Gauvin-Dubeau's formula and value function

References:
J. Gauvin, F. Dubeau (1982) "Differential properties pf the marginal function in mathematical programming". Math. Prog. Study pp 101-119
K. Shimizu, Y. Ishizuka, J. F. Bard. "Nondifferentiable and Two-level Mathematical Programming". Kluwer Academic Publishers 1997. (Cap 6)

Consider the following parametrized problem:

$$\begin{array}{lll} V(y) = & \mathsf{Min}_{\,v} & f(v,y) \\ & (P) & h(v,y) = 0 \quad \mid \lambda \quad \Rightarrow \; S^*(y) \\ & & g(v,y) \geq 0 \quad \mid \mu \end{array}$$

where $S^*(y)$ is the set of solutions of the problem for the parameters of the problem fixed to y. $V(y)$ is known as the value function of problem (P).

$$\boxed{L(v,y,\lambda,\mu) = f - \lambda^\top h - \mu^\top g}$$

The Kuhn-Tucker set is then defined as:

$$K(v,y) \triangleq \{(\lambda,\mu) \in R^p \times R^q \mid \text{verifying the K-T conditions}\}$$

$K(v,y)$ is a polyhedron. Now, for fixed v consider the set: $\boxed{KT(y) \triangleq \bigcup_{v\in S^*(y)} K(v,y)}$

Assume f continuous and differentiable on $\mathsf{R}^n \times \{y\}$

Let $v^* \in S(\bar{y})$ be a local optimum of problem (P) for $y = \bar{y}$ and assume that

$$\text{(H)} \qquad K(v^*,y) \neq \emptyset \text{ compact and convex } \forall y \in E(\bar{y}) \;\; (\text{a neighborhood of } \bar{y})$$

<u>**Result**</u> **1** (Gauvin-Dubeau's formula) If problem (P) is convex on $E(\bar{y})$ and either 1 and 2 or 3 hold, then $\partial\, V(y)$ can be evaluated by formula (1) :

1. the solution set of problem (P) at $\bar{y}$ is a singleton: $S(\bar{y}) = \{\bar{v}\}$
2. At $(\bar{v},\ \bar{y})$ condition (H) holds (i.e., $K(\bar{v},\bar{y}) \neq \emptyset$ is compact and convex)
3. $f(.,.),\ g(.,.)$ are convex functions and h is affin.

$$\partial\, V(y) = \mathsf{Hull}\Big(\bigcup_{(\lambda,\mu)\in KT(\bar{y})} \nabla_y\, L(\bar{v},\bar{y},\lambda,\mu) \Big) \tag{1}$$

<u>**Result**</u> **2**: Let Y be a convex set and consider the following problem with value function $p_A(y)$ defined on Y. Let also X be a convex set.

$$p_A(y) = Min_{\,x\in X} \quad \begin{array}{l} f(x) \\ Ax = y \end{array} \quad \mapsto P_A$$

assume also that f is a convex function on $\mathcal{F} + \varepsilon\, B$,

$$\text{where} \qquad \mathcal{F} = \{\, x \mid \exists y \in Y,\ Ax = y \,\} = \mathsf{Imf}\, A(y)$$

$$(\ A(y) = \{ \ x \in R^n \mid Ax = y \ \} \ : \ Y \mapsto R^n)$$

Then, if Lips $f = \hat{k} < +\infty$, on $\mathcal{F} + \varepsilon \ B$, the value function $p_A(y)$ is convex and the solution set $P_A(y)$ is convex.