Introduction to Dantzig-Wolfe decomposition (1961)

Application to lagrangian duality to linear and quadratic problems (6.6 Bazaraa)

Desc. D-W: Price decomposition

The question is the application of lagrangian duality to linear problems of the type:

$$\begin{array}{lrl} & Min_x & c^\top x \\ x \in \mathbb{R}^n & s.t. & Ax = b \\ & & Dx = d;\ x \geq 0 \end{array}$$

where D is a block diagonal matrix so that the dualization of constraints $Ax = b$ is advantageous allowing for the decomposition of the problem:

$$D = \begin{bmatrix} \boxed{D_1} & & & \\ & \boxed{D_2} & & 0 \\ & 0 & \ddots & \\ & & & \boxed{D_K} \end{bmatrix},\ A = [A_1 A_2 \ldots A_K]$$

$$d = (d_1, d_2, \ldots, d_K), \quad \begin{array}{l} c^\top = (c_1^\top, c_2^\top, \ldots, c_k^\top) \\ x = (x_1, x_2, \ldots, x_k) \end{array}$$

Explicitly, the program is then,

$$\begin{array}{lrl} & Min_x & \sum_{l=1}^K c_\ell^\top x_\ell \\ (P) & s.t. & \sum_{l=1}^K A_\ell x_\ell = b \\ & & \left.\begin{array}{l} D_\ell x_\ell = d_\ell \\ x_\ell \geq 0 \end{array}\right\} \ell = 1, 2, \ldots, K \end{array}$$

For comfort, let now consider the following polyhedra:

$$X_\ell = \{x_\ell \in R^{n_\ell} \mid D_\ell x_\ell = d_\ell,\ x_\ell \geq 0\}$$

(P) is then rewritten as:

$$\begin{array}{ll} Min_x & \sum_{\ell=1}^K c_\ell^\top x_\ell \\ & \sum_{\ell=1}^K A_\ell x_\ell = b \ \ |\lambda \\ & x_\ell \in X_\ell,\ \ \ell = 1, 2, \ldots, K \end{array}$$

and, by dualizing $Ax = b$, the dual lagrangian function results in:

$$\begin{array}{ll} w(\lambda) = \ Min_x & \sum_{l=1}^K c_\ell^\top x_\ell - \lambda^\top (\sum_{l=1}^K A_\ell x_\ell - b) \\ & x_\ell \in X_\ell,\ \ell = 1, 2, \ldots, K \end{array}$$

$$\begin{array}{lll} w(\lambda) = \ Min_x & \sum_{l=1}^{K} (c_\ell - A_\ell^\top \lambda)^\top x_\ell - \lambda^\top b & \rightarrow \ S_D^*(\lambda) \\ (D) & x_\ell \in X_\ell, \ \ell = 1,2,...,K & \end{array}$$

$\boxed{Max_\lambda \ w(\lambda)}$ (LD) $\rightarrow \lambda^*$

w is concave and non-differentiable . (null duality gap $\rightarrow$ because the corresponding value function for the rhs of the dualized constraints is convex)

If (P) presents a bounded optimal solution then there exists a saddle point so that:

$\forall x_D^* \in S_D(\lambda^*), \ \lambda^* \in S_{LD}$

$f(x_D^*) = w(\lambda^*) = f^* = f(x^*)$

additionally if $x_d^* \in S_D^*(\lambda)$ then

$\partial w(\lambda) = \{b - Ax_D^* \mid x_D^* \in S_D^*(\lambda)\}$

The advantage of the decomposition relies in that problem (D) splits in K smaller problems of the type:

$$\begin{array}{lll} Min_x & \widetilde{c}_\ell^\top x_\ell & \ell = 1,2,...,K \\ & x_\ell \in X_\ell & \end{array}$$

$$(\widetilde{c}_\ell = c_\ell - A_\ell^\top \lambda)$$

Let us con sider now Y_ℓ = the set (finite) of vertexes X_ℓ

$$Y = \{y \in R^n \times ... \times R^n \ | y^\top = (y_1^\top ... y_K^\top), \ y_\ell \in Y_\ell \ l = 1,2,...K\}$$

Thus, the dual lagrangian function can be rewritten as:

$$\boxed{w(\lambda) = \lambda^\top b + Min_{y \in Y} (c - A^\top \lambda)^\top y}$$

or equivalently:

$$\begin{array}{llll} w(\lambda) \le \lambda^\top B & + & (c - A^\lambda)^\top g_1 & \\ w(\lambda) \le \lambda^\top B & + & (c - A^\lambda)^\top g_2 & g_\ell \in Y \\ & \vdots & & (q = |Y|) \\ w(\lambda) \le \lambda^\top B & + & (c - A^\lambda)^\top g_q & \end{array}$$

and also:

$$\boxed{\begin{array}{ll} Max_{\theta,\lambda} & \theta \\ & \theta \le \lambda^\top b + (c - A^\top \lambda)^\top g_\ell, \ \ l = 1,2,...,q \end{array}}$$

(with a very large number of constraints)

Let us rewrite:

$$(LD)\quad \begin{array}{ll} Max_{\theta,\lambda} & \theta \\ & \theta \le \lambda^\top \gamma_\ell + c_\ell^\top y_\ell, \;\; \ell = 1,2,...,q \quad |\alpha_\ell \end{array}$$

(α_ℓ dual variable) ($\gamma_\ell \stackrel{\triangle}{=} b - Ay_\ell$)

Also:

$$Min\,_{\theta,\lambda} \; -\theta$$

$$(LD)\quad \begin{pmatrix} 1-\gamma_1^\top \\ 1-\gamma_2^\top \\ \vdots \\ 1-\gamma_q^\top \end{pmatrix} \begin{pmatrix} \theta \\ \lambda \end{pmatrix} \le \begin{pmatrix} c_1^\top y_1 \\ c_2^\top y_2 \\ \vdots \\ c_q^\top y_q \end{pmatrix}$$

$$\begin{array}{ll} Max_\alpha & \sum_{\ell=1}^q (c^\top y_\ell)\alpha_\ell \qquad\qquad \to \sum_{\ell=1}^q (c^\top y_\ell)\alpha_\ell = -\theta \\ & \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -\gamma_1 & -\gamma_2 & \cdots & -\gamma_q \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ & \alpha_\ell \le 0 \end{array}$$

$$\left[(P') \quad \begin{array}{llll} Min_\beta & \sum_{\ell=1}^q (c^\top y_\ell)\beta_\ell = Z^* = \theta & & (\beta_\ell = -\alpha_\ell) \\ & \sum_{\ell=1}^q \beta_\ell = 1, \;\; \gamma_\ell \ge 0 & |\,\theta & Master \\ & \sum_{\ell=1}^q \beta_\ell \gamma_\ell = 0, & |\,\lambda & program \end{array} \right.$$

Very large number of variables (vertexes of X. $|Y| = q$)

It must be remarked that problem (P') is just a way to express the original problem (P) by means of set of vertexes Y.
Assume now that we know a subset of vertexes Y', $|Y'| = q' < q$. Then, problem (P') can be stated for Y' and the objective should give a value $z^* \ge z^*$. Because the duality gap is 0.

It must be noticed that the dual of (P') must be precisely LD and hence, lagrange multipliers (θ, λ) for (P') must be the variables of (LD)

Dantzig - Wolfe's algorithm. (Price decomposition)

0) Calculate an initial set of vertexes $Y^{(0}$. At step $(n$:
1) Solve problem (P') for $Y^{(n}$, $|Y^{(n}| = q_n$

$$
(P') \qquad \begin{array}{ll} Min_\beta & \sum_{\ell=1}^{q_n} (c^\top y_\ell)\beta_\ell \\ & \sum_{\ell=1}^{q_n} \beta_\ell \gamma_\ell = 0 \quad | \, \lambda^{(n} \\ & \sum_{\ell=1}^{q_n} \beta_\ell = 1 \qquad | \, \theta^{(n} \\ & \beta_\ell \geq 0 \end{array}
$$

Master problem

$(\gamma_\ell = b - Ay_\ell, \ y_\ell \in Y^{(n)})$

If not STOP **then**:
2) With the obtained LM's $\lambda^{(n}, \theta^{(n}$ by solving (P').
Calculate the value of the dual lagrangian function

$$
w(\lambda^{(n)}) = b^\top \lambda^{(n} + \quad \begin{array}{ll} Min_x & \sum_{i=1}^{k} (c_i - A^\top \lambda^{(n)})^\top x_i \\ & x_i \in X_i \end{array}
$$

(by decomposition.) $\swarrow$

$$
\boxed{\begin{array}{ll} Min_{x_i} & (c_i - A_i^\top \lambda^{(n)})^\top x_i \\ & x_i \in X_i \\ & i = 1, 2, ..., K \end{array}} \longrightarrow \text{new vertex, } \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_K \end{pmatrix}
$$

$w(\lambda^{(n)}) = b^\top \lambda^{(n} + \sum_{i=1}^{K} (c_i - A^\top \lambda^{(n)})\hat{x}_i = c^\top \hat{x} + \hat{\gamma}\lambda^{(n}; (\hat{\gamma} = b - A\hat{x})$

$Y^{(n+1} = Y^{(n} \cup \{\hat{x}\}$

- If $w(\lambda^{(n)}) \geq \theta^{(n}$: STOP = true

$n \leftarrow n + 1$;
EndIf
GO TO 1

At the end, the solution x^* is obtained by: $x^* = \sum_{\ell=1}^{q_n} \beta_\ell y_\ell$

$$
w(\lambda^{(n)}) = \sum_{i=1}^{K} c_i \hat{x}_i^*, \qquad \text{solution } \ x^* = \begin{pmatrix} \hat{x}_1^* \\ \hat{x}_2^* \\ \vdots \\ \hat{x}_K^* \end{pmatrix}
$$

$\nearrow$
(nul duality gap)

In practice the algorithm stops when: $\theta^{(n} - w(\lambda^{(n)}) \leq \varepsilon$