Koç CSSM Bootcamp 2021 **Problem Set 5**

Homework X

NAME: ANSWER KEY

For the following exercises, read the problems carefully and show all your work. Attach more pages if necessary. Avoid using a calculator or the computer to solve the exercises. Please, staple your homework.

1 Limits

Solve the following limits or show they do not exist.

1. $\lim_{x\to\infty} \dfrac{e}{x}$

$$\begin{aligned}\lim_{x\to\infty} \frac{e}{x} &= \lim_{x\to\infty} e \cdot \lim_{x\to\infty} \frac{1}{x} \\ &= e \cdot 0 \\ &= 0\end{aligned}$$

2. $\lim_{x\to-\infty} \dfrac{e}{x}$

$$\begin{aligned}\lim_{x\to-\infty} \frac{e}{x} &= \lim_{x\to-\infty} e \cdot \lim_{x\to-\infty} \frac{1}{x} \\ &= e \cdot 0 \\ &= 0\end{aligned}$$

3. $\lim_{x\to 3} \dfrac{x}{x^3-27}$

$$\begin{aligned}\lim_{x\to 3} \frac{x}{x^3-27} &= \frac{\lim_{x\to 3} x}{\lim_{x\to 3} x^3 - 27} \\ &= \frac{3}{0},\end{aligned}$$

so we see this function has a vertical asymptote at 3. The student may show either graphically or numerically that $f(x)$ becomes arbitrarily large and positive when approaching 3 from above and becomes arbitrarily large and negative when approaching 3 from below; therefore $\lim_{x\to 3+} \neq \lim_{x\to 3-}$ and the limit $\lim_{x\to 3} \dfrac{x}{x^3-27}$ does not exist.

4. $\lim_{x\to\infty} \dfrac{x}{x^3-27}$

$$\lim_{x\to\infty} \frac{x}{x^3-27} = \lim_{x\to\infty} \frac{x^3 x^{-2}}{x^3(1-27x^{-3})}$$

$$= \lim_{x\to\infty} \frac{x^{-2}}{1-27x^{-3}}$$

$$= \frac{0}{1-0}, \text{ since } \lim_{x\to\infty} x^a = 0 \text{ for } a < 0$$

$$= 0$$

5. $\lim_{x\to 3} \dfrac{x-3}{x^3-27}$

   The student could apply L'Hospital's rule:

$$\lim_{x\to 3} \frac{x-3}{x^3-27} = \lim_{x\to 3} \frac{\frac{d}{d(}x-3)}{\frac{d}{d(}x^3-27)}$$

$$= \lim_{x\to 3} \frac{1}{3x^2}$$

$$= \frac{1}{27}$$

   Or they could factor first, leading directly to the solution:

$$\lim_{x\to 3} \frac{x-3}{x^3-27} = \lim_{x\to 3} \frac{x-3}{(x-3)(x^2+3x+9)}$$

$$= \lim_{x\to 3} \frac{1}{x^2+3x+9}$$

$$= \frac{1}{27}$$

6. $\lim_{x\to\infty} \dfrac{x+1}{2x}$

$$\lim_{x\to\infty} \frac{x+1}{2x} = \lim_{x\to\infty} \frac{1}{2}(1+1/x)$$

$$= \lim_{x\to\infty} \frac{1}{2} + \lim_{x\to\infty} \frac{1}{x}$$

$$= \frac{1}{2} + 0$$

$$= \frac{1}{2}$$

7. $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x$

$$\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = 0$$

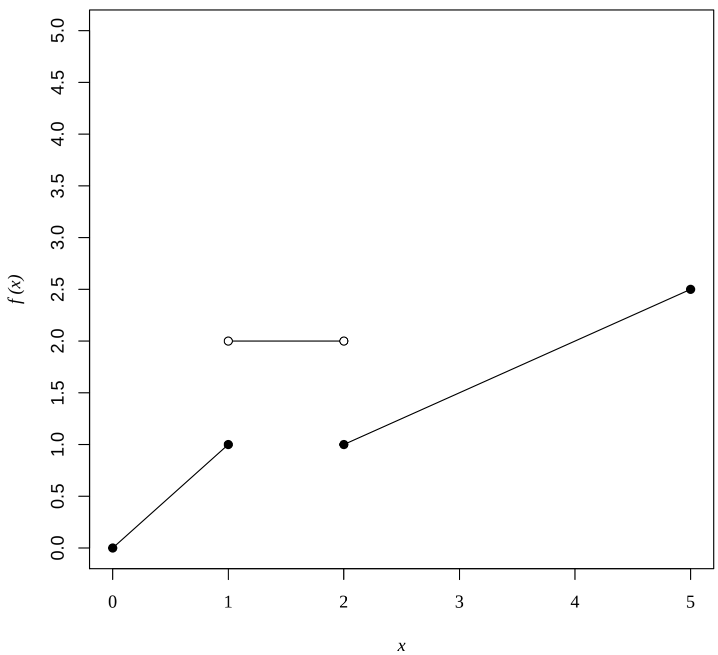
8. $\lim_{x\to\infty} \dfrac{3x^3 + 2x^2 - x + 3}{4x^4 + 3x^3 + 2x^2 + x + 4}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^3 + 2x^2 - x + 3}{4x^4 + 3x^3 + 2x^2 + x + 4} &= \lim_{x\to\infty} \frac{x^4(3x^{-1} + 2x^{-2} - x^{-3} + 3x^{-4})}{x^4(4 + 3x^{-1} + 2x^{-2} + 4x^{-4} + 4x^{-4})} \\
&= \lim_{x\to\infty} \frac{3x^{-1} + 2x^{-2} - x^{-3} + 3x^{-4}}{4 + 3x^{-1} + 2x^{-2} + 4x^{-4} + 4x^{-4}} \\
&= \frac{0}{4}, \text{ since } \lim_{x\to\infty} x^a = 0 \text{ for } a < 0 \\
&= 0
\end{aligned}$$

9. $\lim_{x\to 0} \dfrac{1}{x^2}$

   Since $\lim_{x\to 0} x^2 = 0$, there is a vertical asymptote at $x = 0$. However, unlike with exercise three, the function becomes arbitrarily large and positive when approaching $x = 0$ from either side. (This can be shown either numerically or graphically). Therefore the limit exists and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$.

10. The limit as x approaches (a) 1, (b) 2, and (c) 5, for the following function defined on $x \in [0, 5]$:



(a) The limit at $x = 1$ does not exist as $\lim_{x\to1+} = 2$ and $\lim_{x\to1-} = 1$.

(b) The limit at $x = 2$ does not exist as $\lim_{x\to1+} = 1$ and $\lim_{x\to1-} = 2$.

(c) The limit at $x = 5$ is 2.5.

2 Continuity

Identify which of the following functions are continuous. For functions that are not continuous, identify the points of discontinuity.

1. $f(x) = x^2$

   As a polynomial, this function is continuous.

2. $f(x) = \dfrac{1}{x}$

   This function has a discontinuity at 0.

3. $f(x) = \dfrac{x-3}{x^3-27}$

   This function has a removable discontinuity at 3.

4. $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ x & \text{for } x \geq 1 \end{cases}$

   This function is continuous.

5. The function depicted below:

   (Plot omitted)

   This function has discontinuities at 1 and 2.

3 Sequences

For each of the following cases, state whether the sequence $\{u_n\}$ converges to a limit, and if so, find the limit:

Honestly, in hindsight this section was a little unfair given what they'd learned about sequences in class.

1. ∞ no limit
2. $-\infty$ no limit
3. 0
4. 0
5. ∞ no limit
6. 4
7. no limit