Discrete well-balanced WENO finite difference schemes: global-flux quadrature method with multi-step ODE integrator weights

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Setting: balance laws

We want to solve numerically hyperbolic systems of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

Typical examples that we will use here

- Burger's equations
- Shallow water equations with topography

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Shopping list

- Arbitrary high order
- Possibly monotonicity/positivity preserving
- Steady state preserving

Objective

We want to solve numerically hyperbolic systems of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

The objective of well balanced schemes is to preserve exactly or with enhanced accuracy some of the steady state solutions.

We focus here on the development of well-balanced high order FD methods for general systems of balance laws that preserve all the stationary solutions.

Some approaches in the Bibliography for 1D

There are numberless approaches, and vast literature. Related work:

1. Reconstruction/evolution of fluctuations wrt discrete equilibria (**approximate full well balanced**):
   - Castro & Parés J.Sci.Comp. 2020, Gómez-Bueno et al, Appl.Math.Comp. 2021, Gómez-Bueno et al athematics 2021, Guerrero Fernández et al Mathematics 2021, Gómez-Bueno et al, Appl.Num.Math. 2023, etc. etc.
2. Well balanced via integration of the source term and global fluxes
   - Gascón & Corberán JCP 2001, Donat & Martinez-Gavara J.Sci.Comp. 2011, Chertock et al JCP 2018, Cheng et al J.Sci.Comp. 2019, Mantri&Noelle JCP 2021, Ciallella et al. J.Sci.Comp. 2023, Xu and Shu J.Sci.Comp. to appear, etc. etc.
3. Other approaches
   - Well balanced via reconstruction/evolution of fluctuations wrt to a given equilibrium
   - Preserving $V(u, \phi(x)) = V_0$ via WB differencing or generalized polynomial approximations
   - Fully well balanced Riemann solver with 0-wave to enforce integral steady balance.

Exactly well balanced

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2}\right) = 0$$

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1. Look for the solution of the Cauchy problem $U_i^*(x)$

$$\begin{aligned} F(U_i^*)_x &= S(U_i^*(x))H_x \\ U_i^*(x_i) &= U_i. \end{aligned} \tag{1}$$

2. Define $\mathcal{F}_j = F(U_j) - F(U_i^*(x_j))$, $j \in \{i-r-1, \cdots, i-s-1\}$
3. Compute $\hat{\mathcal{F}}_{i,i+1/2}$, $\hat{\mathcal{F}}_{i,i-1/2}$ through a flux-reconstruction operator, eg WENO reconstruction on the stencil $\mathcal{S}_i = i-r, \cdots, i+s$
4. Find the numerical fluxes using (for eg.) an upwind scheme.

Exactly well balanced

We seek solutions of the hyperbolic system of balance laws

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Some options:

1. The solution $U_i^*(x)$ of (1) if known or easy to find.
2. Solve (1) numerically w.r.t U i.e $U_x = J(U)^{-1}S(U)H_x$
3. or..

Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{G}_{i+1/2} - \widehat{G}_{i-1/2}\right) = 0$$

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In this approach we seek to embed directly in the discrete equation a consistency condition with a discrete approximation of the flux form of the Cauchy problem. So we integrate directly the latter with some approach.

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Main result

Proposition (Discrete steady state). *The WENO-FD scheme with global flux quadrature preserves exactly <u>continuous</u> discrete steady states* $U_i^* = U(F_i)$ *with* F *obtained by integrating the ODE*

$$F' = S(U(F))\partial_x H$$

using the multi-step ODE integrator with weights $\{\omega_q\}_{q\geq 0}$ *on spatial slabs of size* h.
If $U(F)$ *is a one to one mapping,* $U^*(x)$ *verifies the consistency estimates of the multi-step scheme.*

Multi-step schemes so far: Adams methods ABn and AMn

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1. Compute approximations $R_j = R_{j-1} + \mathcal{I}_{j-1/2}(U)$, $\forall i$ where

$$\mathcal{I}_{j-1/2} \approx \int_{x_{j-1}}^{x_j} S(U) H_x dx$$

2. Define $\mathcal{F}_j = F(U_j) - R_j$, $j \in \{i-r-1, \cdots, i-s-1\}$
3. Reconstruct WENO polynomials $G^{\pm}_{i+1/2}(x)$ using $\mathcal{F}_j$
4. Compute upwind fluxes $\widehat{G}_{i+1/2} = P^{+}_{i+1/2} G^{-}_{i+1/2}(x_{i+1/2}) + P^{-}_{i+1/2} G^{+}_{i+1/2}(x_{i+1/2})$

Difference with exactly well balanced: The way in which the reference flux subtracted is obtained- Integration of the source

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We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U)\partial_x H$$

Here we compute

$$\mathcal{I}_{j-1/2} \approx \int_{x_{j-1}}^{x_j} S(U) H_x dx$$

using Adams ODE Integrators ie:

$$\mathcal{I}_{j-1/2} = \sum_{k=-m1}^{m2} \beta_k S(U_{j+k}) H_x(x_{j+k})$$

Since we will work with schemes of formal accuracy $2k+1 \geq 3$ we consider ODE solvers whose solutions have accuracy ar least $2k+2 \geq 4$.

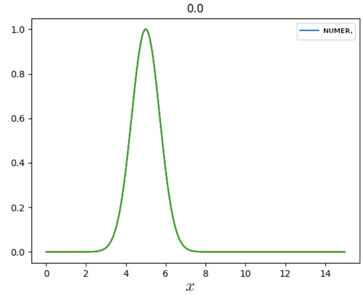
- Adams-Bashforth : $m_1 = 4, 6, 8$ and $m_2 = -1$
- Adams-Moulton: $m_1 = 3, 5, 7$ and $m_2 = 0$

Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

Moving solution (safety check): $S(U) = U - C$ and $H(x,t) = e^{-(x-x_0-Ct)^2}$, $U_{\text{ex}} = H(x,t)$



For WENOn-ABm or WENOn-AMn

we expect convergence rates

$$\epsilon = \Delta x^{\min(m,n)}$$

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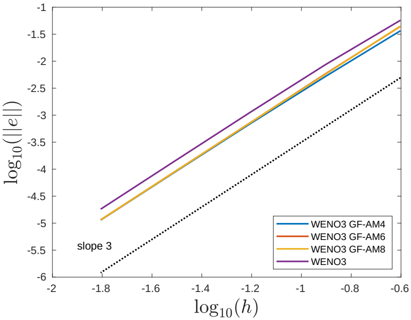

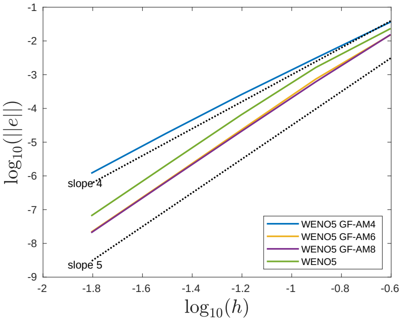

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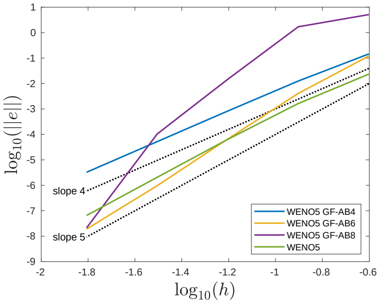
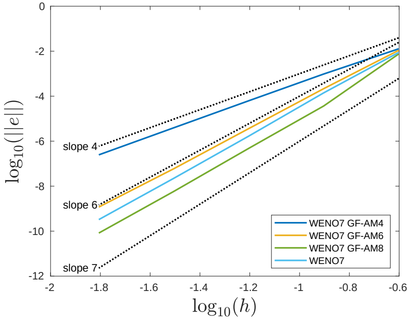


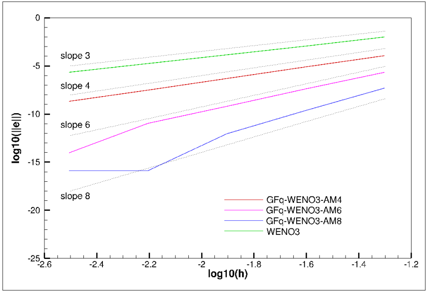


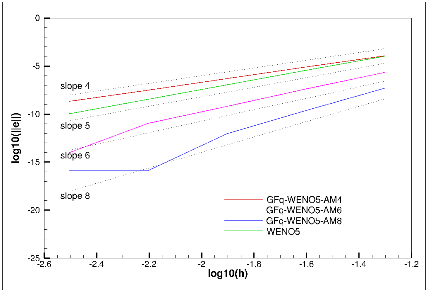
Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

Steady state: $S(U) = U^2$ and $H(x) = x$, exact solution $U_{\text{ex}}(x) = e^x$



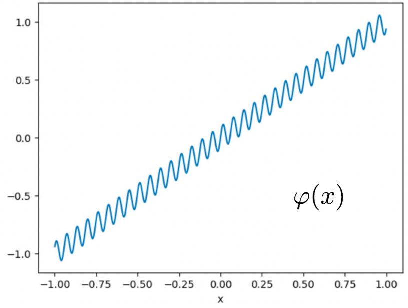


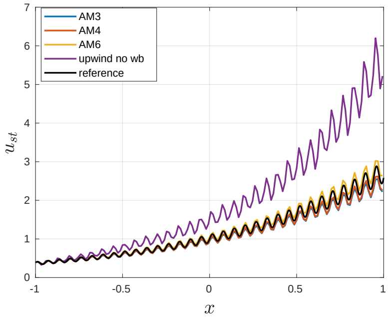
Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

Steady state: $S(U) = U^2$ and $\varphi = x + 100 \sin x$



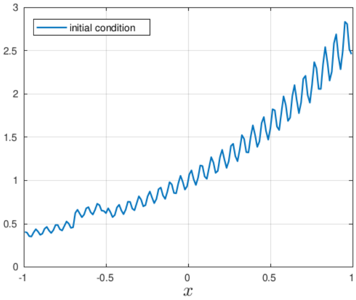


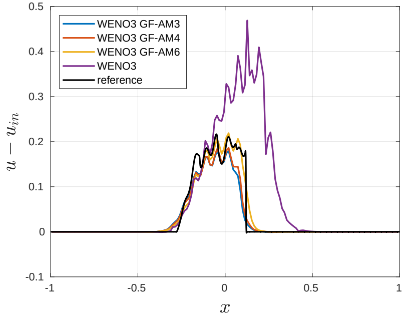
Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

Perturbed steady state: $S(U) = U^2$ and $\varphi = x + 100\sin x$ + top hat $\delta u = 0.2\chi_{[-0.7,-0.5]}$





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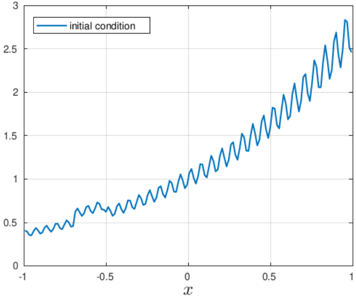
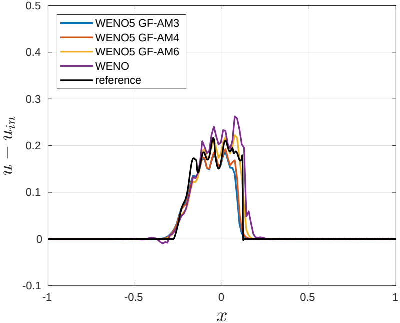


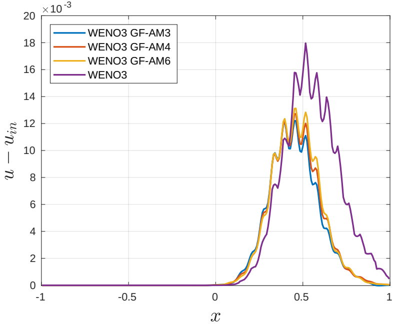


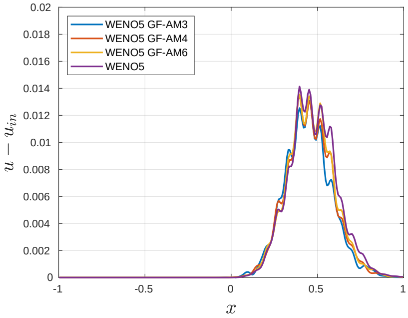
Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

Perturbed steady state: $S(U) = U^2$ and $\varphi = x + 100 \sin x$ + Gaussian with amplitude $\alpha = 0.005$



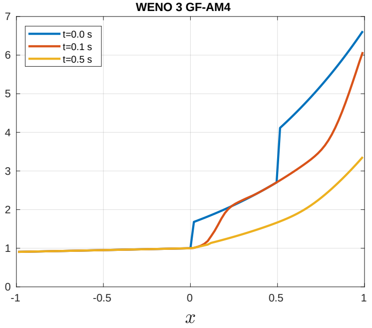


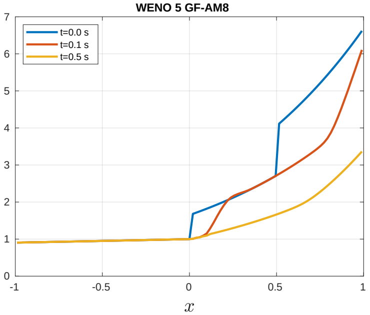
Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U)\partial_x H$$

Discontinuous H: $S(U) = U^2$, $H(x) = 0.1x|_{(x\leq 0)}, 0.9 + x|_{(x>0.5)}, (0.5+x)|_{(0<x\leq 0.5)}$, $U_{\text{ex}}(x) = e^H$





Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

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Discontinuous H: $S(U) = U^2$, $H(x) = 0.1x|_{(x\leq 0)}, 0.9 + x|_{(x>0.5)}$, $U_{\text{ex}}(x) = e^H$

Lets assume that the discontinuity is at the middle of the cell $[x_{i-1},\ x_i]$

- Far from the discontinuity we use our standard n-step selected method.
- In the cell $[x_{i-1},\ x_i]$ we set $I_{i-1/2} = \tilde{S}_{i-1/2}[[H]]_{i-1/2}$ where $\tilde{S}_{i-1/2}$ linearization, such as $[[F]]_{i-1/2} = \tilde{S}_{i-1/2}[[H]]_{i-1/2}$.
- For n cells after the discontinuity we use AM2.

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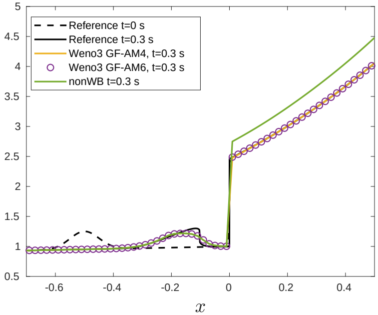

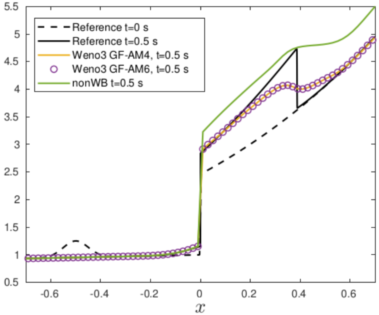

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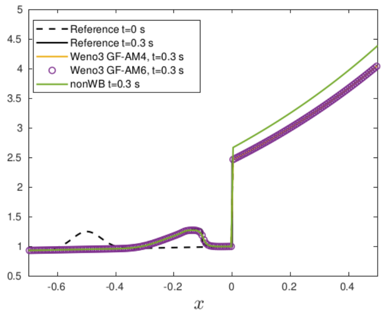
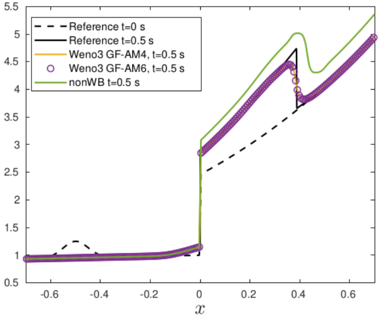


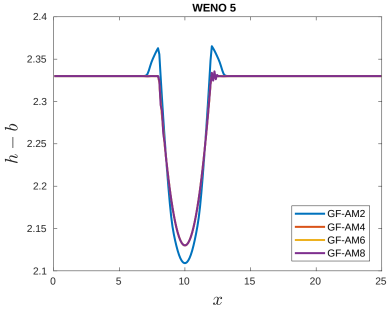


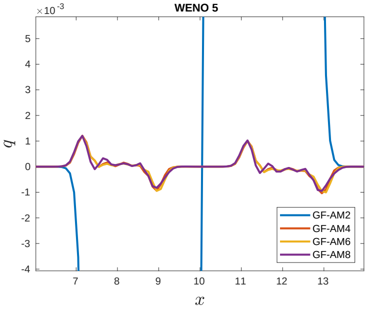
Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} = -\begin{bmatrix} 0 \\ h \end{bmatrix} b'(x)$$

Lake at rest on a parabolic hump.





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Method modification (with both AB and AM):

$$\hat{b}(x) = \sum_m L_m(x) b(x_m)$$

with $\{x_m\}$ the points generating the AMm method. We proceed as follows

$$\int_{x_i}^{x_{i+1}} ghb'(x)dx \approx \int_{x_i}^{x_{i+1}} g\eta\hat{b}'(x)dx - \int_{x_i}^{x_{i+1}} g\left(\frac{\hat{b}^2}{2}\right)' dx = g\sum_m \omega_m \eta_m \hat{b}'(x_m) - g\left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2}\right)$$

Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t \left[\begin{array}{c} h \\ hu \end{array} \right] + \partial_x \left[\begin{array}{c} hu \\ hu^2 + gh^2/2 \end{array} \right] = - \left[\begin{array}{c} 0 \\ h \end{array} \right] b'(x)$$

For lake at rest:

Then it easy to show that

$$F_{i+1} - F_i + g \sum_m \omega_m \eta_m \hat{b}'(x_m) - g \left(\frac{b_{i+1}^2}{2} - \frac{b_i^2}{2} \right) = 0$$

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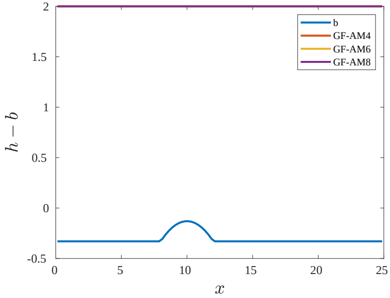
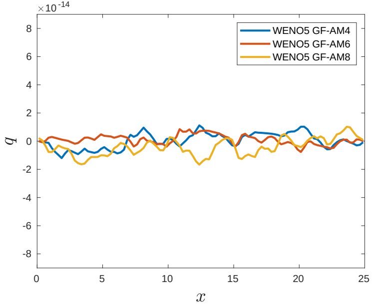
Lake at rest





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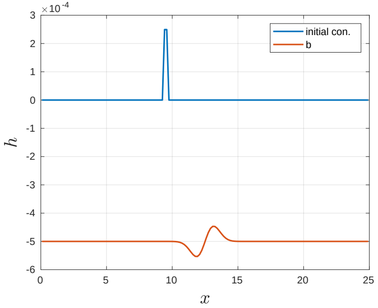
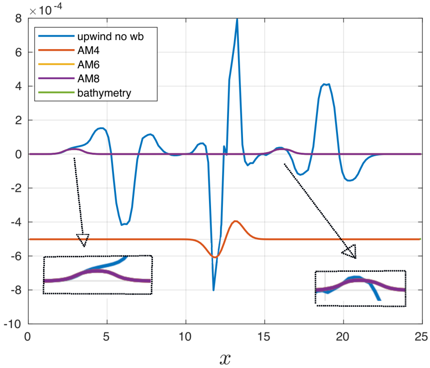
Lake at rest with perturbation





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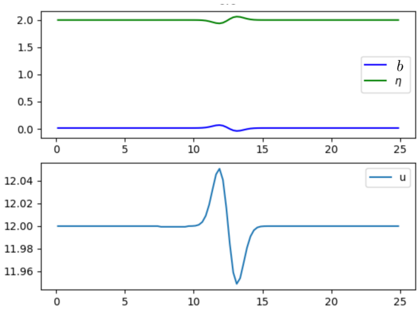
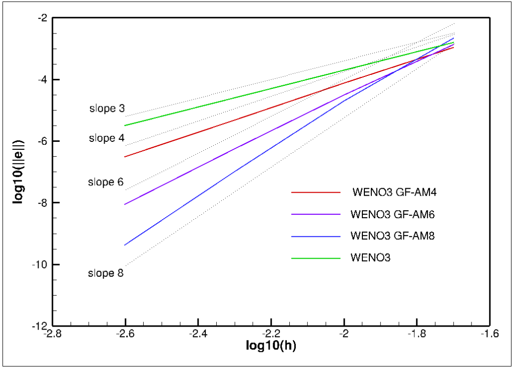
Supercritical flow on a smooth hump





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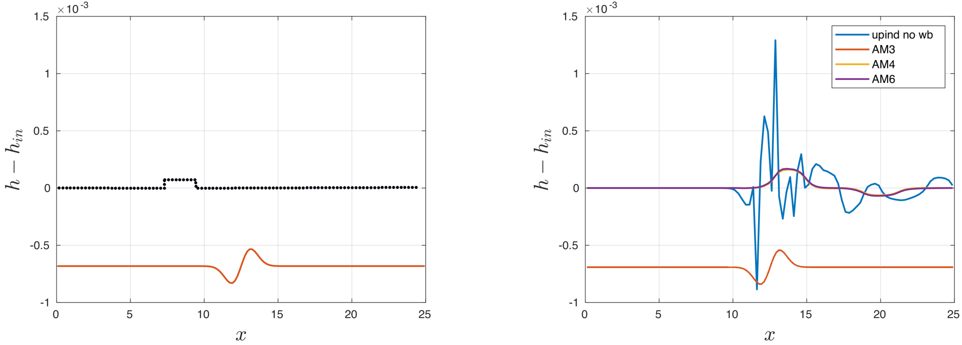
Supercritical flow with perturbation



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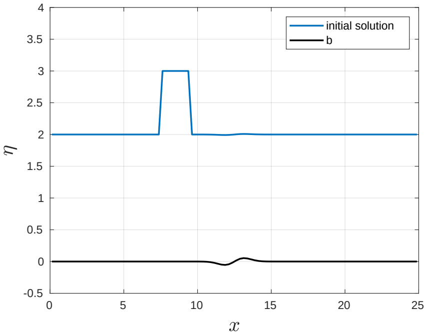

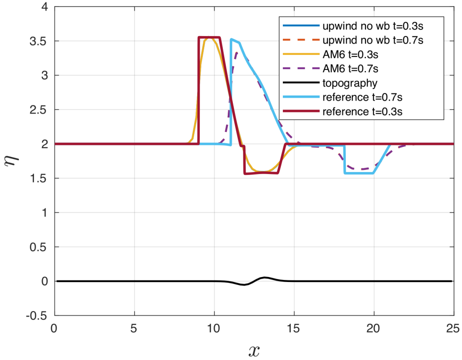

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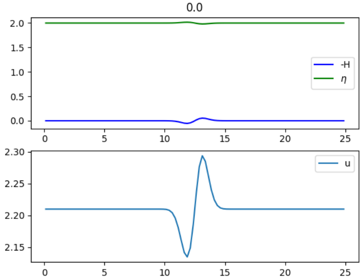
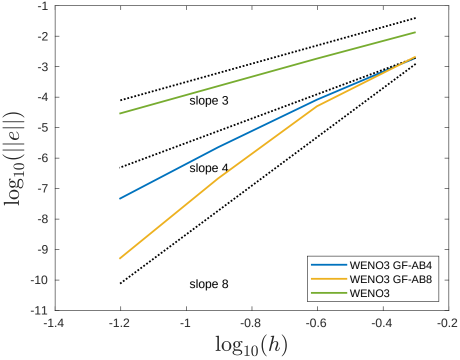
Subcritical flow on a smooth hump



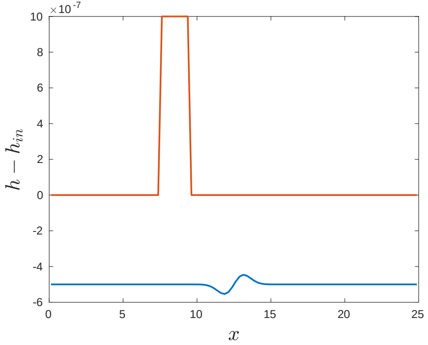


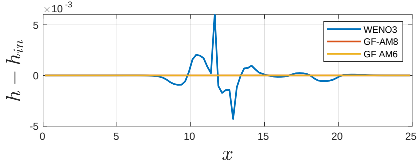
Global Flux Quadrature with WENO approximation

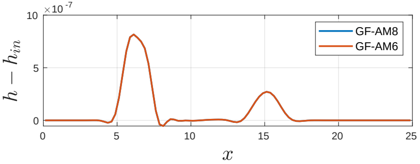
We seek solutions of the hyperbolic system of balance laws

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} = - \begin{bmatrix} 0 \\ h \end{bmatrix} b'(x)$$

Subcritical flow with perturbation







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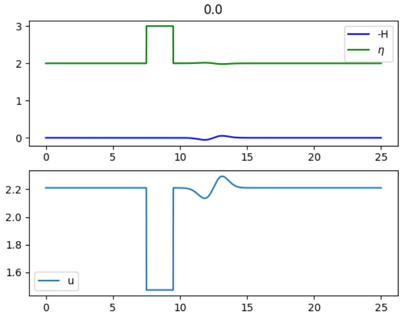

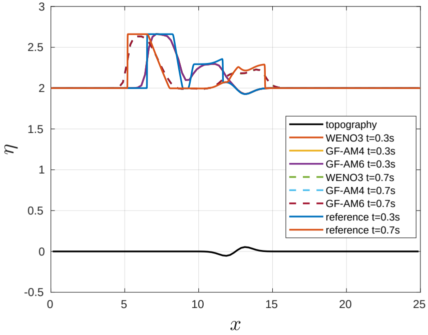

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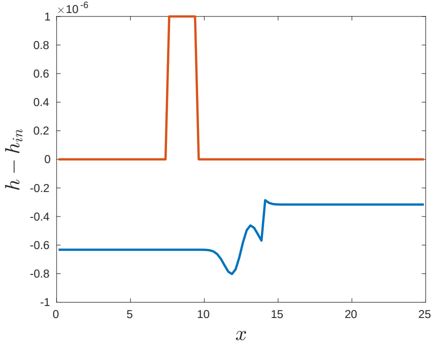
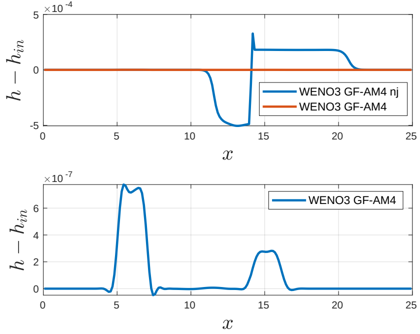
Subcritical flow with perturbation and Discontinuous H





Global Flux Quadrature with WENO approximation

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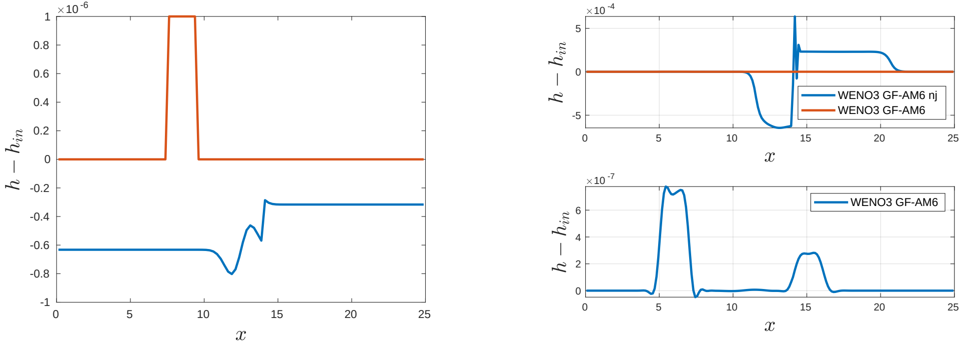

Global Flux Quadrature with WENO approximation

- ✓ No a-priori knowledge of equilibrium, all steady states!
- ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
- ✓ Convergence at steady state arbitrarily accurate by changing ODE weights
- ✓ Discrete initial state can be generated
- × Non-compact quadrature

Take home message

- GFq in 1D: clever quadrature of the source based on high order ODE integrators
- Tremendous error reduction at stady-state (super-covergence)
- In one dimension discrete equilibria can be generated a-priori if necessary with the ODE solver

Future

- Sonic points ?
- adaptive ODE weights
- multiD

Thank you!