Integrals: Part 1

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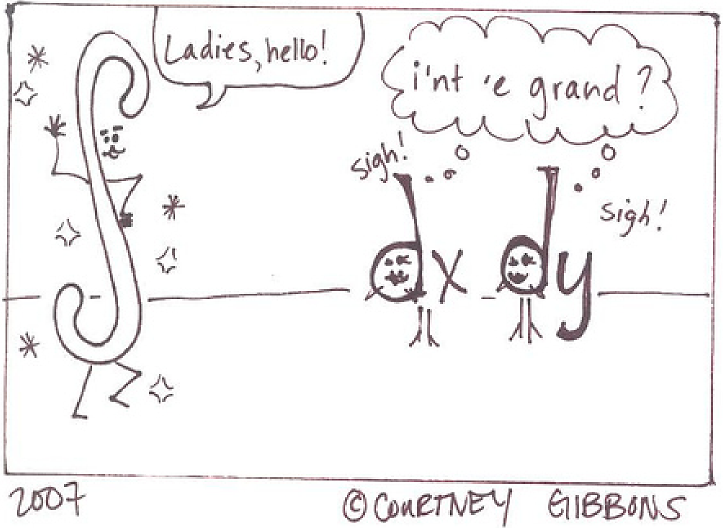
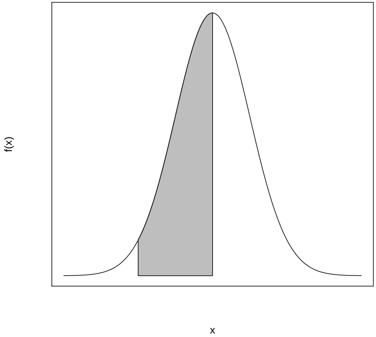
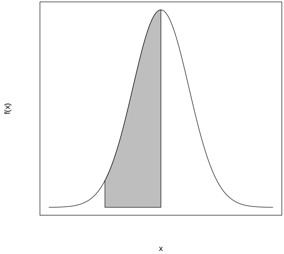


Some motivation

If you were trying to find the area of a rectangle or in a trapezoid, you would have no problem. But what if you were trying to find the area under a curve like this?





- This is an important question, because the above function is a drawing of the normal distribution – the most commonly used probabilty function in all of statistics.
- The above picture is essentially asking the question: what is the probabilty (i.e., $f(x)$) of observing a value of x between -2 and 0?

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- The answer, as it turns out, is sometimes very difficult to get. Nonetheless it is very important.
- You will be doing *a lot* of integration in statistics (especially if you venture into Bayesian statistics). And there are many, many applications of this technique in game theory.
- But in general, you will *always* be using these techniques for one of these two goals:
  - Finding the area under a curve
  - Finding a function given its derivative.

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- Sometimes it is *impossible*.
  - There are many important functions (e.g., the normal probability density function) whose indefinite integral has never been derived.
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  - Focus on understanding the basic concept.
  - Then starting develop a library of "tricks" that mathematicians frequently use to solve these kinds of problems.
  - You will get better with practice.

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- Let DF be the derivative of F. And let $DF(x)$ be the derivative of F evaluated at x.
  - Then the antiderivative is denoted by D^{-1} (i.e., the inverse derivative). If $DF = f$, then $F = D^{-1}f$.
  - In words, we need something that will go backwards. It takes in a function, and tells you what it was before it was differentiated.

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- **Indefinite Integral**: Equivalently, if F is the antiderivative of f, then F is also called the indefinite integral of f and written

$$F(x) = \int f(x)dx$$

.

Examples:

$$\int \frac{1}{x^2}dx = -\frac{1}{x} + c$$

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- Notice that while there is only a single derivative for any function, there are multiple antiderivatives: one for any arbitrary constant c. c just shifts the curve up or down on the y-axis.
- If more info is present about the antiderivative — e.g., that it passes through a particular point — then we can solve for a specific value of c.

$$\int 3e^{3x}\,dx =$$

$$\int 3e^{3x}\,dx = e^{3x} + c$$

$$\int 3e^{3x} dx = e^{3x} + c$$

$$\int (x^2 - 4) dx =$$

$$\int 3e^{3x}dx = e^{3x} + c$$

$$\int (x^2 - 4)dx = \frac{1}{3}x^3 - 4x + c$$

Common rules of integration:

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4. $$\int e^x dx = e^x + c$$

5.

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$$\int e^{f(x)} f'(x) dx = e^{f(x)} + c$$

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7.

$$\int [f(x)]^n f'(x) dx = \frac{1}{n+1} [f(x)]^{n+1} + c$$

8.

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$$

Examples

$$\int 3x^2 dx = 3 \int x^2 dx$$

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$$\begin{aligned}\int 3x^2 dx &= 3\int x^2 dx \\ &= 3\left(\frac{1}{3}x^3\right) + c\end{aligned}$$

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$$
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\end{aligned}
$$

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Discussion

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- You will want to do a lot to "simplify" the problems (in your head or on paper) and take things in stages.
- So you can make this

$$\int e^x e^{e^x} dx = e^{e^x} + c$$

  into this

$$\int f(x) e^{f(x)} dx = e^{f(x)} + c$$

Solve these indefinite integrals

1. $$\int 3x^{1/3} dx$$

2. $$\int -1 dx$$

3. $$\int -3 + 4x dx$$

4. $$\int 4x + 3 dx$$

5. $$\int 3x^2 dx$$

6.
$$\int -2x + 3 - 4x^3 dx$$

7.
$$\int 5x^4 - x - 4 dx$$

8.
$$\int 5x^5 dx$$

9.
$$\int 4x^4 + 3x^3 + 2x^2 + x + 1 dx$$

10.
$$\int x^{-1} + 3x^2 dx$$

11.

$$\int e^{5x} dx$$

12.

$$\int \frac{5}{x^3} + \frac{5}{x} + e^x dx$$

13.

$$\int x^{100} + 3e^x - 7(4^x) dx$$

The definite integral

The other major way we are going to use integrals, is to find the area under a curve between two specific points.

Reimann Sum

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- One way to calculate the area would be to divide the interval $a \leq x \leq b$ into n subintervals of length Δx and then approximate the region with a series of rectangles, where the base of each rectangle is Δx and the height is $f(x)$ at the midpoint of that interval.

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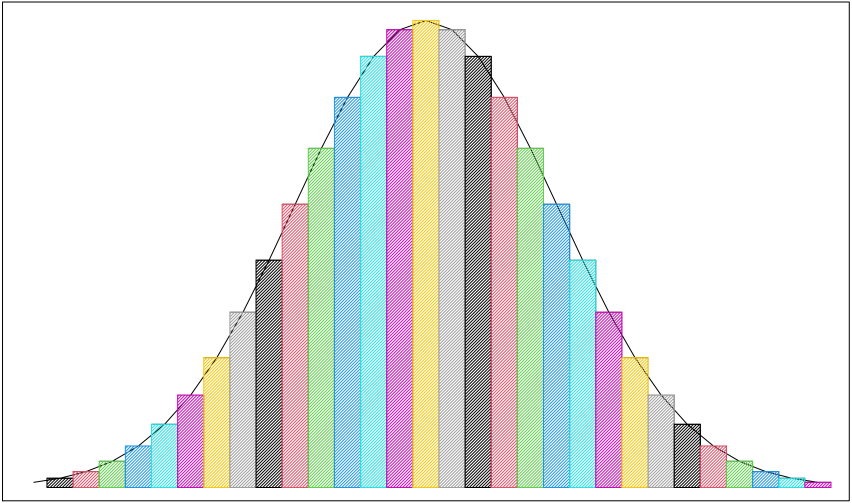
- $A(R)$ would then be approximated by the area of the union of the rectangles, which is given by

$$S(f, \Delta x) = \sum_{i=1}^{n} f(x_i)\Delta x$$

and is called a Riemann sum.



Thinking of integrals as summations

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$$A(R) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

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- **Riemann Integral**: If for a given function f the Riemann sum approaches a limit as $\Delta x \to 0$, then that limit is called the Riemann integral of f from a to b. Formally,

$$\int_a^b f(x) dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) \Delta x$$

- **Definite Integral**: We use the notation $\int_a^b f(x)dx$ to denote the definite integral of f from a to b.

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###The fundamental Theorem(s) of calculus

- **First Fundamental Theorem of Calculus**: Let the function f be bounded on $[a, b]$ and continuous on (a, b). Then the function

$$F(x) = \int_a^x f(s)ds, \quad a \leq x \leq b$$

has a derivative at each point in (a, b) and

$$F'(x) = f(x), \quad a < x < b$$

Thus, differentiation is the inverse of integration.

- **Second Fundamental Theorem of Calculus**: Let the function f be bounded on $[a, b]$ and continuous on (a, b). Let F be any function that is continuous on $[a, b]$ such that $F'(x) = f(x)$ on (a, b). Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

- This gives us a procedure to calculate a “simple" definite integral $\int_a^b f(x)dx$:
  1. Find the indefinite integral $F(x)$.
  2. Evaluate $F(b) - F(a)$.

Examples:

$$\int_{1}^{3} 3x^2 dx =$$

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$$\int_1^3 3x^2 dx = 3\left(\frac{1}{3}x^3\right)\Big|_1^3$$

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$$= (3)^3 - (1)^3 = 26$$

$$\int_{-2}^{2} e^{x} e^{e^{x}} \, dx =$$

$$\int_{-2}^{2} e^{x} e^{e^{x}}\, dx = e^{e^{x}} \Big|_{-2}^{2}$$

$$\int_{-2}^{2} e^x e^{e^x} dx = e^{e^x}\Big|_{-2}^{2}$$

$$= e^{e^2} - e^{e^{-2}}$$

$$\int_{-2}^{2} e^x e^{e^x} dx = e^{e^x} \Big|_{-2}^{2}$$

$$= e^{e^2} - e^{e^{-2}} = 1617.033$$

Properties of Definite Integrals:

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4.

$$\int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx = \int\limits_a^c f(x)dx$$

Examples:

$$\int_{1}^{1} 3x^2 dx = x^3 \Big|_{1}^{1}$$

Examples:

$$\int_1^1 3x^2dx = x^3\Big|_1^1$$

$$= (1)^3 - (1)^3 = 0$$

$$\int_0^4 (2x+1)dx = 2\int_0^4 x dx + \int_0^4 1 dx$$

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$$= x^2\Big|_0^4 + x|_0^4$$

$$\int_0^4 (2x+1)dx = 2\int_0^4 xdx + \int_0^4 1dx$$

$$= \left. x^2 \right|_0^4 + \left. x \right|_0^4$$

$$= (16-0) + (4-0) = 20$$

$$\int_0^4 (2x+1)dx = 2\int_0^4 x dx + \int_0^4 1 dx$$

$$= x^2\Big|_0^4 + x\Big|_0^4$$

$$= (16-0) + (4-0) = 20$$

What would happen if we reversed 4 and 0?

$$\int_{-2}^{0} e^x e^{e^x}\,dx + \int_{0}^{2} e^x e^{e^x}\,dx =$$

$$\int_{-2}^{0} e^{x} e^{e^{x}} dx + \int_{0}^{2} e^{x} e^{e^{x}} dx = \left. e^{e^{x}} \right|_{-2}^{0} + \left. e^{e^{x}} \right|_{0}^{2}$$

$$\int_{-2}^{0} e^x e^{e^x} dx + \int_{0}^{2} e^x e^{e^x} dx = \left. e^{e^x} \right|_{-2}^{0} + \left. e^{e^x} \right|_{0}^{2}$$

$$= e^{e^0} - e^{e^{-2}} + e^{e^2} - e^{e^0}$$

$$\int_{-2}^{0} e^x e^{e^x} dx + \int_{0}^{2} e^x e^{e^x} dx = \left. e^{e^x} \right|_{-2}^{0} + \left. e^{e^x} \right|_{0}^{2}$$

$$= e^{e^0} - e^{e^{-2}} + e^{e^2} - e^{e^0}$$

$$= e^{e^2} - e^{e^{-2}} = 1617.033$$

Solve the following definite integrals:

1. $$\int_{-1}^{1} x^3 dx$$

2. $$\int_{0}^{0.1} x^2 dx$$

3. $$\int_{0}^{9} 1 dx$$

5.

$$\int_{1}^{9} 2y^5 dy$$

6.

$$\int_{-1}^{0} 3x^2 - 1 dx$$

7.

$$\int_{-1}^{1} 14 + x^2 dx$$

8.

$$\int_{1}^{-1} 14 + x^2 dx$$

9.

$$\int_1^2 \frac{1}{x}\,dx$$

10.

$$\int_1^2 \frac{1}{x^2}\,dx$$

11.

$$\int_2^\infty \frac{12}{t^2}\,dt$$

12.

$$\int_3^{y^2} \sqrt{z}\,dz$$