Preliminary Results

Large Scale Optimization

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Contents

1. Convexity. Basic concepts

Convex sets and convex functions

Definition of a convex set C

A set $C \subseteq \mathbb{R}^n$ is convex if, given $x^1, x^2 \in C$, the closed segment $\overline{x^1, x^2}$ is contained in C, i.e.: $\overline{x^1, x^2} \subseteq C$,

$$(\overline{x^1, x^2} \triangleq \{x \in \mathbb{R}^n \,|\, x = \alpha x^1 + (1-\alpha)x^2,\ 0 \leq \alpha \leq 1\})$$

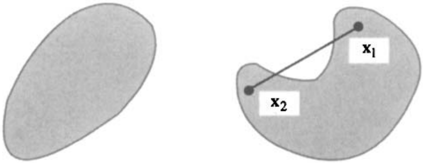


Figure: A convex set (left). A non-convex set (right)

If C_1, C_2 are convex sets, then $C_1 \cap C_2$, $C_1 \oplus C_2$, $C_1 \ominus C_2$ are also convex sets.

Convex sets and convex functions

Definition of a convex function f on P, convex set

$$f(\alpha x^1 + (1-\alpha)x^2) \leq \alpha f(x^1) + (1-\alpha) f(x^2),\ 0 \leq \alpha \leq 1,\ x^1, x^2 \in P$$

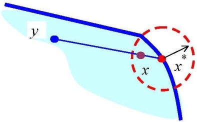
If f is a convex function $f(x) \geq f(x_1) + (x - x_1)^\top \nabla f(x_1)$.

Igualmente, si f convexa sobre P conjunto convexo:
$f(x) \geq f(x_1) + (x - x_1)^\top \nabla f(x_1),\ \forall\, x, x_1 \in P$

Si x^* es mínimo local de f convexa sobre P convexo $\Rightarrow$
$\Rightarrow$ x^* es mínimo global de f sobre P.

$\nabla f(x^*)^\top (x - x^*) \geq 0,\ \forall x \in B_\delta(x^*) \cap P$
Si $y \in P,\ y \notin B_\delta(x^*) \cap P\ \ y\bar{x}^* \subset P$
$f(y) - f(x^*) \geq \nabla f(x^*)^\top (y - x^*) = \alpha \nabla f(x^*)^\top (x - x^*) \geq 0$

(f differentiable on P)



Polyhedral Sets and extreme points

Relevant examples of convex sets

- **Hyperplane** $S = \{x \in \mathbb{R}^n |\ a^\top x = \alpha\}$, where $a \neq 0$ and $\alpha \in \mathbb{R}$
- **Half-Space** $S = \{x \in \mathbb{R}^n |\ a^\top x \leq \alpha\}$, where $a \neq 0$ and $\alpha \in \mathbb{R}$
- **Polyhedral Set** $S = \{x \in \mathbb{R}^n |\ Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Finite intersection of m half-spaces.
- **Polyhedral Cone** $C = \{x \in \mathbb{R}^n |\ Ax \leq 0\}$, where $A \in \mathbb{R}^{m \times n}$. Finite intersection of m half-spaces, each of them containing $0 \in \mathbb{R}^m$.

Convex Hulls

Convex Hull of a finite set of points $S = \{x^1, ..., x^k\}$ (**Polytope**)

The convex hull of S, denoted by *Hull*$\{S\}$ (also as *Conv*$\{S\}$), is defined as:

$$\textit{Hull}\{S\} \triangleq \left\{ x \in \mathbb{R}^n \mid x = \sum_{\ell=1}^{k} \lambda_\ell x^\ell,\ \sum_{\ell=1}^{k} \lambda_\ell = 1,\ \lambda_\ell \geq 0,\ 1 \leq \ell \leq k \right\}$$

Hull$\{S\}$ for a general subset $S \subset \mathbb{R}^n$.

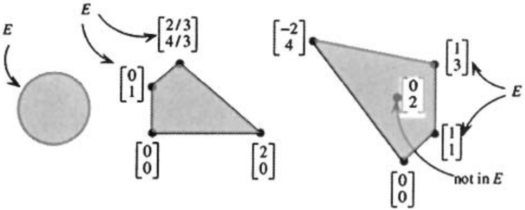
$$x \in \textit{Hull}\{S\} \Leftrightarrow \begin{cases} \exists \{x^1, ..., x^k\} \subseteq S, k > 0,\ \textit{and}\ \lambda_1, ...\lambda_k \geq 0, \\ \textit{such that} \sum_{\ell=1}^{k} \lambda_\ell = 1 \\ x = \sum_{\ell=1}^{k} \lambda_\ell x^\ell, \end{cases}$$

Convex Hulls

Extreme points (vertexes) of a Convex Set

$\hat{x} \in S$ is a vertex of S iff. for any two points $x^1 \neq x^2 \in S$ so that $\hat{x} = \alpha x^1 + (1-\alpha)x^2$, $0 \le \alpha \le 1$ either $\hat{x} = x^1$ or $\hat{x} = x^2$.

$\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\}$,
$\{x \in \mathbb{R}^2 \mid x_1 + x_2 \le 2, -x_1 + 2x_2 \le 2, x \ge 0\}$,
$Hull\{(0,0),(1,1),(1,3),(-2,4),(0,2)\}$



2. Useful concepts from the Simplex Algorithm

Standard form and bfs

STANDARD FORM OF AN L.P.

After suitable transformations any L.P. can be expressed as:

$$
\begin{array}{ll}
Min_x & c_1 \cdot x_1 + \dots + c_n \cdot x_n \\
s.a: & a_{11} \cdot x_1 + \dots + a_{1n} \cdot x_n = b_1 \\
 & a_{21} \cdot x_1 + \dots + a_{2n} \cdot x_n = b_2 \\
 & \dots \\
 & a_{m1} \cdot x_1 + \dots + a_{mn} \cdot x_n = b_m \\
 & x_1 \geq 0, \ \dots x_n \geq 0
\end{array}
\longrightarrow
\begin{array}{ll}
Min_x & c^\top x \\
s.a: & Ax = b \\
 & x \geq 0
\end{array}
$$

$(m \leq n)$

- All the variables x_i are subject to $x_i \geq 0, \ i = 1, 2, \dots n$
- Any rhs term b_i are non-negative: $b_i \geq 0, \ i = 1, 2, \dots m$
- Matrix A is full rank:

Therea are m columns of A such that when building a squared matrix B with them, B is inversible.

Any LP solver converts automatically to the standard form.

Standard form and bfs

Example:

$$\boxed{\begin{array}{lll} Min & -x_1+3x_2+4x_3 & \\ s.a: & -x_1-x_2-x_3 & \geq -5 \\ & 2x_1+3x_2+x_3 & \geq 6 \\ & x_2,\ x_3 \geq 0 & \end{array}}$$

$$\begin{array}{lll} Min & -x_1+3x_2+4x_3 & \\ s.a: & -x_1-x_2-x_3-x_4 & = -5 \\ & 2x_1+3x_2+x_3 \quad -x_5 & = 6 \\ & x_2,\ x_3,\ x_4,\ x_5 \geq 0 & \end{array}$$

$$x_1 = 5-2x_2-x_3-x_4$$

$$c^\top x = -5+2x_2-x_3-x_4+3x_2+4x_3 = \cancel{-5}+5x_2+5x_3+x_4$$

$$2(5-x_2-x_3-x_4)+3x_2+x_3-x_5 = 6$$

$$\boxed{\begin{array}{lll} Min & 5x_2+5x_3+x_4 & \\ s.a: & x_2+x_3+2x_4+x_5 & = 4 \\ & x_2,\ x_3,\ x_4,\ x_5 \geq 0 & \end{array}}$$

Standard form and bfs

DEFINITION OF A FEASIBLE BASIS

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + \tfrac{1}{3}x_2 + x_4 &= 1 \\ -x_1 + \tfrac{2}{3}x_2 + x_5 &= 4 \\ x_i \geq 0 & \quad (\, i = 1, \ldots 5 \,) \end{aligned} \longrightarrow A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & \frac{1}{3} & 0 & 1 & 0 \\ -1 & \frac{2}{3} & 0 & 0 & 1 \end{pmatrix}$$

System $Ax = b, \; x \geq 0$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

DEFINITION: B **is a feasible basis if** $\boxed{B^{-1}b \geq 0}$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \geq 0$$

B **is a basis assocaoted to the index set {1, 4, 5}**

Standard form and bfs

System $Ax = b, \; x \geq 0$

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + \tfrac{1}{3}x_2 + x_4 &= 1 \\ -x_1 + \tfrac{2}{3}x_2 + x_5 &= 4 \end{aligned}$$

$$x_i \geq 0 \qquad (\, i = 1, \ldots 5 \,)$$

B **basis associated to I_B = {1, 4, 5}**

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \geq 0$$

$$x_{Reord} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \\ \hline x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_B \\ \hline x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ \hline 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ \hline 0 \\ 0 \end{pmatrix} \geq 0$$

Canonical form of $Ax = b, x \geq 0$

CANONICAL FORM OF $Ax=b$ For the index set $I_B = \{1, 4, 5\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$B \; x_B \; + \; N \; x_N \; = \; b$$

$$B^{-1} (B \; x_B \; + \; N \; x_N) \; = \; B^{-1} b$$

$$x_B \; + \; B^{-1} N \; x_N \; = \; B^{-1} b$$

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$x_B \; + \; Y \; x_N \; = \; y_0$$

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{-2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

1	4	5	2	3	
1	0	0	−1	1	1
0	1	0	−2/3	1	2
0	0	1	−1/3	1	5

Canonical form of $Ax = b, x \geq 0$

CANONICAL FORM OF A LINEAR SYSTEM $Ax=b$

For the set of indexes associated to a basis B, $I_B=\{i_1, i_2, \ldots, i_m\}$

$$x_B + Y x_N = y_0$$

$$x_B = \begin{pmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_m} \end{pmatrix}, \quad x_N = \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \vdots \\ x_{j_{n-m}} \end{pmatrix}$$

$$\begin{array}{ccc|ccc|c}
i_1 & \ldots & i_m & j_1 & \ldots & j_{n-m} & 0 \\
\hline
1 & \ldots & 0 & y_{1,1} & \ldots & y_{1,n-m} & y_{1,0} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 1 & y_{m,1} & \ldots & y_{m,n-m} & y_{m,0}
\end{array}$$

Basic columns

Non-basic columns

≥ 0 if B is a feasible basis

3. Geometrical aspects

Vertexes $\longleftrightarrow$ bfs. Connection between vertices and bfs

$P = \{x \in \mathbb{R}^n \mid Ax = b,\, x \geq 0\}$ in standard form

- **If $\bar{x}$ is a bfs of P** with basic indexes $I_B \Rightarrow$ **is a vertex of P** :

  $\bar{x}_{Reord} = \begin{pmatrix} \bar{x}_B \\ 0 \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$. *Assume that it is not a vertex P. Then,* $\exists x^1, x^2 \in F$, $x^1 \neq x^2$, $\bar{x} = \alpha x^1 + (1-\alpha)x^2,\ 0 < \alpha < 1$.
  $\begin{pmatrix} \bar{x}_B \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} x_B^1 \\ x_N^1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} x_B^2 \\ x_N^2 \end{pmatrix} \Rightarrow\ x_N^1 = x_N^2 = 0$, *but, as* x^1, x^2 *are feasible:*
  $Bx_B^1 = b,\ \ Bx_B^2 = b \Rightarrow\ x_B^1 = x_B^2 = \bar{x}_B \Rightarrow\ x^1 = x^2 = \bar{x}$.

- **For each vertex $\bar{x} \in F$ exists at least a basic set I_B** and a basis B such that: $\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$.

- **Different vertices are represented by different basis.**

Vertexes $\longleftrightarrow$ bfs

Existence of vertices and Fundamental Theorem in LP

Consider $P = \{x \in \mathbb{R}^n \,|\, Ax = b,\, x \geq 0\ \}$ in standard form.

Theorem. (2.6.5 in Bazaraa's text) If $P \neq \emptyset$, then it has at least always a vertex.

Fundamental Theorem in LP (Section 2.4 in Luenberger's text). If the feasible region of the LP is not empty, there exists always at least a basic feasible solution. Moreover, if LP has solutions, then at least one of them is a vertex.

$$
\begin{array}{lll}
\text{Min}_x & c^\top x & \\
s.t.: & Ax = b & \text{(standard form)} \\
 & x \geq 0 &
\end{array}
$$

Extreme directions

Concept of feasible direction at $x \in S$

A vector d is called a **feasible direction** at a point $x \in S$ if $x + \alpha d \in S$, for any $\alpha > 0$. (S an arbitrary set)

Definition of a Cone C with vertex 0^n

C is a cone with vertex 0^n iff. $\forall\, x \in C$ there follows that $\mu x \in C$, $\forall\, \mu > 0$.

Polarity

Definition of Polar Cone. Given an arbitrary set S, the polar cone of S is denoted by S^* (it has vertex at 0^n) and it is defined by:

$$S^* \triangleq \{v \in \mathbb{R}^n \,|\, v^\top x \leq 0,\ \forall\, x \in S\}$$

S^* is closed and convex, $S \subseteq S^{**}$; $S_1 \subseteq S_2 \Rightarrow S_2^* \subseteq S_1^*$.
If C is a closed and convex cone, then $C = C^{**}$

Extreme directions

Positive Hull of a finite set of directions $D = \{d^1, ..., d^q\}$ (Closed, convex cone)

The positive hull of D, denoted by *Pos*$\{D\}$ (also as *Cone*$\{D\}$), is defined as:

$$\mathit{Pos}\{S\} \triangleq \left\{ d \in \mathbb{R}^n \mid d = \sum_{\ell=1}^{q} \mu_j d^j,\ \mu_j \geq 0,\ 1 \leq j \leq q \right\}$$

$\mathit{Pos}\{D\}$ for a general subset $S \subset \mathbb{R}^n$.

$$d \in \mathit{Pos}\{D\} \Leftrightarrow \begin{cases} \exists \{d^1, ..., d^q\} \subseteq D, q > 0,\ \textit{and}\ \mu_1, ...\mu_q \geq 0, \\ d = \displaystyle\sum_{\ell=1}^{k} \mu_j d^j, \end{cases}$$

As it happened with vertices (or extreme points) in a convex set, a convex cone may have directions which may not be expressed using other directions of the convex cone. These are referred to as extreme directions.

Extreme direction $\vec{d}$ of a convex cone D

$\vec{d} \in S$ is an extreme direction of D iff. for any two directions $d^1 \neq d^2 \in D$ so that $\vec{d} = \alpha d^1 + (1-\alpha)d^2$, $0 \leq \alpha \leq 1$, either $\vec{d} = d^1$ or $\vec{d} = d^2$.

Characterization of vertices and extreme directions

Directions in a Polyhedron P

d is a feasible direction of the polyhedron P iff. $\forall\, x \in P$, $x + \mu d \in P, \forall \mu > 0$.

- Clearly, the existence of feasible directions in a Polyhedron implies its unboundedness.
- If they exist, the set of feasible directions D of a Polyhedron is a **closed and convex Cone**
- If $P = \{x \in \mathbb{R}^n \,|\, Ax = b,\, x \geq 0\}$ (standard form), then its cone D of feasible directions must verify: $Ad = 0, d \geq 0$

Characterization of vertices and extreme directions

(Theorem 2.6.6 in Bazaraa's text) Characterization of Extreme directions of a Polyhedron using a canonical form of $Ax = b, x \geq 0$.

Assume that the canonical form (below) of the system $Ax = b, x \geq 0$ associated to a feasible basis B is known and that columns of A are reordered such that $A = (B|N)$.

$$x_B + Yx_N = Y_0, \;\; Y = B^{-1}N$$

Then, the Polyhedron P has a non-null cone of feasible directions iff. for some feasible basis, the corresponding matrix Y of the canonical form has at least a non-positive column ℓ, $1 \leq \ell \leq n - m$. Furthermore, in that case, associated to the base B (and to the corresponding vertex of the Polyhedron), extreme directions are of the form:

$$\vec{d} = \left(\frac{-B^{-1}a_{j_\ell}}{e_\ell} \right) = \left(\frac{-Y_\ell}{e_\ell} \right), \text{ with } Y_\ell \leq 0, \; j_\ell \in I_N$$

Can you write the components of the extreme direction?

CANONICAL FORM OF $Ax=b$ For the index set $I_B = \{1, 4, 5\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$B \; x_B + N \; x_N = b$$

$$B^{-1}(B \; x_B + N \; x_N) = B^{-1} b$$

$$x_B + B^{-1}N \; x_N = B^{-1} b$$

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$x_B + Y \; x_N = y_0$$

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{-2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

1	4	5	2	3	
1	0	0	-1	1	1
0	1	0	$-2/3$	1	2
0	0	1	$-1/3$	1	5

4. Relevant theorems

Minkowsky-Weyl's Representation Theorem for Polyhedrons

Theorem 2.6.7 in Bazaraa's text

Let $V = \{\hat{v}^1, ..., \hat{v}^k\}$, the set of vertexes of a polyhedron P, $(k > 0, i.e., V \neq \emptyset)$, and let $D = \{\vec{d}^1, ..., \vec{d}^q\}$ its (possibly empty) set of extreme directions $(q \geq 0)$.
Then P can be written in the form:

$$P = \mathsf{Hull}(V) \oplus \mathsf{Pos}(D) \;\equiv \begin{cases} x \in P \Leftrightarrow x = \displaystyle\sum_{\ell=1}^{k} \lambda_\ell \hat{v}^\ell + \sum_{j=1}^{q} \mu_j \vec{d}^j \\ \left(\displaystyle\sum_{\ell=1}^{k} \lambda_\ell = 1, \;\; \lambda_\ell \geq 0, \;\; \mu_j \geq 0 \right) \end{cases}$$

Results related to LP duality

Farkas-Minkowsky's lemma. Theorem 2.4.5 in Bazaraa's text

Let $A \in \mathbb{R}^{m\times n}$ 1) and $b \in \mathbb{R}^m$. Then, 1) is equivalent to 2)

1. $P = \{x \in \mathbb{R}^n \,|\, Ax = b,\ x \geq 0\} \neq \emptyset$
2. $\forall u \in \{v \in \mathbb{R}^m \,|\, A^\top v \geq 0\}$, it is verified that $b^\top u \geq 0$

Gale's theorem

Either 1) or 2) holds but not both:

1. $A^\top u \leq c$ has a solution u.
2. $Ad = 0$, $c^\top d = -1$, $d \geq 0$ has a solution d.

5. A worked example

5. Calculate the basic feasible solutions for the set of constraints:

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + \tfrac{1}{3}x_2 + x_4 &= 1 \\ -x_1 + \tfrac{2}{3}x_2 + x_5 &= 4 \\ x_i \geq 0 & \qquad (\, i = 1, \ldots 5 \,) \end{aligned}$$

starting from the basis $I_B = \{\, 3, 4, 5 \,\}$.

3	4	5	1	2	
1	0	0	1	-1	1
0	1	0	-1	$1/3$	1
0	0	1	-1	$2/3$	4

$$I_2^+ = \{i_4 = 2, \; i_5 = 3\}$$

$$\hat{\varepsilon} = Min_{\,i \in I_2^+} \{ \frac{y_{i,0}}{y_{i,2}} \} = Min\{ \frac{1}{1/3} , \frac{4}{2/3} \} = \frac{1}{1/3} = 3 \Rightarrow i_p = 2 \Rightarrow p = 4$$

exiting variable x_4.

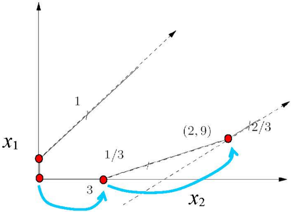
$$\begin{array}{ccccc|c} 3 & 4 & 5 & 1 & 2 & \\ \hline 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1/3 & 1 \\ 0 & 0 & 1 & -1 & 2/3 & 4 \end{array} \rightarrow \begin{array}{ccccc|c} 3 & 4 & 5 & 1 & 2 & \\ \hline 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 3 & 0 & -3 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2/3 & 4 \end{array}$$

$$\begin{array}{ccccc|c} 3 & 4 & 5 & 1 & 2 & \\ \hline 1 & 3 & 0 & -2 & 0 & 4 \\ 0 & 3 & 0 & -3 & 1 & 3 \\ 0 & -2 & 1 & 1 & 0 & 2 \end{array}$$

New basis $I_B = \{3, 2, 5\}$

$$I_1^+ = \{i_5 = 3\}$$

$$\hat{\varepsilon} = Min_{\ i \in I_1^+} \left\{ \frac{y_{i,0}}{y_{i,2}} \right\} = Min\left\{ \frac{2}{1} \right\} = 2 \Rightarrow$$



$$I_B = \{3, 2, 1\}$$

3	4	5	1	2	
1	−1	2	0	0	8
0	−3	3	0	1	9
0	−2	1	1	0	2

$$d^\top = (d_3, d_2, d_1, d_4, d_5) = (1, 3, 2, 1, 0)$$

Could you apply Minkowsky-Weyl's theorem ?

$$\begin{array}{rrrrr|r} 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & -1 & 1 & 0 & 0 & 1 \\ -1 & 1/3 & 0 & 1 & 0 & 1 \\ -1 & 2/3 & 0 & 0 & 1 & 4 \end{array} \quad \text{Enters } x_1 \Rightarrow \text{Exits } x_3 \Rightarrow \quad \begin{array}{rrrrr|r} 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & -2/3 & 1 & 1 & 0 & 2 \\ 0 & -1/3 & 1 & 0 & 1 & 5 \end{array}$$

$$d^\top = (d_1, d_4, d_5, d_2, d_3) = (1, 2/3, 1/3, 1, 0)$$