An Introduction to Bayesian Statistics

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- 6) Repeat 3—5 as necessary

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  - Step 2: Prior information
  - Step 5: Prior information $\rightarrow$ posterior information

Economic Applications of Bayesian Statistics

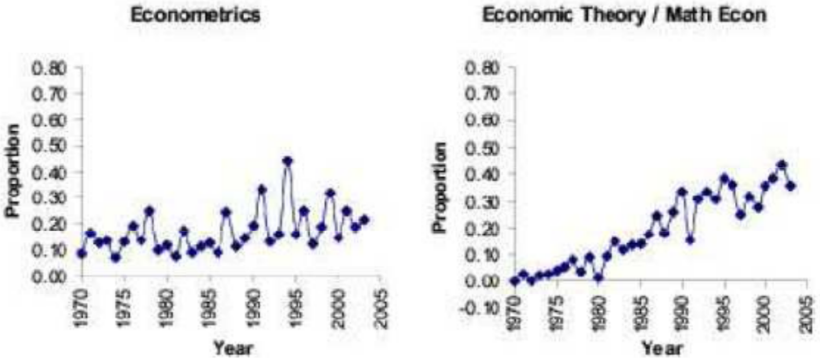

Figure 5: Econometrica Containing "Bayes" or "Bayesian"

Common Applications of Bayesian Statistics

- Economics

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- Economics
- Marketing

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- Once an outcome is revealed, prior information is updated

Table of Frequentist vs. Bayesian Interpretations

	Frequentist statistics	Bayesian statistics
Definition of the p value	The probability of observing the same or more extreme data assuming that the null hypothesis is true in the population	The probability of the (null) hypothesis
Large samples needed?	Usually, when normal theory-based methods are used	Not necessarily
Inclusion of prior knowledge possible?	No	Yes
Nature of the parameters in the model	Unknown but fixed	Unknown and therefore random
Population parameter	One true value	A distribution of values reflecting uncertainty
Uncertainty is defined by	The sampling distribution based on the idea of infinite repeated sampling	Probability distribution for the population parameter
Estimated intervals	Confidence interval: Over an infinity of samples taken from the population, 95% of these contain the true population value	Credibility interval: A 95% probability that the population value is within the limits of the interval

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- Prior reflects knowledge about parameters before observing current data
- Science can be accumulative!

Bayes' Theorem

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A) \times \mathrm{P}(A)}{\mathrm{P}(B)}$$

An Example of Bayes' Theorem

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- How should we update the uncertainty after a test?

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f(\theta|y) &= \frac{f(\theta, y)}{f(y)} \\
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- Posterior $\propto$ likelihood $\times$ prior

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Some Common Conjugate Priors

Likelihood	Prior	Posterior
$X\|\theta \sim \mathcal{N}(\theta, \sigma^2)$	$\theta \sim \mathcal{N}(\mu, \tau^2)$	$\theta\|X \sim \mathcal{N}(\frac{\tau^2}{\sigma^2+\tau^2}X + \frac{\sigma^2}{\sigma^2+\tau^2}\mu, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2})$
$X\|\theta \sim \mathcal{B}(n, \theta)$	$\theta \sim \mathcal{B}e(\alpha, \beta)$	$\theta\|X \sim \mathcal{B}e(\alpha + x, n - x + \beta)$
$X_1, \ldots, X_n\|\theta \sim \mathcal{P}(\theta)$	$\theta \sim \mathcal{G}a(\alpha, \beta)$	$\theta\|X_1, \ldots, X_n \sim \mathcal{G}a(\sum_i X_i + \alpha, n + \beta)$.
$X_1, \ldots, X_n\|\theta \sim \mathcal{NB}(m, \theta)$	$\theta \sim \mathcal{B}e(\alpha, \beta)$	$\theta\|X_1, \ldots, X_n \sim \mathcal{B}e(\alpha + mn, \beta + \sum_{i=1}^n x_i)$
$X \sim \mathcal{G}(n/2, 2\theta)$	$\theta \sim \mathcal{IG}(\alpha, \beta)$	$\theta\|X \sim \mathcal{IG}(n/2 + \alpha, (x/2 + \beta^{-1})^{-1})$
$X_1, \ldots, X_n\|\theta \sim \mathcal{U}(0, \theta)$	$\theta \sim \mathcal{P}a(\theta_0, \alpha)$	$\theta\|X_1, \ldots, X_n \sim \mathcal{P}a(\max\{\theta_0, x_1, \ldots, x_n\}\alpha + n)$
$X\|\theta \sim \mathcal{N}(\mu, \theta)$	$\theta \sim \mathcal{IG}(\alpha, \beta)$	$\theta\|X \sim \mathcal{IG}(\alpha + 1/2, \beta + (\mu - X)^2/2)$
$X\|\theta \sim \mathcal{G}a(\nu, \theta)$	$\theta \sim \mathcal{G}a(\alpha, \beta)$	$\theta\|X \sim \mathcal{G}a(\alpha + \nu, \beta + x)$

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- Data: An experiment is conducted and 9 heads and 3 tails are observed

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- Bayesian method has no difficulty $\rightarrow$ the same conclusion under both scenarios

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Simulate Normal Random Variables: Box–Muller Transformation

- We require two random variables, U and V, uniformly distributed on $[0, 1]$. Set

$$R = \sqrt{-2 \log V},$$
$$\theta = 2\pi U,$$

  and

$$Z_1 = R \cos \theta,$$
$$Z_2 = R \sin \theta.$$

  Then they are independent standard normal variables. To obtain two standard normal variables with correlation ρ, take

$$X = Z_1$$
$$Y = \rho Z_1 + \sqrt{1 - \rho^2} Z_2$$

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Frequentist Inference

- Ordinary least squares

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$
$$\hat{\sigma}^2 = s^2 = \frac{1}{n-k}(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})$$

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- Marginal posterior of $\boldsymbol{\beta}|\boldsymbol{y}$ is the multivariate t-distribution with $n-k$ degrees of freedom

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  - Hierarchical models can often fit data with a small number of parameters but can also do well in prediction

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- Observed datum is $\boldsymbol{X}$

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  - Third-stage prior: $\alpha \sim \mathit{Exp}(1)$

Hierarchical Linear Model

$$
\begin{aligned}
Y|X,\beta,\Sigma &\sim \mathcal{N}(X\beta,\Sigma) \\
\beta|X_\beta,\alpha,\Sigma_\beta &\sim \mathcal{N}(X_\beta\alpha,\Sigma_\beta) \\
\alpha|\alpha_0,\Sigma_\alpha &\sim \mathcal{N}(\alpha_0,\Sigma_\alpha)
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  - Draw θ_k from $p(\theta_k | \theta_1, \theta_2, \ldots, \theta_{k-1}, y)$

Gibbs Sampler (cont.)

- Full conditional distribution $p(\theta_j|\theta_{-j}, y)$, where $\theta_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$

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- Each θ_j is updated conditional on the latest values of θ

Example: Simulate from a Bivariate Normal Distribution

S

- Joint distribution

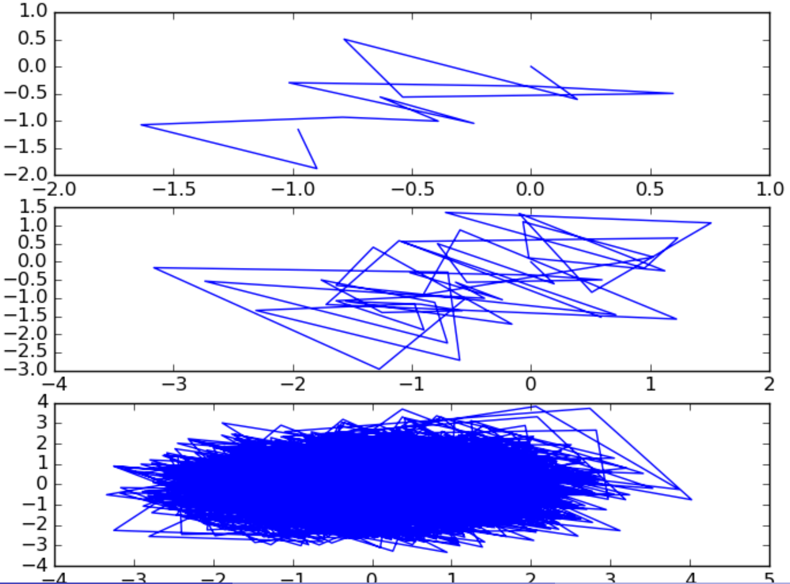
$$\boldsymbol{Z} = (X, Y)' \sim \mathcal{N}(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix})$$

Python Code

```
import numpy as np
x = [0]
y = [0]
rho = .9
c = np.sqrt(1 - rho**2)
for i in range (6000):
    x.append(rho*y[i - 1] + c*np.random.normal(0, 1, 1))
    y.append(rho*x[i] + c*np.random.normal(0, 1, 1))

plt.style.use('classic')
plt.figure()
plt.subplot(3, 1, 1)
plt.plot(x[0:14], y[0:14], '-')
plt.subplot(3, 1, 2)
plt.plot(x[0:49], y[0:49], '-')
plt.subplot(3, 1, 3)
plt.plot(x, y, '-');
```

The Resulting Posterior



General Property of Gibbs Sampler

- Output: A dependent sequence

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\theta^{(1)} &= \{\theta_1^{(1)}, \ldots, \theta_p^{(1)}\} \\
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- This is called a Markov property, and the sequence a Markov chain
- For the models in this class, the sampling distribution of $\theta^{(S)}$ approaches the target distribution as $S \to \infty$, regardless of starting value

$$Pr(\theta^{(S)} \in A \to \int_A p(\theta) d\theta \text{ as } S \to \infty$$

General Property of Gibbs Sampler (cont.)

- More importantly, for most functions g of interest,

$$\frac{1}{S}\sum_{s=1}^{S} g(\theta^{(s)}) \rightarrow E[g(\theta)] = \int g(\theta)p(\theta)d\theta \text{ as } S \rightarrow \infty$$

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- Hence, we call this method Markov chain Monte Carlo (MCMC)

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods

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  - Model specification
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  - Posterior summary
- When the posterior distribution is complicated, we can "look at" the posterior by studying Monte Carlo samples from the posterior

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- Estimation: How we use $p(\theta|y)$ to make inferences about θ
- Approximation: The use of Monte Carlo procedures to approximate integrals

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- We will deal with these later