

### Ecuación de Lagrange

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad i = 1, \dots, n$$

Lagrangiano:

$$L(q_i, \dot{q}_i) = T - V$$

T → Energía Cinética del sistema.

V → Energía Potencial del sistema.

D → Función de disipación de energía del sistema.

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} L \dot{q}^2 + K_{m,e} \dot{q} x + K_{m,e} \dot{q} \theta$$

$$V = mgh + \frac{1}{2} kx^2 + \frac{1}{2} K \theta^2 + \frac{1}{2C} q^2$$

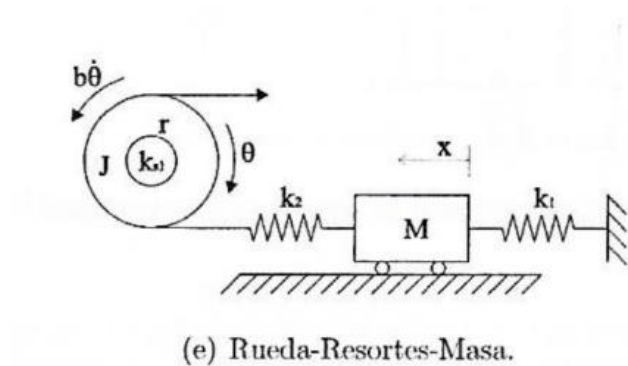
$$D = \frac{1}{2} b \dot{x}^2 + \frac{1}{2} B \dot{\theta}^2 + \frac{1}{2} R \dot{q}^2$$

Sistemas mecánicos

Sistemas eléctricos

Sistemas electromecánicos

### Ejemplo sistema mecánico (Ejercicio 1e del taller)



$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad i = 1, \dots, n$$

Ecuaciones de Lagrange:

$$(1) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0$$

$$(2) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = Fr$$

**Lagrangiano:**

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2$$

$$V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 (x - r\theta)^2 + \frac{1}{2} k_s \theta^2$$

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 - \left( \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 (x - r\theta)^2 + \frac{1}{2} k_s \theta^2 \right)$$

$$D = \frac{1}{2} b \dot{\theta}^2$$

**Encontrando las derivadas que están en las ecuaciones de Lagrange**

$$\frac{\partial L}{\partial \dot{x}} = M \dot{x}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = M \ddot{x}$$

$$\frac{\partial L}{\partial x} = -k_1 x - k_2 (x - r\theta)$$

$$\frac{\partial D}{\partial \dot{x}} = 0$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0$$

$$\boxed{(1) M \ddot{x} + k_1 x + k_2 (x - r\theta) = 0}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = Fr$$

$$\frac{\partial L}{\partial \dot{\theta}} = J \dot{\theta}$$

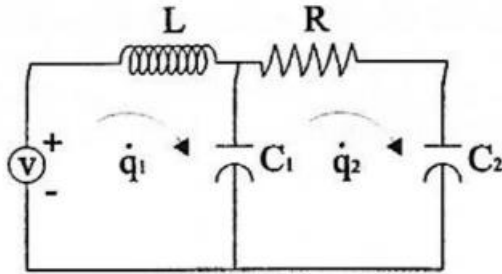
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = J \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = r k_2 (x - r\theta) - k_s \theta$$

$$\frac{\partial D}{\partial \dot{\theta}} = b \dot{\theta}$$

$$\boxed{(2) J \ddot{\theta} - k_2 r (x - r\theta) + k_s \theta + b \dot{\theta} = Fr}$$

**Ejemplo de sistema eléctrico (Corrección del ejemplo 2.8 del Libro: NARANJO, Freddy P., Control Lineal Moderno, análisis y diseño en el espacio de estados, Cap. 2)**



**Ecuaciones de Lagrange:**

$$(1) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} = v$$

$$(2) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_2} \right) - \frac{\partial L}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} = 0$$

**Lagrangiano:**

$$L = T - V$$

$$T = \frac{1}{2} L \dot{q}_1^2$$

$$V = \frac{1}{2C_1} (q_1 - q_2)^2 + \frac{1}{2C_2} q_2^2$$

$$L = \frac{1}{2} L \dot{q}_1^2 - \left( \frac{1}{2C_1} (q_1 - q_2)^2 + \frac{1}{2C_2} q_2^2 \right)$$

$$D = \frac{1}{2} R \dot{q}_2^2$$

**Encontrando las derivadas que están en las ecuaciones de Lagrange**

$$\frac{\partial L}{\partial \dot{q}_1} = L \dot{q}_1$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_1} \right) = L \ddot{q}_1$$

$$\frac{\partial L}{\partial q_1} = - \frac{(q_1 - q_2)}{C_1}$$

$$\frac{\partial D}{\partial \dot{q}_1} = 0$$

$$(1) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} = v$$

$$(1) L \ddot{q}_1 + \frac{1}{C_1} (q_1 - q_2) = v$$

$$\frac{\partial L}{\partial \dot{q}_2} = 0$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_2} \right) = 0$$

$$\frac{\partial L}{\partial q_2} = \frac{q_1 - q_2}{C_1} - \frac{q_2}{C_2}$$

$$\frac{\partial D}{\partial \dot{q}_2} = R \dot{q}_2$$

$$(2) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_2} \right) - \frac{\partial L}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} = 0$$

$$(2) -\frac{1}{C_1} (q_1 - q_2) + \frac{1}{C_2} q_2 + R \dot{q}_2 = 0$$

Para el código:

$$L \ddot{q}_1 + \frac{1}{C_1} (q_1 - q_2) = v$$

$$x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2$$

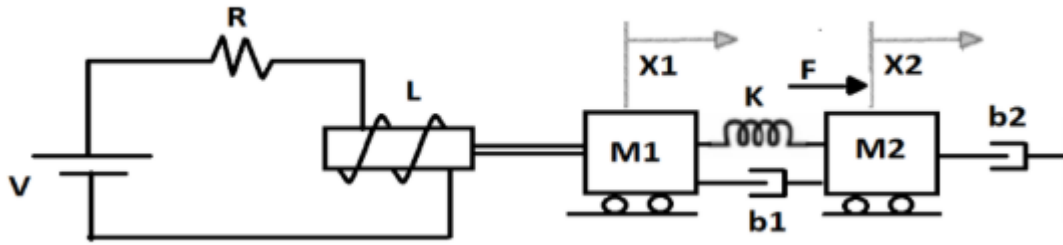
$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{\left( v - \frac{1}{C_1} (x_1 - x_3) \right)}{L}$$

$$\frac{dx_3}{dt} = \frac{\left( \frac{1}{C_1} (x_1 - x_3) - \frac{1}{C_2} x_3 \right)}{R}$$

### Ejemplo de sistema electromecánico (Ejercicio 10.a del taller)

a)



### Ecuaciones de Lagrange

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{q}} \right) - \frac{\partial L_L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = v$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{x}_1} \right) - \frac{\partial L_L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = 0$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{x}_2} \right) - \frac{\partial L_L}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = F$$

### Lagrangiano:

$$L_L(q_i, \dot{q}_i) = T - V$$

$$T = \frac{1}{2} L \dot{q}^2 + \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 + K_{m,e} \dot{q} x_1$$

$$V = \frac{1}{2} k (x_2 - x_1)^2$$

$$L_L = \frac{1}{2} L \dot{q}^2 + \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 + K_{m,e} \dot{q} x_1 - \frac{1}{2} k (x_2 - x_1)^2$$

$$D = \frac{1}{2} R \dot{q}^2 + \frac{1}{2} b_1 (\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2} b_2 \dot{x}_2^2$$

### Empezar a derivar

Para q:

$$\frac{\partial L_L}{\partial \dot{q}} = L \dot{q} + K_e x_1$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{q}} \right) = L \ddot{q} + K_e \dot{x}_1$$

$$\frac{\partial L_L}{\partial q} = 0$$

$$\frac{\partial D}{\partial \dot{q}} = R\dot{q}$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{q}} \right) - \frac{\partial L_L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = v$$

$$\boxed{L\ddot{q} + K_e\dot{x}_1 + R\dot{q} = v} \quad (1)$$

Para x1:

$$\frac{\partial L_L}{\partial \dot{x}_1} = M_1\dot{x}_1$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{x}_1} \right) = M_1\ddot{x}_1$$

$$\frac{\partial L_L}{\partial x_1} = K_m\dot{q} + k(x_2 - x_1)$$

$$\frac{\partial D}{\partial \dot{x}_1} = b_1(\dot{x}_1 - \dot{x}_2)$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{x}_1} \right) - \frac{\partial L_L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = 0$$

$$\boxed{M_1\ddot{x}_1 - K_m\dot{q} + k(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2) = 0} \quad (2)$$

Para x2:

$$\frac{\partial L_L}{\partial \dot{x}_2} = M_2\dot{x}_2$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{x}_2} \right) = M_2\ddot{x}_2$$

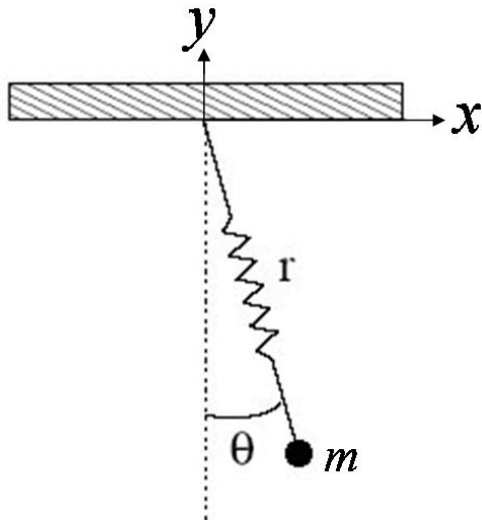
$$\frac{\partial L_L}{\partial x_2} = -k(x_2 - x_1)$$

$$\frac{\partial D}{\partial \dot{x}_2} = -b_1(\dot{x}_1 - \dot{x}_2) + b_2\dot{x}_2$$

$$\frac{d}{dt} \left( \frac{\partial L_L}{\partial \dot{x}_2} \right) - \frac{\partial L_L}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = F$$

$$\boxed{M_2\ddot{x}_2 + k(x_2 - x_1) + b_1(\dot{x}_2 - \dot{x}_1) + b_2\dot{x}_2 = F} \quad (3)$$

### Ejemplo sistema mecánico- Péndulo de resorte



Ecuaciones de Lagrange:

$$(1) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$(2) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0$$

Lagrangiano:

$$L = T - V$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$x = r \sin \theta$$

$$y = -r \cos \theta$$

$$\dot{x} = \frac{dx}{dt} = \dot{r} \sin \theta + \dot{\theta} r \cos \theta$$

$$\dot{y} = \frac{dy}{dt} = -\dot{r} \cos \theta + \dot{\theta} r \sin \theta$$

$$T = \frac{1}{2} m \left( (\dot{r} \sin \theta + \dot{\theta} r \cos \theta)^2 + (\dot{\theta} r \sin \theta - \dot{r} \cos \theta)^2 \right)$$

$$T = \frac{1}{2} m (\dot{r}^2 \sin^2 \theta + 2\dot{r}\dot{\theta} r \sin \theta \cos \theta + \dot{\theta}^2 r^2 \cos^2 \theta + \dot{\theta}^2 r^2 \sin^2 \theta - 2\dot{r}\dot{\theta} r \sin \theta \cos \theta + \dot{r}^2 \cos^2 \theta)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$V = \frac{1}{2} k (r - l)^2 - mgr \cos \theta$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r - l)^2 + mgr\cos\theta$$

**Derivar**

Para  $\theta$ :

$$(1) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$(1)m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) + mgr\sin\theta = 0$$

$$\boxed{(1) r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0}$$

Para  $r$ :

$$(2) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = -k(r - l) + mg\cos\theta + mr\dot{\theta}^2$$

$$(2)m\ddot{r} - mr\dot{\theta}^2 + k(r - l) - mg\cos\theta = 0$$

$$\boxed{(2)\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r - l) - g\cos\theta = 0}$$