**NAME:** ANSWER KEY

For the following exercises, read the problems carefully and show all your work. Attach more pages if necessary. Avoid using a calculator or the computer to solve the exercises. Please, staple your homework.

# 1 Extrema

For each of the functions from the previous section, find any critical points and classify them as maxima, minima, or neither.

1. Since $f'(x) = x^2$, the only point where $f'(x) = 0$ is $x = 0$. This critical point is neither a minimum nor a maximum. First, consider that $f''(x) = 2x$, so $f''(0) = 0$. Since we cannot tell the nature of the critical point using the second derivative test, we use the extremum test. The derivative order at which we reach a non-zero result at the critical point is the third derivative ($f'''(x) = 2$, so $f'''(0) = 2$), so the critical point is an inflection point.

2. Since $f'(x) = \frac{1-x}{e^x}$, the only point where $f'(x) = 0$ is $x = 1$. The second derivative test confirms this point is a maximum. First note $f''(x) = \frac{x-2}{e^x}$:

$$\begin{aligned} f''(x) &= \frac{-e^x - e^x(1-x)}{(e^x)^2} \\ &= \frac{e^x(-1-(1-x))}{(e^x)^2} \\ &= \frac{-1-(1-x)}{e^x} \\ &= \frac{x-2}{e^x} \end{aligned}$$

   Then $f''(1) = \frac{-1}{e^x} < 0$.

3. Since $f'(x) = 1$, there are no critical points.

4. Since $f'(x) = 3x^2 - 2x$, one point where $f'(x) = 0$ is $x = 0$. Another point where $f'(x) = 0$ is $\frac{2}{3}$:

$$\begin{aligned} 3x^2 - 2x &= 0 \\ 3x^2 &= 2x \\ 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

   The second derivative test will be sufficient to determine the nature of both critical points, so first note $f''(x) = 6x - 2$. Then $x = 0$ is a (local) maximum, as $f''(0) = -2$, and $x = \frac{2}{3}$ is a (local) minimum, as $f''(\frac{2}{3}) = 6(\frac{2}{3}) - 2 = 4 - 2 = 2$.

5. Since $f'(x) = 2e^{2x} - 12e^{-4x}$, the only point where $f'(x) = 0$ is $x = \frac{\ln(6)}{6}$:

$$\begin{aligned}
2e^{2x} - 12e^{-4x} &= 0 \\
2e^{2x} &= 12e^{-4x} \\
e^{2x} &= 6e^{-4x} \\
\ln(e^{2x}) &= \ln(6e^{-4x}) \\
2x &= \ln(6) - 4x \\
6x &= \ln(6) \\
x &= \frac{\ln(6)}{6}
\end{aligned}$$

   The second derivative test confirms this point is a minimum. First note $f''(x) = 4e^{2x} + 48e^{-4x}$. Then notice that since $e^{cx}$ must be positive for all values of $c$ and $x$, $f''(x)$ must be positive for all $x$. Therefore, the critical point must be a minimum.

6. Since $f'(x) = e^{2x}(2x + 1)$, the only point where $f'(x) = 0$ is $x = \frac{-1}{2}$. The second derivative test confirms this point is a minimum. First note $f''(x) = 4e^{2x}(x + 1)$:

$$\begin{aligned}
f''(x) &= 2e^{2x}(2x + 1) + 2e^{2x} \\
&= 4xe^{2x} + 4e^{2x} \\
&= 4e^{2x}(x + 1)
\end{aligned}$$

   Then since $4e^{2x}$ is positive for any $x$ and $(x + 1)$ is positive for $x = \frac{-1}{2}$, $f''(\frac{-1}{2})$ must be positive.

7. Since $f'(x) = \dfrac{6}{3x - 1}$, there are no critical points. A student may try to claim $x = \frac{1}{3}$ as a critical point, as $f'(\frac{1}{3})$ does not exist; however, $\frac{1}{3}$ is not in the domain of $f$, as $f(\frac{1}{3}) = \ln((3(\frac{1}{3}) - 1)^2) = \ln(0)$ is undefined.

8. Since $f'(x) = \ln(5)5^{x-1}$, there are no critical points.

9. Since $f'(x) = 6x(1 + x^2)^2$, the only point where $f'(x) = 0$ is $x = 0$. Using the second derivative test, we see this is a minimum. $f''(x) = 6(1 + 6x^2 + 5x^4)$, which is positive for all $x$.

10. Since $f'(x) = \dfrac{2}{x}$, there are no critical points. The only point where $f'(x) = 0$ or $f'(x)$ does not exist is at $x = 0$, but as discussed above $x = 0$ is not in the domain of $f$.

## 2 Concavity

For each of the functions from the previous sections, identify, if any exist, the intervals on which the function is convex, and the intervals on which it is concave.

1. Recall the function's second derivative is $f''(x) = 2x$. Then $f''(x) \leq 0$ for all $x \leq 0$, and $f$ is therefore concave for all $x \leq 0$, and $f''(x) \geq 0$ for all $x \geq 0$, and $f$ is therefore convex for all $x \geq 0$.

2. Recall $f''(x) = \frac{x-2}{e^x}$. Then $f''(x) \leq 0$ for all $x \leq 2$, and $f$ is therefore concave for all $x \leq 2$, and $f''(x) \geq 0$ for all $x \geq 2$, and $f$ is therefore convex for all $x \geq 2$.

3. Since the function is linear, it is both convex and concave on its entire domain ($\mathbb{R}$).

4. Recall $f''(x) = 6x - 2$. Then $f''(x) \leq 0$ for all $x \leq 1/3$, and $f$ is therefore concave for all $x \leq 1/3$, and $f''(x) \geq 0$ for all $x \geq 1/3$, and $f$ is therefore convex for all $x \geq 1/3$.

5. Recall $f''(x) = 4e^{2x} + 48e^{-4x}$. Then $f''(x) \geq 0$ for all $x$, and $f$ is therefore convex for all $x$.

6. Recall $f''(x) = 4e^{2x}(x+1)$. Then $f''(x) \leq 0$ for all $x \leq -1$, and $f$ is therefore concave for all $x \leq -1$, and $f''(x) \geq 0$ for all $x \geq -1$, and $f$ is therefore convex for all $x \geq -1$.

7. Note $f''(x) = \frac{-18}{(3x-1)^2}$. Then $f''(x) \leq 0$ for all $x \neq \frac{1}{3}$ and $f$ is therefore concave for all $x \neq \frac{1}{3}$.

8. Recall $f'(x) = \ln(5)5^{x-1}$. Then $f''(x) = \ln(5)^2 5^{x-1}$, so that $\forall\, x, f''(x) \geq 0$. Therefore, $f$ is convex for all $x$.

9. Recall $f''(x) = 6(1 + 6x^2 + 5x^4)$. Then $f''(x) \geq 0$ for all $x$, and $f$ is therefore convex for all $x$.

10. Note $f''(x) = \dfrac{-2}{x^2}$. Then $f''(x) \leq 0$ for all $x \neq 0$ and $f$ is therefore concave for all $x \neq 0$.

## 3 L'Hospital's Rule

Find the following limits:

1. $\displaystyle\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

$$\begin{aligned}
\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9} \frac{\frac{d}{dx}x-9}{\frac{d}{dx}\sqrt{x}-3} \\
&= \lim_{x\to 9} \frac{1}{1/2x^{-1/2}} \\
&= 2\sqrt{9} \\
&= 6
\end{aligned}$$

2. $\displaystyle\lim_{x\to 0} \frac{8^x - 4^x}{x^3 - x^2 - x}$

$$\begin{aligned}
\lim_{x\to 0} \frac{8^x - 4^x}{x^3 - x^2 - x} &= \lim_{x\to 0} \frac{\frac{d}{dx}8^x - 4^x}{\frac{d}{dx}x^3 - x^2 - x} \\
&= \lim_{x\to 0} \frac{8^x \ln(8) - 4^x \ln(4)}{3x^2 - 2x - 1} \\
&= \ln(4) - \ln(8) \\
&= -\ln(2) \text{ or } \ln(1/2)
\end{aligned}$$

3. $\displaystyle\lim_{x\to\infty} \frac{e^{2x}}{2x+144}$

$$\begin{aligned}\lim_{x\to\infty}\frac{e^{2x}}{2x+144} &= \lim_{x\to\infty}\frac{\frac{d}{dx}e^{2x}}{\frac{d}{dx}2x+144}\\ &= \lim_{x\to\infty}\frac{2e^{2x}}{2}\\ &= \infty\end{aligned}$$

4. $\lim_{x\to\infty}\frac{2+\ln(x)}{x^2+3}$

$$\begin{aligned}\lim_{x\to\infty}\frac{2+\ln(x)}{x^2+3} &= \lim_{x\to\infty}\frac{\frac{d}{dx}2+\ln(x)}{\frac{d}{dx}x^2+3}\\ &= \lim_{x\to\infty}\frac{1}{x}\frac{1}{2x}\\ &= 0\end{aligned}$$

# 4 An Applied Problem

Political scientists often employ rational choice theory to study politics. Political actors such as legislators are assumed to have goals, and to choose actions designed to achieve them. This is operationalized by defining an actor's *utility functions* and *feasible actions*, and determining which feasible action maximizes her utility.

For example, say a legislator $i$'s utility function $u$ was defined by $u_i(c) = v - c^2$, where $v$ is the legislator's vote share in an election, and $c$ is the portion of her wealth the legislator spent on the campaign. That is, the legislator gains utility from gaining votes, but loses utility from spending her wealth to get them. Now say vote share was determined entirely by campaign spending such that $v = c$; what level of campain spending maximizes the legislator's utility?

We simply find the value of $c$ for which $u_i(c)$ achieves its maximum:

$$\begin{aligned}u_i(c) &= v - c^2\\ &= c - c^2\\ u_i'(c) &= 1 - 2c\\ 0 &\equiv 1 - 2c\\ 2c &= 1\\ c &= \frac{1}{2}\\ u_i''(c) &= -2\end{aligned}$$

We see that $c = \frac{1}{2}$ is a critical point of $u_i(c)$ and by the second derivative test it is a maximum; so in this example, the legislator maximizes her utility by allocating half her wealth for campaign spending.