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Problem #1

a) $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\rightsquigarrow$ To find characteristic polynomial:

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda-1 & -4 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix}$$

$\therefore$ Characteristic polynomial $= (\lambda-1)(\lambda-2)^2$

$\Rightarrow$ Eigenvalues : $\lambda_1 = 1$, $\lambda_2 = 2$
$\Rightarrow$ Algebraic Mult : $r_1 = 1$, $r_2 = 2$

$E(\lambda_1)$: $A - \lambda_1 I_3 = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = 0 \rightsquigarrow$ By inspection: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or any 'a'

LI and in null!

* Since the geometric multiplicity of λ_1 is upper bounded by $r_1 = 1$, we don't need to search anymore. $\{v_1\}$ is a basis for $E(\lambda_1)$

$E(\lambda_2)$: $A - \lambda_2 I_3 = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0 \rightsquigarrow$ By inspection: $v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

LI & in null!

* Since the geometric multiplicity of λ_2 is upper bounded by $r_2 = 2$, we don't need to search anymore. $\{v_2, v_3\}$ is a basis for $E(\lambda_2)$.

$\Rightarrow m_1 = 1$, $m_2 = 2$ $\rightarrow$ Since $r_j = m_j$ for $j \in \{1, 2\}$, A is diagonizable.

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\rightsquigarrow$ $\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix} = \lambda^2$ characteristic polynomial

$\Rightarrow$ Eigenvalue : $\lambda_1 = 0$
$\Rightarrow$ Algebraic Multiplicity : $r_1 = 2$

$E(\lambda_1)$: $A - \lambda_1 I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v = 0 \rightsquigarrow v_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$ any integer/value; LI and in null space

Geometric multiplicity, $m_1 = 1$

$\therefore$ Since $r_j \neq m_j$ for $j \in \{1\}$, A is NOT diagonizable.

c) When $A = I_n \rightarrow$ characteristic polynomial $= (\lambda - 1)^n$

$\Rightarrow$ Eigenvalue : $\lambda_1 = 1$
$\Rightarrow$ Algebraic multiplicity : $r_1 = n$
$\Rightarrow$ Geometric multiplicity : $m_1 = n$

$E(\lambda_1) = A - \lambda_1 I_n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n \times n} v = 0$

Basis of nullspace:

$\left\{ \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right\} \rightarrow$ n vectors in $\mathbb{R}^n$

$\therefore$ A is diagonizable since $r_1 = m_1$!

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Problem #2

Using results from Problem 1:

a) $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad V = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $V^{-1} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $\left.\begin{matrix} \lambda_1 = 1 \text{ with } r_1 = 1 \\ \lambda_2 = 2 \text{ with } r_2 = 2 \end{matrix}\right\} \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$\therefore \quad e^{At} = V \begin{bmatrix} e^{\lambda_{11}t} & 0 & 0 \\ 0 & e^{\lambda_{22}t} & 0 \\ 0 & 0 & e^{\lambda_{33}t} \end{bmatrix} V^{-1} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow e^{At} = \begin{bmatrix} e^t & -4e^t(1-e^t) & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$A^{100} = V \Lambda^{100} V = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow A^{100} = \begin{bmatrix} 1 & -4 + 4 \cdot 2^{100} & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix}$$

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

↳ Do not know of any distinct methods to solve for e^{At} of non-diagonizable matrices... used MATLAB to find solution:

```
syms t;
A = [0 1; 0 0]
expm(A*t)
```

$\Longrightarrow \rightsquigarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

~(see attached MATLAB code)

* Knowing that $A^{100} = A \cdot A \cdot A^{98} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} A^{98}$

$$A^{100} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} A^{98}$$

Anything multiplied by $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ ~ $\therefore A^{100} = 0$

Problem #3

$\dot{x}(t) = Ax(t)$, $t \in \mathbb{R}$, $A \in \mathbb{R}^{2\times 2}$ constant (scalar) matrix
$x(t) = e^{At}x(0)$

experiment #1 : $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2e^{-2t} - e^{-3t} \\ e^{-3t} \end{bmatrix} = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

experiment #2 : $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} e^{-3t} \\ e^{-3t} \end{bmatrix} = e^{At}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

let $e^{At} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then rewrite equation:

$$\begin{bmatrix} 2e^{-2t} - e^{-3t} & e^{-3t} \\ e^{-3t} & -e^{-3t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^{-1} = \underbrace{\frac{1}{(-1\cdot 1)-(1\cdot 1)}}_{-\frac{1}{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - e^{-3t} & e^{-3t} \\ e^{-3t} & -e^{-3t} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} - \cancel{\tfrac{1}{2}e^{-3t}} + \cancel{\tfrac{1}{2}e^{-3t}} & e^{-2t} - \tfrac{1}{2}e^{-3t} - \tfrac{1}{2}e^{-3t} \\ \cancel{\tfrac{1}{2}e^{-3t}} - \cancel{\tfrac{1}{2}e^{-3t}} & \tfrac{1}{2}e^{-3t} + \tfrac{1}{2}e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} = \Phi(t,0) = \text{STM}$$

Check: Does $\Phi(0,0) = I_2$? $\Phi(0,0) = \begin{bmatrix} e^0 & e^0 - e^0 \\ 0 & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

To find A : $\frac{d}{dt}\Phi(t,s) = A\,\Phi(t,s)$

$$\frac{d}{dt}\Phi(t,0) = \begin{bmatrix} -2e^{-2t} & -2e^{-2t} + 3e^{-3t} \\ 0 & -3e^{-3t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

Solve equations: $-2e^{-2t} = e^{-2t}a_{11} + 0\cdot a_{12} \Rightarrow a_{11} = -2$

$\therefore \cancel{-2e^{-2t}} + 3e^{-3t} = \cancel{-2e^{-2t}} + 2e^{-3t} + a_{12}e^{-3t}$

$3e^{-3t} = 2e^{-3t} + a_{12}e^{-3t} \Rightarrow a_{12} = 1$

$0 = a_{21}\cdot e^{-2t} + a_{22}\cdot 0 \Rightarrow a_{21} = 0$

$\therefore -3e^{-3t} = 0(e^{-2t} - e^{-3t}) + a_{22}\cdot e^{-3t} \Rightarrow a_{22} = -3$

$$\approx \quad A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}$$

Problem #4

Using 2 different methods, find unit step response of:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad , \quad y = [2 \;\; 3]\, x \quad \Rightarrow \text{Assume } x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(zero conditions)

① Laplace domain calculation:

$y(t) = \mathcal{L}^{-1}\{\hat{g}(s)\,\hat{u}(s)\}$ where $\hat{u}(s) = \frac{1}{s}$ for a step function.

$$\hat{g}(s) = C(sI - A)^{-1}B + D = 5 * \frac{s}{s^2+2s+2} = \hat{g}(s) \Rightarrow \text{MATLAB}$$

$$\therefore \hat{y}(s) = g(s)\,u(s) = \frac{5}{s^2+2s+2} = 5 \times \frac{1}{(s+1)^2+1}$$

~ Using the fact that: $\frac{b}{(s+a)^2+b^2} \overset{\mathcal{L}}{\Longleftrightarrow} e^{-at}\sin(bt)$

$y(t) = 5e^{-t}\sin t$ for $t \geq 0$ ☑

② Using the exponential integral definition of e^{At}, we know that:

$$y(t) = C(t)\exp\{(t-t_0)A\}x\{t_0\} + C(t)\int_{t_0}^{t} \exp\{(t-\tau)A\}B(\tau)u(\tau)\,d\tau + D(t)u(t)$$

Knowing that $t_0 = 0$, $D = [0]$, $u(\tau)$ is '1' for a step (in time domain), and $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we simplify to:

$$y(t) = C\int_0^t e^{A(t-\tau)}B\,d\tau = \underbrace{[2 \;\; 3]\int_0^t e^{A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau}$$

Evaluating this integral, as seen in MATLAB, shows that:

$y(t) = 5e^{-t}\sin t$ for $t \geq 0$ ☑

```matlab
% Carlos Lazo
% EC504 - Homework #5
% Due: 10/13/09

clc; clear all; close all;

%% Problem #2

% Part(a)

% From the eigenvectors of Problem 1, we define our V & Lamdba
matrices:

V     = [1 4 0; 0 1 0; 0 0 1];
V_inv = inv(V);

% Define derived eigenvalues:

L1 = 1;
L2 = 2;

% Compute L & exp(L):

syms t;

L     = diag([L1 L2 L2]);

L_exp = diag([exp(L1*t) exp(L2*t) exp(L2*t)]);

A1_et  = V * L_exp * V_inv;

display('A^et for Problem 2, Part 1 is:');
display(' ');

pretty(A1_et);

% Symbolic variable 'b' will represent base 2:

syms b;

L     = diag([(L1^100) b b])

A_100 = V * (L^100) * V_inv;

display('A^100 for Problem 2, Part 1 is:');
display(' ');

pretty(A_100);

% Part(b)

A2 = [0 1; 0 0]
```

```
A2_et = expm(A2*t);

display('A^et for Problem 2, Part 2 is:');
display(' ');

pretty(A2_et);

%% Problem #4

clc; clear all; close all;

% Solution 1

A = [0 1; -2 -2];
B = [1; 1];
C = [2 3];
D = [0];
I = eye(2);

syms s;

g_s = (C * (inv((s*I) - A)) * B) + D;

display('g(s) for Problem #4 in the Laplace domain is:');
display(' ');

simplify(g_s)

% Solution 2

syms t; syms T;

integrand = (expm(A*(t-T))) * B;

int_eval = int(integrand, T, 0, t);

display('With integral computation using the exp(A*t) definition, y(t)
=');
display (' ');

y_t = C * int_eval
```

Output for Comment Block #1:

g(s) for Problem #4 in the Laplace domain is:

ans =

5*s/(s^2+2*s+2)

y_t =

5*exp(-t)*sin(t)

A^et for Problem 2, Part 1 is:

```
[exp(t)    -4 exp(t) + 4 exp(2 t)      0   ]
[                                   ]
[  0            exp(2 t)             0   ]
[                                   ]
[  0               0            exp(2 t)]
```

L =

```
[ 1, 0, 0]
[ 0, b, 0]
[ 0, 0, b]
```

A^100 for Problem 2, Part 1 is:

```
[            100      ]
[1    -4 + 4 b      0 ]
[                     ]
[        100          ]
[0       b          0 ]
[                     ]
[                  100]
[0         0        b ]
```

A2 =

```
0   1
0   0
```

A^et for Problem 2, Part 2 is:

```
[1    t]
```

```
[     ]
[0   1]
```

Output for Comment Block #2:

g(s) for Problem #4 in the Laplace domain is:

ans =

5*s/(s^2+2*s+2)

With integral computation using the exp(A*t) definition, y(t) =

y_t =

5*exp(-t)*sin(t)