Discrete well-balanced WENO finite difference schemes: global-flux quadrature method with multi-step ODE integrator weights

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Converence name
Conference adress
Conference date

Setting: balance laws

We want to solve numerically hyperbolic systems of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- GPR model and hyperbolic reformulations of viscous/dispersive systems
- etc.

Setting: balance laws

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Shopping list

- Arbitrary high order
- Possibly monotonicity/positivity preserving
- Steady state preserving

Objective

We want to solve numerically hyperbolic systems of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

Steady state preserving: on a given mesh we want to the scheme embed in the discretization a

Definition of $\partial_x F_{\mathrm{h}} = S_{\mathrm{h}}$ such that data verifying it is exactly preserved

Projection operator providing such data, possibly different from the scheme

Characterization of the above projection: e.g enhanced consistency, well posedness

Bibliography (incomplete) for 1D

There are numberless approaches, and vast literature. Related work:

1. Reconstruction/evolution of fluctuations wrt discrete equilibria (**approximate full well balanced**):
   - Castro & Parés J.Sci.Comp. 2020, Gómez-Bueno et al, Appl.Math.Comp. 2021, Gómez-Bueno et al Mathematics 2021, Guerrero Fernández et al Mathematics 2021, Gómez-Bueno et al, Appl.Num.Math. 2023, etc. etc.
2. Well balanced via integration of the source term and global fluxes
   - Gascón & Corberán JCP 2001, Donat & Martinez-Gavara J.Sci.Comp. 2011, Chertock et al JCP 2018, Cheng et al J.Sci.Comp. 2019, Mantri&Noelle JCP 2021, Ciallella et al. J.Sci.Comp. 2023, Xu and Shu J.Sci.Comp. to appear, etc. etc.
3. Other approaches
   -

1 Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

Approach 1.[1]

$$\frac{d\overline{U}_i}{dt} + \frac{1}{\Delta x}\left(\widehat{G}_{i+1/2} - \widehat{G}_{i-1/2}\right) = 0$$

[1] M. Ciallella, D. Torlo, MR, Arbitrary High Order WENO Finite Volume Scheme with Flux Globalization for Moving Equilibria Preservation, *J.Sci.Comp.* 96, 2023

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Approach 1.[1]

$$\frac{d\overline{U}_i}{dt} + \frac{1}{\Delta x}\left(\widehat{G}_{i+1/2} - \widehat{G}_{i-1/2}\right) = 0$$

1. Reconstruct WENO polynomials $U_i(x)$
2. Compute cell averaged source primitive $\overline{R}_i = R^+_{i-1/2} - \sum_q \omega_q \int_{x_{i-1/2}}^{x_q} S(U_i(x), \varphi(x))dx$
3. Compute cell averaged fluxes $\overline{F}_i = \sum_q \omega_q F(U_i(x_q), \varphi(x_q))dx$
4. Reconstruct WENO polynomials $G_i(x) = (F + R)_i(x)$
5. Compute upwind fluxes $\widehat{G}_{i+1/2} = (A^+A^{-1})_{i+1/2}G_i(x_{i+1/2}) + (A^-A^{-1})_{i+1/2}G_{i+1}(x_{i+1/2})$

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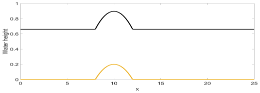

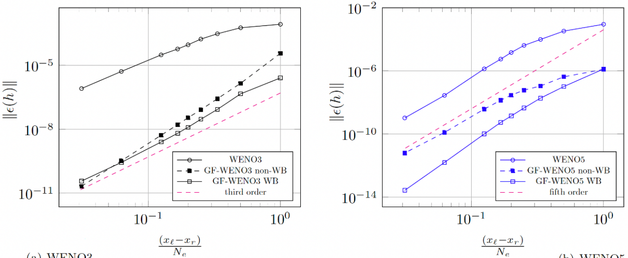

(a) WENO3 (b) WENO5

Supercritical flow: convergence tests

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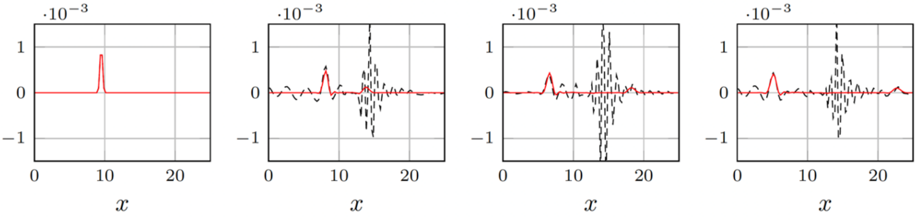
Approach 1.[1]

Small perturbation of steady state



(a) $t = 0$ (b) $t = 0.66$ (c) $t = 1.33$ (d) $t = 2$

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Approach 1.[1]

- ✓ No a-priori knowledge of equilibrium, all steady states!
- ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
- ✓ Considerable accuracy enhancements at steady state (3 to 4 orders of magnitude)
- × No super convergence
- × Very hard to charcterize the steady state and generate one

$$\overline{R}_i = R^{+}_{i-1/2} - \sum_q \omega_q \int_{x_{i-1/2}}^{x_q} S(U_i(x), \varphi(x))dx$$

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New approach

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{G}_{i+1/2} - \widehat{G}_{i-1/2}\right) = 0$$

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New approach

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{G}_{i+1/2} - \widehat{G}_{i-1/2}\right) = 0$$

1. ~~Reconstruct WENO polynomials~~ $\cancel{U_i(x)}$
2. Compute nodal source primitive $R_i = R_{i-1} - \Delta x \sum_q \omega_q S(U_{i-q}, \varphi(x_{i-q}))$
3. ~~Compute cell averaged fluxes~~ $\cancel{\overline{F}_i = \sum_q \omega_q F(U_i(x_q), \varphi(x_q))dx}$
4. Reconstruct WENO polynomials $G^{\pm}_{i+1/2}(x) = (F+R)^{\pm}_{i+1/2}(x)$
5. Compute upwind fluxes $\widehat{G}_{i+1/2} = P^{+}_{i+1/2}G^{-}_{i+1/2}(x_{i+1/2}) + P^{-}_{i+1/2}G^{+}_{i+1/2}(x_{i+1/2})$

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New approach

Main result

Proposition (Discrete steady state). *The WENO-FD scheme with global flux quadrature preserves exactly continuous discrete steady states* $U_i^* = U(F_i)$ *with* F *obtained by integrating the ODE*

$$F' = S(U(F))\partial_x H$$

using the multi-step ODE integrator with weights $\{\omega_q\}_{q\geq 0}$ *on spatial slabs of size* h.
If $U(F)$ *is a one to one mapping,* $U^*(x)$ *verifies the consistency estimates of the multi-step scheme.*

Multi-step schemes so far: Adams methods ABn and AMn

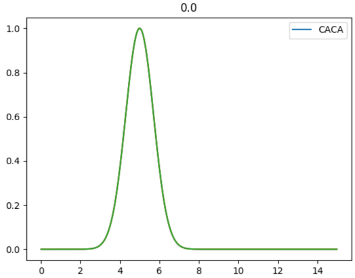
1 Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

New approach

Moving solution (safety check): $S(U) = U - C$ and $H(x,t) = e^{-(x-x_0-Ct)^2}$, $U_{\text{ex}} = H(x,t)$



For WENOn-ABm or WENOn-AMn

we expect convergence rates

$$\epsilon = \Delta x^{\min(m,n)}$$

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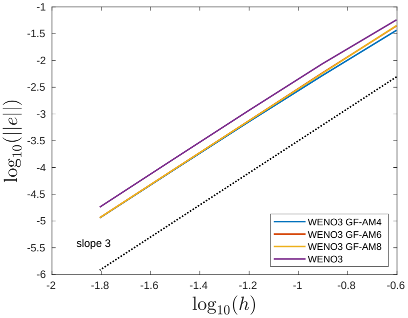

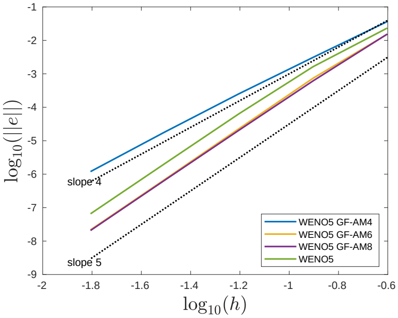

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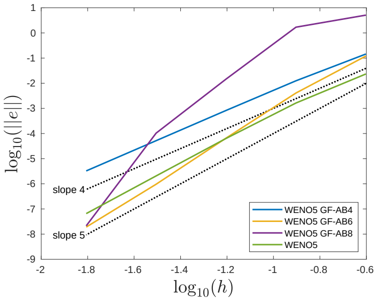

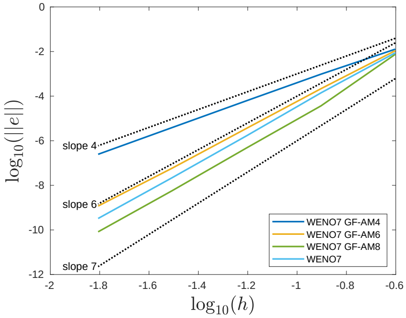

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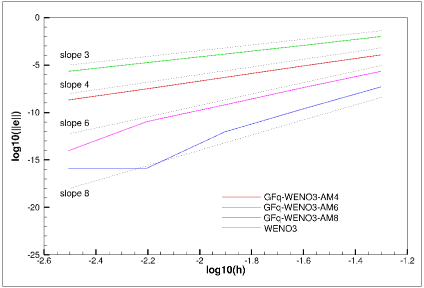
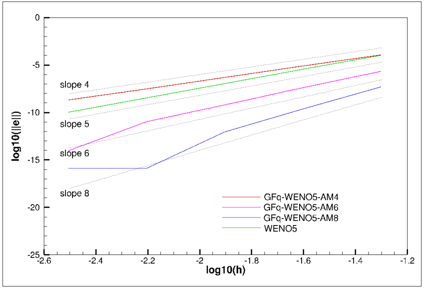
New approach

Steady state: $S(U) = U^2$ and $H(x) = x$, exact solution $U_{\text{ex}}(x) = e^x$





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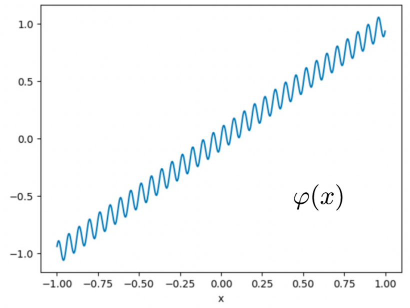
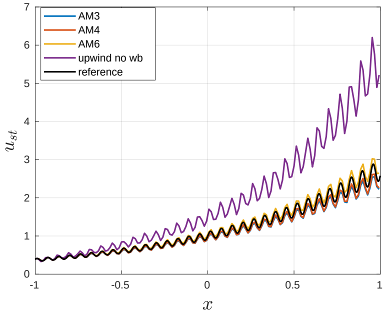
$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

New approach

Steady state: $s(U) = U^2$ and $\varphi = x + 100 \sin x$





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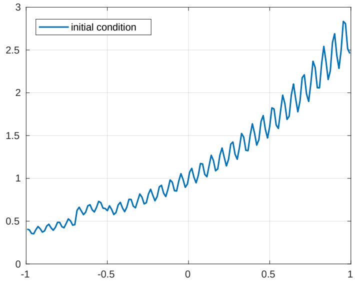
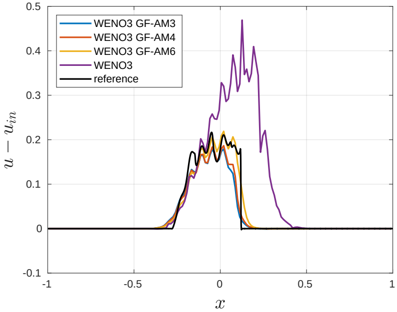
$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

New approach

Perturbed steady state: $s(U) = U^2$ and $\varphi = x + 100\sin x$ + top hat $\delta u = 0.2\chi_{[-0.7,-0.5]}$





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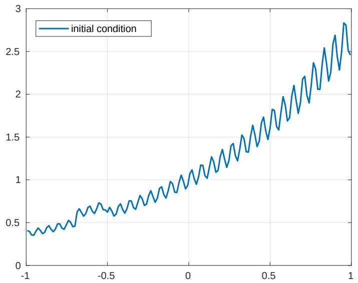

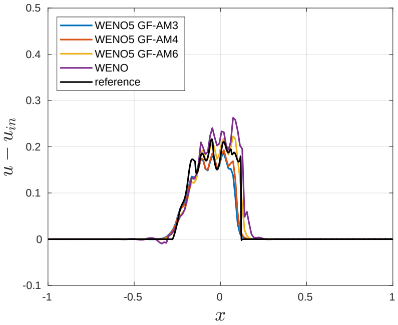

WENO-FD-GFq

Kazolea, Parés, Ricchiuto

Intro

WENO GFq

End

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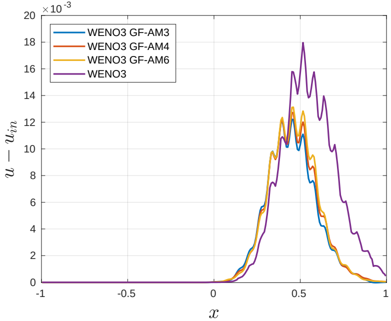
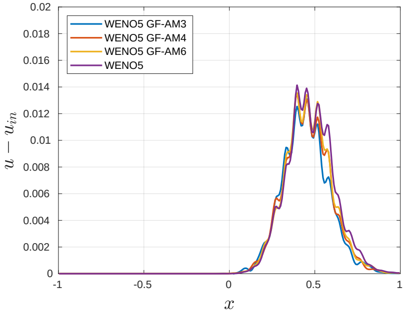
$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

New approach

Perturbed steady state: $s(U) = U^2$ and $\varphi = x + 100 \sin x$ + Gaussian with amplitude $\alpha = 0.005$





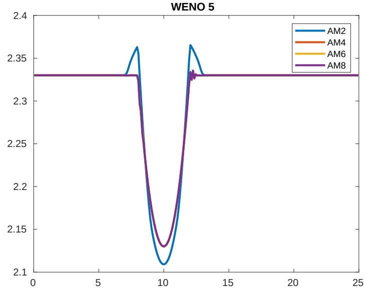
1 Global Flux Quadrature with WENO approximation

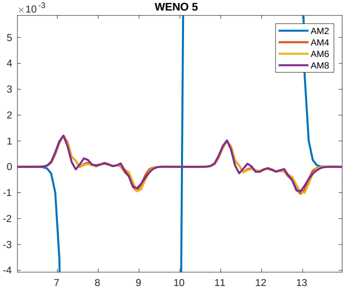
We seek solutions of the hyperbolic system of balance laws

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} = -\begin{bmatrix} 0 \\ h \end{bmatrix} b'(x)$$

New approach

Lake at rest on a parabolic hump. Out of the box method





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New approach

Method modification (with both AB and AM):

$$\hat{b}(x) = \sum_m L_m(x) b(x_m)$$

with $\{x_m\}$ the points generating the AMm method. We proceed as follows

$$\int_{x_i}^{x_{i+1}} ghb'(x)dx \approx \int_{x_i}^{x_{i+1}} g\eta\hat{b}'(x)dx - \int_{x_i}^{x_{i+1}} g\left(\frac{\hat{b}^2}{2}\right)' dx = g\sum_m \omega_m \eta_m \hat{b}'(x_m) - g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right)$$

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New approach

For lake at rest:

$$\begin{aligned}
F_{i+1} - F_i + g\sum_m \omega_m \eta_m \hat{b}'(x_m) - g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right) &= \\
g\left(\frac{h_{i+1}^2}{2} - \frac{h_i^2}{2}\right) + g\eta_0 \sum_m \omega_m \hat{b}'(x_m) - g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right) &= \\
g\left(\frac{(\eta_0 - b_{i+1})^2}{2} - \frac{(\eta_0 - b_i)^2}{2}\right) + g\eta_0 \int_{x_i}^{x_{i+1}} \hat{b}'(x)dx - g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right) &= \\
-g\eta_0(b_{i+1} - b_i) + g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right) + g\eta_0(b_{i+1} - b_i) - g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right) &= 0
\end{aligned}$$

We seek solutions of the hyperbolic system of balance laws

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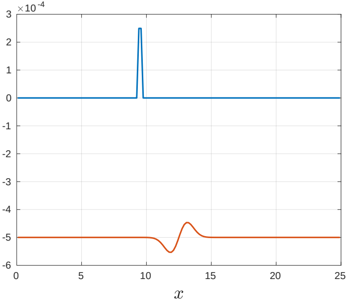
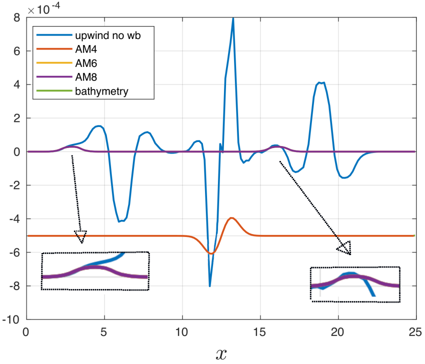
New approach

Lake at rest perturbation





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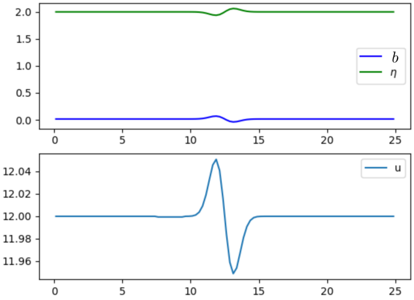
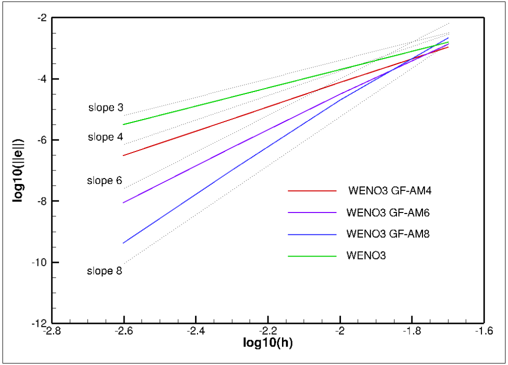
New approach

Supercritical flow on a smooth hump





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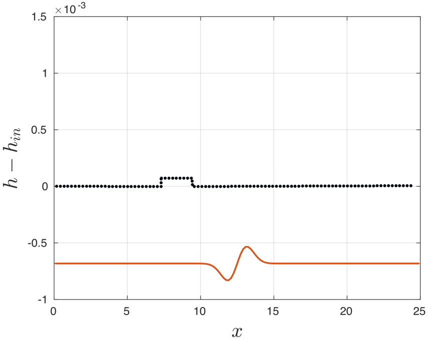

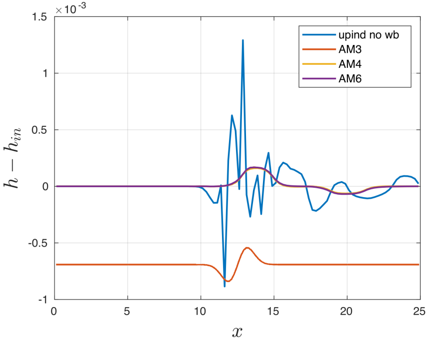

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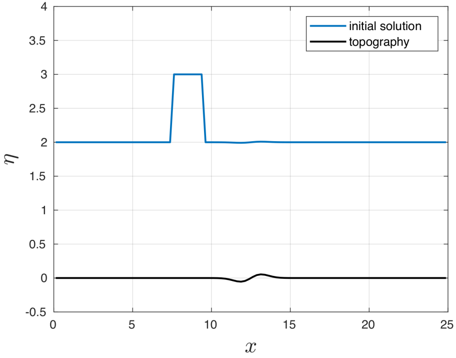

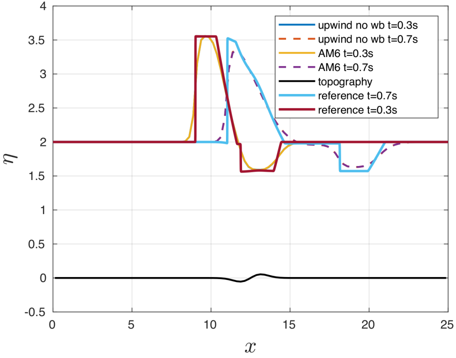

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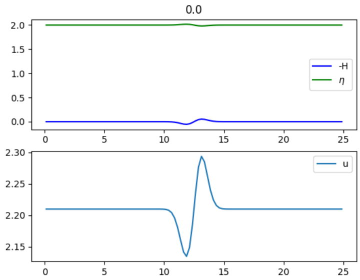

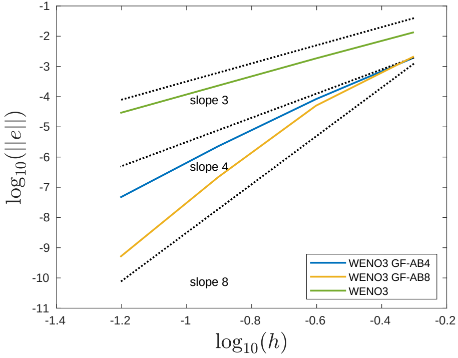

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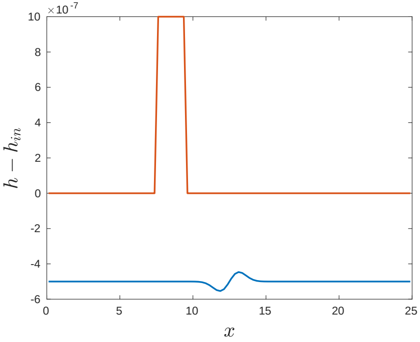
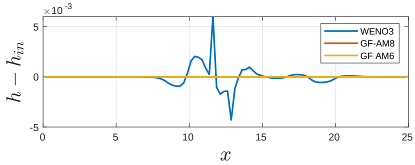
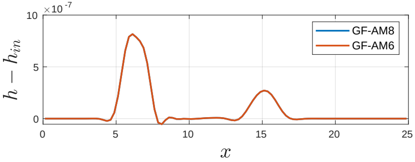




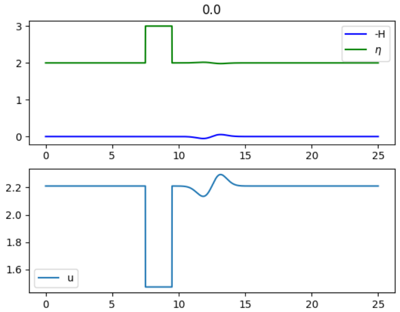


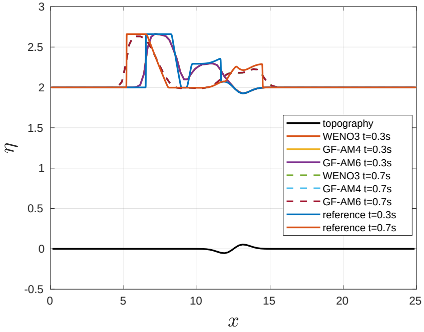
1 Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} = -\begin{bmatrix} 0 \\ h \end{bmatrix} b'(x)$$

New approach





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New approach

- ✓ No a-priori knowledge of equilibrium, all steady states!
- ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
- ✓ Convergence at steady state arbitrarily accurate by changing ODE weights
- ✓ Discrete initial state can be generated
- × Non-compact quadrature

This is the last slide

Take home message

- GFq in 1D: clever quadrature of the source based on high order ODE integrators
- Tremendous error reduction at stady-state (super-covergence)
- In one dimension discrete equilibria can be generated a-priori if necessary with the ODE solver

Future

- Sonic points ?
- discontinuous data ?
- adaptive ODE weights
- multiD