Gamma and Beta Functions

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Overview

Introduction

- Gamma function Γ is a single variable function.
- Γ is like a factorial for natural numbers, its extension to positive real numbers makes it useful for modelling situations with continuous change, differential equations, complex analysis and statistics.
- Beta function β is a dual variable function.
- β is used for integral calculus, probability theory and statistics, and has applications in physical and engineering problems, such as in the conetxt of fluid dynamics, quantum mechanics, and signal processing.

Definition of Gamma function I

Definition

The Gamma function, denoted by Γ, is defined by the formula

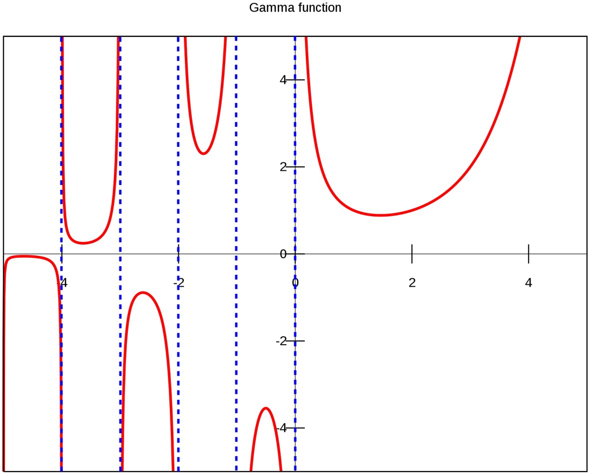
$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds \quad \Re(x) > 0$$

which is defined for all complex numbers except the nonpositive integers.

It is also known as Euler's integral of second kind and it serves as an extension of the factorial only for the positive integers.

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}^+$$

Definition of Gamma function II



Properties of Gamma function I

Some properties of the Gamma function are:

1. $\Gamma(1) = 1$
2. $\Gamma(x+1) = x\Gamma(x)$
3. $\Gamma\left(x\right) = z^x \displaystyle\int_0^\infty s^{x-1} e^{-zs} ds \quad x, z > 0$
4. $\Gamma\left(x\right) = \displaystyle\int_0^1 \left(\log \frac{1}{y}\right)^{x-1} dy$
5. $\Gamma\left(x+1\right) = \displaystyle\int_0^\infty e^{-y^{\frac{1}{x}}} dy$

Proof of first property

- $\Gamma(1) = 1$

We know that

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$$

Then

$$\begin{aligned}\Gamma(1) &= \int_0^\infty s^{1-1} e^{-s} ds \\ &= \int_0^\infty e^{-s} ds \\ &= -e^{-s}\Big|_0^\infty = -0 + 1 = 1\end{aligned}$$

Proof of second property I

- $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$$

Now, putting $x = x+1$, we get

$$\Gamma(x+1) = \int_0^\infty s^{x} e^{-s} ds$$

Integrating by parts,

$$\begin{aligned}\Gamma(x+1) = \int_0^\infty s^{x} e^{-s} ds &= -s^x e^{-s}\Big|_0^\infty + x\int_0^\infty s^{x-1} e^{-s} ds \\ &= (0+0) + x\Gamma(x) = x\Gamma(x)\end{aligned}$$

Proof of second property II

From this property, we get

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

If $x = 0$, then $\Gamma(0) = \dfrac{1}{0} = \infty$
And if $x = -1$, we can use that $\Gamma(x) = (x-1)\Gamma(x-1)$

$$\Gamma(0) = -1\Gamma(-1)$$

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = \infty$$

Proceeding in the same way, we get

$$\Gamma(-n) = \infty, \quad \forall n \in \mathbb{N}$$

So this definition is not valid for zero or negative integers.

Proof of third property

- $\Gamma(x) = z^x \int_0^\infty s^{x-1} e^{-zs} ds \quad x, z > 0$

By definition of the Gamma function, we get

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$$

We apply the change of variable $s = zy$ or $ds = zdy$. In this case, the limit doesn't change,

$$\begin{aligned}\Gamma(x) &= \int_0^\infty (zy)^{x-1} e^{-zy} zdy \\ &= \int_0^\infty z^{x-1} y^{x-1} e^{-zy} zdy \\ &= z^x \int_0^\infty y^{x-1} e^{-zy} dy\end{aligned}$$

Proof of fourth property

- $\Gamma(x) = \int_0^1 \left(\log \frac{1}{s}\right)^{x-1} ds$

We know that,

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$$

We apply the change of variable $s = \log \frac{1}{y}$ or $y = e^{-s}$, $dy = -e^{-s} ds$.
Now, when $s = 0, y = 1$ and when $s = \infty, y = 0$. Then,

$$\begin{aligned}\Gamma(x) &= \int_1^0 (\log \frac{1}{y})^{x-1}(-dy) \\ &= \int_0^1 (\log \frac{1}{y})^{x-1} dy\end{aligned}$$

Proof of fifth property

- $\Gamma(x+1) = \int_0^\infty e^{-y^{\frac{1}{x}}}\, dy$

We know that,

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$$

Let $s = y^{1/x} \iff s^x = y$ or $xs^{x-1} ds = dy \iff s^{x-1} ds = dy/x$.
The limits will not be changed. Then,

$$\Gamma(x) = \int_0^\infty e^{-y^{\frac{1}{x}}} \frac{dy}{x} = \frac{1}{x} \int_0^\infty e^{-y^{\frac{1}{x}}} dy$$

$$x\Gamma(x) = \int_0^\infty e^{-y^{\frac{1}{x}}} dy \Rightarrow \Gamma(x+1) = \int_0^\infty e^{-y^{\frac{1}{x}}} dy$$

Euler's form I

For a fixed integer m, as the integer n increases, we have that

$$\lim_{n\to\infty} \frac{n!(n+1)^m}{(n+m)!} = 1$$

We can get a unique extension of the factorial function to the non-integers by insisting that this equation continue to hold when the arbitrary integer m is replaced by an arbitrary complex number z,

$$\lim_{n\to\infty} \frac{n!(n+1)^z}{(n+z)!} = 1$$

Euler's form II

Multiplying both sides by $(z-1)!$ gives

$$\begin{aligned}
\Gamma(z) &= (z-1)! \\
&= \frac{1}{z}\lim_{n\to\infty} n!\frac{z!}{(n+z)!}(n+1)^z \\
&= \frac{1}{z}\lim_{n\to\infty}(1\cdot 2\cdot \ldots \cdot n)\frac{1}{(1+z)\ldots(n+z)}\left(\frac{2}{1}\frac{3}{2}\ldots\frac{n+1}{n}\right)^z \\
&= \frac{1}{z}\prod_{n=1}^{\infty}\left[\frac{1}{1+\frac{z}{n}}\left(1+\frac{1}{n}\right)^z\right]
\end{aligned}$$

This infinite product converges for all complex numbers z except the non-positive integers, which fail because of a division by zero.

Euler's reflection formula I

Theorem

Let Γ *denote the gamma function. Then*

$$\forall z \notin \mathbb{Z} : \Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}$$

We have the Weierstrass products:

$$sin(\pi z) = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) exp\left(\frac{z}{n}\right)$$

From the Weierstrass form of the Gamma function:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) exp\left(-\frac{z}{n}\right)$$

Euler's reflection formula II

from which:

$$
\begin{aligned}
\frac{1}{-z\Gamma(z)\Gamma(-z)} &= \frac{-z^2 exp(\gamma z) exp(-\gamma z)}{-z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right) exp\left(\frac{z}{n}-\frac{z}{n}\right) \\
&= z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) \\
&= \frac{sin(\pi z)}{\pi}
\end{aligned}
$$

whence:

$$
\begin{aligned}
\Gamma(z)\Gamma(1-z) &= -z\Gamma(z)\Gamma(-z) \\
&= \frac{\pi}{sin(\pi z)}
\end{aligned}
$$

Evaluating the Gamma function at 1/2

By definition of the Gamma function,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}dx$$
$$= 2\int_0^\infty e^{-x}\frac{1}{2\sqrt{x}}dx$$

Performing the substitution $u = \sqrt{x}$ or $du = \frac{1}{2\sqrt{x}}dx$, so

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}du = \int_{\mathbb{R}} e^{-u^2s}du$$

,
where the last equality holds because e^{-u^2} is an even function. Since the area under the bell curve is $\sqrt{\pi}$, it follows that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Gamma Function of Positive Half-Integer I

Theorem

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{(2m)!}{2^{2m}m!}\sqrt{\pi}$$

where $m + 1$ is a half-integer such that $m > 0$ and Γ denotes the Gamma function.

Proof by induction:
For all $m \in \mathbb{Z}_{>0}$, let $P(m)$ be the proposition:

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{(2m)!}{2^{2m}m!}\sqrt{\pi}$$

Gamma Function of Positive Half-Integer II

Basis of the induction:
$P(1)$ is the case:

$$\begin{aligned}
\Gamma\left(1+\frac{1}{2}\right) &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}}{2} \\
&= \frac{2}{4}\sqrt{\pi} \\
&= \frac{2!}{2^{2\cdot 1}1!}\sqrt{\pi} \\
&= \frac{(2m)!}{2^{2m}m!}\sqrt{\pi}
\end{aligned}$$

And so $P(1)$ holds.

Gamma Function of Positive Half-Integer III

Induction hypothesis:
Now we need to show that, if $P(k)$ is true, where $k \geq 1$, then it logically follows that $P(k+1)$ is true.
So this is the induction hypothesis:

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$$

Then we need to show:

$$\Gamma\left(k+1+\frac{1}{2}\right) = \frac{(2(k+1))!}{2^{2(k+1)}(k+1)!}\sqrt{\pi}$$

Gamma Function of Positive Half-Integer IV

Induction Step:

$$
\begin{aligned}
\Gamma\left(k+1+\frac{1}{2}\right) &= \left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right) \\
&= \left(k+\frac{1}{2}\right)\frac{(2k)!}{2^{2k}k!}\sqrt{\pi} \\
&= \frac{(2k+1)(2k)!}{2\cdot 2^{2k}k!}\sqrt{\pi} \\
&= \frac{(2k)!(2k+1)(2k+2)}{2(2k+2)2^{2k}k!}\sqrt{\pi} \\
&= \frac{(2k+2)!}{2^{2(k+1)}(2(k+1))k!}\sqrt{\pi} \\
&= \frac{(2(k+1))!}{2^{2(k+1)}(k+1)!}\sqrt{\pi}
\end{aligned}
$$

So $P(k) \Rightarrow P(k+1)$ and the result follows by the Principle of Induction.

Definition of Beta function

Definition

The Beta function, denoted as β, is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m,n > 0$$

This is also the Euler's integral of the first kind.

Properties of Beta function

These three properties are widely used:

1. It is symmetric, $\beta(m, n) = \beta(n, m)$
2. $\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$
3. $\beta(m, n) = 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2m-1} (cos\theta)^{2n-1} d\theta$

Proof of first property

- $\beta(m, n) = \beta(n, m)$

We know that

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

We know that for definite integral

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

so using this property in the above equation, we get

$$\begin{aligned}
\beta(m, n) &= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1} dx \\
&= \int_0^1 x^{n-1}(1-x)^{m-1} dx \\
&= \beta(n, m)
\end{aligned}$$

Proof of second property

- $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Let $x = \frac{1}{1+y} \Rightarrow dx = \frac{-1}{(1+y)^2} dy$ or $(1+y) = \frac{1}{x}$.
Now, when $x = 0, y = \infty$ and when $x = 1, y = 0$. Then,

$$\begin{aligned}
\beta(m,n) &= \int_\infty^0 (\frac{1}{1+y})^{m-1}(1 - \frac{1}{1+y})^{n-1} \frac{-dy}{(1+y)^2} \\
&= \int_0^\infty \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(y+1)^{n-1}} \frac{dy}{(y+1)^2} \\
&= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy
\end{aligned}$$

Proof of third property

- $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2m-1}(cos\theta)^{2n-1} d\theta$

We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Let $x = sin^2\theta$ or $dx = 2sin\theta cos\theta d\theta$.
When $x = 0, \theta = 0$ and when $x = 1, \theta = \pi/2$.

$$\begin{aligned}
\beta(m,n) &= \int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1}(1-sin^2\theta)^{n-1} 2sin\theta cos\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2m-2}(cos\theta)^{2n-2} sin\theta cos\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2m-1}(cos\theta)^{2n-1} d\theta
\end{aligned}$$

Relation between Beta and Gamma functions I

Theorem

Beta and Gamma function can be related by the relation

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m, n > 0$$

We know from the definition of Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Now using the third property of the Gamma function, we write

$$\Gamma(n) = z^n \int_0^\infty e^{-zx} x^{n-1} dx$$

Relation between Beta and Gamma functions II

Multiplying both sides by $e^{-z}z^{m-1}$ and then integrating with respect to z from $z = 0$ to $z = \inf$, we get

$$\Gamma(n)\int_0^{\infty} e^{-z}x^{m-1}dz = \int_0^{\infty} e^{-z}z^{m-1}\left[z^n\int_0^{\infty} e^{-zx}x^{n-1}dx\right]dz$$

$$\Gamma(n)\Gamma(m) = \int_0^{\infty}\int_0^{\infty} e^{-z(1+x)}z^{m+n-1}x^{n-1}dxdz$$

Changing the order of integration,

$$\Gamma(n)\Gamma(m) = \int_0^{\infty} x^{n-1}\left[\int_0^{\infty} e^{-z(1+x)}z^{m+n-1}dz\right]dx$$

$$= \int_0^{\infty} x^{n-1}\left[\frac{\Gamma(m+n)}{(1+x)^{m+n}}\right]dx$$

Relation between Beta and Gamma functions III

Finally,

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$

Binomial Coefficient I

Theorem

Let $\binom{r}{k}$ *denote a binomial coefficient. Then,*

$$\binom{r}{k} = \frac{1}{(r+1)\beta(k+1, r-k+1)}$$

$$\begin{aligned}
\binom{r}{k} = \frac{r!}{k!(r-k)!} &= \frac{\Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} \\
&= \frac{\Gamma(r+2)}{+1} \frac{1}{\Gamma(k+1)\Gamma(r-k+1)} \\
&= \frac{1}{r+1} \frac{\Gamma(r+2)}{\Gamma(k+1)\Gamma(r-k+1)} \\
&= \frac{1}{r+1} \frac{1}{\beta(k+1, r-k+1)}
\end{aligned}$$