

Discrete well-balanced WENO finite difference schemes: global-flux quadrature method with multi-step ODE integrator weights

Maria Kazolea*, Carlos Parés[†], Mario Ricchiuto*

*Team CARDAMOM, Inria research center at University of Bordeaux (France)

[†] EDANYA Group, Universidad de Málaga (Spain)

Conference name

Conference address

Conference date

We want to solve numerically hyperbolic systems of balance laws

$$\partial_t U + \partial_x F(U) = S(U) \partial_x H$$

Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- GPR model and hyperbolic reformulations of viscous/dispersive systems
- etc.

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Shopping list

- Arbitrary high order
- Possibly monotonicity/positivity preserving
- Steady state preserving

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$$\partial_t U + \partial_x F(U) = S(U) \partial_x U$$

Steady state preserving: on a given mesh we want to the scheme embed in the discretization a

Definition of $\partial_x F_h = S_h$ such that data verifying it is exactly preserved

Projection operator providing such data, possibly different from the scheme

Characterization of the above projection: e.g enhanced consistency, well posedness

There are numberless approaches, and vast literature. Related work:

① Reconstruction/evolution of fluctuations wrt discrete equilibria (**approximate full well balanced**):

- Castro & Parés J.Sci.Comp. 2020, Gómez-Bueno et al, Appl.Math.Comp. 2021, Gómez-Bueno et al Mathematics 2021, Guerrero Fernández et al Mathematics 2021, Gómez-Bueno et al, Appl.Num.Math. 2023, etc. etc.

② Well balanced via integration of the source term and global fluxes

- Gascón & Corberán JCP 2001, Donat & Martinez-Gavara J.Sci.Comp. 2011, Chertock et al JCP 2018, Cheng et al J.Sci.Comp. 2019, Mantri&Noelle JCP 2021, Ciallella et al. J.Sci.Comp. 2023, Xu and Shu J.Sci.Comp. to appear, etc. etc.

③ Other approaches

-

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U) \partial_x H$$

Approach 1.¹

$$\frac{d\bar{U}_i}{dt} + \frac{1}{\Delta x} \left(\hat{G}_{i+1/2} - \hat{G}_{i-1/2} \right) = 0$$

¹M. Ciallella, D. Torlo, MR, Arbitrary High Order WENO Finite Volume Scheme with Flux Globalization for Moving Equilibria Preservation, *J.Sci.Comp.* 96, 2023

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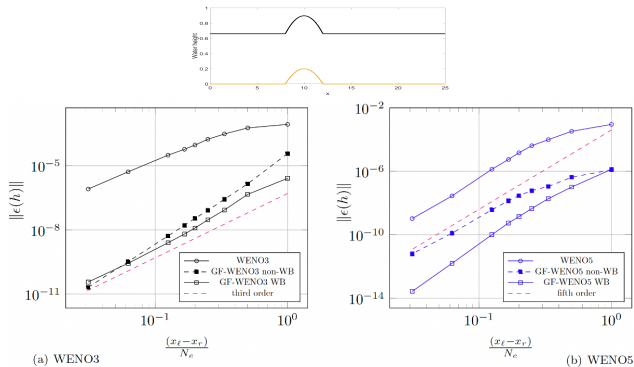
- 1 Reconstruct WENO polynomials $U_i(x)$
- 2 Compute cell averaged source primitive $\bar{R}_i = R_{i-1/2}^+ - \sum_q \omega_q \int_{x_{i-1/2}}^{x_q} S(U_i(x), \varphi(x)) dx$
- 3 Compute cell averaged fluxes $\bar{F}_i = \sum_q \omega_q F(U_i(x_q), \varphi(x_q)) dx$
- 4 Reconstruct WENO polynomials $G_i(x) = (F + R)_i(x)$
- 5 Compute upwind fluxes $\hat{G}_{i+1/2} = (A^+ A^{-1})_{i+1/2} G_i(x_{i+1/2}) + (A^- A^{-1})_{i+1/2} G_{i+1}(x_{i+1/2})$

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Approach 1.¹



Supercritical flow: convergence tests

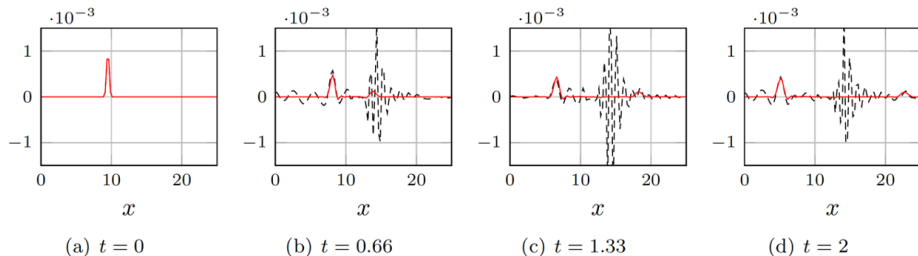
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Approach 1.¹

Small perturbation of steady state



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$$\partial_t U + \partial_x F(U) = S(U) \partial_x H$$

Approach 1.¹

- ✓ No a-priori knowledge of equilibrium, all steady states!
- ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
- ✓ Considerable accuracy enhancements at steady state (3 to 4 orders of magnitude)
- ✗ No super convergence
- ✗ Very hard to characterize the steady state and generate one

$$\overline{R}_i = R_{i-1/2}^+ - \sum_q \omega_q \int_{x_{i-1/2}}^{x_q} S(\mathbf{U}_i(x), \varphi(x)) dx$$

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$$\partial_t U + \partial_x F(U) = S(U) \partial_x H$$

New approach

$$\frac{dU_i}{dt} + \frac{1}{\Delta x} \left(\hat{G}_{i+1/2} - \hat{G}_{i-1/2} \right) = 0$$

1 Global Flux Quadrature with WENO approximation

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U) \partial_x H$$

New approach

$$\frac{dU_i}{dt} + \frac{1}{\Delta x} \left(\hat{G}_{i+1/2} - \hat{G}_{i-1/2} \right) = 0$$

- 1 Reconstruct ~~WENO~~ polynomials $U_i(x)$
- 2 Compute nodal source primitive $R_i = R_{i-1} - \Delta x \sum_q \omega_q S(U_{i-q}, \varphi(x_{i-q}))$
- 3 Compute cell averaged fluxes $\bar{F}_i = \sum_q \omega_q F(\hat{U}_i(x_q), \varphi(x_q)) dx$
- 4 Reconstruct WENO polynomials $G_{i+1/2}^\pm(x) = (F + R)_{i+1/2}^\pm(x)$
- 5 Compute upwind fluxes $\hat{G}_{i+1/2} = P_{i+1/2}^+ G_{i+1/2}^-(x_{i+1/2}) + P_{i+1/2}^- G_{i+1/2}^+(x_{i+1/2})$

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x F(U) = S(U) \partial_x H$$

New approach

Main result

Proposition (Discrete steady state). *The WENO-FD scheme with global flux quadrature preserves exactly continuous discrete steady states $U_i^* = U(F_i)$ with F obtained by integrating the ODE*

$$F' = S(U(F)) \partial_x H$$

using the multi-step ODE integrator with weights $\{\omega_q\}_{q \geq 0}$ on spatial slabs of size h .

If $U(F)$ is a one to one mapping, $U^(x)$ verifies the consistency estimates of the multi-step scheme.*

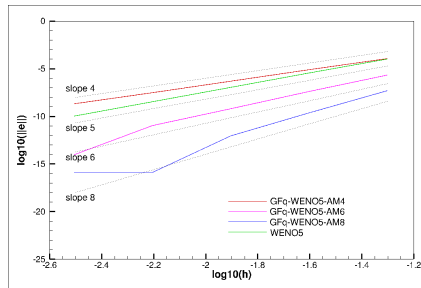
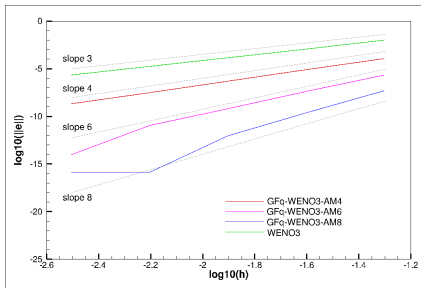
Multi-step schemes so far: Adams methods AB_n and AM_n

We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U) \partial_x H$$

New approach

For $S(U) = U^2$ and $H(x) = x$ exact steady state $u = e^x$

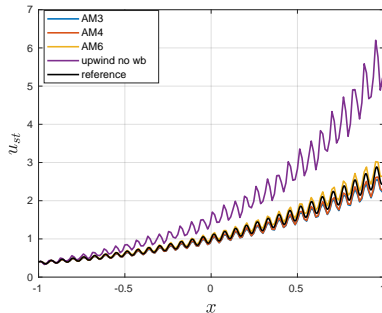
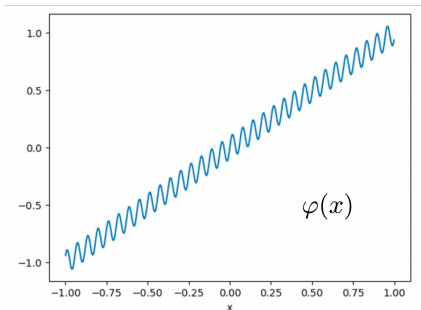


We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U) \partial_x H$$

New approach

For $s(U) = U^2$ and $\varphi = x + 100 \sin x$

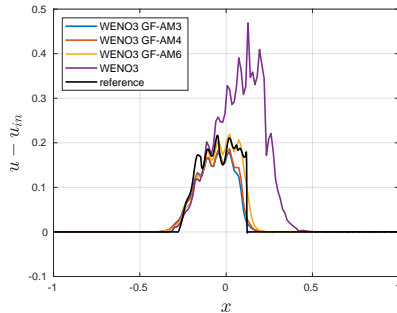
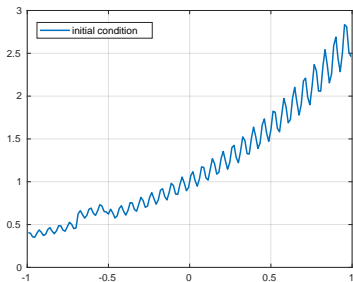


We seek solutions of the hyperbolic system of balance laws

$$\partial_t U + \partial_x \frac{U^2}{2} = S(U) \partial_x H$$

New approach

For $S(U) = u^2$ and $H = x + 100 \sin x + \text{top hat of perturbation } \delta u = 0.2 \chi_{[-0.7, -0.5]}$

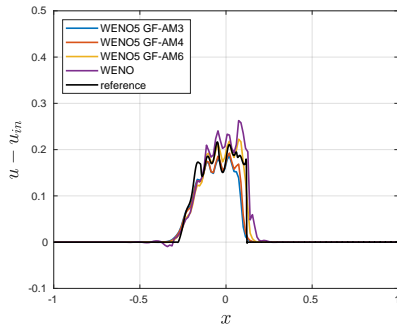
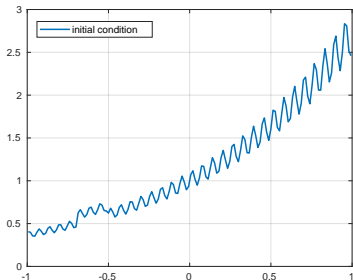


We seek solutions of the hyperbolic system of balance laws

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} = - \begin{bmatrix} 0 \\ h \end{bmatrix} b'(x)$$

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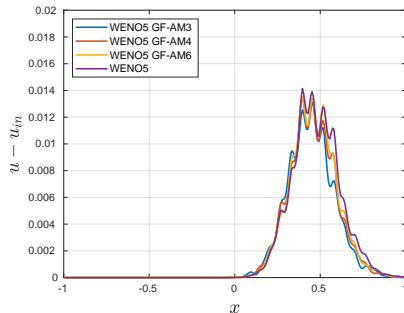
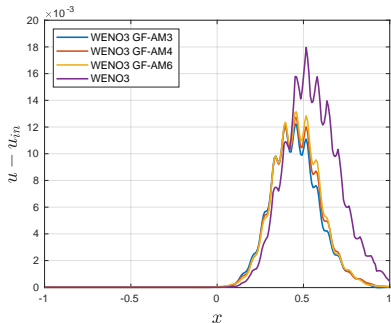


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For $S(U) = u^2$ and $H = x + 100 \sin x + \text{top hat of Gaussian perturbation}$ $\alpha = 0.005$

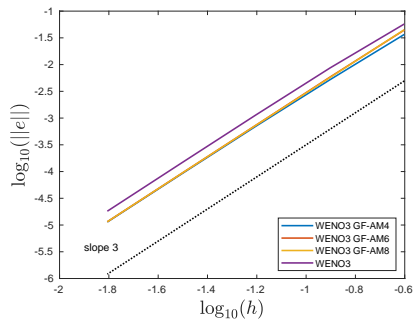
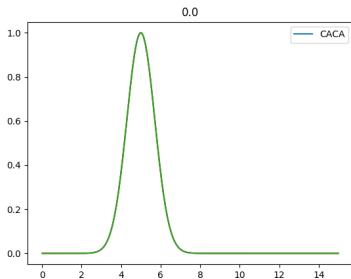


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New approach

For $S(U) = u - 1$ and $H(x) = e^{-(x-x_0-ct)^2}$, $u(x) = e^{-(x-x_0-ct)^2}$. The wave is centered at $x_0 = 0.5$.

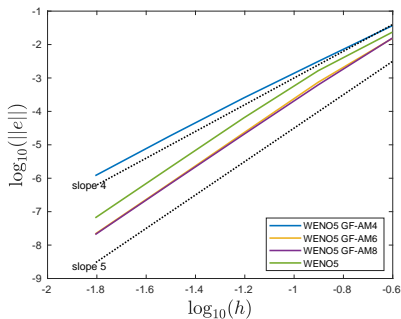
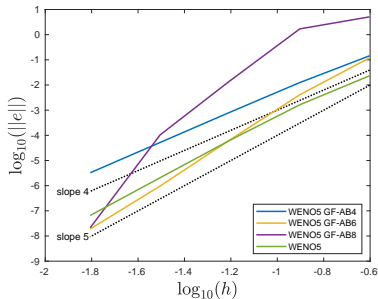


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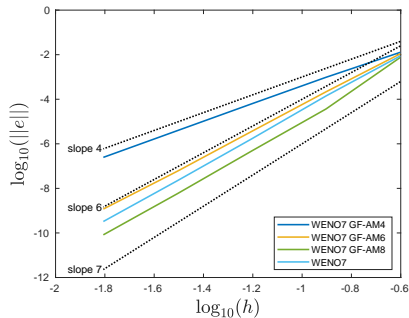
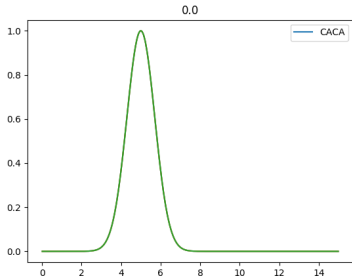


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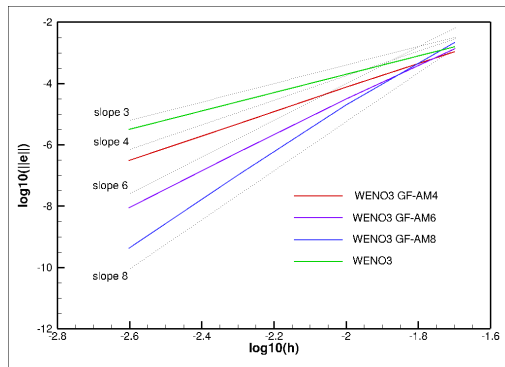
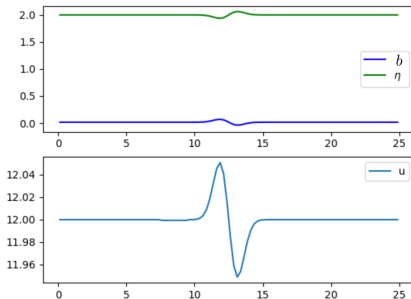


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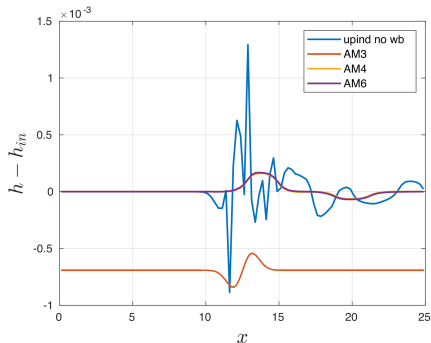
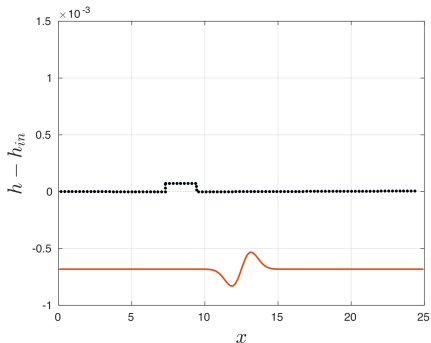
Supercritical flow on a smooth hump



We seek solutions of the hyperbolic system of balance laws

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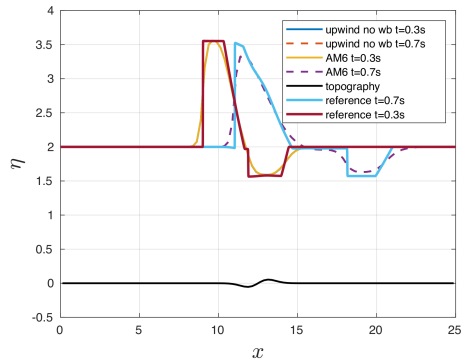
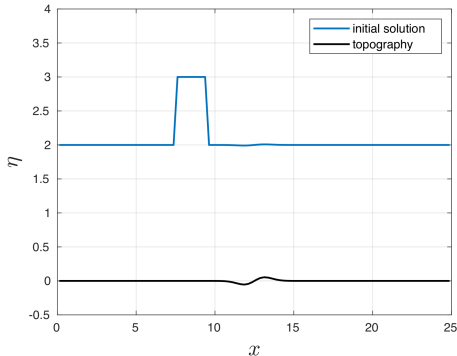


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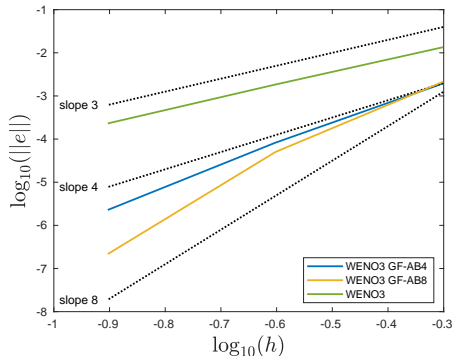
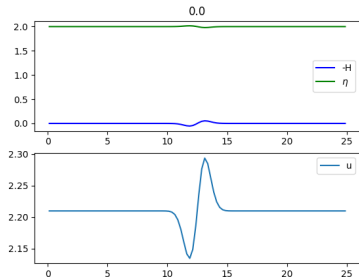


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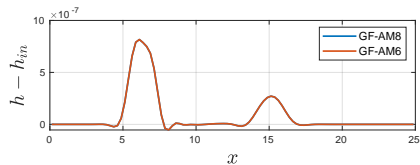
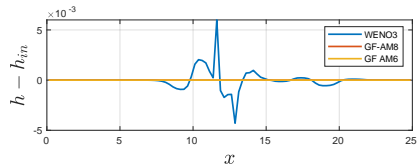
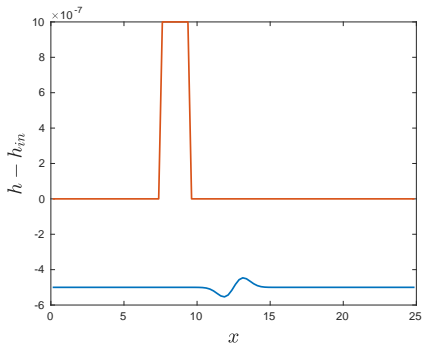
Subcritical flow on a smooth hump



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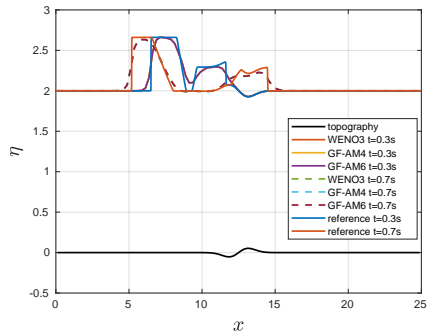
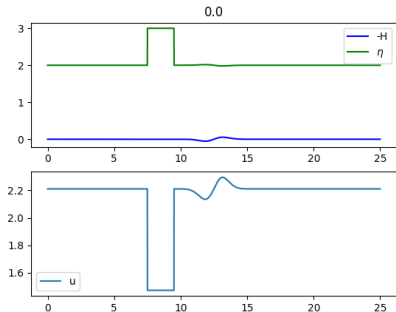
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New approach

- ✓ No a-priori knowledge of equilibrium, all steady states!
- ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
- ✓ Convergence at steady state arbitrarily accurate by changing ODE weights
- ✓ Discrete initial state can be generated
- ✗ Non-compact quadrature

Take home message

- GFq in 1D: clever quadrature of the source based on high order ODE integrators
- Tremendous error reduction at stady-state (super-convergence)
- In one dimension discrete equilibria can be generated a-priori if necessary with the ODE solver

Future

- Sonic points ?
- discontinuous data ?
- adaptive ODE weights
- multiD