8 A computer generates random numbers which are uniformly distributed on the interval [0, 1]. Find an increasing function $g(x)$, such that, if X is uniform on [0,1], then the r.v. $Y = g(X)$ has a Rayleigh density:

$$f_Y(y) = \begin{cases} 0 & , \ y < 0 \\ \frac{y}{\alpha} e^{-\frac{y^2}{2\alpha}} & , \ y \geq 0 \end{cases} \qquad {}^*(\alpha = \sigma^2)$$

~ Knowing that $Y = g(X)$.

Since $F_Y[y = g(x)] = F_X(x) \Rightarrow y = g(x) = F_Y^{-1}(x)$

$\therefore$ Find $F_Y(y) = x$ and solve for y.

Find $F_Y(y) = \int_{-\infty}^{+\infty} f_Y(y)\, dy = \begin{cases} 0 & , \ y < 0 \\ 1 - e^{-\frac{y^2}{2\alpha}} & , \ y \geq 0 \end{cases}$

$\Rightarrow F_Y(y) = 1 - e^{-\frac{y^2}{2\alpha}} = x$ from $0 < x < 1$

Solve for y:
$(y = g(x))$

$$-e^{-\frac{y^2}{2\alpha}} = x - 1$$
$$e^{-\frac{y^2}{2\alpha}} = 1 - x$$
$$-\frac{y^2}{2\alpha} = \ln(1-x)$$
$$y^2 = -2\alpha \ln(1-x)$$

$$\boxed{y = g(x) = \sqrt{-2\alpha \ln(1-x)}}$$

9 Let X be exponentially distributed with mean $\mu = \lambda^{-1}$. Find and sketch the distribution functions for the r.v.'s : a) $Y = e^{X}$ b) $Z = \min(X, 3)$

* Note : $f_X(x) = \lambda e^{-\lambda x} u(x)$; $F_X(x) = [1 - e^{-\lambda x}]\, u(x)$

* AND : $F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$

a) Knowing that $y = e^{x} = g(x)$: $g^{-1}(y) = x = \ln y$

$\therefore\ F_Y(y) = F_X(\ln y) = (1 - e^{-\lambda \ln y})\, u(\ln y)$ $\quad e^{0} = y = 1$

$F_Y(y) = 1 - e^{\ln y^{-\lambda}}$

Step turns on when $\ln y = 0$

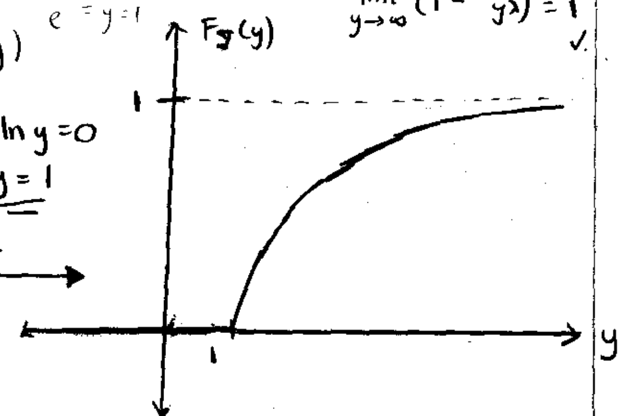
At $\underline{y = 1}$

NEW BOUND!

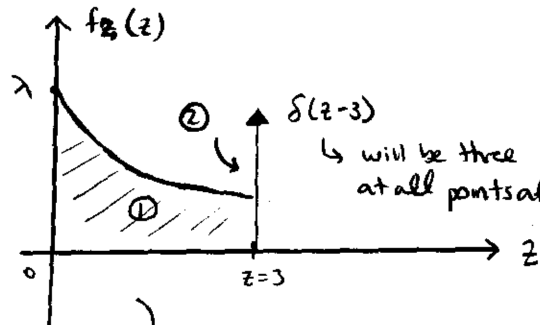
$$\boxed{F_Y(y) = \begin{cases} 1 - y^{-\lambda}, & y \ge 1 \\ 0, & \text{else} \end{cases}}$$

CDF PLOT

$\lim_{y \to \infty} (1 - \frac{1}{y^{\lambda}}) = 1$ ✓



b) Looking at the pdf of Z:



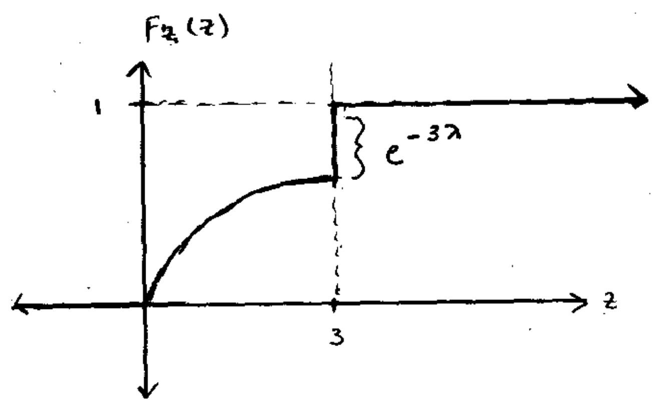
↳ will be three at all points afterwards.

To find magnitude of $\delta(z-3)$

$\int f_Z(z) = 1 \ \therefore\ A(\delta) = 1 - ①$

$② = 1 - \int_0^3 \lambda e^{-\lambda z} dz = \underline{e^{-3\lambda}}$

Integrate area ① and plot from $0 \to 3$



$$F_Z(z) = \begin{cases} 1 - e^{-\lambda z}, & 0 \le z < 3 \\ e^{-3\lambda} u(z-3), & z \ge 3 \\ 0, & \text{elsewhere} \end{cases}$$