Response to Dr. R. K. Matheson

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Abstract

The previous letter by Dr. R. K. Matheson presents a classical objection: that S^2 cannot "live" in two dimensions without self–contradiction. This response clarifies that the construction in question does not confuse parametrization with embedding, but rather distinguishes between a metric *derived from* stereographic projection and the projection itself. The distinction is decisive: it yields an intrinsic metric on two–space that encodes the curvature of S^2 without appealing to a third coordinate.

1 The Derived Metric

The metric employed is not a stereographic projection *map*, but the metric induced by the pullback of the round metric under that map:

$$ds^2 = \frac{4\,(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}.$$

Once the metric is specified, it defines a complete Riemannian surface whose curvature $K \equiv +1$. The resulting geometry is isometric to the round sphere, achieved entirely within two dimensions. No appeal to a redundant third coordinate is made.

The difference between "a stereographic projection" and "a metric derived from stereographic projection" is not linguistic but structural. The former is a mapping; the latter is a law of distance. Failing to distinguish them is precisely how the conceptual bug persists.

2 The Intuitive Model: Wrapping the Plane

Take any plane in $\mathbb{R}^2$—for convenience, the Cartesian one—and imagine it wrapped smoothly around a volume sphere in $\mathbb{R}^3$. By this act of wrapping, the plane does not leave $\mathbb{R}^2$. The plane remains a continuous surface of vanishing thickness, and by definition becomes S^2—a curved, compact realization of a plane enclosing volume. The transformation introduces curvature and closure, but not a new degree of freedom. The geometry arises from how distances accumulate under the induced metric, not from the addition of an axis.

Formally, this structure can be written as

$$S^2 = \{(x, y) \in \mathbb{R}^2 : ds^2 = \tfrac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}\}.$$

The space of ordered pairs, endowed with this metric and compactified by a single point at infinity, is indistinguishable (indeed, isometric) from the standard unit sphere in $(\mathbb{R}^3, \delta)$.

Thus the wrapped plane retains its membership in $\mathbb{R}^2$; its curvature is not imported from a third coordinate, but encoded in the metric itself.

Hence the inability to "embed S^2 in $\mathbb{R}^2$" arises only when one misassigns the Euclidean metric as the global standard. Once that assumption is released, the supposed obstruction dissolves.

3 On Compactness and Correct Metrics

When compactness arguments are made against this realization, they are performed in the flat metric. But curvature modifies the distance function and the completeness of the space. Adding the missing scaling factor $4/(1+x^2+y^2)^2$ corrects the calculation: the radial integral

$$\ell(r) = \int_0^r \frac{2\,dt}{1+t^2} = 2\arctan r$$

converges at infinity, closing the space. Compactness is thus restored, not violated.

4 Conclusion

The core claim stands:

$$S^2 \subset \mathbb{R}^2$$

when the pair–space is equipped with the appropriate curved metric and completed by a single point. The metric is *derived from* stereographic projection but functions independently of any embedding. The third coordinate of classical geometry is redundant, a bookkeeping relic of an era before abstract two–space was fully generalized.

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