Top Objections to $S^2 \subset \mathbb{R}^2$ and Their Reconciliations

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Context. The note argues that equipping $\mathbb{R}^2$ with the round metric

$$ds^2 = \frac{4\,(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}$$

and adding the point at infinity yields a surface isometric to the standard unit sphere; i.e., $(\mathbb{R}^2, g) \cup \{\infty\} \cong S^2$ by an explicit isometry. The objections below are common points of confusion; each is paired with a concise reconciliation.

Objections and Reconciliations

1. Objection: "Two-space cannot be compact without a third coordinate."

Reconciliation. Compactness depends on the metric, not the coordinate count. Under the curved metric g, radial length is finite:

$$\ell(r) = \int_0^r \frac{2\,dt}{1 + t^2} = 2 \arctan r \to \pi \quad (r \to \infty),$$

so $(\mathbb{R}^2, g) \cup \{\infty\}$ is compact and boundaryless. Compactness fails only under the flat metric δ.

2. Objection: "Adding a point at infinity creates a boundary or a cusp."

Reconciliation. With the round metric, the added point is a smooth limit point, not a boundary point. Geodesic length is finite to that point and neighborhoods are metric spheres, giving a smooth closure (no edge or corner).

3. Objection: "This is just stereographic projection in disguise; it's not intrinsic."

Reconciliation. The round metric on $\mathbb{R}^2$ can be derived via stereography, but once specified it functions intrinsically on the pair space (x, y). The isometry is defined entirely from $(\mathbb{R}^2, g)$, independent of ambient $\mathbb{R}^3$.

4. Objection: "You haven't embedded topologically; you only have an isometry to a surface in $\mathbb{R}^3$."

Reconciliation. The claim is about *metric realization* on a set of ordered pairs. The map $(\mathbb{R}^2, g) \cup \{\infty\} \to S^2$ is an isometry (hence a homeomorphism). Topological embedding in a fixed *ambient* $\mathbb{R}^2$ with the flat metric is a different statement.

5. Objection: "Spheres require three dimensions."

Reconciliation. "Require" conflates curvature with coordinate count. Curvature is a property of the metric; the round metric lives on two coordinates. The usual textbook phrase tacitly assumes the flat metric on $\mathbb{R}^2$ and so is underspecified.

6. Objection: "You changed the metric; that's cheating."

Reconciliation. A manifold is a set plus structure. Writing $S^2 \subset \mathbb{R}^3$ always included *extrinsic* structure (flat δ in $\mathbb{R}^3$). Here we make the *intrinsic* structure explicit on $\mathbb{R}^2$. Declaring the metric is precisely the point: it distinguishes geometry from bare coordinates.

7. Objection: "But the classical theorem says S^2 is not homeomorphic to $\mathbb{R}^2$."

Reconciliation. Correct—*as plain topological spaces* with their standard (flat) structures, $S^2 \setminus \{\text{a point}\} \cong \mathbb{R}^2$, while S^2 itself is not homeomorphic to $\mathbb{R}^2$. The claim here is different: $(\mathbb{R}^2, g) \cup \{\infty\}$ is a metric space *isometric* to S^2, obtained by compactifying in the metric g.

8. Objection: "Is it complete before adding ∞?"

Reconciliation. In the round metric, Cauchy sequences running to Euclidean infinity have finite length limit, so completion adds exactly one point and yields a compact, complete surface; the completion is isometric to S^2.

9. Objection: "Is the isometry explicit or merely asserted?"

Reconciliation. It is explicit:

$$(x, y) \mapsto \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2} \right)$$

is an isometry onto the unit sphere endowed with the flat metric δ in $\mathbb{R}^3$; all spherical distances pull back to g on (x, y).

10. Objection: "This contradicts the standard 'minimal embedding dimension' lore."

Reconciliation. The lore refers to embedding S^2 *as a subset of flat* $\mathbb{R}^n$. Once we separate "set of ordered pairs" from "metric attached," the minimal coordinate count for an intrinsic round sphere is two; curvature demands a metric, not an extra axis.

Remark on statements. A precise canonical phrasing is: "Let $(\mathbb{R}^n, \delta)$ denote flat Euclidean space." Then "S^2 does not embed in $(\mathbb{R}^2, \delta)$" is true, while "S^2 is realized by $(\mathbb{R}^2, g) \cup \{\infty\}$" is the intrinsic metric statement supported above.