : \Delta_2, \mathbf{p}_5 \vdash \mathbf{p}_5}^{\bullet \mathbf{h}_1 : \Delta_2, \mathbf{p}_5 \vdash \mathbf{p}_5} I \quad \frac{\mathbf{h}_6 : \Delta_4, \mathbf{p}_7, \mathbf{p}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6 : \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \quad \underbrace{-: (\Delta_2, \mathbf{p}_5), \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8}_{-: \Delta_2, \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \quad \text{ax/W}}_{\bullet} \quad Cut$$

• Case rule  $\wedge_R$ 

$$\frac{ \underbrace{ \begin{array}{c} \bullet \mathbf{h}_1 : \Delta_2, \mathbf{p}_5 \vdash \mathbf{p}_5 \\ - : (\Delta_2, \mathbf{p}_5), \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ - : (\Delta_2, \mathbf{p}_5), \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ - : \Delta_2, \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \end{array}}_{\mathbf{d} \times \mathbb{A}/\mathbb{W}} \ \mathbf{Cut} \\ \\ \frac{\bullet \mathbf{h}_1 : \Delta_2, \mathbf{p}_5 \vdash \mathbf{p}_5}{- : \Delta_2, \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8} \ \mathbf{ax/W} \\ \end{array}$$

• Case rule  $\vee_1$ 

$$\frac{ \underbrace{ \begin{array}{c} \bullet_{h_1} : \Delta_2, p_5 \vdash p_5 \end{array}}_{} I \quad \frac{h_6 : \Delta_4, p_5 \vdash F_7}{\bullet h_6 : \Delta_4, p_5 \vdash F_7 \vee F_8} \quad \begin{array}{c} \vee_1 \\ - : (\Delta_2, p_5), \Delta_4 \vdash F_7 \vee F_8 \\ \longrightarrow \\ \hline - : \Delta_2, \Delta_4, p_5 \vdash F_7 \vee F_8 \end{array} \quad \text{ax/W} \end{array}}_{} \quad \text{Cut}$$

• Case rule  $\vee_2$ 

$$\frac{ \frac{\mathbf{h}_6: \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_8}{\mathbf{e}\mathbf{h}_1: \Delta_2, \mathbf{p}_5 \vdash \mathbf{p}_5} \ I \ \frac{\mathbf{h}_6: \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8}{\mathbf{e}\mathbf{h}_6: \Delta_4, \mathbf{p}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \ \frac{\mathbf{v}_2}{\mathbf{cut}} }{ \frac{-: (\Delta_2, \mathbf{p}_5), \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8}{\mathbf{o}}} \ \frac{\mathbf{v}_2}{\mathbf{cut}}$$

• Case rule  $\rightarrow_L$ 

$$\begin{array}{c} \underbrace{ \begin{array}{c} \bullet \mathbf{h}_1 : \Delta_2, \mathbf{p}_4 \vdash \mathbf{p}_4 \\ - : (\Delta_2, \mathbf{p}_4 \vdash \mathbf{p}_4) \end{array} I \begin{array}{c} \mathbf{h}_5 : \Delta_9, \mathbf{p}_4, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_6 & \mathbf{h}_5 : \Delta_9, \mathbf{F}_7, \mathbf{p}_4 \vdash \mathbf{F}_8 \\ \bullet \mathbf{h}_5 : (\Delta_9, \mathbf{F}_6 \to \mathbf{F}_7), \mathbf{p}_4 \vdash \mathbf{F}_8 \\ - : (\Delta_2, \mathbf{p}_4), \Delta_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \to \\ \hline - : \Delta_2, \Delta_9, \mathbf{p}_4, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \end{array}} \begin{array}{c} \mathsf{Cut} \\ \bullet \mathsf{ax/W} \end{array}$$

• Case rule  $\wedge_L$ 

$$\frac{\underbrace{\bullet \mathbf{h}_1:\Delta_2,\mathbf{p}_4\vdash\mathbf{p}_4}_{-:(\Delta_2,\mathbf{p}_4),\Delta_9,\mathbf{f}_6,\mathbf{f}_7,\mathbf{p}_4\vdash\mathbf{f}_8}_{-:(\Delta_2,\mathbf{p}_4),\Delta_9,\mathbf{f}_6\land\mathbf{f}_7\vdash\mathbf{f}_8} \cap_{\text{Cut}} \frac{-:(\Delta_2,\mathbf{p}_4),\Delta_9,\mathbf{f}_6\land\mathbf{f}_7\vdash\mathbf{f}_8}{-:\Delta_2,\Delta_9,\mathbf{p}_4,\mathbf{f}_6\land\mathbf{f}_7\vdash\mathbf{f}_8} \xrightarrow{\text{ax/W}}$$

• Case rule  $\vee_L$ 

$$\frac{\underbrace{\bullet\mathbf{h}_1:\Delta_2,\mathbf{p}_4\vdash\mathbf{p}_4}_{} \ I \ \frac{\mathbf{h}_5:\Delta_9,\mathbf{F}_6,\mathbf{p}_4\vdash\mathbf{F}_8 \quad \mathbf{h}_5:\Delta_9,\mathbf{F}_7,\mathbf{p}_4\vdash\mathbf{F}_8}{\bullet\mathbf{h}_5:(\Delta_9,\mathbf{F}_6\vee\mathbf{F}_7),\mathbf{p}_4\vdash\mathbf{F}_8}_{} \ \mathbf{Cut}} \\ \frac{-:(\Delta_2,\mathbf{p}_4),\Delta_9,\mathbf{F}_6\vee\mathbf{F}_7\vdash\mathbf{F}_8}{-} \ \frac{}{-:\Delta_2,\Delta_9,\mathbf{p}_4,\mathbf{F}_6\vee\mathbf{F}_7\vdash\mathbf{F}_8}_{} \ \frac{}{} \ \mathbf{ax/W}}$$

• Case rule  $\perp_L$ 

$$\begin{array}{c|c} \hline \\ \bullet \mathbf{h}_1 : \Delta_2, \mathbf{p}_4 \vdash \mathbf{p}_4 & I & \hline \\ \bullet \mathbf{h}_5 : (\bot, \Delta_7), \mathbf{p}_4 \vdash \mathbf{F}_6 \\ \hline \\ - : (\Delta_2, \mathbf{p}_4), \bot, \Delta_7 \vdash \mathbf{F}_6 \\ \hline \\ \hline \\ - : \bot, \Delta_2, \Delta_7, \mathbf{p}_4 \vdash \mathbf{F}_6 \end{array} \stackrel{\bot_L}{\leftarrow} \mathbf{Cut}$$

 $\bullet\,$  Case rule I

$$\begin{array}{c|c} \hline \bullet_{\mathbf{h}_1}: \Delta_2, \mathbf{p}_4 \vdash \mathbf{p}_4 & I & \hline \bullet_{\mathbf{h}_5}: (\Delta_7, \mathbf{p}_6), \mathbf{p}_4 \vdash \mathbf{p}_6 \\ \hline -: (\Delta_2, \mathbf{p}_4), \Delta_7, \mathbf{p}_6 \vdash \mathbf{p}_6 \\ \hline -: \Delta_2, \Delta_7, \mathbf{p}_4, \mathbf{p}_6 \vdash \mathbf{p}_6 & I \\ \hline \hline \bullet_{\mathbf{h}_1}: \Delta_2, \mathbf{p}_6 \vdash \mathbf{p}_6 & I & \hline \bullet_{\mathbf{h}_4}: \Delta_5, \mathbf{p}_6 \vdash \mathbf{p}_6 & I \\ \hline -: (\Delta_2, \mathbf{p}_6), \Delta_5 \vdash \mathbf{p}_6 & Cut \\ \hline \hline -: \Delta_2, \Delta_5, \mathbf{p}_6 \vdash \mathbf{p}_6 & I \\ \hline \end{array}$$

$$\frac{ \frac{\mathbf{h}_{1}:\Delta_{2},\mathbf{p}_{4}\vdash\mathbf{p}_{4}}{\bullet} \ I \ \frac{\mathbf{h}_{5}:\Delta_{7},\mathbf{p}_{4}\vdash\mathbf{F}_{6}}{\bullet\mathbf{h}_{5}:(\top,\Delta_{7}),\mathbf{p}_{4}\vdash\mathbf{F}_{6}}}{ -:(\Delta_{2},\mathbf{p}_{4}),\top,\Delta_{7}\vdash\mathbf{F}_{6}} \ \frac{\top_{L}}{\bullet} \ \mathbf{Cut} } \\ \frac{-:(\Delta_{2},\mathbf{p}_{4}),\top,\Delta_{7}\vdash\mathbf{F}_{6}}{-:\top,\Delta_{2},\Delta_{7},\mathbf{p}_{4}\vdash\mathbf{F}_{6}} \ \mathbf{ax/W}$$

## 7.11 Status of $\top_L$ : OK

• Case rule  $\top_R$ 

$$\frac{ \begin{array}{l} \mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_5 \\ \bullet \mathbf{h}_1 : \top, \Delta_2 \vdash \mathbf{F}_5 \end{array} \top_L \quad \begin{array}{l} \bullet \mathbf{h}_6 : \Delta_4, \mathbf{F}_5 \vdash \top \\ \hline - : (\top, \Delta_2), \Delta_4 \vdash \top \\ \hline - : \top, \Delta_2, \Delta_4 \vdash \top \end{array} \quad \begin{array}{l} \top_R \\ \mathsf{Cut} \end{array} }$$

• Case rule  $\rightarrow_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \mathbf{F}_5} \quad \top_L \quad \frac{\mathbf{h}_6: \Delta_4, \mathbf{F}_5, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \quad \xrightarrow{\bullet}_R \\ \hline -: (\top, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8 \\ \hline \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet} \quad \text{ax/W} \quad \frac{\bullet}{\bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \to \mathbf{F}_8} \\ -: \top, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \to \mathbf{F}_8 \end{array} \quad \begin{array}{c} \mathbf{ax/W} \\ \mathsf{hCut} \end{array}$$

• Case rule  $\wedge_R$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \mathbf{F}_5} \ \, \top_L \ \, \frac{\mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \quad \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8} \ \, \mathbf{Cut} \\ \\ -: (\top, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \\ \hline \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8} \ \, \frac{\mathsf{ax/W}}{\bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \land \mathbf{F}_8} \\ \hline -: \top, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \land \mathbf{F}_8 \end{array} \right. \\ \mathbf{hCut}$$

• Case rule  $\vee_1$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \mathbf{F}_5} & \top_L & \frac{\mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7}{\bullet \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \hline -: (\top, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \hline \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} & \frac{\mathbf{ax/W}}{\bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \hline -: \top, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 & \mathbf{ax/W} \\ \hline \end{array}$$

• Case rule  $\vee_2$ 

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \mathbf{F}_5} \quad \top_L \quad \frac{\mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_6: \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \\ -: (\top, \Delta_2), \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \\ \\ \underline{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_5} \quad \frac{\mathsf{ax/W}}{\bullet \mathbf{h}_6: \top, \Delta_4, \mathbf{F}_5 \vdash \mathbf{F}_7 \vee \mathbf{F}_8} \\ \\ -: \top, \Delta_2, \Delta_4 \vdash \mathbf{F}_7 \vee \mathbf{F}_8 \end{array} \quad \frac{\mathsf{ax/W}}{\mathsf{hCut}}$$

• Case rule  $\rightarrow_L$ 

$$\begin{array}{c|c} \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\top,\Delta_2 \vdash \mathbf{F}_4} & \top_L & \frac{\mathbf{h}_5:\Delta_9, \mathbf{F}_4, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_6 & \mathbf{h}_5:\Delta_9, \mathbf{F}_4, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5:(\Delta_9, \mathbf{F}_6 \to \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8} & \mathbf{Cut} \\ \hline & -:(\top,\Delta_2),\Delta_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ & \xrightarrow{\bullet} \\ \hline & \frac{\mathbf{ax}/\mathbb{W}}{\bullet \mathbf{h}_5:\top,\Delta_9, \mathbf{F}_4, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8} & \mathbf{ax}/\mathbb{W} \\ & -:\top,\Delta_2,\Delta_9, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{ax}/\mathbb{W} \\ \hline & \bullet \mathbf{h}_5:\top,\Delta_9, \mathbf{F}_4, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4 : \Delta_5, \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5, \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5, \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & -:(\top,\Delta_2),\Delta_5 \vdash \mathbf{F}_8 & \rightarrow \\ \hline & \bullet \mathbf{h}_4:\Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5, \mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{h}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4:\Delta_5,\mathbf{F}_6 \to \mathbf{F}_7 \vdash \mathbf{h}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{h}_5 \to \mathbf{h}_7 \vdash \mathbf{h}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{h}_5 \to \mathbf{h}_7 \vdash \mathbf{h}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{h}_5 \to \mathbf{h}_7 \vdash \mathbf{h}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{h}_5 \to \mathbf{h}_7 \vdash \mathbf{h}_8 \\ \hline & \bullet \mathbf{h}_4:\Delta_5,\mathbf{h}_5 \to \mathbf{h}_7 \vdash \mathbf{h}_8 \\ \hline$$

• Case rule  $\wedge_L$ 

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\top,\Delta_2 \vdash \mathbf{F}_4} \quad \top_L \quad \frac{\mathbf{h}_5:\Delta_9, \mathbf{F}_4, \mathbf{F}_6, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5:(\Delta_9, \mathbf{F}_6 \land \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8} \quad \overset{\wedge_L}{\leftarrow} \\ \frac{-:(\top,\Delta_2),\Delta_9, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8}{\to} \quad \overset{\text{ax/W}}{\bullet} \\ \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\to} \quad \frac{\mathbf{ax/W}}{\bullet \mathbf{h}_5:\top,\Delta_9, \mathbf{F}_4, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \mathbf{hCut} \\ \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7}{\bullet \mathbf{h}_1:\top,\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7} \quad \top_L \quad \frac{\mathbf{h}_4:\Delta_5, \mathbf{F}_6, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_4:\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\wedge_L}{\leftarrow} \\ \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_6 \land \mathbf{F}_7}{\to} \quad \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5, \mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{F}_6 \land \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4:\top,\Delta_5,\mathbf{h}_5 \vdash \mathbf{h}_6 \land \mathbf{F}_7 \vdash \mathbf{h}_8} \quad \overset{\text{ax/W}}{\leftarrow} \\ \frac{\rightarrow}{\bullet \mathbf{h}_4$$

### • Case rule $\vee_L$

$$\begin{array}{c|c} \frac{\mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1 : \top, \Delta_2 \vdash \mathbf{F}_4} & \top_L & \frac{\mathbf{h}_5 : \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \vdash \mathbf{F}_8 \quad \mathbf{h}_5 : \Delta_9, \mathbf{F}_4, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_5 : (\Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7), \mathbf{F}_4 \vdash \mathbf{F}_8} \quad \mathbf{Cut} \\ \hline & -: (\top, \Delta_2), \Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline & \frac{\mathbf{h}_1 : \Delta_2 \vdash \mathbf{F}_4}{\bullet} \quad \mathbf{ax/W} & \frac{\bullet}{\bullet \mathbf{h}_5 : \top, \Delta_9, \mathbf{F}_4, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \mathbf{ax/W} \\ \hline & -: \top, \Delta_2, \Delta_9, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8 & \mathbf{h}_4 : \Delta_5, \mathbf{F}_7 \vdash \mathbf{F}_8 \\ \hline \bullet \mathbf{h}_1 : \top, \Delta_2 \vdash \mathbf{F}_6 \vee \mathbf{F}_7 & \top_L \quad \frac{\mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \vdash \mathbf{F}_8 \quad \mathbf{h}_4 : \Delta_5, \mathbf{F}_7 \vdash \mathbf{F}_8}{\bullet \mathbf{h}_4 : \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \mathbf{Cut} \\ \hline & -: (\top, \Delta_2), \Delta_5 \vdash \mathbf{F}_8 \\ \hline & \frac{\bullet}{\bullet \mathbf{h}_4 : \top, \Delta_5, \mathbf{F}_6 \vee \mathbf{F}_7 \vdash \mathbf{F}_8} \quad \mathbf{ax/W} \\ \hline & -: (\top, \Delta_2, \Delta_5 \vdash \mathbf{F}_8 \\ \hline & -: (\top, \Delta_2, \Delta_5 \vdash \mathbf{F}_8 \\ \hline \end{array}$$

### • Case rule $\perp_L$

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2\vdash\bot}{\bullet\mathbf{h}_1:\top,\Delta_2\vdash\bot} \;\; \top_L \;\; \frac{\bullet\mathbf{h}_4:\Delta_5,\bot\vdash \mathbf{F}_6}{\bullet\mathbf{h}_4:\Delta_5,\bot\vdash \mathbf{F}_6} \;\; \overset{\bot_L}{\mathsf{Cut}} \\ \hline -:(\top,\Delta_2),\Delta_5\vdash \mathbf{F}_6 \\ \hline \frac{\overset{}{\mathbf{h}_1:\Delta_2\vdash\bot}}{-:\top,\Delta_2\vdash\bot} \;\; \frac{\to}{\bullet\mathbf{h}_4:\bot,\top,\Delta_5\vdash \mathbf{F}_6} \;\; \frac{\bot_L}{\mathsf{hCut}} \\ \hline -:\top,\Delta_2,\Delta_5\vdash \mathbf{F}_6 \end{array} \quad \frac{\mathbf{h}_1:\Delta_2\vdash \mathbf{F}_4}{\bullet\mathbf{h}_1:\top,\Delta_2\vdash \mathbf{F}_4} \;\; \top_L \;\; \frac{\bullet\mathbf{h}_5:(\bot,\Delta_7),\mathbf{F}_4\vdash \mathbf{F}_6}{\bullet\mathbf{h}_5:(\bot,\Delta_7),\mathbf{F}_4\vdash \mathbf{F}_6} \;\; \overset{\bot_L}{\mathsf{Cut}} \\ \hline -:(\top,\Delta_2),\bot,\Delta_7\vdash \mathbf{F}_6 \\ \hline -:\bot,\top,\Delta_2,\Delta_7\vdash \mathbf{F}_6 \;\; \overset{\bot_L}{\smile} \end{array}$$

### ullet Case rule I

$$\begin{array}{c} \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \mathbf{F}_4} \quad \top_L \quad & \frac{\bullet \mathbf{h}_5: (\Delta_7, \mathbf{p}_6), \mathbf{F}_4 \vdash \mathbf{p}_6}{\bullet \mathbf{h}_5: (\Delta_7, \mathbf{p}_6), \mathbf{F}_4 \vdash \mathbf{p}_6} \quad \mathbf{Cut} \\ & -: (\top, \Delta_2), \Delta_7, \mathbf{p}_6 \vdash \mathbf{p}_6 \\ & -: \top, \Delta_2, \Delta_7, \mathbf{p}_6 \vdash \mathbf{p}_6 \end{array} \quad I \\ \\ \frac{\mathbf{h}_1: \Delta_2 \vdash \mathbf{p}_6}{\bullet \mathbf{h}_1: \top, \Delta_2 \vdash \mathbf{p}_6} \quad \top_L \quad & \frac{\bullet \mathbf{h}_4: \Delta_5, \mathbf{p}_6 \vdash \mathbf{p}_6}{\bullet \mathbf{h}_4: \Delta_5, \mathbf{p}_6 \vdash \mathbf{p}_6} \quad \mathbf{Cut} \\ & -: (\top, \Delta_2), \Delta_5 \vdash \mathbf{p}_6 \\ & -: \top, \Delta_2, \Delta_5 \vdash \mathbf{p}_6 \end{array} \quad \mathbf{ax/W}$$

### • Case rule $\top_L$

$$\begin{array}{c} \frac{\mathbf{h}_1:\Delta_2 \vdash \top}{\bullet \mathbf{h}_1:\top,\Delta_2 \vdash \top} \;\; \top_L \quad \frac{\mathbf{h}_4:\Delta_5 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_4:\Delta_5,\top \vdash \mathbf{F}_6} \quad \top_L \\ \hline -:(\top,\Delta_2),\Delta_5 \vdash \mathbf{F}_6 \\ \hline -:\top,\Delta_2,\Delta_5 \vdash \mathbf{F}_6 \quad \mathbf{ax/W} \\ \hline \\ \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\top,\Delta_2 \vdash \mathbf{F}_4} \;\; \top_L \quad \frac{\mathbf{h}_5:\Delta_7,\mathbf{F}_4 \vdash \mathbf{F}_6}{\bullet \mathbf{h}_5:(\top,\Delta_7),\mathbf{F}_4 \vdash \mathbf{F}_6} \quad \top_L \\ \hline -:(\top,\Delta_2),\top,\Delta_7 \vdash \mathbf{F}_6 \quad \mathbf{Cut} \\ \hline \\ \frac{\mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4}{\bullet \mathbf{h}_1:\Delta_2 \vdash \mathbf{F}_4} \quad \mathbf{ax/W} \quad \frac{\bullet}{\bullet \mathbf{h}_5:\top,\top,\Delta_7,\mathbf{F}_4 \vdash \mathbf{F}_6} \quad \mathbf{ax/W} \\ \hline -:\top,\top,\Delta_2,\Delta_7 \vdash \mathbf{F}_6 \quad \mathbf{hCut} \\ \hline \end{array}$$