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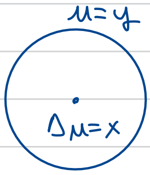

$x^2+y^2 \leq 1$

$\Delta u = x$

$u = y$

$r=1$

$x = r\cos\varphi$

$y = r\sin\varphi$

$r^2 = x^2+y^2$

$\varphi = \operatorname{arctg}(y/x)$

$$\Omega \to M = \{ r \in [0,1] \ , \ \varphi \in [0, 2\pi] \}$$

$$\Delta u = \frac{\partial^2 u}{r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$$

$$\Delta u = x \longrightarrow u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = r\cos\varphi$$

Boundary conditions: if $r=1 \Rightarrow u = r\sin\varphi \Rightarrow u(1,\varphi) = \sin\varphi$

Now we calculate Fourier decomposition of u by sin and cos:

$$u = \sum_{k=0}^{\infty} V_k(r)\cos(k\varphi) + \sum_{k=1}^{\infty} W_k(r)\sin(k\varphi)$$

Since u should be continuous: $u(r,\varphi) \xrightarrow[r\to 0]{} u^*$, $V_k(0) = W_k(0) = 0 \quad k \neq 0$

① $V_k'' + \frac{1}{r}V_k' - \frac{k^2}{r^2}V_k = \begin{cases} r & , k=1 \\ 0 & , k \neq 1 \end{cases} \Rightarrow \begin{cases} V_0(0) \text{ is bounded} \\ V_k(0) = 0 \quad\quad k = 0, 1, \dots \\ V_k(1) = 0 \end{cases}$

② $W_k'' + \frac{1}{r}W_k' - \frac{k^2}{r^2}W_k = 0 \Rightarrow \begin{cases} W_k(0) = 0 \quad\quad k = 1, 2, \dots \\ W_k(1) = \begin{cases} 1 & , k=1 \\ 0 & , k \neq 1 \end{cases} \end{cases}$

① $V_K'' + \frac{1}{r} V_K' - \frac{K^2}{r^2} V_K = 0$

$r^2 V_K'' + r V_K' - K^2 V_K = 0$

$\lambda(\lambda - 1) + \lambda - K^2 = 0 \longrightarrow \lambda^2 - K^2 = 0 \longrightarrow \lambda_1, \lambda_2 = \pm K$

• if $K = 0 \Rightarrow V_0 = c_1 + c_2 t = c_1 + c_2 \ln(r)$

$V_0(0)$ bounded $\Leftrightarrow \lim_{r \to 0} c_1 + c_2 \ln(r) < \infty \Leftrightarrow c_2 = 0$

$$\left.\begin{array}{c} V_K(0) = 0 \\ \text{and} \\ V_0(0) = c_1 = 0 \end{array}\right\} \Rightarrow V_0 = 0$$

• if $K \neq 1 \Rightarrow V_K = C_1 r^K + C_2 r^{-K}$

$$\left.\begin{array}{l} V_K(0) = 0 \Leftrightarrow \lim_{r \to 0} c_1 r^K + c_2 r^{-K} = 0 \Leftrightarrow C_2 = 0 \\ V_K(1) = 0 \Leftrightarrow C_1 \cdot r = 0 \Leftrightarrow C_1 = 0 \end{array}\right\} V_K = 0 \quad K \neq 1$$

• if $K = 1 \Rightarrow V_1'' + \frac{1}{r} V_1' - \frac{1}{r^2} V_1 = r$

$r^2 V_1'' + r V_1' - V_1 = r^3$

$V_1 = C_1 r + \frac{C_2}{r} + V_1^*$

$V_1^* = A r^3 \rightarrow (V_1^*)' = 3Ar^2 \longrightarrow (V_1^*)'' = 6Ar$

$6Ar^3 + 3Ar^3 - Ar^3 = r^3 \Leftrightarrow A = \frac{1}{8}$

$\Rightarrow V_1 = C_1 r + \frac{C_2}{r} + \frac{r^3}{8}$

$$v_1(0)=0 \Leftrightarrow \lim_{r\to 0} c_1 r + \frac{c_2}{r} + \frac{1}{8} r^3 = 0 \Leftrightarrow c_2 = 0$$

$$v_1(1)=0 \Leftrightarrow c_1 + \frac{1}{8} = 0 \Leftrightarrow c_1 = -\frac{1}{8}$$

$$\left.\right\} v_1 = -\frac{r}{8} + \frac{r^3}{8}$$

② $$w_K'' + \frac{1}{r} w_K' - \frac{K^2}{r^2} w_K = 0$$

- $w_K(0)=0 \Rightarrow \lim_{r\to 0} c_1 \cdot r^K + c_2 r^{-K} = 0 \Leftrightarrow c_2 = 0 \quad \forall K$

- if $K \neq 1 \Rightarrow w_K = c_1 r^K$

  $w_K(1) = c_1 \cdot 1^K = 0 \Leftrightarrow c_1 = 0 \Rightarrow w_K = 0$

- if $K = 1 \Rightarrow w_1 = c_1 \cdot r$

  $w_1(1) = c_1 = 1 \Rightarrow w_1 = r$

Finally, $v_1 = -\frac{1}{8} r + \frac{1}{8} r^3$ and $w_1 = r$ and all other are zero.

From Fourier decomposition

$$u = \left(-\frac{1}{8} r + \frac{1}{8} r^3\right) \cos\varphi + r \sin\varphi = \frac{1}{8} r \cos\varphi \,(r^2 - 1) + r \sin\varphi$$

$$u = \frac{x}{8}(x^2 + y^2 - 1) + y$$