Derivatives: Part 2

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2021

Derivatives 2: Advanced topics

Chain rule

Composite functions

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$$(f \circ g)(x) = f[g(x)]$$

- To form $f[g(x)]$, the range of g must be contained (at least in part) within the domain of f.
- The domain of $f \circ g$ consists of all the points in the domain of g for which $g(x)$ is in the domain of f.

Examples:

$f(x) = \ln x$, $0 < x < \infty$
$g(x) = x^2 \; -\infty < x < \infty$

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$(f \circ g)(x) = \ln x^2,\ -\infty < x < \infty - \{0\}$
$(g \circ f)(x) = [\ln x]^2,\ 0 < x < \infty$

Discussion

Notice that $f \circ g$ and $g \circ f$ are not the same functions.

$f(x) = 4 + \sin x$, $-\infty < x < \infty$

$g(x) = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$

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$(f \circ g)(x) = 4 + \sin \sqrt{1 - x^2}$, $-1 \leq x \leq 1$
$(g \circ f)(x)$ does not exist, since the range of f, $[3, 5]$, has no points in common with the domain of g.

The Chain Rule

Let $y = f(z)$ and $z = g(x)$.

- Then, $y = (f \circ g)(x) = f[g(x)]$ and the derivative of y with respect to x is

$$\frac{d}{dx}\{f[g(x)]\} = f'[g(x)]g'(x)$$

  which can also be written as

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

- Note: the above does not imply that the dz's cancel out, as in fractions. They are part of the derivative notation and have no separate existence.

- The chain rule can be thought of as the derivative of the "outside" times the derivative of the "inside,"" remembering that the derivative of the outside function is evaluated at the value of the inside function.

The Generalized Power Rule

- If $y = [g(x)]^k$, then $dy/dx = k[g(x)]^{k-1}g'(x)$.
- How is this related to the chain rule?

Example

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Find dy/dx for $y = (3x^2 + 5x - 7)^6$.
Let $f(z) = z^6$ and $z = g(x) = 3x^2 + 5x - 7$. Then, $y = f[g(x)]$ and

$$\begin{aligned}
\tfrac{dy}{dx} &= f'(z)g'(x) \\
&= \left(6z^5\right)(6x+5) \\
&= 6\left(3x^2 + 5x - 7\right)^5 (6x+5)
\end{aligned}$$

Find the derivatives for the following functions

1. $f(x) = e^{x^2}$
2. $f(x) = ln(x^2 + 9)$
3. $f(x) = ln(ln(x))$
4. $f(x) = e^{ln(2x)}$
5. $f(x) = e^{x - ln(x) + 5}$
6. $f(x) = \sqrt{x}e^{\sqrt{x}}$
7. $f(y) = \sqrt{\frac{(y^4 - 3y^2)ln(7y-4)}{e^{y^3-2y}}}$
8. $f(z) = ln(z^3 + 2z)\exp(1/z^2 + 2z - 2)$

Higher order derivatives

- One conceptual leap you need to make is that $f'(x)$ is just another function of x. An input value for x gives an output value determined by $f'(x)$.

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- One conceptual leap you need to make is that $f'(x)$ is just another function of x. An input value for x gives an output value determined by $f'(x)$.
- For some people it might help to change the name so that $h(x) = f'(x)$.
- The important thing to understand is that just as $f(x)$ can have a derivative, $f'(x)$ can also have have a derivative.

- We can keep applying the differentiation process to functions that are themselves derivatives.
- The derivative of $f'(x)$ with respect to x, would then be

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

and so on.

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  and so on.
- Similarly, the derivative of $f''(x)$ would be denoted $f'''(x)$.
- **First derivative:** $f'(x)$, y', $\frac{df(x)}{dx}$, $\frac{dy}{dx}$

  Second derivative: $f''(x)$, y'', $\frac{d^2f(x)}{dx^2}$, $\frac{d^2y}{dx^2}$

  nth derivative: $\frac{d^nf(x)}{dx^n}$, $\frac{d^ny}{dx^n}$

Example

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$f'(x) = 3x^2$

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$f'''(x) = 6$
$f''''(x) = 0$

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$$f'(x) = 16x^3 - 24x + 1$$

$$f''(x) = 48x^2 - 24$$

$$f'''(x) = 96x$$

Example

$$f(x) = 4x^4 + 12x^2 + x - 2$$

$$f'(x) = 16x^3 - 24x + 1$$

$$f''(x) = 48x^2 - 24$$

$$f'''(x) = 96x$$

$$f''''(x) = 96$$

Example

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$$f'''(x) = 96x$$

$$f''''(x) = 96$$

$$f'''''(x) = 0$$

Find the second and third derivatives for the following functions

1. $f(x) = ln(x)$
2. $f(x) = (x^2 + 1)(x^3 - 1)$
3. $f(x) = x^6 + 5x^5 - 2x^2 + 8$

Finding maxima and minima

The most frequent way you will encouter a derivative in your training is when you need to find either the maximum or minimum of some function. That is, you need to find the value of x that will maximize (minimize) $f(x)$.

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- You are building a theoretical model of how some agent makes a decision, and you need to find the value of x that maximizes their utility.
- You have built a statistical model for your data, and you need to find the value of θ that will maximize the likelihood of $f(\theta)$

The role of the first derivative

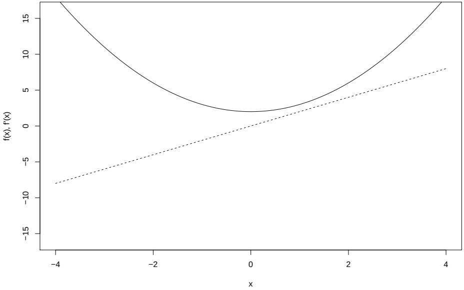
The first derivative $f'(x)$ identifies whether the function $f(x)$ at the point x is

1. **Increasing:**
   $f'(x) > 0$
2. **Decreasing:**
   $f'(x) < 0$
3. **Extremum/Saddle**:
   $f'(x) = 0$

Examples

$f(x) = x^2 + 2$, $f'(x) = 2x$

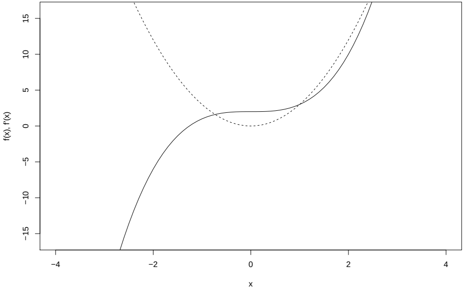
```
x<-seq(-4, 4, by=.1); fx<-x^2+2; fx.prime<-2*x
plot(x, fx, xlab="x", ylab="f(x), f'(x)",
     type="l", ylim=c(-16, 16))
lines(x, fx.prime, lty=2)
```



Examples

$f(x) = x^3 + 2$, $f'(x) = 3x^2$

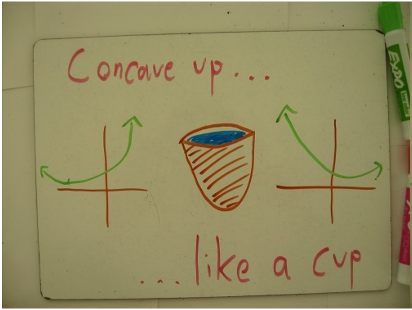
```
x<-seq(-4, 4, by=.1); fx<-x^3+2; fx.prime<-3*x^2
plot(x, fx, xlab="x", ylab="f(x), f'(x)",
     type="l", ylim=c(-16, 16))
lines(x, fx.prime, lty=2)
```

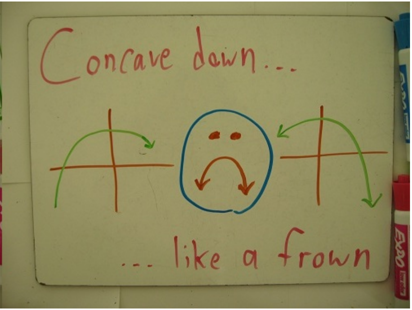


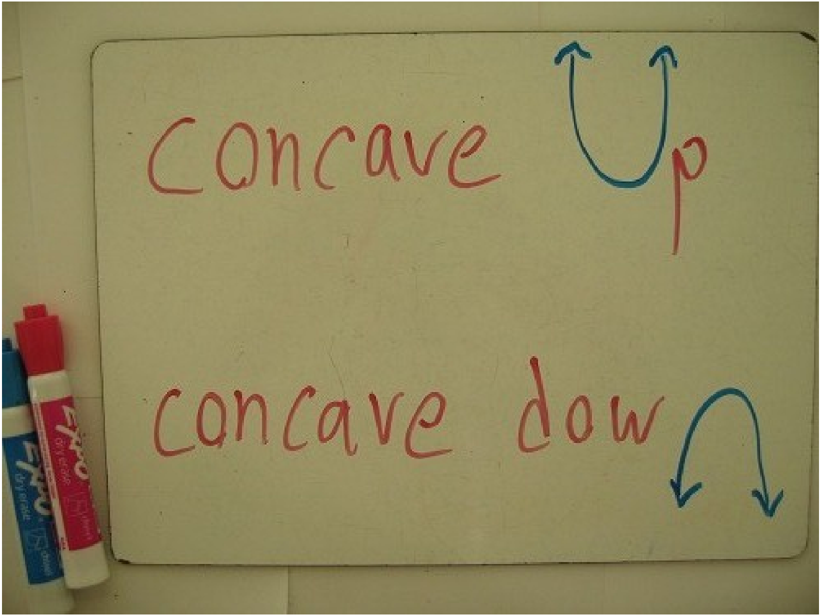
The role of the second derivative

The second derivative $f''(x)$ identifies whether the function $f(x)$ at the point x is

- **Concave down:** $f''(x) < 0$
- **Concave up:** $f''(x) > 0$


Concave up...
...like a cup

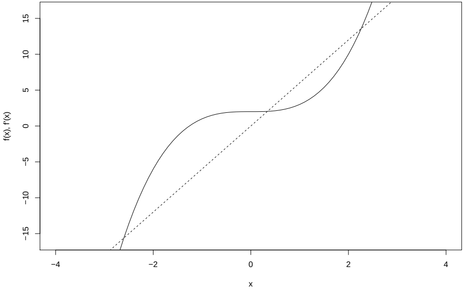

Concave down...
...like a frown


concave Up
concave dow
EXPO

Example

$f(x) = x^3 + 2$, $f''(x) = 6x$

```
x<-seq(-4, 4, by=.1); fx<-x^3+2; fx.prime2<-6*x
plot(x, fx, xlab="x", ylab="f(x), f'(x)",
     type="l", ylim=c(-16, 16))
lines(x, fx.prime2, lty=2)
```



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- x_0 is a **global** minimum if $f(x_0) < f(x)$ for all x in the domain of f.

Critical points

- The maxima and minima will be a subset of something called critical points.
- Given the function f defined over domain D, all of the following are critical points:
  - Any interior point of D where $f'(x) = 0$.
  - Any interior point of D where $f'(x)$ does not exist.
  - Any endpoint that is in D.

Putting it all together

Combined, the first and second derivatives can tell us whether a point is a maximum or minimum of $f(x)$.

- **Local Maximum:** $f'(x) = 0$ **and** $f''(x) < 0$
- **Local Minimum:** $f'(x) = 0$ **and** $f''(x) > 0$
- **Need more info:** $f'(x) = 0$ **and** $f''(x) = 0$

Note on finding global max/min

Sometimes no global max or min exists — e.g., $f(x)$ not bounded above or below. However, three situations where we can fairly easily identify global max or min.

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1. **Functions with only one critical point.** If x_0 is a local maximum of f and it is the only critical point, then it is a global maximum.
2. **Globally concave up or concave down functions.** If f'' is never zero, then there is at most one critical point, which is a global maximum if $f'' < 0$ and a global minimum if $f'' > 0$. Why?

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3. **Functions over closed and bounded intervals** must have both a global maximum and a global minimum. BUT REMEMBER TO CHECK THE END POINTS OF THE INTERVAL.

Examples

$$f(x) = x^2 + 2$$

$$f'(x) = 2x$$

$$f''(x) = 2$$

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1. Find the values of x where $f'(x) = 0$.
2. Note that the second derivative shows it to be globally concave up.
3. Conclude that $x = 0$ is a global minimum.

$$f(x) = x^3 + 2$$
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2. Substitute that number (0) into $f''(x)$. We get zero, so we need more info.

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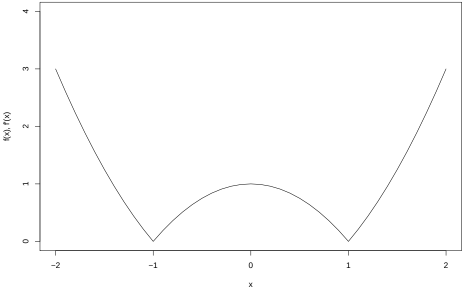
$$f''(x) = 6x$$

1. Find the values of x where $f'(x) = 0$.
2. Substitute that number (0) into $f''(x)$. We get zero, so we need more info.
3. If we examined either the graph or values of $f''(x)$ around $x = 0$, we would find that $x = 0$ is in fact a saddle point. Since $f(x)$ is unbounded above or below, there are no maxima or minima.

$f(x) = |x^2 - 1| \; x \in [-2, 2]$

```
x<-seq(-2, 2, by=.1); fx<-abs(x^2-1)
plot(x, fx, xlab="x", ylab="f(x), f'(x)",
     type="l", ylim=c(0, 4))
```



$$f'(x) = \begin{cases} 2x & -2 < x < -1, 1 < x < 2 \\ -2x & -1 < x < 1 \end{cases}$$

$$f''(x) = \begin{cases} 2 & -2 < x < -1, 1 < x < 2 \\ -2 & -1 < x < 1 \end{cases}$$

1. There are five critical points.
   - The two endpoints are $(-2, 3)$ and $(2, 3)$.
   - $f'(x)$ is not defined for $x = \pm 1$, so $(-1, 0)$ and $(1, 0)$ are also critical points.
   - Additionally, $f'(x) = 0$ for $x = 0$, so $(0, 1)$ is a critical point.

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   - Additionally, $f'(x) = 0$ for $x = 0$, so $(0, 1)$ is a critical point.
2. Since $f''(0)$ is negative, $(0, 1)$ is a local maximum.
3. Comparing $f(x)$ for these five points, we conclude that $(-2, 3)$ and $(2, 3)$ are global maxima and $(-1, 0)$ and $(1, 0)$ are global minima.

Find the global minimum and global maximum of the following functions over the stated domains:

1. $f(x) = 3x^4 - 4x^3 - 36x^2$, where $x \in [-4, 4]$
2. $g(x) = xln(x) - x$, where $x \in (0, 3]$
3. $f(x) = x^3 - \frac{15}{2}x^2 + 12x + 8$, where $x \in [0, 6]$

L'Hospital's Rule

- In studying limits, we saw that $\lim_{x \to c} f(x)/g(x) = \left(\lim_{x \to c} f(x)\right) / \left(\lim_{x \to c} g(x)\right)$, provided that $\lim_{x \to c} g(x) \neq 0$, which will cause the limit to be unbounded.
- If both $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then we get an **indeterminate form** of the type $0/0$ as $x \to c$.

However, we can still analyze such limits using L'Hospital's rule.

L'Hospital's Rule: Suppose f and g are differentiable on $a < x < b$ and that either

1. $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$, or
2. $\lim_{x \to a^+} f(x) = \pm\infty$ and $\lim_{x \to a^+} g(x) = \pm\infty$

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Suppose further that $g'(x)$ is never zero on $a < x < b$ and that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Examples

$$\lim_{x \to 0^+} \frac{\ln(1+x^2)}{x^3}$$

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- Let $f(x) = \ln(1+x^2)$ and $g(x) = x^3$.
- Then $f'(x) = 2x/(1+x^2)$ and $g'(x) = 3x^2$.

Examples

$$\lim_{x \to 0^+} \frac{\ln(1+x^2)}{x^3}$$

- Let $f(x) = \ln(1+x^2)$ and $g(x) = x^3$.
- Then $f'(x) = 2x/(1+x^2)$ and $g'(x) = 3x^2$.
- Using L'Hospital's rule,

$$\lim_{x \to 0^+} \frac{2x/(1+x^2)}{3x^2} = \lim_{x \to 0^+} \frac{2}{3x(1+x^2)}$$

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$$= \infty$$

$$\lim_{x \to 0^+} \frac{e^{1/x}}{1/x}$$

- Let $f(x) = e^{1/x}$ and $g(x) = 1/x$.

$$\lim_{x \to 0^+} \frac{e^{1/x}}{1/x}$$

- Let $f(x) = e^{1/x}$ and $g(x) = 1/x$.
- Then $f'(x) = -\frac{1}{x^2}e^{1/x}$ and $g'(x) = -1/x^2$.

$$\lim_{x \to 0^+} \frac{e^{1/x}}{1/x}$$

- Let $f(x) = e^{1/x}$ and $g(x) = 1/x$.
- Then $f'(x) = -\frac{1}{x^2} e^{1/x}$ and $g'(x) = -1/x^2$.
- Using L'Hospital's rule,

$$\lim_{x \to 0^+} \frac{-\frac{1}{x^2} e^{1/x}}{-1/x^2} = \lim_{x \to 0^+} e^{1/x}$$

$$= \infty$$

$$\lim_{x \to 2} \frac{x - 2}{(x + 6)^{1/3} - 2}$$

- Let $f(x) = x - 2$ and $g(x) = (x + 6)^{1/3} - 2$.
- Then $f'(x) = 1$ and $g'(x) = \frac{1}{3}(x + 6)^{-2/3}$.

$$\lim_{x \to 2} \frac{x-2}{(x+6)^{1/3}-2}$$

- Let $f(x) = x - 2$ and $g(x) = (x+6)^{1/3} - 2$.
- Then $f'(x) = 1$ and $g'(x) = \frac{1}{3}(x+6)^{-2/3}$.
- Using L'Hospital's rule,

$$\begin{aligned}\lim_{x \to 2} \frac{1}{\frac{1}{3}(x+6)^{-2/3}} &= 3(8)^{2/3} \\ &= 12\end{aligned}$$

1. $$\lim_{x \to \infty} \frac{ln(x)}{x}$$

2. $$\lim_{x \to \infty} \frac{3^x}{x^3}$$

3. $$\lim_{x \to \infty} \frac{3e^y}{y^3}$$

4. $$\lim_{x \to \infty} \frac{xln(x)}{x + log_{10}(x)}$$