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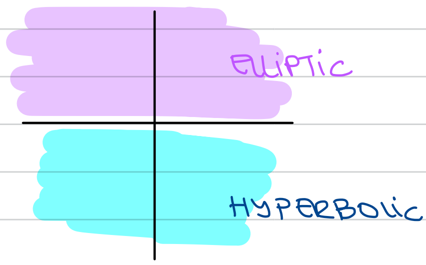
$$\boxed{u_{xx} - y\,u_{yy} = 0} \longrightarrow \text{reduce to the canonical form}$$

We have that $a_{11} = 1$, $a_{12} = 0$, $a_{22} = -y$, $b_1 = b_2 = c = 0$

Then $D(x,y) = y \Rightarrow$ The equation is $\begin{cases} \text{hyperbolic when } y > 0 \\ \text{elliptic when } y < 0 \end{cases}$



1) $y > 0$, $\lambda^2 - y = 0$ and $\lambda_{1,2} = \pm\sqrt{y}$

- $y' = \sqrt{y} \Rightarrow \dfrac{dy}{dx} = \sqrt{y} \Rightarrow \displaystyle\int \frac{dy}{\sqrt{y}} = \int dx \Rightarrow 2\sqrt{y} = x + c$

$$\Rightarrow \xi(x,y) = 2\sqrt{y} - x$$

- $y' = -\sqrt{y} \Rightarrow \dfrac{dy}{dx} = -\sqrt{y} \Rightarrow \displaystyle\int \frac{dy}{\sqrt{y}} = \int - dx \Rightarrow 2\sqrt{y} = -x + c$

$$\Rightarrow \eta(x,y) = 2\sqrt{y} + x$$

$$\boxed{\begin{array}{ll} \xi_x = -1 & \eta_x = 1 \\ \xi_y = \dfrac{1}{\sqrt{y}} & \eta_y = \dfrac{1}{\sqrt{y}} \end{array}}$$

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi}\cdot\frac{1}{y} + u_{\xi\eta}\cdot\frac{2}{y} + u_{\eta\eta}\cdot\frac{1}{y} + \frac{1}{2\sqrt{y}}(u_\xi + u_\eta)$$

$$\boxed{\begin{array}{l} \xi_{xx} = 0 = \eta_{xx} \\ \xi_{xy} = 0 = \eta_{xy} \\ \xi_{yy} = \dfrac{1}{2\sqrt{y}} = \eta_{yy} \end{array}}$$

$$u_{xx} - y\,u_{yy} = 0$$

$$\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} - \cancel{u_{\xi\xi}} - 2u_{\xi\eta} - \cancel{u_{\eta\eta}} - \frac{1}{2}\sqrt{y}\,(u_\xi + u_\eta) = 0$$

$$-4u_{\xi\eta} - \frac{1}{2}\sqrt{y}\,(u_\xi + u_\eta) = 0$$

Canonical form: $u_{\xi\eta} + \frac{1}{8}\sqrt{y}\,(u_\xi + u_\eta) = 0$

$$\boxed{u_{\xi\eta} + \frac{1}{32}(\xi + \eta)(u_\xi + u_\eta) = 0}$$

2) $y < 0$, $\lambda_{1,2} = \pm i\sqrt{-y}$

- $y' = i\sqrt{-y} \Rightarrow \frac{dy}{dx} = i\sqrt{-y} \Rightarrow \int \frac{dy}{\sqrt{-y}} = \int i\,dx \Rightarrow -2\sqrt{-y} = ix + c$

$c = -2\sqrt{-y} - ix$

$$\Rightarrow \begin{cases} \varepsilon(x,y) = -2\sqrt{-y} \\ \eta(x,y) = -x \end{cases}$$

$$\boxed{\begin{array}{ll} \varepsilon_x = 0 & \eta_x = -1 \\ \varepsilon_y = 1/\sqrt{-y} & \eta_y = 0 \end{array}}$$

$A_{11} = A_{22} = -y \cdot \left(\frac{1}{\sqrt{-y}}\right)^2 = -y \cdot \left(\frac{1}{-y}\right) = 1$

$B_1 = -y \cdot \frac{-1}{2\sqrt{-y}} = \cancel{\sqrt{(-y)}^2} \cdot \frac{1}{-2\cancel{\sqrt{-y}}} = -\frac{1}{2}\sqrt{-y}$

$$\boxed{\begin{array}{ll} \varepsilon_{xx} = 0 & \eta_{xx} = 0 \\ \varepsilon_{yy} = \frac{-1}{2\sqrt{-y}} & \eta_{yy} = 0 \\ \varepsilon_{xy} = 0 & \eta_{xy} = 0 \end{array}}$$

Canonical form: $u_{\varepsilon\eta} + u_{\eta\eta} + \frac{1}{2}\sqrt{-y}\, u_\varepsilon = 0$

$$\boxed{u_{\varepsilon\eta} + u_{\eta\eta} - \frac{1}{4}\varepsilon \cdot u_\varepsilon = 0}$$