Ecuación de Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \qquad i = 1, \dots, n$$

Lagrangiano:

$$L(q_i, \dot{q}_i) = T - V$$

T → Energía Cinética del sistema.

V→ Energía Potencial del sistema.

D→ Función de disipación de energía del sistema.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}L\dot{q}^2 + K_{m,e}\dot{q}x + K_{m,e}\dot{q}\theta$$

$$V = mgh + \frac{1}{2}kx^2 + \frac{1}{2}K\theta^2 + \frac{1}{2C}q^2$$

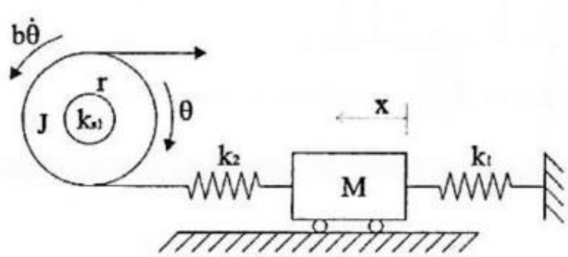
$$D = \frac{1}{2}b\dot{x}^2 + \frac{1}{2}B\dot{\theta}^2 + \frac{1}{2}R\dot{q}^2$$

Sistemas mecánicos

Sistemas eléctricos

Sistemas electromecánicos

Ejemplo sistema mecánico (Ejercicio 1e del taller)



(e) Rueda-Resortes-Masa.

Ecuaciones de Lagrange:

$$(1)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0$$

$$(2)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = Fr$$

Lagrangiano:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

$$V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2(x - r\theta)^2 + \frac{1}{2}k_s\theta^2$$

$$\boldsymbol{L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 - \left(\frac{1}{2}k_1x^2 + \frac{1}{2}k_2(x - r\theta)^2 + \frac{1}{2}k_s\theta^2\right)}$$

$$D = \frac{1}{2}b\dot{\theta}^2$$

Encontrando las derivadas que están en las ecuaciones de Lagrange

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x}$$

$$\frac{\partial L}{\partial x} = -k_1x - k_2(x - r\theta)$$

$$\frac{\partial D}{\partial \dot{x}} = 0$$

$$\boxed{(1)\boldsymbol{M\ddot{x} + k_1x + k_2(x - r\theta) = 0}}$$

$$\frac{\partial L}{\partial \dot{\theta}} = J\dot{\theta}$$

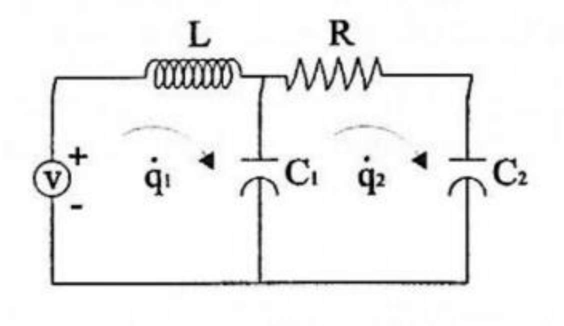
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = k_2r(x - r\theta) - k_s\theta$$

$$\frac{\partial D}{\partial \dot{\theta}} = b\dot{\theta}$$

$$\boxed{(2)\boldsymbol{J\ddot{\theta} - k_2r(x - r\theta) + k_s\theta + b\dot{\theta} = Fr}}$$

Ejemplo de sistema eléctrico (Corrección del ejemplo 2.8 del Libro: NARANJO, Freddy P., Control Lineal Moderno, análisis y diseño en el espacio de estados, Cap. 2)



Ecuaciones de Lagrange:

$$(1)\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_1}}\right)-\frac{\partial L}{\partial q_1}+\frac{\partial D}{\partial \dot{q_1}}=v$$

$$(2)\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_2}}\right)-\frac{\partial L}{\partial q_2}+\frac{\partial D}{\partial \dot{q_2}}=0$$

Lagrangiano:

$$L=T-V$$

$$T=\frac{1}{2}L\dot{q}_1^2$$

$$V=\frac{1}{2C_1}(q_1-q_2)^2+\frac{1}{2C_2}q_2^2$$

$$L=\frac{1}{2}L\dot{q}_1^2-\left(\frac{1}{2C_1}(q_1-q_2)^2+\frac{1}{2C_2}q_2^2\right)$$

$$D=\frac{1}{2}R\dot{q}_2^2$$

Encontrando las derivadas que están en las ecuaciones de Lagrange

$$\frac{\partial L}{\partial \dot{q_1}}=L\dot{q_1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_1}}\right) \quad =L\ddot{q}_1$$

$$\frac{\partial L}{\partial q_1}=-\frac{1}{C_1}(q_1-q_2)$$

$$\frac{\partial D}{\partial \dot{q_1}}=0$$

$$\boxed{\mathbf{(1)}\boldsymbol{L\ddot{q}_1}\ +\frac{\mathbf{1}}{\boldsymbol{C_1}}(\boldsymbol{q_1}-\boldsymbol{q_2})=\boldsymbol{v}}$$

$$\frac{\partial L}{\partial \dot{q_2}} = 0$$

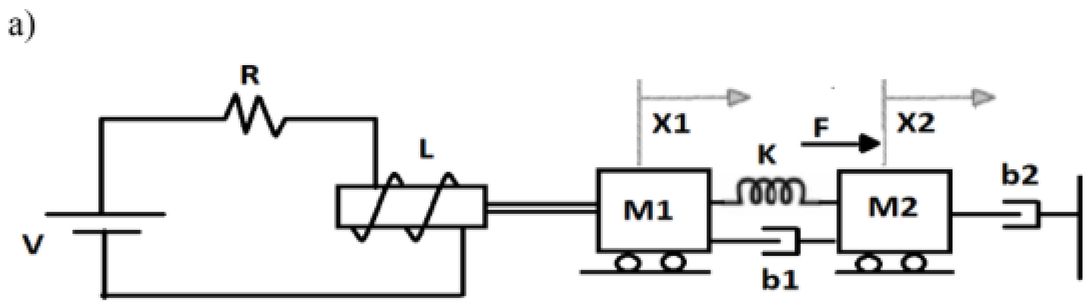
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_2}}\right) = 0$$

$$\frac{\partial L}{\partial q_2} = \frac{1}{C_1}(q_1 - q_2) - \frac{1}{C_2}q_2$$

$$\frac{\partial D}{\partial \dot{q}_2} = R\dot{q}_2$$

$$\boxed{(\mathbf{2}) - \frac{\mathbf{1}}{\boldsymbol{C_1}}(\boldsymbol{q_1} - \boldsymbol{q_2}) + \frac{\mathbf{1}}{\boldsymbol{C_2}}\boldsymbol{q_2} + \boldsymbol{R\dot{q}_2} = \mathbf{0}}$$

Ejemplo de sistema electromecánico (Ejercicio 10.a del taller)



Ecuaciones de Lagrange

$$\frac{d}{dt}\left(\frac{\partial L_L}{\partial \dot{q}}\right) - \frac{\partial L_L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = v$$

$$\frac{d}{dt}\left(\frac{\partial L_L}{\partial \dot{x}_1}\right) - \frac{\partial L_L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L_L}{\partial \dot{x}_2}\right) - \frac{\partial L_L}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = F$$

Lagrangiano:

$$L_L(q_i, \dot{q_\imath}) = T - V$$

$$T = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}M_1\dot{x}_1{}^2 + \frac{1}{2}M_2\dot{x}_2{}^2 + K_{m,e}\dot{q}x_1$$

$$V = \frac{1}{2}k(x_2 - x_1)^2$$

$$L_L = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}M_1\dot{x}_1{}^2 + \frac{1}{2}M_2\dot{x}_2{}^2 + K_{m,e}\dot{q}x_1 - \frac{1}{2}k(x_2 - x_1)^2$$

$$D = \frac{1}{2}R\dot{q}^2 + \frac{1}{2}b_1(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}b_2{\dot{x}_2}^2$$

Empezar a derivar

Para q:

$$\frac{\partial L_L}{\partial \dot{q}} = L\dot{q} + K_e x_1$$

$$\frac{d}{dt}\left(\frac{\partial L_L}{\partial \dot{q}}\right) = L\ddot{q} + K_e \dot{x}_1$$

$$\frac{\partial L_L}{\partial q} = 0$$

$$\frac{\partial D}{\partial \dot{q}} = R\dot{q}$$

$$\boxed{\boldsymbol{L\ddot{q} + K_e\dot{x}_1 + R\dot{q} = v}} \quad (1)$$

Para x1:

$$\frac{\partial L_L}{\partial \dot{x}_1} = M_1\dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial L_L}{\partial \dot{x}_1}\right) = M_1\ddot{x}_1$$

$$\frac{\partial L_L}{\partial x_1} = K_m\dot{q} + k(x_2 - x_1)$$

$$\frac{\partial D}{\partial \dot{x}_1} = b_1(\dot{x}_1 - \dot{x}_2)$$

$$\boxed{\boldsymbol{M_1\ddot{x}_1 - K_m\dot{q} + k(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2) = 0}} \quad (2)$$

Para x2:

$$\frac{\partial L_L}{\partial \dot{x}_2} = M_2\dot{x}_2$$

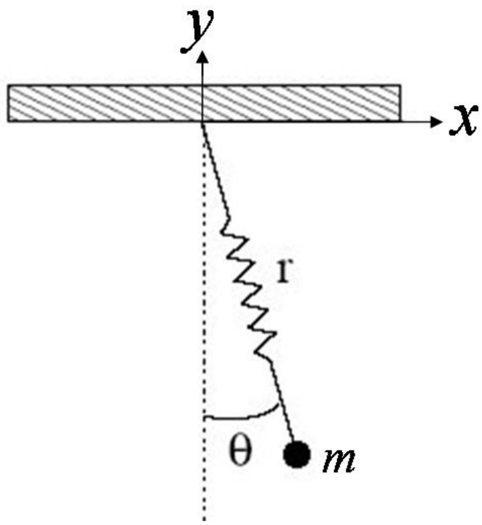
$$\frac{d}{dt}\left(\frac{\partial L_L}{\partial \dot{x}_2}\right) = M_2\ddot{x}_2$$

$$\frac{\partial L_L}{\partial x_2} = -k(x_2 - x_1)$$

$$\frac{\partial D}{\partial \dot{x}_2} = -b_1(\dot{x}_1 - \dot{x}_2) + b_2\dot{x}_2$$

$$\boxed{\boldsymbol{M_2\ddot{x}_2 + k(x_2 - x_1) + b_1(\dot{x}_2 - \dot{x}_1) + b_2\dot{x}_2 = F}} \quad (3)$$

Ejemplo sistema mecánico- Péndulo de resorte



Ecuaciones de Lagrange:

$$(1)\,\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

$$(2)\,\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0$$

Lagrangiano:

$$\boldsymbol{L=T-V}$$

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$x=rsin\theta$$

$$y=-rcos\theta$$

$$\dot{x}=\frac{dx}{dt}=\dot{r}sin\theta+r\dot{\theta}cos\theta$$

$$\dot{y}=\frac{dy}{dt}=-\dot{r}cos\theta+r\dot{\theta}sin\theta$$

$$T=\frac{1}{2}m\left(\left(\dot{r}sin\theta+r\dot{\theta}cos\theta\right)^2+\left(r\dot{\theta}sin\theta-\dot{r}cos\theta\right)^2\right)$$

$$T=\frac{1}{2}m\left(\dot{r}^2\sin^2\theta+2\dot{r}sin\theta r\dot{\theta}cos\theta+r^2\dot{\theta}^2\cos^2\theta+r^2\dot{\theta}^2\sin^2\theta-2r\dot{\theta}sin\theta\dot{r}cos\theta+\dot{r}^2\cos^2\theta\right)$$

$$\boldsymbol{T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)}$$

$$V = \frac{1}{2}k(r-l)^2 - mgrcos\theta$$

$$\boldsymbol{L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}k(r-l)^2 + mgrcos\theta}$$

Derivar

Para θ:

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

$$\frac{\partial L}{\partial \theta} = -mgrsin\theta$$

$$(1)m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + mgrsin\theta = 0$$

$$\boxed{\boldsymbol{(1)\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + gsin\theta = 0}}$$

Para r:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k(r-l) + mgcos\theta$$

$$(2)m\ddot{r} - mr\dot{\theta}^2 + k(r-l) - mgcos\theta = 0$$

$$\boxed{\boldsymbol{(2)\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r-l) - gcos\theta = 0}}$$