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ECE 504
Homework # 6

1) Chen Problem 5.10

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

To test for (internal) marginal stability, look @ eigenvalues of A:

Characteristic polynomial $= \det(\lambda I_3 - A) = \det \begin{pmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

$= (\lambda+1)\lambda^2$

$\therefore \lambda_1 = -1 \rightarrow r_1 = 1 \quad \Rightarrow Re(\lambda_1) \leq 0$? $-1 \leq 0$ ✓ We are good.

$\lambda_2 = 0 \rightarrow r_2 = 2 \quad \Rightarrow Re(\lambda_2) = 0 \ldots$ need to make sure that $r_2 == m_2$

$E(\lambda_2) = \text{nullspace}(A - \lambda_2 I_3) = \text{nullspace}(A) \simeq \underset{3\times3}{\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \underset{3\times1}{v} = 0$

$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ Both vectors are in the nullspace of A... therefore ; $m_2 = 2$

$\Rightarrow$ this implies that the system IS marginally stable!

$\Rightarrow$ the system IS NOT asymptotically stable since $Re(\lambda_i)$ is NOT < 0 for λ_1 and λ_2.

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2) Chen Problem 5.13

$$x[k+1] = \begin{bmatrix} 0.9 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x[k]$$

Test for marginal stability by looking at eigenvalues of A.

Characteristic polynomial = $\det(\lambda I_3 - A) = \det \begin{pmatrix} \lambda - .9 & 0 & -1 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{pmatrix}$

$= (\lambda - .9)(\lambda - 1)^2$

$\therefore \quad \lambda_1 = .9 \longrightarrow r_1 = 1 \Rightarrow |\lambda_1| \leq 1 \ ? \quad .9 \leq 1$ ✓ we are good.

$\lambda_2 = 1 \longrightarrow r_2 = 2 \Rightarrow$ Since $|\lambda_2| = 1$, need to check if $r_2 = m_2$

$E(\lambda_2) = \text{nullspace}(A - \lambda_2 I) = \text{nullspace}(A - I) = \begin{pmatrix} -0.1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} v = 0$

Only $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is in the nullspace of $\lambda_2 \rightarrow m_2 = 1$

Since $m_2 \neq r_2$, the system <u>IS NOT</u> marginally stable.

The system <u>IS NOT</u> asymptotically stable since $\lambda_2 = 1$ and $|\lambda_j| < 1$ for all $j \in \{1, 2\}$.

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3) Chen Problem 5.18

Given : $A^T M + MA + 2\mu M = -N$

~> Perform a substitution of variables where $C = A + \mu I$. Transforming the equation yields:

$$A^T M + MA + 2\mu I = -N$$

~ through matrix properties:

$$(A+\mu I)^T M + C(A+\alpha I) = -N$$

~ use substitution:

$$C^T M + MC = -N$$

~> Assuming that λ is an eigenvalue of A; we take a look at the nullspace representation:

We know that $Av = \lambda v \Rightarrow (A+\mu I)v = (\lambda+\mu)v$
↳ This implies that $(\lambda+\mu)$ is an eigenvalue of C!

* This implies that all λ_i values of C have real parts < 0, implying that C is a stable matrix. If B is a stable matrix, this implies that A is a stable matrix.

~ According to the Lyapunov theorem, for any positive definite symmetric matrix N:

$C^T M + MC = -N$ will have a unique, positive definite solution M iff C is stable.

* this shows that all e-values of A have real parts less than $-\mu < 0$.

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4) Chen Problem 5.23

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} x$$

Find the eigenvalues of A:

$$\det(\lambda I - A(t)) = \det \begin{pmatrix} \lambda+1 & 0 \\ e^{-3t} & \lambda \end{pmatrix} = (\lambda+1)(\lambda) = 0$$

$$\left.\begin{array}{ll} \lambda_1 = -1 & \rightarrow r_1 = 1 \\ \lambda_2 = 0 & \rightarrow r_2 = 1 \end{array}\right\}$$

Using the following MATLAB code:

```
syms t;

A = [-1 0 ; -1 * exp(-1 * 3 * t) 0];

phi = expm (A * t)
```

$\rightarrow$

$$\Phi(t,0) = \begin{bmatrix} e^{-t} & 0 \\ e^{-4t} - e^{-3t} & 1 \end{bmatrix}$$

Since $\|\Phi(t,0)\| \rightarrow 0$ as $t \rightarrow \infty$ (as seen by the decaying e^{-t} terms), the equation is asymptotically stable AND marginally stable.

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5) $\sim$

a) Knowing that A is a diagonal matrix, we can factor out the P matrix:

$$P(A^T + A) = -Q \rightsquigarrow P = -Q(A^T + A)^T$$

b) If A is diagonizable, this implies that the null space of A causes $m_j = r_j$ for all $j \in \{1, 2, \dots n\}$ for a size n matrix

This implies... (?)