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ECE 504
Homework #8

Problem #1 - Chen 6.1

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

Is the state equation controllable? observable?

$$y = \underbrace{[1 \ 2 \ 1]}_{C} x$$

- Since $\bar{x}$ is controllable iff $\bar{x}$ is reachable, we can check the reachability matrix for full rank:

$$Q_r = [B \ AB \ A^2B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

By inspection, it can be seen that rank$(Q_r) = 3$, implying a full rank.

A full rank $\Rightarrow$ reachability, which also implies controllabity. (system is controllable).

- To find observability, check for full rank in the observability matrix:

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

By inspection, it can be seen that rank$(Q_o) = 1$ $\rightsquigarrow$ columns 2 & 3 are dependent on the first!

Since rank$(Q_o) \neq n$, the system is unobservable.

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Problem #2 - Chen 6.3

Let: $Q_r = [B \; AB \; A^2B \dots A^{n-1}B]$
$Q_r^* = [AB \; A^2B \; A^3B \dots A^nB]$

The rank(Q_r) does not always equal the rank(Q_r^*)!

→ This condition will exist if the following cases are avoided:

① $A = \text{zeros}(n,n)$. This implies that the state will not be changing as a function of time. Assuming that B is non-zero, the following is seen:

$\text{rank}(Q_r) = 1$, $\text{rank}(Q_r^*) = \emptyset$ (null)

② $A = \text{ones}(n,n)$. This implies that starting after entry AB, a linear (exponentially increasing) dependence will exist between the column vectors...

$\text{rank}(Q_r) = 2 \rightsquigarrow [\overset{1)}{B} \; \overset{2)}{AB} \; A^2B \; A^3B \dots]$
$\text{rank}(Q_r^*) = 1 \rightsquigarrow [\overset{1)}{AB} \; A^2B \; A^3B \; A^4B \dots]$

→ Rank(Q_r) will always equal Rank(Q_r^*) if $A = I_n$.

$$Q_r = [B \; AB \; A^2B \dots A^{n-1}B] = [B \; B \; B \dots B]$$
$$Q_r^* = [AB \; A^2B \; A^3B \dots A^nB] = [B \; B \; B \dots B]$$
Rank are equal!

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Problem #3 – Chen 6.8

In order to reduce the state-equations, convert to a transfer function and check for pole cancellations:

$$G = C(sI-A)^{-1}B + D = [1\ 1]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [0]$$

$$\hat{G}(s) = \frac{2}{s-3}$$ ⇝ this means that the system can be represented with one state!

$$\hat{g}(s) = \frac{2}{s-3} = \frac{N(s)}{D(s)}$$

set: $\hat{v}(s) = \frac{1}{D(s)} \cdot \hat{u}(s)$ ⇝ $\hat{v}(s)D(s) = \hat{u}(s)$

$\hat{v}(s)(s-3) = \hat{u}(s)$

$\downarrow \mathcal{L}^{-1}$

$\dot{v}(t) - 3v(t) = u(t) \longrightarrow \dot{v}(t) = 3v(t) + u(t)$ *

set: $x(t) = [v(t)] \Rightarrow \dot{x}(t) = [\dot{v}(t)]$

$\therefore \underbrace{\dot{x}(t) = [3]x(t) + [1]u(t)}_{A \qquad B}$ *

set: $\hat{y}(s) = \hat{g}(s)\hat{u}(s) = N(s)\hat{v}(s) \longrightarrow \hat{y}(s) = (2)\hat{v}(s)$

$\downarrow$

$y(t) = 2v(t) = 2x(t)$

$\therefore \underbrace{y(t) = [2]x(t) + [0]u(t)}_{C \qquad D}$

Is system controllable? Check for reachability. $Q_r = [B] = [1]$ ⇝ $\text{rank}(Q_r) = 1 = \text{full}$
~ full rank ⇒ reachability ⇒ controllability

Is system observable? Check observability matrix. $Q_o = [C] = [2]$ ⇝ $\text{rank}(Q_o) = 1 = \text{full}$
~ full rank ⇒ observability

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Problem #4 - Chen 6.15

In checking for controllability, find the reachability matrix:

$$Q_r = [B \; AB \; A^2B \; A^3B \; A^4B] = \begin{bmatrix} b_{11} & b_{12} & b_{11}+b_{21} & b_{12}+b_{22} & b_{11}+2b_{21} & b_{12}+2b_{22} & \\ b_{21} & b_{22} & b_{21} & b_{22} & b_{21} & b_{22} & \\ b_{31} & b_{32} & b_{31}+b_{41} & b_{32}+b_{42} & b_{31}+2b_{41} & b_{32}+2b_{42} & \cdots \\ b_{41} & b_{42} & b_{41} & b_{42} & b_{41} & b_{42} & \\ b_{51} & b_{52} & b_{51} & b_{52} & b_{51} & b_{52} & \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ (5) & (6) \end{array}$$

VERIFIED THROUGH MATLAB.

It can be seen that columns 1-4 are linearly independant, ensuring a rank of atleast 4... it can be shown that (5) = 2 × (3) - 1 × (1) AND (6) = 2 × (4) - 1 × (2) ... this will be the case in increasing powers of n for the entire reachability matrix. This means that the rank, for any choices of be, will only reach 4 at greatest. Since rank$(Q_r) \neq 10$, then system is not reachable ⇒ not controllable!

In checking for observability, find the observability matrix:

$$Q_o = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{11} & c_{11}+c_{12} & c_{13} & c_{13}+c_{14} & c_{15} \\ c_{21} & c_{21}+c_{22} & c_{23} & c_{23}+c_{24} & c_{25} \\ c_{31} & c_{31}+c_{32} & c_{33} & c_{33}+c_{34} & c_{35} \\ c_{11} & 2c_{11}+c_{12} & c_{13} & 2c_{13}+c_{14} & c_{15} \\ c_{21} & 2c_{21}+c_{22} & c_{23} & 2c_{23}+c_{24} & c_{25} \\ c_{31} & 2_{31}+c_{32} & c_{33} & 2c_{33}+c_{34} & c_{35} \\ c_{11} & 3c_{11}+c_{12} & c_{13} & 3c_{13}+c_{14} & c_{15} \\ c_{11} & 3c_{21}+c_{22} & c_{23} & 3c_{23}+c_{24} & c_{25} \\ c_{31} & 3c_{31}+c_{32} & c_{33} & 3c_{33}+c_{34} & c_{35} \\ c_{11} & 4c_{11}+c_{12} & c_{13} & 4c_{13}+c_{14} & c_{15} \\ c_{21} & 4c_{21}+c_{22} & c_{23} & 4c_{23}+c_{24} & c_{25} \\ c_{31} & 4c_{31}+c_{32} & c_{33} & 4c_{33}+c_{34} & c_{35} \end{bmatrix}$$

→ In analyzing the matrix symbolically, it is seen that the rank$(Q_o) = 5$, implying that the matrix is observable, * in most cases.*

If $c_{i,j} \; \forall (i,j) = 1$, then columns (1) and (5) would be linearly dependent, implying an unobservable system.

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Problem #5 – Chen 7.2

$\hat{g}(s) = \dfrac{s-1}{s^3+2s^2-s-2}$ ⇝ Find the observable canonical form.

Mapping this equation : $N(s) = \quad + \beta_1 s^2 + \beta_2 s + \beta_3$
$D(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$

→ The variables end up being : $\beta_1 = 0, \ \beta_2 = 1, \ \beta_3 = -1$
$\alpha_1 = 2, \ \alpha_2 = -1, \ \alpha_3 = -2$

Looking @ pg. 188, put the variables into observable canonical form :

$$\dot{x} = \begin{bmatrix} -\alpha_1 & 1 & 0 \\ -\alpha_2 & 0 & 1 \\ -\alpha_3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u \rightsquigarrow \dot{x} = \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{[1 \ 0 \ 0]}_{C} x + \underbrace{[0]}_{D} u$$

Check for observability :

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}$$

→ rank$(Q_o) = 3 =$ full, implying system is observable !

Check for reachability :

$$Q_r = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -3 \\ -1 & 0 & -1 \end{bmatrix}$$

⇝ rank$(Q_r) = 3 =$ full, implying system is reachable

⇓

System is controllable.