Lec 6 10/13/09 → oct 6 slides

Slide 33/53

Proof Sketch: Assume $m_j = r_j$ for $j \in \{1, \dots, s\}$

"big beta"

- for each j let $\mathcal{B} = \{\underbrace{V_{j1}, V_{j2}, \dots, V_{jr_j}}_{\text{All off these vectors are linearly independent}}\}$ be a basis for the eigen space associated w/ the e-value λ_j

- String all of the bases together to form

$$\mathcal{B} = \{\underbrace{\mathcal{B}_1}_{r_1 \text{ vectors}}, \dots, \underbrace{\mathcal{B}_s}_{r_s \text{ vectors}}\}$$

- total of vectors in $\mathcal{B} = \sum_j r_j = n$

Facts:

- each $\mathcal{B}_j$ is a basis for some subspace
- E-vectors corresponding to different e-values are linearly independent

Form V using vectors from $\mathcal{B}$; V has dimensions

$$V = [v_{11} \cdots v_{1r_1}, v_{21} \cdots v_{2r_2}, v_{s1} \cdots v_{sr_s}] \in \mathbb{C}^{n \times n}$$

- Is V invertable? Yes because the rank is n
- we can also say $AV = V \underbrace{\begin{bmatrix} \lambda_1 I_{r_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_s I_{r_s} \end{bmatrix}}_{\Lambda} \Rightarrow V^{-1}AV = \Lambda$

  $\Rightarrow$ A is a diagonizable matrix

Slide 34/53 ex

- E-Vectors corresponding to different e-values must be linearly independent.

Assume V_1 and V_2 corresponding to λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$

And $C_1V_1 + C_2V_2 = 0$ (V_1 & V_2 are linearly depend.) for some $C_1 \neq 0$ and $C_2 \neq 0$

Take (1) multiply by $A \Rightarrow A(C_1V_1 + C_2V_2) = 0$

$\Rightarrow C_1AV_1 + C_2AV_2 = 0$ eq (1)

$(C_1\lambda_1V_1 + C_2\lambda_2V_2) = 0$ eq (2)

Multiply (1) by λ_2 and subtract from (2)

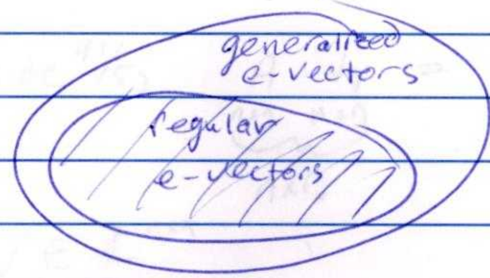
$$C_1\lambda_1V_1 + \cancel{C_2\lambda_2V_2} - (C_1\lambda_2V_2 + \cancel{C_2\lambda_2V_2}) = 0$$

$\Rightarrow C_1(\lambda_1 - \lambda_2)V_1 = 0$ Contradiction

OK, let $C_1 = 0$

V cant be a zero e-vector

Slide 37/53 ① yes ② no



Slide

$\mathcal{E}(\lambda_0)$ regular eigen space corresponding to evalue λ

$\mathcal{F}(\lambda_0)$ generalized eigen space

$\mathcal{E}(\lambda_0) \subseteq \mathcal{F}(\lambda_0)$

ex $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ Char. poly. $\det(\lambda I_2 - A) = (\lambda - 1)$

$\lambda_1 = 1 \quad r_1 = 2$

# of distinct e-values → $s = 1$

look @ $A - \lambda_1 I_2 = \underset{\text{Rank}=1}{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}}$ we want $(A - \lambda_1 I_2)V$

$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ e-vector

$m_1 = 1 < r_1$ So not diagonizable

generalized e-vector $(A - \lambda_1 I_2)^2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Rank =

$\underbrace{V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{generalized eigen vectors}}$

Slide $4/53$ ① n

Slide $41/53$ $\underbrace{\underset{n\times n}{A}\ \underset{n\times r_j}{V_j}}_{n\times r_j} = \underbrace{\underset{n\times r_j}{V_j}\ \underset{r_j\times r_j}{Q_j}}_{n\times r_j}$

$V \in \mathbb{C}^{n\times n}$

ex

$$Q = \begin{bmatrix} \begin{pmatrix}1&1\\1&1\end{pmatrix} & \begin{matrix}0&0\\0&0\end{matrix} \\ \begin{matrix}0&0\\0&0\end{matrix} & \begin{pmatrix}1&1\\1&1\end{pmatrix}\end{bmatrix}$$ Block Diagonal Matrix $n\times n$

Slide $44/53$ ex Assume

$(Q_j - \lambda_j I_{r_j})^{r_j}$ ~~we~~ we know $V^{-1}AV = Q$

$$= \begin{bmatrix} Q_1 & & 0 \\ 0 & \ddots & 0 \\ 0 & & Q_s \end{bmatrix} \Rightarrow V^{-1}(A-\lambda_1 I_n)V$$

$$= \begin{bmatrix} Q_1 - \lambda_1 I_{r_1} & \\ & \ddots \end{bmatrix}$$

Still block diagram

$$V^{-1}(A-\lambda I_n)^{r_j} V = V = \begin{bmatrix} (Q - \lambda_1 - I_{r_1})^{r_j} & & \\ & \ddots & \\ & & \text{stuff} \end{bmatrix}$$

first vectors ~~of~~ of V matrix

Still block diagonlizabale

$\Rightarrow$ First r_j columns of RHS have to $= \phi$

Slide 45/53 ① no ② yes ③ yes

Slide 47/53 • If A cant be diagonalized use nillpotent to do it.

$V^{-1}AV = Q = \Lambda + \hat{N}$

$A^k = (V(\Lambda+\hat{N})V^{-1})(V(\Lambda+\hat{N})V^{-1}) \cdots (V(\Lambda+\hat{N})V^{-1})$

$= V(\Lambda+\hat{N})^k V^{-1}$

Slide 48/53 $\exp(t\hat{N})$ will go to zero or "truncate"

ex

$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ Char poly $= (\lambda-3)^2$ implies that only 1 distinct e-value exists.

Now check if A is diagonalizable?

$A - \lambda I_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\lambda_1 = 3, \; r_1 = 2, \; S = 1$

→ only 1 dimension nullspace $\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basis for $\mathcal{E}(\lambda_1)$

$\Rightarrow m_1 = 1 < r_1 \Rightarrow$ Not diagonalizable

Now

generalized e-values $(A - \lambda_1 I_2)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \mathcal{F}(\lambda_1)$ is all of $\mathbb{R}^2$

$\Rightarrow \{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is a basis for $\mathcal{F}(\lambda_1)$

$A = V(\Lambda + \hat{N})V^{-1}$

$\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$V^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\hat{N} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$V^{-1}AV = \underbrace{\Lambda}_{\text{diagonal bits}} + \underbrace{\hat{N}}_{\text{left overs}}$

do this

$e^{tA} = \overset{I}{\cancel{V}} e^{t\Lambda} e^{t\hat{N}} \overset{I}{\cancel{V^{-1}}} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} [\,?\,] = \begin{pmatrix} e^{3t} & 2t \\ 0 & e \end{pmatrix}$

$e^{t\hat{N}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \hat{N}^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix} + \text{zeros} = \begin{pmatrix} 1 & \\ 0 & \end{pmatrix}$

Slide 51/53 #6 has typo exp(~) =

missing equals

Midterm: 90 minutes

Part I: Mathematical representations of systems

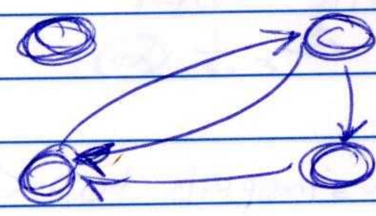
Lec 1-3
- TF
- Impulse response
- SS
- DE

• Pros/cons
• Know what system to use for each situation.

ex last year midterm (2)
e^{tA} for LTI System SS

- Characterization of systems → lumped distribution / memoryless / time inv.

- going from physical model to math
Circuits ⟹ math

-  Transition from one Rep. to another.

* Know all linear Alg. * (det, Adj, etc...)

- No Jordan form ("cononical forms")

Know this
$$\hat{g}(s) = C(sI-A)^{-1}B + D$$

- Linearization

Part 2: Quantitative & Qualitative Analysis

Lec 4-5

- Focus on SS descriptions
  - Solutions to LTI / LTV CT/DT SS descri
  - STM
- Diagonalizability
  - LTI: A^k, e^{tA}
  - "If A matrix is a nillpotent then Peano Baker will cancel out"
  - Typically use fundemental Matrix unless its obvious A will cancel out w/ Peano Baker
  - Know fundemental Matrix & Peano
  - Know simple transform tables
    ex) $\frac{1}{s+3} \Leftrightarrow e^{3t}$ etc... δ impulse

If A is not diagonizable by $\mathcal{E}(\lambda)$ regular ~~Diagon~~ the use $\mathcal{F}(\lambda)$ to get generalized

generalized $e^{tA} = \Phi(t, t_0) = e^{t\Lambda} e^{t\hat{N}}$

- Last year. Midterm #4 b check answer
  - by setting $t=0 \rightarrow e^{tA} = I_n$
  - $\frac{\partial}{\partial t}(e^{tA}) = Ae^{tA}$

c) A^{100} $P = \Lambda$ $Q = \hat{N}$ or $P = \hat{N}$ $Q = \Lambda$

$\leftrightarrow$ Commutative $P + Q$

$A^{100} = V(\Lambda + \hat{N})^{100} V^{-1} \Rightarrow A^{100} = \Lambda + \hat{N}$

V because $= I_n$

Fundemental Matrix theorem

Chen 4.16

ex

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix}$$

$\boxed{\text{Step1: Solve DE}}$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Rightarrow \begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= t x_2(t) \end{aligned}$$

← do first not dependent on $x_1(t)$ only $x_2(t)$

try $x_2(t) = e^{t^2/2}$ $\quad \frac{\partial}{\partial t} x_2(t) = e^{t^2/2}\left(\frac{\partial}{\partial t} \frac{t^2}{2}\right) = t e^{t^2/2}$

$\Rightarrow x_2(t) = c e^{t^2/2} \qquad c = x_2(0)$

$\dot{x}_2(t) = t c e^{t^2/2}$

now solve for $\dot{x}_1(t) = c e^{t^2/2}$

constant $x_1(0)$

Solve for $x_1(t) = \int_0^t c e^{t^2/2}\,\partial t + d$

Generate Initial conditions

$x_p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_q(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

must be linearly independent

Solve initial conditions $\quad x_p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} d &= 1 \\ c &= 0 \end{aligned}$

$x_q(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} d &= 0 \\ c &= 1 \end{aligned}$

$$X(t) = \begin{bmatrix} 1 & \int_0^t e^{t^2/2}\,\partial t \\ 0 & e^{t^2/2} \end{bmatrix}$$

$$\boxed{\Phi(t, t_0) = X(t) X^{-1}(t_0)}$$

fund mtx: must always be invertable otherwise something is wrong