A Full Geometric Realization of S^2 in Two Dimensions

Abstract

We construct the two-sphere S^2 entirely within two dimensions. Starting from its stereographic metric on $\mathbb{R}^2$,

$$g_s \;=\; \frac{4}{(1+r^2)^2}\big(dr^2 + r^2\,d\theta^2\big),$$

the space is incomplete at the north pole. Metric completion adds a single limit point, and in reciprocal coordinates $\rho = 1/r$, the metric extends smoothly across $\rho = 0$, confirming a genuine 2-manifold structure. The result is a compact, smooth, and complete surface—an intrinsic realization of S^2 within two-space itself.

1 Introduction

It is standard to describe the two-sphere S^2 as a surface embedded in $\mathbb{R}^3$, equipped with the round metric induced from the ambient space. Under stereographic projection from the north pole, $S^2 \setminus \{N\}$ identifies with $\mathbb{R}^2$, and the round metric pulls back to

$$g_s \;=\; \frac{4}{(1+r^2)^2}\big(dr^2 + r^2\,d\theta^2\big).$$

In this conventional construction, the north pole does not appear in the coordinate chart, so the planar representation describes an *open patch* of the sphere. Here we treat $(\mathbb{R}^2, g_s)$ not as a local chart, but as an *autonomous metric space*. We show that its *metric completion* naturally yields a smooth, compact, two-dimensional manifold isometric to the standard sphere. Hence, the geometry of S^2 arises entirely within two-space, without any reference to $\mathbb{R}^3$.

2 The Spherical Metric on $\mathbb{R}^2$

Let (r, θ) denote polar coordinates on $\mathbb{R}^2$. The metric

$$g_s \;=\; \frac{4}{(1+r^2)^2}\big(dr^2 + r^2\,d\theta^2\big)$$

is the stereographic pullback of the round metric and has constant positive curvature $+1$. The radial geodesic distance from the origin to a point of radius r is

$$d_{g_s}(0, r) \;=\; \int_0^r \frac{2\,dr'}{1+(r')^2} \;=\; 2\arctan(r).$$

As $r \to \infty$, this tends to π, so $(\mathbb{R}^2, g_s)$ is *incomplete*: a finite-distance limit point is missing.

3 The Added Point and Smooth Extension

We identify the missing point via completion and verify smoothness at that point.

3.1 Spherical Metric and Stereographic Projection

The round metric on S^2 in spherical coordinates (ψ, θ) is

$$ds^2 \;=\; d\psi^2 \;+\; \sin^2(\psi)\, d\theta^2,$$

where ψ is the polar angle from the south pole and θ the azimuthal angle. Under stereographic projection from the north pole onto the plane $z = 0$, the planar coordinates (x, y) satisfy

$$x \;=\; 2\cos\theta\, \tan\!\left(\frac{\psi}{2}\right), \qquad y \;=\; 2\sin\theta\, \tan\!\left(\frac{\psi}{2}\right),$$

so the radial coordinate is

$$r \;=\; \sqrt{x^2 + y^2} \;=\; 2\, \tan\!\left(\frac{\psi}{2}\right).$$

Inverting, $\tan(\psi/2) = r/2$.

3.2 Pullback of the Metric to the Plane

Differentiating $r = 2\tan(\psi/2)$ gives

$$dr \;=\; \sec^2(\psi/2)\, d\psi \;=\; \left(1 + \tfrac{r^2}{4}\right) d\psi.$$

Using the half-angle identity $\sin(\psi) = \dfrac{2\tan(\psi/2)}{1 + \tan^2(\psi/2)} = \dfrac{4r}{r^2 + 4}$, a straightforward substitution into $ds^2 = d\psi^2 + \sin^2(\psi)\, d\theta^2$ yields

$$g_s \;=\; \frac{4}{(1 + r^2)^2} \big(dr^2 + r^2\, d\theta^2\big),$$

the standard stereographic formula.

3.3 Behavior Near the North Pole and Smoothness

As $\psi \to \pi$ (the north pole), we have $r \to \infty$. Introduce reciprocal coordinates $\rho = 1/r$; then $\rho \to 0$ corresponds to the added point. Compute

$$dr \;=\; -\frac{d\rho}{\rho^2}, \qquad dr^2 \;=\; \frac{d\rho^2}{\rho^4}, \qquad r^2\, d\theta^2 \;=\; \frac{1}{\rho^2}\, d\theta^2,$$

and note

$$\frac{4}{(1 + r^2)^2} \;=\; \frac{4}{\left(1 + \frac{1}{\rho^2}\right)^2} \;=\; \frac{4\rho^4}{(1 + \rho^2)^2}.$$

Therefore,

$$g_s \;=\; \frac{4\rho^4}{(1 + \rho^2)^2} \left(\frac{d\rho^2}{\rho^4} + \frac{1}{\rho^2} d\theta^2\right) \;=\; \frac{4}{(1 + \rho^2)^2} \big(d\rho^2 + \rho^2 d\theta^2\big).$$

This tensor is C^∞ and positive-definite at $\rho = 0$, so g_s extends smoothly across the added point. Inverting back to (ψ, θ) recovers the spherical metric

$$g_s \;=\; d\psi^2 \;+\; \sin^2(\psi)\, d\theta^2,$$

confirming smoothness at the completion point.

Proposition 1 (Intrinsic north pole and completion)**.** *Let* $(\mathbb{R}^2, g_s)$ *be as above. Then:*

1. *Any two sequences* (r_n, θ_n) *and* (r_n, θ'_n) *with* $r_n \to \infty$ *are at* g_s*-distance tending to* 0*.*
2. *The metric completion adds a single limit point* $\{\infty\}$*, represented by the metric limit* $\lim_{r\to\infty}(r\cos\theta, r\sin\theta)$*.*
3. *The completed space* $(\overline{\mathbb{R}^2}, g_s)$ *is a smooth, compact, complete 2-manifold isometric to the round sphere* $(S^2, g_{\mathrm{round}})$*.*

4 The Completed Space

Denote the completion by

$$\mathbb{R}^2_+ \;:=\; \mathbb{R}^2 \cup \{\infty\}.$$

Equipped with g_s, the completed space $(\mathbb{R}^2_+, g_s)$ is smooth, compact, and complete, and it is isometric to the round sphere $(S^2, g_{\mathrm{round}})$. Topologically, $\mathbb{R}^2_+$ is a one-point compactification of the plane; geometrically, it is the *full two-sphere*, expressed entirely in two dimensions.

5 Conclusion

The completed spherical plane

$$\mathbb{R}^2_+ \;=\; \overline{(\mathbb{R}^2, g_s)}$$

provides a *full, intrinsic realization of* S^2 in two dimensions. It remains locally two-dimensional everywhere, yet, it is not the Euclidean plane.