Sequences, Limits, Etc.

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$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

LEFT-HANDED LIMIT (by a right handed mathematician!)

2007

What's a limit?

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The limit of a function

Let f be defined at each point in some open interval containing the point c, although possibly not defined at c itself. Then $\lim\limits_{x\to c} f(x) = L$ if for any (small positive) number ϵ, there exists a corresponding number $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Basic examples

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- $\lim_{x \to c} x = c$
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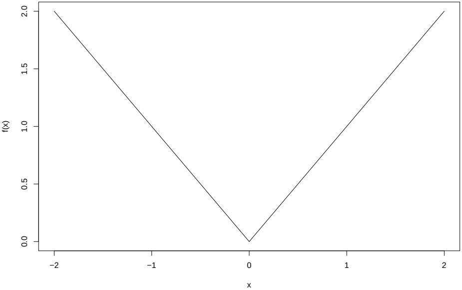
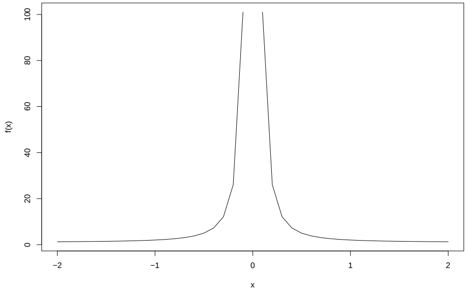
```
plot(seq(-2, 2, by=.1), abs(seq(-2, 2, by=.1)),
     xlab="x", ylab="f(x)", type="l")
```



- $\lim_{x \to 0} \left(1 + \frac{1}{x^2}\right) = \infty$

```
plot(seq(-2, 2, by=.1), 1+1/(seq(-2, 2, by=.1))^2,
     xlab="x", ylab="f(x)", type="l")
```



Properties of a limit

Let f and g be functions with $\lim_{x \to c} f(x) = A$ and $\lim_{x \to c} g(x) = B$.

- $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = A + B$

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- $\lim_{x \to c} \alpha f(x) = \alpha \lim_{x \to c} f(x) = \alpha A$
- $\lim_{x \to c} f(x)g(x) = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = AB$
- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{A}{B}$, provided $B \neq 0$

Examples

$$\lim_{x\to 2}(2x-3) = 2\lim_{x\to 2} x - 3\lim_{x\to 2} 1 = 2\times 2 - 3\times 1 = 1$$

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$$\lim_{x\to c} x^n = [\lim_{x\to c} x]\cdots[\lim_{x\to c} x] = c\cdots c = c^n$$

Other types of limits:

- **Right-hand limit**: $\lim_{x \to c^+} f(x) = L$, if $c < x < c + \delta \Longrightarrow |f(x) - L| < \epsilon$

Example: $\lim_{x \to 0^+} \sqrt{x} = 0$

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- **Left-hand limit**: $\lim_{x \to c^-} f(x) = L$, if

  $c - \delta < x < c \Longrightarrow |f(x) - L| < \epsilon$

- **Infinity**: $\lim_{x \to \infty} f(x) = L$, if $x > N \Longrightarrow |f(x) - L| < \epsilon$
- $-$**Infinity**: $\lim_{x \to -\infty} f(x) = L$, if $x < -N \Longrightarrow |f(x) - L| < \epsilon$

Example: $\lim_{x \to \infty} 1/x = \lim_{x \to -\infty} 1/x = 0$

Caution

In some situations, you will not be able to calculate a limit. For instance, $\lim_{x\to\infty} \frac{x}{-x}$. The numerator is headed towards ∞ while the denominator is headed towards $-\infty$. In this case the limit does not exist. In other circumstances, the limit may exist but additional steps need to be taken.

Existence of a limit

The limit of a function exists only if the left and right hand limits are equal to the same value.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

Find the limits (if they exist):

1. $$\lim_{x \to 5} \frac{1}{(x-5)^2}$$

2. $$\lim_{x \to 5} 2x^2 - 5x + 7$$

3. $$\lim_{y \to \infty} \frac{1}{y^6}$$

4. $$\lim_{y \to 0} \frac{1}{y^6}$$

5. $$\lim_{x \to 20} \frac{2x+3}{5x^2}$$

Continuity

Continuity: Suppose that the domain of the function f includes an open interval containing the point c. Then f is continuous at c if $\lim_{x \to c} f(x)$ exists and if $\lim_{x \to c} f(x) = f(c)$.

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Further, f is continuous on an open interval (a, b) if it is continuous at each point in the interval.

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Examples: Discontinuous functions.

$f(x) = \text{floor}(x)$
$f(x) = 1 + \frac{1}{x^2}$

Basic properties of continuous functions

If f and g are continuous at point c, then

- $f + g$, $f - g$, $f \times g$, $|f|$, and αf are continuous.

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- $f+g$, $f-g$, $f \times g$, $|f|$, and αf are continuous.
- f/g is continuous, provided $g(c) \neq 0$.

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If f is continuous on the closed bounded interval $[a, b]$, then there is a number K such that $|f(x)| \leq K$ for each x in $[a, b]$.

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Max/min:

If f is continous on the closed bounded interval $[a, b]$, then f has a maximum and a minimum on $[a, b]$, possibly at the end points. The range of a closed bounded interval $[a, b]$ under a continuous function f is also a closed bounded interval $[m, M]$.

Some practice

1. Show whether $f(x) = x + x^3$ has a limit at $x = 3$ and, if so, the value of the limit.
2. Find $\lim_{x \to 4}(x-3)(x+5)$
3. Find $\lim_{x \to 2} \frac{3x^2-12}{x-2}$
4. Find $\lim_{x \to 2} \frac{x^3-4}{x-2}$
5. Is the function $f(x) = \frac{ln(x)}{x}$ continuous for $x \in [2, \infty)$?
6. Is the function below continuous? If not, what could be done to make it so?

$$f(x) = \begin{cases} x^3 - 3x + 4 & x \leq 3 \\ x^2 & x > 3 \end{cases}$$

Sequences

A **sequence** $\{y_n\} = \{y_1, y_2, y_3, \ldots, y_n\}$ is an ordered set of real numbers, where y_1 is the first term in the sequence and y_n is the nth term. Generally, a sequence extends to $n = \infty$. We can also write the sequence as $\{y_n\}_{n=1}^{\infty}$.

Example sequences

Think of sequences like functions. Before, we had $y = f(x)$ with x specified over some domain. Now we have $\{y_n\} = \{f(n)\}$ with $n = 1, 2, 3, \ldots$.

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1. $\{y_n\} = \left\{2 - \frac{1}{n^2}\right\} = \left\{1, \frac{7}{4}, \frac{17}{9}, \frac{31}{16}, \ldots\right\}$
2. $\{y_n\} = \left\{\frac{n^2+1}{n}\right\} = \left\{2, \frac{5}{2}, \frac{10}{3}, \ldots\right\}$
3. $\{y_n\} = \left\{(-1)^n \left(1 - \frac{1}{n}\right)\right\} = \{0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, \ldots\}$

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- Sequences like 1 above that converge to a limit.
- Sequences like 2 above that increase without bound.
- Sequences like 3 above that neither converge nor increase
- Why? Let's draw them.

Defining some terms

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- **Monotone Increasing**: $y_{n+1} > y_n$ for all n
- **Monotone Decreasing**: $y_{n+1} < y_n$ for all n
- **Subsequence**: Choose a (possibly infinite) collection of entries from $\{y_n\}$, retaining their order.

Limits of a sequence

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The sequence $\{y_n\}$ has the **limit** L, that is $\lim_{n \to \infty} y_n = L$, if for any $\epsilon > 0$ there is an integer N (which depends on ϵ) with the property that $|y_n - L| < \epsilon$ for each $n > N$. $\{y_n\}$ is said to converge to L. If the above does not hold, then $\{y_n\}$ diverges.

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Limits of a vector of sequences

Finding the limit of a sequence in $\mathbf{R}^n$ is similar to that in $\mathbf{R}^1$.

- The sequence of vectors $\{\mathbf{y_n}\}$ has the limit $\mathbf{L}$, that is $\lim_{n\to\infty} \mathbf{y_n} = \mathbf{L}$, if for any ϵ there is an integer N where $||\mathbf{y_n} - \mathbf{L}|| < \epsilon$ for each $n > N$. The sequence of vectors $\{\mathbf{y_n}\}$ is said to converge to the vector $\mathbf{L}$ — and the distances between $\mathbf{y_n}$ and $\mathbf{L}$ converge to zero.

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- Think of each coordinate of the vector $\mathbf{y_n}$ as being part of its own sequence over n. Then a sequence of vectors in $\mathbf{R}^n$ converges if and only if all n sequences of its components converge.

Examples:

1. The sequence $\{y_n\}$ where $y_n = \left(\frac{1}{n}, 2 - \frac{1}{n^2}\right)$ converges to $(0, 2)$.
2. The sequence $\{y_n\}$ where $y_n = \left(\frac{1}{n}, (-1)^n\right)$ does not converge, since $\{(-1)^n\}$ does not converge.

- $\lim_{n\to\infty} \left\{2 - \frac{1}{n^2}\right\} = ?$
- $\lim_{n\to\infty} \left\{\frac{4^n}{n!}\right\} = ?$
- Show $\sum_{k=0}^{4} ar^k$. Then find $\lim_{K\to\infty} \sum_{k=0}^{K} ar^k$, where $|r| < 1$.