NAME: ANSWER KEY

For the following exercises, read the problems carefully and show all your work. Attach more pages if necessary. Avoid using a calculator or the computer to solve the exercises. Please, staple your homework.

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \mathbf{C} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{E} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 8 \end{bmatrix}$$

1. For each of them, identify whether the matrix is: square, symmetric, triangular, idempotent, identity, $\mathbf{J}$, $\mathbf{0}$, or none of the above.

   A is square, symmetric, identity, idempotent, and triangular. **B** is square. **C** is square. **D** is **J**, square, and symmetrical. **E** is triangular and square.

2. Calculate $\text{Tr}(\mathbf{A})$

$$\text{Tr}(\mathbf{A}) = 1 + 1 = 2$$

3. Calculate $5(\text{Tr}(\mathbf{B}) + \text{Tr}(\mathbf{E}))$

$$5(\text{Tr}(\mathbf{B}) + \text{Tr}(\mathbf{E})) = 5(15 + 12) = 5(27) = 135$$

Now consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & -2 & -1 \\ 5 & -1 & 2 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{D} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \mathbf{E} = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 1 & -1 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

4. Is $\mathbf{E}'\mathbf{E}$ square, symmetric, triangular, idempotent, identity, $\mathbf{J}$, $\mathbf{0}$, or none of the above?

   It is square and symmetric.

5. For each of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, and $\mathbf{E}$, find the trace.

   (a) $\text{Tr}(\mathbf{A} = 0$

   (b) $\text{Tr}(\mathbf{B} = 7$

   (c) $\text{Tr}(\mathbf{C} = 3$

   (d) $\text{Tr}(\mathbf{D} = -1$

   (e) Tr($\mathbf{E}$ cannot be computed as $\mathbf{E}$ is not a square matrix.

Invert the following matrices or give a reason why you cannot:

6. $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

   This matrix can be inverted using Gauss-Jordan elimination:

$$\left[\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 1 & \frac{3}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 5-5 & 7-\frac{15}{2} & 1+0 & 0-\frac{5}{2} \\ 1 & \frac{3}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 0 & -\frac{1}{2} & 1 & -\frac{5}{2} \\ 1 & \frac{3}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -\frac{5}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -2 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{2}-\frac{3}{2} & 0+3 & \frac{1}{2}-\frac{15}{2} \\ 0 & 1 & -2 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

   Since it is a 2×2 matrix, it can also be inverted using a simpler method. First find the determinant: $5(3) - 7(2) = 1$. Then rearrange the matrix:

$$\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

And divide each element by the determinant, yielding

$$\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

7. $\begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}$

   This matrix cannot be inverted because its determinant is 0.

8. $\begin{bmatrix} 9 & 5 & 6 \\ 8 & 2 & 7 \\ 4 & 3 & 1 \end{bmatrix}$

   The matrix can be inverted via Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 8 & 2 & 7 & 0 & 1 & 0 \\ 4 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 0 & -4 & 5 & 0 & 1 & -2 \\ 4 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 0 & -4 & 5 & 0 & 1 & -2 \\ 36 & 27 & 9 & 0 & 0 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 0 & -4 & 5 & 0 & 1 & -2 \\ 0 & 7 & -15 & -4 & 0 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 0 & -4 & 5 & 0 & 1 & -2 \\ 0 & -5 & 0 & -4 & 3 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 0 & -5 & 0 & -4 & 3 & 3 \\ 0 & -4 & 5 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 5 & 6 & 1 & 0 & 0 \\ 0 & -5 & 0 & -4 & 3 & 3 \\ 0 & 0 & 5 & 16/5 & -7/5 & -22/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 0 & 6 & -3 & 3 & 3 \\ 0 & -5 & 0 & -4 & 3 & 3 \\ 0 & 0 & 5 & 16/5 & -7/5 & -22/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 0 & 0 & -171/25 & 117/25 & 207/25 \\ 0 & -5 & 0 & -4 & 3 & 3 \\ 0 & 0 & 5 & 16/5 & -7/5 & -22/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 0 & 0 & -171/25 & 117/25 & 207/25 \\ 0 & 1 & 0 & 4/5 & -3/5 & -3/5 \\ 0 & 0 & 5 & 16/5 & -7/5 & -22/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 9 & 0 & 0 & -171/25 & 117/25 & 207/25 \\ 0 & 1 & 0 & 4/5 & -3/5 & -3/5 \\ 0 & 0 & 1 & 16/25 & -7/25 & -22/25 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -19/25 & 13/25 & 23/25 \\ 0 & 1 & 0 & 4/5 & -3/5 & -3/5 \\ 0 & 0 & 1 & 16/25 & -7/25 & -22/25 \end{array}\right]$$

The student could also divide the elements of the adjoint by the determinant.

9. $\begin{bmatrix} 11 & 3 & 5 \\ 3 & 2 & 19 \\ 0 & 0 & 0 \end{bmatrix}$

This matrix cannot be inverted. Students may offer a number of reasons for this: it has a row of 0s, its determinant is 0, etc.

10. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assuming $ad \neq bc$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

11. $\begin{bmatrix} -2/27 & -16/81 & -11/81 \\ 5/18 & 11/27 & 5/54 \\ -1/6 & -4/9 & -1/18 \end{bmatrix}$

Provide a solution to the following systems of linear equations, or explain why you cannot:

12. $x = 1, y = -7, z = -1$

13. Cannot be solved; the system is singular.

An Applied Problem

Often political scientists use linear regression to study political phenomena. We may think about this method using matrix algebra. Essentially, the researcher has a vector $\mathbf{y}$ of outcome observations, and a matrix $\mathbf{X}$ of explanatory variable observations. That is, each row of $\mathbf{y}$ is the observed oucome for the corresponding row of obvservations for the independent variables in $\mathbf{X}$. The researcher's goal is to find the vector $\mathbf{b}$ of coefficients; that is, we assume that each explanatory variable has a linear effect on the outcome such that in expectation $\mathbf{y} = \mathbf{Xb}$. Given the equation $\mathbf{y} = \mathbf{Xb}$, find the formula for $\mathbf{b}$. Be sure to account for the fact that $\mathbf{X}$ may not be a square matrix. Are there any situations where the formula you found will not work?

If $\mathbf{X}$ is a square, non-singular matrix, the solution is apparent:

$$\begin{aligned} \mathbf{y} &= \mathbf{Xb} \\ \mathbf{X}^{-1}\mathbf{y} &= \mathbf{X}^{-1}\mathbf{Xb} \\ \mathbf{X}^{-1}\mathbf{y} &= \mathbf{b} \end{aligned}$$

However, if $\mathbf{X}$ is not a square matrix, we can obtain a square matrix by pre-multiplying it by its transpose:

$$\begin{aligned} \mathbf{y} &= \mathbf{Xb} \\ \mathbf{X}'\mathbf{y} &= \mathbf{X}'\mathbf{Xb} \\ (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Xb} \\ \mathbf{b} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \end{aligned}$$

This formula to find $\mathbf{b}$ only works if $(\mathbf{X}'\mathbf{X})$ is invertible.