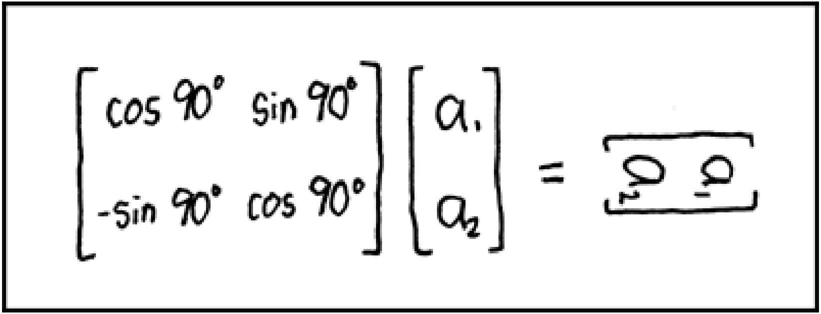
Linear Algebra: The Basics

David Carlson

2021

Linear Algebra: The Basics

The mighty matrix



Some motivation

The purpose of the next several lectures is to present some basic concepts and results concerning matrices, linear algebra, and vector geometry.

The reasons you are studying these areas are:

- matrices provide a natural notation for much of statistics, and almost all of your proofs in statistics classes will be in the form of linear algebra;

Some motivation

The purpose of the next several lectures is to present some basic concepts and results concerning matrices, linear algebra, and vector geometry.

The reasons you are studying these areas are:

- matrices provide a natural notation for much of statistics, and almost all of your proofs in statistics classes will be in the form of linear algebra;
- linear algebra is the most natural way to approach large sets of simultaneous equations.

Some motivation

The purpose of the next several lectures is to present some basic concepts and results concerning matrices, linear algebra, and vector geometry.

The reasons you are studying these areas are:

- matrices provide a natural notation for much of statistics, and almost all of your proofs in statistics classes will be in the form of linear algebra;
- linear algebra is the most natural way to approach large sets of simultaneous equations.
- vector spaces provide a powerful conceptual tool for understanding many aspects of both linear models and several families of game theoretic models. (Although I am not covering this.)

- At the most basic level, we use matrix algebra because a dataset that looks like this:

Nation	Taxation in 1981	Taxation in 1995
Australia	30	31
Austria	44	42
$\vdots$	$\vdots$	$\vdots$
United States	29	28

- At the most basic level, we use matrix algebra because a dataset that looks like this:

Nation	Taxation in 1981	Taxation in 1995
Australia	30	31
Austria	44	42
$\vdots$	$\vdots$	$\vdots$
United States	29	28

Can also look like this:

$$\mathbf{A} = \begin{pmatrix} 30 & 31 \\ 44 & 42 \\ \vdots & \vdots \\ 29 & 28 \end{pmatrix}$$

Goals

- The goal for these lecture is to get you familiar with many of the most basic concepts of linear algebra.
- The focus is on how to think about matrices and how to extend basic algebraic concepts to matrices.

Goals

- The goal for these lecture is to get you familiar with many of the most basic concepts of linear algebra.
- The focus is on how to think about matrices and how to extend basic algebraic concepts to matrices.
- In your graduate studies you will almost *never* actually do any of these calculations yourself. Except for the kinds of toy examples we study here, matrix algebra will be done by your computer.

Goals

- The goal for these lecture is to get you familiar with many of the most basic concepts of linear algebra.
- The focus is on how to think about matrices and how to extend basic algebraic concepts to matrices.
- In your graduate studies you will almost *never* actually do any of these calculations yourself. Except for the kinds of toy examples we study here, matrix algebra will be done by your computer.
- However, *you still need to understand what your computer is doing*.

Working with vectors

- **Vector**: A vector in n-space is an ordered list of n numbers.
- These numbers can be represented as either a row vector or a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Working with vectors

- **Vector**: A vector in n-space is an ordered list of n numbers.
- These numbers can be represented as either a row vector or a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- We can also think of a vector as defining a point in n-dimensional space, usually $\mathbf{R}^n$; each element of the vector defines the coordinate of the point in a particular direction.

Vector addition

- Vector addition is defined for two vectors **u** and **v** iff they have the same number of elements:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 & \cdots & u_k + v_n \end{pmatrix}$$

Scalar Multiplication

- The product of a scalar c and vector $\mathbf{v}$ is:

$$c\mathbf{v} = \begin{pmatrix} cv_1 & cv_2 & \dots & cv_n \end{pmatrix}$$

Scalar Multiplication

The product of a scalar c and vector $\mathbf{v}$ is:

$$c\mathbf{v} = \begin{pmatrix} cv_1 & cv_2 & \dots & cv_n \end{pmatrix}$$

Vector Inner Product

- The inner product (also called the dot product or scalar product) of two vectors $\mathbf{u}$ and $\mathbf{v}$ is again defined iff they have the same number of elements

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Vector Inner Product

- The inner product (also called the dot product or scalar product) of two vectors **u** and **v** is again defined iff they have the same number of elements

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

- If $\mathbf{u} \cdot \mathbf{v} = 0$, the two vectors are **orthogonal** (or perpendicular).

Vector Norm

- The norm of a vector is a measure of its length.

Vector Norm

- The norm of a vector is a measure of its length.
- There are many different norms, the most common of which is the Euclidean norm (which corresponds to our usual conception of distance in three-dimensional space):

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1 v_1 + v_2 v_2 + \cdots + v_n v_n}$$

Vector Norm

- The norm of a vector is a measure of its length.
- There are many different norms, the most common of which is the Euclidean norm (which corresponds to our usual conception of distance in three-dimensional space):

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1 v_1 + v_2 v_2 + \cdots + v_n v_n}$$

- This would be how far away some point described by **v** is from the origin.

Example

$$||\mathbf{v}|| \quad = \quad || \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} || \tag{1}$$

Example

$$
\begin{aligned}
||\mathbf{v}|| &= ||\begin{pmatrix} 2 & 0 & 1 \end{pmatrix}|| & (1) \\
&= \sqrt{2^2 + 0^2 + 1^2} & (2)
\end{aligned}
$$

Example

$$
\begin{aligned}
||\mathbf{v}|| &= ||\begin{pmatrix} 2 & 0 & 1 \end{pmatrix}|| && (1)\\
&= \sqrt{2^2 + 0^2 + 1^2} && (2)\\
&= \sqrt{5} && (3)
\end{aligned}
$$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$
- Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$
- Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$
- Difference Norm: $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$
- Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$
- Difference Norm: $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$
  $= ||\mathbf{u}||^2 - 2||\mathbf{u}|| \cdot ||\mathbf{v}||(\cos\theta) + ||\mathbf{v}||^2$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$
- Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$
- Difference Norm: $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$
  $= ||\mathbf{u}||^2 - 2||\mathbf{u}|| \cdot ||\mathbf{v}||(\cos\theta) + ||\mathbf{v}||^2$
- Triangle Inequality: $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$
- Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$
- Difference Norm: $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$
  $= ||\mathbf{u}||^2 - 2||\mathbf{u}|| \cdot ||\mathbf{v}||(\cos\theta) + ||\mathbf{v}||^2$
- Triangle Inequality: $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$
- Cauchy-Schwartz Inequality: $||\mathbf{u} \cdot \mathbf{v}|| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$

Let $\mathbf{a} = \begin{pmatrix} 10 \\ 2 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 15 \\ 6 \\ 8 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 2 & 6 & 8 \end{pmatrix}$, and $\mathbf{d} = \left(1, 15, 12\right)$.

Calculate each of the following (indicate ones that cannot be done).

1. $\mathbf{a} + \mathbf{b}$
2. $\mathbf{a} + \mathbf{c}$
3. $\mathbf{c} - \mathbf{d}$
4. $15\mathbf{c}$
5. $||\mathbf{b}||$
6. $||\mathbf{c} + \mathbf{d}||$
7. $||\mathbf{c} - \mathbf{d}||$
8. $\mathbf{a} \cdot \mathbf{b}$
9. $\mathbf{c} \cdot \mathbf{d}$
10. $||\mathbf{c} \cdot \mathbf{d}||$

Matrix algebra

- **Matrix**: A matrix is an array of $m \times n$ real numbers arranged in m rows by n columns.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Note that you can think of vectors as special cases of matrices; a column vector of length k is a $k \times 1$ matrix, while a row vector of the same length is a $1 \times k$ matrix.
- We describe the above vector as an "m by n" matrix.

Thinking about a matrix

- You can also think of larger matrices as being made up of a collection of *column vectors*. For example,

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{pmatrix}$$

$$\mathbf{a}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \cdots \\ a_{im} \end{pmatrix}$$

- Alternatively, we can think of a matrix as a set of *row vectors*. For example,

$$\mathbf{A} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$$

  where

$$\mathbf{b}_j = \begin{pmatrix} b_{1j} & b_{2j} & \dots & b_{nj} \end{pmatrix}$$

- It is also possible to partion the matrix into subsets of columns and rows or subsections of each. For example,

$$\underset{(4\times 3)}{\mathbf{A}} = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \hline a_{41} & a_{42} & a_{43} \end{array}\right) = \begin{pmatrix} \underset{(3\times 2)}{\mathbf{A}} & \underset{(3\times 1)}{\mathbf{A}} \\ \underset{(1\times 2)}{\mathbf{A}} & \underset{(1\times 1)}{\mathbf{A}} \end{pmatrix}$$

Matrix Addition

- Let **A** and **B** be two $m \times n$ matrices. Then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

- Note that matrices **A** and **B** must be the same size, in which case they are **conformable for addition**.

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 4 \\ 6 & 6 & 8 \end{pmatrix}$$

Scalar Multiplication

- Given the scalar s, the scalar multiplication of $s\mathbf{A}$ is

$$s\mathbf{A} = s\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} sa_{11} & sa_{12} & \cdots & sa_{1n} \\ sa_{21} & sa_{22} & \cdots & sa_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ sa_{m1} & sa_{m2} & \cdots & sa_{mn} \end{pmatrix}$$

Example:

$$s = 2, \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$s\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

Matrix Multiplication

- If $\mathbf{A}$ is an $m \times k$ matrix and $\mathbf{B}$ is a $k \times n$ matrix, then their product $\mathbf{C} = \mathbf{AB}$ is the $m \times n$ matrix where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Examples:

1. $$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}$$

Examples:

1. $$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}$$

2. $$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} 1(-2) + 2(4) - 1(2) & 1(5) + 2(-3) - 1(1) \\ 3(-2) + 1(4) + 4(2) & 3(5) + 1(-3) + 4(1) \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 16 \end{pmatrix}$$

Conformability

- Note that the number of columns of the first matrix must equal the number of rows of the second matrix, in which case they are **conformable for multiplication**.
- The sizes of the matrices (including the resulting product) must be

$$(m \times k)(k \times n) = (m \times n)$$

- Given $\mathbf{AB}$, say $\mathbf{B}$ is pre-multiplied by $\mathbf{A}$

Conformability

- Note that the number of columns of the first matrix must equal the number of rows of the second matrix, in which case they are **conformable for multiplication**.
- The sizes of the matrices (including the resulting product) must be
$$(m \times k)(k \times n) = (m \times n)$$
- Given $\mathbf{AB}$, say $\mathbf{B}$ is pre-multiplied by $\mathbf{A}$, $\mathbf{B}$ is left-multiplied by $\mathbf{A}$

Conformability

- Note that the number of columns of the first matrix must equal the number of rows of the second matrix, in which case they are **conformable for multiplication**.
- The sizes of the matrices (including the resulting product) must be

$$(m \times k)(k \times n) = (m \times n)$$

- Given $\mathbf{AB}$, say $\mathbf{B}$ is pre-multiplied by $\mathbf{A}$, $\mathbf{B}$ is left-multiplied by $\mathbf{A}$, $\mathbf{A}$ is post-multiplied by $\mathbf{B}$

Conformability

- Note that the number of columns of the first matrix must equal the number of rows of the second matrix, in which case they are **conformable for multiplication**.
- The sizes of the matrices (including the resulting product) must be

$$(m \times k)(k \times n) = (m \times n)$$

- Given $\mathbf{AB}$, say $\mathbf{B}$ is pre-multiplied by $\mathbf{A}$, $\mathbf{B}$ is left-multiplied by $\mathbf{A}$, $\mathbf{A}$ is post-multiplied by $\mathbf{B}$, $\mathbf{A}$ is right-multiplied by $\mathbf{B}$

What does it all mean?

- Let's pause for a moment to think about why matrix multiplication works this way.
- The idea is to find ways to reduce large sets of equations into easy to handle objects.
- For instance, imagine we have a set of two linear equations with two variables. Each of these two equations represents a line (hence "linear") in a two-dimensional space.

$$2x_1 + 5x_2 = 4 \tag{4}$$

$$x_1 + 3x_2 = 5 \tag{5}$$

- We can re-write these equations as:

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\underset{(2\times 2)}{\mathbf{A}}\underset{(2\times 1)}{\mathbf{x}} = \underset{(2\times 1)}{\mathbf{b}}$$

- Now we can solve this system of equations for **x** using matrix manipulation rather than solving the set of equations line by line.
- Solving line by line isn't so difficult with the toy examples in your book and in these notes. But even with reasonably sized datasets, dealing with data in anything but matrix form becomes increasingly difficult and computationally intensive.

Laws of Matrix Algebra

1. Associative:
$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$
$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$$
2. Commutative:
$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$
3. Distributive:
$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}$$

- Note that Commutative law for multiplication does not necessarily hold – the order of multiplication matters in many cases:

$$\mathbf{AB} \neq \mathbf{BA}$$

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -2 & 2 \end{pmatrix}, \qquad \mathbf{BA} = \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix}$$

The Transpose

The transpose of the $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}^T$ (sometimes written $\mathbf{A}'$) obtained by interchanging the rows and columns of $\mathbf{A}$.

Examples:

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}, \qquad \mathbf{A}^T = \begin{pmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad \mathbf{B}^T = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$$

Rules for transposes

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{A}^T)^T = \mathbf{A}$
3. $(s\mathbf{A})^T = s\mathbf{A}^T$
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Example

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix}$$

Example

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix}$$

$$(\mathbf{AB})^T = \left[\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix}\right]^T = \begin{pmatrix} 12 & 7 \\ 5 & -3 \end{pmatrix}$$

Example

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix}$$

$$(\mathbf{AB})^T = \left[\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix} \right]^T = \begin{pmatrix} 12 & 7 \\ 5 & -3 \end{pmatrix}$$

$$\mathbf{B}^T\mathbf{A}^T = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 7 \\ 5 & -3 \end{pmatrix}$$

Given the followign matrices, perform the calculations below.

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 \\ 6 & 2 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 4 & 5 \\ -2 & -3 & 6 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2 \\ -5 & 4 \\ -3 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

1. $\mathbf{A} + \mathbf{C}'$
2. $(\mathbf{A} - \mathbf{B})^T$
3. $\mathbf{A} + 5\mathbf{B}$
4. $2\mathbf{B} - 5\mathbf{A}$
5. $\mathbf{B}^T - \mathbf{C}$

6. $\mathbf{BA}$
7. $(\mathbf{DA})^T$
8. $\mathbf{AD}$
9. $\mathbf{CD}$
10. $\mathbf{BC}$
11. $\mathbf{CB}$