Top Objections to $S^2 \subset \mathbb{R}^2$ and Their Reconciliations

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Context. The note argues that equipping $\mathbb{R}^2$ with the round metric

$$g \;=\; \frac{4\,(dx^2+dy^2)}{(1+x^2+y^2)^2}$$

and adding one point at infinity yields a surface isometric to the unit 2-sphere; i.e., $(\mathbb{R}^2, g) \cup \{\infty\} \cong S^2$ by an explicit isometry. The objections below are common points of confusion; each is paired with a short reconciliation.

Objections and Reconciliations

1. Objection: "Two-space cannot be compact without a third coordinate."

Reconciliation. Compactness depends on the metric, not the coordinate count. Under g, radial length to Euclidean infinity is finite:

$$\ell(r) = \int_0^r \frac{2\,dt}{1+t^2} = 2\arctan r \xrightarrow[r\to\infty]{} \pi,$$

so $(\mathbb{R}^2, g) \cup \{\infty\}$ is compact and boundaryless. Compactness fails only under the flat metric δ.

2. Objection: "Adding a point at infinity creates a boundary or a cusp."

Reconciliation. With the round metric, the added point is a smooth limit point, not a boundary. Geodesic length to that point is finite and neighborhoods are metric balls, so the closure is smooth (no edge).

3. Objection: "This is just stereographic projection; it's not intrinsic."

Reconciliation. The round metric can be *derived* via stereography, but once specified, it is intrinsic on $\mathbb{R}^2$. The isometry is defined from $(\mathbb{R}^2, g)$ alone, independent of an ambient $\mathbb{R}^3$ description.

4. Objection: "You have an isometry to a surface in $\mathbb{R}^3$, not an embedding in flat $\mathbb{R}^2$."

Reconciliation. The claim concerns *metric realization* on ordered pairs. The map $(\mathbb{R}^2, g) \cup \{\infty\} \to S^2$ is an isometry (hence a homeomorphism). Embedding S^2 as a subset of $(\mathbb{R}^2, \delta)$ is a different statement and remains false.

5. Objection: "Spheres require three dimensions."

Reconciliation. That conflates curvature with coordinate count. Curvature is metric data; the round metric lives on two coordinates. The textbook mantra silently fixes the flat metric on $\mathbb{R}^2$.

6. Objection: "Changing the metric is cheating."

Reconciliation. A manifold is a set *plus structure*. Writing $S^2 \subset \mathbb{R}^3$ uses extrinsic structure (flat δ in $\mathbb{R}^3$). Here we state the intrinsic structure explicitly on $\mathbb{R}^2$: that is the point.

7. Objection: "But the classical theorem says $S^2 \not\cong \mathbb{R}^2$."

Reconciliation. As plain topological spaces with standard structures, $S^2 \not\cong \mathbb{R}^2$ while $S^2 \setminus \{*\} \cong \mathbb{R}^2$. Our statement is different: $(\mathbb{R}^2, g) \cup \{\infty\}$ is *isometric* to S^2.

8. Objection: "Is $(\mathbb{R}^2, g)$ complete before adding ∞?"

Reconciliation. Cauchy sequences escaping to Euclidean infinity have finite round length; the completion adds exactly one point and becomes compact and complete, isometric to S^2.

9. Objection: "Is the isometry explicit or merely asserted?"

Reconciliation. It is explicit:

$$\Phi(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}\right)$$

is an isometry onto the unit sphere with the ambient Euclidean metric; spherical distances pull back to g.

10. Objection: "This contradicts minimal embedding-dimension lore."

Reconciliation. That lore refers to embedding S^2 as a subset of flat Euclidean space. Once we separate "points" from "metric," the intrinsic round sphere has two coordinates; curvature requires a metric, not an extra axis.

Remark. A precise canonical phrasing is: "Let $(\mathbb{R}^n, \delta)$ denote flat Euclidean space." Then "S^2 does not embed in $(\mathbb{R}^2, \delta)$" is true, while "S^2 is realized by $(\mathbb{R}^2, g) \cup \{\infty\}$" is the intrinsic metric statement supported above.

Appendix: Explicit Isometry and Length/Compactness Calculations

A.1 Inverse stereographic map and pullback metric

Let $S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ with the ambient Euclidean metric δ. The inverse stereographic map from the north pole $N = (0, 0, 1)$ sends $(x, y) \in \mathbb{R}^2$ to

$$\Phi(x, y) = \left(\frac{2x}{1 + r^2}, \frac{2y}{1 + r^2}, \frac{r^2 - 1}{1 + r^2}\right), \qquad r^2 = x^2 + y^2.$$

A direct differential computation yields the pullback

$$\Phi^*\delta \;=\; \frac{4}{(1 + x^2 + y^2)^2}\,(dx^2 + dy^2) \;=\; g.$$

Hence $\Phi : (\mathbb{R}^2, g) \to S^2 \setminus \{N\}$ is a local isometry. It is bijective with smooth inverse (stereographic projection), so Φ is an isometry onto $S^2 \setminus \{N\}$.

A.2 Finite distance to the added point and completion

For a radial path $t \mapsto (t, 0)$, $t \geq 0$, we have

$$ds = \frac{2\,dt}{1+t^2}, \qquad \ell(0 \to R) = \int_0^R \frac{2\,dt}{1+t^2} = 2 \arctan R \xrightarrow[R\to\infty]{} \pi.$$

Thus Euclidean infinity lies at finite g-distance π. Any Cauchy sequence escaping to Euclidean infinity converges in the metric completion to a single added point ∞. The completed space $(\mathbb{R}^2, g) \cup \{\infty\}$ is compact (closed and totally bounded) and the extension of Φ by $\Phi(\infty) = N$ is an isometry onto all of S^2.

A.3 Topology and smoothness at ∞

Metric spheres in $(\mathbb{R}^2, g)$ correspond under Φ to geodesic spheres on S^2; neighborhoods of ∞ pull back from neighborhoods of N. Hence ∞ is neither a boundary nor a cone point; the completed surface is smooth and homeomorphic (indeed isometric) to S^2.

Conclusion. The map Φ exhibits an explicit isometry $(\mathbb{R}^2, g) \cup \{\infty\} \cong S^2$, resolving the objections by separating intrinsic metric realization from extrinsic flat embedding.