ECE504 Homework Assignment Number 2
Due by 8:50pm on 30-Sep-2008

Tips: Make sure your reasoning and work are clear to receive full credit for each problem.

1. 3 pts. For $\boldsymbol{Q} \in \mathbb{C}^{n \times n}$ and $\alpha \in \mathbb{C}$, compute $\det(\alpha \boldsymbol{Q})$ and $\operatorname{adj}(\alpha \boldsymbol{Q})$ in terms of α, $\det(\boldsymbol{Q})$ and $\operatorname{adj}(\boldsymbol{Q})$. What does this say about $(\alpha \boldsymbol{Q})^{-1}$ for $\alpha \neq 0$ and $\det(\boldsymbol{Q}) \neq 0$?

2. 4 pts. Suppose you are given a lumped discrete-time LTI system described by the state space equations

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}u[k] \\ y[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}u[k]. \end{aligned} \tag{1}$$

Now suppose we define $\boldsymbol{v}[k] = \boldsymbol{P}\boldsymbol{x}[k]$ for all k where $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ is known and $\boldsymbol{P}$ is invertible such that $\boldsymbol{x}[k] = \boldsymbol{P}^{-1}\boldsymbol{v}[k]$.

(a) Find $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{B}}$, $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{D}}$, in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{P}$ such that

$$\begin{aligned} \boldsymbol{v}[k+1] &= \bar{\boldsymbol{A}}\boldsymbol{v}[k] + \bar{\boldsymbol{B}}u[k] \\ y[k] &= \bar{\boldsymbol{C}}\boldsymbol{v}[k] + \bar{\boldsymbol{D}}u[k]. \end{aligned} \tag{2}$$

(b) Show that (1) and (2) have the same transfer function by showing that

$$\bar{\boldsymbol{C}}(z\boldsymbol{I} - \bar{\boldsymbol{A}})^{-1}\bar{\boldsymbol{B}} + \bar{\boldsymbol{D}} = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}.$$

Hint: Recall that $(\boldsymbol{X}\boldsymbol{Y})^{-1} = \boldsymbol{Y}^{-1}\boldsymbol{X}^{-1}$.

3. 6 pts. Suppose

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix},$$

$\boldsymbol{B} = [0, 0, \dots, 0, 1]^\top$, $\boldsymbol{C} = [b_0, b_1, \dots, b_{n-1}]$, and $\boldsymbol{D} = 0$. Also suppose that

$$\hat{g}(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}.$$

(a) Is $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ is a state-space realization for the single-input single-output transfer function $\hat{g}(s)$?

(b) Let $\bar{\boldsymbol{A}} = \boldsymbol{A}^\top$, $\bar{\boldsymbol{B}} = \boldsymbol{C}^\top$, $\bar{\boldsymbol{C}} = \boldsymbol{B}^\top$, and $\bar{\boldsymbol{D}} = \boldsymbol{D}$. Show that $\{\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{C}}, \bar{\boldsymbol{D}}\}$ is a state-space realization for the single-input single-output transfer function $\hat{g}(s)$.

(c) Find two different state-space realizations for

$$\hat{g}(s) = \frac{s^3}{s^3 + 2s^2 - s + 2}.$$

(d) Find two different state-space realizations for the discrete time system

$$\hat{g}(z) = \frac{z^{-1}}{z^{-2} + 2z^{-1} - 3}.$$

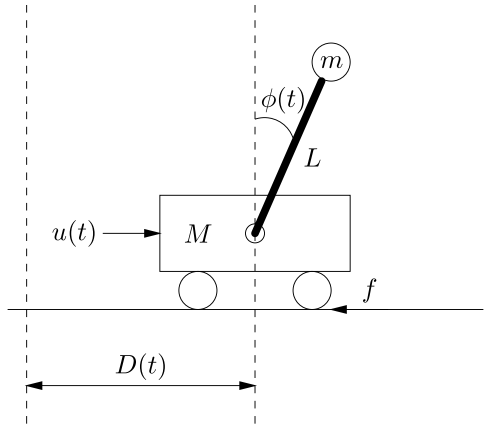

Figure 1: Inverted pendulum on moving carriage.

4. 6 pts. Consider the mechanical system described by Figure 1. Define the states $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = D$, and $x_4 = \dot{D}$. Analysis of Figure 1 (and application of some reasonable approximations) reveals that

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{g}{L}\sin x_1(t) - \frac{1}{LM}(-f x_4 + u(t))\cos x_1(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= -\frac{f}{M}x_4(t) + \frac{1}{M}u(t)
\end{aligned}$$

Suppose also that the output of this system is $y(t) = \tan x_1$.

(a) Observe that $x_1(t) = x_2(t) = x_3(t) = x_4(t) = u(t) = 0$ is a solution to this set of differential equations. Linearize this system around this solution and find $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ such that $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$ and $y(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t)$.

(b) Observe that $x_1(t) = \pi$ and $x_2(t) = x_3(t) = x_4(t) = u(t) = 0$ is another solution to this set of differential equations. Linearize this system around this solution and find $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ such that $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$ and $y(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t)$.

5. 3 pts. For each of the following, find a solution for $\boldsymbol{x}$. If a solution does not exist then show why. If the solution is not unique then mathematically describe the set of all possible solutions. Justify your answers.

(a)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \\ 4 & 7 & 13 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \\ 4 & 7 & 13 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 3 & 6 & 10 & 13 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

6. 3 pts. Given arbitrary $\boldsymbol{P} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{Q} \in \mathbb{R}^{n\times k}$. Show that, if the columns of $\boldsymbol{PQ}$ are linearly independent, then so are the columns of $\boldsymbol{Q}$. Give an example to show that the converse of this statement is not true, in general. Hint: Write $\boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k]$. If the columns of $\boldsymbol{Q}$ are linearly *dependent* then there exists a set of k scalars $\{\alpha_i\}_{i=1}^k$ such that $\alpha_1\boldsymbol{q}_1 + \alpha_2\boldsymbol{q}_2 + \cdots + \alpha_k\boldsymbol{q}_k = 0$.

7. 5 pts. Given the following discrete time, LTI, state-space system description,

$$\begin{aligned} \boldsymbol{x}[k+1] &= \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boldsymbol{x}[k] + u[k] \end{aligned}$$

(a) Find a different state-space realization of this system that has the same impulse response as this system.

(b) For $k \geq 0$, explicitly compute the zero-input response of the system for the following cases:

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

(c) Use your result from part (a) to derive a general expression for the zero-input response of the system when the initial state $\boldsymbol{x}(0) = [\gamma_1, \gamma_2, \gamma_3]^\top$.