Analysis and Enhancement of the Simulated Binary Crossover*

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Abstract—Most recombination operators are designed to provide an adaptive behavior with the principle of altering its exploration capabilities depending on the diversity of the population. However, depending only on the content of the population might be a drawback in long-term executions because diversity could not be large enough and the search process might prematurely converge. Based on the previous scenario, a novel recombination operator is designed for tackling continuous Multi-objective Optimization Problems (MOPs), which works effectively to enhance the search capability of Multi-Objective Evolutionary Algorithms (MOEAs). Particularly, this operator extends the Simulated Binary Crossover (SBX) by considering the stopping criterion to alter its internal operation. In order to validate the effectiveness of our proposal, it is studied by substituting the original recombination operators in three state-of-the-art MOEAs: NSGA-II, MOEA/D and SMS-EMOA. The popular DTLZ, WFG and UF benchmark problems are taken into account. Experimental validation shows a significant improvement in the performance of all the MOEAs when applying the novel crossover operator. Additionally, our proposal is also tested against state-of-the-art differential evolution operators, providing quite competitive results.

I. INTRODUCTION

Evolutionary Algorithms (EA) have become a promising alternative in several real-world problems where the deterministic approches are not suitable. Multi-objective Optimization Problems (MOPs) involves the simultaneous optimization of two or more objective functions that are usually in conflict. A continuous minimization multi-objective problem can be defined as follows:

$$minimize \quad F(x) = (f_1(x), f_2(x), ..., f_m(x))$$
$$subject \quad to \quad x \in \Omega \quad (1)$$

where $x = (x_1, ..., x_n) \in R^n$ indicate a decision variable vector, n correspond to the number of decision variables, Ω is the feasible space, $F : \Omega \rightarrow R^m$ which consist of m objective functions and R^m is known as the *objective space*.

In a minimization MOP with m objective functions, and given two solutions $x, y \in \Omega$, x dominates y, denoted by $x \prec y$, if $f_i(x) \leq f_i(y)$ for all objectives $\{1, ..., m\}$, and $F(x) \neq F(y)$. This means that the solution x is not worse than y in any of the objectives and x is strictly better than y in at least one objective. The Pareto dominance definition states that the best solutions of a multi-objective optimization problem are those whose objective vectors are not dominated by any other feasible vector. A solution $x^* \in \Omega$ is known as Pareto optimal solution if no other solution $x \in \Omega$ dominates

x^*. The Pareto set is the set of all the Pareto optimal solutions and the Pareto front are the images of the Pareto set. The goal of multi-objective optimization approaches is to obtain a proper approximation of the Pareto front. Particularly, a set of solutions that are diverse and close to the Pareto front are desired.

In the last decade several categories of Multi-Objective Evolutionary Algorithms (MOEAs) have been arising [1], [2], among them, the Non-Dominated Sorting Genetic Algortihm II (NSGA-II) [3], the MOEA based on decomposition (MOEA/D) [4], the S-metric Selection Evolutionary Multi-objective Optimization Algorithm (SMS-EMOA) [5], these representative methods are considered as the state-of-the-art.

Differential Evolution (DE) is a popular and efficient Evolutionary Algorithm (EA) which has yielded better results than Genetic Algorithms (GAs) [6], [7]. However, in the literature is well known that several componentes of DE induces an aggresive convergence leading to several drawbacks in different scenarios such as in long-term exetucions. Usually, the aggressive behaviour of DE provokes that the algorithm stagnates in some sub-optimal regions at the first stages of the execution.

To deal with these drawbacks, the most recent algorithms are designed with strategies where is considered the criteria stop, such as adaptive parameters [8], Linear Population Size Reduction (LPSR) [9] and several mutation strategies [10]. Principally, these adaptive strategies are oriented to deal with the substantial disadvantage of setting the control parameters. These are the crossover probability (CR) and the mutation factor scale (F). Also, is important to take into account the implications between single-objective and multi-objective problems which are related with the decision variable space diversity [11].

On the other hand, GAs imply a less aggressive and flexible behavior provoking lowest quality solutions than DE in short-term executions both in continuous and discrete domains [6], [7]. Nevertheless, the GAs have interesting behaviours when are considered in long-term executions. Principally, because several componentes are based in a probability density distribution which provides an extensive search process.

In this paper, a novel recombination operator, is designed for tackling continuous multi-objective optimization problems (MOPs), which works effectively to enhance the search capability, guiding to a gradual change between exploration

to intensification. Therefore, several components of the Simulated Binary Crossover (SBX) are analyzed and modified, which aims to offer a suitable performance in long-term executions.

The rest of this paper is organized as follows. In Section II is revised the literature related with the SBX operator, and a brief description of the state-of-the-art of MOEAs. In the section III is realized an empirical analyzes to each component of this operator. Thereafter, our proposal is designed based in this analyzes. The experimental validation of the proposal and some DE variants are showed in Section IV. Finally, conclusions and some lines of future work are given in Section V.
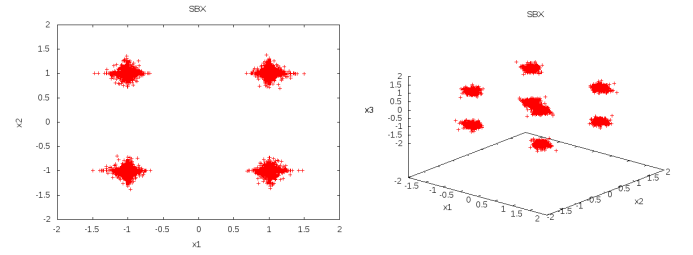

Fig. 1. Simulations of the SBX operator with a distribution index of 20, considering the parents $P_1 = (-1.0, -1.0)$ and $P_2 = (1.0, 1.0)$ and $P_1 = (-1.0, -1.0, -1.0)$ and $P_2 = (1.0, 1.0, 1.0)$ for two and three variables respectively.

II. LITERATURE REVIEW

This section is devoted to review some of the most important works that are highly related to the research presented in this paper. Firstly, the most important MOEAs paradigms are defined. Thereafter, some relevant classifications of crossover operators are discussed. Finally, the popular Simulated Binary Crossover (SBX) operator, which is used extensively in this paper, is explained in detail.

A. Multi-objective Evolutionary Algorithms

Through the last years has been created a large number of MOEAs that follows different design principles. In order to better classify them, several taxonomies have been proposed [12]. Attending to the principles of design, MOEAs can be based on Pareto dominance, indicators and/or decomposition [13]. Currently, none of them have reported a clear advantage over the other ones. Particularly, the experimental validation has been carry out by including the Non-Dominated Sorting Genetic Algorithm (NSGA-II) [3], the MOEA based on Decomposition [4], and the S-Metric Selection Evolutionary Multi-objective Optimization Algorithm (SMS-EMOA) [5]. Therefore, they are representative methods of the domination-based, decomposition-based and indicator-based paradigms, respectively. The following subsections briefly describe each one of these paradigms, as well as the selected methods.

1) Domination Based MOEAs - NSGA-II: One of the most recognized paradigms are the domination based algorithms, particularly this family are based on the application of the dominance relation to design different components of the EAs. Since that the dominance relation does not inherently promotes diversity in the objective space, auxiliary techniques such as niching crowding and/or clustering are usually integrated to obtain an acceptable spread and diversity in the objective space. A critic drawback to implement the dominance relation is that it is seriously affected as the number of objectives increases. Due to this the selection pressure is decremented substantially (exponentially) Some strategies have been developed to deal with this issue,

A principall drawback of implement the dominance relation is tharelated with the number of objectives, therefore if the number of function objective increase the selection pressure is decremented substantially, based in this, some strategies have been developed to deal with the many-objective problems.

Probably, the most popular technique of this group is the NSGA-II. This algorithm [3] implements a special parent selection operator. This operator is based on two mechanisms: fast-non-dominated-sort and crowding. The first one tends to provide a convergence to the Pareto front and the second one promotes the preservation of diversity in the objective space. A current version is the NSGA-III which is designed to deal with many-objective problems [14].

2) Decomposition Based MOEAs - MOEA/D: Decomposition-based MOEAs [4] transform a MOP in a set of single-objective optimization problems that are considered simultaneously. This transformation can be achieved through several approaches. The most popular of them is based in a weighted Tchebycheff function, therefore it is necessary provide weight vectors that are well distributed in the $m - 1$ simplex that aims well-spread solutions. However, an important drawback of this kind of approaches is that the quality of the approximations is highly related with the Pareto front shape and the weight vectors.

MOEA/D [4] is a recently designed decomposition-base MOEA. Its main principles include problem decomposition, weighted aggregation of objectives and mating restrictions through the use of neighborhoods. Particularly, the neighborhoods are considered in the variation operators. A very used variant of the MOEA/D is the MOEA/D-DE, which use the DE operators [15] and the polynomial mutation operator [16] in the reproduction phase, also it has two extra measures for maintaining the population diversity [17].

3) Indicator Based MOEAs - SMS-EMOA: In multi-objective optimization several quality indicators have been developed to compare the performance of MOEAs. Since these indicators measure the quality of the approximations attained by MOEAs, a paradigm based on the application of these indicators was proposed. Particularly, instead of dominance concept, the indicators are used in the MOEAs to guide the optimization process. Among the different indicators, hypervolume is a widely accepted Pareto-compliance quality indicator [18]. The principal advantage of this algorithms is that the indicator usually takes into account both the quality and diversity of the solutions.
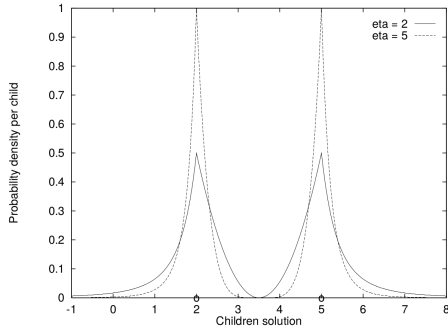
Fig. 2. Probability density function of the SBX operator with indexes of distribution 2 and 5. The parents are located in 2 and 5 respectively.
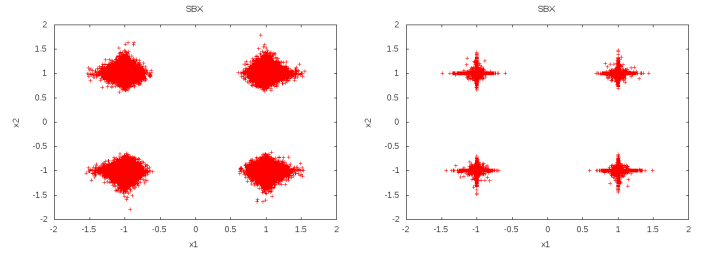

Fig. 3. Simulations of the SBX operator with a distribution index of 20, the parents are located in $P_1 = (-1.0, -1.0)$ and $P_2 = (1.0, 1.0)$. The left simulation corresponds to a probability of altering a variable (δ_1 in Algorithm 1) to 1.0 and in the right corresponds to 0.1.

A popular and extensively used indicator-based algorithm is the SMS-EMOA [5]. This algorithm might be considered as hybrid, since it involves an indicator and dominance concepts. Essentially, it integrates the non-dominated sorting method with the use of the hypervolume metric. Thus, SMS-EMOA uses the hypervolume as a density estimator which results in a computationally extensive task. Particularly, the replacement phase erases the individual of the worst ranked front with the minimum contribution to the hypervolume. Taking into account the promising behavior of SMS-EMOA, it has been used in our experimental validation.

B. Crossover operators

Crossover operators are designed to generate offspring solutions using information of the parent solutions. They combine the feature of two or more parent solutions to form children solutions. Since several crossover operators have been proposed, some taxonomies have also been provided. The taxonomies are based on features such as the location of new generated solutions or the relation among the variables.

A popular taxonomy classifies crossover operators into variable-wise operators and vector-wise operators. In the variable wise category, each variable from parent solutions is recombined independently with a certain pre-specified probability to create new values. These operators are specially suitable to deal with separable problems. Some operators belonging to this category are the Blend Crossover (BLX) [19], and the SBX [20]. On the other hand, the vector-wise recombination operators are designed to take into account the linkage among variables. They usually perform a linear combination of the variable vectors. Some operators belonging to this category are the Unimodal Normally Distributed Crossover (UNDX) [21], and the simplex crossover (SPX) [22]. Additionally, the crossover operators can be classified as Parent-Centric and Mean-Centric [23]. In Parent-Centric operators, children solutions are created around one of the parent solutions, whereas in Mean-Centric operators, children solutions are created mostly around the mean of the participating parent solutions. Among the crossover operators, SBX is probably the most frequently used operator, so this research focuses on this crossover.

1) The Simulated Binary Crossover - SBX: The reproduction operators are one of the most relevant components that influence the search process of the GAs. Specifically, the crossover and mutation operators are highly related with the diversity affecting directly the quality of the solutions.

The Simulated Binary Crossover (SBX) [24] is popularly implemented in GAs [3], [5] and is classified as Parent-Centric, meaning that two children values (c_1 and c_2) are created around the parent values (p_1 and p_2). Also the process of generate the child values is based in a probability distribution. This distribution is defined by a non-dimensional variable, better known as the spread factor $\beta = |c_1 - c_2|/|p_1 - p_2|$, indicating the ratio of the spread children values related with the parent values.

Additionally, this density function uses a distribution index η_c (user-defined control parameter) that alters the exploration capability of the operator. Specifically, a small index induces a larger probability of building children values more dissimilar than parents values, whereas with a high index the probabilities of generating children solutions that are similar to the parents increase as is showed in the Figure 2.

Principally, the SBX has non-zero probability of creating any number in the search space by recombining any two parent values from the search space. The probability distribution to create an offspring value is defined as a function of a non-dimensionalized parameter $\beta \in [0, \infty]$ as follows:

$$P(\beta) = \begin{cases} 0.5(\eta_c + 1)\beta^{\eta_c}, & \text{if} \quad \beta \leq 1 \\ 0.5(\eta_c + 1)\frac{1}{\beta^{\eta_c+2}}, & \text{otherwise} \end{cases} \quad (2)$$

Based in the mean-preserving property of children values and parent values, the distribution probability has the following properties:

- Both offspring values are equi-distant from parent values.
- There exist a non-zero probability to create offspring solutions in the entire feasible space from any two parent values.
- The overall probability of creating a pair offspring values within the range of parent values is identical to the overall probability of creating two offspring values outside the range of parent values.

Algorithm 1 Simulated Binary Crossover (SBX)

1: Input: Parents (P_1, P_2), Distribution index (η_c), Probability distribution (P_c).
2: Output: Children (C_1, C_2).
3: **if** $U[0,1] \leq P_c$ **then**
4:    **for** each variable d **do**
5:       **if** $U[0,1] \leq \delta_1$ **then**
6:          Generate $C_{1,d}$ with Equations (5) and (6).
7:          Generate $C_{2,d}$ with Equations (5) and (7).
8:          **if** $U[0,1] \geq \delta_2$ **then**
9:             Swap $C_{1,d}$ with $C_{2,d}$.
10:       **else**
11:          $C_{1,d} = P_{1,d}$.
12:          $C_{2,d} = P_{2,d}$.
13: **else**
14:    $C_{1,d} = P_{1,d}$.
15:    $C_{2,d} = P_{2,d}$.

Therefore, considering two participating parent values (p_1 and p_2), two offspring values (c_1 and c_2) can be created as linear combination of parent values with a random number $u \in [0,1]$, as follows:

$$c_1 = 0.5(1 + \beta(u))p_1 + 0.5(1 - \beta(u))p_2$$
$$c_2 = 0.5(1 - \beta(u))p_1 + 0.5(1 + \beta(u))p_2 \quad (3)$$

The parameter $\beta(u)$ depends on the random number u, as follows:

$$\beta(u) = \begin{cases} (2u)^{\frac{1}{\eta_c+1}}, & \text{if} \quad u \leq 0.5, \\ \left(\frac{1}{2(1-u)}\right)^{\frac{1}{\eta_c+1}}, & \text{otherwise} \end{cases} \quad (4)$$

The above equation only considers a optimization problem having no variable bounds. In most practical problems, each variable is bounded within a lower and upper bound. Thus, Deb and Beyer in 1999 [25] proposed a modification of the probability distribution showed and is in the Equation (5).

$$\beta(u) = \begin{cases} (2u(1-\gamma))^{\frac{1}{\eta_c+1}}, & \text{if} \quad u \leq 0.5/(1-\gamma), \\ \left(\frac{1}{2(1-u(1-\gamma))}\right)^{\frac{1}{\eta_c+1}}, & \text{otherwise} \end{cases} \quad (5)$$

$$c_1 = 0.5(1 + \beta(u))p_1 + 0.5(1 - \beta(u))p_2 \quad (6)$$

$$c_2 = 0.5(1 + \beta(u))p_1 + 0.5(1 - \beta(u))p_2 \quad (7)$$

where the child c_1 which is nearest to p_1 is calculated according the Equation (6). Therefore, for $p_1 < p_2$ and the lower bound a is closer to p_1 than to p_2, thus $\gamma = 1/(\alpha^{\eta_c+1})$, where $\alpha = 1 + (p_1 - a)/(p_2 - p_1)$. Similarly, the second child c_2 is computed with $\alpha = 1 + (b - p_2)/(p_2 - p_1)$, where b correspond to the upper bound. Then, the second child is computed as is indicated in the Equation (7). In the literature [20] is not entirely studied the SBX extension to multi-variables problems, in fact the authors implemented a similar mechanism to the uniform crossover for multiple variables [21] for choosing which variables to cross. However those authors recognized the important implications with the linkage issues, therefore it does not alleviate the linkage problem of some MOPs.

2) Implementation and analyses of SBX operator: This section discusses the principal characteristics of SBX operator. Essentially, the behavior of operator is directly affected by three key components. Firstly, it applies a probability of altering each variable which is fixed to 0.5, therefore in average the half of each parent is copied to each child. Increasing this probability value induces more dissimilar children values, since that in average more variables are modified. An appropriate setting of this probability is related with the MOP, therefore a high probability value is better with objective functions that have high dependence level in the parameters, due that it provides a rotationally invariant behavior [15], this can be noticed in the Figure 3, where a low probability provokes a bias to the axis, therefore it is an ideal approach for separable problems a difference with a high probability (right simulation) where this could be better in non-separable problems. Also this probability value is related with the distribution index, in fact both components have a direct effect in the similarity between parents and children. Otherwise, a low probability value is suitable for objective functions that are separable [26], due that few decision variables are modified by the crossover operation.

The second key component consists that two offspring values are interchanged through a probability of 0.5, in some contexts this probability is known as "Variable uniform crossover probability" [6] or "Discrete Recombination" [27]. Although that in single-objective this action provides auto-adaptive behavior, in multi-objective optimization could not provide a desirable effect at first stages, the principal reason is that it could be highly disruptive. This components has serial implications, since that interchange variables between the children has the effect of multiple "reflections" in the feasible space. However, increasing the dimensions of the decision variables has the effect of increase exponentially the number of reflections ($2^n - 2$) as is showed in the figure 1 where is considered two and three decision variables.

Finally, the last component is the distribution index, which plays an important role, since a low index results in a greater exploration level than considering a high distribution index, in fact a distribution index of the unity has a similar effect of the Fuzzy Recombination Operator [28], the effect of the index can be analyzed in the figure 4 where in the left is a simulation with a low index which implies to create more dissimilar children, also can be noticed that it has a bias of create children according to the axis.

For simplicity the SBX implementation is showed in the Algorithm 1, which is based in the most frequently used implementation integrated in the NSGA-II code published by Deb et al. [3]. It requires two parents (P_1 and P_2) and create two children (C_1 and C_2).

Principally, the first and second key components correspond to the lines 5 and 8 respectively. As is usual, the SBX is configured with $\delta_1 = \delta_2 = 0.5$ and $\eta_c = 20$. It is important take into account that this configuration neither considers the dimension of the decision variables space or the criteria stop.

|	NSGA-II						MOEA/D						SMS-EMOA					
	1	2	3	4	5	DE	1	2	3	4	5	DE	1	2	3	4	5	DE
Average HV	0.88	0.90	0.90	0.91	0.93	**0.94**	0.87	0.87	0.87	0.90	**0.91**	0.91	0.88	0.89	0.87	0.91	0.92	**0.93**
Best Counts HV	2	1	0	1	8	**11**	2	0	2	2	8	**9**	0	1	1	5	6	**10**
Average Best Difference HV	0.068	0.057	0.053	0.039	0.019	**0.017**	0.053	0.048	0.049	0.024	**0.013**	0.014	0.074	0.064	0.081	0.045	0.028	**0.019**
Average IGD+	0.12	0.09	0.11	0.07	0.06	**0.05**	0.14	0.12	0.14	0.09	0.08	**0.07**	0.13	0.11	0.14	0.08	0.07	**0.05**
Best Counts IGD+	2	1	1	1	8	**10**	3	0	2	3	6	**9**	0	2	0	3	9	**9**
Average Best Difference IGD+	0.086	0.052	0.077	0.035	0.021	**0.016**	0.075	0.059	0.072	0.025	0.019	**0.008**	0.093	0.071	0.101	0.038	0.030	**0.017**
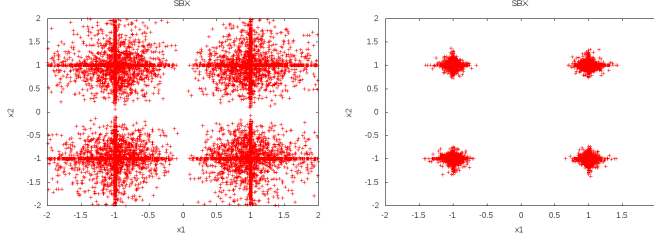
Fig. 4. Simulation the SBX operator with $10,000$ children values, where the parents are located in $P_1 = (-1.0, -1.0)$ and $P_2 = (1.0, 1.0)$. The left and right are with a distribution index of 2 and 20 respectively.

III. PROPOSAL

A. Implementing the linear dynamic behavior

Based in the previous analysis, to foment a balance between exploration at first stages and intensification at the end of the execution, the following modifications are proposed. First, the probability of modify a variable (δ_1) change among the execution, thus at first generations in average almost all variables are modified or sampled by the distribution of SBX, and in the last stages less variables are modified. This change is based in a linear decreasing model, where initially is fixed to 1.0 and it is decreasing to the half of total generations to 0.5 which is maintained until the end of the execution, thus from half of executions there is present the standard SBX, as is showed in the Equation (8), where $T_{Elapsed}$ is the current generation (Elapsed time) and T_{End} is the number of generations (Total time). In a similar way, the second change is related with the "variable uniform crossover probability" which is incremented from 0.0 to 0.5 as is indicated in the Equation (8). This modification is motivated to avoid the disruptive behavior of interchange the variables at first generations, once that the individuals converged to certain grade (to half of total generations) this probability is fixed to 0.5.

$$\delta_1 = \delta_2 = max\left(0.5, 1.0 - \frac{T_{Elapsed}}{T_{End}}\right) \quad (8)$$

Finally, the distribution index changes among the execution, where at the first stages a low distribution index induces an open distribution curve and it is linearly incremented producing a close distribution curve, this linear increment is indicated in the Equation (9). Particularly, this dynamic change has been showed in the literature with the polynomial mutation [29]. Also has been shown that a low distribution index provides reasonable quality solutions [16].

$$\eta_c = 2 + 20 \times \left(\frac{T_{Elapsed}}{T_{End}}\right) \quad (9)$$

B. The relevance of isolated components

In this section we show the effect of each of the three components previously mentioned of the SBX. In general is applied the same configuration as is indicated in the Section IV. Particularly, the jMetalcpp [30] framework was used to perform our executions. Taking into account the stochastic behavior of MOEAs, 35 independent executions were run. In all of them, the stopping criteria was set to $25,000$ generations and the size of the population was fixed to 100. The effect of each component is analyzed through four cases, based in the Algorithm 1 each case is described as follows:

- **Case 1**: The standard SBX operator where $\delta_1 = \delta_2 = 0.5$ and $\eta_c = 20$.
- **Case 2**: δ_1 changes according the Equation (8), $\delta_2 = 0.5$ and $\eta_c = 20$.
- **Case 3**: δ_2 changes according the Equation (8), $\delta_1 = 0.5$ and $\delta_c = 20$.
- **Case 4**: The distribution index changes according the Equation (9), $\delta_1 = \delta_2 = 0.5$.

In order to analyze the performance of each Case (Case 5 is discussed in section IV), the Tables I, II and III shows information of the Normalized Hyper-volume (HV) [29] and the Inverted Generational Distance Plus (IGD+) [18]. In the Table III is showed a summary of the statistical tests, where are considered the HV and IGD+ both in two and three objectives. The Case 4 which correspond to the dynamic distribution index, presents the best results than Case 1, Case 2 and Case 3, due that it has more wins in the state-of-the-art MOEAs, also it has a less number of lost. Thus, increasing the distribution index provokes an suitable behavior to aim a balance between exploration and intensification, due that even if at the first generations the diversity is not enough, it could be increased since that the curve of distribution is open, hence it might generate dissimilar individuals and increase the diversity of the population. In the second place is the Case 3, it increases the probability of interchange a variable based in a linear model, this might occurs because in the first stages is avoided a disruptive behavior, and in a gradual way the search process explores the interest regions, although that this case is not the best, it still outperforms the standard SBX (Case 1). The average of HV and IGD+ for two and three objectives are showed in the Tables I and II, it shows that the Case 4 outperforms the Case 1, Case 2 and Case 3 with

two and three objectives. On the other hand, just considering the averages, the Case 2 is worse than the Case 1, this might occurs because altering almost all the variables with the SBX distribution could induce very dissimilar and distant children. Perhaps, modifying this linear model (i.e. from 0.4 to 0.5) could provide a better behavior. It is important to take into account that in some situations the statistical test indicate a superiority, however the average of the metrics are of low quality, this occurs because a determined case provide high quality solutions for some problems and very low quality solutions for other problems, generally speaking in long-term executions is desirable to obtain stable and well converged solutions.

Based in the previously analyses, a variant of the SBX is proposed, where the Case 3 and Case 4 are joined. Therefore, based in Algorithm 1 our proposal is configured as follows. As is usual the parameter δ_1 is fixed to 0.5, based in the Case 3 the δ_2 is changed according the Equation (8) and based in the Case 3 the distribution index (η_c) is changed according the Equation (9). Hence, an important advantage is that our proposal will be more stable in relation with the distribution index.

IV. EXPERIMENTAL VALIDATION

This section is devoted to validate our proposal named as "Case 5". The WFG [31], DTLZ [32] and UF [17] test problems have been used for our purpose. Our experimental validation includes the Differential Evolution based variants DEMO such strategy is explained in [6]. Given that all of them are stochastic algorithms, each execution was repeated 35 times with different seeds. The common configuration in all of them was the following: the stopping criterion was set to $25,000$ generations, the population size was fixed to 100, the WFG test problems were configured with two and three objectives, setting 24 parameters, where 20 of them are distance parameters and 4 are position parameters. Specifically, in the DTLZ test instances, the number of decision variables is set to $n = M + r - 1$, where $r = \{5, 10, 20\}$ for DTLZ1, DTLZ2 to DTLZ6 and DTLZ7 respectively, as is suggested by the authors [32]. The UF benchmark is composed of ten test instances, where the first seven are of two objectives and the rest with three objectives, the number of decision variables is assigned to $n = 10$. In general, the crossover and mutation operators are SBX and polynomial respectively, with a crossover probability of 0.9 and mutation probability of $1/n$, also the crossover and mutation distribution indexes were assigned to 20 and 50 respectively. The extra parametrization of each algorithm is as follows:

- **DE-Variants**: CR = 0.3 and F = 0.5.
- **SMS-EMOA**: offset = 100.
- **MOEA/D**: size of neighborhood = 10, max updates by sub-problem (nr) = 2 and $\delta = 0.9$.

Our experimental analysis has been performed in base of the Normalized HV and IGD+. The reference points implemented

in the hypervolume indicator are showed in the Table IV as used in [33], [34].

In order to statistically compare the hypervolume results, a similar guideline than the proposed in [35] was used. First a Shapiro-Wilk test was performed to check whatever or not the values of the results followed a Gaussian distribution. If, so, the Levene test was used to check for the homogeneity of the variances. If samples had equal variance, an ANOVA test was done; if not, a Welch test was performed. For non-Gaussian distributions, the nonparametric Kruskal-Wallis test was used to test whether samples are drawn from the same distribution. An algorithm X is said to win algorithm Y when the differences between them are statistically significant, if the mean and median obtained by X are higher than the mean and median achieved by Y.

In addition, pair-wise statistical tests with HV and IGD+ were performed (Table III). For each instance, the column "↑" reports the number of comparisons where the statistical tests confirmed the superiority of the MOEA listed in the corresponding group, whereas the column "↓" reports the number of cases where it was inferior and "←→" indicates that are not significantly different and is considered as a tie.

Based in the statistical tests[1] (Table III), generally speaking our proposal (Case 5) has the best scores. Our proposal is outperformed by the Case 4 considering three objectives by the NSGA-II, this might occurs due that this MOEA applies a binary tournament selection where is considered the dominance concept, thus some level of diversity is maintained, therefore in some problems some optimal regions are reached, and with other problems the solutions could be far from the Pareto front. Although that in average the metrics are similar between the Case 4 and our proposal with three objectives, the average difference with the best indicates that our proposal is the closest to the best since that it shows in the "Average Best Difference" IGD+ with 0.020 against the Case 4 with 0.023. Also, inspecting the statistical test, the Case 4 has more lost than our proposal 10 against 9 and 15 against 9 of the HV and IGD+ respectively. However, the MOEA/D and SMS-EMOA which are more elitist algorithms and do not consider the dominance concept at all, are not affected in this sense. Anyway, our proposal and the Case 3 reports significantly better results that the normal SBX (Case 1), in fact this could evidence that variate the distribution index among the execution has an important effect.

In order to better understand the behavior of our proposal, the Tables I and II shows a summary average of Normalized HV and IGD+ considering all the instances[2]. Also is counted the number of instances which attain the best results labeled by "Best Counts". Additionally, the average of difference with the best result is showed, this aims to provide a metric which indicates the difference with the best result attained in each problem. Although that DE improves our proposal with two

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[1]DE is not considered in the statistical tests.
[2]The detailed results can be consulted in the web page https://github.com/joelchaconcastillo/SBX_CEC2018.

TABLE II
STATISTICAL INFORMATION OF METRICS WITH THREE OBJECTIVES

	NSGA-II						MOEA/D						SMS-EMOA					
	1	2	3	4	5	DE	1	2	3	4	5	DE	1	2	3	4	5	DE
Average HV	**0.87**	0.84	**0.87**	**0.87**	**0.87**	0.85	0.84	0.84	0.84	**0.86**	**0.86**	0.85	0.90	0.89	0.88	**0.91**	**0.91**	**0.91**
Best Counts HV	1	2	1	4	4	**7**	1	2	1	2	5	**8**	3	2	0	2	5	**7**
Average Best Difference HV	0.019	0.047	0.020	**0.014**	**0.014**	0.032	0.036	0.041	0.038	0.016	**0.013**	0.027	0.038	0.038	0.049	**0.019**	0.027	**0.019**
Average IGD+	0.13	0.16	0.13	**0.12**	**0.12**	0.13	0.15	0.14	0.15	**0.11**	**0.11**	0.13	0.11	0.11	0.13	**0.09**	**0.09**	0.13
Best Counts IGD+	0	2	2	4	3	**8**	2	2	0	2	4	**9**	1	3	0	3	5	**7**
Average Best Difference IGD+	0.029	0.061	0.027	0.023	**0.020**	0.032	0.053	0.048	0.053	**0.015**	**0.015**	0.030	0.047	0.040	0.062	**0.020**	0.024	0.069

TABLE III
SUMMARY OF STATISTICAL TESTS

NSGA-II																
	1			2			3			4			5		
	↑	↓	←→	↑	↓	←→	↑	↓	←→	↑	↓	←→	↑	↓	←→
HV-2obj	16	29	47	6	61	25	28	19	45	31	23	38	**54**	3	35
HV-3obj	15	19	42	12	50	14	17	15	44	**33**	10	33	26	9	41
IGD-2obj	14	30	48	4	60	28	25	17	50	33	19	40	**52**	2	38
IGD-3obj	14	18	44	13	44	19	18	15	43	**33**	15	28	23	9	44

MOEA/D																
	1			2			3			4			5		
	↑	↓	←→	↑	↓	←→	↑	↓	←→	↑	↓	←→	↑	↓	←→
HV-2obj	15	33	44	10	60	22	25	26	41	39	18	35	**57**	9	26
HV-3obj	10	22	44	12	39	25	11	19	46	24	10	42	**38**	5	33
IGD-2obj	16	31	45	9	60	23	23	27	42	37	17	38	**57**	7	28
IGD-3obj	12	22	42	13	43	20	13	24	39	30	9	37	**40**	10	26

SMS-EMOA																
	1			2			3			4			5		
	↑	↓	←→	↑	↓	←→	↑	↓	←→	↑	↓	←→	↑	↓	←→
HV-2obj	9	35	48	7	43	42	16	31	45	41	9	42	**53**	8	31
HV-3obj	7	21	48	9	35	32	13	21	42	27	6	43	**31**	4	41
IGD-2obj	10	34	48	15	48	29	12	33	47	41	12	39	**55**	6	31
IGD-3obj	8	20	48	13	30	33	9	19	48	22	5	49	**27**	5	44

TABLE IV
REFERENCES POINTS FOR THE HV INDICATOR

Instances	Reference Point
WFG1-WFG9	$[2.1, ..., 2m + 0.1]$
DTLZ 1, 2, 4	$[1.1, ..., 1.1]$
DTLZ 3, 5, 6	$[3, ..., 3]$
DTLZ7	$[1.1, ..., 1.1, 2m]$
UF 1-10	$[2, ..., 2]$

objectives, in the Table I is showed that our proposal provides near results than the bests.

On the other hand our proposal provides the best results considering three objectives, it is interesting remark that the performance of DE is seriously deteriorated according increases the number of objectives. Despite the fact that DE variants have high "Best Count" values, in average it is improved by the Case 4 and Case 5, this might occurs because DE is directly influence by the parameters (CR and FR), therefore in some instances the DE variants attains the best results, and in other instances the solutions are far from the Pareto front.

On the other hand, our proposal shows a robust behavior in the state-of-the-art MOEAs, in fact the "Average Best Difference" is low both with two objectives and three objectives that confirms the stability and superiority of our proposal.

V. CONCLUSIONS

In the usual scheme of EAs, the crossover is one of the most important operators. In most cases, crossover operators provide a large degree of exploration when the content of the population is diverse. However, when a low diversity degree is reached, they tend to promote intensification. Since the behavior of operators does not usually depend on the stopping criterion, losing diversity in a gradual way might be a complex task. Thus, some parameterizations might be suitable for some stopping criterion but not for others. This paper proposes extending the well-known SBX to incorporate the stopping criterion and elapsed generations as one of its inputs. First, the standard version of SBX and some actions that can be done to alter its exploration capabilities are identified. Specifically, the SBX is composed by three key components that are related with diversity issues. The first controls the amount of variables that are inherited intact. The second is the probability of interchanging a given variable between offspring. Finally, the last component is related with the aperture of the distribution index. These tree features are adapted by taking into account both the stopping criterion and elapsed generations, with the aim of inducing an additional degree of exploration in the first stages of the optimization. The experimental validation is carried out with long-term executions and the popular WFG, DTLZ and UF problems. This validation shows that a dynamic distribution index provides significantly better

and more robust results than the standard SBX with all the tested MOEAs. Additionally, adapting the probability of interchanging variables between offspring provides benefits and by adapting this probability simultaneously with the distribution index performance can be further improved. In the case of three objectives, state-of-the-art DE operators could also be outperformed by our proposal. No additional parameterizations were required to devise the proposed operators, meaning that robust results could be obtained with the additional user efforts.

Several lines of the future work might be explored. First, we would like to devise some strategies to adaptively manage the distribution index. Meassuring the diversity of the population to alter the behavior of the operators seems a plausible approach. Additionally, using these kinds of operators together with some of the specific methods that has been devised to avoid premature convergence might bring additional benefits. Finally, we plan to use the principles that governed the design of our proposals to devise novel vector-wise operators.

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