Derivatives: Part 1

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Encounters with basic concepts from calculus will be a daily event in your methods training. In the case of derivatives, you will primarily be using them to:

- Find the maximum and minimum values of specific functions including utility and likelihood functions;
- Derive a probability density function given its cumulative density function

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- These concepts were developed in the 17th century to understand physical motion.
- If a function describes the location of a baseball flying through the air, the first derivative describes the velocity of the ball, and the second derivative represents the acceleration.

Defining derivatives

- The derivative of f at x is its rate of change at x — i.e., how much $f(x)$ changes with a change in x.
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- More formally, let f be a function whose domain includes an open interval containing the point x. The derivative of f at x is given by

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

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- Imagine that you want to calculate how a function changes between time period 1 and period 2. With a little thought we would calculate this as:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- This is same "rise over run" calculation we used to calculate the slope of a line from before.

Derivative (Version 1)

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- What we want to calculate is the "instantaneous" rate of change.
- That is, we want to know the rate of change as x_1 and x_2 get closer and closer together such that time period between when we measure them moves towards zero. This is the conceptual leap that it takes a genius like Newton to discover, but which we can accept with a bit of thought.
- So one way to think about this is:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Derivative (Version 2)

- We could also think of it this way

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- This is just a re-parameterized version of the same thing.

###Examples: Solve using version 1

$$f(x) = 3x^2 + 4x - 6$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

###Examples: Solve using version 2

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$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note on notation

For historical reasons, there are multiple notations for derivatives.

- (Prime or Lagrange Notation)
- (Leibniz's Notation)

Using either version, find the derivatives for the following functions. I need one person to volunteer to draw each function and its derivative.

1. $f(x) = c$
2. $f(x) = 2x$
3. $f(x) = x^2$
4. $f(x) = x^3$

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- The process of calculating $f'(x)$ is called **differentiation**.
- A function is *monotonically increasing* in a domain if it has a positive derivative over that domain. Likewise, it is *monotically decreasing* if it has a negative derivative.

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- The classic case where this “pencil rule” does not work are:
  - $$f(x) = |x|$$
  - $$f(x) = \frac{1}{x-1}$$
  - $$f(x) = 1, x \in [0, 1]$$

Determine whether or not the following functions are differentiable. If not, at what value of x do they fail?

1. $$f(x) = |x|$$

2. $$f(x) = \begin{cases} x^2 & x \leq 0 \\ -x & x < 0 \\ 0 & x = 0 \end{cases}$$

3. $$f(x) = \sqrt(x)$$

4. $$f(x) = x^{1/3}$$

5. $$f(x) = 1/x$$

Properties of derivatives

Suppose that f and g are differentiable at x and that α is a constant. Then the functions $f \pm g$, αf, fg, and f/g (provided $g(x) \neq 0$) are also differentiable at x.

Rules: Making derivatives easier

Doing this calculation for every function is cumbersome, and sometimes difficult. Fortunately, hundreds of mathematical graduate students and professors have developed several “rules” that help us calculate derivatives quickly. All of these rules come with proofs, but for our purposes we usually just memorize them and use them. The first (and easiest) is that the derivative of a constant is zero.

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- **Quotient rule:** $[f(x)/g(x)]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$

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$$f'(x) = (2x^7)' = 14x^6$$

Derivatives of exp

The derivative of the exp is worth just memorizing. This will make a bit more sense when we cover the chain rule, but for now:

- $\frac{d}{dx}\alpha e^x = \alpha e^x$
- $\frac{d^n}{dx^n}\alpha e^x = \alpha e^x$

Derivatives of exponential functions with other bases

$$\frac{d}{dx}(a^x) = (\ \ln(a)\ )\ a^x$$

Derivatives of the natural log

Same thing for the ln() function

- $\frac{d}{dx} \ln x = \frac{1}{x}$
- $\frac{d}{dx} \ln x^k = \frac{d}{dx} k \ln x = \frac{k}{x}$

Find the derivatives for the following functions:

1. $f(x) = 3x^{1/3}$
2. $f(t) = 14t - 7$
3. $f(y) = y^3 + 3y^2 - 12$
4. $f(x) = \frac{1}{100}x^{25} - \frac{1}{10}x^{0.25}$
5. $f(x) = (x^2 + 1)(x^3 - 1)$
6. $f(y) = (1 - 1/y^2)$
7. $f(y) = (y^3 - 7)(1 - 1/y^2)$
8. $f(x) = ln(2\pi x^2)$
9. $f(y) = (y - y^{-1})(y - y^{-2})$
10. $f(x) = x^6 + 5x^5 - 2x^2 + 8$

11. $g(y) = 3e^y - \sqrt{y}$
12. $h(z) = ln(z) + 1/z + 3^z$
13. $f(x) = (x+3)^7(3x^4 - 2x^2 - 8)$

For these functions, find the derivative at $x = 1$ and $x = 3$

14. $f(x) = 2x^2 + 7$
15. $f(x) = x^3 - x + 1$

Find the derivatives for the following functions:

16. $y = 27x^3 + 5x^2 - x + 13$
17. $y = 81x^2 + 10x - 1$
18. $y = 162x + 10$