An Introduction to Bayesian Statistics

David Carlson

May 16, 2022

The Scientific Method

- 1) Ask a question or pose a problem

The Scientific Method

- 1) Ask a question or pose a problem
- 2) Collect the existing relevant information

The Scientific Method

- 1) Ask a question or pose a problem
- 2) Collect the existing relevant information
- 3) Based on information, design an investigation or experiment to address the question

The Scientific Method

- 1) Ask a question or pose a problem
- 2) Collect the existing relevant information
- 3) Based on information, design an investigation or experiment to address the question
- 4) Carry out the investigation or experiment

The Scientific Method

- 1) Ask a question or pose a problem
- 2) Collect the existing relevant information
- 3) Based on information, design an investigation or experiment to address the question
- 4) Carry out the investigation or experiment
- 5) Use the evidence to update previously known information, then draw conclusions

The Scientific Method

- 1) Ask a question or pose a problem
- 2) Collect the existing relevant information
- 3) Based on information, design an investigation or experiment to address the question
- 4) Carry out the investigation or experiment
- 5) Use the evidence to update previously known information, then draw conclusions
- 6) Repeat 3—5 as necessary

Where Does Statistics Fit In?

- Central to steps 2, 3, and 5

Where Does Statistics Fit In?

- Central to steps 2, 3, and 5
  - Experimental design, survey sampling

Where Does Statistics Fit In?

- Central to steps 2, 3, and 5
  - Experimental design, survey sampling
  - Statistical inference

Where Does Statistics Fit In?

- Central to steps 2, 3, and 5
  - Experimental design, survey sampling
  - Statistical inference
- Bayesian statistics is particularly well-suited for step 2 and 5

Where Does Statistics Fit In?

- Central to steps 2, 3, and 5
  - Experimental design, survey sampling
  - Statistical inference
- Bayesian statistics is particularly well-suited for step 2 and 5
  - Step 2: Prior information

Where Does Statistics Fit In?

- Central to steps 2, 3, and 5
  - Experimental design, survey sampling
  - Statistical inference
- Bayesian statistics is particularly well-suited for step 2 and 5
  - Step 2: Prior information
  - Step 5: Prior information $\rightarrow$ posterior information

Economic Applications of Bayesian Statistics

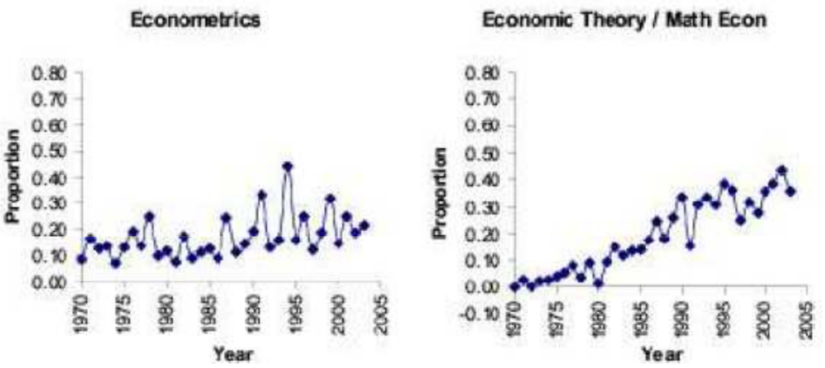

Figure 5: Econometrica Containing "Bayes" or "Bayesian"

Common Applications of Bayesian Statistics

- Economics

Common Applications of Bayesian Statistics

- Economics
- Marketing

Common Applications of Bayesian Statistics

- Economics
- Marketing
- Education

Common Applications of Bayesian Statistics

- Economics
- Marketing
- Education
- Medical research

Common Applications of Bayesian Statistics

- Economics
- Marketing
- Education
- Medical research
- Genomics

Common Applications of Bayesian Statistics

- Economics
- Marketing
- Education
- Medical research
- Genomics
- Weather

Common Applications of Bayesian Statistics

- Economics
- Marketing
- Education
- Medical research
- Genomics
- Weather
- The list goes on...

Theoretical Differences to Frequentist Approaches

- Probability as the subjective experience of uncertainty

Theoretical Differences to Frequentist Approaches

- Probability as the subjective experience of uncertainty
- No notion of infinitely repeating an event of interest

Theoretical Differences to Frequentist Approaches

- Probability as the subjective experience of uncertainty
- No notion of infinitely repeating an event of interest
- Using as much prior information as possible as well as personal judgment (placing a bet)

Theoretical Differences to Frequentist Approaches

- Probability as the subjective experience of uncertainty
- No notion of infinitely repeating an event of interest
- Using as much prior information as possible as well as personal judgment (placing a bet)
- Once an outcome is revealed, prior information is updated

Table of Frequentist vs. Bayesian Interpretations

	Frequentist statistics	Bayesian statistics
Definition of the p value	The probability of observing the same or more extreme data assuming that the null hypothesis is true in the population	The probability of the (null) hypothesis
Large samples needed?	Usually, when normal theory-based methods are used	Not necessarily
Inclusion of prior knowledge possible?	No	Yes
Nature of the parameters in the model	Unknown but fixed	Unknown and therefore random
Population parameter	One true value	A distribution of values reflecting uncertainty
Uncertainty is defined by	The sampling distribution based on the idea of infinite repeated sampling	Probability distribution for the population parameter
Estimated intervals	Confidence interval: Over an infinity of samples taken from the population, 95% of these contain the true population value	Credibility interval: A 95% probability that the population value is within the limits of the interval

Bayesian vs. Frequentist

- Frequentist: Parameter is unknown, but fixed

Bayesian vs. Frequentist

- Frequentist: Parameter is unknown, but fixed
- Bayesian: Unknown parameters are treated as uncertain and described by probability distribution

Bayesian vs. Frequentist

- Frequentist: Parameter is unknown, but fixed
- Bayesian: Unknown parameters are treated as uncertain and described by probability distribution
- Frequentist: 95 of 100 replications of same experiment capture the fixed but unknown parameter

Bayesian vs. Frequentist

- Frequentist: Parameter is unknown, but fixed
- Bayesian: Unknown parameters are treated as uncertain and described by probability distribution
- Frequentist: 95 of 100 replications of same experiment capture the fixed but unknown parameter
- Bayesian: Probability that a parameter lies in the credible interval

Priors

- Understanding of parameters of interest given our current data depends on our prior knowledge about the parameters of interest weighted by the current evidence given those parameters of interest

Priors

- Understanding of parameters of interest given our current data depends on our prior knowledge about the parameters of interest weighted by the current evidence given those parameters of interest
- Priors: Equally important to quantify ignorance as to quantify cumulative understanding of a problem

Priors

- Understanding of parameters of interest given our current data depends on our prior knowledge about the parameters of interest weighted by the current evidence given those parameters of interest
- Priors: Equally important to quantify ignorance as to quantify cumulative understanding of a problem
- Objective prior: Pure ignorance

Priors

- Understanding of parameters of interest given our current data depends on our prior knowledge about the parameters of interest weighted by the current evidence given those parameters of interest
- Priors: Equally important to quantify ignorance as to quantify cumulative understanding of a problem
- Objective prior: Pure ignorance
- Prior reflects knowledge about parameters before observing current data

Priors

- Understanding of parameters of interest given our current data depends on our prior knowledge about the parameters of interest weighted by the current evidence given those parameters of interest
- Priors: Equally important to quantify ignorance as to quantify cumulative understanding of a problem
- Objective prior: Pure ignorance
- Prior reflects knowledge about parameters before observing current data
- Science can be accumulative!

Bayes' Theorem

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A) \times \mathrm{P}(A)}{\mathrm{P}(B)}$$

An Example of Bayes' Theorem

- Enzyme-Linked Immuno Sobert Assay (ELISA)

An Example of Bayes' Theorem

- Enzyme-Linked Immuno Sobert Assay (ELISA)
  - Diagnosis tool for HIV

An Example of Bayes' Theorem

- Enzyme-Linked Immuno Sobert Assay (ELISA)
  - Diagnosis tool for HIV
  - Low cost

An Example of Bayes' Theorem

- Enzyme-Linked Immuno Sobert Assay (ELISA)
  - Diagnosis tool for HIV
  - Low cost
  - Less accurate compared to more expensive tools

An Example of Bayes' Theorem

- Enzyme-Linked Immuno Sobert Assay (ELISA)
  - Diagnosis tool for HIV
  - Low cost
  - Less accurate compared to more expensive tools
- Let $A = \{\text{the patient is positive}\}$. Is it proper to use $\Pr(A)$?

An Example of Bayes' Theorem

- Enzyme-Linked Immuno Sobert Assay (ELISA)
  - Diagnosis tool for HIV
  - Low cost
  - Less accurate compared to more expensive tools
- Let A $=$ {the patient is positive}. Is it proper to use Pr(A)?
- How should we update the uncertainty after a test?

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$\mathrm{P}(A|+) = \frac{\mathrm{P}(+|A)\mathrm{P}(A)}{\mathrm{P}(+|A)\mathrm{P}(A) + \mathrm{P}(+|\neg A)\mathrm{P}(\neg A)}$$

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$\mathrm{P}(A|+) = \frac{\mathrm{P}(+|A)\mathrm{P}(A)}{\mathrm{P}(+|A)\mathrm{P}(A) + \mathrm{P}(+|\neg A)\mathrm{P}(\neg A)}$$

- Sensitivity: $\mathrm{P}(+|A) = 0.95$

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$\mathrm{P}(A|+) = \frac{\mathrm{P}(+|A)\mathrm{P}(A)}{\mathrm{P}(+|A)\mathrm{P}(A) + \mathrm{P}(+|\neg A)\mathrm{P}(\neg A)}$$

- Sensitivity: $\mathrm{P}(+|A) = 0.95$
- Specificity: $\mathrm{P}(-|\neg A) = 1 - \mathrm{P}(+|\neg A) = 0.98$

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$\mathrm{P}(A|+) = \frac{\mathrm{P}(+|A)\mathrm{P}(A)}{\mathrm{P}(+|A)\mathrm{P}(A) + \mathrm{P}(+|\neg A)\mathrm{P}(\neg A)}$$

- Sensitivity: $\mathrm{P}(+|A) = 0.95$
- Specificity: $\mathrm{P}(-|\neg A) = 1 - \mathrm{P}(+|\neg A) = 0.98$
- Population prevalance: $\mathrm{P}(A) = 0.01$

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$\mathrm{P}(A|+) = \frac{\mathrm{P}(+|A)\mathrm{P}(A)}{\mathrm{P}(+|A)\mathrm{P}(A) + \mathrm{P}(+|\neg A)\mathrm{P}(\neg A)}$$

- Sensitivity: $\mathrm{P}(+|A) = 0.95$
- Specificity: $\mathrm{P}(-|\neg A) = 1 - \mathrm{P}(+|\neg A) = 0.98$
- Population prevalance: $\mathrm{P}(A) = 0.01$
- $\mathrm{P}(A|+) = 0.32$

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$P(A|+) = \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|\neg A)P(\neg A)}$$

- Sensitivity: $P(+|A) = 0.95$
- Specificity: $P(-|\neg A) = 1 - P(+|\neg A) = 0.98$
- Population prevalance: $P(A) = 0.01$
- $P(A|+) = 0.32$
- False positive rate: $P(\neg A|+) = 1 - P(A|+) = 0.68$

An Example of Bayes' Theorem (cont.)

- Data: Test result (positive/negative)
- Update the uncertainty using Bayes' theorem

$$\mathrm{P}(A|+) = \frac{\mathrm{P}(+|A)\mathrm{P}(A)}{\mathrm{P}(+|A)\mathrm{P}(A) + \mathrm{P}(+|\neg A)\mathrm{P}(\neg A)}$$

- Sensitivity: $\mathrm{P}(+|A) = 0.95$
- Specificity: $\mathrm{P}(-|\neg A) = 1 - \mathrm{P}(+|\neg A) = 0.98$
- Population prevalance: $\mathrm{P}(A) = 0.01$
- $\mathrm{P}(A|+) = 0.32$
- False positive rate: $\mathrm{P}(\neg A|+) = 1 - \mathrm{P}(A|+) = 0.68$
- False negative rate: $\mathrm{P}(A|-) = 0.00052$

General Form of Bayes' Theorem

- Model: $y|\theta \sim f(y|\theta)$

General Form of Bayes' Theorem

- Model: $y|\theta \sim f(y|\theta)$
  - y: data

General Form of Bayes' Theorem

- Model: $y|\theta \sim f(y|\theta)$
  - y: data
  - θ: parameter

General Form of Bayes' Theorem

- Model: $y|\theta \sim f(y|\theta)$
  - y: data
  - θ: parameter
- Bayes' theorem

$$
\begin{aligned}
f(\theta|y) &= \frac{f(\theta, y)}{f(y)} \\
&= \frac{f(y|\theta)f(\theta)}{f(y)} \\
&\propto f(y|\theta)f(\theta)
\end{aligned}
$$

General Form of Bayes' Theorem

- Model: $y|\theta \sim f(y|\theta)$
  - y: data
  - θ: parameter
- Bayes' theorem

$$
\begin{aligned}
f(\theta|y) &= \frac{f(\theta, y)}{f(y)} \\
&= \frac{f(y|\theta)f(\theta)}{f(y)} \\
&\propto f(y|\theta)f(\theta)
\end{aligned}
$$

- Posterior $\propto$ likelihood $\times$ prior

Key Issues in Bayesian Statistics

- How to specify prior distributions?

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?
  - In general we need to know the marginal distribution $f(y) = \int f(y|\theta)f(\theta)d\theta$

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?
  - In general we need to know the marginal distribution $f(y) = \int f(y|\theta)f(\theta)d\theta$
  - Direct computation of this integral is difficult especially when θ is high-dimensional

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?
  - In general we need to know the marginal distribution $f(y) = \int f(y|\theta)f(\theta)d\theta$
  - Direct computation of this integral is difficult especially when θ is high-dimensional
  - Simulation is the general solution

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?
  - In general we need to know the marginal distribution $f(y) = \int f(y|\theta)f(\theta)d\theta$
  - Direct computation of this integral is difficult especially when θ is high-dimensional
  - Simulation is the general solution
- How to summarize the posterior?

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?
  - In general we need to know the marginal distribution $f(y) = \int f(y|\theta)f(\theta)d\theta$
  - Direct computation of this integral is difficult especially when θ is high-dimensional
  - Simulation is the general solution
- How to summarize the posterior?
  - Posterior mean, median, or mode

Key Issues in Bayesian Statistics

- How to specify prior distributions?
- How to compute the posterior $f(\theta|y)$?
  - In general we need to know the marginal distribution $f(y) = \int f(y|\theta)f(\theta)d\theta$
  - Direct computation of this integral is difficult especially when θ is high-dimensional
  - Simulation is the general solution
- How to summarize the posterior?
  - Posterior mean, median, or mode
  - Credible interval

Comparison of Bayesian and Frequentist Methods

- Interval estimation

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval
- Hypothesis testing

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval
- Hypothesis testing
  - Consider testing $H_0 : \theta = 0.5$ vs. $H_1 : \theta > 0.5$

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval
- Hypothesis testing
  - Consider testing $H_0 : \theta = 0.5$ vs. $H_1 : \theta > 0.5$
  - Frequentist:

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval
- Hypothesis testing
  - Consider testing $H_0 : \theta = 0.5$ vs. $H_1 : \theta > 0.5$
  - Frequentist:
    - *p*-value can only be computed when the paramter value is given under the null

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval
- Hypothesis testing
  - Consider testing $H_0 : \theta = 0.5$ vs. $H_1 : \theta > 0.5$
  - Frequentist:
    - p-value can only be computed when the paramter value is given under the null
    - If we switch the role of two hypotheses, frequentists cannot compute p-value

Comparison of Bayesian and Frequentist Methods

- Interval estimation
  - Frequentist: Confidence interval
    - The experiment should be repeatable
    - Not conditional on the data
  - Bayesian: Credible interval
- Hypothesis testing
  - Consider testing $H_0 : \theta = 0.5$ vs. $H_1 : \theta > 0.5$
  - Frequentist:
    - p-value can only be computed when the paramter value is given under the null
    - If we switch the role of two hypotheses, frequentists cannot compute p-value
  - Bayesian: No trouble

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult
- Advantages of Bayesian

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult
- Advantages of Bayesian
  - Can easily incorporate prior information

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult
- Advantages of Bayesian
  - Can easily incorporate prior information
  - Inferences are conditional on the actual data

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult
- Advantages of Bayesian
  - Can easily incorporate prior information
  - Inferences are conditional on the actual data
  - Results are more easily interpretable

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult
- Advantages of Bayesian
  - Can easily incorporate prior information
  - Inferences are conditional on the actual data
  - Results are more easily interpretable
  - Inferences are based on posterior, which is conceptually simple

Advantages and Disadvantages of Bayesian

- Disadvantages of Bayesian
  - Subjective choice of prior
  - Computation of posterior can be difficult
- Advantages of Bayesian
  - Can easily incorporate prior information
  - Inferences are conditional on the actual data
  - Results are more easily interpretable
  - Inferences are based on posterior, which is conceptually simple
  - In principle, all problems can be solved

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior
  - Histogram

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior
  - Histogram
  - Parameters in priors using moments or quantiles

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior
  - Histogram
  - Parameters in priors using moments or quantiles
  - Interactive computer programs

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior
  - Histogram
  - Parameters in priors using moments or quantiles
  - Interactive computer programs
  - Using a hierarchical structure is often helpful

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior
  - Histogram
  - Parameters in priors using moments or quantiles
  - Interactive computer programs
  - Using a hierarchical structure is often helpful
- Noninformative prior

Priors

- Mixture of conjugate prior $\rightarrow$ mixture of conjugate posterior
  - Prior $\times$ posterior takes known form
  - Computationally efficient
  - Analytically tractible
- Elicited (informative) prior
  - Histogram
  - Parameters in priors using moments or quantiles
  - Interactive computer programs
  - Using a hierarchical structure is often helpful
- Noninformative prior
- Empirical Bayes

Noninformative Prior

- Flat (uniform)

Noninformative Prior

- Flat (uniform)
  - Improper when the support is not finite

Noninformative Prior

- Flat (uniform)
  - Improper when the support is not finite
  - Improper prior can lead to proper posterior

Noninformative Prior

- Flat (uniform)
  - Improper when the support is not finite
  - Improper prior can lead to proper posterior
  - Usually not flat after transformation

Noninformative Prior

- Flat (uniform)
  - Improper when the support is not finite
  - Improper prior can lead to proper posterior
  - Usually not flat after transformation
- Jeffrey's prior: Invariant under one-to-one transformation

Noninformative Prior

- Flat (uniform)
  - Improper when the support is not finite
  - Improper prior can lead to proper posterior
  - Usually not flat after transformation
- Jeffrey's prior: Invariant under one-to-one transformation
- Location invariant prior for location family

Noninformative Prior

- Flat (uniform)
  - Improper when the support is not finite
  - Improper prior can lead to proper posterior
  - Usually not flat after transformation
- Jeffrey's prior: Invariant under one-to-one transformation
- Location invariant prior for location family
- Scale invariant prior for scale family

Some Basic Models

- Beta-binomial model

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model
  - $y_1, \ldots, y_n|\theta \sim \text{Poisson}$

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model
  - $y_1, \ldots, y_n|\theta \sim$ Poisson
  - Prior: gamma

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model
  - $y_1, \ldots, y_n|\theta \sim \text{Poisson}$
  - Prior: gamma
  - Marginal model: Negative-binomial

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model
  - $y_1, \dots, y_n|\theta \sim$ Poisson
  - Prior: gamma
  - Marginal model: Negative-binomial
- Exponential model

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model
  - $y_1, \ldots, y_n|\theta \sim \text{Poisson}$
  - Prior: gamma
  - Marginal model: Negative-binomial
- Exponential model
  - $y_1, \ldots, y_n|\theta \sim \text{Exp}(\theta)$

Some Basic Models

- Beta-binomial model
  - $y|\theta \sim \text{binomial}(n, \theta)$
  - $\theta \sim \text{beta}(a, b)$
  - Posterior: beta
- Poisson model
  - $y_1, \ldots, y_n|\theta \sim \text{Poisson}$
  - Prior: gamma
  - Marginal model: Negative-binomial
- Exponential model
  - $y_1, \ldots, y_n|\theta \sim \text{Exp}(\theta)$
  - Prior: gamma

Some Common Conjugate Priors

Likelihood	Prior	Posterior
$X\|\theta \sim \mathcal{N}(\theta, \sigma^2)$	$\theta \sim \mathcal{N}(\mu, \tau^2)$	$\theta\|X \sim \mathcal{N}(\frac{\tau^2}{\sigma^2+\tau^2}X + \frac{\sigma^2}{\sigma^2+\tau^2}\mu, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2})$
$X\|\theta \sim \mathcal{B}(n, \theta)$	$\theta \sim \mathcal{B}e(\alpha, \beta)$	$\theta\|X \sim \mathcal{B}e(\alpha + x, n - x + \beta)$
$X_1, \ldots, X_n\|\theta \sim \mathcal{P}(\theta)$	$\theta \sim \mathcal{G}a(\alpha, \beta)$	$\theta\|X_1, \ldots, X_n \sim \mathcal{G}a(\sum_i X_i + \alpha, n + \beta)$.
$X_1, \ldots, X_n\|\theta \sim \mathcal{NB}(m, \theta)$	$\theta \sim \mathcal{B}e(\alpha, \beta)$	$\theta\|X_1, \ldots, X_n \sim \mathcal{B}e(\alpha + mn, \beta + \sum_{i=1}^n x_i)$
$X \sim \mathcal{G}(n/2, 2\theta)$	$\theta \sim \mathcal{IG}(\alpha, \beta)$	$\theta\|X \sim \mathcal{IG}(n/2 + \alpha, (x/2 + \beta^{-1})^{-1})$
$X_1, \ldots, X_n\|\theta \sim \mathcal{U}(0, \theta)$	$\theta \sim \mathcal{P}a(\theta_0, \alpha)$	$\theta\|X_1, \ldots, X_n \sim \mathcal{P}a(\max\{\theta_0, x_1, \ldots, x_n\}\alpha + n)$
$X\|\theta \sim \mathcal{N}(\mu, \theta)$	$\theta \sim \mathcal{IG}(\alpha, \beta)$	$\theta\|X \sim \mathcal{IG}(\alpha + 1/2, \beta + (\mu - X)^2/2)$
$X\|\theta \sim \mathcal{G}a(\nu, \theta)$	$\theta \sim \mathcal{G}a(\alpha, \beta)$	$\theta\|X \sim \mathcal{G}a(\alpha + \nu, \beta + x)$

Likelihood Principle

- Likelihood principle: In the inference about θ, after y is observed, all relevant experimental information is contained in the likelihood function for the observed y. Furthermore, two likelihood functions contain the same information about θ if they are proportional to each other

Likelihood Principle

- Likelihood principle: In the inference about θ, after y is observed, all relevant experimental information is contained in the likelihood function for the observed y. Furthermore, two likelihood functions contain the same information about θ if they are proportional to each other
- Consider testing the fairness of a coin:

$$H_0 : \theta = \frac{1}{2} \text{ vs. } H_1 : \theta > \frac{1}{2}$$

Likelihood Principle

- Likelihood principle: In the inference about θ, after y is observed, all relevant experimental information is contained in the likelihood function for the observed y. Furthermore, two likelihood functions contain the same information about θ if they are proportional to each other
- Consider testing the fairness of a coin:

$$H_0 : \theta = \frac{1}{2} \text{ vs. } H_1 : \theta > \frac{1}{2}$$

- Data: An experiment is conducted and 9 heads and 3 tails are observed

Two Possible Experiments

- Binomial: 12 toss in total

Two Possible Experiments

- Binomial: 12 toss in total
- Negative binomial: keep tossing until get three tails

Two Possible Experiments

- Binomial: 12 toss in total
- Negative binomial: keep tossing until get three tails
- Likelihoods are proportional

Two Possible Experiments

- Binomial: 12 toss in total
- Negative binomial: keep tossing until get three tails
- Likelihoods are proportional
- Conclusions based on p-values are contradictive $\rightarrow$ violation of likelihood principle

Two Possible Experiments

- Binomial: 12 toss in total
- Negative binomial: keep tossing until get three tails
- Likelihoods are proportional
- Conclusions based on p-values are contradictive $\rightarrow$ violation of likelihood principle
- Bayesian method has no difficulty $\rightarrow$ the same conclusion under both scenarios

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$
  - $var(\theta) = E(var(\theta|y)) + var(E(\theta|y)) \geq E(var(\theta|y))$

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$
  - $var(\theta) = E(var(\theta|y)) + var(E(\theta|y)) \geq E(var(\theta|y))$
- Sequential updates in Bayesian inference

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$
  - $var(\theta) = E(var(\theta|y)) + var(E(\theta|y)) \geq E(var(\theta|y))$
- Sequential updates in Bayesian inference
  - Prior: $p(\theta)$

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$
  - $var(\theta) = E(var(\theta|y)) + var(E(\theta|y)) \geq E(var(\theta|y))$
- Sequential updates in Bayesian inference
  - Prior: $p(\theta)$
  - After first batch of data $y_1 \rightarrow p(\theta|y_1) \propto p(y_1|\theta)p(\theta)$

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$
  - $var(\theta) = E(var(\theta|y)) + var(E(\theta|y)) \geq E(var(\theta|y))$
- Sequential updates in Bayesian inference
  - Prior: $p(\theta)$
  - After first batch of data $y_1 \rightarrow p(\theta|y_1) \propto p(y_1|\theta)p(\theta)$
  - After second batch y_2 (assume it is conditionally independent from y_1) $\rightarrow p(\theta|y_1, y_2) \propto p(y_2|\theta)p(\theta|y_1) \propto p(y_2|\theta)p(y_1\theta)p(\theta) = p(y_1, y_2|\theta)p(\theta)$

Some General Facts About Bayesian Inference

- On average, posterior distribution is less variable than the prior distribution
  - Prior variance: $var(\theta)$
  - Posterior variance: $var(\theta|y)$
  - $var(\theta) = E(var(\theta|y)) + var(E(\theta|y)) \geq E(var(\theta|y))$
- Sequential updates in Bayesian inference
  - Prior: $p(\theta)$
  - After first batch of data $y_1 \to p(\theta|y_1) \propto p(y_1|\theta)p(\theta)$
  - After second batch y_2 (assume it is conditionally independent from y_1) $\to p(\theta|y_1, y_2) \propto p(y_2|\theta)p(\theta|y_1) \propto p(y_2|\theta)p(y_1\theta)p(\theta) = p(y_1, y_2|\theta)p(\theta)$
  - This is the same as if we observed both batches together

Simulate Normal Random Variables: Box–Muller Transformation

- We require two random variables, U and V, uniformly distributed on $[0,1]$. Set

$$R = \sqrt{-2\log V},$$
$$\theta = 2\pi U,$$

and

$$Z_1 = R\cos\theta,$$
$$Z_2 = R\sin\theta.$$

Then they are independent standard normal variables. To obtain two standard normal variables with correlation ρ, take

$$X = Z_1$$
$$Y = \rho Z_1 + \sqrt{1-\rho^2}Z_2$$

Multiple Linear Regression

- Response: y

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$
- Goal: Find $y = f(\mathbf{X})$

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$
- Goal: Find $y = f(\mathbf{X})$
- Data: $(\mathbf{x}_i, y_i), i = 1, \ldots, n$

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$
- Goal: Find $y = f(\mathbf{X})$
- Data: $(\mathbf{x}_i, y_i), i = 1, \ldots, n$
- Model: $y_i = f(\mathbf{x}_i) + \epsilon_i$,

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$
- Goal: Find $y = f(\mathbf{X})$
- Data: $(\mathbf{x}_i, y_i), i = 1, \ldots, n$
- Model: $y_i = f(\mathbf{x}_i) + \epsilon_i$,
- Typical assumption for normal linear model:

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2) iid$$

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$
- Goal: Find $y = f(\mathbf{X})$
- Data: $(\mathbf{x}_i, y_i), i = 1, \ldots, n$
- Model: $y_i = f(\mathbf{x}_i) + \epsilon_i,$
- Typical assumption for normal linear model:

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2) iid$$

- $E(\mathbf{y}|\mathbf{X}) = f(\mathbf{X})$

Multiple Linear Regression

- Response: y
- Explanatory variables: $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$
- Goal: Find $y = f(\mathbf{X})$
- Data: $(\mathbf{x}_i, y_i), i = 1, \ldots, n$
- Model: $y_i = f(\mathbf{x}_i) + \epsilon_i$,
- Typical assumption for normal linear model:

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2) iid$$

- $E(\mathbf{y}|\mathbf{X}) = f(\mathbf{X})$
- $\mathbf{y}|\mathbf{X} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2 I)$

Frequentist Inference

- Ordinary least squares

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$
$$\hat{\sigma}^2 = s^2 = \frac{1}{n-k}(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})$$

Bayesian Inference

- Noninformative prior:

$$f(\beta, \sigma^2 | \mathbf{X}) \propto \sigma^{-2}$$

Bayesian Inference

- Noninformative prior:

$$f(\boldsymbol{\beta}, \sigma^2 | \mathbf{X}) \propto \sigma^{-2}$$

- Posterior:

$$\boldsymbol{\beta}, \sigma^2 | \mathbf{y} \sim \mathcal{N}(\hat{\boldsymbol{\beta}}, V_\beta \sigma^2)$$
$$\frac{(n-k)s^2}{\sigma^2} | \mathbf{y} \sim \chi^2_{n-k}$$
$$\sigma^2 | \mathbf{y} \sim Inv - \chi^2(n-k, s^2)$$

Bayesian Inference

- Noninformative prior:

$$f(\boldsymbol{\beta}, \sigma^2 | \mathbf{X}) \propto \sigma^{-2}$$

- Posterior:

$$\begin{aligned} \boldsymbol{\beta}, \sigma^2 | \mathbf{y} &\sim \mathcal{N}(\hat{\boldsymbol{\beta}}, V_\beta \sigma^2) \\ \frac{(n-k)s^2}{\sigma^2} | \mathbf{y} &\sim \chi^2_{n-k} \\ \sigma^2 | \mathbf{y} &\sim Inv - \chi^2(n-k, s^2) \end{aligned}$$

- Marginal posterior of $\boldsymbol{\beta}|\mathbf{y}$ is the multivariate t-distribution with $n-k$ degrees of freedom

Hierarchical Model

- Powerful technique for describing complex models. Idea is to break the model down into smaller easier understood pieces, which when put together describes the joint distribution of all data and parameters

Hierarchical Model

- Powerful technique for describing complex models. Idea is to break the model down into smaller easier understood pieces, which when put together describes the joint distribution of all data and parameters
- Why go hierarchical?

Hierarchical Model

- Powerful technique for describing complex models. Idea is to break the model down into smaller easier understood pieces, which when put together describes the joint distribution of all data and parameters
- Why go hierarchical?
  - Non-hierarchical models with few parameters generally don't fit the data well

Hierarchical Model

- Powerful technique for describing complex models. Idea is to break the model down into smaller easier understood pieces, which when put together describes the joint distribution of all data and parameters
- Why go hierarchical?
  - Non-hierarchical models with few parameters generally don't fit the data well
  - Non-hierarchical models with many parameters tend to fit the data well, but have poor predictive ability (overfitting)

Hierarchical Model

- Powerful technique for describing complex models. Idea is to break the model down into smaller easier understood pieces, which when put together describes the joint distribution of all data and parameters
- Why go hierarchical?
  - Non-hierarchical models with few parameters generally don't fit the data well
  - Non-hierarchical models with many parameters tend to fit the data well, but have poor predictive ability (overfitting)
  - Hierarchical models can often fit data with a small number of parameters but can also do well in prediction

A Simple Example

- Observed datum is $\mathbf{X}$

$$\mathbf{X}|\theta \sim \mathcal{N}(\theta, 1)$$

A Simple Example

- Observed datum is **X**

$$\mathbf{X}|\theta \sim \mathcal{N}(\theta, 1)$$

- First-stage prior: $\theta|\tau^2 \sim \mathcal{N}(0, \tau^2)$

A Simple Example

- Observed datum is **X**

$$\mathbf{X}|\theta \sim \mathcal{N}(\theta, 1)$$

- First-stage prior: $\theta|\tau^2 \sim \mathcal{N}(0, \tau^2)$
- Second-stage prior: $\tau^2 \sim \pi$ where π is completely specified

A Simple Example

- Observed datum is $\mathbf{X}$

$$\mathbf{X}|\theta \sim \mathcal{N}(\theta, 1)$$

- First-stage prior: $\theta|\tau^2 \sim \mathcal{N}(0, \tau^2)$
- Second-stage prior: $\tau^2 \sim \pi$ where π is completely specified
- Possibly more levels

A Simple Example

- Observed datum is **X**

$$\mathbf{X}|\theta \sim \mathcal{N}(\theta, 1)$$

- First-stage prior: $\theta|\tau^2 \sim \mathcal{N}(0, \tau^2)$
- Second-stage prior: $\tau^2 \sim \pi$ where π is completely specified
- Possibly more levels
  - Second-stage prior: $\tau^2|\alpha \sim \mathit{gamma}(\alpha, 1)$

A Simple Example

- Observed datum is **X**

$$\mathbf{X}|\theta \sim \mathcal{N}(\theta, 1)$$

- First-stage prior: $\theta|\tau^2 \sim \mathcal{N}(0, \tau^2)$
- Second-stage prior: $\tau^2 \sim \pi$ where π is completely specified
- Possibly more levels
  - Second-stage prior: $\tau^2|\alpha \sim \mathit{gamma}(\alpha, 1)$
  - Third-stage prior: $\alpha \sim \mathit{Exp}(1)$

Hierarchical Linear Model

$$
\begin{aligned}
Y|X, \beta, \Sigma &\sim \mathcal{N}(X\beta, \Sigma) \\
\beta|X_\beta, \alpha, \Sigma_\beta &\sim \mathcal{N}(X_\beta\alpha, \Sigma_\beta) \\
\alpha|\alpha_0, \Sigma_\alpha &\sim \mathcal{N}(\alpha_0, \Sigma_\alpha)
\end{aligned}
$$

Simple Random Effects Model

- J groups

Simple Random Effects Model

- J groups
- Data in group j: $Y_{1j}, \ldots, Y_{n_j j}$

Simple Random Effects Model

- J groups
- Data in group j: $Y_{1j}, \ldots, Y_{n_j j}$
- $Y_{ij} | \beta_j, \sigma^2 \sim \mathcal{N}(\beta_j, \sigma^2)$ independent, $j = 1, \ldots, J$, $i = 1, \ldots, n_j$

Simple Random Effects Model

- J groups
- Data in group j: $Y_{1j}, \ldots, Y_{n_j j}$
- $Y_{ij} | \beta_j, \sigma^2 \sim \mathcal{N}(\beta_j, \sigma^2)$ independent, $j = 1, \ldots, J$, $i = 1, \ldots, n_j$
- Random effects: $\beta_j | \alpha, \sigma_\beta^2 \sim \mathcal{N}(\alpha, \sigma_\beta^2)$ iid

Simple Random Effects Model

- J groups
- Data in group j: $Y_{1j}, \ldots, Y_{n_j j}$
- $Y_{ij} | \beta_j, \sigma^2 \sim \mathcal{N}(\beta_j, \sigma^2)$ independent, $j = 1, \ldots, J$, $i = 1, \ldots, n_j$
- Random effects: $\beta_j | \alpha, \sigma_\beta^2 \sim \mathcal{N}(\alpha, \sigma_\beta^2)$ iid
- $\alpha \sim \mathcal{N}(\alpha_0, \sigma_\alpha^2)$

Posterior Inference by Simulation

- Approach 1: Independence sampling

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \ldots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \ldots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$
  - This is all we have discussed so far

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \dots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$
  - This is all we have discussed so far
- Approach 2: Markov Chain Monte Carlo (MCMC)

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \ldots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$
  - This is all we have discussed so far
- Approach 2: Markov Chain Monte Carlo (MCMC)
  - Draw $\theta^{(i+1)}$ from $g(\theta^{(i+1)}|\theta^{(i)}$ such that

$$f(\theta^{(i+1)}) \to f(\theta|y)$$

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \ldots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$
  - This is all we have discussed so far
- Approach 2: Markov Chain Monte Carlo (MCMC)
  - Draw $\theta^{(i+1)}$ from $g(\theta^{(i+1)}|\theta^{(i)}$ such that

$$f(\theta^{(i+1)}) \rightarrow f(\theta|y)$$

  - Gibbs sampling

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \ldots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$
  - This is all we have discussed so far
- Approach 2: Markov Chain Monte Carlo (MCMC)
  - Draw $\theta^{(i+1)}$ from $g(\theta^{(i+1)}|\theta^{(i)}$ such that

$$f(\theta^{(i+1)}) \to f(\theta|y)$$

  - Gibbs sampling
  - Metropolis-Hastings

Posterior Inference by Simulation

- Approach 1: Independence sampling
  - Simulate independent samples from the posterior distributions
  - Draw $\theta^{(1)}, \ldots, \theta^{(M)}$ iid from the posterior distribution $f(\theta|y)$
  - This is all we have discussed so far
- Approach 2: Markov Chain Monte Carlo (MCMC)
  - Draw $\theta^{(i+1)}$ from $g(\theta^{(i+1)}|\theta^{(i)}$ such that

$$f(\theta^{(i+1)}) \to f(\theta|y)$$

  - Gibbs sampling
  - Metropolis-Hastings
  - Hamiltonian dynamics

Gibbs Sampler

- Used for multiparameter models

Gibbs Sampler

- Used for multiparameter models
- Parameter: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$

Gibbs Sampler

- Used for multiparameter models
- Parameter: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$
- An iterative algorithm

Gibbs Sampler

- Used for multiparameter models
- Parameter: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$
- An iterative algorithm
  - Draw θ_1 from $p(\theta_1 | \theta_2, \theta_3, \ldots, \theta_k, y)$

Gibbs Sampler

- Used for multiparameter models
- Parameter: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$
- An iterative algorithm
  - Draw θ_1 from $p(\theta_1 | \theta_2, \theta_3, \ldots, \theta_k, y)$
  - Draw θ_2 from $p(\theta_2 | \theta_1, \theta_3, \ldots, \theta_k, y)$

Gibbs Sampler

- Used for multiparameter models
- Parameter: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$
- An iterative algorithm
  - Draw θ_1 from $p(\theta_1 | \theta_2, \theta_3, \ldots, \theta_k, y)$
  - Draw θ_2 from $p(\theta_2 | \theta_1, \theta_3, \ldots, \theta_k, y)$
  - $\ldots$

Gibbs Sampler

- Used for multiparameter models
- Parameter: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$
- An iterative algorithm
  - Draw θ_1 from $p(\theta_1|\theta_2, \theta_3, \ldots, \theta_k, y)$
  - Draw θ_2 from $p(\theta_2|\theta_1, \theta_3, \ldots, \theta_k, y)$
  - $\ldots$
  - Draw θ_k from $p(\theta_k|\theta_1, \theta_2, \ldots, \theta_{k-1}, y)$

Gibbs Sampler (cont.)

- Full conditional distribution $p(\theta_j|\theta_{-j}, y)$, where $\theta_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$

Gibbs Sampler (cont.)

- Full conditional distribution $p(\theta_j|\theta_{-j}, y)$, where $\theta_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$
- In iteration t, draw $\theta_j^t = p(\theta_j|\theta_{-j}^t, y)$, where $(\theta_1^t, \ldots, \theta_{j-1}^t, \theta_{j+1}^{t-1}, \ldots, \theta_k^{t-1})$

Gibbs Sampler (cont.)

- Full conditional distribution $p(\theta_j|\theta_{-j}, y)$, where $\theta_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$
- In iteration t, draw $\theta_j^t = p(\theta_j|\theta_{-j}^t, y)$, where $(\theta_1^t, \ldots, \theta_{j-1}^t, \theta_{j+1}^{t-1}, \ldots, \theta_k^{t-1})$
- Each θ_j is updated conditional on the latest values of θ

General Property of Gibbs Sampler

- Output: A dependent sequence

$$\begin{aligned}\theta^{(1)} &= \{\theta_1^{(1)}, \ldots, \theta_p^{(1)}\} \\ \theta^{(2)} &= \{\theta_1^{(2)}, \ldots, \theta_p^{(2)}\} \\ &\vdots \\ \theta^{(S)} &= \{\theta_1^{(S)}, \ldots, \theta_p^{(S)}\}\end{aligned}$$

General Property of Gibbs Sampler

- Output: A dependent sequence

$$
\begin{aligned}
\theta^{(1)} &= \{\theta_1^{(1)}, \dots, \theta_p^{(1)}\} \\
\theta^{(2)} &= \{\theta_1^{(2)}, \dots, \theta_p^{(2)}\} \\
&\vdots \\
\theta^{(S)} &= \{\theta_1^{(S)}, \dots, \theta_p^{(S)}\}
\end{aligned}
$$

- $\theta^{(S)}$ depends on $\theta^{(0)}, \dots, \theta^{(S-1)}$ only through $\theta^{(S-1)}$

General Property of Gibbs Sampler

- Output: A dependent sequence

$$\theta^{(1)} = \{\theta_1^{(1)}, \ldots, \theta_p^{(1)}\}$$
$$\theta^{(2)} = \{\theta_1^{(2)}, \ldots, \theta_p^{(2)}\}$$
$$\vdots$$
$$\theta^{(S)} = \{\theta_1^{(S)}, \ldots, \theta_p^{(S)}\}$$

- $\theta^{(S)}$ depends on $\theta^{(0)}, \ldots, \theta^{(S-1)}$ only through $\theta^{(S-1)}$
- $\theta^{(S)}$ is conditionally independent of $\theta^{(0)}, \ldots, \theta^{(S-2)}$ given $\theta^{(S-1)}$

General Property of Gibbs Sampler

- Output: A dependent sequence

$$
\begin{aligned}
\theta^{(1)} &= \{\theta_1^{(1)}, \ldots, \theta_p^{(1)}\} \\
\theta^{(2)} &= \{\theta_1^{(2)}, \ldots, \theta_p^{(2)}\} \\
&\vdots \\
\theta^{(S)} &= \{\theta_1^{(S)}, \ldots, \theta_p^{(S)}\}
\end{aligned}
$$

- $\theta^{(S)}$ depends on $\theta^{(0)}, \ldots, \theta^{(S-1)}$ only through $\theta^{(S-1)}$
- $\theta^{(S)}$ is conditionally independent of $\theta^{(0)}, \ldots, \theta^{(S-2)}$ given $\theta^{(S-1)}$
- This is called a Markov property, and the sequence a Markov chain

General Property of Gibbs Sampler

- Output: A dependent sequence

$$\theta^{(1)} = \{\theta_1^{(1)}, \ldots, \theta_p^{(1)}\}$$
$$\theta^{(2)} = \{\theta_1^{(2)}, \ldots, \theta_p^{(2)}\}$$
$$\vdots$$
$$\theta^{(S)} = \{\theta_1^{(S)}, \ldots, \theta_p^{(S)}\}$$

- $\theta^{(S)}$ depends on $\theta^{(0)}, \ldots, \theta^{(S-1)}$ only through $\theta^{(S-1)}$
- $\theta^{(S)}$ is conditionally independent of $\theta^{(0)}, \ldots, \theta^{(S-2)}$ given $\theta^{(S-1)}$
- This is called a Markov property, and the sequence a Markov chain
- For the models in this class, the sampling distribution of $\theta^{(S)}$ approaches the target distribution as $S \to \infty$, regardless of starting value

$$Pr(\theta^{(S)} \in A \to \int_A p(\theta) d\theta \text{ as } S \to \infty$$

General Property of Gibbs Sampler (cont.)

- More importantly, for most functions g of interest,

$$\frac{1}{S}\sum_{s=1}^{S} g(\theta^{(s)}) \rightarrow E[g(\theta)] = \int g(\theta)p(\theta)d\theta \text{ as } S \rightarrow \infty$$

General Property of Gibbs Sampler (cont.)

- More importantly, for most functions g of interest,

$$\frac{1}{S}\sum_{s=1}^{S} g(\theta^{(s)}) \to E[g(\theta)] = \int g(\theta)p(\theta)d\theta \text{ as } S \to \infty$$

- One can approximate $E[g(\theta)]$ with sample average of $\{g(\theta^{(1)}), \ldots, g(\theta^{(S)})\}$. This is the Monte Carlo part

General Property of Gibbs Sampler (cont.)

- More importantly, for most functions g of interest,

$$\frac{1}{S}\sum_{s=1}^{S} g(\theta^{(s)}) \to E[g(\theta)] = \int g(\theta)p(\theta)d\theta \text{ as } S \to \infty$$

- One can approximate $E[g(\theta)]$ with sample average of $\{g(\theta^{(1)}), \ldots, g(\theta^{(S)})\}$. This is the Monte Carlo part
- Hence, we call this method Markov chain Monte Carlo (MCMC)

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part
  - Numerical approximation: The Monte Carlo part

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part
  - Numerical approximation: The Monte Carlo part
- Ingredients of Bayesian data analysis

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part
  - Numerical approximation: The Monte Carlo part
- Ingredients of Bayesian data analysis
  - Model specification

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part
  - Numerical approximation: The Monte Carlo part
- Ingredients of Bayesian data analysis
  - Model specification
  - Prior specification

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part
  - Numerical approximation: The Monte Carlo part
- Ingredients of Bayesian data analysis
  - Model specification
  - Prior specification
  - Posterior summary

Distinguishing Parameter Estimation from Posterior Approximation

- Bayesian data analysis using Monte Carlo methods
  - Data analysis: The statistical part
  - Numerical approximation: The Monte Carlo part
- Ingredients of Bayesian data analysis
  - Model specification
  - Prior specification
  - Posterior summary
- When the posterior distribution is complicated, we can "look at" the posterior by studying Monte Carlo samples from the posterior

Distinguishing Parameter Estimation from Posterior Approximation

- Monte Carlo and MCMC sampling algorithms

Distinguishing Parameter Estimation from Posterior Approximation

- Monte Carlo and MCMC sampling algorithms
  - Are not models

Distinguishing Parameter Estimation from Posterior Approximation

- Monte Carlo and MCMC sampling algorithms
  - Are not models
  - They do not generate more information than is in y and $p(\theta)$

Distinguishing Parameter Estimation from Posterior Approximation

- Monte Carlo and MCMC sampling algorithms
  - Are not models
  - They do not generate more information than is in y and $p(\theta)$
  - They are simply ways of looking at $p(\theta|y)$

Distinguishing Parameter Estimation from Posterior Approximation

- Monte Carlo and MCMC sampling algorithms
  - Are not models
  - They do not generate more information than is in y and $p(\theta)$
  - They are simply ways of looking at $p(\theta|y)$
- Estimation: How we use $p(\theta|y)$ to make inferences about θ

Distinguishing Parameter Estimation from Posterior Approximation

- Monte Carlo and MCMC sampling algorithms
  - Are not models
  - They do not generate more information than is in y and $p(\theta)$
  - They are simply ways of looking at $p(\theta|y)$
- Estimation: How we use $p(\theta|y)$ to make inferences about θ
- Approximation: The use of Monte Carlo procedures to approximate integrals

Additional Issues

- Length of a chain

Additional Issues

- Length of a chain
- Multiple chains (in parallel)

Additional Issues

- Length of a chain
- Multiple chains (in parallel)
- Burn-in

Additional Issues

- Length of a chain
- Multiple chains (in parallel)
- Burn-in
- Reduce autocorrelation by thinning

Additional Issues

- Length of a chain
- Multiple chains (in parallel)
- Burn-in
- Reduce autocorrelation by thinning
- Convergence assessment

Additional Issues

- Length of a chain
- Multiple chains (in parallel)
- Burn-in
- Reduce autocorrelation by thinning
- Convergence assessment
- We will deal with these later