Linear Algebra 2: Special Matrices and Matrix Operations

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2021

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Square Matrices

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- This may seem a bit odd at first. After all, we usually have more observations in our datasets (rows) than variables (columns).
- However, one of the most common tricks or operations we do with matrices is to turn them into square matrices. For instance, if we had a non-square matrix $\underset{(m \times n)}{\mathbf{X}}$, we change it into a square matrix using the following operation $\mathbf{X}^T\mathbf{X}$.

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- This may seem a bit odd at first. After all, we usually have more observations in our datasets (rows) than variables (columns).
- However, one of the most common tricks or operations we do with matrices is to turn them into square matrices. For instance, if we had a non-square matrix $\underset{(m \times n)}{\mathbf{X}}$, we change it into a square matrix using the following operation $\mathbf{X}^T\mathbf{X}$.
- The result is an $n \times n$ square matrix.
- In fact, this matrix will be a symmetric square matrix (Try it!).

Basics for square matrices

- **Square** matrices have the same number of rows and columns
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- **Square** matrices have the same number of rows and columns
- A $k \times k$ square matrix is referred to as a matrix of order k.
- The **diagonal** of a square matrix is the vector of matrix elements that have the same subscripts.
- If $\mathbf{A}$ is a square matrix of order k, then its diagonal is $[a_{11}, a_{22}, \ldots, a_{kk}]'$.

The Trace

- **Trace**: The trace of a square matrix $\mathbf{A}$ is the sum of the diagonal elements:

$$\mathrm{tr}(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{kk}$$

- Properties of the trace operator: If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of order k, then
  1. $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$
  2. $\mathrm{tr}(\mathbf{A}^T) = \mathrm{tr}(\mathbf{A})$
  3. $\mathrm{tr}(s\mathbf{A}) = s\mathrm{tr}(\mathbf{A})$
  4. $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$

Special types of square matrices

There are several important types of square matrix:

Symmetric Matrix

- A matrix $\mathbf{A}$is symmetric if $\mathbf{A} = \mathbf{A}'$
- this implies that $a_{ij} = a_{ji}$ for all i and j.

Examples:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \mathbf{A}', \qquad \mathbf{B} = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{pmatrix} = \mathbf{B}'$$

Diagonal Matrix

- A matrix $\mathbf{A}$ is diagonal if all of its non-diagonal entries are zero
- Formally, if $a_{ij} = 0$ for all $i \neq j$

Examples:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Triangular Matrix

- A matrix is triangular one of two cases.
  - If all entries below the diagonal are zero ($a_{ij} = 0$ for all $i > j$), it is **upper triangular**.
  - Conversely, if all entries above the diagonal are zero ($a_{ij} = 0$ for all $i < j$), it is **lower triangular**.

Examples

$$\mathbf{A}_{LT} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ -3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{A}_{UT} = \begin{pmatrix} 1 & 7 & -4 \\ 0 & 3 & 9 \\ 0 & 0 & -3 \end{pmatrix}$$

Special Square Matrices

There are a number of specific square matrices that you need to add to your vocabulary

Identity Matrix

The $n \times n$ identity matrix $\mathbf{I}_n$ is the matrix whose diagonal elements are 1 and all off-diagonal elements are 0.

Examples:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- You will see this matrix all of the time. Usually it is used in one of two ways.
  1. First, it plays a role much like the number 1 in scalar algebra.
     - If you multiply any matrix by a comformable identity matrix, you will get back the original matrix.
     - That is $\mathbf{AI} = \mathbf{A}$.

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  1. First, it plays a role much like the number 1 in scalar algebra.
     - If you multiply any matrix by a comformable identity matrix, you will get back the original matrix.
     - That is $\mathbf{AI} = \mathbf{A}$.
  2. Second, the identity matrix is often a convenient way to denote a diagonal matrix with the same element on each row.
     - For instance,

$$\sigma \mathbf{I}_3 = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

The J Matrix

$$\mathbf{1_3} = (1, 1, 1)'$$

$$\underset{(3\times 4)}{\mathbf{J}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{J}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{J}_2\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$$

0 Matrix

- This is a matrix of zeroes that plays the same role as the scalar zero.

Idempotent

- A square matrix $\mathbf{A}$ is said to be idempotent if:

$$\mathbf{A}^2 = \mathbf{A}$$

- Which matrices that we just discussed are idempotent?

Identify the following matrices as diagonal, identity, square, symmetric, triangular, or none of the above.

1. $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 5 \\ 1 & -2 & -1 \\ 5 & -1 & 2 \end{pmatrix}$$

2. $$\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$$

3. $$\mathbf{B}^T\mathbf{B}$$

Identify the following matrices as diagonal, idenitity, square, symmetric, triangular, or none of the above.

4.

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5.

$$\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$$

6.

$$\mathbf{E} = \begin{pmatrix} 0 & 1 & 2 \\ 5 & 1 & -1 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

7.

$$\mathbf{E^T E}$$

8. For each of the square matrices identified above, find the associated trace.

Matrix Inversion

- In scalar algebra, if we want to solve a simple equation we often use the scalar inverse. For example,

$$6x = 12$$
$$x \times 6 \times 6^{-1} = 12 \times 6^{-1}$$
$$x = \frac{12}{6} = 2$$

Matrix Inversion

- In scalar algebra, if we want to solve a simple equation we often use the scalar inverse. For example,

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\begin{aligned}
6x &= 12 \\
x \times 6 \times 6^{-1} &= 12 \times 6^{-1} \\
x &= \frac{12}{6} = 2
\end{aligned}
$$

- The key thing you are doing in this operation is mutiplying both sides of the equation by the inverse of 6 so that you have isolated x by itself on one side of the equation.
- This is possible because $6 \times 6^{-1} = 1$.
- If we have a matrix version of the above situation, we want an inverse that accomlishes the same thing.

Basic motivation for inverses

- In the case of *square matrices* we want to find an inverse function that allows us to "eliminate" a matrix from one side of an equation.
- For instance,

$$\mathbf{XA} = \mathbf{B}$$

$$\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$$

$$\mathbf{XI} = \mathbf{BA}^{-1}$$

$$\mathbf{X} = \mathbf{BA}^{-1}$$

- So what we really want is find a an inverse matrix such that:

$$\mathbf{AA}^{-1} = \mathbf{I}$$

Note on generalized inverses

- There are also generalized inverses for non-square matrices that can be derived and calculated.
- We won't cover this much in part because it is far more complicated, but also because you will almost never actually need to do it.
- However, note that these kinds of inverses are often denoted $\mathbf{A}^{-}$ rather than $\mathbf{A}^{-1}$.

How to find the inverse

- Figuring out how to invert large matrices is a big topic in both linear algebra, statistics, and computer science.

How to find the inverse

- Figuring out how to invert large matrices is a big topic in both linear algebra, statistics, and computer science.
- If you were a graduate student 30 years ago, this would be an important thing to understand in much greater detail. However, modern computer powers make the inversion of all but gigantic matrices a standard operation and you will not really need to understand that much about what the computer is doing – with one *very* important exception.
- It is possible (and even common) that a matrix may either not have an inverse or may not be invertible by particular methods.

- We will talk about one approach for calculating a matrix today (more later).
- We begin by adjoining this matrix ($\mathbf{A}$) to an identity matrix (an augmented matrix):

$$\left(\begin{array}{ccc|ccc} 2 & -2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 4 & 4 & -4 & 0 & 0 & 1 \end{array}\right)$$

- We want to find the set of operations that will turn the matrix on the left into the matrix on the right.

- We perform operations to both sides.
- When the left hand side is the identity matrix, the right hand side will be the inverse ($\mathbf{A}^{-1}$).

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- When the left hand side is the identity matrix, the right hand side will be the inverse ($\mathbf{A}^{-1}$).
- We perform three basic operations:
  1. Multiply a whole row by a non-zero scalar constant
  2. Add a scalar multiple of one row to another.
  3. Exchange two rows in the matrix.

- This is the same doing one of the following to both sides of the equation:
  1. Scalar multiplication of both sides
  2. Add a matrix to both sides. For instance if we wanted to subtract row 1 from row 2 we could add the following matrix to both sides:
  $$\begin{pmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
  3. Multiply both sides by a "transition matrix" that allows you to exchange rows. For instance, if we want to exchange the position of rows 1 and 2 we can multiply both sides by the matrix:
  $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 1

Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

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$$\left(\mathbf{A}|\mathbf{I}_2\right) \quad = \quad \begin{pmatrix} 1 & 1 & | & 1 & 0 \\ 2 & 3 & | & 0 & 1 \end{pmatrix}$$

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$$\begin{aligned} \left(\mathbf{A}|\mathbf{I}_2\right) &= \begin{pmatrix} 1 & 1 & | & 1 & 0 \\ 2 & 3 & | & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{pmatrix} \end{aligned}$$

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&= \begin{pmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & | & 3 & -1 \\ 0 & 1 & | & -2 & 1 \end{pmatrix}
\end{aligned}
$$

Check of answer

$$
\begin{aligned}
\mathbf{AA}^* &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathbf{I}_2 \\
\mathbf{A}^*\mathbf{A} &= \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathbf{I}_2 \\
\mathbf{A}^* &= \mathbf{A}^{-1}
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Example 2

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$$\left(\mathbf{A} | \mathbf{I}_3\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

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&= \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 2 & 3 & | & 0 & 1 & 0 \\ 0 & 0 & -4 & | & -5 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 2 & 3 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & \frac{5}{4} & 0 & -\frac{1}{4} \end{pmatrix}
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$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right)$$

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&= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right)
\end{aligned}
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Check of answer

$$\mathbf{A}\mathbf{A}^* = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{pmatrix}$$

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Laws of Matrix Inversion

1. $$\mathbf{I}^{-1} = \mathbf{I}$$

2. $$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

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4. $$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

5. $$(c\mathbf{A})^{-1} = c^{-1}\mathbf{A}^{-1}$$

6. If $\mathbf{A} = diag(a_1, a_2, \ldots, a_n)$ and all elements d_i are nonzero, then

$$\mathbf{A}^{-1} = diag(1/a_1, 1/a_2, \ldots, 1/d_n)$$

.

6. If $\mathbf{A} = diag(a_1, a_2, \ldots, a_n)$ and all elements d_i are nonzero, then

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7. The inverse of a symmetric matrix is symmetric.

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$$\mathbf{A}^{-1} = diag(1/a_1, 1/a_2, \ldots, 1/d_n)$$

   .
7. The inverse of a symmetric matrix is symmetric.
8. If the inverse exists (if the matrix is **nonsingular**), then it is unique.

If it exists, find the inverse of the following matrices

1. $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. $$\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$$

3. $$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

4.

$$\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$$

5.

$$\begin{pmatrix} 3 & 6 & 6 \\ -3 & 4 & 6 \\ -9 & 0 & 6 \end{pmatrix}$$

6.

$$\begin{pmatrix} 3 & 8 & 6 \\ 0 & -3 & -5 \\ -9 & 0 & 4 \end{pmatrix}$$

Advanced matrix operations

- Our final topic for today will cover some more advanced matrix operations that you may occasionally see.
- My goal here is that you should be familliar with these concepts.
- Mastery of these concepts is beyond the scope of this class, but you should know how to calculate a determinant for a small matrix and what it means.

Determinants

- Determinants can be used to *determine* whether a square matrix can be inverted (whehter or not it is nonsingular).
- A square matrix is nonsingular iff its determinant is not zero.

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- For a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{A}$ is nonsingular only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$ (check by doing Gaussian inverse to find the inverse of a 2×2 matrix).

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- We then define the determinant of a 2×2 matrix $\mathbf{A}$ as:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\
&= a_{11}|a_{22}| - a_{12}|a_{21}|
\end{aligned}
$$

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Kronecker Product

- Denoted $\mathbf{A} \otimes \mathbf{B}$ and sometimes called the direct product.
- Most commonly used to keep notation in some proof from getting out of hand.

Example 1:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \left(\begin{array}{cc|cc|cc} \sigma_1^2 & \sigma_{12} & 0 & 0 & 0 & 0 \\ \sigma_{12} & \sigma_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & \sigma_{12} & 0 & 0 \\ 0 & 0 & \sigma_{12} & \sigma_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1^2 & \sigma_{12} \\ 0 & 0 & 0 & 0 & \sigma_{12} & \sigma_2^2 \end{array}\right)$$

Example 2

$$\mathbf{A} \otimes \mathbf{B} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{(m \times n)} \otimes \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}}_{(p \times q)} \tag{1}$$

$$= \underbrace{\begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix}}_{(mp \times nq)} \tag{2}$$

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  $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -2 \cdot 1 - 1 \cdot 0 & 1 \cdot 2 - 3 \cdot 1 & 3 \cdot 0 - (-2) \cdot 2 \end{pmatrix}$
  $= \begin{pmatrix} -2 & -1 & 4 \end{pmatrix}$

Calculate the determinants for the following matrics and decide whether or not the inverse of each matrix exists.

1.
$$\begin{pmatrix} 6 & -9 & -8 \\ -5 & 4 & 10 \\ -9 & 8 & 6 \end{pmatrix}$$

2.
$$\begin{pmatrix} -9 & 8 & -9 \\ 1 & -3 & 4 \\ 15 & -7 & 6 \end{pmatrix}$$

3.

$$\begin{pmatrix} 3 & 0 & -2 \\ 5 & -6 & -6 \\ -2 & -4 & -8 \end{pmatrix}$$

4. Find the values of λ that make the following a singular matrix

$$\begin{pmatrix} 8-\lambda & 7 \\ 7 & 8-\lambda \end{pmatrix}$$