From Algebra to Abstract Machine: A Verified Generic Construction

TyDe'18

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To put it another way ...

Verified tail-recursive folds through dissection

Motivation

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data Expr : Set where
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  Add : Expr → Expr → Expr
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\quad \mathsf{Val}\ :\ \mathbb{N} \to \mathsf{Expr} \\
\quad \mathsf{Add}\ :\ \mathsf{Expr} \to \mathsf{Expr} \to \mathsf{Expr}
\end{array}
$$

and we write an evaluator for it.

$$
\begin{array}{l}
\mathsf{eval}\ :\ \mathsf{Expr} \to \mathbb{N} \\
\mathsf{eval}\ (\mathsf{Val}\ n) = n \\
\mathsf{eval}\ (\mathsf{Add}\ e_1\ e_2) = \mathsf{eval}\ e_1 + \mathsf{eval}\ e_2
\end{array}
$$

A Problem with eval


```
> eval (Add (Add (Add ... (Add (Val 1) (Val 2)))))   -- large expression
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What happened?

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What happened? A **well-typed** program *went wrong*.

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$***$ *Exception : stack overflow*

What happened? A **well-typed** program *went wrong*.

- eval evaluates both subtrees before reducing _+_ further.
- Record unevaluated subtrees on the stack.
- On large inputs, the **stack** might **overflow**.

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We proceed in *three steps*:

- Make the underlying stack explicit.
- Define a *tail-recursive* function over the stack.
- Show that it is equivalent to eval.

A tail-recursive evaluator

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Stack}\ :\ \mathsf{Set}\ \mathsf{where} \\
\quad \mathsf{Top}\ \ \ :\ \mathsf{Stack} \\
\quad \mathsf{Left}\ \ :\ \mathsf{Expr} \to \mathsf{Stack} \to \mathsf{Stack} \\
\quad \mathsf{Right}\ :\ \mathbb{N} \ \ \ \ \to \mathsf{Stack} \to \mathsf{Stack}
\end{array}
$$

A tail-recursive evaluator

```
data Stack : Set where
  Top   : Stack
  Left  : Expr → Stack → Stack
  Right : ℕ    → Stack → Stack

mutual
  load : Expr → Stack → ℕ
  load (Val n)       stk = unload n stk
  load (Add e₁ e₂) stk = load e₁ (Left e₂ stk)

  unload : ℕ → Stack → ℕ
  unload v Top              = v
  unload v (Left e stk)     = load e (Right v stk)
  unload v (Right v' stk)   = unload (v' + v) stk
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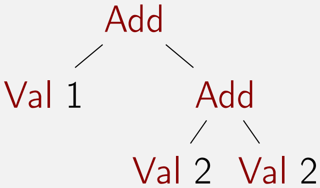
tail-rec-eval : Expr → ℕ
tail-rec-eval e = load e Top
```

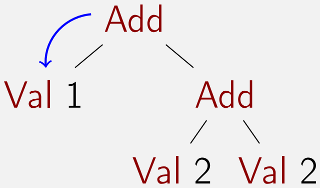
tail-rec-eval in action

load (Add (Val 1) (Add (Val 2) (Val 2))) <u>Top</u>



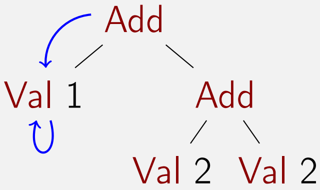
tail-rec-eval in action

load (Val 1) (Left (Add (Val 2) (Val 2)) Top)



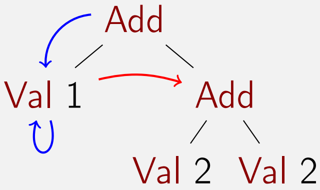
tail-rec-eval in action

unload 1 (Left (Add (Val 2) (Val 2)) Top)



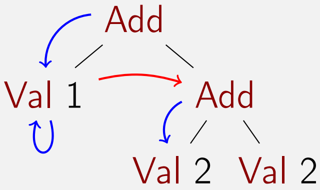
tail-rec-eval in action

load (Add (Val 2) (Val 2)) (Right 1 Top)



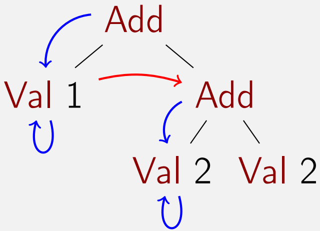
tail-rec-eval in action

load (Val 2) (Left (Val 2) (Right 1 Top))



tail-rec-eval in action

unload 2 (Left (Val 2) (Right 1 Top))



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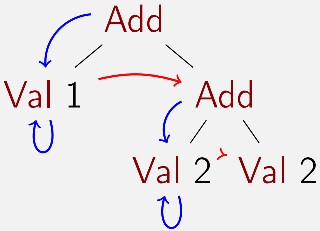

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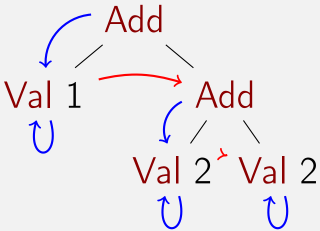
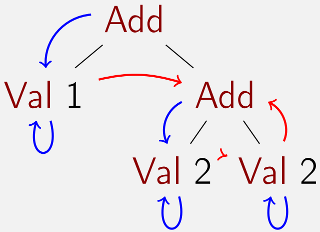


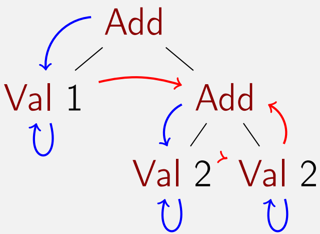
tail-rec-eval in action

unload (2 + 2) (Right 1 Top)



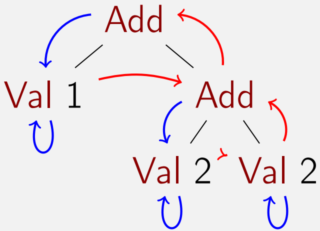
tail-rec-eval in action

unload 4 (Right 1 Top)



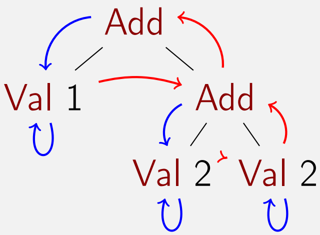
tail-rec-eval in action

unload $(1 + 4)$ Top



tail-rec-eval in action

unload 5 Top



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Have we **actually** solved the problem?

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mutual

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\begin{aligned}
&\boxed{\mathsf{load}} : \mathsf{Expr} \to \mathsf{Stack} \to \mathbb{N}\\
&\ldots\\
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$$\begin{array}{lll}
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\end{array}$$

We **lost** termination guarantees.

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Termination We **construct** a tail-recursive evaluator that terminates for every input.

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That's not all

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Both versions of the function are <u>the same</u>.

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Moreover:

- Expr is an example of a **regular** type.
- fold is an instance of a **catamorphism**.

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Everything is machine-checked by *Agda*.

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4. Prove that the resulting function is equivalent to eval.

Breaking down mutual recursion

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\begin{aligned}
&\mathsf{unload}^{+} : (\mathbb{N} \times \mathsf{Stack}) \to (\mathbb{N} \times \mathsf{Stack}) \uplus \mathbb{N} \\
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Now, both functions obviously terminate.

A step back

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$$\mathsf{unload}^{+} : (\mathbb{N} \times \mathsf{Stack}) \rightarrow (\mathbb{N} \times \mathsf{Stack}) \uplus \mathbb{N}$$

- unload^{+} is the “step“ function of an <u>abstract machine</u>!

A step back

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- unload^+ is the “step“ function of an abstract machine!
- Configurations: $\mathsf{Config} = \mathbb{N} \times \mathsf{Stack}$
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- No infinitely long **decreasing chains**
  $\Longrightarrow$ recursion eventually terminates.
- In *Agda* we encode such property as an accessibility predicate.
- A relation is **well-founded** if every element is accessible.

Another tail-recursive evaluator §3

Iterate unload$^+$ until a value is produced.

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```
tail-rec-eval : Expr → ℕ
tail-rec-eval e with load e Top
  ... | inj₁ c = rec (<-Well-founded c) c
  where
    rec : (c : Config) → Acc _<_ c → ℕ
    rec c (acc rs) with unload⁺ c
    ... | inj₁ c' = rec (rs ...) c'
    ... | inj₂ r  = r
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- `_<_ : Config → Config → Set`

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```

Still, we have to:

- $\_{<}\_ : \text{Config} \to \text{Config} \to \text{Set}$
- $\forall\, (c\ c' : \text{Config}) \to \text{unload}^{+}\ c \equiv \text{inj}_1\ c' \to c' < c$

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```

Still, we have to:

- $\_<\_ \; : \; \mathsf{Config} \to \mathsf{Config} \to \mathsf{Set}$
- $\forall \, (c \; c' \; : \; \mathsf{Config}) \to \mathsf{unload}^{+} \; c \equiv \mathsf{inj}_1 \; c' \to c' < c$
- $\mathsf{Well\text{-}founded} \; \_<\_$

The relation

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\begin{aligned}
&\text{data } \_<\_ : \text{Config} \to \text{Config} \to \text{Set where} \\
&\quad \text{<-StepR} : (t_1 , s_1) < (t_2 , s_2) \\
&\qquad\qquad \to (t_1 , \text{Right } l\ n\ eq\ s_1) < (t_2 , \text{Right } l\ n\ eq\ s_2) \\
&\quad \text{<-StepL} : (t_1 , s_1) < (t_2 , s_2) \\
&\qquad\qquad \to (t_1 , \text{Left } r\ s_1) < (t_2 , \text{Left } r\ s_2) \\
&\quad \text{<-Base} : (t_1 , \text{Right } n\ e_1\ eq\ s_1) < (t_2 , \text{Left } e_2\ s_2)
\end{aligned}
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The details

- Two distinct values of Config **can** represent folds over different Expressions.

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$$\begin{array}{l} \text{data Config}_{\Downarrow}\ (e : \text{Expr}) : \text{Set where} \\ \quad \_,\_ : (c : \text{Config}) \rightarrow \text{plugC}_{\Downarrow}\ c \equiv e \rightarrow \text{Config}_{\Downarrow}\ e \end{array}$$

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- It is needed to show the relation is **well-founded**.

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- Proof that unload$^+$ delivers a smaller result: tedious but easy.
- Proof of correctness: follows from well-founded recursion.

Motivation
Contributions
Solving the problem
Generalization
Conclusion

Generalization

- Universe of **regular** datatypes of which Expr is an instance.

Generalization

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Generalization

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- McBride's dissection to calculate generic Stacks.
- Generalize load^G and unload^G.
- Generic well-founded relation.
- unload^G decreases.
- Assemble everything together.

regular universe

$$
\begin{array}{l}
\text{data Reg : Set}_1\ \text{where} \\
\quad \mathbb{0} \quad : \text{Reg} \\
\quad \mathbb{1} \quad : \text{Reg} \\
\quad \mathsf{I} \quad : \text{Reg} \\
\quad \mathsf{K} \quad : (A : \text{Set}) \to \text{Reg} \\
\quad \_\oplus\_ : (R\ Q : \text{Reg}) \to \text{Reg} \\
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$$
\begin{array}{lll}
[\![\_]\!] : \mathsf{Reg} \to (\mathsf{Set} \to \mathsf{Set}) & & \\
[\![\ \mathbb{0}\ ]\!]\ X & = & \bot \\
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$$
\begin{array}{l}
\mathsf{cata} : (R : \mathsf{Reg}) \to ([\![\, R \,]\!]\ X \to X) \to \mu\ R \to X \\
\mathsf{cata}\ R\ \psi\ (\mathsf{In}\ r) = \psi\ (\mathsf{fmap}\ R\ (\mathsf{cata}\ R\ \psi)\ r)
\end{array}
$$

An example

- Original type:

$$
\begin{array}{l}
\text{data } \mathsf{Expr} \;:\; \mathsf{Set} \text{ where} \\
\quad \mathsf{Val} \;:\; \mathbb{N} \to \mathsf{Expr} \\
\quad \mathsf{Add} \;:\; \mathsf{Expr} \to \mathsf{Expr} \to \mathsf{Expr}
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- Generic representation:

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- Isomorphic:

$$\mathsf{Expr} \simeq \mu\ \mathsf{exprF}$$

- "One hole" context of a functor.

$$
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&\nabla : (R : \mathsf{Reg}) \to (\mathsf{Set} \to \mathsf{Set} \to \mathsf{Set})\\
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Dissections

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- An example:

$$\mathsf{Stack} \simeq \mathsf{List}\ (\nabla\ \mathsf{exprF}\ \mathbb{N}\ (\mu\ \mathsf{exprF}))$$

Our results

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A generic tail-recursive evaluator

$$\text{tail-rec-cata} : (R : \mathsf{Reg}) \to (\psi : [\![R]\!]\, X \to X) \to \mu\, R \to X$$

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$$\mathsf{tail\text{-}rec\text{-}cata} : (R : \mathsf{Reg}) \to (\psi : [\![R]\!]\, X \to X) \to \mu\, R \to X$$

and its correctness proof.

$$\begin{aligned}\mathsf{correctness}^{G} &: \forall\, (R : \mathsf{Reg})\, (\psi : [\![R]\!]\, X \to X)\, (t : \mu\, R) \\ &\to \mathsf{cata}\; R\; \psi\; t \equiv \mathsf{tail\text{-}rec\text{-}cata}\; R\; \psi\; t\end{aligned}$$

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For more details read the paper
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Thank you very much for your attention!