BAE SYSTEMS

Digital Communications

Digital Communications Problem Set

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Course: TDC1
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Problem # 1

Determine the impulse response $h(t)$ corresponding to $H(f)$.

a) $H(f) = \dfrac{1 - e^{-j2\pi f T}}{j2\pi f}$

- To get to a point where the inverse transform can be taken, we first manipulate the equation into an Euler format:

$$H(f) = \frac{e^{-j\pi f T}}{\pi f}\left[\frac{(e^{j\pi f T} - e^{-j\pi f T})}{j2}\right]$$ } Known sine transform of the Euler equation!

$$H(f) = e^{-j\pi f T} \times \boxed{\frac{\sin(\pi f T)}{\pi f} \times \frac{T}{T}}$$ } $\dfrac{\sin x}{x} = \mathrm{sinc}(x)$ → substitute!

$$H(f) = \left[e^{-j\pi f T}\right] T\, \mathrm{sinc}(\pi f T) \longrightarrow \text{Substitute } 2\pi f = \omega$$

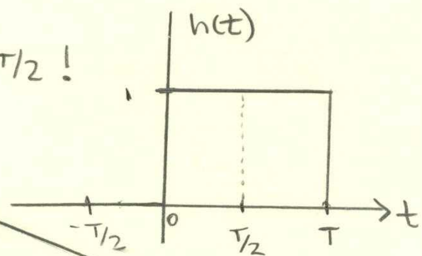
$$H(\omega) = \underbrace{\left[e^{-j\omega\frac{T}{2}}\right]}_{1}\ \underbrace{T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)}_{2}$$

KNOWN INVERSE FOURIER TRANSFORMS

1. $x(t - t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0}$
2. $\mathrm{rect}(-T/2, T/2) \longleftrightarrow T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$

$\therefore\ H(\omega) \longleftrightarrow h(t) = \mathrm{rect}(-T/2, T/2)$ shifted by $+T/2$!

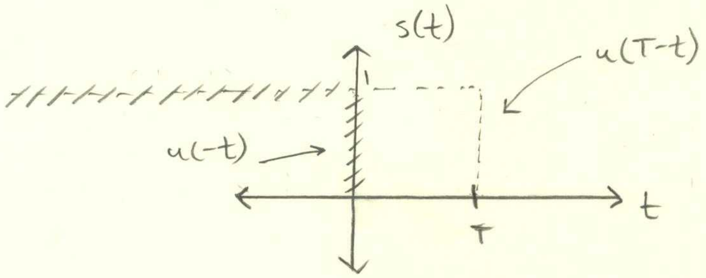
$$h(t) = u(t + T/2 - (T/2)) - u(t - T/2 - (T/2))$$
$$h(t) = u(t) - u(t - T)$$



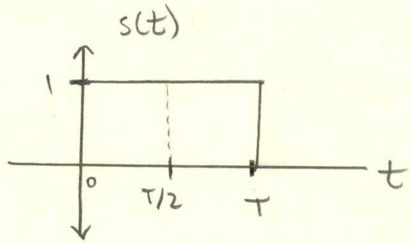
b) To find the signal $s(t)$, simply invert variables knowing the reflective relationship:

$$h(t) = s(T - t) \longrightarrow s(t) = h(T - t)$$

$$\therefore\ s(t) = u((T-t) + T/2 - (T/2)) - u((T-t) - T/2 - (T/2))$$
$$s(t) = u(T - t) - u(-t)$$ → Shift, Scale, Reflect.



$\Longrightarrow$

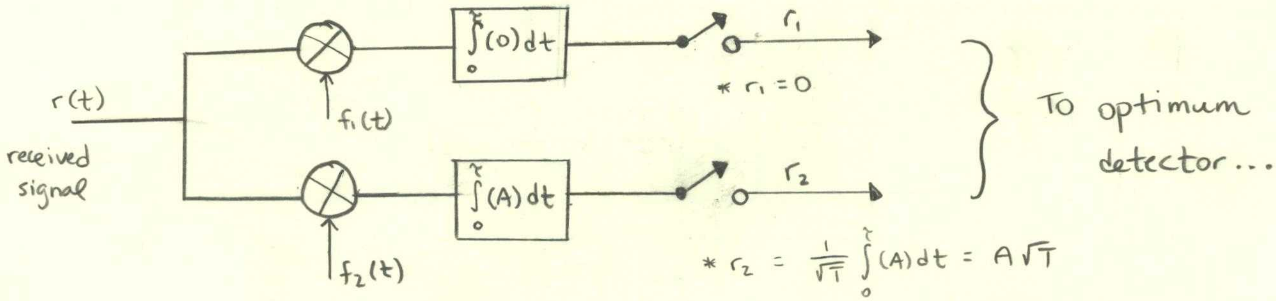


Problem #2

a) Determine block diagram of the demodulator and optimum detector...

$s_0(t) = 0 \;,\; 0 \le t \le T$
$s_1(t) = A \;,\; 0 \le t \le T$

• we first begin by deriving the block diagram of the demodulator:



$* \; r_2 = \frac{1}{\sqrt{T}} \int_0^\tau (A)\,dt = A\sqrt{T}$

• the basis functions mimic a filter with the

$$* \quad f_m(t) = \begin{cases} \frac{1}{\sqrt{T}} \;, & 0 < \overset{t}{\text{range}}(t) < T \\ 0 \;, & \text{otherwise} \end{cases}$$

• $m = 1, 2$
• $n = \#$ of dimensions $= 2$

• we are given that the power spectral density is $\frac{N_0}{2}$. Given the formula:

$$p(r|s_m) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{K=1}^{N} \frac{(r_K - s_{mK})^2}{N_0}\right] \;,$$

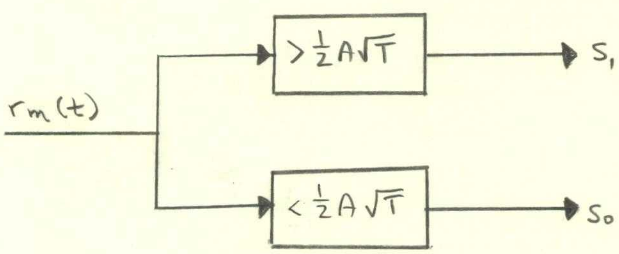
$N = 1$ since we have one noise channel.

• Computing the $p(r|s_m)$ for each signal yields the following results:

$$p(r|s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \quad \Big\} \quad p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r - A\sqrt{T})^2}{N_0}}$$

• Since we know that these signals are equally probable and that $\Sigma(r_1 + r_2) = A\sqrt{T}$, it <u>must</u> be the case that the probability $= \frac{1}{2}A\sqrt{T}$

→ we can now construct our optimum detector diagram as follows:



$\boxed{\text{Optimal threshold} = \frac{1}{2}A\sqrt{T}}$

b) Given that the probabilities of error are the same, we can see that:

$$P(e) = \tfrac{1}{2} P(\text{error}|s_0) + \tfrac{1}{2} P(\text{error}|s_1)$$

• Knowing that $P(e|s_1) = \int_{-\infty}^{0} p(r|s_1)\,dr$ ... we substitute using probability expressions in part a).

$$P(e) = \tfrac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} P(r|s_0) + \tfrac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} P(r|s_1)$$

$\rightarrow$ These regions are opposite those of our optimum detector, indicating that these are errors. Substitute.

$$P(e) = \underbrace{\tfrac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}}\,dr}_{A} + \underbrace{\tfrac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}}\,dr}_{B}$$

we want to make these resemble gaussian PDF's. Perform variable substitution!

A $\rightarrow$ Set $r = \dfrac{x}{\sqrt{\frac{2}{N_0}}}$, then $dr = dx\left(\dfrac{1}{\sqrt{\frac{2}{N_0}}}\right)$

Put lower bound in terms of x; substitute bound for r & solve for x:

$$\tfrac{1}{2}A\sqrt{T} = \frac{x}{\sqrt{\frac{2}{N_0}}} \Rightarrow x = \tfrac{1}{2}A\sqrt{T}\sqrt{\tfrac{2}{N_0}}$$

B $\rightarrow$ Set $r = \dfrac{x}{\sqrt{\frac{2}{N_0}}} + A\sqrt{T}$, Then $dr = dx\left(\dfrac{1}{\sqrt{\frac{2}{N_0}}}\right)$

Put upper bound in terms of x; substitute bound for r and solve for x:

$$\tfrac{1}{2}A\sqrt{T} = \frac{x}{\sqrt{\frac{2}{N_0}}} - A\sqrt{T} \Rightarrow x = -\tfrac{1}{2}A\sqrt{T}\sqrt{\tfrac{2}{N_0}}$$

Rewrite integral: $$P(e) = \tfrac{1}{2}\int_{\frac{1}{2}A\sqrt{T}\sqrt{\frac{2}{N_0}}}^{+\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{x^2}{2}}\,dx + \tfrac{1}{2}\int_{-\infty}^{-\frac{1}{2}A\sqrt{T}\sqrt{\frac{2}{N_0}}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{x^2}{2}}\,dx$$

≈ Through gaussian integration (WIKIPEDIA + NOTES), the expression simplifies:

$$P(e) = Q\left[\tfrac{1}{2}\sqrt{\tfrac{2}{N_0}}A\sqrt{T}\right]$$

$\rightarrow$ Also seen in slide 50 of notes! $P(e) = Q$ (limits of integration, non-∞)

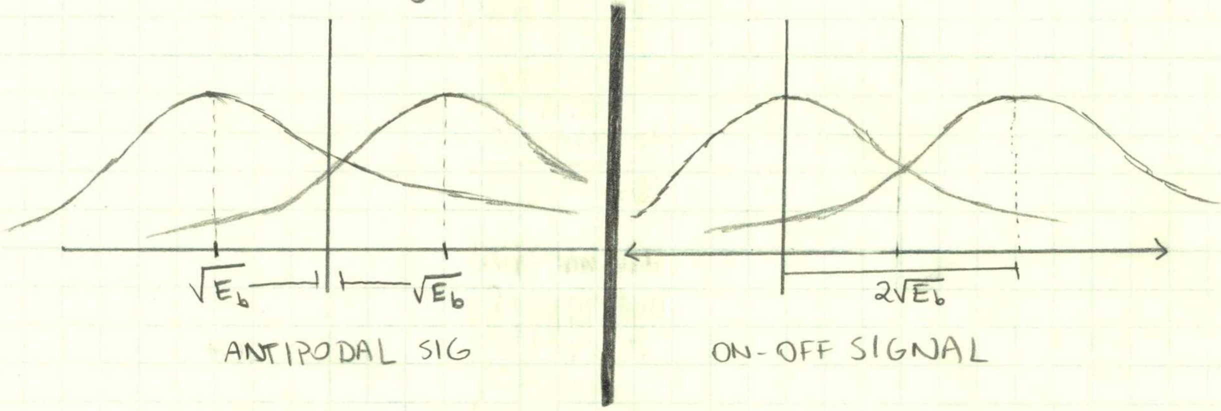
& $$P(e) = Q\left[\sqrt{\frac{2E_b}{N_0}}\right] = Q\left[\sqrt{SNR_0}\right], \quad SNR_0 = \frac{2E_b}{N_0}$$

(Slide 49) $\rightarrow$ Slide 42

To write in terms of SNR, equate first and second expressions:

$$\tfrac{1}{2}\sqrt{\tfrac{2}{N_0}}A\sqrt{T} = \sqrt{SNR_0} \Rightarrow \boxed{SNR = \frac{A^2T}{2N_0}}$$

Looking at the difference between the on-off switching and the antipodal signal representations:



$\Rightarrow$ As seen through the graphs & based on the calculated SNR, we see that <u>on-off signaling requires 2x more energy</u> to achieve the same error performance as antipodal signaling.

$\boxed{\text{* on-OFF requires 2x more E than antipodal *}}$

$$s_1(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \qquad s_2(t) = -s_3(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}T \\ -1, & \frac{1}{2}T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

3.)

A) since $s_2(t) = -s_3(t)$, the dimensionality will be 2.

B)

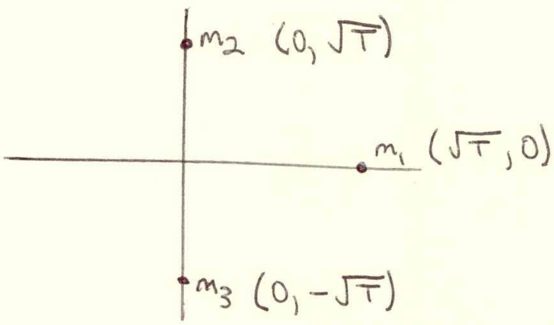
$$f_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad f_2(t) = -f_3(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{\sqrt{T}} \int_0^T 1\, dt = \frac{T}{\sqrt{T}} \frac{\sqrt{T}}{\sqrt{T}} = \sqrt{T}$$

$$\frac{1}{\sqrt{T}} \int_0^{T/2} 1\, dt + \frac{-1}{\sqrt{T}} \int_{T/2}^{T} -1 = \sqrt{T}$$

$$m_1 = [\sqrt{T}, 0] \qquad m_2 = [0, \sqrt{T}] \qquad m_3 = [0, -\sqrt{T}]$$

c)



3D) Given our orthonormal basis, our output will add noise components in both dimensions. The three outputs are:

$(\sqrt{T} + n_1, n_2)$

$(n_1, \sqrt{T} + n_2)$

$(n_1, -\sqrt{T} + n_2)$

<u>R_1/R_2 boundary</u>

$$\sqrt{(\sqrt{T}+n_1)^2 + (n_2)^2} = \sqrt{(n_1)^2 + (\sqrt{T}+n_2)^2}$$

$$\sqrt{T + 2n_1\sqrt{T} + n_1^2 + n_2^2} = \sqrt{n_1^2 + T + 2n_2\sqrt{T} + n_2^2}$$

$$\cancel{T} + 2n_1\sqrt{T} + \cancel{n_1^2} + \cancel{n_2^2} = \cancel{n_1^2} + \cancel{T} + 2n_2\sqrt{T} + \cancel{n_2^2}$$

$$2n_1\sqrt{T} = 2n_2\sqrt{T}$$

$$n_1 = n_2$$

$$(n_1, n_2)$$

<u>R_1/R_3 boundary</u>

$$\sqrt{(\sqrt{T}+n_1)^2 + (n_2)^2} = \sqrt{(n_1)^2 + (-\sqrt{T}+n_2)^2}$$

$$\sqrt{T + 2n\sqrt{T} + n_1^2 + n_2^2} = \sqrt{n_1^2 + T - 2n_2\sqrt{T} + n_2^2}$$

$$2n_1\sqrt{T} = -2n_2\sqrt{T}$$

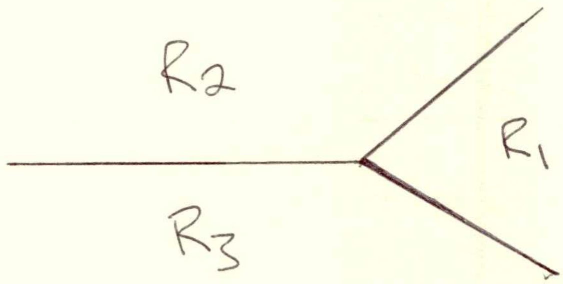
$$n_1 = -n_2$$

$$(n_1, -n_2)$$

<u>R_2/R_3 boundary</u>

Since $s_2(t) = -s_3(t)$, R_2 and R_3 are on opposite sides of the y-axis.

3D) continued...



This is a graph depicting the optimal decision regions.

$R_1 \Rightarrow m_1$

$R_2 \Rightarrow m_2$

$R_3 \Rightarrow m_3$

3e) R_2 and R_3 are each $3/8$ of the total signal space.
R_1 is 25% of the total space.

Each message (m_1, m_2, m_3) is equally probable and has $33\frac{1}{3}$ % chance of being received.

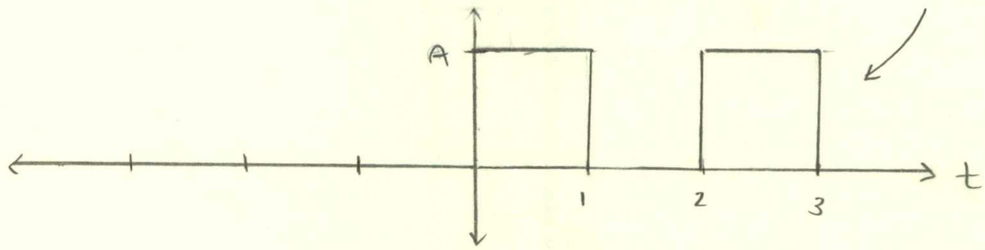
R_1 is most vulnerable to error because m_1 has an equal probability of being received and corresponds to the smallest region, R_1.

Problem #4

a) Sketch the impulse response of the matched filter to s(t).

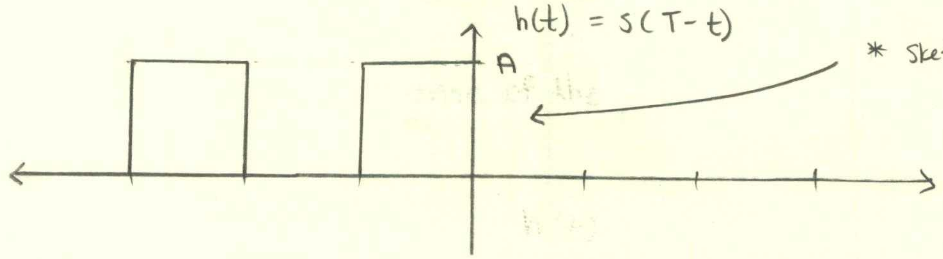
$h(t) = s(T-t)$

* Sketched for $T=3$



$h(t) = s(T-t)$

* Sketched for $T=0$



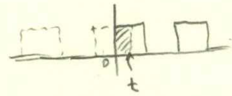
b) Sketch output of matched filter to input s(t) over interval $-2 \leftrightarrow +8$.

$$y(t) = s(t) * s(t) = \int_0^t s(\tau)\, s(T-t+\tau)\, d\tau$$

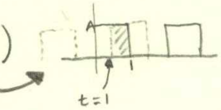
· Interval

$t < 0 \longrightarrow y(t) = \underline{0}$

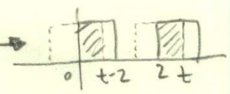
$0 \le t < 1 \longrightarrow \int_0^t A \times A \, dt = A^2 t \Big|_0^t = \underline{A^2 t} \longrightarrow$



$1 \le t < 2 \longrightarrow \int_{t-1}^{1} A \times A \, dt = A^2 t \Big|_{t-1}^{1} = A^2 (1-(t-1)) = \underline{A^2 (2-t)}$

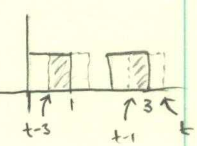


$2 \le t < 3 \longrightarrow \int_2^t A \times A \, dt + \int_0^{t-2} A \times A \, dt = A^2(t-2) + A^2(t-2) = \underline{2A^2(t-2)} \longrightarrow$

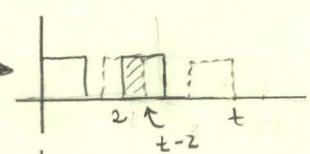


$3 \le t < 4 \longrightarrow \int_{t-1}^{3} A \times A \, dt + \int_{t-3}^{1} A \times A \, dt = A^2(3-(t-1)) + A^2(1-(t-3))$

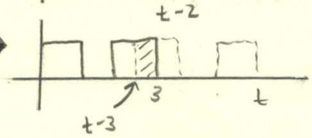
$= A^2(4-t) + A^2(4-t) = \underline{2A^2(4-t)} \longrightarrow$



$4 \le t < 5 \longrightarrow \int_2^{t-2} A \times A \, dt = A^2(t-2(-2)) = \underline{A^2(t-4)} \longrightarrow$



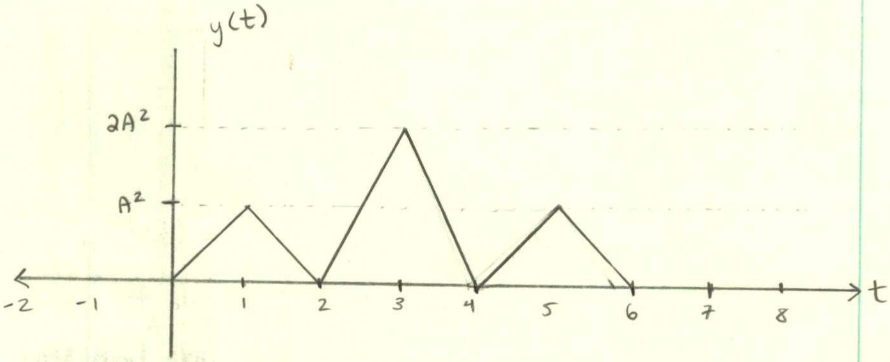
$5 \le t < 6 \longrightarrow \int_{t-3}^{3} A \times A \, dt = A^2(3-(t-3)) = \underline{A^2(6-t)} \longrightarrow$



$t \ge 6 \longrightarrow y(t) = 0$

Putting together the piecewise and sketching yields:

$$y(t) = \begin{cases} 0 & , \quad t < 0 \\ A^2 t & , \quad 0 \le t < 1 \\ A^2(2-t) & , \quad 1 \le t < 2 \\ 2A^2(t-2) & , \quad 2 \le t < 3 \\ 2A^2(4-t) & , \quad 3 \le t < 4 \\ A^2(t-4) & , \quad 4 \le t < 5 \\ A^2(6-t) & , \quad 5 \le t < 6 \\ 0 & , \quad t \ge 6 \end{cases}$$



c) Determine the variance of the noise at the output of the matched filter @ $t = 3$.

→ From slide 42 in the notes, we know that the variance of the noise is expressed by the following formula, @ $t = T = 3$.

$$\sigma_{n_T}^2 = \int_0^T \int_0^T E[n(\tau)\, n(t)]\, s(\tau)\, s(t)\, dt\, d\tau$$

$$= \frac{1}{2} N_o \underbrace{\int_0^T s^2(t)\, dt}_{2A^2 \text{ as seen in convolution}} \quad \Big\} \quad \boxed{\sigma_{n_T}^2 = N_o A^2}$$

d) Determine P_e as a function of A and N_o.

* we know the following relationships:
$$\begin{cases} ① \quad SNR_o = \dfrac{2E_b}{N_o} = \dfrac{y_s^2(T)}{E[y_n^2(T)]} \\ ② \quad P_e = Q\left(\sqrt{\dfrac{2E_b}{N_o}}\right) \end{cases}$$

Combining ① & ②, we see that:

$$P_e = Q(\sqrt{SNR_o}) = Q\left(\sqrt{\frac{y_s^2(T)}{E[y_n^2(T)]}}\right) \quad \begin{cases} y_s^2(T) @ T=3 = (2A^2)^2 = 4A^4 \\ E[y_n^2(T) @ T = \sigma_{T=3}^2 = N_o A^2 \quad \text{(from part c)} \end{cases}$$

$$\therefore \quad \boxed{P_e = Q\left(\sqrt{\frac{4A^4}{N_o A^2}}\right) = Q\left(\sqrt{\frac{4A^2}{N_o}}\right)}$$