Multivariate Calculus: The Basics

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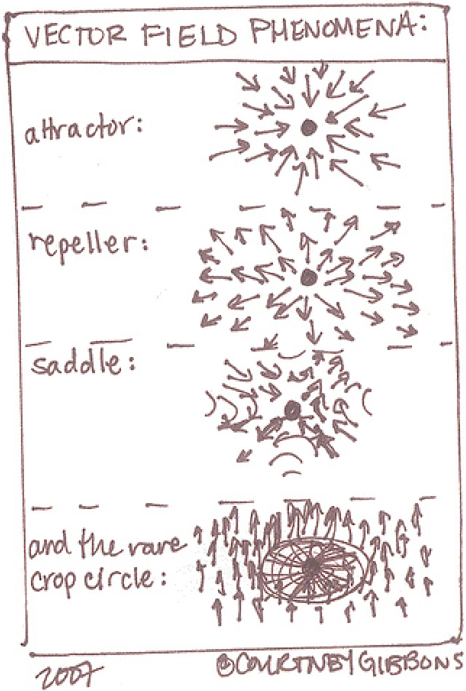
Multivariate Calculus: The Basics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Multivariate calculus: An overview

- This is where things get cool (but also a bit tricky).
- Just like we often have too many equations and variables to deal with efficiently in separate pieces for basic algebraic operations, we will often want a better way to do calculus with systems of equations.

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- This is where things get cool (but also a bit tricky).
- Just like we often have too many equations and variables to deal with efficiently in separate pieces for basic algebraic operations, we will often want a better way to do calculus with systems of equations.
- In these situations, very smart people have developed matrix methods for handling differentiation equivalent to first (gradients) and second (Hessians) derivatives.
- This is pretty useful for finding global maxima and minimum in multi-dimensional spaces.

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- Unfortunately, there aren't a set of equally clean methods for dealing with multidimensional integration.
- That's good news for you today (since you don't have to learn about it), but bad news for the rest of your life since you will spend a lot of your time working with imperfect numerical approximations of high-dimensional integrals.

Differentiation in several variables

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Differentiation in several variables

- Suppose we have a function f now of two (or more) variables and we want to determine the rate of change relative to one of the variables.
- To do so, we would find its *partial derivative*, which is defined similar to the derivative of a function of one variable.

Partial derivative

- Let f be a function of the variables $(x_1, \dots, x_n)$.
- The partial derivative of f with respect to x_i is

$$\frac{\partial f}{\partial x_i}(x_1, \dots, x_n) = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

- Only the ith variable changes — the others are treated as constants.
- We can take higher-order partial derivatives, like we did with functions of a single variable, except now we the higher-order partials can be with respect to multiple variables.

Example 1:

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$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 0$$

Example 2:

$$f(x,y) = x^3y^4 + e^x - \ln y$$
$$\frac{\partial f}{\partial x}(x,y) = 3x^2y^4 + e^x$$
$$\frac{\partial f}{\partial y}(x,y) = 4x^3y^3 - 1/y$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6xy^4 + e^x$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 12x^2y^3$$

What does it mean?

- Start at some point, how fast does $f(x, y)$ change when move in x direction? How fast when in y direction?
- For any given value of y, what is the slope of the hyperplane along the x-axis.

Standard regression models often look something like this:

$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2$$

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2. Interpret both.

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$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2 + \beta_4 x_1 x_2$$

3. Find the partial derivatives in terms of x_1 and x_2.
4. Interpret both.

For the following equations, find the first and second order partial derivatives in terms of x and y

5.
$$\exp x^2 + y^2 - 2x + 5y + 7$$

6.
$$ln(x + \sqrt{y})$$

7.
$$(x + y)\sqrt{x - y}$$

Integration with several variables

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- Take the total function and
  1. choose one of the variables (although you will want to be strategic about this choice).
  2. Perform the integration, while *treating the other variable as a constant*.
  3. Make sure you keep track of the ∂x_i symbols.

Example

$$\int\int(2x+2y)\partial x\partial y = \int x^2+2xy\partial y + c$$

$$= yx^2+xy^2 + ?$$

Handling constants

- This is not as straight-forward as it seems because indefinite integrals are only correct up to a constant.
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or (if we are treating y as a constant)

$$= \frac{1}{4}x^4 - e^y$$

- Likewise:

$$\int\int 12x^2y^3\partial x\partial y$$

- Likewise:

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- Usually we get around this by making use of the fact that we know that funciton we are working with integrates to a known constant or when the integrated function passes through a specific point.
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- For instance, we know that any probability function must integrate to 1.
- But often, even this doesn't help that much when we are integrating many times across many dimensions.
- None of this is particularly important for anything you will be doing soon. Just something to keep in mind as you work towards more advanced methods.

The area under a (hyper)plane

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$$\int_a^b \int_c^d f(x,y) dy dx = \int_a^b \left[\int_c^d f(x,y) \mathrm{d}y \right] dx$$

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  - Take all air under sheet

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  - Take all air under sheet
  - Count it

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$$= \int_2^3 x^2 \left[\frac{1}{4} (1 - 0) \right] dx \quad (2)$$

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\begin{aligned}
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$$= \frac{19}{12} \tag{6}$$

Example in reverse

$$\int_0^1 \int_2^3 x^2 y^3 \mathrm{d}x \mathrm{d}y \quad = \quad \int_0^1 y^3 \left[\frac{1}{3} x^3 |_2^3 \right] \mathrm{d}y \tag{7}$$

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&= \frac{19}{3} \frac{1}{4} y^4 |_0^1 && (10)
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1. If the region over which you're integrating is rectangular, no. (but sometimes $\mathrm{d}x$ may be easier to find than $\mathrm{d}y$.)
2. Yes. The limits may need to be adjusted. Consider

$$\int_0^1 \int_1^{e^y} 1 dx dy$$

$$\int_1^e \int_{\log x}^1 1 dy dx$$

Solve:

1. $$\int_{2}^{3}\int_{0}^{1} x^{2}y^{3}\,dydx$$

2. $$\int_{2}^{4}\int_{3}^{5} dydx$$

3. $$\int_{0}^{1}\int_{0}^{1} x^{3/2}y^{2/3}\,dxdy$$

4.
$$\int_1^3 \int_1^x \frac{x}{y} dydx$$

5.
$$\int_0^1 \int_0^x (x+y^2) dydx$$

6.
$$\int_1^2 \int_1^{\sqrt{2-y}} ydxdy$$

7.
$$\int_0^1 \int_0^{2-2x} \int_0^{x^2y^2} dzdydx$$

Vector/matrix representation of calculus

- The function $y = f(x_1, x_2, \ldots, x_n)$ of the independnet variable $x_1, x_2, \ldots, x_n$ can be written as the function $y = f(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$.
- The **gradient** is the vector of partial derivaitves and is denoted:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{pmatrix}$$

- The **Hessian** $\mathbf{H}(\mathbf{x})$ is an $n \times n$ matrix, where the (i, j)th element is the second order partial derivative of $f(\mathbf{x})$ with respect to x_i and x_j:

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

Find the gradient and Hessian of the following functions:

1. $$f(x,y) = (x^2 - y^2)ln(x+y)$$

2. $$f(x,y) = -x^2 + xy - y^2 + 2x + y$$

3. $$f(x,y) = \frac{3}{2}x^2 - 2xy - 5x + 2y^2 - 2y$$