High-Energy Astroparticle Theory
Exercises

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1 Radiative Processes in Astroparticle Physics

1.1 Synchrotron energetics and Electron Cooling

Consider a population of electrons described by a power-law distribution in terms of their Lorentz factor γ, given as $N(\gamma)d\gamma = \gamma^{-p}d\gamma$, where p is the power-law index and the distribution extends from $\gamma_{\min}$ to $\gamma_{\max}$.

- Derive the expression for the total energy density U_e of electrons within the specified γ range. Show that it can be approximated (assuming $\gamma_{\max} \gg \gamma_{\min}$) by

$$U_e = \left(\frac{p-1}{p-2}\right)\gamma_{\min} n_e m_e c^2$$

  where n_e is the physical number density.
- With $p = 2.5$, calculate the energy loss timescale (incorrectly known as *cooling time*) for electrons due to synchrotron radiation or inverse Compton scattering. Express your answer in terms of $\gamma_{\min}$ and $\gamma_{\max}$, presenting the timescale in Myr and the energy density in 10^{-10} erg/cm^3 (*Hint:* compute $\tau_{\rm loss} = U_e/P$ where P is the power per unit of volume emitted via IC or synchrotron).
- Calculate the loss timescale for mildly-relativistic electrons ($\gamma_{\min} \simeq \gamma_{\max} \sim 1$) via inverse Compton scattering off of CMB photons. Estimate at what redshift this timescale becomes shorter than the age of the Universe (*Hints*: Approximate the age of the universe at redshift z as $t_{\rm age} = t_0(1+z)^{-3/2}$, where t_0 is the current age).

1.2 Energy Loss and Diffusion of Electrons in the Galactic Environments

- Determine the magnetic field strength that would result in *synchrotron* energy losses for an electron equivalent to those experienced via Inverse Compton (IC) scattering on the CMB. Then, using this magnetic field strength, calculate the synchrotron energy loss timescale for a relativistic electron as a function of its energy E.
- Assuming that the interstellar radiation field (ISRF) can be described as the sum of 3 gray-bodies: UV ($\rho_{\rm UV} = 0.37$ eV/cm^3, $T_{\rm UV} = 3000$ K), Optical ($\rho_\star = 0.055$ eV/cm^3,

$T_\star = 300$ K) and IR ($\rho_{\rm IR} = 0.25$ eV/cm^3, $T_{\rm IR} = 30$ K), compute the total energy loss rate assuming Thomson scattering.

- Identify the component of the ISRF where the transition to the Klein-Nishina (KN) scattering regime occurs at the *lowest* electron energy. Calculate the specific electron energy threshold at which this transition to the KN regime is expected.
- Calculate the maximum distance an electron with energy E can diffuse before significantly losing its energy through synchrotron radiation and IC scattering, under the assumption of Bohm diffusion. Given that electrons are observed via their synchrotron emission approximately a kpc away from the galactic disc, infer implications for the diffusion coefficient in the Milky Way.

1.3 Characteristic Energy Loss Time for Cosmic Ray Electrons

Cosmic ray electrons lose energy primarily through synchrotron radiation and inverse Compton scattering, described by the rate of energy loss

$$\frac{dE}{dt} = -AE^2$$

where A is a positive constant.

- Derive the expression for the energy $E(t)$ of a cosmic ray electron as a function of time t, assuming it starts with an initial energy E_0 at time $t = 0$.
- Use your result to demonstrate that $\frac{E}{|dE/dt|}$ provides a reliable estimate for the time scale over which the electron significantly loses its energy.
- Consider an alternate scenario where the energy loss mechanism is governed by $\frac{dE}{dt} = -BE$, with B being a positive constant, and derive $E(t)$. Identify a physical process that results in energy loss following this law.

1.4 Low diffusivity around TeV halos

TeV halos, extended regions of very high-energy ($E_\gamma \sim$ TeV) gamma-ray emission, have been observed surrounding few middle-aged pulsars, such as Geminga.

- Utilizing the known distance to Geminga[1] and given that the angular extension of its TeV halo is approximately $\theta \sim 5.5^\circ$, calculate the halo's physical size.
- Assuming electrons are initially emitted from the center of the halo, estimate the local diffusion coefficient, D, using the formula $D \sim H^2/\tau$, where H represents the halo size, and τ is the energy loss timescale. Consider energy losses primarily due to IC scattering on CMB.

[1] https://en.wikipedia.org/wiki/Geminga

- Compare the result with the Bohm diffusion coefficient in a $\sim 1\mu$G magnetic field, which is the smaller possible diffusion coefficient.
- Discuss the scenario in which the gamma-ray emission occurs in the Klein-Nishina regime. Explain the conditions under which the photon field would result in this regime being applicable.

1.5 Luminosity Ratio of Cosmic Ray Protons and Electrons

Consider a cosmic source, like a supernova remnant, with a gas density of approximately $n \sim 3$ cm^{-3}. This source contains cosmic ray protons and electrons, each with an identical spectral energy distribution. Protons have a spectrum $N_p(E) = N_{0,p}(E/\text{TeV})^{-2.4}$ for $E \gtrsim m_p c^2$, while electrons have a spectrum $N_e(E) = N_{0,e}(E/\text{TeV})^{-2.4}$ for $E \gtrsim m_e c^2$. The total energies contained in cosmic ray protons and electrons within the source are $W_{CR,p}$ and $W_{CR,e}$, respectively.

Cosmic ray protons produce gamma rays due to proton-proton interactions in the ambient gas, and the resulting luminosity is $Q_{p\gamma}(E_\gamma)E_\gamma^2$. Cosmic ray electrons produce gamma rays due to inverse Compton scattering in the CMB radiation, and the resulting luminosity is $Q_{e\gamma}(E_\gamma)E_\gamma^2$.

Compute the ratio W_e/W_p that would satisfy the condition: $Q_{p\gamma}(E_\gamma)E_\gamma^2 = Q_{e\gamma}(E_\gamma)E_\gamma^2$ at $E_\gamma = 1$ TeV.

1.6 Threshold of UHECR Photo-disintegration

The process of photo-disintegration, where a nucleus releases a nucleon (either a proton or a neutron) upon interaction with a photon, plays a pivotal role in our understanding of UHECRs. This interaction is represented by the equation:

$$A + \gamma \rightarrow A' + N$$

where N denotes the nucleon released during the process.

- Calculate the energy threshold required for a nucleus with mass number A to undergo photo-disintegration, resulting in the emission of a neutron, in terms of the nucleus's binding energy B_A^Z (*Hint:* Model the nuclear masses using the formula $M(A, Z) = Zm_p + (A - Z)m_n - B_A^Z$, and apply the approximation $m_p \approx m_n$ for simplification).
- Show that the threshold Lorentz factor (Γ_{th}) for photo-disintegration is weakly dependent of the nucleus's mass.
- Provide an estimate of this threshold specifically for interactions with cosmic microwave background (CMB) photons.

- Estimate the mean free path for photo-disintegration on CMB photons at the threshold energy.

1.7 Cosmological horizons

- The Intergalactic Medium (IGM) at redshifts $z \sim 6$ is observed to be highly ionized, likely due to radiation from galaxies and quasars. Post-recombination at $z \sim 10^3$, the IGM was almost completely neutral. This observation indicates that reionization of the IGM occurred somewhere $z_r \gtrsim 10$, although the exact timing of this crucial transition remains unknown.

  – An ionized IGM scatters CMB photons by Thomson scattering. Under the assumption of a uniform Universe with a specified baryon fraction Ω_b in units of the critical density Ω_c, derive the relation between τ_r and z_r (*Hint:* Notice that the contribution to τ is dominated by electrons at high redshifts, so you are allowed to drop Ω_Λ).

  – The inferred τ_r from observation of the CMB anisotropy by the Planck satellite [**?**] is $\tau_r = 0.063$. For $H_0 = 70$ km/s/Mpc, $\Omega_{\rm m} = 0.3$, and $\Omega_b = 0.048$, determine z_r.

- The Extragalactic Background Light (EBL) is a significant factor in the absorption of γ-rays from distant astronomical objects, such as blazars, through the mechanism of pair production.

  – Utilize observations of γ-rays with energies $E_\gamma \sim$ TeV from a blazar at a given redshift to outline a method to determine a conservative upper limit for the average EBL intensity as a function of z. Assume that dt/dz can be approximated by H_0^{-1}, and that all EBL photons have the energy where the pair-production cross section is maximized (monochromatic approximation).

2 Particle Transport and Acceleration in Galactic environments

2.1 Cosmic Ray Dynamics in a Starburst Galaxy Nucleus

Consider a starburst galaxy nucleus modeled as a cylindrical region with a radius $R = 500$ pc, a height $H = 500$ pc, and a mean particle density $n = 300\ \mathrm{cm}^{-3}$. Within this volume, supernovae occur at a rate of $0.1\ \mathrm{yr}^{-1}$, each releasing 10^{50} erg of energy primarily as cosmic ray protons. The inelastic scattering cross-section for proton-proton collisions is given as $\sigma = 3 \times 10^{-26}\ \mathrm{cm}^2$, assumed constant across energyies. The diffusion coefficient is modeled as $D(E) = 3 \times 10^{26}(E/\mathrm{GeV})^{1/3}\ \mathrm{cm}^2/\mathrm{s}$, with diffusion occurring solely along the cylinder's axis.

- Derive the equilibrium spectrum of cosmic rays within the starburst nucleus. Solve the transport equation in the z direction (perpendicular to the disk) under a free escape boundary condition at $|z| = H$.
- Determine the spectrum of cosmic rays escaping from the starburst nucleus.
- Compare the diffusive escape timescale with the inelastic loss timescale of an Iron nucleus within the same environment, considering its spallation cross-section is $45\ \mathrm{mb} \times A^{0.7}$.

2.2 Cosmic Ray Energetics in the Milky Way

In the simplified model of our Galaxy, we consider that supernova (SN) remnants, located in an infinitely thin disk, act as sources of cosmic rays. These remnants contribute with a fraction $\epsilon < 1$ of the SN kinetic energy ($E = 10^{51}$ erg) to cosmic rays. Supernovae occur at a rate of 1 every 30 years. The galaxy features a halo of size $H = 5$ kpc and an ordered magnetic field with strength $B_0 = 1\mu$G. The power spectrum of magnetic field irregularities, $P(k) = Ak^{-5/3}$, is normalized so that the integral of $P(k)$ over wave number k from $1/L$ to infinity equals 10^{-2} of the ordered magnetic energy density. The energy-containing scale L is 50 pc.

- Using quasi-linear theory, calculate the diffusion coefficient and the escape timescale for cosmic rays propagating through the galactic magnetic field.
- Determine the acceleration efficiency required to achieve a cosmic ray energy density of 1 eV/cm^3 for CRs with energy greater than 1 GeV at the disc's plane ($z = 0$).

2.3 Primary Positrons from Galactic Pulsars

Galactic pulsars, particularly those associated with bow shocks, are believed to be the main contributors to cosmic-ray positrons.

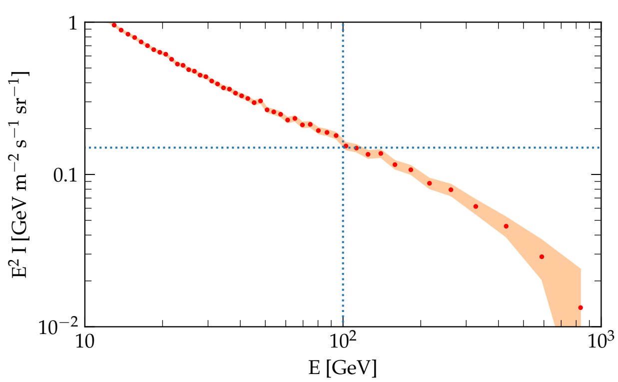
The luminosity of bow-shock pulsars, in terms of pairs, is given as a function of time (t):

$$\mathcal{L}_{\rm bs}(t) = \frac{1}{2} I \Omega_0^2 \frac{1}{\tau_0} \frac{1}{\left(1 + \frac{t}{\tau_0}\right)^2}$$

where $I = \frac{2}{5} M R^2$ is the moment of inertia, $\Omega_0 = \frac{2\pi}{P_0}$ is the angular frequency, and τ_0 is the spin-down age.

The cosmic-ray positron flux at $E = 100$ GeV is measured to be (see plot):

$$E^2 \Phi \approx 0.15\,\text{GeV m}^{-2}\text{s}^{-1}\text{sr}^{-1}$$



- Compute the total energy in the form of positrons released by the pulsars $E_{\rm bs}$ for $t \gg \tau_0$, assuming $P_0 = 0.1$ s^{-1}, $M = 1.4\,M_\odot$, $R = 10$ km.
- With a given rate of $\mathcal{R} \sim 2$/century and an efficiency $\xi < 1$, estimate the local energy density of positrons at the same energy $E = 100$ GeV. Assume a diffusion coefficient $D(E) = 3 \times 10^{28} (E/\text{GeV})^{1/2}$ cm^2/s, halo size $H = 5$ kpc, and consider energy losses due to Inverse Compton scattering on the CMB and synchrotron radiation in a 3μG magnetic field. The spectrum released by the pulsars is assumed to be $\propto E^{-1}$ up to a maximum energy of $E = 100$ GeV.
- Derive the value of ξ by comparison with the measured value.

2.4 Cosmic Ray Dynamics and Gravitational Effects in a Galaxy

Consider a galaxy characterized by an infinitely thin gas disc with a radius $R_d = 15$ kpc and a magnetic halo extending up to $H = 5$ kpc. The galaxy experiences a supernova rate of 1 every 50 years, with each supernova contributing 10^{51} erg of kinetic energy.

The galaxy resides within a dark matter halo of $M_{\rm DM} = 10^{12}$ solar masses, significantly influencing the gravitational dynamics. The dark matter density profile is taken as $\rho(r) = \rho_0(r/R_0)^{-1}$ with $R_0 = 10$ kpc.

- Find the normalization density ρ_0 assuming that $M_{\rm DM}$ is the mass contained within the virial radius. *Hint: the virial radius is that one for which the mean density* $\bar{\rho}_{\rm DM} = \frac{M_{\rm DM}}{\frac{4}{3}\pi R_{\rm vir}^3}$ *is 200 times the cosmological critical density.*
- Solve the transport equation with a free escape boundary condition at $|z| = H$ and determine the acceleration efficiency required to achieve a cosmic ray energy density of 1 eV/cm^3 for CRs with energy greater than 1 GeV at the disc's plane ($z = 0$), taking into account a diffusion coefficient $D(E) = 3 \times 10^{28}(E/\text{GeV})^{1/3}$ cm^2/s.
- Calculate the gravitational force exerted by the dark matter on a ISM parcel $m_{\rm ISM}$ at a *radial* distance r from the center of the galaxy. Use the given dark matter density profile to perform this calculation.
- Compare the gravitational force due to dark matter with the force arising from the gradient of cosmic ray pressure $\nabla P_{CR}(z)$ as a function of the distance z from the disc, perpendicular to the disc plane at the Sun position $R_\odot = 8.5$ kpc. *Hint: assume that* $z \ll R_\odot$.

2.5 Maximum Energies for Particle Acceleration in a Supernova Remnant

Consider a Supernova explosion that releases an energy of 10^{51} erg in the form of kinetic energy of the ejecta, that consist of 5 solar masses of material.

The shell expands in an interstellar medium with constant density 0.5 cm^{-3}, magnetic field of 3 μG and temperature of 10^4 K.

- Calculate the Mach number of the shock that accompanies the expansion of the shell and the slope of the spectrum of the associated non-thermal particles accelerated at the shock.
- Assuming Bohm diffusion upstream and downstream with the same magnetic field, calculate the maximum energy of the accelerated particles at the end of the ejecta dominated phase, by comparing the acceleration time with the relevant time.

- Calculate the spectrum of electrons at the same shock by introducing synchrotron energy losses of electrons of energy E at a rate $dE/dt = -2.5 \times 10^{-18} B_{\mu\mathrm{G}}^2 E_{\mathrm{GeV}}^2$ GeV/s. All accelerated particles can be assumed relativistic.
- What is the maximum energy of electrons determined by?

3 Bibliography