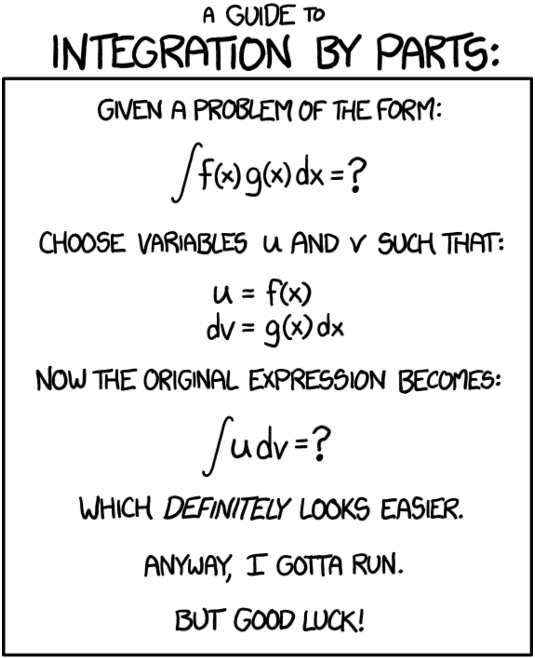
Integrals 2: The Integrand Strikes Back

David Carlson

2021

Integrals 2: The integrand strikes back


A GUIDE TO
INTEGRATION BY PARTS:
GIVEN A PROBLEM OF THE FORM:
$\int f(x)\,g(x)\,dx = ?$
CHOOSE VARIABLES u AND v SUCH THAT:
$u = f(x)$
$dv = g(x)\,dx$
NOW THE ORIGINAL EXPRESSION BECOMES:
$\int u\,dv = ?$
WHICH *DEFINITELY* LOOKS EASIER.
ANYWAY, I GOTTA RUN.
BUT GOOD LUCK!

Introduction

- It turns out that for some functions, staring at the function until you see the answer is just too hard.
- In this lecture, we'll give you a couple of tricks to help you along with the harder integral.
- This is hard and very frustrating at times. Prepare your soul.

Integration by substitution (or u-substitution)

- Sometimes the integrand doesn't appear integrable using common rules and antiderivatives. A method one might try is **integration by substitutions**, which is related to the Chain Rule.

Integration by substitution (or u-substitution)

- Sometimes the integrand doesn't appear integrable using common rules and antiderivatives. A method one might try is **integration by substitutions**, which is related to the Chain Rule.
- Suppose we want to find the indefinite integral $\int g(x)dx$ and assume we can identify a function $u(x)$ such that $g(x) = f[u(x)]u'(x)$. Let's refer to the antiderivative of f as F.
- Then the chain rule tells us that $\frac{d}{dx}F[u(x)] = f[u(x)]u'(x)$. So, $F[u(x)]$ is the antiderivative of g.
- We can then write

$$\int g(x)dx = \int f[u(x)]u'(x)dx = \int \frac{d}{dx}F[u(x)]dx = F[u(x)]+c$$

Indefinite integral

- Procedure to determine the indefinite integral $\int g(x)dx$ by the method of substitions:
  1. Identify some part of $g(x)$ that might be simplified by substituting in a single variable u (which will then be a function of x).
  2. Determine if $g(x)dx$ can be reformulated in terms of u and du.
  3. Solve the indefinite integral.
  4. Substitute back in for x

Definite integral

- Substitution can also be used to calculate a definite integral. Using the same procedure as above,

$$\int_a^b g(x)dx = \int_c^d f(u)du = F(d) - F(c)$$

where $c = u(a)$ and $d = u(b)$.

Example 1

$$\int x^2\sqrt{x+1}dx$$

- The problem here is the $\sqrt{x+1}$ term. However, if the integrand had $\sqrt{x}$ times some polynomial, then we'd be in business.

Example 1

$$\int x^2\sqrt{x+1}dx$$

- The problem here is the $\sqrt{x+1}$ term. However, if the integrand had $\sqrt{x}$ times some polynomial, then we'd be in business.
- Let's try $u = x+1$. Then $x = u-1$ and $dx = du$.
- Substituting these into thevabove equation, we get

$$\begin{aligned}\int x^2\sqrt{x+1}dx &= \int (u-1)^2\sqrt{u}du \\ &= \int (u^2-2u+1)u^{1/2}du \\ &= \int (u^{5/2}-2u^{3/2}+u^{1/2})du\end{aligned}$$

- We can easily integrate this, since it's just a polynomial. Doing so and substituting $u = x + 1$ back in, we get

$$\int x^2\sqrt{x+1}dx = 2(x+1)^{3/2}\left[\frac{1}{7}(x+1)^2 - \frac{2}{5}(x+1) + \frac{1}{3}\right] + c$$

- For the above problem, we could have also used the substitution $u = \sqrt{x+1}$. Then $x = u^2 - 1$ and $dx = 2udu$.

- For the above problem, we could have also used the substitution $u = \sqrt{x+1}$. Then $x = u^2 - 1$ and $dx = 2udu$.
- Substituting these in, we get

$$\int x^2\sqrt{x+1}dx = \int (u^2-1)^2 u2udu$$

- For the above problem, we could have also used the substitution $u = \sqrt{x+1}$. Then $x = u^2 - 1$ and $dx = 2udu$.
- Substituting these in, we get

$$\int x^2\sqrt{x+1}dx = \int (u^2-1)^2 u 2udu$$

which when expanded is again a polynomial and gives the same result as above.

Example 2

$$\int_0^1 \frac{5e^{2x}}{(1+e^{2x})^{1/3}}\,dx$$

- When an expression is raised to a power, it's often helpful to use this expression as the basis for a substitution. So, let $u = 1 + e^{2x}$.

Example 2

$$\int_0^1 \frac{5e^{2x}}{(1+e^{2x})^{1/3}}\,dx$$

- When an expression is raised to a power, it's often helpful to use this expression as the basis for a substitution. So, let $u = 1 + e^{2x}$.
- Then $du = 2e^{2x}dx$ and we can set $5e^{2x}dx = 5du/2$.

Example 2

$$\int_0^1 \frac{5e^{2x}}{(1+e^{2x})^{1/3}}\,dx$$

- When an expression is raised to a power, it's often helpful to use this expression as the basis for a substitution. So, let $u = 1 + e^{2x}$.
- Then $du = 2e^{2x}dx$ and we can set $5e^{2x}dx = 5du/2$.
- Additionally, $u = 2$ when $x = 0$ and $u = 1 + e^2$ when $x = 1$.

- Substituting all of this in, we get

$$\begin{aligned}\int_0^1 \frac{5e^{2x}}{(1+e^{2x})^{1/3}}dx &= \frac{5}{2}\int_2^{1+e^2} \frac{du}{u^{1/3}} \\ &= \frac{5}{2}\int_2^{1+e^2} u^{-1/3}du \\ &= \frac{15}{4}u^{2/3}\Big|_2^{1+e^2} \\ &= 9.53\end{aligned}$$

Solve the following using u-substition

1. $$\int (10x^9 - 20x^3 + 3)(5x^{10} - 25x^4 + 15x)^7 dx$$

2. $$\int_0^3 x \exp(x^2 - 2) dx$$

3. $$\int \frac{4x + 3}{4x^2 + 6x - 11} dx$$

4. $$\int_0^2 2x\sqrt{x^2 + 1} dx$$

Integration by parts

- Another useful integration technique is **integration by parts**, which is related to the Product Rule of differentiation.

Integration by parts

- Another useful integration technique is **integration by parts**, which is related to the Product Rule of differentiation.
- The product rule states that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integration by parts

- Another useful integration technique is **integration by parts**, which is related to the Product Rule of differentiation.
- The product rule states that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Integrating this and rearranging (see pg. 178 in Kropko), we get

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int v(x)\frac{du(x)}{dx}dx$$

or

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Integration by parts

- Another useful integration technique is **integration by parts**, which is related to the Product Rule of differentiation.
- The product rule states that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Integrating this and rearranging (see pg. 178 in Kropko), we get

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int v(x)\frac{du(x)}{dx}dx$$

or

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

- More frequently remembered as

$$\int udv = uv - \int vdu$$

where $du = u'(x)dx$ and $dv = v'(x)dx$.

- For definite integrals: $\int\limits_a^b u\frac{dv}{dx}dx = uv|_a^b - \int\limits_a^b v\frac{du}{dx}dx$

Example 1

$$\int xe^{ax}\,dx$$

Example 1

$$\int xe^{ax}\,dx$$

- Let $u = x$ and $dv = e^{ax}\,dx$.

Example 1

$$\int xe^{ax}\,dx$$

- Let $u = x$ and $dv = e^{ax}\,dx$.
- Then $du = dx$ and $v = (1/a)e^{ax}$.

Example 1

$$\int xe^{ax}dx$$

- Let $u = x$ and $dv = e^{ax}dx$.
- Then $du = dx$ and $v = (1/a)e^{ax}$.
- Substituting this into the integration by parts formula, we obtain

$$\int xe^{ax}dx = uv - \int vdu$$

Example 1

$$\int xe^{ax}dx$$

- Let $u = x$ and $dv = e^{ax}dx$.
- Then $du = dx$ and $v = (1/a)e^{ax}$.
- Substituting this into the integration by parts formula, we obtain

$$\begin{aligned} \int xe^{ax}dx &= uv - \int vdu \\ &= x\left(\frac{1}{a}e^{ax}\right) - \int \frac{1}{a}e^{ax}dx \end{aligned}$$

Example 1

$$\int xe^{ax}dx$$

- Let $u = x$ and $dv = e^{ax}dx$.
- Then $du = dx$ and $v = (1/a)e^{ax}$.
- Substituting this into the integration by parts formula, we obtain

$$\begin{aligned}\int xe^{ax}dx &= uv - \int vdu \\ &= x\left(\frac{1}{a}e^{ax}\right) - \int \frac{1}{a}e^{ax}dx \\ &= \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax} + c\end{aligned}$$

Example 2

$$\int x^n e^{ax} dx$$

Example 2

$$\int x^n e^{ax} dx$$

- As in the first problem, let's let $u = x^n$ and $dv = e^{ax} dx$.

Example 2

$$\int x^n e^{ax} dx$$

- As in the first problem, let's let $u = x^n$ and $dv = e^{ax} dx$.
- Then $du = nx^{n-1} dx$ and $v = (1/a)e^{ax}$.

Example 2

$$\int x^n e^{ax} dx$$

- As in the first problem, let's let $u = x^n$ and $dv = e^{ax} dx$.
- Then $du = nx^{n-1} dx$ and $v = (1/a)e^{ax}$.
- Substituting these into the integration by parts formula gives

$$\int x^n e^{ax} dx = uv - \int v du$$

Example 2

$$\int x^n e^{ax} dx$$

- As in the first problem, let's let $u = x^n$ and $dv = e^{ax} dx$.
- Then $du = nx^{n-1} dx$ and $v = (1/a)e^{ax}$.
- Substituting these into the integration by parts formula gives

$$\begin{aligned} \int x^n e^{ax} dx &= uv - \int v du \\ &= x^n \left(\frac{1}{a} e^{ax} \right) - \int \frac{1}{a} e^{ax} n x^{n-1} dx \end{aligned}$$

Example 2

$$\int x^n e^{ax} dx$$

- As in the first problem, let's let $u = x^n$ and $dv = e^{ax} dx$.
- Then $du = nx^{n-1} dx$ and $v = (1/a)e^{ax}$.
- Substituting these into the integration by parts formula gives

$$\begin{aligned}
\int x^n e^{ax} dx &= uv - \int v du \\
&= x^n \left(\frac{1}{a} e^{ax}\right) - \int \frac{1}{a} e^{ax} n x^{n-1} dx \\
&= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx
\end{aligned}$$

- Notice that we now have an integral similar to the previous one, but with x^{n-1} instead of x^n.
- For a given n, we would repeat the integration by parts procedure until the integrand was directly integrable — e.g., when the integral became $\int e^{ax}\,dx$.
- Let's do this with $n = 2$.

Example 3

$$\int x^3 e^{-x^2} dx$$

Example 3

$$\int x^3 e^{-x^2} dx$$

- We could, as before, choose $u = x^3$ and $dv = e^{-x^2} dx$. But we can't then find v — i.e., integrating $e^{-x^2} dx$ isn't possible.

Example 3

$$\int x^3 e^{-x^2} dx$$

- We could, as before, choose $u = x^3$ and $dv = e^{-x^2} dx$. But we can't then find v — i.e., integrating $e^{-x^2} dx$ isn't possible.
- Instead, notice that $d(e^{-x^2})/dx = -2xe^{-x^2}$, which can be factored out of the original integrand

$$\int x^3 e^{-x^2} dx = \int x^2 (xe^{-x^2}) dx$$

Example 3

$$\int x^3 e^{-x^2} dx$$

- We could, as before, choose $u = x^3$ and $dv = e^{-x^2} dx$. But we can't then find v — i.e., integrating $e^{-x^2} dx$ isn't possible.
- Instead, notice that $d(e^{-x^2})/dx = -2xe^{-x^2}$, which can be factored out of the original integrand

$$\int x^3 e^{-x^2} dx = \int x^2 (xe^{-x^2}) dx$$

- We can then let $u = x^2$ and $dv = xe^{-x^2} dx$.

Example 3

$$\int x^3 e^{-x^2} dx$$

- We could, as before, choose $u = x^3$ and $dv = e^{-x^2}dx$. But we can't then find v — i.e., integrating $e^{-x^2}dx$ isn't possible.
- Instead, notice that $d(e^{-x^2})/dx = -2xe^{-x^2}$, which can be factored out of the original integrand

$$\int x^3 e^{-x^2} dx = \int x^2 (xe^{-x^2}) dx$$

- We can then let $u = x^2$ and $dv = xe^{-x^2}dx$.
- Then $du = 2xdx$ and $v = -\frac{1}{2}e^{-x^2}$.

- Substituting these in, we have

$$
\begin{aligned}
\int x^3 e^{-x^2} dx &= uv - \int v du \\
&= x^2 \left(-\frac{1}{2} e^{-x^2} \right) - \int \left(-\frac{1}{2} e^{-x^2} \right) 2x dx \\
&= -\frac{1}{2} x^2 e^{-x^2} + \int x e^{-x^2} dx \\
&= -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} + c
\end{aligned}
$$

and the rest is algebra.

Solve the following integrals using integration by parts.

1.
$$\int_1^4 x\sqrt{x+5}dx$$

2.
$$\int 3xe^x dx$$

3.
$$\int_1^e x\ln(x)dx$$

4.
$$\int x^2 e^x dx$$