Quantifying Views - Modified Black-Litterman Model from Probablistic Perspective

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Abstract

Ever since the birth of Modern Portfolio Theory, building the appropriate model to estimate the expected return has become an everlasting topic. The Black-Litterman Model is one of the most practical and widely discussed models for expected return estimation. Useful as this model is, it requires the portfolio manager to subjectively choose the key statistic – views on the relative change of stock prices. However, different managers may have completely distinct or even reverse views, leading to different portfolio management strategies. To deal with the subjectiveness issue of the traditional Black-Litterman Model, this paper endeavors to create an objective measure of views from the probabilistic perspective, under the stock pool which has the mean-reversion property suggested by the literature. Generally speaking, the higher the relative deviation from the mean level of such stock returns, the more likely its return will go back to the mean level. Therefore, we construct views using the stocks whose returns are most positively deviated and negatively deviated from the mean level. After constructing the view, this paper uses the traditional Black-Litterman model to estimate the expected return and construct a portfolio strategy using optimization. It turns out that the portfolio strategy constructed in this way is sparse, captures more volatility, and is robust to certain hyper-parameter changes. This paper provides intuition on the objectiveness measure of views and the improvement of the Black-Litterman Model. It also provides insights into sparse portfolio management literature.

Keywords: Black-Litterman model, portfolio management, views, mean-reversion stocks, sparse allocation

1 Intoduction

1.1 Background and Motivation

In portfolio management, modeling the relationship between the expected return and risk has been a classical issue. Markowitz proposed the renowned mean-variance portfolio (1952, p77-91), which is the first model indicating the effects of diversification. However, the assumptions of Markowitz’s model are too strong thus its values are limited to the literature. To apply the model to reality, the most famous modification of the Markowitz model is the Black-Litterman model, which integrates the active managers’ views into portfolio management (1992, p28-43). This model turns out to be more realistic and useful than Markowitz’s theoretical model. However, in Black-Litterman’s model, the key component “view” is entirely determined by managers’ subjective perceptions. Different managers may have distinct views about the relative changes in the stock returns, leading to different estimations of the expected return of the portfolio. The subjectiveness of views makes asset pricing hard. To deal with such issues, this paper tries to introduce objective measures of the views using probabilistic intuition and utilizing the mean-reversion characteristics of long-term stock return suggested by James and Lawrence (1988, p28).

1.2 Literature Review

Since the born of Black-Litterman Model, financial analysts have modified the model from various perspectives. For instance, Dimitris, Vishal, and Ioannis improved the model from an optimization perspective. (2012, p1390). Inspired by CAPM, they utilize inverse optimizations to model the returns. Andrzej and Jan (2018) tried to improve the model by assuming the stable fat-tail distribution of the returns instead of the normality distribution used in the original model (p708-720). Rockafellar, Uryasev, and Zabarankin (2006) even propose a distribution-free model to eliminate the effects of the posterior distribution-selection error (p51). The abovementioned literature indeed increases the accuracy and the universality of the Black-Litterman model. However, they still use the managers’ subjective options as the views, leading to potential deviations from reality.

In fact, several analysts have realized this issue and tried to measure the views in qualitative ways. For instance, Mankert and Seiler (2006) used behavioral finance to predict the behaviors of managers in different market segmentations (p99-122). Nevertheless, he still assumed the magnitude of the views (the change in returns) to be given and only focused on selecting the stock pairs whose returns are expected to be changed.

To deal with the abovementioned issues regarding the lack of objective view measure, this paper endeavors to quantify the views in the Black-Litterman Model from a probabilistic perspective. The core assumption behind this intuition is that some stock returns have the mean-reversion behavior in the long run. Based on this, the stock with the "most unlikely" current returns today has the highest potential to go back to the average level. This result can be used for view measure construction. It is worth mentioning that the mean-reversion behavior is not generally true for every stock. However, according to James and Lawrence's famous paper in 1988, under certain circumstances, and over long run, the mean-reversion behavior can be distinctly observable. On top of that, Sandrip (2011, p22-27),Spierdijk and Nikker (2017, p119-139), Daniel(2012, p1-6) all found empirical evidence of stock's mean-reversion behavior. The literature has provided solid support for the use of the mean-reversion assumption.

In summary, current literature lacks the quantitative measure of views, which is catastrophic for portfolio management since different managers may have distinct views on the return changes thus, they may have completely different stock selections. To address issues from the literature, this paper will make the following contributions to modify the Black-Litterman model:

1. Verifying the mean-reversion behaviors for the selected stocks from the Chinese Stock market.

2. Determining the stocks whose relative returns are expected to change based on probability and constructing the view measure using historical data.

2 Model Construction and Analysis

2.1 The Black-Litterman Model

The classical B-L model takes the Bayesian probability structure that takes the expected return and the views as prior distributions and the realized return as the posterior distribution. Basically, it has the following assumptions.

Assumption 1. *The prior distribution is normality, say*

$$\mu \sim N(\Pi, \tau\Sigma) \tag{1}$$

where μ is the prior expected return, Π is the implied return, Σ is the covariance matrix of returns and τ is some constant.

Inherited from Modern Portfolio Theory, the implied price is given by:

$$\Pi = r_f + \beta(E[R_m] - r_f) \tag{2}$$

As for views, in original B-L model, they are basically the opinions from the managers about the relative changes between stock returns. For instance, to construct a portfolio from

5 stocks, a manager predicts the first stock will outperform the second stock 5% and third stock will outperform the fourth stock 6% in the future, then we write the view vector

$$Q = \begin{bmatrix} 5\% \\ 6\% \end{bmatrix}$$

and we write the link matrix:

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \tag{3}$$

each element in Q is the magnitude of relative change and each row in P indicates the stocks to long or short based on views.

Assumption 2. *The views are random and follow a normal distribution with mean vector Q, say*

$$\tilde{Q} \sim N(Q, \Omega) \tag{4}$$

where Ω measures the uncertainty of the managers' view.

Based on these two assumptions, according to Allaj (2013, p220), the expected return of the stocks in the pool is given by the vector

$$\mu = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q) \tag{5}$$

Assumption 3. *The posterior distribution of the returns are given by*

$$r \sim N(\mu_p, \Sigma) \tag{6}$$

Based on this assumption , assume the weights vector of stocks is w, we have the optimization problem:

$$\begin{aligned} &\min_w \quad \frac{1}{2} w^T \Sigma w \\ s.t. \quad &\sum_{i=1}^{n} w_i = 1, \qquad \mu_p := \mu * w = z \end{aligned} \tag{7}$$

where z is a constant given beforehand. Solving the optimization problem, we can have optimal w as desired.

2.2 Mean-Reversion Behavior

To quantify the views, we need an important assumption:

Assumption 4. The return of the stock has mean-reversion behavior. That is, over the long-run, the returns will go back to the mean level.

This assumption is crucial since it guarantees that the stock with an absurdly high deviation from the mean level will have a high potential to go back to its normal stage. Based on this assumption, we can construct standardized measures for the deviations of different stocks (Refer to section 2.3), then choose the one with the largest intrinsic to increase and the one with the smallest intrinsic to increase as our long and short portfolio, respectively.

This assumption has abundant theoretical support. The most famous empirical research is the distinguishable paper by James and Lawrence, which reveals that the majority of the stocks have gradually smaller variants to mean (1988, p53). Subsequently Sandrip (2011, p100), Spierdijk and Bikker(2017, p119-139), Daniel(2012,p217-251) all found similar patterns in stock returns. Solid as evidence is, nearly all of them claimed that there are still minor exceptions in the stock market that are not stationary, challenging the validity of our assumption.

Based on this, it is exceedingly crucial to select stocks with the mean-reversion property. Since the literature suggests the long-run mean-reversion is more evident, we will select stocks which were issued at least 15 years from now. To ensure the validity of this assumption, we will also plot and examine their returns' time series.

2.3 Modelling the Views from a Probabilistic Perspective

Assumption 3 inserts that all the returns are given by a normal distribution. Assume that

$$r \sim N(\mu_r, \sigma_r^2) \tag{8}$$

Then, we shall have

$$\frac{r - \mu_r}{\sigma_r} \sim N(0,1) \tag{9}$$

Hence we have

$$\frac{(r - \mu_r)^2}{\sigma_r^2} \sim \chi(1) \tag{10}$$

It is clear that,

$$G(y) = P(r \geq x) = P(\chi(1) \geq y), \qquad where \qquad y = \frac{(x - \mu_r)^2}{\sigma_r^2} \tag{11}$$

is an strictly decreasing function with respect to y. Therefore, given current return x_r, the higher $y = \frac{(x_r - \mu_r)^2}{\sigma_r^2}$ is, the less likely that we can find another return r as extreme as x_r. In other words, It is hard for the stock with high $\frac{(x_r-\mu_r)^2}{\sigma_r^2}$ to maintain at current level, hence it is very likely for the stock to change.

By assumption 4, the stocks have mean-reversion trends. For a stock to change its current return. the most likely direction is to go back to the normal level. In other words, we have

$r > \mu_r$, the stock return will shift downwards.

$r < \mu_r$, the stock return will shift upwards.

On top of that, since the returns are expected to go back to the mean level, hence the higher deviations are, the larger the changes are expected. We can measure the scale of change by how many standard deviations from its normal level.

Combining the abovementioned arguments, we construct $\frac{r-\mu_r}{\sigma_r}$ as our statistics for determining the direction of views. The higher the statistic is, the more likely it will decrease, and its decreasing scale will be larger. Similarly, the lower the statistic is (even negative is possible), the less likely it will decrease (hence more likely it will increase). Therefore, we expect the stock with the smallest $\frac{r-\mu_r}{\sigma_r}$ to increase and the stock with the largest $\frac{r-\mu_r}{\sigma_r}$ to decrease.

We formulate this process as an algorithm. Suppose there are N views. We can repeat selection procedure for N times , each time selecting the one with the largest and smallest $\frac{r-\mu_r}{\sigma_r}$ to decrease and increase. The selection is without replacement. Assume for the kth selection, we have

$$i_k = \arg\min_{1\leq i\leq n} \frac{r-\mu_r}{\sigma_r} \qquad j_k = \arg\max_{1\leq i\leq n} \frac{r-\mu_r}{\sigma_r} \tag{12}$$

Then, the link matrix P's k-th row m-th column element is given by

$$(P)_{km} = \begin{cases} 1 & if \quad m = i_k \\ -1 & if \quad m = j_k \\ 1 & otherwise \end{cases} \tag{13}$$

As for the magnitude of the view measure, we go back to its original definition. The view is given by the relative percentage change in returns between two stocks. Therefore, under the mean-reverting assumption, we assume that the return will go back to the mean in the next period, then, for the stock whose return will increase more, we calculate the percentage change

$$q_1 = \frac{\mu_1 - r_1}{r_1} \tag{14}$$

similarly, for the stock which return will increase less, we calculate the percentage change

$$q_2 = \frac{\mu_2 - r_2}{r_2} \tag{15}$$

Then the view measure is given by

$$q = q_1 - q_2 \tag{16}$$

Hence We construct the N-dimensional vector Q with elements q(N is the number of views).

2.4 Estimation of key statistics in the model

2.4.1 Estimation of mean, variance, and covariance matrix of return

The mean and variance are estimated by their unbiased estimator:

$$\hat{E}(\mu) = \frac{1}{n}\sum_{i=1}^{n} r_i \tag{17}$$

$$\hat{V}ar(r) = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{E}(\mu))^2 \tag{18}$$

This estimation applies to all the cases. The covariance matrix is estimated by

$$\hat{\Sigma} = \frac{1}{n}\sum_{j=1}^{n}(R_j - \hat{E}(\mu_j))(R_M - \hat{}E(\mu_M))^T \tag{19}$$

2.4.2 Estimation of implied return

Firstly, we shall estimate the $beta$ from the single factor model

$$r_i - rf = \beta(r_M - rf) + \epsilon_i \tag{20}$$

Regression gives the analytic solution of $beta$:

$$\beta_i = \frac{Cov(r_i - rf, r_M - rf)}{Var(r_M - rf)} \tag{21}$$

We estimate the β by estimate the covariance and variance with their unbiased estimator, say

$$C\hat{o}v(r_i - rf, r_M - rf) = \frac{1}{n-1}\sum_{j=1}^{n}(r_j - \hat{E}(\mu_j))(r_M - \hat{}E(\mu_M)) \tag{22}$$

estimation for variance is given by (18). Hence we have

$$\hat{\beta}_i = \frac{C\hat{o}v(r_i - rf, r_M - rf)}{V\hat{a}r(r_M - rf)} \tag{23}$$

The estimated implied return is given by

$$\hat{\Pi} = r_f + \hat{\beta}(\hat{E}[R_m] - r_f) \tag{24}$$

2.4.3 Estimation of view matrix

Use the estimated expectation and variance, we can estimate the value

$$\frac{r_i - \hat{E}(\mu_i)}{\sqrt{\hat{V}ar(r_i)}} := \frac{r_i - \hat{E}(\mu_i)}{\hat{\sigma}_i} \tag{25}$$

Then, follow the algorithm developed in section 2.3, by (12), (13), given N views we can obtain the estimation of an $N \times n$ link matrix $\hat{P}$ and view vector $\hat{Q}$.

According to Black and Litterman (1992, p78), the estimation for the Ω matrix is appropriately given by

$$\hat{\Omega} = diag(\tau \hat{P}^T \hat{\Sigma} \hat{P}) \tag{26}$$

Hence we have the estimation of the expected return of single stocks:

$$\hat{\mu} = ((\tau\hat{\Sigma})^{-1} + \hat{P}^T\hat{\Omega}^{-1}\hat{P})^{-1}((\tau\hat{\Sigma})^{-1}\hat{\Pi} + \hat{P}^T\hat{\Omega}^{-1}\hat{Q}) \tag{27}$$

The expected mean and variance of the portfolio given allocation weight w is given by:

$$\hat{\mu}_p = w^T \hat{\mu} \tag{28}$$

$$\hat{\sigma}_p^2 = w^T \hat{\Sigma} w \tag{29}$$

3 Data Source and Selection

3.1 Overview of Data and Variables

The data is obtained from China Stock Market & Accounting Research (CSMAR) Database (2022). In this research, we collect the monthly data, including the stock's code, stock's return, market return, risk free rate from 2012-01 to 2021-12. All the returns and rates are expressed on a monthly basis.

To have a relatively stable distribution, all the individual stocks are selected from the main board of the Shanghai and Shenzhen Stock Exchange. To represent the market rate, we choose CSI300 (or Hu Shen 300 in Chinese) as a proxy, since it contains the stocks with the highest market capitalization and liquidity, hence their risks are more close to the risk solely caused by volatility. The risk free rate is chosen as overnight Shibor rate (Shanghai Interbank Offered Rate) by convention.

The below table summarizes the details of the variables used in the dataset:

Table 1: Description of variables

Variable	Description
$r_{j,t}$	The j-th stock's return on month t, reported on a monthly basis
$r_{m,t}$	CSI300 Index on month t, reported on a monthly basis
$r_{f,t}$	Overnight Shibor rateon month t, reported on a monthly basis

We will summarize these data in section 3.4

3.2 Data Pre-Processing

Firstly, to simplify the analysis, we require the panel data to be balanced. In other words, for every stock and every month, there are available return data. It requires the selected stocks cannot delist during the observation period. Hence, we abandon all the stocks which lack certain data, ridding them of the portfolio selection pool.

Secondly, the Shibor rate is expressed on a yearly basis. To transform the data into the monthly basis, consider that

$$(1 + r_{f,monthly})^{12} \approx 1 + 12r_{f,monthly} = 1 + r_{f,yearly} \tag{30}$$

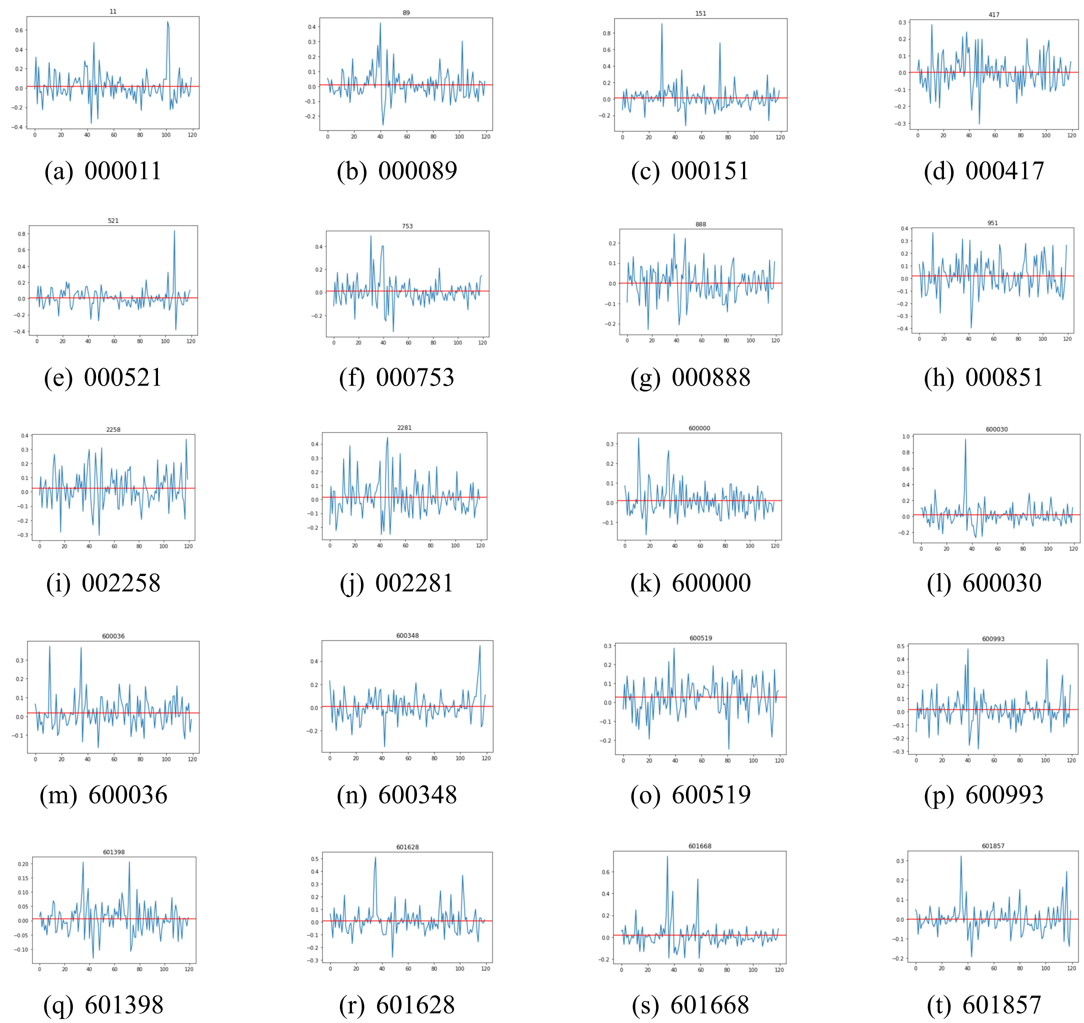
Hence, the monthly rate is approximately $\frac{1}{12}$ of the yearly Shibor Rate, say

$$r_{f,monthly} = \frac{1}{12} r_{f,yearly} \tag{31}$$

3.3 Stock Selection and mean-reversion Validation

Recall that in section 2.2, we discussed the importance and existence of mean-reversion stocks. Literature suggests we could try stocks issued for a long time since they may have a stable mean over the period. Based on this criterion, we randomly select 20 stocks from all the A shares from the main board with issurance date dating back to 15 years ago at least, and select those who have explicitly stable mean by observing their return-time relationship plots. Please refer to the next page for the plots (Figure 1).

Based on the plots, clearly, some stocks have mean-shifting behavior. For instance, the stock 600000 has a downward shifting mean. Hence we avoid such stocks and select the ten stocks with the most stable mean and mean-reversion patterns. Their stock id are: 000089,000151,000417,000521,000753,600030,600036,600348,600519,601398. Those stocks form our needed portfolio selection pool for empirical design.



(a) 000011 (b) 000089 (c) 000151 (d) 000417

(e) 000521 (f) 000753 (g) 000888 (h) 000851

(i) 002258 (j) 002281 (k) 600000 (l) 600030

(m) 600036 (n) 600348 (o) 600519 (p) 600993

(q) 601398 (r) 601628 (s) 601668 (t) 601857

Figure 1: The relationship between return and time

3.4 Summary of Basic Statistics

Here, we give the following summary of the data:

StockCode	mean	var	std	max	min	count
89	0.009579	0.008804	0.093831	0.423804	-0.260393	120
151	0.010147	0.021828	0.147744	0.911227	-0.326923	120
417	0.000167	0.009876	0.099379	0.284926	-0.306496	120
521	0.008141	0.015846	0.125880	0.836364	-0.387789	120
753	0.009068	0.013795	0.117451	0.490291	-0.342679	120
600030	0.018046	0.018918	0.137544	0.964079	-0.266022	120
600036	0.018250	0.006700	0.081854	0.372256	-0.166759	120
600348	0.006159	0.012888	0.113526	0.534787	-0.341463	120
600519	0.026823	0.007719	0.087855	0.287150	-0.248082	120
601398	0.006394	0.002753	0.052468	0.206452	-0.132505	120

Figure 2: Summary of Individual Stocks Returns

	count	mean	std	min	25%	50%	75%	max
Rm	120.0	0.008297	0.064989	-0.210376	-0.022448	0.004908	0.041925	0.258075
Rf	120.0	0.001676	0.000593	0.001241	0.001241	0.001241	0.002466	0.002871

Figure 3: Summary of market return and risk free rate

4 Empirical Design

4.1 Obtain the Expected Return

In this research, we set the hyper parameter $\tau = 0.025$ and the number of views $N = 3$. We will do sensitivity analysi on the hyper-parameters in section 4.5

We use the data from 2012-01 to 2021-11 to do estimations as section 2.4 suggests, and use the data on 2021-12 as the new return to determine the portfolio estimation. Basically, we present the results:

4.1.1 Estimation of Implied Returns

The estimated values of betas are given by:

	89	151	417	521	753	600030	600036	600348	600519	601398
0	0.930648	0.836959	1.155758	0.937641	0.956118	1.756295	0.9354	0.808322	0.739861	0.491098

Figure 4: Estimation of $\hat{\beta}$

Based on Figure (3), $\hat{E}[r_m] = 0.008297$. On 2021.12, we have $r_f = 0.0012$. We can derive the implied return on 2021.12:

	89	151	417	521	753	600030	600036	600348	600519	601398
0	0.007808	0.007147	0.009396	0.007857	0.007988	0.013634	0.007841	0.006945	0.006462	0.004706

Figure 5: Estimation of implied returns $\hat{\Pi}$

4.1.2 Estimation of $\hat{\Sigma}$

	89	151	417	521	753	600030	600036	600348	600519	601398
89	0.008804	0.004360	0.005701	0.004433	0.006562	0.005671	0.002711	0.003180	0.002791	0.001626
151	0.004360	0.021828	0.005128	0.005309	0.009297	0.006331	0.003020	0.004310	0.002313	0.002232
417	0.005701	0.005128	0.009876	0.005624	0.007092	0.008297	0.003859	0.005095	0.002920	0.002072
521	0.004433	0.005309	0.005624	0.015846	0.006641	0.005428	0.001628	0.004614	0.003387	0.000418
753	0.006562	0.009297	0.007092	0.006641	0.013795	0.003704	0.002290	0.004800	0.003032	0.001302
600030	0.005671	0.006331	0.008297	0.005428	0.003704	0.018918	0.007693	0.006737	0.004276	0.004230
600036	0.002711	0.003020	0.003859	0.001628	0.002290	0.007693	0.006700	0.002757	0.002640	0.003004
600348	0.003180	0.004310	0.005095	0.004614	0.004800	0.006737	0.002757	0.012888	0.001150	0.001709
600519	0.002791	0.002313	0.002920	0.003387	0.003032	0.004276	0.002640	0.001150	0.007719	0.001626
601398	0.001626	0.002232	0.002072	0.000418	0.001302	0.004230	0.003004	0.001709	0.001626	0.002753

Figure 6: Estimation of $\hat{\Sigma}$

4.1.3 Estimation of View

Followed by (25)∼(29) in section 2.4, we have

	89	151	417	521	753	600030	600036	600348	600519	601398
view1	0	0	1	0	0	0	0	0	-1	0
view2	0	0	0	0	0	0	-1	1	0	0
view3	0	0	0	1	0	0	0	0	0	-1

Figure 7: Estimation of link matrix $\hat{P}$

	view1	view2	view3
0	0.188192	0.653074	0.151776

Figure 8: Estimation of view vector $\hat{Q}$

	view1	view2	view3
view1	0.000294	0.000000	0.000000
view2	0.000000	0.000352	0.000000
view3	0.000000	0.000000	0.000444

Figure 9: Estimation of uncertainty $\hat{\Omega}$

4.1.4 Estimation of Expected Returns

Plug all the above data into (27), we have the expected returns:

89	151	417	521	753	600030	600036	600348	600519	601398
0.036815	0.054516	0.077522	0.105367	0.092035	0.01432	-0.076415	0.259304	-0.049847	-0.025007

Figure 10: Estimation of Expected Realized Returns

4.2 Validation of Assumptions

4.2.1 Validation of Mean-Reversion Characteristics

The validation of the mean reversion of the stock returns is guaranteed by the selection process. This part has been finished in section 3.3. Please refer to the details in section 3.3 and Figure 1.

4.2.2 Validation of Normality Assumption

In section 2.3, we require that $\frac{r-\mu_r}{\sigma_r}$ follows the standard normal distribution. Here, we validate that our data indeed have such property in reality. We pool all the stock returns, standardize them by $\frac{r-\mu_r}{\sigma_r}$, and plot the density histogram. The plot is given by:

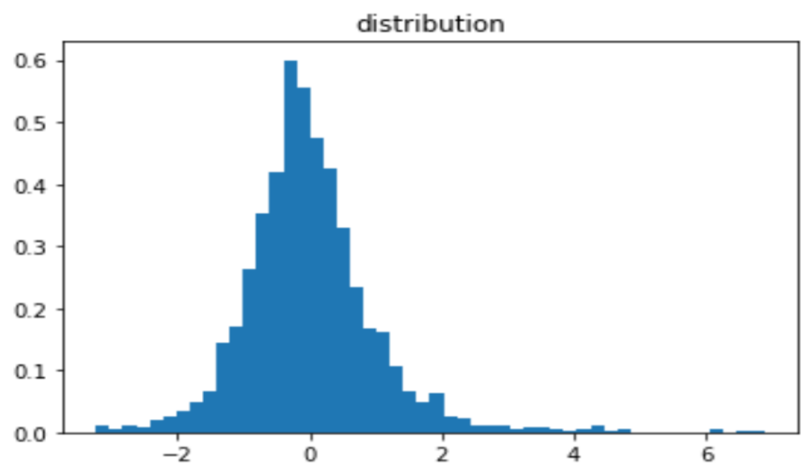


Figure 11: histogram of standardized returns

From the histogram, indeed we can find out a normality pattern of the pooled data. Furthermore, we can compute the mean and standard deviation of the standardized data, the results are:

$$\hat{E}[\frac{r-\mu_r}{\sigma_r}] = 0, \qquad \hat{\sigma}[\frac{r-\mu_r}{\sigma_r}] = 0.9958246164193104 \approx 1, \tag{32}$$

The approximately normal shape, zero mean and extremely close to 1 standard deviation provide solid evidence for the validity of Assumption 3.

4.3 Optimization for Asset Allocation

Assume the allocation weight is given by $w = (w_1, w_2, \cdots, w_n)$. Under CAPM settings, accoring to Fama (1970,p388), the portfolio's beta is additive, say

$$\beta_p = \sum_{i=1}^{n} w_i \beta_i := w^T \beta \tag{33}$$

The expected return of the portfolio is given by

$$\mu_p = r_f + \beta_p * (\mu_m - r_f) \tag{34}$$

Note that in section 4.1.4, we obtain the expected returns for stocks in the pool. Note that accoring to Fama (1970, p388), the portfolio's expected return is also additive, say

$$\mu_p = \sum_{j=1}^{n} w_i \mu_j := w^T \mu_p \tag{35}$$

Hence we have the following optimization

$$\begin{aligned} &\min_{w} \quad \frac{1}{2} w^T \Sigma w \\ s.t. \quad &\sum_{i=1}^{n} w_i = 1, \qquad w^T \mu = \mu_p \end{aligned} \tag{36}$$

Solving this optimization problem, we have the optimal weights given by:

89	151	417	521	753	600030	600036	600348	600519	601398
0.039226	0.0	0.0	0.089435	0.013239	0.0	0.0	0.088153	0.070175	0.699772

Figure 12: Optimal weights of the portfolio pool

From the weights, we can find that for our modified Black-Litterman model, the result is sparse. This is exceeding useful in portfolio management since sparse results can filter certain high-correlated stocks and have better persistence. Sparse allocation can also reduce the number of stocks to be invested, hence reducing the management costs and transaction costs. Literature has endeavored to find sparse allocation plans. For instance, Yumin (2016, p1259) used norm penalties to make allocation sparse. Our modified model can provide a new perspective for sparse allocation in the literature.

4.4 Efficient Frontier for in-sample data

We can derive both the mean-variance efficient frontiers under classical CAPM and our modified B-L model. given expected return z, we have the optimization problem

$$\begin{aligned} \min_{w} \quad & \frac{1}{2} w^T \Sigma w \\ s.t. \quad & \sum_{i=1}^{n} w_i = 1, \qquad w^T \mu_p = z \end{aligned} \tag{37}$$

where μ_p is the only difference for the two models. The results are given by

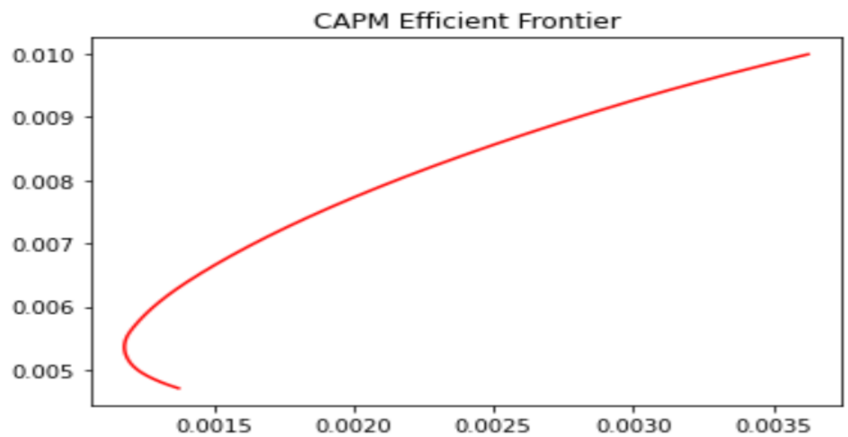


Figure 13: Efficient Frontier under CAPM

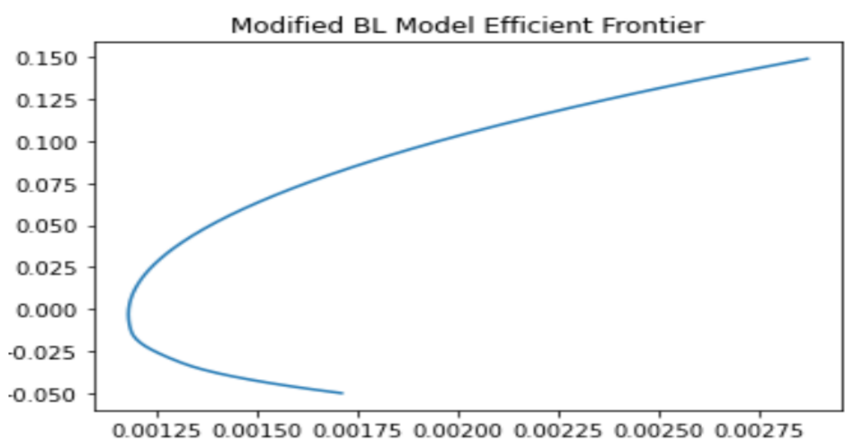


Figure 14: Efficient Frontier under Modified BL Model

It can be observed that, the modified B-L model has a much wider height, ranging from -0.05 to 0.15, which is almost 10 times the CAPM model. Obviously, the modified BL model is way closer to reality. From the out-of-sample data observed, in different periods, the expected returns can be widely spread. Even the market monthly expected return in certain periods can reach up to 0.1. Our modified BL Model captures the more volatile stock market.

4.5 Sensitivity Analysis

Note that we adopted the settings of hyperparameters $\tau = 0.025$ and number of views $N = 3$. We want to conduct sensitivity analysis on the hyperparameter to show our allocation plan (weights on the stocks in the pool) is robust.

4.5.1 Sensitivity Analysis on N

Here, we change N to be 2 and 4 respectively. The allocation weights are given by:

89	151	417	521	753	600030	600036	600348	600519	601398
0.039202	0.0	0.0	0.086419	0.013389	0.0	0.0	0.090157	0.069456	0.701378

Figure 15: The allocation weights with 2 views

89	151	417	521	753	600030	600036	600348	600519	601398
0.040051	0.0	0.0	0.089443	0.010264	0.0	0.0	0.082501	0.072409	0.705333

Figure 16: The allocation weights with 4 views

Note that, both two weights are close to each other and are similar to the allocation plan when $N = 3$ as shown in Figure 12. It means that our model can effectively construct strong views. Hence our allocation is relatively robust and invariant to the change in the number of views.

4.5.2 Sensitivity Analysis on τ

Here, we change the value of τ from 0.025 to 0.5. all the values in this range are frequently used in B-L Model. Hence we need to confirm our robustness on the choice of τ. The allocation weights for different τ are given by:

	89	151	417	521	753	600030	600036	600348	600519	601398
0.025	0.040051	0.0	0.0	0.089443	0.010264	0.0	0.0	0.082501	0.072409	0.705333
0.050	0.040227	0.0	0.0	0.087199	0.008714	0.0	0.0	0.075698	0.076427	0.711737
0.075	0.040244	0.0	0.0	0.086310	0.008100	0.0	0.0	0.073213	0.077968	0.714164
0.100	0.040241	0.0	0.0	0.085829	0.007771	0.0	0.0	0.071903	0.078798	0.715459
0.125	0.040236	0.0	0.0	0.085526	0.007565	0.0	0.0	0.071089	0.079318	0.716266
0.150	0.040230	0.0	0.0	0.085318	0.007424	0.0	0.0	0.070534	0.079675	0.716819
0.175	0.040225	0.0	0.0	0.085167	0.007321	0.0	0.0	0.070131	0.079936	0.717220
0.200	0.040220	0.0	0.0	0.085051	0.007244	0.0	0.0	0.069824	0.080135	0.717526
0.225	0.040217	0.0	0.0	0.084960	0.007183	0.0	0.0	0.069583	0.080291	0.717766
0.250	0.040213	0.0	0.0	0.084887	0.007134	0.0	0.0	0.069388	0.080418	0.717960
0.275	0.040210	0.0	0.0	0.084826	0.007093	0.0	0.0	0.069228	0.080522	0.718120
0.300	0.040208	0.0	0.0	0.084776	0.007059	0.0	0.0	0.069094	0.080610	0.718254
0.325	0.040206	0.0	0.0	0.084733	0.007030	0.0	0.0	0.068979	0.080684	0.718368
0.350	0.040204	0.0	0.0	0.084695	0.007006	0.0	0.0	0.068881	0.080748	0.718466
0.375	0.040202	0.0	0.0	0.084663	0.006984	0.0	0.0	0.068795	0.080804	0.718551
0.400	0.040201	0.0	0.0	0.084635	0.006965	0.0	0.0	0.068720	0.080853	0.718626
0.425	0.040199	0.0	0.0	0.084609	0.006948	0.0	0.0	0.068654	0.080897	0.718693
0.450	0.040198	0.0	0.0	0.084587	0.006933	0.0	0.0	0.068595	0.080935	0.718752
0.475	0.040197	0.0	0.0	0.084567	0.006920	0.0	0.0	0.068541	0.080970	0.718805
0.500	0.040196	0.0	0.0	0.084549	0.006908	0.0	0.0	0.068494	0.081002	0.718852

Figure 17: The allocation weights for different τ

Still, even ranging from 0.025 to 0.5 with the latter τ being 20 times of the initial τ, the allocation plan still changes exceedingly tiny. This result shows our model can provide an allocation plan invariant to the choices of τ, which strengthens our model's robustness.

4.6 Validation Test for Out-of-Sample Data

Consider another random drawn set of stocks, say the portfolio formed by stock id: 600360, 600674, 600976, 600108, 600201, 600783, 600889 follow exactly the same procedure as before, we can derive the optimal weights for current stock pool:

600360	600674	600976	600108	600201	600783	600889
0.114099	0.565494	0.072301	0.147068	0.101039	0.0	0.0

Figure 18: Optimal weights of the portfolio pool

Still, the result is sparse, which is coherent with the discussion in section 4.3. Our model preserves sparsity even out of sample.

We can also derive the mean-variance efficient frontiers under classical CAPM and our modified B-L model, the results are given by:

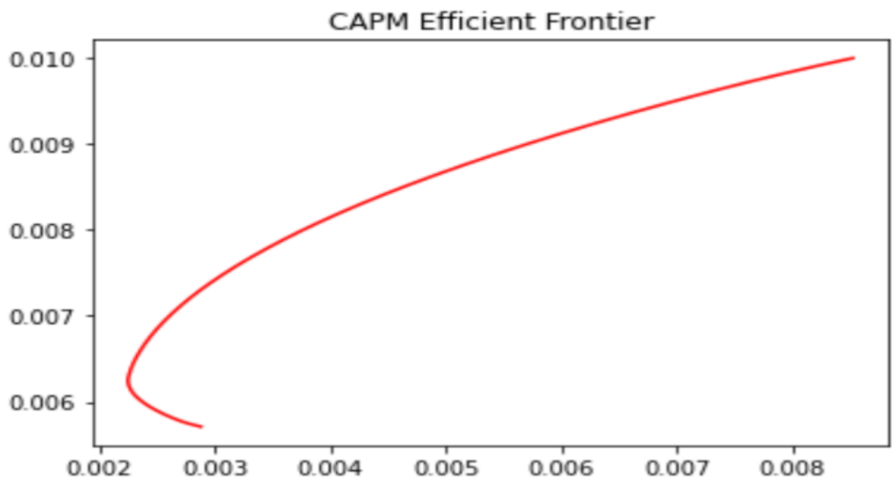


Figure 19: Efficient Frontier under CAPM

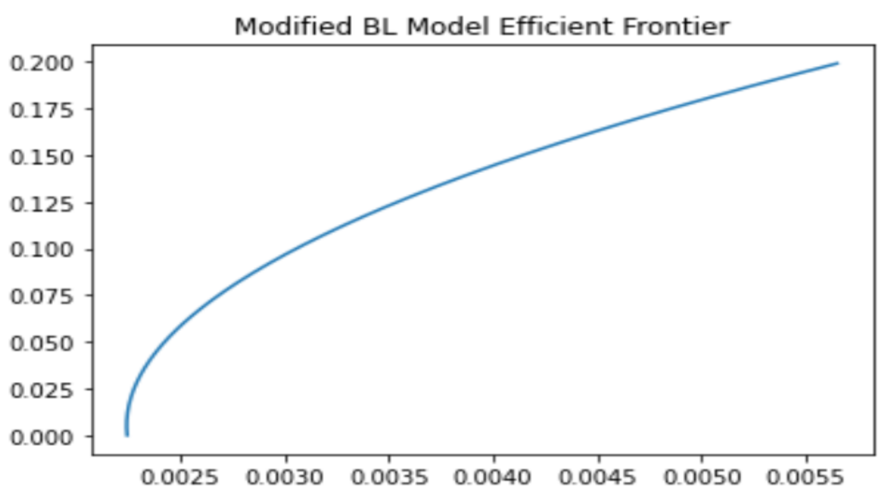


Figure 20: Efficient Frontier under Modified BL Model

Still, we can find that our model captures a larger range of expected return. which is more practical.

5 Conclusion

In the classical Black-Litterman Model, the views are considered as subjective options from managers' options, leading to huge inaccuracy and uncertainty in asset pricing and portfolio management. The literature has tried different methods to improve the accuracy. Useful as they are, they seldom consider dealing with the subjectiveness issue in the view. This research constructs an objective view measure based on historical data. On top of the classical Modern Portfolio Theory (MPT) Assumptions, we further assume that the majority of stock returns preserve a mean-reversion pattern. This pattern has been supported by the literature and validated by this paper. Based on this assumption, under the normality assumption of returns from MPT (also verified in this paper), we derived an algorithm determining the relative changes in stock returns from the perspective of probability, thus constructing the view vector and link matrix in classical Black-Litterman Model using historical data instead of subjective opinions. On top of the objectiveness of view measure, based on our modified Black-Litterman model, portfolio management can attain two concomitant advantages:

1. The optimal weights in asset allocation are sparse, which can reduce the correlation between assets and provide another perspective for sparse allocation literature.

2. Compared with the traditional CAPM model, our model captures more volatility of expected returns, which is more realistic and caters to existing data.

Our model can survive the out-of-sample validity examination sensitivity analysis. This also provides support for the application of our modified model.

Based on our model, practitioners can take the results as a reference to make a more objective portfolio plan. Our model can effectively reduce the influence of the subjective moods of managers and make the result more stable. Besides, our model provides an automatic algorithm for portfolio management. Managers no longer need to continuously update the market information. They only need to select the stock pools for portfolio construction, and our model can help them to get a sparse and objective allocation weight.

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