Remember:

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \Gamma(n+1) = n\cdot\Gamma(n) = n!$$

Change of variable:

$$\boxed{\begin{array}{l} t = \frac{1}{2}(1+x) \Rightarrow dt = \frac{1}{2}dx \\ x = 2t-1 \Rightarrow dx = 2\,dt \\ \text{When } x=-1,\ t=0 \\ \text{when } x=1,\ t=1 \end{array}}$$

a) $$\int_{-1}^1 (1-x)^{1/3} (1+x)^{-1/2}\, dx$$

$$\int_0^1 (1-2t+1)^{1/3} (1+2t-1)^{-1/2} \cdot 2\, dt =$$

$$= 2\int_0^1 (2-2t)^{1/3} (2t)^{-1/2}\, dt =$$

$$= 2\int_0^1 2^{\frac{1}{3}} (1-t)^{1/3} \cdot 2^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}}\, dt = 2^{\frac{5}{6}} \int_0^1 t^{-\frac{1}{2}} \cdot (1-t)^{\frac{1}{3}}\, dt =$$

$$= 2^{5/6} \cdot \beta\left(\frac{1}{2}, \frac{4}{3}\right) = 2^{5/6} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{6}\right)} = 2^{5/6} \cdot \frac{\sqrt{\pi}\cdot\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{6}\right)} =$$

$$= \sqrt[6]{2^5} \cdot \frac{2\sqrt{\pi}}{5} \cdot \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} \simeq 2.99809$$

- $\Gamma\left(\frac{4}{3}\right) = \Gamma\left(1+\frac{1}{3}\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$
- $\Gamma\left(\frac{11}{6}\right) = \Gamma\left(1+\frac{5}{6}\right) = \frac{5}{6}\Gamma\left(\frac{5}{6}\right)$

b) $\int_0^1 (1-\sqrt{t})^{1/3} \, dt$

$x = \sqrt{t} \Rightarrow dx = \frac{1}{t} \, dt$
$t = x^2 \Rightarrow dt = 2x \, dx$
when $x=0$, $t = 0^2 = 0$
when $x=1$, $t = 1^2 = 1$

$$\int_0^1 (1-x)^{1/3} \cdot 2x \, dx =$$

$$= 2 \int x^1 \cdot (1-x)^{1/3} \, dx =$$

$$= 2 \cdot \beta\left(2, \frac{4}{3}\right) = 2 \cdot \frac{\Gamma(2) \cdot \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{10}{3}\right)} = 2 \cdot \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{10}{3}\right)} = \frac{9}{14}$$

- $\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{10}{3}\right)} = \frac{\Gamma\left(\frac{4}{3}\right)}{\frac{7}{3} \cdot \Gamma\left(\frac{7}{3}\right)} = \frac{\Gamma\left(\frac{4}{3}\right)}{\frac{7}{3} \cdot \frac{4}{3} \cdot \Gamma\left(\frac{4}{3}\right)} = \frac{9}{28}$

c) $$\int_0^{\frac{\pi}{2}} (tg\ x)^{\frac{1}{3}}\, dx =$$

$$= \int_0^{\pi/2} \left(\frac{\text{sen } x}{\cos x}\right)^{1/3} dx =$$

$$= \int_0^{\pi/2} (\text{sen} x)^{1/3} \cdot (\cos x)^{-1/3} dx =$$

$$y = \cos^2 x$$

$$dy = -2\cos x \,\text{sen} x \, dx$$

$$dx = \frac{dy}{-2\sqrt{y(1-y)}}$$

when $x=0$, $y=1$
when $x=\frac{\pi}{2}$, $y=0$

$$= \int_0^{\pi/2} (1-\cos^2 x)^{1/6} (\cos^2 x)^{-1/6} dx =$$

$$= \int_1^0 (1-y)^{1/6}\, y^{-1/6} \cdot \frac{1}{(-2)\, y^{1/2} (1-y)^{1/2}} dy =$$

$$= \frac{1}{2}\int_0^1 (1-y)^{-1/3}\, y^{-2/3}\, dy = \frac{1}{2}\, \beta\left(\frac{1}{3}, \frac{2}{3}\right) =$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\pi = \frac{\pi}{\sqrt{3}}$$

- we know that $\Gamma(n)\cdot\Gamma(1-n) = \frac{\pi}{\sin(\pi\cdot n)}$ (Euler's reflection)

$$\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{1}{3}\right) = \Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(1-\frac{2}{3}\right) = \frac{\pi}{\sin\left(\pi\cdot\frac{2}{3}\right)} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$$

d) $$\int_0^1 (-\ln x)^{3/2}\, dx =$$

$$x = e^y \Rightarrow dx = e^y\, dy$$

when $x = 0,\ y = -\infty$

when $x = 1,\ y = 0$

$$= \int_{-\infty}^{0} \left(-\ln (e^y)\right)^{3/2} \cdot e^y\, dy =$$

$$y = -z \Rightarrow dy = -dz$$

when $y = -\infty,\ z = \infty$

when $y = 0,\ z = 0$

$$= \int_{-\infty}^{0} (-y)^{3/2}\, e^y\, dy =$$

$$= -\int_{\infty}^{0} z^{3/2}\, e^{-z}\, dz = \int_0^{\infty} z^{3/2} \cdot e^{-z}\, dz = \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

- $$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\underset{\substack{\uparrow \\ m=2}}{2} + \frac{1}{2}\right) = \frac{(2m)!}{2^{2m} \cdot m!}\sqrt{\pi} = \frac{4!}{2^4 \cdot 2!}\sqrt{\pi} = \frac{4 \cdot 3 \cdot \cancel{2}}{2^4 \cdot \cancel{2}}\sqrt{\pi} =$$

$$= \frac{3}{4}\sqrt{\pi}$$

e) $$\int_0^{+\infty} \frac{\sqrt[4]{x}}{(1+x)^2}\, dx =$$

$$\boxed{\begin{array}{l} y = \frac{1}{1+x} \iff x = \frac{1}{y} - 1 \\ dx = -\frac{1}{y^2}\, dy \\ \text{when } x=0,\ y=1 \\ \text{when } x=+\infty,\ y=0 \end{array}}$$

$$= \int_1^0 \sqrt[4]{\frac{1}{y} - 1} \cdot y^2 \cdot \left(-\frac{1}{y^2}\right) dy =$$

$$= \int_1^0 \left(\frac{1}{y} - 1\right)^{1/4} \cdot (-1)\, dy =$$

$$= (-1) \cdot \int_1^0 \frac{\sqrt[4]{1-y}}{\sqrt[4]{y}}\, dy = \int_0^1 y^{-\frac{1}{4}} \cdot (1-y)^{\frac{1}{4}}\, dy =$$

$$= \beta\left(\frac{3}{4}, \frac{5}{4}\right) = \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)}{\Gamma(2)} = \frac{\sqrt{2}\,\pi}{\Gamma(1/4)} \cdot \frac{\Gamma(1/4)}{4} = \frac{\pi}{2\sqrt{2}}$$

- $$\Gamma\left(n + \frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right) \cdot \frac{(4n-3)!!!!}{4^n}$$

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1 + \frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right) \cdot \frac{1!!!!}{4} = \frac{\Gamma\left(\frac{1}{4}\right)}{4}$$

- We know that $\Gamma(n) \cdot \Gamma(1-n) = \frac{\pi}{\sin(\pi \cdot n)}$ (Euler's reflection)

$$\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(1 - \frac{1}{4}\right) = \frac{\pi}{\sin \cdot \left(\frac{\pi}{4}\right)} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,\pi$$

$$\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)}$$

f) $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt[4]{(2+x)^3(2-x)}} =$

$$y = \frac{2+x}{4}$$

$$x = 4y-2 \Rightarrow dx = 4\,dy$$

when $x=-2$, $y=0$
when $x=2$, $y=1$

$$= \int_{-2}^{2} (2+x)^{-\frac{3}{4}} (2-x)^{-\frac{1}{4}}\, dx =$$

$$= \int_{0}^{1} (4y)^{-\frac{3}{4}} (2-4y+2)^{-\frac{1}{4}}\, dx =$$

$$= \int_{0}^{1} 4^{-\frac{3}{4}} \cdot y^{-\frac{3}{4}} \cdot 4^{-\frac{1}{4}} (1-y)^{-\frac{1}{4}} \cdot 4\, dy =$$

$$= \frac{4}{4} \int_{0}^{1} y^{-3/4} \cdot (1-y)^{-1/4}\, dy = \beta\left(\frac{1}{4}, \frac{3}{4}\right) =$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$$

- We know that $\Gamma(n)\cdot\Gamma(1-n) = \dfrac{\pi}{\sin(\pi\cdot n)}$ (Euler's reflection)

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,\pi$$

$$g)\ \int_0^{\pi/2} (\sin x)^4 \ (\cos x)^6 \ dx =$$

$$\boxed{\begin{array}{l} y = \cos^2 x \Rightarrow \cos x = \sqrt{y} \\ dy = -2\cos x \cdot \operatorname{sen} x \ dx \\ dx = \dfrac{dy}{-2\sqrt{y}\cdot\sqrt{1-y}} \\ \text{when } x=0,\ y=1 \\ \text{when } x=\frac{\pi}{2},\ y=0 \end{array}}$$

$$= \int_0^{\pi/2} (\sin^2 x)^2 \ (\cos^2 x)^3 \ dx =$$

$$= \int_0^{\pi/2} (1-\cos^2 x)^2 \ (\cos^2 x)^3 \ dx =$$

$$= \int_1^0 (1-y)^2 \cdot y^3 \cdot \frac{1}{-2\sqrt{y(1-y)}} \ dy =$$

$$= -\frac{1}{2}\int_1^0 \frac{(1-y)^2}{(1-y)^{1/2}} \cdot \frac{y^3}{y^{1/2}} \ dy =$$

$$= -\frac{1}{2}\int_1^0 (1-y)^{3/2} \ y^{5/2} \ dy =$$

$$= \frac{1}{2}\int_0^1 y^{5/2} \ (1-y)^{3/2} \ dy = \frac{1}{2} \ \beta\left(\frac{7}{2}, \frac{5}{2}\right) =$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)} = \frac{1}{2}\cdot\frac{\frac{15}{8}\cdot\frac{3}{4}}{5!}\pi = \frac{3}{512}\pi$$

- $$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\underset{\substack{\uparrow \\ m=3}}{3} + \frac{1}{2}\right) = \frac{(2m)!}{2^{2m} \quad m!}\sqrt{\pi} = \frac{6\cdot5\cdot4\cdot\cancel{3}\cdot\cancel{2}}{2^6\cdot\cancel{3}\cdot\cancel{2}}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

- $$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

even function

h) $$\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^{n+1}}\,dx =$$

$$y = \frac{1}{(1+x^2)} \Rightarrow x = \sqrt{\frac{1}{y}-1}$$

$$dx = \frac{-y^{-2}}{2\sqrt{\frac{1}{y}-1}}\,dy$$

when $x=0$, $y=1$
when $x=+\infty$, $y=0$

$$= 2\cdot\int_0^{+\infty} \frac{1}{(1+x^2)^{n+1}}\,dx =$$

$$= \cancel{2}\cdot\int_1^0 y^{n+1}\cdot\frac{-1}{\cancel{2}\cdot y^2\cdot\sqrt{\frac{1}{y}-1}}\,dy = -\int_1^0 y^{n-1}\cdot\left(\frac{1}{y}-1\right)^{-1/2} dy =$$

$$= \int_0^1 y^{n-1}\cdot\left(\frac{1-y}{y}\right)^{-1/2} dy = \int_0^1 y^{n-1+\frac{1}{2}}\cdot(1-y)^{-1/2}\,dy =$$

$$= \int_0^1 y^{n-\frac{1}{2}}\cdot(1-y)^{-1/2}\,dy = \beta\left(n+\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(n+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} =$$

$$= \sqrt{\pi}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$$

i) $$\int_0^{+\infty} \frac{1}{1+x^{\pi}}\,dx =$$

$$\boxed{\begin{array}{l} y = \dfrac{1}{1+x^{\pi}} \Rightarrow x = \left(\dfrac{1}{y} - 1\right)^{1/\pi} \\[2ex] dx = \dfrac{1}{\pi}\left(\dfrac{1}{y}-1\right)^{\frac{1-\pi}{\pi}} \cdot (-1) \cdot \dfrac{1}{y^2} \\[2ex] \text{when } x = 0,\ y = 1 \\ \text{when } x = +\infty,\ y = 0 \end{array}}$$

$$= -\int_1^0 y \cdot \frac{1}{\pi} \cdot \left(\frac{1}{y} - 1\right)^{\frac{1-\pi}{\pi}} \cdot \frac{1}{y^2}\,dy =$$

$$= \frac{1}{\pi} \int_0^1 \frac{1}{y} \cdot \left(\frac{1-y}{y}\right)^{\frac{1-\pi}{\pi}} dy = \frac{1}{\pi} \cdot \int_0^1 y^{-1-\frac{1-\pi}{\pi}} \cdot (1-y)^{\frac{1-\pi}{\pi}}\,dy =$$

$$= \frac{1}{\pi} \int_0^1 y^{\frac{-\cancel{\pi}-1+\cancel{\pi}}{\pi}} \cdot (1-y)^{\frac{1}{\pi}-1}\,dy = \frac{1}{\pi}\,\beta\left(-\frac{1}{\pi}+1, \frac{1}{\pi}\right) =$$

$$= \frac{1}{\pi}\,\beta\left(1 - \frac{1}{\pi}, \frac{1}{\pi}\right) = \frac{1}{\pi} \cdot \frac{\Gamma\left(1-\frac{1}{\pi}\right) \cdot \Gamma\left(\frac{1}{\pi}\right)}{\Gamma(1)} = \frac{1}{\pi} \cdot \frac{\pi}{\sin(1)} = \boxed{\frac{1}{\sin(1)}}$$

- We know that $\Gamma(n) \cdot \Gamma(1-n) = \dfrac{\pi}{\sin(\pi \cdot n)}$

$$\Gamma\left(\frac{1}{\pi}\right) \cdot \Gamma\left(1 - \frac{1}{\pi}\right) = \frac{\pi}{\sin\left(\pi \cdot \frac{1}{\pi}\right)} = \frac{\pi}{\sin(1)}$$