Semantic Analysis of Normalization by Evaluation for Fitch-Style Modal Lambda Calculi

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Fitch-style modal lambda calculi (Borghuis 1994; Clouston 2018) provide a solution to programming necessity modalities (denoted by a $\Box$) in a typed lambda calculus by extending the typing context with a delimiting operator (denoted by a 🔒). In this work, we perform a semantic analysis of *normalization by evaluation* (NbE) (Berger and Schwichtenberg 1991) for Fitch-style modal lambda calculi by beginning with the calculus λ_{IK}—a system for the most basic modal logic IK (for "intuitionistic" and "Kripke")—as our object of study. We construct an NbE model for λ_{IK}, and show that it is an instance of the possible-worlds semantics for IK. The presented NbE procedure has been formalized (Valliappan 2020–2021) in the proof assistant Agda (Abel et al. 2005–2021).

The Fitch-style modal lambda calculus under consideration. IK extends intuitionistic propositional logic with the *necessity modality* $\Box$, the *necessitation rule* (if $\cdot \vdash A$ then $\Gamma \vdash \Box A$) and the *K axiom* ($\Gamma \vdash \Box(A \to B) \to \Box A \to \Box B$). Correspondingly, λ_{IK} extends the simply-typed lambda calculus (STLC) with the typing rules in Figure 1. The rules for λ-abstraction and function application are formulated in the usual way—but note the modified variable rule!

$$\text{Ty} \quad A ::= \ldots \mid \Box A \qquad\qquad \text{Ctx} \quad \Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \text{🔒}$$

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A}\ \text{🔒} \notin \Gamma' \qquad \frac{\Gamma, \text{🔒} \vdash t : A}{\Gamma \vdash \mathrm{box}\ t : \Box A} \qquad \frac{\Gamma \vdash t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash \mathrm{unbox}\ t : A}\ \text{🔒} \notin \Gamma'$$

Figure 1: Typing rules for λ_{IK} (omitting λ-abstraction and application)

The NbE model for $\boldsymbol{\lambda_{\mathrm{IK}}}$. NbE is the process of *evaluating*, or interpreting, terms of a calculus in a suitable model and then *reifying*, or extracting, normal forms from values in that model. NbE for STLC can be performed by interpreting types and contexts as *covariant* presheaves over the category W of contexts Γ, Δ and order-preserving embeddings (OPEs) $e : \Gamma \leq \Delta$, and terms as natural transformations (Altenkirch, Hofmann, and Streicher 1995). Given that the category of presheaves $\widehat{W}$ is a cartesian closed category (CCC), the evaluation function $(\!|\_|\!) : \Gamma \vdash A \to [\![\Gamma]\!] \dot{\to} [\![A]\!]$ is given by the standard interpretation of STLC in a CCC. The reification function, on the other hand, is given by a family of natural transformations $\downarrow_A : [\![A]\!] \dot{\to} \mathrm{Nf}\ A$, where the presheaf $\mathrm{Nf}\ A$ denotes normal forms of type A.

To achieve NbE for λ_{IK}, we define a new category $W_{\text{🔒}}$ akin to W by requiring that morphisms additionally preserve locks and refer to the resulting notion of context embedding as *OLPE*. Note that whenever there is an OLPE $e : \Gamma \leq \Delta$ then Δ has the exact same number of locks as Γ. Further, we extend the interpretation of types and contexts to the type former $\Box$ and the context operator 🔒. Clouston (2018) shows that λ_{IK} can be soundly interpreted in a CCC equipped with an adjunction $\mathrm{Lock} \dashv \mathrm{Box}$ of endofunctors by interpreting $\Box$ by the right adjoint Box and 🔒 by the left adjoint Lock. Following this soundness result, we can use the CCC $\widehat{W_{\blacksquare}}$ as our new NbE model, after equipping it with an adjunction. By virtue of our definition of this adjunction (given in Figure 2), the evaluation of box and unbox is given by the *generic* interpreter of Clouston (2018), and we can construct natural transformations $\downarrow_{\Box A} : \mathrm{Box}\ [\![A]\!] \dot{\to} \mathrm{Nf}\ \Box A$, for every type A—thus retaining reification.

We summarize the data part of the NbE model for the modal fragment of λ_{IK} in Figure 2 as definitions in a constructive type-theoretic metalanguage. A presheaf $\mathcal{A}$ over $W_{\blacksquare}$ consists of a family of sets $\mathcal{A}_\Gamma$ indexed by contexts Γ, and a family of functions $\mathrm{wk}_e : \mathcal{A}_\Gamma \to \mathcal{A}_{\Gamma'}$ indexed by OLPEs $e : \Gamma \leq \Gamma'$. The *reflection* function $\uparrow_A$ defines a natural transformation from the presheaf of neutral terms $\mathrm{Ne}\, A$, and can be used to construct an element $\uparrow_\Gamma(\overline{\Gamma}) : [\![\Gamma]\!]_\Gamma$ from $\overline{\Gamma} : \mathrm{Ne}_\Gamma\, \Gamma$. Normalization for a term $\Gamma \vdash t : A$ is then given by $\downarrow_A((\!|\mathrm{t}|\!)(\uparrow_\Gamma(\overline{\Gamma})))$.

$$\frac{x : \mathcal{A}_{\Gamma,\blacksquare}}{\mathbf{box}\ x : \mathrm{Box}_\Gamma\, \mathcal{A}} \qquad\qquad \frac{x : \mathcal{A}_\Gamma}{\mathbf{lock}\ x : \mathrm{Lock}_{\Gamma,\blacksquare,\Gamma'}\, \mathcal{A}}\ \blacksquare \notin \Gamma'$$

$$\begin{array}{lll}
[\![_]\!] : \mathrm{Ty} \to \widehat{W_{\blacksquare}} & (\!|_|\!) : \Gamma \vdash A \to [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta & \downarrow_A : [\![A]\!]_\Gamma \to \mathrm{Nf}_\Gamma\, A \\
[\![\Box A]\!]_\Gamma = \mathrm{Box}_\Gamma\ [\![A]\!] & (\!|\mathrm{box}\, \mathrm{t}|\!)\ \gamma \quad = \mathbf{box}\ ((\!|\mathrm{t}|\!)\, \gamma) & \downarrow_{\Box A}\ (\mathbf{box}\ x) = \mathrm{box}\ (\downarrow_A x) \\
[\![_]\!] : \mathrm{Ctx} \to \widehat{W_{\blacksquare}} & (\!|\mathrm{unbox}\, \mathrm{t}|\!)\ \langle \gamma, _ \rangle \quad = (\!|\mathrm{unbox}\, \mathrm{t}|\!)\, \gamma & \\
[\![\Delta, \blacksquare]\!]_\Gamma = \mathrm{Lock}_\Gamma\ [\![\Delta]\!] & (\!|\mathrm{unbox}\, \mathrm{t}|\!)\ (\mathbf{lock}\ \gamma) = \mathrm{wk}\, x & \uparrow_A : \mathrm{Ne}_\Gamma\, A \to [\![A]\!]_\Gamma \\
 & \quad \text{where } \mathbf{box}\ x = (\!|\mathrm{t}|\!)\, \gamma & \uparrow_{\Box A}\ n = \mathbf{box}\ (\uparrow_A(\mathrm{unbox}\, n))
\end{array}$$

Figure 2: NbE for the modal fragment of λ_{IK}

Connection with possible-worlds semantics. Analogously to how the NbE model for STLC can be seen as an instance of the Kripke semantics of IPL, the NbE model we present here can be seen as an instance of the possible-worlds semantics of IK. Hence, the observation that the NbE model construction for STLC corresponds to the completeness proof for Kripke semantics (C. Coquand 1993; T. Coquand and Dybjer 1997) carries over to the setting here.

The possible-worlds semantics for IK is parameterized by a *frame*, i.e. a type W together with two binary relations $\leq$ and R on W which are required to satisfy certain conditions (Božić and Došen 1984; Došen 1985; Simpson 1994): 1. $\leq$ is reflexive, 2. $\leq$ is transitive, 3. if $w \leq w'$ and $w'\ R\ v'$ then there exists $v : W$ such that $w\ R\ v$ and $v \leq v'$, and 4. if $w\ R\ v$ and $v \leq v'$ then there exists $w' : W$ such that $w \leq w'$ and $w'\ R\ v'$. An element $w : W$ can be thought of as a representation of the "knowledge state" about some "possible world" at a certain point in time; $w \leq w'$ as representing an increase in knowledge; and $w\ R\ v$ as specifying accessibility of worlds from one another.

Given a frame $(W, \leq, R)$, the possible-worlds semantics interprets a formula A at $w : W$ as the presheaf $\mathcal{A}(w)$ over $(W, \leq)$. The interpretation of $\Box A$ at w is the type of functions p assigning an element $p(v) : \mathcal{A}(v)$ to every $v : W$ such that $w\ R\ v$. Note that, by virtue of the frame conditions, the interpretation of $\Box$ extends to a functor Box on the category of presheaves and that Box has a left adjoint Lock. Hence, the possible-worlds semantics fits into the semantic framework of Clouston (2018). The left adjoint Lock can be described directly as mapping $\mathcal{A}$ and $w : W$ to the type of pairs $\langle v, a \rangle$ where $v : W$ such that $v\ R\ w$ and $a : \mathcal{A}(v)$

Now, we observe that the NbE model for λ_{IK} can be seen as the possible-worlds model where we pick Fitch-style contexts for W, OLPEs for $\leq$, extensions by a $\blacksquare$ for R, i.e. $\Gamma\ R\ \Delta$ if and only if there exists Γ' such that $\blacksquare \notin \Gamma'$ and $\Delta = \Gamma, \blacksquare, \Gamma'$ (cf. Figures 1 and 2), and normal forms as the interpretation of base types. Note that the required frame conditions are satisfied.

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