Examining the primal solution after a Lagrangian dual maximization

What must be done after applying a Lagrangian dual maximization?
Assume that the Lagragian dual problem has been solved:

$$\begin{array}{ll} Max & w(\lambda, \mu) \\ & \lambda,\ \mu \in D \end{array} \rightarrow \lambda^*, \mu^*$$

This means that in the final iteration the following problem has been solved:

$$\widetilde{x} \leftarrow \begin{array}{ll} Min & f(x) - \lambda^{*\top} h(x) - \mu^{*\top} g(x) \\ & x \in X \end{array}$$

If a non-null duality gap is obtained, then x will not be a solution of the original primal problem (P). But even in the case that a null gap is attained, $\tilde{x}$ may be not a solution as well. The following result provides an immediate way to recognize whether a null gap is attained or not.

<u>(Theorem 6.5.1 Bazaraa)</u>
Let $(\widetilde{\lambda}, \widetilde{\mu})$ with $\widetilde{\mu} \geq 0$ and $\widetilde{x}$ a solution of problem:

$$\begin{array}{ll} Min & f(x) - \lambda^{*\top} h(x) - \mu^{*\top} g(x) \\ & x \in X \end{array} \rightarrow \widetilde{x}$$

Then $\widetilde{x}$ is also a solution of problem:

$$\begin{array}{ll} Min_x & f(x) \\ \widetilde{F} \rightarrow & \left\{ \begin{array}{l} h(x) = h(\widetilde{x}) \\ g_i(x) \geq g_i(\widetilde{x}),\ i \in I_+(\mu) \end{array} \right. \\ & x \in X \end{array}$$

($I_+(\mu) \triangleq \{1 \leq \ell \leq q \mid \mu_\ell > 0\}$)

It is obvious to check that if $g(\widetilde{x}) \geq 0$, $h(\widetilde{x}) = 0$ and, additionally, $\widetilde{\mu} g(\widetilde{x}) = 0$ then

a) $\widetilde{x}$ is a solution of (P)

b) $\widetilde{\lambda}, \widetilde{\mu}$ is a solution of (LD)

<u>Proof of theorem 6.5.1</u>

Because $\widetilde{x}$ solves the relaxation: $\begin{array}{ll} Min & f(X) - \lambda^{*\top} h(x) - \mu^{*\top} g(x) \\ & x \in X \end{array}$

$f(x) - \widetilde{\lambda}^\top h(x) - \widetilde{\mu}^\top g(x) \geq f(\widetilde{x}) - \widetilde{\lambda}^\top h(\widetilde{x}) - \widetilde{\mu}^\top g(\widetilde{x})\ \ \forall x \in \widetilde{F}$. Also, as $h(x) = h(\widetilde{x})$, $\forall x \in \widetilde{F}$, then

$$f(x) - \widetilde{\mu}^\top g(x) \geq f(\widetilde{x}) - \widetilde{\mu}^\top g(\widetilde{x})$$
$$f(x) + \widetilde{\mu}^\top (g(\widetilde{x}) - g(x)) \geq f(\widetilde{x})$$

but as $x \in \widetilde{F}$ is $g_\ell(\widetilde{x}) - g_\ell(x) \leq 0, \ \forall \ell \in I_+(\mu) \Rightarrow, \Rightarrow \mu^\top(g(\widetilde{x}) - g(x)) \leq 0 \ \Rightarrow \ f(x) \geq f(\widetilde{x})$. Thus $\widetilde{x}$ is a solution of $Min_x\{f(x) \mid h(x) = h(\widetilde{x}), \ g_\ell(x) \geq g_\ell(\widetilde{x}); \ \ell \in I_+(\mu)\}$. $\square$

In summary, once the lagrangian relaxation has been solved and $(\widetilde{x}, \widetilde{\lambda}, \widetilde{\mu})$ have been calculated, one needs to check for:

a) The feasibility of $\widetilde{x}$: $h(\widetilde{x}) = 0, \ \ g(\widetilde{x}) \geq 0$

b) The complementarity $\widetilde{\mu}^\top g(\widetilde{x}) = 0$