NAME: ANSWER KEY

For the following exercises, read the problems carefully and show all your work. Attach more pages if necessary. Avoid using a calculator or the computer to solve the exercises. Please, turn in ONE pdf.

1 Indefinite Integrals

Solve the following indefinite integrals; i.e., find the anti-derivative:

1. $\int 3x^3 + 2x^2 - e^x \; dx$

$$\begin{aligned}\int 3x^3 + 2x^2 - e^x \; dx &= \int 3x^3 \; dx + \int 2x^2 \; dx - \int e^x \; dx \\ &= \frac{3x^4}{4} + \frac{2x^3}{3} - e^x + C\end{aligned}$$

2. $\int \frac{2x}{x^2} \; dx$

This can be solved via u-substitution. Let $u = x^2$. Then $\frac{du}{dx} = 2x$, so $du = 2x \; dx$. Then

$$\begin{aligned}\int \frac{2x}{x^2} \; dx &= \int \frac{1}{u} \; du \\ &= \ln|u| + C \\ &= \ln(x^2) + C\end{aligned}$$

3. $\int \frac{1}{x^2} \; dx$

$$\begin{aligned}\int \frac{1}{x^2} \; dx &= \int x^{-2} \; dx \\ &= -\frac{1}{x} + C\end{aligned}$$

4. $\int 2x(x^2 - 64)^2 \; dx$

This can be solved via u-substitution. Let $u = x^2 - 64$. Then $\frac{du}{dx} = 2x$, so $du = 2x \; dx$. Then

$$\begin{aligned}\int 2x(x^2 - 64)^2 \; dx &= \int u^2 \; du \\ &= \frac{1}{3}u^3\end{aligned}$$

$$= \frac{1}{3}(x^2 - 64)^3 + C$$

5. $\displaystyle\int \frac{1}{x\ln(x)}\ dx$

   Notice $\dfrac{1}{x\ln(x)} = \dfrac{1/x}{\ln(x)}$. Then $\displaystyle\int \frac{1}{x\ln(x)}\ dx = \int \frac{f'(x)}{f(x)}\ dx$, where $f(x) = \ln(x)$, so

$$\begin{aligned}\int \frac{1}{x\ln(x)}\ dx &= \ln(f(x)) + c \\ &= \ln(\ln(x)) + c\end{aligned}$$

6. $\displaystyle\int \exp(5x^3)x^2 - x + 2\ dx$

   This can be solved via u-substitution. Let $u = 5x^3$. Then $\dfrac{du}{dx} = 15x^2$, so $du = 15x^2\ dx$. Then

$$\begin{aligned}\int \exp(5x^3)x^2 - x + 2\ dx &= \int e^u\ \frac{1}{15}du - \int x\ dx + \int 2\ dx \\ &= \frac{1}{15}e^u - \frac{1}{2}x^2 + 2x + c \\ &= \frac{1}{15}e^{5x^3} - \frac{1}{2}x^2 + 2x + c\end{aligned}$$

7. $\displaystyle\int (10 - x)^{10}\ dx$

   This can be solved via u-substitution. Let $u = 10 - x$. Then $\dfrac{du}{dx} = -1$, so $du = -dx$. Then

$$\begin{aligned}\int (10 - x)^{10}\ dx &= \int -u^{10} dx \\ &= -\frac{1}{11}u^{11} + c \\ &= -\frac{1}{11}(10 - x)^{11} + c\end{aligned}$$

2 Definite and Improper Integrals

1. $\displaystyle\int_4^5 2x\ dx$

   $\int 2x\ dx = x^2$. Then

$$\int_4^5 2x\ dx = 5^2 - 4^2$$
$$= 25 - 16$$
$$= 9$$

2. $\displaystyle\int_{e^{\sqrt{2}}}^{e^2} \frac{\ln(x)}{x}\ dx$

First note we can find $F(x)$ via u substitution. Let $u = \ln(x)$. Then $du = \dfrac{1}{x}dx$, and

$$\int \frac{\ln(x)}{x}\ dx = \int u\ du$$
$$= \frac{u^2}{2} + C$$
$$= \frac{\ln(x)^2}{2} + C$$

Then

$$\int_{e^{\sqrt{2}}}^{e^2} \frac{\ln(x)}{x}\ dx = \frac{\ln(e^2)^2}{2} - \frac{\ln(e^{\sqrt{2}})^2}{2}$$
$$= \frac{2^2}{2} - \frac{\sqrt{2}^2}{2}$$
$$= 2 - 1$$
$$= 1$$

3. $\displaystyle\int_{-\infty}^{0} e^x\ dx$

First note $\displaystyle\int e^x\ dx = e^x + C$. Then

$$\int_{-\infty}^{0} e^x\ dx = e^0 - \lim_{x\to-\infty} e^x$$
$$= 1 - 0$$
$$= 1$$

4. $\displaystyle\int_2^\infty \frac{2x-1}{(x^2-x)^2}dx$

Use u-substitution to find $F(x)$. Let $u = x^2 - x$. Then

$$\int \frac{2x-1}{(x^2-x)^2}dx = \int \frac{1}{u^2}du$$

$$= \int u^{-2} du$$
$$= -u^{-1}$$
$$= -\frac{1}{x^2 - x}$$

and

$$\begin{aligned}\int_2^\infty \frac{2x-1}{(x^2-x)^2} dx &= \lim_{x\to\infty} -\frac{1}{x^2-x} + \frac{1}{2^2-2} \\ &= \lim_{x\to\infty} -\frac{1}{x^2-x} + \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

5. $\int_1^9 2y^5 dy$

$$\begin{aligned}\int_1^9 2y^5 dy &= \left[\frac{1}{3}y^6\right]_1^9 \\ &= \frac{1}{3}(9^6 - 1)\end{aligned}$$

6. $\int_{-1}^0 3x^2 - 1dx$

$$\begin{aligned}\int_{-1}^0 3x^2 - 1dx &= \left[x^3 - x\right]_{-1}^0 \\ &= (0-0) - ((-1)^3 - (-1)) \\ &= 0\end{aligned}$$

7. $\int_{-1}^1 14 + x^2 dx$

$$\begin{aligned}\int_{-1}^1 14 + x^2 dx &= \left[14x + \frac{1}{3}x^3\right]_{-1}^1 \\ &= (14 + \frac{1}{3}) - (14(-1) + \frac{1}{3}(-1)^3) \\ &= 28 + \frac{2}{3}\end{aligned}$$

8. $\int_1^{-1} 14 + x^2 dx$

$$
\begin{aligned}
\int_{1}^{-1} 14 + x^2 dx &= \left[14x + \frac{1}{3}x^3\right]_{1}^{-1} \\
&= (14(-1) + \frac{1}{3}(-1)^3) - (14 + \frac{1}{3}) \\
&= -28 - \frac{2}{3}
\end{aligned}
$$

3 Integration by Parts

1. $\int \frac{\ln(x)}{x^3}\ dx$

   Let $u = \ln(x)$ and $dv = x^{-3}\ dx$. Then $du = x^{-1}\ dx$ and $v = -0.5x^{-2}$, and

$$
\begin{aligned}
\int \frac{\ln(x)}{x^3}\ dx &= -0.5x^{-2}\ln(x) - \int -0.5x^{-2}x^{-1}\ dx \\
&= -0.5x^{-2}\ln(x) + 0.5\int x^{-3}\ dx \\
&= -0.5x^{-2}\ln(x) + 0.5(-0.5x^{-2}) \\
&= -0.5x^{-2}\ln(x) - 0.25x^{-2} \\
&= -0.25x^{-2}(2\ln(x) + 1) + c
\end{aligned}
$$

2. $\int x^2 e^x\ dx$

   Let $u = x^2$ and $dv = e^x\ dx$. Then $du = 2x\ dx$ and $v = e^x$, and

$$
\begin{aligned}
\int x^2 e^x\ dx &= x^2 e^x - \int 2xe^x\ dx \\
&= x^2 e^x - 2\int xe^x\ dx
\end{aligned}
$$

   Now let $u = x$ and $dv = e^x\ dx$. Then $du = dx$ and $v = e^x$, and

$$
\begin{aligned}
\int xe^x\ dx &= xe^x - \int e^x\ dx \\
&= xe^x - e^x
\end{aligned}
$$

   Therefore,

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - 2(xe^x - e^x) \\ &= e^x(x^2 - 2x + 2) + c\end{aligned}$$

3. $\int_1^e x \ln(x) \, dx$

Let $u = \ln(x)$ and $dv = x \, dx$. Then $du = x^{-1} \, dx$ and $v = 0.5x^2$, and

$$\begin{aligned}\int x \ln(x) \, dx &= 0.5x^2 \ln(x) - \int 0.5x^2 x^{-1} \, dx \\ &= 0.5x^2 \ln(x) - 0.5 \int x \, dx \\ &= 0.5x^2 \ln(x) - 0.25x^2 \\ &= 0.25x^2(2\ln(x) - 1)\end{aligned}$$

so

$$\begin{aligned}\int_1^e x \ln(x) \, dx &= 0.25e^2(2\ln(e) - 1) - 0.25(1^2)(2\ln(1) - 1) \\ &= 0.25e^2(2(1) - 1) - 0.25(2(0) - 1) \\ &= 0.25e^2 + 0.25\end{aligned}$$

4. $\int \frac{x^3}{(x^2+7)^2} \, dx$

Let $u = x^2$ and $dv = x(x^2+7)^{-2}$. Then $du = 2x \, dx$ and $v = -\frac{1}{2(x^2+7)}$, and

$$\begin{aligned}\int \frac{x^3}{(x^2+7)^2} \, dx &= -\frac{x^2}{2(x^2+7)} - \int -\frac{1}{2(x^2+7)} 2x \, dx \\ &= -\frac{x^2}{2(x^2+7)} - \int -\frac{x}{x^2+7} \, dx \\ &= -\frac{x^2}{2(x^2+7)} + 0.5 \ln|x^2+7| + c\end{aligned}$$

5. $\int (\ln(x))^2 \, dx$

Let $u = \ln(x)$ and $dv = \ln(x) \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = x\ln(x) - x$, and

$$\int (\ln(x))^2 \, dx = \ln(x)(x\ln(x) - x) - \int \frac{x\ln(x) - x}{x} \, dx$$

$$
\begin{aligned}
&= \ln(x)(x\ln(x) - x) - \int \ln(x) - 1 \; dx \\
&= \ln(x)(x\ln(x) - x) - (x\ln(x) - x - x) \\
&= x\ln(x)^2 - x\ln(x) - x\ln(x) + 2x \\
&= x\ln(x)^2 - 2x\ln(x) + 2x \\
&= x((\ln(x))^2 - 2\ln(x) + 2) + c
\end{aligned}
$$