Linear Equations 1: Solving systems of equations

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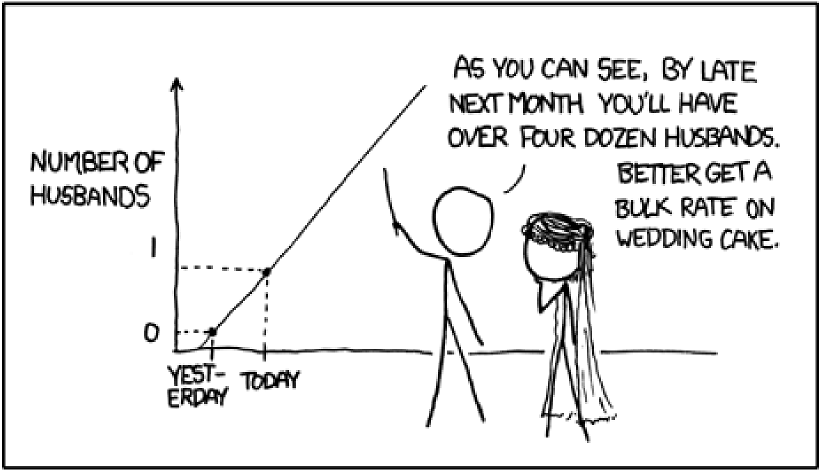
MY HOBBY: EXTRAPOLATING



Thinking about lines and equations

- Another way that matrices are used is to solve large sets of linear equations.
- For small numbers of equations (e.g., two equations and two unknowns) we can use the very simple approaches that you might remember from high-school algebra.
- However, even for modest numbers of equations this becomes more difficult and matrix representations are preferred.
- In addition, in many situations (especially in game theory) our interest is not so much in calculating a solution to a specific set of equations, but rather in specifying the circumstances under which such a solution will exist.

- The goals for today are to help you understand:
    1. How to think about systems of equations using a matrix;
    2. How to figure out if sets of equations have a solution;
    3. The relationship between matrix inversion and solutions to sets of simultaneous equations;
    4. How to find solutions (and inverses).

Linear Equations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

- a_i are parameters or coefficients.
- x_i are variables or unknowns.
- Linear because only one variable per term and degree is at most 1.

- Geometrically, these equations can be thought of as describing
  1. $\mathbf{R}^2$: line
  $$x_2 = \frac{b}{a_2} - \frac{a_1}{a_2} x_1$$
  2. $\mathbf{R}^3$: plane
  $$x_3 = \frac{b}{a_3} - \frac{a_1}{a_3} x_1 - \frac{a_2}{a_3} x_2$$
  3. $\mathbf{R}^n$: hyperplane

Thinking about solutions to systems of equations

- We are often interested in solving linear systems like
$$\begin{array}{rcrcr} 1x_1 & - & 3x_2 & = & -3 \\ 2x_1 & + & 1x_2 & = & 8 \end{array}$$
- More generally, we might have a system of m equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

- A **solution** to a linear system of m equations in n unknowns is a set of n numbers $x_1, x_2, \cdots, x_n$ that satisfy each of the m equations.
    1. $\mathbf{R}^2$: intersection of the lines.
    2. $\mathbf{R}^3$: intersection of the planes.
    3. $\mathbf{R}^n$: intersection of the hyperplanes.
- Note that if there is only one solution to this system, then there must be only one point in space where these intersections overlap.

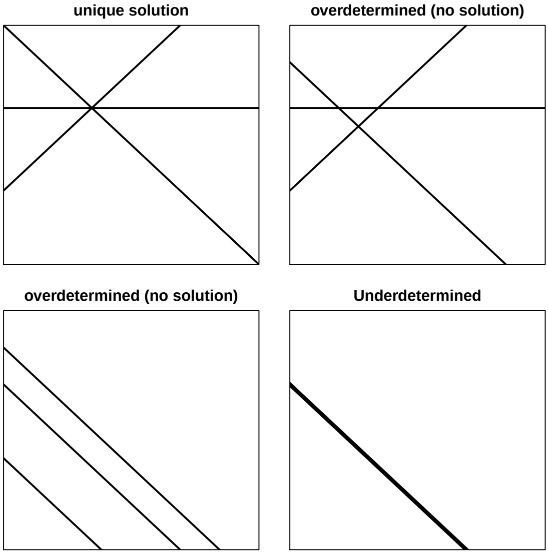
Example

- $$\begin{array}{rcrcr} 1x_1 & - & 3x_2 & = & -3 \\ 2x_1 & + & 1x_2 & = & 8 \end{array}$$
- $x_1 = 3$ and $x_2 = 2$ is the solution to the above 2×2 linear system.

One solution, many, or none?

- Does a linear system have one, no, or multiple solutions?
- For a system of 2 equations in 2 unknowns (i.e., two lines):
  - **One solution:** The lines intersect at exactly one point.
  - **No solution:** The lines are parallel.
  - **Infinite solutions:** The lines coincide.

Visualizing possibilities with three lines



Solving systems of equations

- Methods to solve linear systems:
  1. Substitution
  2. Elimination of variables
  3. Matrix methods

Substitution

- Procedure:
  1. Solve one equation for one variable, say x_1, in terms of the other variables in the equation.
  2. Substitute the expression for x_1 into the other $m-1$ equations, resulting in a new system of $m-1$ equations in $n-1$ unknowns.
  3. Repeat steps 1 and 2 until there is only one equation left in terms of one unknown (say x_n). We now have a value for x_n.
  4. Backward substitution: Substitute x_n into the previous equation (which should be a function of only x_n). Repeat this, usingthe successive expressions of each variable in terms of the other variables, to find the values of all x_i's.

Exercise 1

- Using substitution, solve:

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

Exercise 2

- Using substitution, solve

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & - & 3y & + & 2z & = & 14 \\ 3x & + & y & - & z & = & -2 \end{array}$$

Elimination

- The second way to solve a system of equations is elimination.
- Before we discuss this, however, we need to go over *equation operations*.

- Elementary equation operations are used to transform the equations of a linear system, while maintaining an **equivalent** linear system.
- Equivalent in the sense that the same values of x_j solve both the original and transformed systems.
- These operations are:
  1. Interchanging two equations,
  2. Multiplying two sides of an equation by a constant, and
  3. Adding equations to each other.

Interchanging equations

- Given the linear system

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

we can interchange its equations, resulting in the equivalent linear system

$$\begin{array}{rcrcl} a_{21}x_1 & + & a_{22}x_2 & = & b_2 \\ a_{11}x_1 & + & a_{12}x_2 & = & b_1 \end{array}$$

Multiplying by a constant

- Suppose we had the following equation:

$$2 = 2$$

If we multiply each side of the equation by some number, say 4, we still have an equality:

$$2(4) = 2(4) \quad \Longrightarrow \quad 8 = 8$$

- More generally, we can multiply both sides of any equation by a constant and maintain an equivalent equation.
- For example, the following two equations are equivalent:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ ca_{11}x_1 + ca_{12}x_2 &= cb_1 \end{aligned}$$

Addition equations

- Suppose we had the following two very simple equations:

$$\begin{aligned} 3 &= 3 \\ 7 &= 7 \end{aligned}$$

- If we add these two equations to each other, we get

$$7 + 3 = 7 + 3 \implies 10 = 10$$

- Suppose we now have

$$\begin{aligned} a &= b \\ c &= d \end{aligned}$$

- If we add these two equations to each other, we get

$$a + c = b + d$$

- Extending this, suppose we had the linear system

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

- If we add these two equations to each other, we get

$$(a_{11} + a_{21})x_1 + (a_{12} + a_{22})x_2 = b_1 + b_2$$

Elimination of variables via Gaussian Elimination

- **Gaussian Elimination** is a method by which we start with some linear system of m equations in n unknowns and use the elementary equation operations to eliminate variables, until we arrive at an equivalent system of the form

$$\begin{array}{ccccccccc}
\mathbf{a'_{11}}x_1 & + & a'_{12}x_2 & + & \cdots & + & a'_{1n}x_n & = & b'_1 \\
 & & \mathbf{a'_{22}}x_2 & + & \cdots & + & a'_{2n}x_n & = & b'_2 \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & \mathbf{a'_{mn}}x_n & = & b'_m
\end{array}$$

- a'_{ij} denotes the coefficient of the jth unknown in the ith equation after the above transformation.
- Note that at each stage of the elimination process, we want to change some coefficient of our system to 0 by adding a multiple of an earlier equation to the given equation.
- Once the linear system is in the above reduced form, we then use back substitution to find the values of the x_j's.

- The bolded coefficients $\mathbf{a}'_{11}$, $\mathbf{a}'_{22}$, etc in boxes are referred to as **pivots**, since they are the terms used to eliminate the variables in the rows below them in their respective columns.
- As we'll see, pivots don't need to be on the $ij, i = j$ diagonal. Additionally, sometimes when we pivot, we will eliminate variables in rows above a pivot.

Example 1

- Using Gaussian elimination, solve

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

Example 2

- Using Gaussian elimination, solve

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & - & 3y & + & 2z & = & 14 \\ 3x & + & y & - & z & = & -2 \end{array}$$

Gauss-Jordan Elimination

- The method of **Gauss-Jordan elimination** takes the Gaussian elimination method one step further.
- Once the linear system is in the reduced form shown in the preceding section, elementary row operations and Gaussian elimination are used to
  1. Change the coefficient of the pivot term in each equation to 1 and
  2. Eliminate all terms above each pivot in its column,
- The result is a reduced, equivalent system.

- For a system of m equations in m unknowns, a typical reduced system would be

$$\begin{array}{ccccccc} \boxed{x_1} & & & & & = & b_1^* \\ & \boxed{x_2} & & & & = & b_2^* \\ & & \boxed{x_3} & & & = & b_3^* \\ & & & \ddots & & & \vdots \\ & & & & \boxed{x_n} & = & b_m^* \end{array}$$

which needs no further work to solve for the x_j's.

Example 1

- Using Gauss-Jordan elimination, solve

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

Example 2

- Using Gaussian-Jordan elimination, solve

$$
\begin{array}{rcrcrcr}
x & + & 2y & + & 3z & = & 6 \\
2x & - & 3y & + & 2z & = & 14 \\
3x & + & y & - & z & = & -2
\end{array}
$$

Solving systems of equations using matrix algebra

- This is all well and good, but it can be very cumbersome.
- Matrices are an efficient way to represent linear systems such as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

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\vdots & & & & \vdots & & & \vdots & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

as $\mathbf{Ax} = \mathbf{b}$

Breaking it down: The Coefficient Matrix

- The $m \times n$ **coefficient matrix A** is an array of mn real numbers arranged in m rows by n columns:

Breaking it down: The Coefficient Matrix

- The $m \times n$ **coefficient matrix A** is an array of mn real numbers arranged in m rows by n columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Breaking it down: The Variable Vector

- The unknown quantities are represented by the vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

.

Breaking it down: The Variable Vector

- The right hand side of the linear system is represented by the vector

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Using matrix algebra to solve equations

- We are going to set up an augmented matrix.
- We are going to use elementary row operations to achieve row echelon form (or reduced row echelon form).
- We are going to do the entire thing over simply using the matrix inverse notation.

Augmented matrix

- When we append $\mathbf{b}$ to the coefficient matrix $\mathbf{A}$, we get the augmented matrix $\widehat{\mathbf{A}} = [\mathbf{A}|\mathbf{b}]$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Elementary Row Operations

- Just as we conducted elementary equation operations, we can conduct elementary row operations to transform some augmented matrix representation of a linear system into another augmented matrix that represents an equivalent linear system.
- Since we're really operating on equations when we operate on the rows of the matrix, these row operations correspond exactly to the equation operations:

1. Interchanging two rows.
   $\Longrightarrow$
   Interchanging two equations.
2. Multiplying a row by a constant.
   $\Longrightarrow$
   Multiplying both sides of an equation by a constant.
3. Adding two rows to each other.
   $\Longrightarrow$
   Adding two equations to each other.

Interchanging Rows

- Suppose we have the augmented matrix

$$\widehat{\mathbf{A}} = \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$$

- If we interchange the two rows, we get the augmented matrix

$$\left(\begin{array}{cc|c} a_{21} & a_{22} & b_2 \\ a_{11} & a_{12} & b_1 \end{array}\right)$$

- This represents a linear system equivalent to that represented by matrix $\widehat{\mathbf{A}}$.

Multiplying by a constant

- If we multiply the second row of matrix $\widehat{\mathbf{A}}$ by a constant c, we get the augmented matrix

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ ca_{21} & ca_{22} & cb_2 \end{array}\right)$$

- This represents a linear system equivalent to that represented by matrix $\widehat{\mathbf{A}}$.

Adding Rows

- If we add the first row of matrix $\widehat{\mathbf{A}}$ to the second, we obtain the augmented matrix

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{11}+a_{21} & a_{12}+a_{22} & b_1+b_2 \end{array}\right)$$

- This represents a linear system equivalent to that represented by matrix $\widehat{\mathbf{A}}$.

Row Echelon Form

- We use the row operations to change coefficients in the augmented matrix to 0 — i.e., pivot to eliminate variables — and to put it in a matrix form representing the final linear system of Gaussian elimination.
- An augmented matrix of the form

$$\left(\begin{array}{cccccc|c} \boxed{a'_{11}} & a'_{12} & a'_{13} & \cdots & a'_{1n} & & b'_1 \\ 0 & \boxed{a'_{22}} & a'_{23} & \cdots & a'_{2n} & & b'_2 \\ 0 & 0 & \boxed{a'_{33}} & \cdots & a'_{3n} & & b'_3 \\ 0 & 0 & 0 & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \boxed{a'_{mn}} & & b'_m \end{array}\right)$$

  is said to be in row echelon form — each row has more leading zeros than the row preceding it.

Reduced Row Echelon Form

- Reduced row echelon form is the matrix representation of a linear system after Gauss-Jordan elimination.
- For a system of m equations in m unknowns, with no all-zero rows, the reduced row echelon form would be

$$\left(\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & 0 & b_1^* \\ 0 & \boxed{1} & 0 & 0 & 0 & b_2^* \\ 0 & 0 & \boxed{1} & 0 & 0 & b_3^* \\ 0 & 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & \boxed{1} & b_m^* \end{array}\right)$$

Examples

- Using matrix methods, solve the following linear system by Gaussian elimination and then Gauss-Jordan elimination:

1.

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

2.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & - & 3y & + & 2z & = & 14 \\ 3x & + & y & - & z & = & -2 \end{array}$$

Solving systems of equations using matrix inversion

- Think again about the matrix representation of a linear system

$$\mathbf{Ax} = \mathbf{b}$$

- If $\mathbf{A}$ is an $n \times n$ matrix,then $\mathbf{Ax} = \mathbf{b}$ is a system of n equations in n unknowns.

- Suppose $\mathbf{A}$ is nonsingular $\implies$ $\mathbf{A}^{-1}$ exists.
- To solve this system, we can premultiply each side by $\mathbf{A}^{-1}$ and reduce it as follows:

$$\begin{aligned}\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) &= \mathbf{A}^{-1}\mathbf{b}\\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}\\ \mathbf{I}_n\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}\\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}\end{aligned}$$

- Hence, given $\mathbf{A}$ and $\mathbf{b}$ and given that $\mathbf{A}$ is nonsingular, then $\mathbf{x} = \mathbf{A}^{-\mathbf{1}}\mathbf{b}$ is a unique solution to this system.
- Notice also that the requirements for $\mathbf{A}$ to be nonsingular correspond to the requirements for a linear system to have a unique solution: rows $\mathbf{A}$ = cols $\mathbf{A}$.

1. Solve the following system of equations using gaussian elimination and matrix inversion

$$\begin{array}{rcrcr} 2x & + & 3y & = & 4 \\ 5x & + & 5y & = & 3 \end{array}$$

2. Solve the following system of equations using Gauss-Jordan elimination

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 2 \\ 3x & - & 2y & + & z & = & 1 \\ & & y & - & z & = & 3 \end{array}$$

3. Solve the following system of equations using matrix inversion

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & -8 \\ x & + & 2y & - & z & = & 2 \\ -x & - & 4y & + & z & = & -6 \end{array}$$

4. Solve the following system of equations using matrix inversion

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 2 \\ 4x & + & y & - & 2z & = & 10 \\ x & - & 3y & + & z & = & 0 \end{array}$$