Dantzig's cutting plane algorithm (1959).

The algorithm generates at each iteration a new constraint approximating each time more accurately the problem:

$$
(P_\infty) \qquad \begin{array}{ll} Max_{(z,\lambda,\mu)} & z \\ & z \leq f(x) - \lambda^\top h(x) - \mu^\top g(x);\ \forall x \in X \\ & \mu \geq 0 \end{array}
$$

The algorithm:
0) Determine $x_0 \in X$ feasible $(h(x_0) = 0,\ g(x_0) \geq 0)$ k=0.

1) Solve:

$$
\begin{array}{lll} & Max_{(z,\lambda,\mu)} & z \\ & \swarrow & z \leq f(x_\ell) - \lambda^\top h(x_\ell) - \mu^\top g(x_\ell) \quad \ell = 0, 1, 2, ..., k \\ (z_k, \lambda_k, \mu_k) & & \mu \geq 0 \end{array}
$$

2) Solve:

$$
\begin{array}{lll} & Min & f(x) - \lambda_k^\top h(x) - \mu_k^\top g(x) = w(\lambda_k, \mu_k) \\ & \swarrow & x \in X \\ x_{k+1} & & \end{array}
$$

- If $w(\lambda_k, \mu_k) = f(x_{k+1}) - \lambda_k^\top h(x_{k+1}) - \mu_k^\top g(x_{k+1}) < z_k$ continue
- If $w(\lambda_k, \mu_k) = z_k$ then STOP.

It must be remarked that problem (P_∞) has an infinite number of constraints and that actually it is a representation of the dual lagrangian problem.

$$
\boxed{\begin{array}{ll} Max & w(\lambda, \mu) \\ & (\lambda, \mu) \in D \end{array}}
$$

Generating primal solutions (convex case)

It must be remarked that, after ending, Dantzig's algorithm may not generate a primal feasible solution for the original problem. As a typical example consider the following linear programming problem LP

$$
\begin{array}{ll} Min_{\ x} & C^\top x \\ & Ax = b \\ & x \leq c \\ & x \geq 0 \end{array} \qquad \begin{array}{c} (h \text{ afí}) \\ (\ X = \{\ x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\ \}\ ) \\ (\ g(x) = c - x, \quad h = 0\ ) \end{array}
$$

Remarks: step 0 does determine a feasible point $x_0 \in X$ $g(x) \geq 0$.
Assume then that x_0 is feasible: $x_0 \in X,\ h(x_0) = 0,\ g(x_0) \geq 0$.
Let x_ℓ $\ell = 0, 1, 2, ...k$ points obtained in maximizing the dual lagrangian function (for instance by using Dantzig's cutting plane algorithm); then the solutions of the following problem:

$$
\left.
\begin{array}{ll}
z_k = Min_{\ \alpha} & \displaystyle\sum_{j=0}^{k} \alpha_j f(\hat{x}_j) \\
\\
& \displaystyle\sum_{j=0}^{k} \alpha_j g(\hat{x}_j) \geq 0 \\
& \displaystyle\sum_{j=0}^{k} \alpha_j h(\hat{x}_j) = 0 \\
& \displaystyle\sum_{j=0}^{k} \alpha_j = 1 \ \ , \ \alpha_j \geq 0
\end{array}
\right\} \longrightarrow \ \ \widetilde{x} = \sum_{j=0}^{k} \alpha^* \hat{x}_j
$$

are so that if $z_k - w(\lambda_k, \mu_k) \leq \epsilon$ then $f(\widetilde{x}) \leq z^* + \epsilon$, where z^* is the optimal objective function's value:

$$
\text{i.e.,} \quad z^* = Min_{\ x \in X} \quad
\begin{array}{l}
f(x) \\
g(x) \geq 0 \ \ , \ h(x) = 0
\end{array}
$$