Problem #5

The pdf of an exponentially distributed function looks like:

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & , \ x \geq 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

* To find the mean, we know that:

$$\mu = E[X] = \int_0^\infty x * f(x,\lambda)\, dx = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx = \lambda\left[\int_0^\infty x\, e^{-\lambda x}\right] dx$$

⇝ Integration by parts: $I = u \cdot v - \int v \cdot du$

Set: $u = x \quad \Rightarrow \quad du = dx$

$dv = e^{-\lambda x}\, dx \quad \Rightarrow \quad v = -\frac{e^{-\lambda x}}{\lambda}$

$$\therefore \ \mu = \lambda\left[\left(-\frac{x \cdot e^{-\lambda x}}{\lambda}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{e^{-\lambda x}}{\lambda}\right) dx\right] \quad \approx \text{multiply \& distribute } \lambda$$

$$\mu = \left(-x \cdot e^{-\lambda x}\right)\Big|_0^\infty - \frac{1}{\lambda}\left(e^{-\lambda x}\right)\Big|_0^\infty$$

$$\mu = \left[\left(\overset{0}{-\infty \cdot \frac{1}{e^{-\lambda(\infty)}}}\right) - \left(\overset{0}{-(0) \cdot e^{-\lambda(0)}}\right)\right] - \frac{1}{\lambda}\left[\left(\overset{0}{\frac{1}{e^{\lambda(\infty)}}}\right) - \left(\overset{1}{e^{-\lambda(0)}}\right)\right]$$

$$\mu = -\frac{1}{\lambda}(-1) \quad \Rightarrow \quad \boxed{\mu = \frac{1}{\lambda}}$$

* To find the variance, we know that :

$$Var(X) = E(X^2) - [E(X)]^2$$

Integrate by parts #1:

$$\approx \quad E(X^2) = \lambda \int_0^\infty x^2 e^{-\lambda x}\,dx \quad \Big\} \quad \begin{array}{l} u = x^2 \longrightarrow du = 2x\,dx \\ dv = e^{-\lambda x}dx \rightarrow v = -\frac{e^{-\lambda x}}{\lambda} \end{array}$$

$$\underbrace{\sigma^2 + \mu^2 = \lambda\left[-\frac{x^2 e^{-\lambda x}}{\lambda}\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty x e^{-\lambda x}\,dx\right]} = \left(-x^2 e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x}\,dx$$

Integrate by parts #2:

$$u = x \rightarrow du = dx$$

$$dv = e^{-\lambda x}dx \rightarrow v = -\frac{e^{-\lambda x}}{\lambda}$$

$$\sigma^2 + \mu^2 = \left(-x^2 e^{-\lambda x}\right)\Big|_0^\infty + 2\left[\left(-\frac{x e^{-\lambda x}}{\lambda}\right)\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\,dx\right]$$

$$E(X^2) = \left(-x^2 \cancel{e^{-\lambda x}}\right)\Big|_0^\infty + 2\left(-\frac{x \cancel{e^{-\lambda x}}}{\lambda}\right)\Big|_0^\infty - \frac{2}{\lambda^2}\left(e^{-\lambda x}\right)\Big|_0^\infty$$

0 0

$$E(X^2) = -\frac{2}{\lambda^2}\left[\left(\cancel{\frac{1}{e^{\lambda\infty}}}\right) - \cancel{e^{-\lambda(0)}}\right] = -\frac{2}{\lambda^2}(-1) = \frac{2}{\lambda^2}$$

0 1

$$\therefore \quad Var(X) = \frac{2}{\lambda^2} - \mu^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{Var(X) = \frac{1}{\lambda^2}}$$