Linear Equations 2: Rank, Singularity, and Definiteness

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Linear Equations 2: Rank, Singularity, and Definiteness

Rank

- We previously noted that a 2×2 system had one, infinite, or no solutions if the two lines intersected, were the same, or were parallel, respectively.
- We want to be able to characterize a matrix to determine which case we have before us.
- More generally, to determine whether one, infinite, or no solutions exist, we can use information about:
  1. The number of equations m (the number of rows),
  2. the number of unknowns n (the number of columns), and
  3. the **rank** of the matrix representing the linear system.

Defining Rank

- The **rank** of a matrix is the number of nonzero rows in its row echelon form.
- The rank corresponds to the maximum number of linearly independent row or column vectors in the matrix.
- It also corresponds to the number of dimensions of the *column space* and the *vector space* (the vector space spanned by the column and row vectors respectively).

Examples

1. Rank=3

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

2. Rank=2

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

Examples

1. Rank=3

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

2. Rank=2

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Rank=3

$$\begin{pmatrix} 1 & 2 & 3 & | & b_1 \\ 0 & 4 & 5 & | & b_2 \\ 0 & 0 & 0 & | & b_3 \end{pmatrix}, \quad b_i \neq 0$$

Basic properties of rank

- Let $\mathbf{A}$ be the coefficient matrix and $\widehat{\mathbf{A}} = [\mathbf{A}|\mathbf{b}]$ be the augmented matrix. Then
  1. $\operatorname{rank}\mathbf{A} \leq \operatorname{rank}\widehat{\mathbf{A}}$

     Augmenting $\mathbf{A}$ with $\mathbf{b}$ can never result in more zero rows than originally in $\mathbf{A}$ itself. Suppose row i in $\mathbf{A}$ is all zeros and that b_i is non-zero. Augmenting $\mathbf{A}$ with $\mathbf{b}$ will yield a non-zero row i in $\widehat{\mathbf{A}}$.

2. $\operatorname{rank} \mathbf{A} \leq \operatorname{rows} \mathbf{A}$
   By definition of rank.

3. $\operatorname{rank} \mathbf{A} \leq \operatorname{cols} \mathbf{A}$
   Suppose there are more rows than columns (otherwise the

   previous rule applies). Each column can contain at most one pivot. By pivoting, all other entries in a column below the pivot are zeroed. Hence, there will only be as many non-zero rows as pivots, which will equal the number of columns.

Ranks and solutions to systems

- The real point here is that the rank of a matrix tells us whether a system of equations has one solution, no solutions, or many.
- You can think of rank as telling us how many "real" equations have been included.
- If we have more unknowns than equations, there will be many solutions.
- And if there are more "distinct" equations than unknowns, then it will have no solution.

Exactly one solution

- $\operatorname{rank} \mathbf{A} = \operatorname{rank} \widehat{\mathbf{A}} = \operatorname{rows} \mathbf{A} = \operatorname{cols} \mathbf{A}$

Necessary condition for a system to have a unique solution: that there be exactly as many equations as unknowns. We often term this as being of **full rank**.

Infinite solutions

- $\operatorname{rank} \mathbf{A} = \operatorname{rank} \widehat{\mathbf{A}}$ and $\operatorname{cols} \mathbf{A} > \operatorname{rank} \mathbf{A}$

  If a system has a solution and has more unknowns than equations, then it has infinitely many solutions.

No solution

rank $\mathbf{A} <$ rank $\widehat{\mathbf{A}}$

Then there is a zero row i in $\mathbf{A}$'s reduced echelon that corresponds to a non-zero row i in $\widehat{\mathbf{A}}$'s reduced echelon. Row i of the $\widehat{\mathbf{A}}$ translates to the equation

$$0x_{i1} + 0x_{i2} + \cdots + 0x_{in} = b'_i$$

where $b'_i \neq 0$. Hence the system has no solution.

Find the rank of the following matrices or systems of equations.

1.
$$\begin{pmatrix} 4 & -1 & 9 \\ 2 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}$$

2.
$$\mathbf{D} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

3.
$$\mathbf{E} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

4.

$$
\begin{array}{rcrcrcl}
x & + & y & + & 2z & = & 2 \\
3x & - & 2y & + & z & = & 1 \\
 & & y & - & z & = & 3
\end{array}
$$

Returning to determinants

- Determinants can be used to *determine* whether a square matrix is nonsingular.
- A square matrix is nonsingular iff its determinant is not zero.
- The requirements for $\mathbf{A}$ to be nonsingular correspond to the requirements for a linear system to have a unique solution: $\operatorname{rank}\mathbf{A} = \operatorname{rows}\mathbf{A} = \operatorname{cols}\mathbf{A}$.

Understanding determinants

- Let's try to give you an intuitive understanding of determinants and their relationship to singularity.
- Let
$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
- Find $\mathbf{A}^{-1}$
- Find $|\mathbf{A}|$
- What happens to $\mathbf{A}^{-1}$ when $|\mathbf{A}| = 0$?
- Let's expand this intuition out to larger matrices.

Inductive definition

- Let $\mathbf{A} = a$.
- We want the determinant to equal zero when the inverse does not exist.
- Since the inverse of a, $1/a$, does not exist when $a = 0$, we let the determinant of a be

$$\det(a) = |a| = a$$

- For a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{A}$ is nonsingular only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.
- We then define the determinant of a 2×2 matrix $\mathbf{A}$ as:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\
&= a_{11}|a_{22}| - a_{12}|a_{21}|
\end{aligned}
$$

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Extending this to a 3×3 matrix, we get:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}$$

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$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}$$

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Determinants Defined

- Let's extend this to an $n \times n$ matrix.
- Let $\mathbf{A}_{ij}$ be the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ obtained by deleting row i and column j.
- Let the (i,j)th **minor** of $\mathbf{A}$ be

$$M_{ij} = |\mathbf{A}_{ij}|$$

- Then for any $n \times n$ matrix $\mathbf{A}$

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- Then for any $n \times n$ matrix $\mathbf{A}$

$$|\mathbf{A}| = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n}$$

Example: Does the following matrix have an inverse?

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$$

1. Calculate its determinant.

$$\begin{aligned} |\mathbf{A}| &= 1 \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ 5 & 5 \end{vmatrix} \\ &= 1(2-15) - 1(0-15) + 1(0-10) \\ &= -13 + 15 - 10 \\ &= -8 \end{aligned}$$

2. Since $|\mathbf{A}| \neq 0$, we conclude that $\mathbf{A}$ has an inverse.

Triangular or Diagonal Matrices

- For any upper-triangular, lower-triangular, or diagonal matrix, the determinant is just the product of the diagonal terms.
- Example: Suppose we have the following square matrix in row echelon form (i.e., upper triangular)

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

- Then

$$|\mathbf{R}| = r_{11} \begin{vmatrix} r_{22} & r_{23} \\ 0 & r_{33} \end{vmatrix} = r_{11} r_{22} r_{33}$$

Finding the inverse of a matrix with determinants

- Thus far, we have a number of algorithms to
  1. Find the solution of a linear system,
  2. Find the inverse of a matrix
- But these remain just that — algorithms.
- At this point, we have no way of telling how the solutions x_j change as the parameters a_{ij} and b_i change, except by changing the values and “rerunning" the algorithms.

- With determinants, we can
    1. Provide an explicit formula for the inverse, and
    2. Provide an explicit formula for the solution of an $n \times n$ linear system.
- Hence, we can examine how changes in the parameters ($\mathbf{A}$) and $\mathbf{b}$ affect the solutions ($\mathbf{x}$).

Defining terms: Cofactor and adjoint matrix

- Define the (i, j)th **cofactor** C_{ij} of $\mathbf{A}$ as $(-1)^{i+j} M_{ij}$. Notice that it's just the signed (i, j)th minor.
- Define the **adjoint** of $\mathbf{A}$ as the $n \times n$ matrix whose (i, j)th entry is C_{ji}
  1. Make a "cofactor matrix" by calculating C_{ij} for each element of the original matrix.
  2. The adjoint matrix is just it's transpose.

Determinant formula for the inverse

- Then, the inverse of $\mathbf{A}$ is given by the formula

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\operatorname{adj}\mathbf{A} = \begin{pmatrix} \frac{C_{11}}{|\mathbf{A}|} & \frac{C_{21}}{|\mathbf{A}|} & \cdots & \frac{C_{n1}}{|\mathbf{A}|} \\ \frac{C_{12}}{|\mathbf{A}|} & \frac{C_{22}}{|\mathbf{A}|} & \cdots & \frac{C_{n2}}{|\mathbf{A}|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_{1n}}{|\mathbf{A}|} & \frac{C_{2n}}{|\mathbf{A}|} & \cdots & \frac{C_{nn}}{|\mathbf{A}|} \end{pmatrix}$$

- Notice the switch in indexing for C_{ij} elements.

Example

Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathrm{adj}\,\mathbf{A} \tag{1}$$

$$
\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|}\operatorname{adj}\mathbf{A} \qquad (1)\\
&= \begin{pmatrix}
\frac{(-1)^{1+1}M_{11}}{|\mathbf{A}|} & \frac{(-1)^{2+1}M_{21}}{|\mathbf{A}|} & \cdots & \frac{(-1)^{n+1}M_{n1}}{|\mathbf{A}|} \\
\frac{(-1)^{1+2}M_{12}}{|\mathbf{A}|} & \frac{(-1)^{2+2}M_{22}}{|\mathbf{A}|} & \cdots & \frac{(-1)^{n+2}M_{n2}}{|\mathbf{A}|} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(-1)^{1+n}M_{1n}}{|\mathbf{A}|} & \frac{(-1)^{2+n}M_{2n}}{|\mathbf{A}|} & \cdots & \frac{(-1)^{n+n}M_{nn}}{|\mathbf{A}|}
\end{pmatrix} \qquad (2)
\end{aligned}
$$

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\end{pmatrix} && (2) \\
&= \begin{pmatrix}
\frac{M_{11}}{|\mathbf{A}|} & \frac{(-1)M_{21}}{|\mathbf{A}|} & \cdots & \frac{M_{n1}}{|\mathbf{A}|} \\
\frac{(-1)M_{12}}{|\mathbf{A}|} & \frac{M_{22}}{|\mathbf{A}|} & \cdots & \frac{(-1)M_{n2}}{|\mathbf{A}|} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{M_{1n}}{|\mathbf{A}|} & \frac{(-1)M_{2n}}{|\mathbf{A}|} & \cdots & \frac{M_{nn}}{|\mathbf{A}|}
\end{pmatrix} && (3)
\end{aligned}
$$

$$= \begin{pmatrix} \frac{-13}{-8} & \frac{(-1)(-4)}{-8} & \frac{1}{-8} \\ \frac{(-1)(-15)}{-8} & \frac{-4}{-8} & \frac{(-1)(3)}{-8} \\ \frac{-10}{-8} & \frac{(-1)0}{-8} & \frac{2}{-8} \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} \frac{-13}{-8} & \frac{(-1)(-4)}{-8} & \frac{1}{-8} \\ \frac{(-1)(-15)}{-8} & \frac{-4}{-8} & \frac{(-1)(3)}{-8} \\ \frac{-10}{-8} & \frac{(-1)0}{-8} & \frac{2}{-8} \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{pmatrix} \tag{5}$$

Cramer's Rule

- Cramer's rule extends this approach to finding the solution to a linear system of equations.
- Let $\mathbf{A}_j =$ matrix obtained from $\mathbf{A}$ by replacing the jth column of $\mathbf{A}$ by $\mathbf{b}$.
- Example:

$$\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- Then the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ to the $n \times n$ system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is

$$x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}$$

Class exercise

Find the solution of the following system:

$$
\begin{array}{rcrcrcr}
-2x_1 & + & 3x_2 & - & x_3 & = & 1 \\
x_1 & + & 2x_2 & - & x_3 & = & 4 \\
-2x_1 & - & x_2 & + & x_3 & = & -3
\end{array}
$$

1. Using the determinant method, find the inverse of this matrix.

$$\begin{pmatrix} 3 & 6 & 6 \\ -3 & 4 & 6 \\ -9 & 0 & 5 \end{pmatrix}$$

2. Solve the following set of equations using Cramer's Rule

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 2 \\ 3x & - & 2y & + & z & = & 1 \\ & & y & - & z & = & 3 \end{array}$$

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- More specifically, it can determine whether a system of equations has a global maximum or minimum.
- This is actually used all of the time for reason's we will later this week.

- When some $n \times n$ matrix $\mathbf{A}$ is pre- and post-multiplied by a conformable non-zero matrix $\mathbf{x}$, we get the equation:

$$\mathbf{x}'\mathbf{A}\mathbf{x} = c$$

- In one dimension, this would be:

$$c = xax$$

$$c = ax^2$$

- In more than one dimension it would be a quadratic formula

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  1. $\mathbf{A}$ is said to be **positive definite** if $c > 0$.

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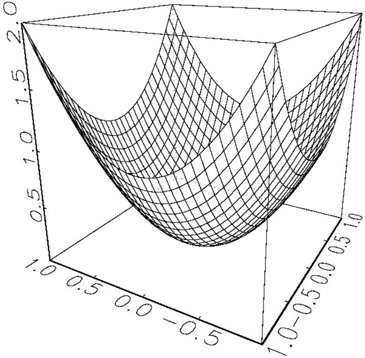
  3. $\mathbf{A}$ is said to be **negative definite** if $c < 0$.
  4. $\mathbf{A}$ is said to be **negative semidefinite** if $c \leq 0$.
  5. $\mathbf{A}$ is **indefinite** if none of these apply.

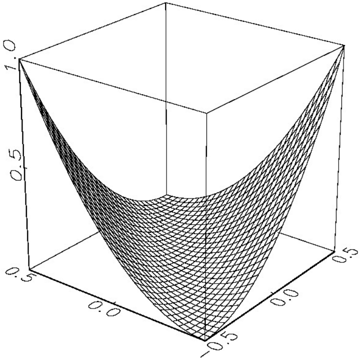
Examples: Positive Definite

- $$\begin{aligned} Q(\mathbf{x}) &= \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \\ &= x_1^2 + x_2^2 \end{aligned}$$



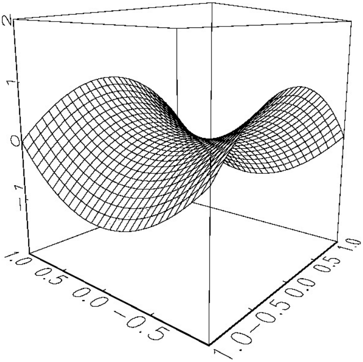
Example: Positive Semidefinite

$$
\begin{aligned}
Q(\mathbf{x}) &= \mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} \\
&= (x_1 - x_2)^2
\end{aligned}
$$



Example: Indefinite

$$
\begin{aligned}
Q(\mathbf{x}) &= \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} \\
&= x_1^2 - x_2^2
\end{aligned}
$$



Tests for definiteness

- Now we have defined it, how do we prove it?
- How can we test if a specific matrix meets one of these criteria?
- We've got one, but first we need to define terms.

Defining terms: Principal minors

- Given an $n \times n$ matrix $\mathbf{A}$, kth order **principal minors** are the determinants of the $k \times k$ submatrices along the diagonal obtained by deleting $n - k$ columns and the same $n - k$ rows from $\mathbf{A}$.

- Example: For a 3×3 matrix $\mathbf{A}$,
  1. First order principal minors:

$$|a_{11}|, \quad |a_{22}|, \quad |a_{33}|$$

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  2. Second order principal minors:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

- Example: For a 3×3 matrix $\mathbf{A}$,
  1. First order principal minors:
  $$|a_{11}|, \quad |a_{22}|, \quad |a_{33}|$$
  2. Second order principal minors:
  $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$
  3. Third order principle minor: $|\mathbf{A}|$

Defining terms: Leading principal minors

- Define the kth **leading principal minor** M_k as the determinant of the $k \times k$ submatrix obtained by deleting the *last* $n - k$ rows and columns from $\mathbf{A}$.

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- Example: For a 3×3 matrix $\mathbf{A}$, the three leading principal minors are

$$M_1 = |a_{11}|, \quad M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad M_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

An algorithm for determining definiteness of a matrix

If $\mathbf{A}$ is an $n \times n$ symmetric matrix, then

1. $M_k > 0,\ k = 1, \ldots, n \qquad \Longrightarrow \qquad$ Positive Definite

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If $\mathbf{A}$ is an $n \times n$ symmetric matrix, then

1. $M_k > 0,\ k = 1, \ldots, n \qquad \Longrightarrow \qquad$ Positive Definite
2. $M_k < 0$, for odd k and $M_k > 0$, for even $k \qquad \Longrightarrow$ Negative Definite
3. $M_k \neq 0,\ k = 1, \ldots, n$, but does not fit the pattern of 1 or 2. $\Longrightarrow \qquad$ Indefinite.

Finding semidefinite matrices

If some leading minor is equal to zero, but the others fit the patterns in 1 or 2 above:

1. Every principal minor $\geq 0 \quad \Longrightarrow \quad$ Positive Semidefinite
2. Every principal minor of odd order ≤ 0 and every principal minor of even order $\geq 0 \quad \Longrightarrow \quad$ Negative Semidefinite

Determine whether the following matrices are positive definite, negative definite, or neither:

1.
$$\begin{pmatrix} 7 & 3 \\ 8 & 9 \end{pmatrix}$$

2.
$$\begin{pmatrix} 6 & 6 \\ 8 & 4 \end{pmatrix}$$

3.
$$\begin{pmatrix} -4 & -1 \\ 3 & 0 \end{pmatrix}$$

4.
$$\begin{pmatrix} 4 & -1 & 9 \\ 2 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}$$